The Alexander polynomial via Fox calculus

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I sincerely apologize for the lack nice knot drawings in my handouts.

The Group ring and Derivatives

Definition 1 (The Group Ring)**.** Let G be any group. We define the group ring of G, denoted by $\mathbb{Z}G$, as the set off all maps

$$v : G \to \mathbb{Z}$$

such that $v(g) = 0$ for all except finitely many $g \in G$. (Sometimes the group ring is also denoted with $\mathbb{Z}[G]$, but I will only use this if confusion may arrise.)

(Note that this is different from $D(G)$ which we defined during the lectures.)

Definition 2. Addition in $\mathbb{Z}G$ is defined by $(v_1 + v_2)(g) = v_1(g) + v_2(g)$, and multiplication by $(v_1v_2)(g) = \sum_{h\in G} v_1(h)v_2(h^{-1}g)$.

Lemma 3. *The group ring $\mathbb{Z}G$ is a ring.*

Proof. Check the axioms. □

Remark 4. There is an injective group homomorphism $G \hookrightarrow \mathbb{Z}G$ by $g \mapsto 1g$. Were $1g$ denotes the map which sends g to 1 and all other elements of the group to 0.
Let e denote the identity element in G. There is also an injective ring homomorphism $\mathbb{Z} \hookrightarrow \mathbb{Z}G$ by $n \mapsto ne$, where ne denote the map which sends e to n and all other elements to 0.

Lemma 5. *The ring $\mathbb{Z}G$ is commutative if and only if G is an abelian group.*

Notation. We write $v : G \to \mathbb{Z}$ as $\sum_{h\in G} v(h)h$ or $\sum_i n_ih_i$,
where $v(h_i) = n_i \in \mathbb{Z}$

With this notation it's easier to proof the previous lemma.

Proof. If G is not abelian, then there exist $a, b \in G$ such that $ab \neq ba$. Let $v_1 = 1a$ and $v_2 = 2b$ then $1ab = v_1v_2 \neq v_2v_1 = 1ba$. Similarly, if G is abelian, then for any $v_1 = \sum_i n_ih_i$ and $v_2 \sum_j n_jh_j$ we have $v_1v_2 = \sum_{i,j} n_in_jh_ih_j = \sum_{i,j} n_jn_ih_jh_i = v_2v_1$. □

Example 6. Let $G = \mathbb{S}^1$, the group of rotations of the plane (denoted by Cartesian coordinates). The element $v \in \mathbb{Z}\mathbb{S}^1$, which satisfies $v((1, 0)) = 15$, $v((0.5\sqrt{2}, 0.5\sqrt{2})) = 1$, $v((0, 1)) = 42$, and is zero for all other elements of $\mathbb{S}^1$. Then v will be denoted by $15(1, 0) + (0.5\sqrt{2}, 0.5\sqrt{2}) + 42(1, 0)$.

Example 7. Let $v_1 = 5\lambda$ and $v_2 = 3\lambda + 4\mu$ for $\lambda, \mu \in G$. then $v_1v_2 = 15\lambda^2 + 20\lambda\mu$

Lemma 8. *Any group homomorphism $\phi : G \to G'$ induces a unique ring homomorphism $\psi : \mathbb{Z}G \to \mathbb{Z}G'$ by $\sum_i n_ih_i \mapsto \sum_i n_ih'_i$. Moreover, if the group homomorphism is surjective/injective, then so is the ring homomorphism*

Definition 9 (The Trivializer)**.** The map $\mathfrak{t} : \mathbb{Z}G \to \mathbb{Z}$ defined by
$\mathfrak{t}(\sum_i n_ih_i) = \sum_i n_i$ is called the trivializer. This map can be extended to a map from $\mathbb{Z}G$ to $\mathbb{Z}G$ by the injection discussed in Remark 1

Definition 10 (A Derivative in a Group Ring)**.** A derivative is a map $D : \mathbb{Z}G \to \mathbb{Z}G$ that satisfies

$$D(v_1 + v_2) = D(v_1) + D(v_2) \tag{$\heartsuit$}$$

$$D(v_1 v_2) = D(v_1)\mathfrak{t}(v_2) + v_1 D(v_2) \tag{$\clubsuit$}$$

Where $\mathfrak{t}$ is now the extended version of the trivializer.

Remark 11. Note that from ($\clubsuit$) we get that
$D(g_1 g_2) = D(g_1) + g_1 D(g_2)$.

Corollary 12. *Let D and D' be any two derivatives and v_0 any element of $\mathbb{Z}G$. The maps $D + D'$ and $D \circ v_0$ are both derivatives, where $(D + D')(v) = D(v) + D'(v)$ and $(D \circ v_0)(v) = D(v) \cdot v_0$.*

Proof. Check if the definitions apply to $D + D'$ and $D \circ v_0$ by using the fact that D and D' are derivatives. $\square$

Corollary 13. *Any derivative D satisfies the following*

1. $D(\sum_i n_i h_i) = \sum_i n_i D(h_i)$
2. $D(ne) = 0$
3. $D(g^{-1}) = -1g^{-1}D(g)$

Proof. The first equality follows from ($\heartsuit$).

For the second equality; start by proving $D(1e) = 0$ by means of the relation $D(1e) = D((1e)\cdot(1e))$. Then use the previous equation.

For the third equality, observe that we have $D(g^{-1}g) = 0$ because of the previous proof. This equation expands to $D(g^{-1})\mathfrak{t}(g) + g^{-1}D(g) = 0 \Leftrightarrow D(g^{-1}) = -g^{-1}D(g)$. $\square$

Lemma 14. *For any derivative D, and any $n \in \mathbb{Z}$, we have the following equalities*

$$D(g^n) = \begin{cases} 0 \text{ if } n = 0 \\ \left(\sum_{i=1}^{n-1} g^i\right) D(g) \text{ if } n > 0 \\ \left(\sum_{i=n}^{-1} g^i\right) D(g) \text{ if } n < 0 \end{cases}$$

Proof. Use induction on n twice. Once going up and once going down. $\square$

Remark 15. By the first equation of Corollary 13 we know that in order to define the derivative, it suffices to define what it does on the group G. Moreover, by Lemma 14, we only need to specify what D does on a generating set of G in order to describe what it does on all of $\mathbb{Z}G$.

The free group

We apply these concepts to the free group with n variables
$F_n := \langle x_1, x_2, \ldots, x_n \rangle$.

Definition 16 (The Free Polynomial Ring)**.** The ring $\mathbb{Z}F_n$ is called the free polynomial ring with n variables. An element of $\mathbb{Z}F_n$ is called a free polynomial.

Remark 17. WARNING!!! Even though the elements are called polynomials, they are not true polynomials. The generators of F_n do not commute and negative powers of x_i may appear in a term.

Definition 18. For each x_j, we define a derivative D_j on $\mathbb{Z}F_n$ by $D_j(x_i) = \frac{\partial x_i}{\partial x_j} = \delta_{i,j}$

Lemma 19. *D_j is well defined. (with which we mean it satisfied the derivative axioms)*

Proof. This is proven in [1]. □

Lemma 20. *D_j is unique.*

Proof. This follows from Remark 15. □

Remark 21. Let $r_1, r_2 \dots$ be any relations. We always have the natural group homomorphism $\gamma : F_n = \langle x_1, x_2, \dots | \emptyset \rangle \to \langle x_1, x_2, \dots | r_1, r_2, \dots \rangle$.

Corollary 22. *For D_j as described above, and for $u, v \in F_n$ we have the following relations $D_j(x_i^{-1}) = -\delta_{i,j} x_i^{-1}$ and $D_j(uv) = D_j(u) + uD_j(v)$.*

Proof. These are direct consequences from Corollary 13 and equation (♣) □

Example 23. $\frac{\partial x_1 x_2}{\partial x_2} = x_1$, but $\frac{\partial x_2 x_1}{\partial x_2} = 1$

Abelinization and the Alexander matrix

Definition 24 (The Commutator Subgroup)**.** Let G be a group. The commutator subgroup is defined as follows

$$[G, G] = \{g_1 g_2 g_1^{-1} g_2^{-1} \in G \,|\, g_1, g_2 \in G\}$$

Definition 25 (The Abelianized Group)**.** Let G be a group. The abelianized group is defined as $G/[G, G]$. The canonical homomorphisms $\mathfrak{a} : G \to G/[G, G]$ is called the abelianizer. We also denote the abelianized group with G^{ab}.

Remark 26. We know that the fundamental group of a knot can be expressed as $\langle x_1, \dots x_n | r_1, \dots, r_s \rangle$ where r_i is an equation of the form $f(x_1, \dots, x_n) = 1$. Because of this we define the following.

Definition 27 (The Alexander matrix)**.** Let G be a group such that $G = \langle x_1, \dots x_n | r_1, \dots, r_s \rangle$. We define the Alexander matrix A of G as follows. First compute $\frac{\partial r_i}{\partial x_j} \in \mathbb{Z}F_n$, then consider them as elements in $\mathbb{Z}G$ and apply the abelianizer. These will be the i, j-th entry of the Alexander matrix. Written concretely we have $A = (a_{i,j})_{i,j}$ where

$$a_{i,j} = \mathfrak{a}\left(\gamma\left(\frac{\partial r_i}{\partial x_j}\right)\right)$$

Remark 28. It should be noted that at this point, A is an $s \times n$ matrix with entries in $\mathbb{Z}G^{\text{ab}}$. Also note that this definition relies on a order of the generators and relations, so we should really say 'a' Alexander matrix. We will later see that this order doesn't matter.

The Abelinization of the Fundamental Group

Theorem 29. *Let K be a knot and D its diagram. The abelianization of the fundamental group of a knot $\mathcal{W}(D)^{ab}$ is isomorphic to $\mathbb{Z}$. (From now own if I'm using D, I mean the diagram and not a derivative.)*

Proof. Recall the fundamental group. It is of the form $\langle x_1, \dots x_n | r_1, \dots, r_s \rangle$. Consider the case where each x_j corresponds to a loop around a different strand. Each strand starts and ends at some under-crossing, the only exception is the unknot whose only stands starts and ends at itself. Hence we must have that there are n crossings. Each crossing corresponds to a different relation r_i. All these r_i are of the form $x_a^- x_b^{\pm} x_c x_b^{\mp} = 1$ for some index a, b, c. But in the abelianization, we have that

$$x_a^- x_b^{\pm} x_c x_b^{\mp} \equiv x_a^- x_b^{\pm} x_c x_b^{\mp} x_b^{\pm} x_c^- x_b^{\mp} x_c = x_a^- x_c.$$

Or to put that last one in other words $x_a = x_c$ This means that loops around "neighboring" strands are actually considered the same in the abelianization. Hence we only have one generator, so $\mathcal{W}(D)^{\text{ab}} \cong \mathbb{Z}$. □

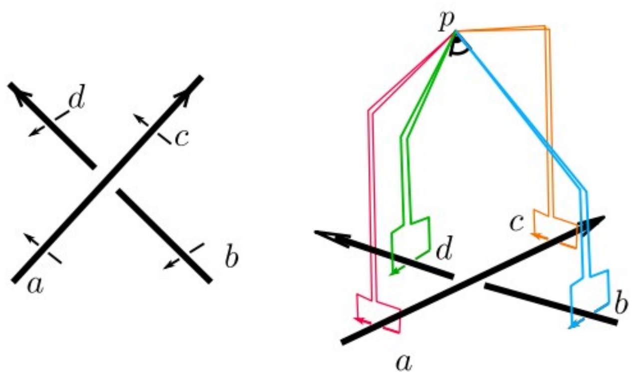

Figure 1: From lecture notes

Remark 30. There is a ring isomorphism $f : \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[t, t^{-1}]$ by
$\sum_i n_i m_i \mapsto \sum_i n_i t^{m_i}$.

Let me now give a small overview of what maps we have

$$F_n \xrightarrow{\gamma} \mathcal{W}(D) \xrightarrow{\mathfrak{a}} \mathcal{W}(D)^{\mathrm{ab}} \quad \cong_{\text{group}} \quad \mathbb{Z}$$

$$\mathbb{Z}F_n \xrightarrow{\frac{\partial r_i}{\partial x_j}} \mathbb{Z}F_n \xrightarrow{\gamma} \mathbb{Z}\mathcal{W}(D) \xrightarrow{\mathfrak{a}} \mathbb{Z}\mathcal{W}(D)^{\mathrm{ab}} \quad \cong_{\text{ring}} \quad \mathbb{Z}[\mathbb{Z}] \quad \cong \quad \mathbb{Z}[t, t^{-1}]$$

And our Alexander matrix lives in $\mathbb{Z}[t, t^{-1}]^{n \times n}$

(If the reader is a fan of commutative diagrams, they may draw some more arrows.)

Equivalences of Matrices

Recall from the proof of Theorem 29 that $s = n$. This means the Alexander matrix is an $n \times n$ matrix. But the exact matrix depended on the order in with we put our generators and relations. We will show that the different matrices we get from permuting the generators and relations are equivalent. Moreover, we can extract some information out of them

Definition 31 (Equivalent Matrices)**.** Let R be any commutative ring and consider any $m \times n$ matrix with entries in R. We say two matrices B and B' are equivalent if one can be obtained by the other via a finite sequence of the following operations:

1. Permuting rows or columns.
2. Adjoining a row of zeroes. i.e $A \mapsto \begin{bmatrix} A \\ \mathbf{0} \end{bmatrix}$.
3. adding a linear combination of rows to a row, or a linear combination of columns to a column.
4. adjoining a new row and column to the matrix in the following way
$A \mapsto \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$.

This is an equivalence relation. Crowell and Fox say transitivity is trivial, which I don't think it is. I might give an example on the board where it might go wrong. Think about how you would deal with a 2×2 matrix and 1×3 matrix that are both equivalent to some 4×4 matrix. Answer: This case is impossible.

Remark 32. The is some intuition behind these operations. For example, if the fundamental group can be expressed as $\langle x_1, \dots, x_n | r_1, \dots, r_n \rangle$, and we introduce a new generator x_{n+1}, then that generator has to come with the relation $x_{n+1} = 1$, otherwise we would end up with a different group. This is the interpretation of operation 4.

Knot Polynomials, Elementary ideals, and the Alexander Polynomial

Lemma 33. *For any free abelian group G of rank $m > 0$, $\mathbb{Z}G$ is a* gcd *domain whose units are $\pm 1g$ for any $g \in G$.*

Definition 34 (The Knot Polynomial)**.** Let $G = \langle x_1, \dots, x_n | r_1, \dots, r_s \rangle$ be a free group and A a Alexander polynomial. For any integer $k \geq 0$ we define the k-th knot polynomial Δ_k to be the gcd of the determinants of all submatrices of A of size $(n-k) \times (n-k)$. We extend this definition with

$$\Delta_k = 0 \text{ if } n - k > m$$
$$\Delta_k = 1 \text{ if } n - k \leq 0.$$

Theorem 35. *The k-th knot polynomials exist and are unique upto multiplication by $\pm t^r$ for some $r \in \mathbb{Z}$.*

For the rest of this section we will try to give an overview of the proof.

Definition 36 (Elementary Ideals)**.** Let R be any commutative ring and consider an $m \times n$ matrix with entries in R. For any integer $k \geq 0$, define the k-th elementary ideal $E_k(A)$ as follows:

- If $0 < n - k \leq m$, then $E_k(A)$ is the ideal generated by the determinants of all submatrices of A of size $(n-k) \times (n-k)$.
- If $n - k > m$, then $E_k(A) = \{0\}$.
- If $n - k \leq 0$, then $E_k(A) = R$.

Theorem 37. *The k-th knot polynomial Δ_k is the generator of the smallest principal ideal containing $E_k(A)$.*

Proof. This is because of the properties of a gcd domain. The gcd of a finite set of elements is the generator of the smallest principle ideal that contains them. □

Lemma 38. *The elementary ideals from a chain.*

$$E_0(A) \subset E_1(A) \subset \cdots \subset E_n(A) = E_{n+1} = \cdots = R.$$

Proof. The determinant of a matrix can be written as an sum over the elements of a row or column, times a smaller matrix. □

Theorem 39. *Equivalent matrices define the same chain of elementary ideals.*

Proof. The proof of this can be found in [1]. □

Corollary 40. *The* gcd *of the determinants of the submatrices of size $(n-k) \times (n-k)$ are unique upto multiplication by a unit in $\mathbb{Z}[t, t^{-1}]$. (And a unit in $\mathbb{Z}[t, t^{-1}]$ is of the form $\pm t^n$ for some $n \in \mathbb{Z}$.)*

Remark 41. Actually it turns out that $E_1(A)$ is always principal ideal, hence any the determinant of any $(n-1) \times (n-1)$ submatix of A gives the same knot polynomial.

Proof. The proof that $E_1(A)$ is principle can be found in [1]. □

Remark 42. The polynomial $\Delta_1$obtained from $E_1(A)$ is not the Alexander polynomial, but we can normalize it to get the Alexander polynomial.

Lemma 43. *The Alexander polynomial Δ^A satisfies the following two relations*

$$\Delta^A(t) = \Delta^A(t^{-1})$$
$$\Delta^A(1) = 1.$$

Remark 44. These two relations allow us

Definition 45 (The Alexander Polynomial)**.** The Alexander polynomial Δ^A is the normalization of Δ_1.

Finally an Example

We going to discuss the example of the trifoil knot, where the group representation is already simplified.

Example 46. Let K be the trefoil knot. Its fundemental group is represented by $\langle x, y | xyx = yxy \rangle$. We compute the Alexander matrix of this representation by first calculation the formal derivatives.

$$\frac{\partial xyxy^{-1}x^{-1}y^{-1}}{\partial x} = 1 + xy - xyxy^{-1}x^{-1}$$
$$\frac{\partial xyxy^{-1}x^{-1}y^{-1}}{\partial y} = x - xyxy^{-1} - xyxy^{-1}x^{-1}y^{-1}$$

This means that the Alexander matrix is equal to $A = \begin{bmatrix} 1 - t + t^2 & -1 + t - t^2 \end{bmatrix}$

We have $n = 2$ so Δ_1 is the gcd of all the determinants of all $(n-1) \times (n-1)$ matices, which in this case are 1×1 matices. We see that the gcd is equal to $-1 + t - t^2$. We now need to normalize this polynomial. We multiply it by $-t^{-1}$ to get $t^{-1} - 1 + t$. Observe that for this polynomial $\tilde{\Delta}_1$, we have $\tilde{\Delta}_1(t) = \tilde{\Delta}_1(t^{-1})$ and $\tilde{\Delta}_1(1) = 1$. Hence we found that the Alexander polynomial is $\tilde{\Delta}_1 = \Delta^A = t^{-1} - 1 + t$, which I have checked is the true Alexander polynomial of the trefoil knot.

The Alexander Matrix for XC-Tangle Diagrams

Yeah... No.

References

[1] Richard H Crowell and Ralph Hartzler Fox. *Introduction to knot theory.* Vol. 57. Springer Science & Business Media, 2012.

[2] Ralph H Fox. "A quick trip through knot theory". In: *Topology of 3-manifolds and related topics* (1962).