INVARIANTS ON FAMILIES OF KNOT DIAGRAMS

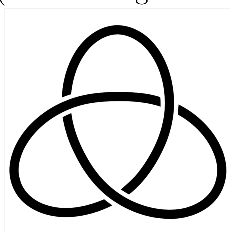
A. LOS

Contents

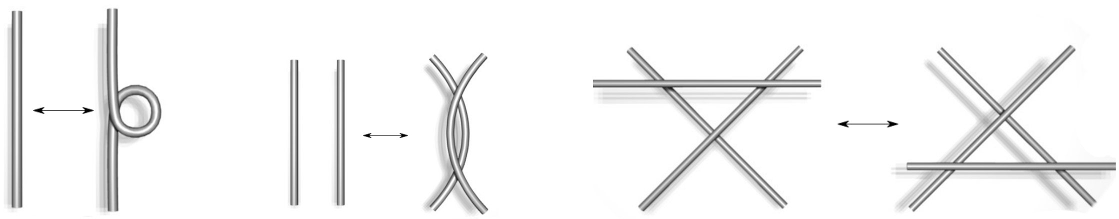
1. Abstract

In this project we study a new way to look at knots, following recent work by Thomas Fiedler [Fie20]. We look at the space of embeddings of the circle in $\mathbb{R}^3$ and paths in that space and investigate integrals along these paths. We aim to survey the theory, investigate its validity and look for applications.

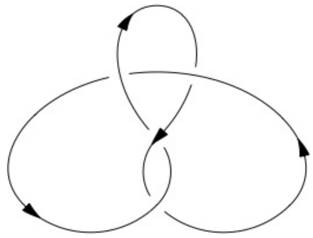
2. Background about Knots

A **Diagram** (or knot diagram or projection) of a knot or link is a two-dimensional representation of the link. 

In manipulating diagrams of a knot we only need the three **Reidemeister moves**. These moves transform an old diagram into a new diagram of the same knot as in the following pictures. The first Reidemeister move (RI) creates or eliminates a curl or loop or twist in a strand. The second Reidemeister move (RII) moves one of two parallel strands completely over the other. The third Reidemeister move (RIII) moves a strand completely over or under a crossing.

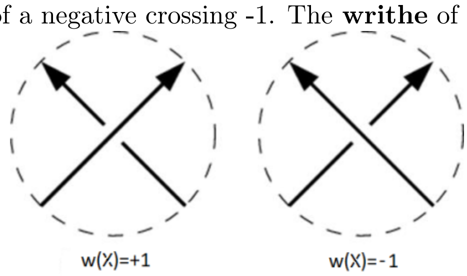


Later on we will talk about special diagrams with a **triple crossing**. By that we mean a crossing where three parts of the knot are projected onto the same point, instead of two as in a **ordinary crossing** or **double crossing**.



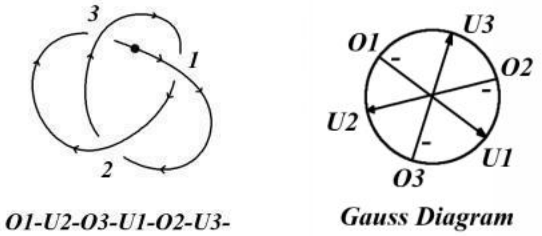
We can give a knot an **orientation**, by choosing a direction of the line.

We can distinct two different oriented crossings in diagrams: A righthanded, or positive, crossing and a lefthanded, or negative, crossing, see the next picture. The **writhe** of an positive crossing is 1 and of a negative crossing -1. The **writhe** of an oriented knot is the sum of the writhes of its crossings.

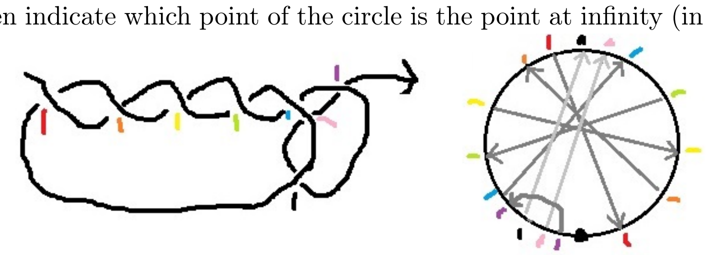


2.1. **Gauss diagrams.** Another way to represent knots. A **Gauss code** of an oriented knot is a sequence of integers indication the order in which we encounter all crossings going along the knot.

With use of this code we can draw a **Gauss diagram** of a knot: circle = knot, chords between points on the circle = crossings. We start at one point on the circle and move counterclockwise. (We associate the different numbers of the Gauss code with different points on the circle in the same order. Then we draw a chord from the point associated to $-n$ to the point n for every n.) We draw arrows from understrand to overstrand, but some authors draw them the other way around (as in this picture).

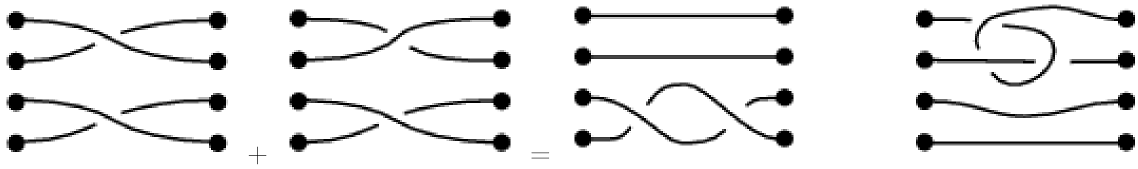


Another way to view knots is as **long knots**. A long knot is knot where we 'blow up' one part of the knot to infinity: It is a line from $-\infty$ to ∞ that is only curved in the interval $[-1, 1]$. Thus it is a circle through infinity. For long knots it is equally possible to draw a Gauss diagram. We then indicate which point of the circle is the point at infinity (in this picture by a thick black dot).



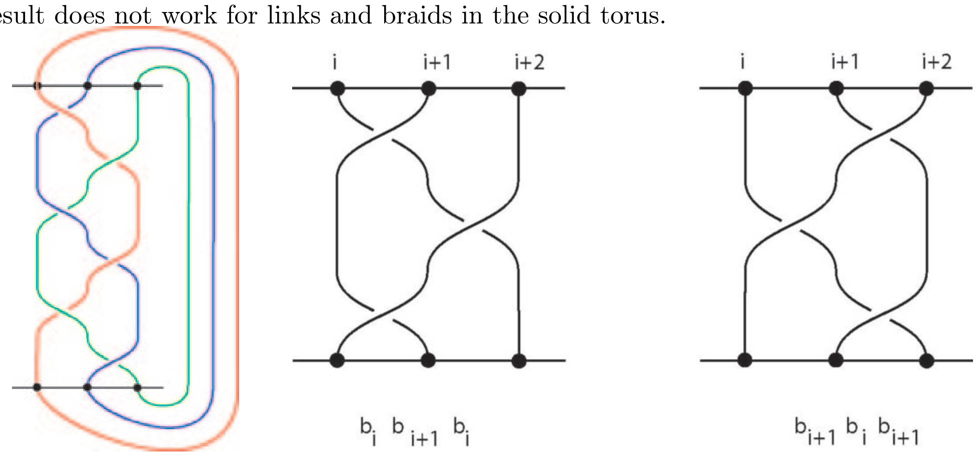
2.2. **Braids.** A **braid** is a set of n strings all of which are attached to a horizontal bar at the top and the bottom. Each string monotonically descends from top to bottom, i.e. each string intersects each horizontal plane between the top and bottom bar exactly once. The endpoints in the top bar should be exactly above the endpoints in the bottom bar. (Or from left to right instead of top to bottom.) The fourth picture below is not an example of a braid. We regard two braids to be equivalent if we can smoothly move the strings of one braid to form the other without moving the endpoints and without passing one strand through another. On equivalence classes of braids we can define the **n-th braid group $\mathbf{B_n}$**. The elemenents are equivalence classes of braids. The operation is the composition of braids, placing them on top of each other. ($\sigma_1\sigma_2$

is obtained by placing σ_2 below σ_1 such that the lower points of σ_1 correspond to the upper points of σ_2.) The identity element is the class of the braid with n straight vertical strings. The inverse of a braid class is the class of braids which undoes what the first class does. See the following pictures. This group is non-commutative for $n \geq 3$. We represent the simplest braid classes by σ_i and σ_i^{-1}, where σ_i is the permutation of strings i and $i+1$ by a lefthanded crossing and σ_i^{-1} is the same permutation by a righthanded crossing. These two are each other's inverse.



The group B_n is generated by $n-1$ elements and their inverses. B_n can be given a finite presentation as follows: $B_n =< \sigma_1, \ldots, \sigma_{n-1} | \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i >$, where $1 \leq i \leq n-2$ and $|i-j| \geq 2$. A **braid word** is a sequence (product) of generating braids (e.g. $w = \sigma_1\sigma_2^{-1}\sigma_1$).

A **closed braid** is constructed if we identify the the top bar with the bottom bar. In this way we obtain a link. It is a theorem (1923) by J.W. Alexander that every link in $\mathbb{R}^3$ can be represented by a closed braid. A link may have many different closed braids as representations. Closed braids represent the same link if they are related by operations of following two types: **conjugation**: premultiplying by σ_i and postmultiplying by σ_i^{-1}, $(w \to \sigma_i w \sigma_i^{-1})$. **stabilisation**: changing an n-braid to an $n+1$-braid by multiplying by σ_n, $(w \to w\sigma_i)$. This representation of links by braids allows us later on to look at all closed braids instead of all links, which is sometimes easier. This result does not work for links and braids in the solid torus.



2.3. **Knot invariants.** A **knot invariant** is a property associated to a knot (often a number or polynomial) that is invariant under isotopies of the knot.

Most invariants are determined by and calculated from a diagram of the knot. The invariant evaluated on different diagrams of the same knot should give the same value. If we can check that the property is preserved under the Reidemeister moves, than all diagrams of a knot have the same property, thus the invariant is well-defined for a knot. Examples of knot invariants:

Number of components. Knots are links of one component. 2-links are links with two components etc.

For (2-)links we can define the **linking number** by associating to each link the number of 'windings' around the other: We look at all the crossing where one component crosses with another component...

The **crossing number** of a link is the minimum number of crossings needed in a diagram.

A **bridge** in a knot diagram is a strand with at least one overcrossing. The **bridge number** of a knot is the minimum number of bridges needed in a diagram.

The **unknotting number** of a knot is the minimum number of times a knot must pass through itself to untie, so the minimum number of times we have to switch a crossing to obtain the unknot.

The **tricolorability** of a knot is the possibility of coloring the strands in some diagram of the knot with three colors such that at every crossing the three strands coming together are either all the same colour or all a different colour.

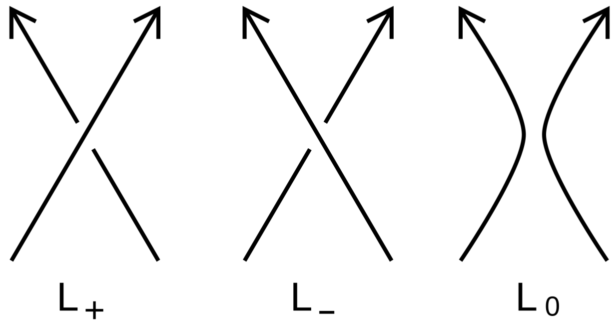
The **stick number** of a knot is the minimum number of straight sticks or lines needed to draw a diagram of the knot.

The **braid index** of a link is the minimum number of strings in a braid corresponding to the closed braid representing the link.

There are different ways to attach a polynomial to a knot (diagram). For example the **Alexander polynomial** or the **Jones polynomial**.

The **HOMFLYPT polynomial**: $P(unknot) = 1$ and $lP(L_+) + l^{-1}P(L_-) + mP(L_0) = 0$

Polynomials use **Skein relations**:

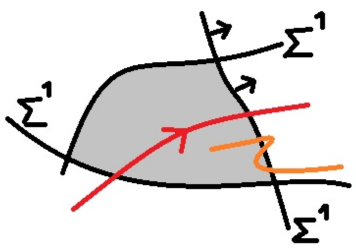


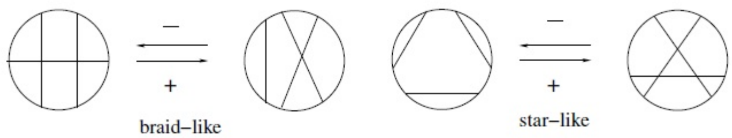
3. New way to look at Knots

3.1. **Introduction.** Idea: (Thomas Fiedler:) Instead of (normal) invariants that associate a value or polynomial to a diagram of a knot we can determine invariants associated to a family of diagrams of a knot. Normally invariants are invariant under different representations. (Or some are associated to a minimum diagram.) By this requirement some info is difficult to extract, e.g. mutant knots are difficult to distinguish (due to commutativity). But loosening that requirement and looking at different (moves between) diagrams, can give more info. So from such a family it can be easier to subtract some information about the knot. We will explain how to construct 1-parameter families/sequences of knot(diagram)s and what invariants on them are interesting.

Knots can be seen as embeddings of the (1-dimensional) circle in 3-space. We can look at all possible embeddings and see these as a space itself, the space of embeddings. Every point in that space corresponds to a specific embedding. A path in that space is a continuous sequence of embeddings. This space is infinite dimensional.

We can also look at the space of all (possibly singular) diagrams of knots. Then every point is a specific diagram and this space is also infinite dimensional. Every knot (type) is represented by a connected part of this space. There are points in the space that are singular diagrams, for example where one of the Reidemeister moves is carried out. The points corresponding to isotopic diagrams form (regular) subspaces of codimension 0, whereas the points corresponding to Reidemeister moves form subspaces of codimension 1. To go from one diagram of a knot to a non-isotopic one, we have to go through/via some singular diagram, so we have to walk a path that crosses at least one of the **singular subspaces** of codimension 1, which we denote by $\Sigma^{(1)}$.



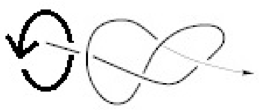


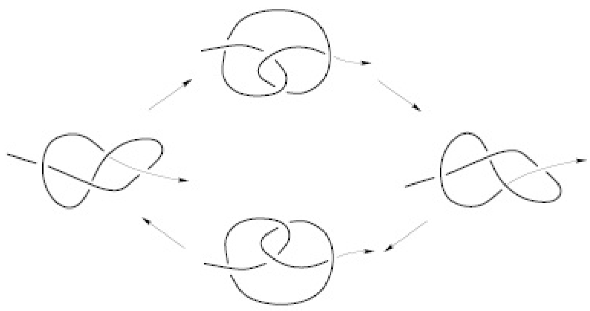
We look at paths in this infinite dimensional space and want to define an invariant on these paths (instead of an invariant on each point). So instead of the requirement that the value has to be invariant under for example a Reidemeister move, we take those moves into account. Of course some paths should be equivalent. Forth and back on the same path should contribute zero. And if we walk a loop, then our invariant may have only an interesting (i.e. nonzero) value if it is a loop around an interesting singularity.

We define the **coorientation** for Reidemeister moves (the black arrows in the picture above): The **sign** of an intersection point p of an oriented arc/loop in the moduli space M with $\Sigma^{(1)}_{tri}$ is defined as an index ± 1 by comparing the orientation of the the arc with the coorientation of $\Sigma^{(1)}_{tri}$. (If they coincide the sign is $+1$, if they are opposite the sign is -1.) See the pictures above for which direction is plus and which direction is minus.

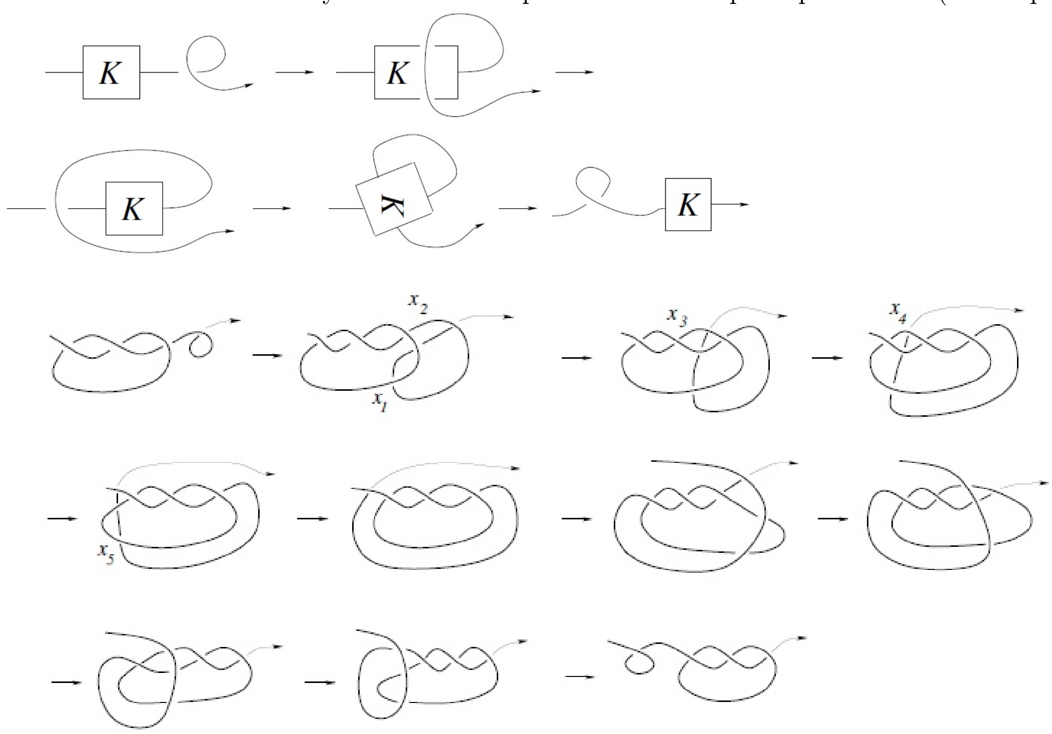
We use singular subspaces up to codimension three in proving invariance. We call these invariants **one-cocycles**.

3.2. **Loops and their representations.** Two examples of loops of diagrams that are/could be interesting are: 1 A loop where we turn around the knot: **Gramain loop**. 2 A loop where we push the knot through itself: **Fox-Hatcher loop**. (Picture 3 is a loop where we push one knot through another.)





But there is an easier way to see these loops. Gramain's loop is equivalent to (mirror picture!):



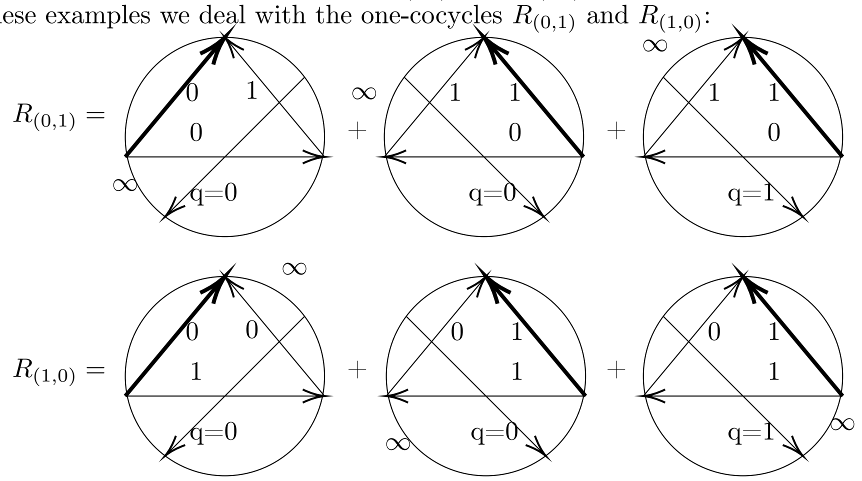
The Fox-Hatcher loop can also be seen as the loop where we take the point at infinity and let it walk over the knot. (So a knot where a point walks one round on the circle.)

4. One-cocycles

As we see in the above pictures, we encounter RII moves and RIII moves in our loop. We will define our invariant one-cocycles as a weighted sum of these moves.

We will just give an example of a one-cocycle and explain its definition.

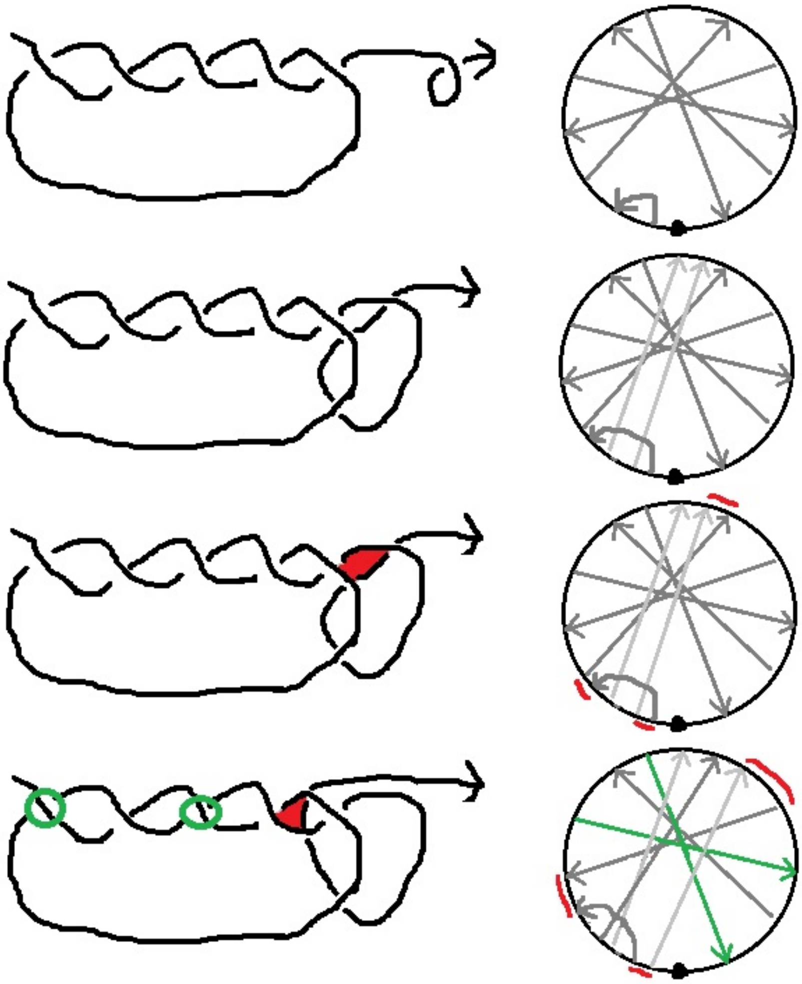
4.1. **Example 1: application of $R_{(0,1)}$ and $R_{(1,0)}$ to rot and fh of $3_1, 4_1, 5_1, 5_2, (2m+1)_1$.** In these examples we deal with the one-cocycles $R_{(0,1)}$ and $R_{(1,0)}$:

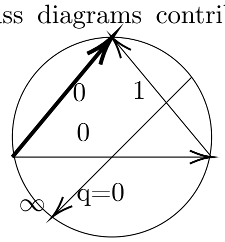


Our cocycles are defined as follows: Take the sum over all diagrams in a loop, for each diagram take the sum over the above configurations. (This is nonzero for only a finite number of points in the loop.) For each occuring configuration multiply the sign of the RIII move with the sign of the fourth crossing. (There are variants we use different weights, but in general we look at which RIII moves occur and consider specific extra arrows in the gauss diagram.) Here the diagram should really be a subdiagram: The whole diagram, without some arrows. The numbers correspond to the 'homological markings' of the arrows, i.e. whether we go along the point at infinity if we move from overstrand to understrand. (But that depends only on the position of the arrow w.r.t. the point at infinity, thus this information occurs twice in the above diagram.)

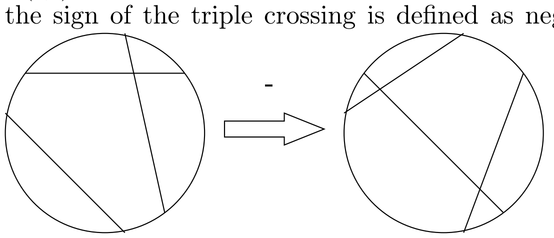
We checked two examples of Fiedler. $R_{(1,0)}(rot(3_1^+)) = 1$. This corresponds with $v_2(3_1)$. And the other is a cocycle on a fox-hatcher loop $R_{(1,0)}(fh(4_1)) = 0$. This corresponds with $6v_3(4_1) - w(k)v_2(4_1)$. Following Arnaud Mortier in [Mor14], his cocycle α_3^1 should have these correspondences (for framed knots?), but that cocycle is Fiedler's cocycle $R_{(0,1)}$. Evaluating on 3_1^+ and 4_1 gives us $R_{(0,1)}(rot(3_1^+)) = -1 = -v_2(3_1)$ and $R_{(1,0)}(fh(4_1)) = 0 = 6v_3(4_1) - w(k)v_2(4_1)$.

We expended the calculation of $R_{(0,1)}$ and $R_{(1,0)}$ to 5_1 and to all $(2,n)$-torus knots with odd $n = 2m+1$, i.e. all knots $(2m+1)_1$. We chose the positive orientation of the knot. First we calculate $rot(2m+1)_1$. This loop is for 5_1 given by the following pictures and corresponding Gauss diagrams.

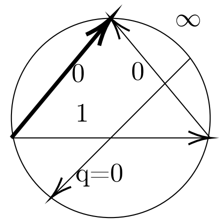


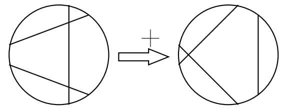
We see that most Gauss diagrams have three zeros (as homological markings) for the the three crossings involved in the RIII move, i.e. the place of the three crossings w.r.t. infinity is not fitting for our cocycles. In the first half of the loop, the other Gauss diagrams contribute to $R_{(0,1)}$ and in the second half of the loop the other Gauss diagrams contribute to $R_{(1,0)}$. For the first half, the contributing diagrams are of the form 

, where the sign of the triple crossing is defined as negative (see our definition of coorientation above):



The writhe of all involved crossings is positive, only the crossing which we indicated in the diagram with a minus has a negative writhe. We see that the number of contributing ordinary crossings decreases from m to 1. This means that $R_{(0,1)}(rot((2m+1)^+_1)) = -m - \ldots - 1 = -m(m+1)/2$.

For the second half, the contributing diagrams are of the form 

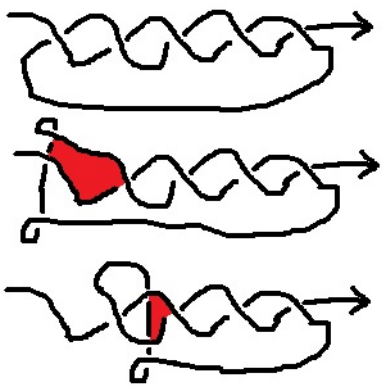
, where the sign of the triple crossing is defined as positive: 

The writhe of all involved crossings is positive, only the crossing which we indicated in the diagram with a minus has a negative writhe. We see that the number of contributing ordinary crossings increases from 1 to m. This means that $R_{(1,0)}(rot((2m+1)^+_1)) = 1+\ldots+m = m(m+1)/2$.

So each cocycles gives the negative of the other, applied to rotation of a $(2, 2m+1)$-torus knot. The values meet our expectations: they correspond with the second order Vasiliev invariant $v_2((2m+1)_1) = m(m+1)/2$.

Taking the mirror knots results in the same total values: The contributing Gauss diagrams interchange with the former non-contributing Gauss diagrams (which had three zeros). They have positive sign in the first half and negative sign in the second half, so all signs are flipped/swapped, but now nearly all the crossings have negative writhe, so are swapped too. Thus in total the values of the respective cocycles stays the same.

Now we calculate $fh(2m+1)_1$. This loop is for 5_1 given by the following pictures and corresponding Gauss diagrams. (We only show the first three pictures.)



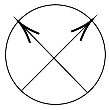
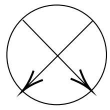
The fox-hatcher loop consists of five $(2m+1)$ times the same arc and that arc has two parts. The first part of the arc contributes $-m-\ldots-1 = -m(m+1)/2$ to $R_{(0,1)}$. The second part contributes $1+\ldots+m = m(m+1)/2$ to $R_{(1,0)}$. So for the fox-hatcher loop our results are: $R_{(0,1)}(fh(2m+1)_1) = -(2m+1)m(m+1)/2$ and $R_{(1,0)}(fh(2m+1)_1) = (2m+1)m(m+1)/2$. We put our results in a table and compare them with $v_2(K), v_3(K)$ and $-6v_3(K) - w(K)v_2(K)$.

Table of results:

K	$R_{(1,0)}(rot)$	$R_{(1,0)}(fh)$	$v_2(K)$	$v_3(K)$	$-6v_3(K) - w(K)v_2(K)$
3_1	1	3	1	-1	3
4_1	..	0	-1	0	0
5_1	3	15	3	-5	15
5_2	..	..	2	-3	
6_1	..	..	-2	1	
6_2	..	..	-1	1	
6_3	..	..	1	0	
7_1	6	42	6	-14	42
9_1	10	90	10	-30	90
$'11'_1$	15	165	...	...	...

Comments about these results: $R_{(0,1)} = -R_{(1,0)}$. Following Fiedler, $R_{(0,1)}$ is the same as Mortier's one-cocycle α^1_3, but to us it seems that only the pictures/formulas are te same, not the interpretation. As far as we know, Mortier uses gauss diagrams with arrows from overstrand to understrand, while Fiedler and we use arrows from understrand to overstrand. Thus α_3^1 is different from $R_{(0,1)}$. If we apply one of the cocycles to a knot and the other to the mirror knot, then they will do the same, up to swapping the sign. Because the only thing that changes then is the sign of the individual crossings, so every term that contributes is multiplied by -1. We know (or do we know only for our specific examples?) that all our one-cocycles give the same values on knots and mirror knots. We can conclude that $R_{(1,0)} = -R_{(0,1)} = \alpha_3^1 = v_2$.

More comments: If we perform the same loop, but in the reverse direction, we encounter the same contributing diagrams, but the signs of the triple crossings are swapped, so in total this causes a swapping of the sign.

4.2. **Proof of relations between $R_{(0,1)}(rot)$ and $R_{(1,0)}(rot)$ and v_2.** We can proof that for a closed braid (which can be viewed as a long knot), $-R_{(0,1)}(rot)$ and $R_{(1,0)}(rot)$ measure the same thing as the gauss diagram formula for v_2 given by Polyak and Viro in [PV94]. They gave the formulas  , which according to them both are measuring the second order Vasiliev invariant of a long knot. Here we have to sum over all possible choices of the two arrows and for each choice multiply the writhes of the two arrows.

We will show that $R_{(0,1)}(rot)$ measures the negative of the first diagram and $R_{(1,0)}$ measures the same as the second diagram.

<Proof: We take a general n-braid which is a knot. We perform rot and look at which RIII-moves occur. We will see that $R_{(0,1)}(rot)$ has contributions from exactly the same pairs of arrows as v_2. >

Taking the mirror knot, switches all crossings in the knot, but not in the curl we add. So (as we will argue) this causes $R_{(0,1)}(rot)$ to measure the negative of second diagram and $R_{(1,0)}(rot)$ to measure the first diagram (of the original knot).

References

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Pictures: https://wildandnoncompactknots.files.wordpress.com/2013/04/project4.jpg
https://girlstalkmath.files.wordpress.com/2018/07/knot-theory-3.png?w=425&h=231
https://www1.cmc.edu/pages/faculty/VNelson/trefoilgd.jpg
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