Universal invariants, perturbed Gaussians and the 2-loop polynomial

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1. What is this about?

In what follows there is a beautiful story relating the universal tangle invariant with respect to a ribbon algebra $\mathbb{D}$, whose truncations can be computed effectively (in polynomial time), with the loop expansion of the Kontsevich invariant and the 2-loop polynomial.

$$\begin{tikzcd}
 & & \text{Kontsevich invariant } Z \arrow[r, leftrightarrow] \arrow[d] & \text{Loop expansion of } Z \arrow[d] \arrow[ddr, bend left] & \\
\text{Universal invariant } Z_{\mathbb{D}} \text{ of } \mathbb{D} \arrow[r, bend left] \arrow[d] & \text{Universal invariant of } U_h(\mathfrak{sl}_2) \arrow[l, dashed, bend left] \arrow[r, leftrightarrow] & \text{Coloured Jones polynomials } J_n \arrow[r, leftrightarrow] & \text{Rational expansion of } J_n \arrow[d] & \\
Z_{\mathbb{D}} \text{ mod } \varepsilon^2 \arrow[r, leftrightarrow] \arrow[d] & (\Delta_K, \rho_K^{1,0}) \arrow[d] \arrow[rr, dashed, no head] & & (\Delta_K, \hat{\Theta}_K) \arrow[d] & (n \leq 2)\text{−loop part of } Z \arrow[l] \arrow[d] \\
Z_{\mathbb{D}} \text{ mod } \varepsilon \arrow[r, leftrightarrow] & \Delta_K \arrow[rr, equal] & & \Delta_K \arrow[r, leftrightarrow] & (n \leq 1)\text{−loop part of } Z
\end{tikzcd}$$

2. Background

Let $\mathcal{T}$ be the strict monoidal category of framed, oriented tangles in a cube. Here we focus on the monoidal subcategory $\mathcal{T}^{\mathrm{up}} \subset \mathcal{T}$ of tangles oriented from bottom to top without closed components, called *upwards tangles*.
The *Kontsevich invariant* is a strong monoidal functor

$$Z : \mathcal{T}_q^{\mathrm{up}} \longrightarrow \mathcal{A}$$

defined on the *non-strictification* $\mathcal{T}_q^{\mathrm{up}}$ of $\mathcal{T}^{\mathrm{up}}$ and with values in the category $\mathcal{A}$ of *Jacobi diagrams*. This invariant is universal among rational knot Vassiliev finite-type invariants, so it determines the coloured Jones polynomials J_n via the so-called $\mathfrak{sl}_2$-weight system. For a (long) knot K, the value $Z(K)$ is an infinite formal linear combination of uni-trivalent graphs, and it is known to admit a *loop expansion*

$$Z(K) = \exp_{\amalg}\left(\textstyle\sum_i \lambda_i D_i + \sum_i \mu_i D_i' + \cdots\right)$$

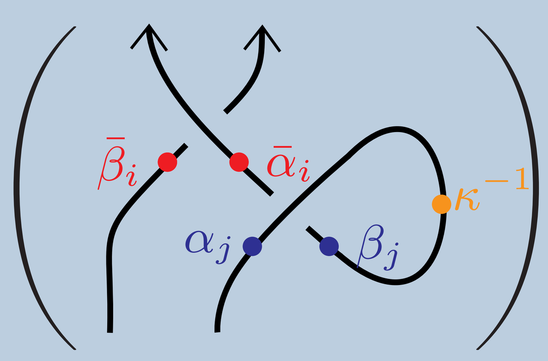
where the first summand (*1-loop* part) is tantamount to the Alexander polynomial Δ_K of K and the second (*2-loop* part) determines a polynomial invariant $\hat{\Theta}_K \in \mathbb{Q}[t, t^{-1}]$ (the one-variable version of the 2-loop polynomial of K). Applying the $\mathfrak{sl}_2$-weight system to the expression above yields the *rational expansion* of the coloured Jones polynomials, a repackaging of the J_n in terms of Δ_K and a family of knot polynomial invariants $P_K^i \in \mathbb{Z}[t, t^{-1}]$, where $P_K^0 = 1$ and $P_K^1 = \hat{\Theta}_K$.

3. The universal tangle invariant

We put the focus on *ribbon* algebras. The main property of a ribbon algebra A is that it possesses two preferred, invertible elements

$$R = \textstyle\sum_i \alpha_i \otimes \beta_i \in A \otimes A \qquad , \qquad \kappa \in A$$

(subject to some axioms). Multiplying copies of $R = \sum_i \alpha \otimes \beta$, $R^{-1} = \sum_i \bar{\alpha}_i \otimes \bar{\beta}_i$ and $\kappa^{\pm 1}$ defines a tangle invariant as follows,

Z_A  $= \sum_{i,j} \bar{\beta}_i \otimes \alpha_j \kappa^{-1} \beta_j \bar{\alpha}_i$

and this gives rise to a strong monoidal functor

$$Z_A : \mathcal{T}^{\mathrm{up}} \longrightarrow \mathcal{P}_A$$

where $\mathcal{P}_A$ is a strict monoidal category that depends on A. For a knot K, the value $Z_A(K) \in A$ dominates the Reshetikhin–Turaev invariants obtained from the representation theory of A, and so the universal invariant with respect to $U_h(\mathfrak{sl}_2)$ determines all coloured Jones polynomials. In fact, work by Habiro shows that the converse is also true.

4. Perturbed Gaussians

A major issue when studying the universal invariant is that the value $Z_A(K)$ can be extremely hard to compute in practice when A is an infinite-dimensional, noncommutative algebra, e.g. when $A = U_h(\mathfrak{sl}_2)$. Bar-Natan and van der Veen [1] constructed a (topological) ribbon algebra $\mathbb{D}$ over the ring $\mathbb{Q}[\varepsilon][[h]]$ such that

- There is a ribbon algebra surjection $\mathbb{D} \longrightarrow U_h(\mathfrak{sl}_2)$, so that $Z_{\mathbb{D}}(K)$ determines $Z_{U_h(\mathfrak{sl}_2)}(K)$ (we expect the converse to hold as well).
- The elements $R^{\pm 1}$ and $\kappa^{\pm 1}$, and hence $Z_{\mathbb{D}}(K)$, can be expressed as *perturbed Gaussians*, that is, expressions of the form
$$e^G(P_0 + P_1\varepsilon + P_2\varepsilon^2 + P_3\varepsilon^3 + \cdots)$$
in commutative variables.
- The truncations $Z_{\mathbb{D}}(K)$ mod ε^N can be computed effectively (in polynomial time).

5. Main results and conjectures

If K is a 0-framed knot, then

- The value $Z_{\mathbb{D}}(K)$ mod ε^N is fully determined by the Alexander polynomial Δ_K and a family of knot polynomial invariants $\rho_K^{i,j} \in \mathbb{Q}[t, t^{-1}]$ for $0 \leq i \leq N$ and $0 \leq j \leq i$ [1, 2],
- $Z_{\mathbb{D}}(K)$ mod ε is tantamount to Δ_K (this means $\rho_K^{0,0} = 1$), and $Z_{\mathbb{D}}(K)$ mod ε^2 is equivalent to the pair $(\Delta_K, \rho_K^{1,0})$ [1],
- $Z_{\mathbb{D}}(K)$ mod ε^2 is tantamount to the triple $(\Delta_K, \rho_K^{1,0}, \rho_K^{2,0})$ [2].

In general, we expect $Z_{\mathbb{D}}(K)$ to be determined by Δ_K and the $\rho_K^{i,0}$. In fact, the polynomials $\rho_K^{i,0}$ seem to be closely related to the P_K^i appearing in §2. More precisely, we expect

$$\rho_K^{1,0} = \hat{\Theta}_K$$

(this has been proven for knots of genus one [3]).

6. References

[1] D. Bar-Natan and R. van der Veen: *Perturbed Gaussian generating functions for universal knot invariants*, arXiv: 2109.02057 (2021)

[2] J. Becerra: *On Bar-Natan - van der Veen's perturbed Gaussians*, arXiv:2302.01124 (2023)

[3] J. Becerra: *A Hopf algebraic construction of the 2-loop polynomial of genus one knots*, in preparation.