Name: Solutions Score: 80

Linear Algebra Test 2a March 1, 2024

Instructions: You may not use your calculator or textbook on this test, but you may use one 3 × 5 inch notecard of hand-written notes. Always show your work and justify your answers. Credit will be given only if your work is clear. Circle your final answers. Good luck! 80 points

1. (6 points) Complete the following definitions using mathematically precise language:

   (a) **Subspace:** A subset W of a vector space V is called a *subspace* of V if ......

   W is a vector space under the same operations of addition and scalar mult defined on V.

   (b) **Linear Combination:**. If $\mathbf{w}$ is a vector in a vector space V, then $\mathbf{w}$ is said to be a *linear combination* of the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ in V if ......

   $\overline{W} = k_1\overline{V_1} + k_2\overline{V_2} + \cdots + k_r\overline{V_r}$

   for some scalars $k_1, k_2, \ldots, k_r \in \mathbb{R}$.

2. (5 points) Do the vectors $\mathbf{v_1} = (1, 0, 1)$, $\mathbf{v_2} = (2, -1, 0)$, and $\mathbf{v_3} = (3, 2, 1)$ span $\mathbb{R}^3$? Show work and justify your answer.

$$k_1(1,0,1) + k_2(2,-1,0) + k_3(3,2,1) = (a,b,c)$$

$$\Rightarrow \left.\begin{array}{r} k_1 + 2k_2 + 3k_3 = a \\ -k_2 + 2k_3 = b \\ k_1 \qquad + k_3 = c \end{array}\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & -1 & 2 & b \\ 1 & 0 & 1 & c \end{array}\right] \xrightarrow[-R2 \to R_2]{-R1+R3} \left[\begin{array}{cccc} 1 & 2 & 3 & a \\ 0 & 1 & -2 & -b \\ 0 & -2 & -2 & -a+c \end{array}\right]$$

$$\xrightarrow{2R2+R3 \to R3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & -2 & b \\ 0 & 0 & -6 & -a-2b+c \end{array}\right]$$

Unique solution for <u>all</u> a, b, c

$\Rightarrow \boxed{\text{YES}}$ The vectors span $\mathbb{R}_3$

3. (10 points) Let W consist of all matrices in M_{22} of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $b = 0$ and $d = 3a - 2c$.

(a) Carefully show that W is a subspace of M_{22}.

Use the Subspace Theorem. Let $k \in \mathbb{R}$ and $\bar{u}, \bar{v} \in W$. Then

$$\bar{u} = \begin{bmatrix} a & 0 \\ c & 3a-2c \end{bmatrix} \text{ and } \bar{v} = \begin{bmatrix} e & 0 \\ g & 3e-2g \end{bmatrix}, \text{ for some } a, c, e, g \in \mathbb{R}.$$

$$a)\ \bar{u} + \bar{v} = \begin{bmatrix} a+e & 0 \\ c+g & 3(a+e)-2(c+g) \end{bmatrix} \underline{\underline{\in W}} \quad \underline{\text{correct form}}$$

$$b)\ k\bar{u} = \begin{bmatrix} ka & 0 \\ ka & 3(ka)-2(kc) \end{bmatrix} \underline{\underline{\in W}} \quad \underline{\text{correct form}}$$

$\Rightarrow$ Both properties of the Subspace Theorem hold.

$\Rightarrow$ W is a subspace of M_{22}.

(b) Find a basis for W.

$$\begin{bmatrix} a & 0 \\ c & 3a-2c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}$$

$$\Rightarrow \text{Basis for } W = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \right\}$$

(c) Fill in the blank: The dimension of W is ___2___.

4. (4 points) Fill in the blank: The vector space P_4 of all polynomials of degree 4 or less has dimension ___5___. Use the definition of *dimension* to justify your answer.

$\{1, x, x^2, x^3, x^4\}$ is a basis for P_4. Thus, P_4 has a basis that consists of $\underline{\underline{5}}$ elements. By the def of dimension, $\dim(P_4) = 5$.

5. (13 points) Let $B = \{1 + 2x + 3x^2,\ x + 3x^2,\ -2 + 4x^2\}$.

(a) Is B a linearly independent set in P_2? YES or NO (circle one). Explain.

$a(1+2x+3x^2) + b(x+3x^2) + c(-2+4x^2) = 0 + 0x + 0x^2$

$$\Rightarrow \left.\begin{aligned} a \quad\;\; - 2c &= 0 \\ 2a + b \quad\;\; &= 0 \\ 3a + 3b + 4c &= 0 \end{aligned}\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 3 & 4 & 0 \end{array}\right]$$

$$\xrightarrow[-3R1+R3 \to R3]{-2R1+R2 \to R2} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 3 & 10 & 0 \end{bmatrix} \xrightarrow{-3R2+R3 \to R3} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$\Rightarrow a = b = c = 0$ is the <u>only</u> solution

$\Rightarrow$ $\boxed{\text{YES, } B \text{ is L.I}}$

(b) Does B span P_2? YES or NO (circle one). Explain.

$\dim(P_2) = 3$ since $\{1, x, x^2\}$ is a basis for P_2.

Thus, any 3 linearly independent polynomials in P_2 are a basis for P_2 and so span P_2.

$\Rightarrow B$ spans P_2 since it consists of 3 linearly independent polynomials

(c) Is B a basis for P_2? YES or NO (circle one). Explain.

A basis for P_2 must span P_2 and be L.I.

B satisfies both properties.

6. (7 points) **Multiple Choice.** Question (a) has exactly one correct answer, but question (b) may have more than one correct answer. Circle all the correct answers. You do not need to justify your reasoning.

$3(-1,0,2)$

$\parallel$

(a) Which of the following describes the span of $\{(-1,0,2), (-3,0,6), (0,1,1)\}$?

i. The zero vector $(0,0,0)$ in $\mathbb{R}^3$.

ii. A single line in $\mathbb{R}^3$.

iii. Two lines in $\mathbb{R}^3$.

iv. Three lines in $\mathbb{R}^3$.

v. A plane in $\mathbb{R}^3$.

vi. All of $\mathbb{R}^3$.

(b) Which of the following sets S are subspaces of the given vector spaces V?

i. S is the set of vectors in the first quadrant of $\mathbb{R}^2$ (i.e., S is the set of vectors of the form (a,b) where $a \geq 0$, $b \geq 0$), $V = \mathbb{R}^2$. Not closed under scalar mult X

ii. $S = \mathbb{R}^2$, $V = \mathbb{R}^3$. $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$ X

iii. $S = P_2$, $V = P_3$.

iv. S is the set of invertible 2×2 matrices, $V = M_{22}$. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin S$ X

v. None of the above sets S is a subspace of the given vector space V.

7. (5 points) Let $\mathbf{v_1} = 10$, $\mathbf{v_2} = \sin^2 x$, and $\mathbf{v_3} = 5\cos^2 x$. Is $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ a linearly independent set in $F(-\infty, \infty)$? YES or NO (circle one). Explain.

$\sin^2 x + \cos^2 x = 1$

$\Rightarrow 10\sin^2 x + 10\cos^2 x = 10$

$\Rightarrow 10\bar{v}_2 + 2\bar{v}_3 = \bar{v}_1$

$\Rightarrow \bar{v}_1$ is a linear combo of $\bar{v}_2$ and $\bar{v}_3$

$\Rightarrow \boxed{\text{NO}}$ S is linearly dependent

8. (16 points) Throughout this problem, let A and U be the following two matrices:

$$A = \begin{bmatrix} 1 & 1 & -3 \\ -3 & -2 & 4 \\ 3 & 0 & 6 \\ 2 & 1 & -1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}$$

Then U is the reduced row-echelon form of the matrix A. You may assume this fact.

(a) Fill in the blanks: The rank of A is ___2___. The nullity of A is ___1___.

(b) Find a basis for the column space of A.

$\{(1,-3,3,2),(1,-2,0,1)\}$

(c) Find a basis for the row space of A.

$\{(1,0,2),(0,1,-5)\}$

(d) Find a basis for the null space of A.

Solve $U\bar{x}=\bar{0}$

$$\bar{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ 5t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} \Rightarrow \text{Basis} = \boxed{\{(-2,5,1)\}}$$

(e) Are the column vectors of A linearly independent? YES or NO (circle one). State reason.

Column 3 = 2(Column 1) − 5(column 2)

The linear relationships of the columns of U also hold for the columns of A.

(f) Find a basis for the range of the linear transformation $T_A : R^3 \to R^4$ induced by A.

$\text{range}(T_A) = CS(A)$ Basis $= \{(1,-3,3,2),(1,-2,0,1)\}$

(g) Describe the range of T_A geometrically.

Plane in $\mathbb{R}^4$ that contains the basis vectors in (f).

(h) Describe the kernel of T_A geometrically (recall that the kernel is the set of all vectors $\mathbf{x}$ such that $T_A(\mathbf{x}) = (0,0,0,0)$.

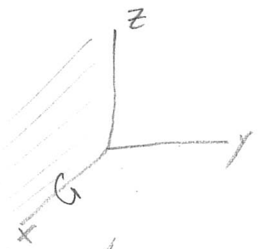
$\text{Kernel}(T_A) = NS(A) \Rightarrow \text{kernel}(T_A)$ is the line in $\mathbb{R}^3$ spanned by $(-2,5,1)$

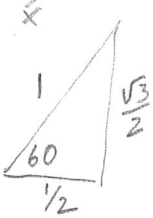
(i) Notice that $T_A(1,1,1) = (-2,-1,9,2)$. Find the general form of a vector $\mathbf{v} \in \mathbb{R}^3$ that $T_A(\mathbf{v}) = (-2,-1,9,2)$. You should just be able to write down the answer – no additional work is needed.

$\bar{v} = (1,1,1) + t(-2,5,1)$, t is any real number.

9. (14 points) Consider the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates a vector $60°$ counterclockwise about the x-axis, then reflects that vector about the xz-plane, and finally dilates that vector by a factor of 2.

(a) Find the standard matrix for T.





$$[T] = \underset{\text{dilation}}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}} \underset{\text{reflection}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{\text{rotation}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & \sqrt{3} \\ 0 & \sqrt{3} & 1 \end{bmatrix}$$

(b) Find $T(0, 1, 2)$. Show work.

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & \sqrt{3} \\ 0 & \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1+2\sqrt{3} \\ \sqrt{3}+2 \end{bmatrix}$$

(c) Is T one-to-one? Explain your reasoning.

$\boxed{\text{Yes}}$ T is one-to-one since it is a product of one-to-one linear operators. You can also check that $[T]$ is invertible:

$$\det([T]) = 2\begin{vmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{vmatrix} = 2(-1-3) = -8 \neq 0 \checkmark$$

(d) Find two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ such that $\mathbf{u} \neq \mathbf{v}$ and $T(\mathbf{u}) = T(\mathbf{v})$, or explain why you know two such vectors do not exist.

No two vectors exist by the def of one-to-one.