Name: Solutions Score: 80

Linear Algebra **Test 2b** March 1, 2024

> **Instructions:** You may not use your calculator or textbook on this test, but you may use one 3×5 inch notecard of hand-written notes. Always show your work and justify your answers. Credit will be given only if your work is clear. Circle your final answers. Good luck! 80 points

1. (5 points) Is $\mathbf{w} = (4,4,1)$ a linear combination of the vectors $\mathbf{v_1} = (1,1,0)$, $\mathbf{v_2} = (1,2,0)$, and $\mathbf{v_3} = (3,5,1)$? YES or NO (circle one). Show work and justify your answer.

$$(4,4,1) = a(1,1,0) + b(1,2,0) + c(3,5,1)$$

$$\Rightarrow \left.\begin{aligned} a + b + 3c &= 4 \\ a + 2b + 5c &= 4 \\ c &= 1 \end{aligned}\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{-R1+R2 \to R2} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{aligned} a &= 3 \\ b &= -2 \\ c &= 1 \end{aligned}$$

$\Rightarrow$ Yes! $\underline{\bar{w} = 3\bar{v}_1 - 2\bar{v}_2 + v_3}$

2. (6 points) Complete the following definitions using mathematically precise language:

   (a) **Span:** Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ be a nonempty set of vectors in a vector space V. Then S is said to *span* V if ......

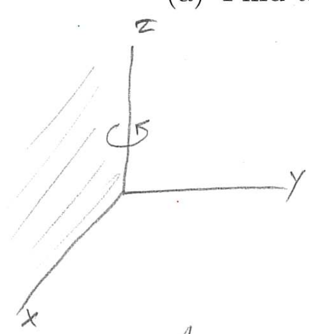
   Every vector in V can be written as a linear combination of the vectors in S

   (b) **Null Space:** Let A be an $n \times m$ matrix. Then the *null space of* A is ...

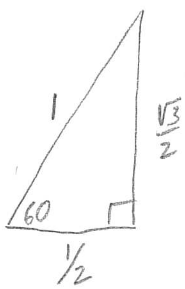
   $NS(A) = \{\bar{x} \in \mathbb{R}^m \mid A\bar{x} = \bar{0}\}$

3. (14 points) Consider the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ that orthogonally projects that vector onto the xz-plane, then dilates that vector by a factor of 2, and finally rotates that vector $60°$ counterclockwise about the z-axis.

(a) Find the standard matrix for T.



$$[T] = \overbrace{\begin{bmatrix} 1/2 & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\text{rotation}} \overbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}^{\text{dilation}} \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\text{projection}}$$



$$= \begin{bmatrix} 1/2 & -\frac{\sqrt{3}}{2} & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(b) Find $T(2, 1, 0)$. Show work.

$$= \begin{bmatrix} 1 & 0 & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2\sqrt{3} \\ 0 \end{bmatrix}$$

(c) Is T one-to-one? Explain your reasoning.

$\det([T]) = 0 \Rightarrow [T]$ is not invertible $\Rightarrow$ $\boxed{\text{NO: } T \text{ is NOT one-to-one}}$

We also know T is not one-to-one because it involves a p

(d) Find a nonzero vector $\mathbf{x}$ in R^3 for which $T(\mathbf{x}) = (0, 0, 0)$, or explain why no such vector $\mathbf{x}$ can exist.

T is NOT one-to-one, so there are infinitely many such vectors $\bar{x} \in \mathbb{R}^3$. Notice:

$$\begin{bmatrix} 1 & 0 & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$ so we can take $\boxed{\bar{x} = (0,1,0)}$

8. (16 points) Throughout this problem, let A and U be the following two matrices:

$$A = \begin{bmatrix} 1 & 1 & -3 \\ -3 & -2 & 4 \\ 3 & 0 & 6 \\ 2 & 1 & -1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}$$

Then U is the reduced row-echelon form of the matrix A. You may assume this fact.

(a) Fill in the blanks: The rank of A is 2. The nullity of A is 1.

(b) Find a basis for the column space of A.

$\{(1,-3,3,2),(1,-2,0,1)\}$

(c) Find a basis for the row space of A.

$\{(1,0,2),(0,1,-5)\}$

(d) Find a basis for the null space of A.

Solve $U\bar{x} = \bar{0}$

$$\bar{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ 5t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} \Rightarrow \text{Basis} = \boxed{\{(-2,5,1)\}}$$

(e) Are the column vectors of A linearly independent? YES or NO (circle one). State reason.

Column 3 = 2(Column 1) − 5(column 2)

The linear relationships of the columns of U also hold for the columns of A.

(f) Find a basis for the range of the linear transformation $T_A : R^3 \to R^4$ induced by A.

$\text{range}(T_A) = CS(A)$ Basis $= \{(1,-3,3,2),(1,-2,0,1)\}$

(g) Describe the range of T_A geometrically.

Plane in $\mathbb{R}^4$ that contains the basis vectors in (F).

(h) Describe the kernel of T_A geometrically (recall that the kernel is the set of all vectors $\mathbf{x}$ such that $T_A(\mathbf{x}) = (0, 0, 0, 0)$.

$\text{kernel}(T_A) = NS(A) \Rightarrow \text{kernel}(T_A)$ is the line in $\mathbb{R}^3$ spanned by $(-2,5,1)$

(i) Notice that $T_A(1, 1, 1) = (-2, -1, 9, 2)$. Find the general form of a vector $\mathbf{v} \in \mathbb{R}^3$ that $T_A(\mathbf{v}) = (-2, -1, 9, 2)$. You should just be able to write down the answer – no additional work is needed.

$\bar{v} = (1,1,1) + t(-2,5,1)$, t is any real number.

5. (10 points) Let W consist of all matrices in M_{22} of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a = 0$ and $d = 2b + 3c$.

(a) Carefully show that W is a subspace of M_{22}.

Use the Subspace Theorem. Let $k \in \mathbb{R}$ and $\bar{u}, \bar{v} \in W$. Then

$$\bar{u} = \begin{bmatrix} 0 & b \\ c & 2b+3c \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 0 & f \\ g & 2f+3g \end{bmatrix}$$

for some $b, c, f, g \in \mathbb{R}$. Thus,

a) $\bar{u} + \bar{v} = \begin{bmatrix} 0 & b+f \\ c+g & 2(b+f)+3(c+g) \end{bmatrix} \underline{\underline{\in W}}$ correct form

b) $k\bar{u} = \begin{bmatrix} 0 & kb \\ kc & 2(kb)+3(kc) \end{bmatrix} \underline{\underline{\in W}}$ correct form

Thus, both properties of the Subspace Theorem hold and W is a subspace of M_{22}.

(b) Find a basis for W.

$$\begin{bmatrix} 0 & b \\ c & 2b+3c \end{bmatrix} = b\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$$

Basis for W: $\boxed{\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix} \right\}}$

(c) Fill in the blank: The dimension of W is 2.

6. (4 points) Fill in the blank: The vector space M_{23} of all 2×3 matrices has dimension 6. Use the definition of *dimension* to justify your answer.

The dimension of a vector space is the number of elements in a basis. The following is a basis for M_{23} that consists of 6 elements:

$$B = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

7. (13 points) Let $B = \{1 + 2x + 3x^2,\ x + 3x^2,\ -2 + 4x^2\}$.

(a) Is B a linearly independent set in P_2? YES or NO (circle one). Explain.

Use the test for LI/LD:

$$a(1+2x+3x^2)+b(x+3x^2)+c(-2+4x^2)=0+0x+0x^2$$

$$\Rightarrow \left.\begin{array}{l} a \quad\quad\quad -2c = 0 \\ 2a + b \quad\quad\;\; = 0 \\ 3a + 3b + 4c = 0 \end{array}\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 3 & 4 & 0 \end{array}\right]$$

$$\xrightarrow[-3R1+R3\to R3]{-2R1+R2\to R2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 3 & 10 & 0 \end{array}\right] \xrightarrow{-3R2+R3\to R3} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$\Rightarrow a=b=c=0$ is the only solution.

$\Rightarrow$ $\boxed{\text{YES, } B \text{ is L.I.}}$

(b) Does B span P_2? YES or NO (circle one). Explain.

$\dim(P_2) = 3$ since $\{1, x, x^2\}$ is a basis for P_2.

Thus any 3 linearly indep polynomials in P_2 are a basis for P_2 and so span P_2.

$\Rightarrow$ $\boxed{B \text{ spans } P_2}$ since it consists of 3 linearly independent polynomials by part (a).

(c) Is B a basis for P_2? YES or NO (circle one). Explain.

A basis for P_2 must span P_2 and be L.I. but set B satisfies both properties, so B is a basis for P_2.

8. (5 points) Let $\mathbf{v_1} = \sin^2 x$, $\mathbf{v_2} = 3\cos^2 x$, and $\mathbf{v_3} = 6$. Is $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ a linearly independent set in $F(-\infty, \infty)$? YES or NO (circle one). Explain.

$\cos^2 x + \sin^2 x = 1$

$\Rightarrow 6\cos^2 x + 6\sin^2 x = 6$

$\Rightarrow 2\bar{v}_2 + 6\bar{v}_1 = \bar{v}_3$

$\Rightarrow$ $\bar{v}_3$ is a linear combo of v_1 and v_2

$\Rightarrow$ S is a linearly dependent set.

9. (7 points) **Multiple Choice.** Question (a) has exactly one correct answer, but question (b) may have more than one correct answer. Circle all the correct answers. You do not need to justify your reasoning.

(a) Which of the following describes the span of $\{(1,0,-3), \underbrace{(-3,0,9), (5,0,-15)}_{\text{all on the same line}}\}$?

$-3(1,0,-3)$ $5(1,0,-3)$

i. ~~The zero vector~~ $(0,0,0)$ in $\mathbb{R}^3$.

ii. ~~Three vectors~~ in $\mathbb{R}^3$.

iii. A single line in $\mathbb{R}^3$.

iv. ~~Three lines~~ in $\mathbb{R}^3$.

v. ~~A plane in~~ $\mathbb{R}^3$.

vi. ~~Three planes~~ in $\mathbb{R}^3$.

vii. ~~All of~~ $\mathbb{R}^3$.

(b) Which of the following sets S are subspaces of the given vector spaces V?

i. S is the set of vectors in the first quadrant of $\mathbb{R}^2$ (i.e., S is the set of vectors of the form (a,b) where $a \geq 0$, $b \geq 0$), $V = \mathbb{R}^2$. Not closed under scalar mult X

ii. $S = \mathbb{R}^2$, $V = \mathbb{R}^3$. $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$ X

iii. $S = P_2$, $V = P_3$.

iv. S is the set of invertible 2×2 matrices, $V = M_{22}$. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin S$ X

v. None of the above sets S is a subspace of the given vector space V.