Knot Theory Seminars: Braided Diagrams

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June 23, 2020

Outline

In this talk I will:

- Introduce the notion of a Hopf algebra
- Introduce the notion of a braided category
- Generalise Hopf algebras to braided categories
- Show how to do algebra with knots (instead of the other way around)

Hopf algebras

A Hopf algebra is:

- An algebra (vector space with multiplication)
- A coalgebra (algebra axioms, but arrows reversed)
- A bialgebra (algebra and coalgebra structures compatible)
- Equipped with an antipode S (generalized inverse)

Hopf algebras

Algebra: product $\cdot\ : A \times A \to A$ that is bilinear.
This is more compactly expressed as $\cdot : A \otimes A \to A$ being linear.

$$\begin{tikzcd}
 & A \otimes A \otimes A \arrow[dl, "\cdot\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{id}\otimes\cdot"] & \\
A \otimes A \arrow[dr, "\cdot"'] & & A \otimes A \arrow[dl, "\cdot"] \\
 & A &
\end{tikzcd}$$

$$\begin{tikzcd}
 & A \otimes A \arrow[dr, "\cdot"] & \\
k \otimes A \arrow[ur, "\eta\otimes\mathrm{id}"] \arrow[rr, phantom, "\cong"] & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \otimes A \arrow[dr, "\cdot"] & \\
A \otimes k \arrow[ur, "\mathrm{id}\otimes\eta"] \arrow[rr, phantom, "\cong"] & & A
\end{tikzcd}$$

Coalgebra: coproduct $\Delta : A \to A \otimes A$

$$
\begin{tikzcd}
& C \otimes C \otimes C & \\
C \otimes C \arrow[ur, "\Delta \otimes \mathrm{id}"] & & C \otimes C \arrow[ul, "\mathrm{id} \otimes \Delta"'] \\
& C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& C \otimes C \arrow[dl, "\epsilon \otimes \mathrm{id}"'] & \\
k \otimes C \arrow[rr, phantom, "\cong"] & & C \arrow[ul, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& C \otimes C \arrow[dl, "\mathrm{id} \otimes \epsilon"'] & \\
C \otimes k \arrow[rr, phantom, "\cong"] & & C \arrow[ul, "\Delta"']
\end{tikzcd}
$$

Compatability: Δ should be an algebra morphism

$$\begin{tikzcd}
H \otimes H \arrow[rr, "\cdot"] \arrow[d, "\Delta\otimes\Delta"'] & & H \arrow[rr, "\Delta"] & & H \otimes H \\
H \otimes H \otimes H \otimes H \arrow[rrrr, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"'] & & & & H \otimes H \otimes H \otimes H \arrow[u, "\cdot\otimes\cdot"']
\end{tikzcd}$$

$$\begin{tikzcd}
H \arrow[rr, "\epsilon"] & & k \\
& H \otimes H \arrow[ul, "\cdot"] \arrow[ur, "\epsilon\otimes\epsilon"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
k \arrow[rr, "\eta"] \arrow[dr, "\eta\otimes\eta"'] & & H \arrow[dl, "\Delta"] \\
& H \otimes H &
\end{tikzcd}$$

Hopf algebras

Antipode: a map $S : H \to H$ that acts sort of as an inverse

$$\begin{tikzcd}
& H \otimes H \arrow[rr, "S\otimes\mathrm{id}"] & & H \otimes H \arrow[dr, "\cdot"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & k \arrow[rr, "\eta"] & & H \\
& H \otimes H \arrow[rr, "\mathrm{id}\otimes S"'] & & H \otimes H \arrow[ur, "\cdot"'] &
\end{tikzcd}$$

Braided categories

A braided category is a category that emulates $\mathtt{Vec}$.
In braided cateogories you can:

- Form tensor products (and there is a unit $\underline{1}$)
- Swap tensor products via a map Ψ

Ψ is a natural transformation and obeys the hexagon identities.

Braided categories

A *natural transformation* $\alpha : F \to G$ consists of the information of a morphism $\alpha_C : \mathscr{F}(C) \to \mathscr{G}(C)$ in $\mathcal{D}$, for each $C \in \operatorname{ob} \mathcal{C}$.

$$\begin{tikzcd}
\mathscr{F}(C) \arrow[r, "\mathscr{F}(f)"] \arrow[d, "\alpha_C"'] & \mathscr{F}(C') \arrow[d, "\alpha_{C'}"] \\
\mathscr{G}(C) \arrow[r, "\mathscr{G}(f)"'] & \mathscr{G}(C')
\end{tikzcd}$$

This is also called *functoriality*.
Ψ is a natural transformation with $\Psi_{V,W} : V \otimes W \to W \otimes V$.

Braided categories

Hexagon identities:

$$
\begin{tikzcd}
 & V \otimes (W \otimes Z) \arrow[dl, "\mathrm{id}\otimes\Psi"'] \arrow[dr, "\Phi^{-1}"] & \\
V \otimes (Z \otimes W) \arrow[d, "\Phi^{-1}"'] & & (V \otimes W) \otimes Z \arrow[d, "\Psi"] \\
(W \otimes Z) \otimes W \arrow[dr, "\Psi\otimes\mathrm{id}"'] & & Z \otimes (V \otimes W) \arrow[dl, "\Phi^{-1}"] \\
 & (Z \otimes V) \otimes W &
\end{tikzcd}
$$

(The other hexagon identity is very similar.)

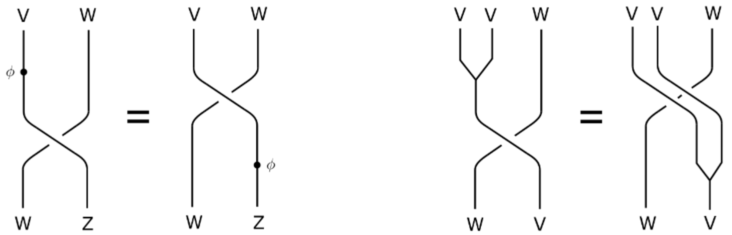
Braided categories

Let's put these things into nice pictures!

$\Psi =$  $\Psi^{-1} =$ 

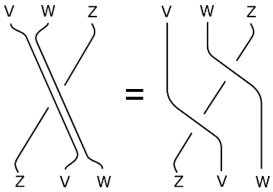
We'll work with strings representing objects. Horizontal juxtaposition of strings means taking $\otimes$. Thus Ψ is a swapping of strings.

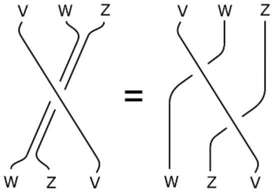
In these 'braided diagrams' functionality looks like this:



On the right hand we see the case that $\phi = \cdot$ for V an algebra living in a braided category.
This is a very important point: functoriality lets us pull morphisms over crossings!

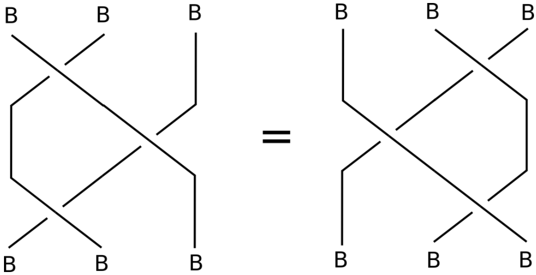
The hexagon identities simply become:





Braided categories

From these axioms one easily derives the *braid relation*



This is why we call them *braided categories*!

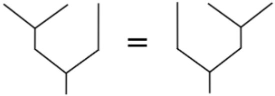
Braided groups

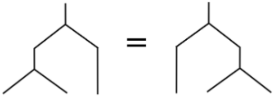
Now onto 'braided groups':
These are Hopf algebra objects in a braided category.
To define them we need the braided diagrams extensively.
Hopf algebras are just braided groups in `Vec`, they have:

- A multiplication $\cdot$, represented by Y
- A unit η: T
- A comultiplication Δ: ⅄
- A counit ϵ: $\perp$
- An antipode S, represented by a circled S

Braided groups

Associativity and coassociativity:

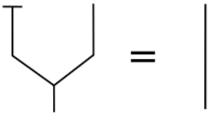


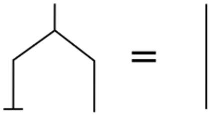


Note the reflection symmetry of braided groups in terms of braided diagrams:

$$\text{Horizontal reflection} \quad \Longleftrightarrow \quad \text{Arrow-reversal}$$

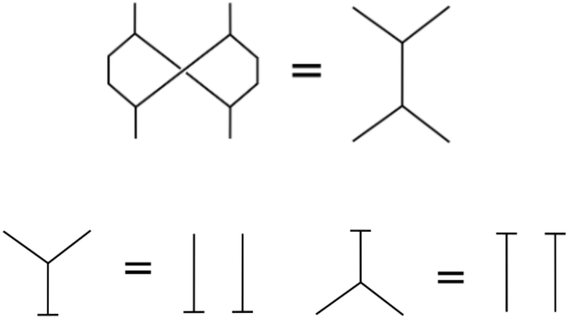
Unit and counit axioms:





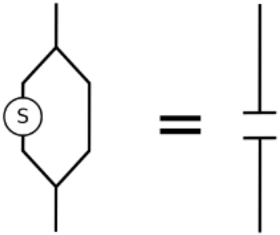
Braided groups

Comultiplication is a braided algebra morphism:



Braided groups

Antipode axiom:



Tangled mathematics

Using braided diagrams instead of commutative diagrams allows us to to proofs by simply treating the strings as physical pieces of knotted string!
Functoriality + braid relation let us treat crossings like physical crossings.
To show off, let's prove the following statement:

$$S \circ \cdot = \cdot \circ \Psi_{B,B} \circ (S \otimes S) \qquad \text{and} \qquad \Delta \circ S = (S \otimes S) \circ \Psi_{B,B} \circ \Delta$$

These statements are the braided group analogue of

$$(gh)^{-1} = h^{-1}g^{-1}$$

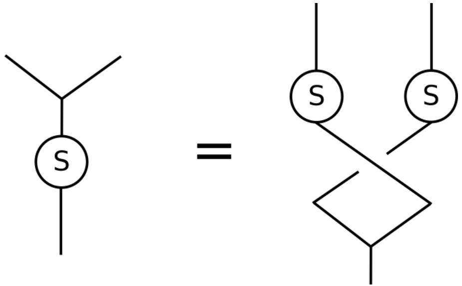
for g, h in a group G.

The proof is a bit involved: it takes up 16 slides.
The idea is analogous to the proof for groups: we sneak in terms $h^{-1}h$ and $g^{-1}g$ and cancel.

$$\begin{aligned}(gh)^{-1} &= h^{-1}h(gh)^{-1} = h^{-1}g^{-1}gh(gh)^{-1} \\ &= (h^{-1}g^{-1})(gh)(gh)^{-1} = h^{-1}g^{-1}.\end{aligned}$$

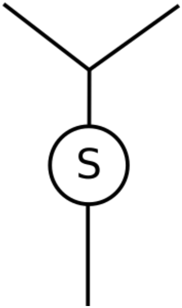
Note that this proof implicitly uses associativity.

Now for the braided group statement.
In a braided group diagram, the statement is:

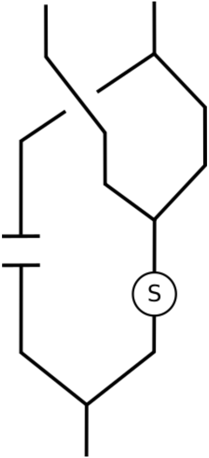


We'll prove it by sneaking in a bunch of stuff, rearranging, and then cancelling out.

Proof:



Tangled mathematics



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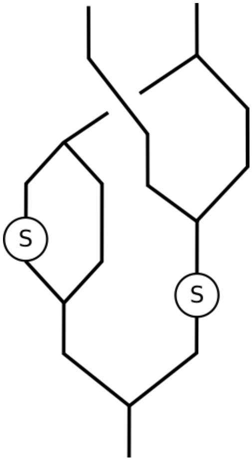

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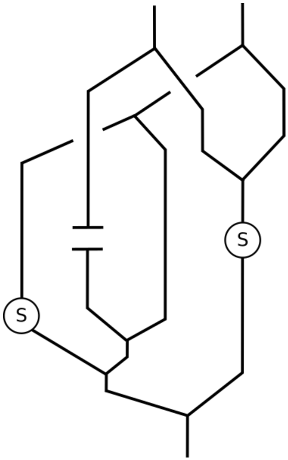

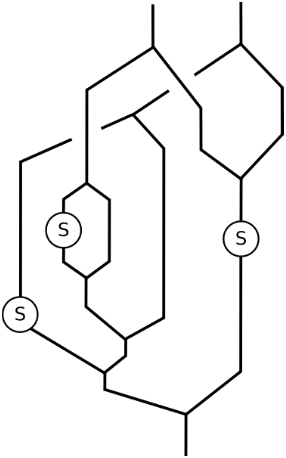
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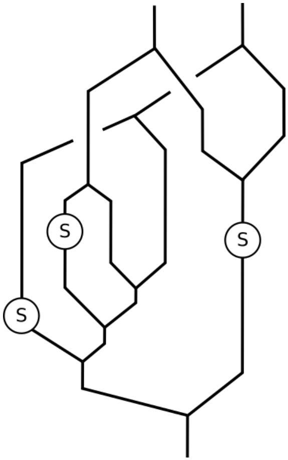
Tangled mathematics



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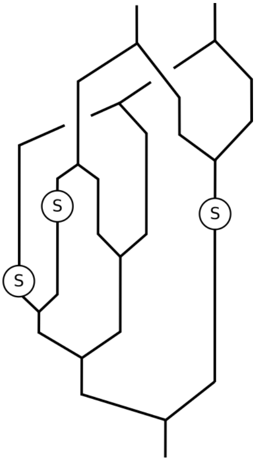

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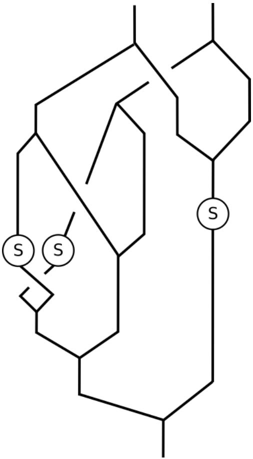

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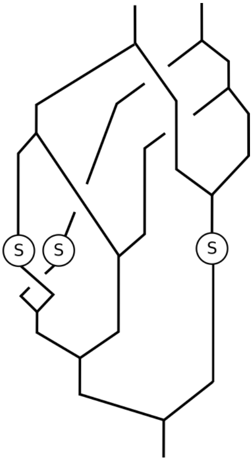

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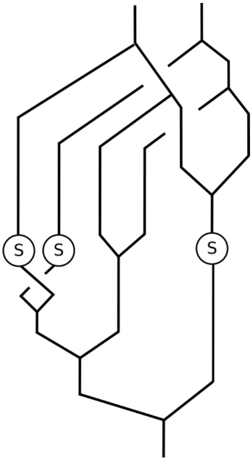

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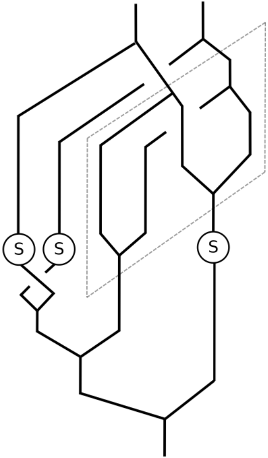

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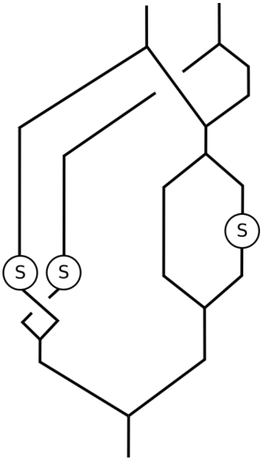

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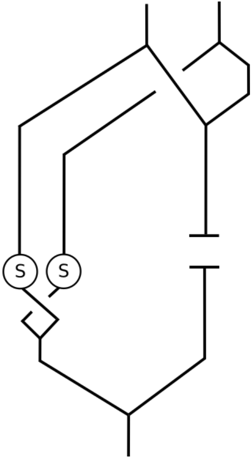

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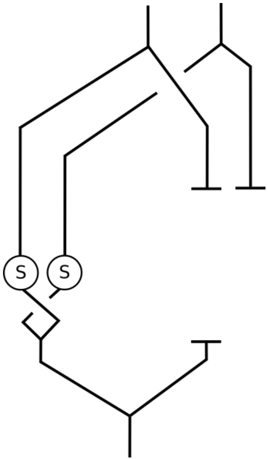

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