LECTURE 2: THE JONES POLYNOMIAL

So far we have only seen one knot invariant, namely the Casson invariant. However, as it is a mere integer, it will fail to distinguish plenty of non-isotopic knots. In the following two lectures we will construct two classical knot polynomial invariants that not only are strong in distinguishing knots but also they enjoy desirable properties.

1. Statement and examples

Recall that if k is a commutative ring, the ring of *Laurent polynomials* in a variable x with coefficients in k is

$$k[x, x^{-1}] := \left\{ \sum_{i=n}^{m} a_i x^i,\ a_i \in k,\ n, m \in \mathbb{Z},\ n \leq m \right\},$$

and it is a ring with the usual sum and multiplication of polynomials.

The goal of this lecture is to show the following theorem:

Theorem 1. *There exists a unique link polynomial invariant*

$$J : \mathcal{L} \longrightarrow \mathbb{Z}[t^{1/2}, t^{-1/2}]$$

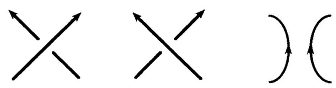
satisfying

(1) $J_{\text{unknot}}(t) = 1$*,*

(2) the following skein relation holds:

$$t^{-1} J_{L_+}(t) - t J_{L_-}(t) = (t^{1/2} - t^{-1/2}) J_{L_0}(t),$$

where L_+, L_-, L_0 *denote three links that are identical except in a neighbourhood of some point where they look like below,*

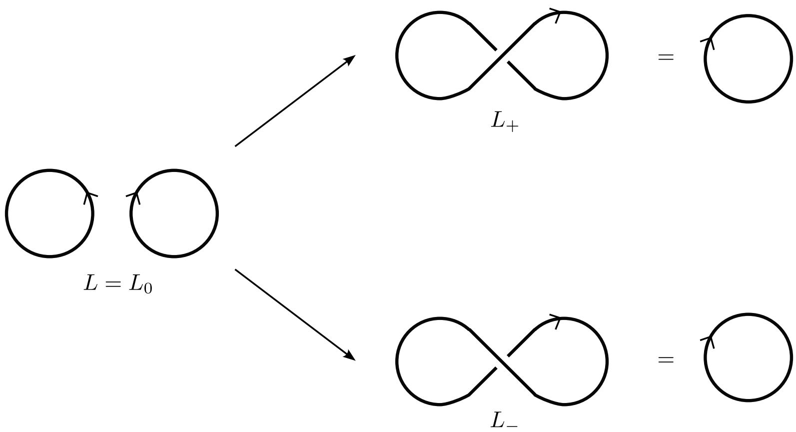


L_+ $\quad$ L_- $\quad$ L_0

Definition 2. The polynomial of the previous theorem is called the *Jones polynomial.* It is coined after Vaughan Jones (1952 – 2020), Fields Medal in 1990.

The way one proves such a statement (at least the existence part) is by means of *providing a construction.* The uniqueness is easier to show. Nevertheless before showing this theorem let us get acquainted with the statement giving some examples.

Example 3. Let L be the trivial 2-component link. We apply the skein relation taking $L = L_0$. The other links (actually knots) involved L_+, L_0 are trivial by a Reidemeister $\Omega 1$ move, so $J_{L_+} = 1 = J_{L_-}$.



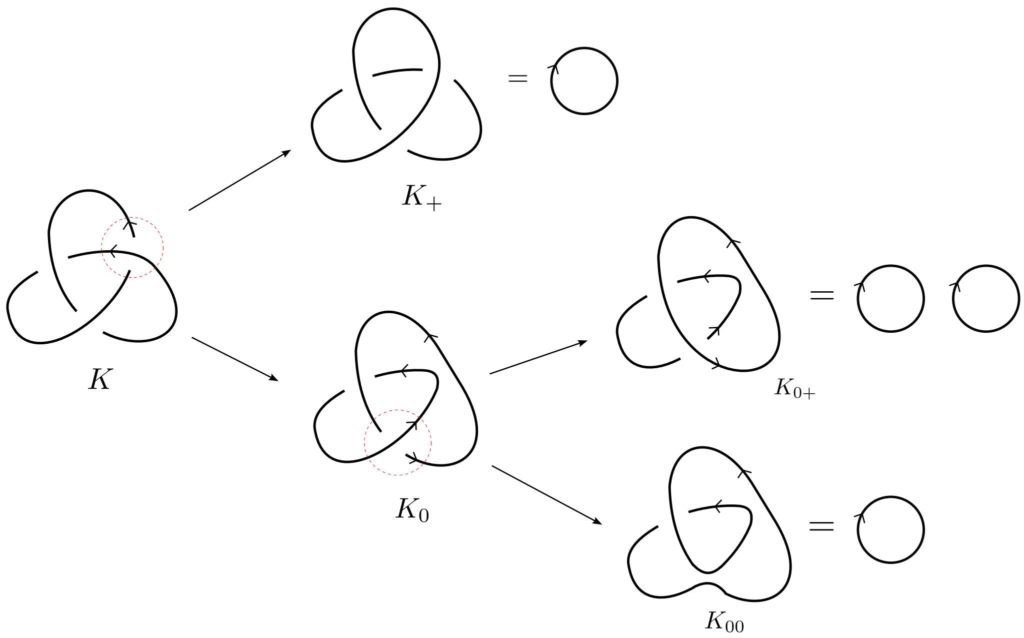
The skein relation then reads

$$t^{-1} - t = J_{L_+}(t) - J_{L_-} = (t^{1/2} - t^{-1/2})J_{L_0}(t)$$

so we conclude that

$$J_L(t) = \frac{t^{-1} - t}{t^{1/2} - t^{-1/2}} = \frac{(t^{-1/2} - t^{1/2})(t^{-1/2} + t^{1/2})}{t^{1/2} - t^{-1/2}} = -t^{-1/2} - t^{1/2}.$$

Example 4. Let us now compute the Jones polynomial of the left-handed trefoil K. The general strategy consists of performing changes at one crossing, as indicated in the skein relation, and keep performing changes in the subsequent links until we reach a link whose Jones polynomial we know. Below we show this for K:



The above figure is sometimes called a *skein tree* for K. Note both K_+ and K_{00} are the unknot, and K_{0+} is isotopic to the trivial 2-component link, whose Jones polynomial we just computed. Using the skein relation, we have

$$t^{-1}J_{K_{0+}} - tJ_{K_0} = (t^{1/2} - t^{-1/2})J_{K_{00}}$$

so

$$\begin{aligned} J_{K_0} &= t^{-1}\Big(t^{-1}(-t^{-1/2} - t^{1/2}) - (t^{1/2} - t^{-1/2})\Big) \\ &= t^{-1}(-t^{-3/2} - t^{1/2}) \\ &= -t^{-5/2} - t^{-1/2} \end{aligned}$$

by the previous example. On the other hand

$$t^{-1} - tJ_K = t^{-1}J_{K_+} - tJ_K = (t^{1/2} - t^{-1/2})J_{K_0} = (t^{1/2} - t^{-1/2})(-t^{-5/2} - t^{-1/2})$$

and therefore

$$\begin{aligned} J_K &= t^{-1}\Big(t^{-1} - (t^{1/2} - t^{-1/2})(-t^{-5/2} - t^{-1/2})\Big) \\ &= -t^{-4} + t^{-3} + t^{-1}. \end{aligned}$$

Exercise 5. Mimicking the above example, show that the Jones polynomial of the trivial n-component link (that is, n disjoint copies of the unknot) equals $(-t^{-1/2} - t^{1/2})^{n-1}$.

2. Proof of the uniqueness

We start by showing first the uniqueness, which is rather easy. The key point of the argument is the following

Lemma 6. *Any knot diagram can be turned into the unknot by changing the sign of some of its crossings, that is, by turning some of its positive/negative crossings into negative/positive.*

More generally, any n-component link can be turned into the trivial n-component link by changing the sign of some of its crossings.

Proof. We only show here the lemma for knots, the case for links is an exercise for the reader.

Let D be a knot diagram and fix a basepoint p. Now traverse the knot according the orientation and change the crossings according to the rule that the first time you hit the crossing you traverse the knot along the understrand of the crossing and the second time is done along the overstrand of the crossing.

Now we claim that the resulting diagram D' represents the unknot. This can be viewed as follows: take a rope, place one of its endpoints on a table and hold the rope from its other endpoints above the table. Starting from the point p, mimic the pattern of D' with the rope, and glue the endpoints. Because of the rule we followed when constructing D', we can simply let the rope lie on the table following the pattern of D'. What this means is that the rope is actually not knotted, ie D' is the unknot. $\square$

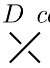
Let us prove now the uniqueness. Given a link L, choose a crossing (say positive) and consider the two other resolutions L_- and L_0. Note L_0 has one fewer crossing, whereas L_- had the sign of the chosen crossing reversed. If any of the two links is the trivial unlink, then we stop; else we chose a different crossing and repeat the previous step. The lemma above (by choosing the crossings that turn L into the unlink) ensures that this process is finite. A repeated application of the skein relation allows to express J_L in terms of $J_{\text{unknot}} = 1$ and copies of $(-t^{-1/2} - t^{1/2})^c$ (with the corresponding products $t^{\pm 1}$ coming from the skein relation), hence the uniqueness.

3. A construction of the Jones polynomial

As we mentioned before, the existence is proved by means of providing a construction. In this lecture we will present the simplest construction possible. However, one of the goals of this lecture course is to recover this polynomial from a completely different setup that gives rise to a large class of invariants.

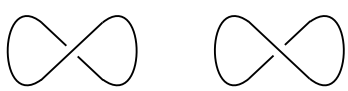
First of all we need two observations. First, recall that a link in $\mathbb{R}^3$ carries a choice of orientation.

Definition 7. If we forget the orientation of the link, we call the resulting embedding $S^1 \hookrightarrow \mathbb{R}^3$ the (underlying) *unoriented link* of L, or simply an unoriented link L.

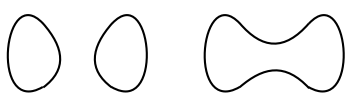
Lemma 8. *Let D be a diagram of a unoriented link. Then D can be turned into a diagram for the trivial unlink (ie, a disjoint union of circles) by replacing*  *by*  *or*  *arbitrarily.*

Proof. Let us argue by induction on the number of crossings n of D.

If $n = 1$, then it is readily seen that D must be one of the following two diagrams (up to planar isotopy):



In both cases, after replacing the crossing as indicated they become



For the general case, take a unoriented link diagram D, fix a crossing and replace the crossing as indicated. The resulting two diagrams have one crossing less, and we conclude by the induction hypothesis. $\square$

The following concept is the first step towards the definition of the Jones polynomial.

Definition 9. Let D be an unoriented link diagram on the plane, and choose an order for its set of crossings. The *Kauffman bracket* of D is the Laurent polynomial

$$\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$$

obtained by recursively applying the following rules:

(i) $\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$

(ii) $\langle D \coprod \bigcirc \rangle = (-A^{-2} - A^{2}) \langle D \rangle$

(iii) $\langle \bigcirc \rangle = 1$

applying (i) according to the chosen order for the crossings.

The definition requires some explanation: as in the skein relation of Theorem 1, the formula in (i) refers to three unoriented link diagrams that are identical except in a neighbourhood of some point where they differ in the indicated way. It is to be understood in the sense that (i) is equivalent to

$$\langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle)(\rangle .$$

On the other hand, (i) and the previous lemma allow to express the Kauffman bracket of a diagram with n crossings as a $\mathbb{Z}[A, A^{-1}]$-linear sum of 2^n diagrams with no crossings, that is, diagrams of trivial unlink diagrams (of different numbers of components). At the same time (ii) and (iii) imply that the bracket of the (unoriented) trivial unlink diagram with c components equals $(-A^{-2} - A^2)^{c-1}$.

The first observation is that the definition is independent of the choice of the order for the crossings.

Lemma 10. *The Kauffman bracket is independent of the choice of the order for the crossings, so it gives a well-defined map*

$$\langle \quad \rangle : \frac{\left\{ \begin{array}{c} \text{link diagrams in} \\ \mathbb{R}^2 \end{array} \right\}}{\text{planar isotopy}} \longrightarrow \mathbb{Z}[A, A^{-1}].$$

Proof. Given two crossings of a unoriented link diagram c_1 and c_2, if we apply (i) first c_1 and second to c_2, we obtain

$$\begin{aligned}
\langle \times \times \rangle &= A \langle)(\times \rangle + A^{-1} \langle \asymp \times \rangle \\
&= A(A\langle)( )(\rangle + A^{-1} \langle)(\asymp \rangle) + A^{-1}(A \langle \asymp)(\rangle + A^{-1} \langle \asymp \asymp \rangle) \\
&= A^2 \langle)( )(\rangle + \langle)(\asymp \rangle + \langle \asymp)(\rangle + A^{-2} \langle \asymp \asymp \rangle,
\end{aligned}$$

and if we apply (i) first to c_2 and second to c_1 we obtain

$$\begin{aligned}
\langle \times \times \rangle &= A \langle \times)(\rangle + A^{-1} \langle \times \asymp \rangle \\
&= A(A\langle)( )(\rangle + A^{-1} \langle \asymp)(\rangle) + A^{-1}(A \langle)(\asymp \rangle + A^{-1} \langle \asymp \asymp \rangle) \\
&= A^2 \langle)( )(\rangle + \langle \asymp)(\rangle + \langle)(\asymp \rangle + A^{-2} \langle \asymp \asymp \rangle,
\end{aligned}$$

that is, the same. $\square$

Next we want to investigate whether the Kauffman bracket descents (or not) to an unoriented link invariant, that is, whether it is invariant under Reidemeister moves. The following proposition tells us that the answer is negative, but the failure is only because of the unoriented Reidemeister 1 move. By "unoriented Reidemeister moves" we mean the Reidemeister moves from Lecture 1 where we forget the orientation of the diagrams.

Proposition 11. *The Kauffman bracket satisfies the following properties:*

(1) If a diagram is changed by a unoriented $\Omega 1$ move, then its Kauffman bracket changes as follows:

$$\langle \gamma \rangle = -A^{3} \langle \frown \rangle \quad , \quad \langle \gamma \rangle = -A^{-3} \langle \frown \rangle$$

(2) The Kauffman bracket is invariant under unoriented $\Omega 2$ *move,*

$$\langle\ \rangle = \langle\ \rangle$$

(3) The Kauffman bracket is invariant under unoriented $\Omega 3$ *move,*

$$\langle\ \rangle = \langle\ \rangle$$

Proof. (1) We simply compute

$$\begin{aligned}\langle\ \rangle &= A\langle\ \rangle + A^{-1}\langle\ \rangle\\ &= (A(-A^{-2} - A^{2}) + A^{-1})\langle\ \rangle\\ &= -A^{-3}\langle\ \rangle\end{aligned}$$

and the second equation follows similarly.

(2) We have

$$\begin{aligned}\langle\ \rangle &= A\langle\ \rangle + A^{-1}\langle\ \rangle\\ &= -A^{-2}\langle\ \rangle + \langle\ \rangle + A^{-2}\langle\ \rangle\\ &= \langle\ \rangle\end{aligned}$$

(3) Lastly we have

$$\begin{aligned}\langle\ \rangle &= A\langle\ \rangle + A^{-1}\langle\ \rangle\\ &= A\langle\ \rangle + A^{-1}\langle\ \rangle\\ &= \langle\ \rangle\end{aligned}$$

where we have used (2) twice in the second equality for the first summand. □

The question now is then: can we modify the Kauffman bracket so that it is invariant under $\Omega 1$ moves - and if so, how?

It turns out that, if our link diagram is oriented, then we fix this problem.

Definition 12. Let D be an (oriented) link diagram. The *writhe* of D is the integer

$$w(D) := \sum_{c} \operatorname{sign}(c),$$

where the sum runs through all crossings c in D and as usual

$$\operatorname{sign}(c) := \begin{cases} +1, & c \text{ is a positive crossing,}\\ -1, & c \text{ is a negative crossing.}\end{cases}$$

It directly follows from the definition that

$$w(\ ) = w(\ ) + 1 \qquad , \qquad w(\ ) = w(\ ) - 1$$

and that the writhe is invariant under Reidemeister moves 2 and 3.

Combining the Kauffman bracket with the writhe, we can define a link invariant:

Theorem 13. *Let L be an (oriented) link and let D be a diagram of L. Then the value*

$$(-A)^{-3w(D)}\langle D\rangle \in \mathbb{Z}[A, A^{-1}]$$

is an invariant of L, and it belongs to $\mathbb{Z}[A^{2}, A^{-2}] \subset \mathbb{Z}[A, A^{-1}]$.

Definition 14. The *Jones polynomial* of a link L is the polynomial invariant

$$J_L(t) := \Big((-A)^{-3w(D)}\langle D\rangle\Big)_{|A^{2}=t^{-1/2}} \in \mathbb{Z}[t^{-1/2}, t^{1,2}]$$

Proof (of Theorem 13) . Since both the Kauffman bracket and the writhe are invariant under $\Omega 2$ and $\Omega 3$ moves, we only have to check the invariance under $\Omega 1$ moves. For the one with the positive crossing, we have

$$(-A)^{-3w(\ )}\langle\ \rangle = (-A)^{-3(w(\ )+1)}(-A^{3})\langle\ \rangle = (-A)^{-3w(\ )}\langle\ \rangle$$

where we have used Proposition 11.(1) in the first equality.

To see that this value is a polynomial in $\mathbb{Z}[A^2, A^{-2}]$, we can argue as follows: the rules for the Kauffman bracket imply (using induction) that an unoriented diagram with an even (resp. odd) number of crossings has Kauffman bracket with only even (resp. odd) powers of A. On the other hand, a diagram with an even (resp. odd) number of crossings must have even (resp. odd) writhe. In both even and odd cases, it follows that $(-A)^{-3w(D)}\langle D\rangle \in \mathbb{Z}[A^2, A^{-2}]$. □

4. Examples revisited

We still have to show that the polynomial invariant we just constructed is indeed the unique polynomial from Theorem 1. Before doing that, let us redo the two examples we computed with the skein relation.

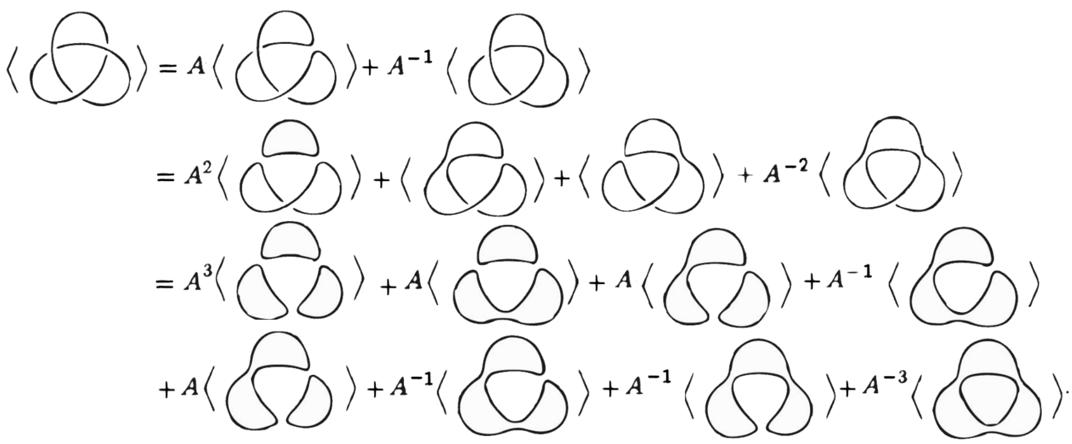
Example 15. Let $L = \bigcirc\,\bigcirc$ be the trivial 2-component link. Its Kauffman bracket is

$$\langle \bigcirc\,\bigcirc \rangle = -A^{-2} - A^2.$$

Since there are no crossings, the writhe of this diagram is 0, so

$$J_L(t) = -t^{-1/2} - t^{1/2}.$$

Example 16. Let K now be the left-handed trefoil. We compute



From here we can compute the Kauffman bracket of the trefoil diagram D as the bracket of a disjoint union of c crossings equals $(-A^{-2} - A^2)^{c-1}$ (alternatively one could apply Proposition 11 to compute the bracket of D).

The upshot is that

$$\langle D\rangle = -A^{-5} - A^3 + A^7.$$

Since $w(D) = -3$, we conclude

$$\begin{aligned} J_K(t) &= \Big((-A)^{-3w(D)}\langle D\rangle\Big)_{|A^2=t^{-1/2}} \\ &= \Big((-A)^9(-A^{-5} - A^3 + A^7)\Big)_{|A^2=t^{-1/2}} \\ &= \Big(A^4 + A^{12} - A^{16}\Big)_{|A^2=t^{-1/2}} \\ &= t^{-1} + t^{-3} - t^{-4}. \end{aligned}$$

5. Proof of existence

Let us check that the Jones polynomial we constructed by modifying the Kauffman bracket with the writhe satisfies the properties of Theorem 1.

That $J_{\text{unknot}}(t) = 1$ trivially follows from $\langle\bigcirc\rangle = 1$. To verify the skein relation, recall that for the Kauffman bracket

$$\langle \times \rangle = A\,\langle\,)(\, \rangle + A^{-1}\langle \asymp \rangle \qquad , \qquad \langle \times \rangle = A\,\langle \asymp \rangle + A^{-1}\langle\,)(\, \rangle.$$

Multiplying the first equation by A, the second by A^{-1} and subtracting we get

$$A\left\langle \times \right\rangle - A^{-1}\left\langle \times \right\rangle = (A^2 - A^{-2})\left\langle)(\right\rangle .$$

If (L_+, L_-, L_0) denotes the triple in the skein relation, we have

$$w(L_+) - 1 = w(L_0) = w(L_-) + 1$$

so multiplying the previous equation by $-(-A)^{-3w(L_0)}$ we obtain

$$A^4 J_{L_+} - A^{-4} J_{L_-} = (A^{-2} - A^2) J_{L_0}$$

and the substitution $t^{-1/2} = A^2$ concludes.

6. Properties of the Jones polynomial

To finish off, we would like to mention some properties of the Jones polynomial.

Definition 17. Let L be a link in $\mathbb{R}^3$, and let K, K' be knots that can be separated by a plane in $\mathbb{R}^3$.

(1) The *orientation-reversal* of L is the link $-L$ obtained by reversing the orientation of every component.
(2) Let $m : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the map $m(x, y, z) := (x, y, -z)$. The *mirror image* of L is the link $\overline{L} := m(L)$.
(3) The *connected sum* $K\#K'$ of K and K' is the knot resulting from removing a small arc in both K and K' and glueing the endpoints of both according to the orientation, as shown below:

$K\#K'$

Proposition 18. *The Jones polynomial satisfies the following properties:*

(1) If L is a link with an odd number of components (in particular if L is a knot), then

$$J_L \in \mathbb{Z}[t, t^{-1}] \subset \mathbb{Z}[t^{1/2}, t^{-1/2}].$$

(2) Let L be a link and denote by $-L$ the orientation reversal of L. Then

$$J_{-L}(t) = J_L(t).$$

(3) Let L be a link and denote by $\overline{L}$ the mirror image of L. Then

$$J_{\overline{L}}(t) = J_L(t^{-1}).$$

(4) If K, K' are knots, then

$$J_{K\#K'}(t) = J_K(t) \cdot J_{K'}(t).$$