# Knot Theory Seminar 20/02/2020

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These notes are based on the book Introduction to Vassiliev Knot Invariants and the notes https://web.northeastern.edu/beasley/MATH7375/schedule.html (Lectures 1 and 2).

## 1 Definition of a knot

Intuitively a knot is a knotted loop of string in space. There are several ways to define a knot, the simplest definition is given as follows:

**Definition 1.1.** A **parametrised knot** is an embedding of the circle $S^1$ into $\mathbb{R}^3$.

Recall that for smooth manifolds $M$ and $N$ an *embedding* $f : M \to N$ is a smooth injective map, such that the differential never vanishes (i.e. injective immersion).

In some cases it is more useful to consider one of the following definitions of a knot:

**Definition 1.2.** Three ways to define a knot:

1. A *topological knot* $K^{\text{top}}$ is a subset of $\mathbb{R}^3$ which is homeomorphic to $S^1$.
2. A *smooth knot* $K^{\text{smooth}}$ is a subset of $\mathbb{R}^3$ which is diffeomorphic to $S^1$.
3. A *piecewise linear knot* $K^{\text{pl}}$ is a closed non-selfintersecting polygonal line in $\mathbb{R}^3$.

**Remark 1.3.** Knots are denoted by capital letters, while the 'corresponding' embedding is given by a lower case letter.

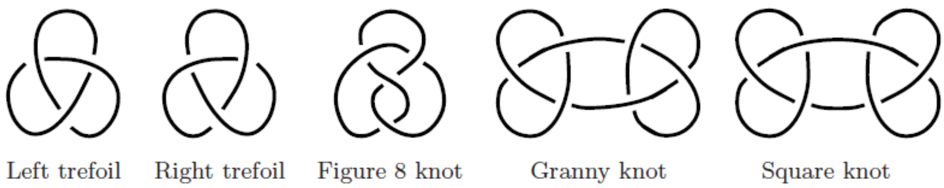
A circle $S^1$ has two choices of an orientation. If we pick one, we give an orientation to all knots. A way to state this more formally,

**Definition 1.4.** An *oriented knot* is an equivalence class of parametrised knots under orientation preserving diffeomorphisms of the parametrised circle. An *unoriented knot knot* is an equivalence class of parametrised knots under all diffeomorphism.

**Remark 1.5.** We will assume a knot is oriented unless stated otherwise. Moreover, we choose the orientation to be counter-clockwise.

It is usually quite hard to draw knots in three dimensions. Therefore we use Knot diagrams. A knot diagram is a plane curve whose only singularities are transversal double points, which we will call crossings, together with a choice of one branch, which we will call an overcrossing and the other one will be called an undercrossing. In other words, a knot diagram is a projection of a knot on the plane. An orientation of a knot will be denoted by arrows.

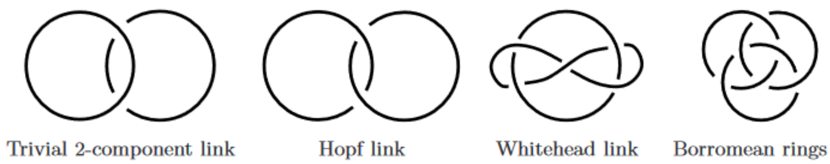
**Example 1.6.** A few examples of knots (and their knot diagrams) are:




Another useful definition is the one of a link.

**Definition 1.7.** A **link** $L$ is a finite disjoint union of knots.

**Example 1.8.** Some examples of links:




In many cases, a result proven for knots will also hold for links.

# 2 Isotopy

We want to consider knots only up to a suitable notion of equivalence. Intuitively, we want to say two knots are equal when we can deform the first knot into the second knot and vice versa by a continuous deformation. This is made formal by a so called isotopy.

**Definition 2.1.** two parametrised (smooth) knots $f_0, f_1 : S^1 \to \mathbb{R}^3$ are isotopic if there is a smooth map $F : S^1 \times I \to \mathbb{R}^3$ such that $F(-,t)$ is an embedding for all $t$ and $F(-,0) = f_0$ and $F(-,1) = f_1$

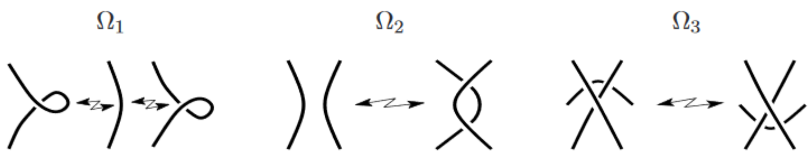
**Remark 2.2.** A knot isotopic to $S^1$ is called the trivial knot or unknot.

**Definition 2.3.** Two parametrised knots, $f$ and $g$ are **ambient isotopic** if there is a smooth map $\Psi : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ with the property that $\Psi(-,t)$ is a diffeomorphism for all $t$ with $\Psi(\theta,t) = \psi_t(\theta)$ such that $\psi_0 = \mathrm{id}$ and $\psi_1 \circ f = g$.

**Remark 2.4.** In the mentioned book is is shown that ambient isotopy is equivalent to the definition of isotopy given in (2.1).

# 3 Reidemeister theorem

**Theorem 3.1 (Reidemeister).** Two unoriented knots $K_1$ and $K_2$ are isotopic if and only if a diagram of $K_1$ can be transformed into a diagram of $K_2$ by a sequence of ambient isotopies of the plane and one of the local moves $\Omega_1, \Omega_2$ and $\Omega_3$ given in the following diagram




in other words,

$$\frac{\{\text{knots}\}}{\{\text{isotopy}\}} \cong \frac{\{\text{knotdiagrams}\}}{\{\text{planarisotopyandReidmeistermoves}\}}$$

**Remark 3.2.** In the case of oriented knots, we have to equip the Reidemeister moves with all possible orientations.

# 4 Connected sum of knots

We can fuse two knots into one new knot by the so called connected sum of knots. This is done as follows.

Consider two oriented knots $K_1$ and $K_2$ such that $K_1$ and $K_2$ can be separated by some sphere $S^2 \subset \mathbb{R}^3$. We then construct the connected sum of $K_1$ and $K_2$ by the following steps:

1. Take two intervals $U_1 \subset K_1$ and $U_2 \subset K_2$.
2. Cut out these intervals from the knot, this leaves a pair of tangles $T_1$ and $T_2$.
3. Joint the loose ends along the boundary of a rectangular strip that intersects the spheres separating $K_1$ and $K_2$ in a single arc. The orientation must stay the same between the two tangles.

The connected sum of knots $K_1$ and $K_2$ is denoted by $K_1 \# K_2$.

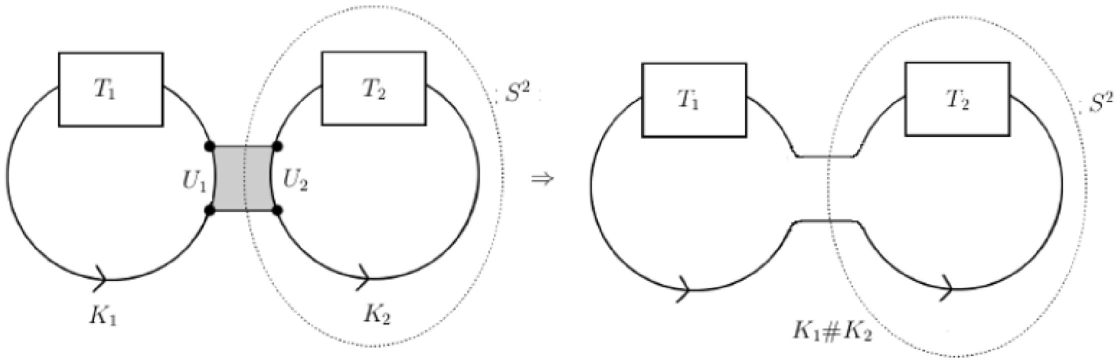



Figure 1: Connected sum of two knots

**Definition 4.1.** A knot $K$ is prime if $K = K_1 \# K_2$ implies that either $K_1$ or $K_2$ is trivial.

# 5 Exercises

- Show connected sum of two knots is well defined and commutative and that $K \# O = K$. Also show that the connected sum of two knots is associative.
- Show that the connected sum of knots is not well defined for non-oriented knots.