Name: Solutions Score: 80

Linear Algebra **Test 1b** February 2, 2024

**Instructions:** You may not use your calculator or textbook on this test, but you may use one $3 \times 5$ inch notecard of hand-written notes. Always show your work and justify your answers. Credit will be given only if your work is clear. Circle your final answers. Good luck! 80 points

1. (8 points) In each part, suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given matrix in row echelon form. Solve the system.

(a)
$$\begin{array}{ccccc} x & y & z & w & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & 6 & 2 \\ 0 & 1 & -2 & 5 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Unique solution: $\boxed{\begin{array}{l} x = 4 \\ y = 2 \\ z = 1 \\ w = 0 \end{array}}$

(b)
$$\begin{array}{ccccc} x & y & z & w & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 0 & 5 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Infinitely many solutions: (2 free variables)

$\boxed{\begin{array}{l} x = 2s + 3t - 16 \\ y = s \\ z = -t + 7 \\ w = t \end{array}}$

2. (6 points) Use the given information to find $A$. (Recall that $A^T$ denotes the transpose of $A$.)

$$(3A^T)^{-1} = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}.$$

$$\Rightarrow 3A^T = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -6 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

$$\Rightarrow A = \left(\frac{1}{3}\begin{bmatrix} 2 & -3 \\ -\frac{1}{2} & 1 \end{bmatrix}\right)^T = \begin{bmatrix} 2/3 & -1/6 \\ -1 & 1/3 \end{bmatrix}$$

3. (7 points) Simplify the following expression assuming that $A$, $B$, and $C$ are invertible $n \times n$ matrices. Simplify your answer as much as possible for full credit.

$$(7AB^{-1}C)^{-1}\left((AC^3)^{-1}\right)^{-1}(AC^2)^{-1}B^T(A^{-1}B^T)^{-1}$$

$$= C^{-1}BA^{-1}\left(\tfrac{1}{7}\right)A\underbrace{C^3C^{-2}}_{C}A^{-1}\underbrace{B^T(B^T)^{-1}}_{I}A$$

$$= \tfrac{1}{7}C^{-1}BC\underbrace{A^{-1}A}$$

$$= \boxed{\tfrac{1}{7}C^{-1}BC}$$

Remember:

1. Order matters with matrix multiplication $AB \neq BA$ (in general)
2. Reverse the order when taking the inverse of a product of matrices: $(AB)^{-1} = B^{-1}A^{-1}$

4. (7 points) Find the inverse of $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$, or show that $A$ is not invertible.

$$\begin{array}{cc} A & I \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R1+R3\to R3]{R2\leftrightarrow R3} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 2 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-2R2+R3\to R3} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & -2 \end{array}\right] \xrightarrow[-2R2+R1\to R1]{R3+R2\to R2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 5 & -2 & 4 \\ 0 & 1 & 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R2+R1\to R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 3 \\ 0 & 1 & 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 & -2 \end{array}\right]$$

$$\begin{array}{cc} I & A^{-1} \end{array}$$

$$\Longrightarrow A^{-1} = \begin{bmatrix} 3 & -1 & 3 \\ -2 & 1 & -1 \\ -2 & 1 & -2 \end{bmatrix}$$

check

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 3 \\ -2 & 1 & -1 \\ -2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

5. (9 points) Let $A = \begin{bmatrix} 0 & 3 & 0 & -3 \\ 2 & 0 & 0 & -1 \\ -4 & 0 & 2 & 0 \\ 0 & 2 & -3 & 1 \end{bmatrix}$.

I will expand along the $1^{st}$ row

(a) Evaluate the determinant of the matrix $A$ by the method of cofactor expansion.

$$\det(A) = -3 \det\begin{bmatrix} 2 & 0 & -1 \\ -4 & 2 & 0 \\ 0 & -3 & 1 \end{bmatrix} - (-3)\det\begin{bmatrix} 2 & 0 & 0 \\ -4 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix}$$

$$= -3\left(-\det\begin{bmatrix} -4 & 2 \\ 0 & -3 \end{bmatrix} + \det\begin{bmatrix} 2 & 0 \\ -4 & 2 \end{bmatrix}\right) + 3(2)\det\begin{bmatrix} 0 & 2 \\ 2 & -3 \end{bmatrix}$$

$$= -3(-12+4) + 6(-4)$$

$$= 24 - 24$$

$$= \boxed{0}$$

(b) Is $A$ invertible? YES or NO (circle one). Use your answer to (a) to explain your reasoning.

By the equivalent conditions:

$\det(A) = 0 \Longleftrightarrow A$ is singular

6. (10 points) Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, and suppose that $\det(A) = 5$. Use this information to find the determinants below. Note that your answers should be numbers.

(a) Find $\det(2A^{-1})$.

Since $A$ is $3\times 3$, $\det(2A^{-1}) = 2^3\left(\frac{1}{\det A}\right) = \boxed{\frac{8}{5}}$

(b) Let $B = \begin{bmatrix} 2d & 2e & 2f \\ a & b & c \\ g-3a & h-3b & i-3c \end{bmatrix}$. Find $\det(B)$.

$R1 \leftrightarrow R2$

$$\det(B) = -\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g-3a & h-3b & i-3c \end{vmatrix}$$

$3R1 + R3 \to R3$

$$= -\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix}$$

$$= -2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -2\det(A) = -2(5) = \boxed{-10}$$

7. (5 points) Decide whether the statement is **true** or **false**. If **true**, give a short explanation that shows it is true. If **false**, give an explanation or counterexample that shows it is false.

(True or False?) If $A$ is a $3\times 3$ matrix such that $A\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then $A$ is the $3\times 3$ zero matrix (*i.e.* every entry of $A$ is equal to 0).

Here the trivial solution $\bar{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not the only solution to $A\bar{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ since $\bar{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is also a solution. Thus, by the equivalent conditions, we know that $A$ is singular but it does not have to be the zero matrix. For example, if $A = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ then $A\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

8. (8 points) Let $\mathbf{v} = (2, -2, -1)$.

(a) Find a unit vector that is perpendicular to $\mathbf{v}$.

$\bar{w} = (1, 1, 0)$ is perpendicular to $\bar{v}$ since $\bar{w} \cdot \bar{v} = 0$

Scale down to get a unit vector.

$$\bar{u} = \frac{1}{\|\bar{w}\|} \cdot \bar{w} = \frac{1}{\sqrt{2}} \bar{w} = \boxed{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}$$

Note: There are infinitely many correct answers.

(b) Find the vector of length 15 that is oppositely directed to $\mathbf{v}$.

$$\|\bar{v}\| = \sqrt{2^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3$$

$$\Longrightarrow \bar{z} = -5\bar{v} = \boxed{(-10, 10, 5)}$$

9. (7 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the matrix (linear) transformation defined by

$$T(x, y, z) = (-2y, x + 3z).$$

(a) Find the standard matrix for $T$.

$$[T] = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

(b) Recall the following theorem:

> **Theorem:** $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix (linear) transformation if and only if the following properties hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and for every scalar $k$:
> (i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
> (ii) $T(k\mathbf{u}) = kT(\mathbf{u})$

Carefully show that **Property (i)** in the theorem above holds for $T$. Note that you do not need to check Property (ii).

Let $k \in \mathbb{R}$ and $\bar{u} = (a, b, c)$, $\bar{v} = (d, e, f)$. Then

$$T(\bar{u} + \bar{v}) = T(a+d, b+e, c+f) = (-2(b+e), a+d+3(c+f))$$

$$\begin{aligned} T(\bar{u}) + T(\bar{v}) &= (-2b, a+3c) + (-2e, d+3f) \\ &= (-2b-2e, a+3c+d+3f) \\ &= (-2(b+e), a+d+3(c+f)) \end{aligned}$$

SAME!

$\Rightarrow$ Property (i) holds.

10. (13 points) (a) Find the conditions on $a$, $b$, $c$, $d$, if any, that guarantee the following linear system is consistent.

$$\begin{aligned} x + 2y + 3z &= a \\ -3x - 6y - 9z &= b \\ x + 2y + 2z &= c \\ 2x + 4y + 8z &= d \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ -3 & -6 & -9 & b \\ 1 & 2 & 2 & c \\ 2 & 4 & 8 & d \end{array}\right] \xrightarrow{\substack{3R1+R2 \to R2 \\ -R1+R3 \to R3 \\ -2R1+24 \to R4}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 0 & 0 & 3a+b \\ 0 & 0 & -1 & -a+c \\ 0 & 0 & 2 & -2a+d \end{array}\right]$$

$$\xrightarrow{\substack{2R3+R4 \to R4 \\ -R3 \to R3 \\ R2 \leftrightarrow R3}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 0 & 1 & a-c \\ 0 & 0 & 0 & 3a+b \\ 0 & 0 & 0 & -4a+2c+d \end{array}\right]$$

The system is consistent $\iff$ $3a+b=0$, $-4a+2c+d=0$

$$\iff \boxed{\begin{array}{l} b = -3a \\ d = 4a-2c \\ \text{No constraints on } a \text{ and } c \end{array}}$$

(b) Note that here we are using the same equation as in part (a). Find the domain, codomain, and range of the linear transformation $T$ defined by

$$T(x, y, z) = (x + 2y + 3z, -3x - 6y - 9z, x + 2y + 2z, 2x + 4y + 8z).$$

Domain of $T$ = $\mathbb{R}^3$ Codomain of $T$ = $\mathbb{R}^4$

Range of $T$ = $\{(a,b,c,d) \mid b=-3a, d=4a-2c\}$

= All vectors in $\mathbb{R}^4$ of the form $(a, -3a, c, 4a-2c)$