Hyperbolic Knot Theory 1

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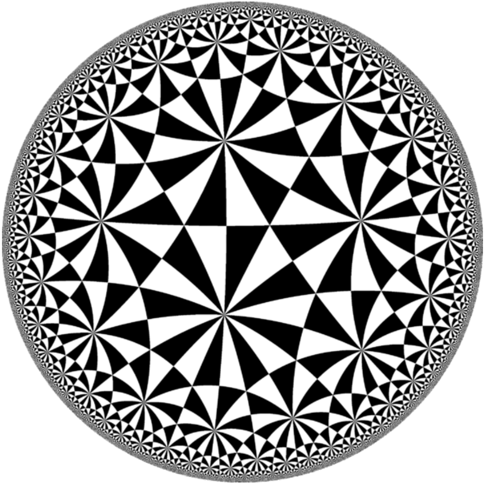
Hyperbolic space



Hyperbolic half plane

$$\mathbb{H}^2 = \{x + iy \in \mathbb{C} : y > 0\},$$

or

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$$

First fundamental form:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

Riemannian geometry

Riemannian metric associates to each point $p \in M$ an inner product $\langle ., . \rangle_p$ on T_pM.

Riemannian geometry

Write $v \in T_pM$ as $v = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$, then we can express the metric on $\mathbb{H}^2$ as

$$\langle v, w \rangle_{(x,y)} = (v_x, v_y) \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix} \begin{pmatrix} w_x \\ w_y \end{pmatrix}$$

Properties of hyperbolic space

Example: Consider a horizontal line at fixed height $h > 0$, running from $(0, h)$ to $(1, h)$.

$$\gamma(t) = (t, h), \text{ where } t \in [0, 1]$$

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Arc length:

$$\begin{aligned} |\gamma| &= \int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} dt \\ &= \frac{1}{h^2} \end{aligned}$$

Properties of hyperbolic space

Example: Consider a vertical line running from (x, a) to (x, b), with $0 < a < b$.

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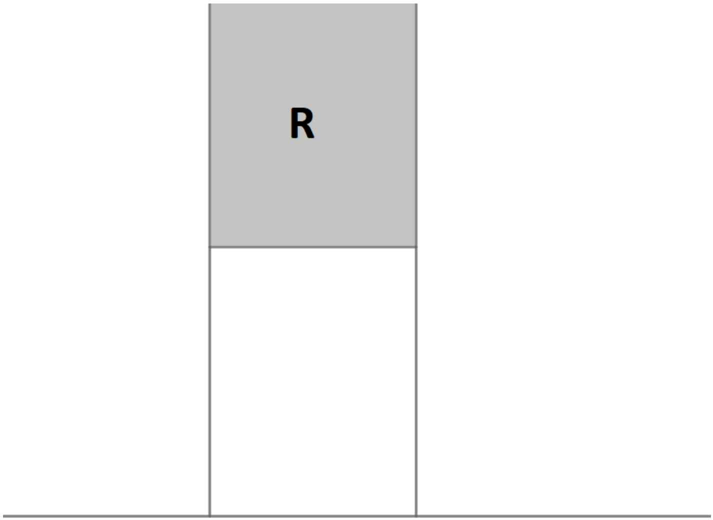
Arc length:

$$\begin{aligned} |\gamma| &= \int_a^b \sqrt{0+1}\frac{1}{t}dt \\ &= \log(\frac{b}{a}) \end{aligned}$$

Properties of hyperbolic space

Example: Consider region R bounded by the lines $x = 0$, $x = 1$, $y = 1$, and the boundary at infinity $\partial\mathbb{H}^2$

Properties of hyperbolic space



Properties of hyperbolic space

$$\begin{aligned}\mathrm{Area}(R) &= \int_R \frac{1}{y^2}\,dxdy \\ &= \int_0^1 \int_1^\infty \frac{1}{y^2}\,dydx \\ &= \int_0^1 dx = 1\end{aligned}$$

Geodesics

Definition A *geodesic* between points p and q is a length minimizing curve between those points.

Geodesics

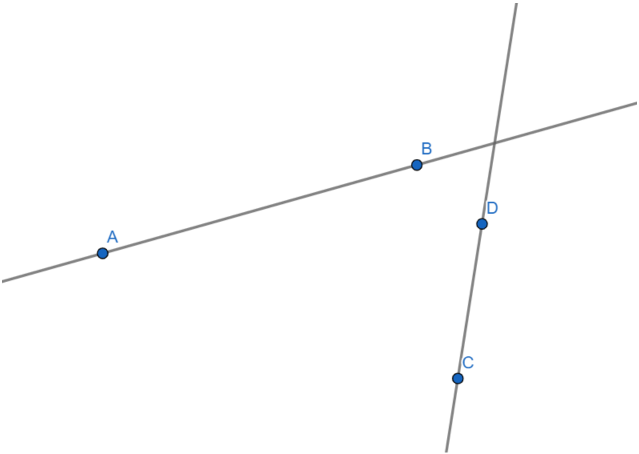


Figure 1: Geodesics in Euclidean space

Geodesics

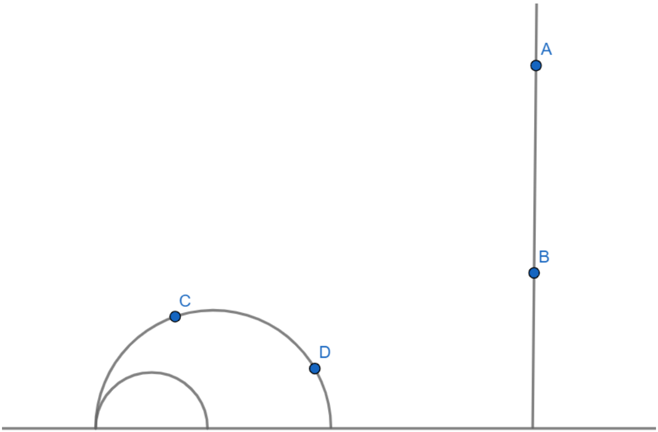


Figure 2: Geodesics in Hyperbolic space

Isometries

Definition: An *isometry* between Riemannian manifolds M and N is a diffeomorphism $f : M \to N$ such that

$$\langle v, w \rangle_p = \langle df_p(v), df_p(w) \rangle_{f(p)}$$

for all $p \in M$ and $v, w \in T_pM$.

Isometries

Theorem: The full group of isometries of $\mathbb{H}^2$ is generated by reflections through geodesics in $\mathbb{H}^2$ and the full group of orientation preserving isometries of $\mathbb{H}^2$ is the group of *linear fractional transformations*

$$z \mapsto \frac{az+b}{cz+d},$$

with $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$.

Isometries

Lemma: Given any three distinct points $z_1, z_2, z_3 \in \partial\mathbb{H}^2$, there exists an orientation preserving isometry of $\mathbb{H}^2$ taking z_3 to ∞ and taking $\{z_1, z_2\}$ to $\{0, 1\}$.

Triangles

Ideal triangle: Edges are geodesics in $\mathbb{H}^2$, vertices lie on $\partial\mathbb{H}^2$

Triangles

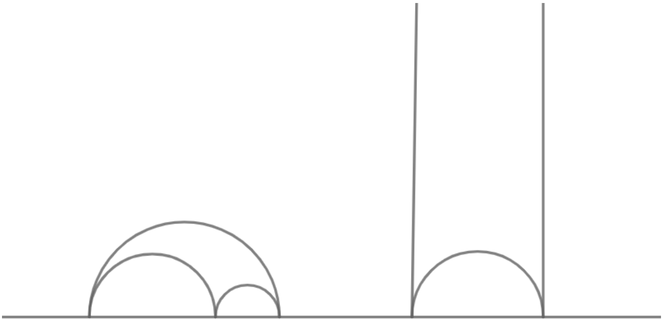


Figure 3: Examples of ideal triangles in $\mathbb{H}^2$

Triangles

Definition: A *horocycle* centered at an ideal point $p \in \partial\mathbb{H}^2$ is defined as a curve perpendicular to all geodesics through p. The interior is called a *horoball*.

Triangles

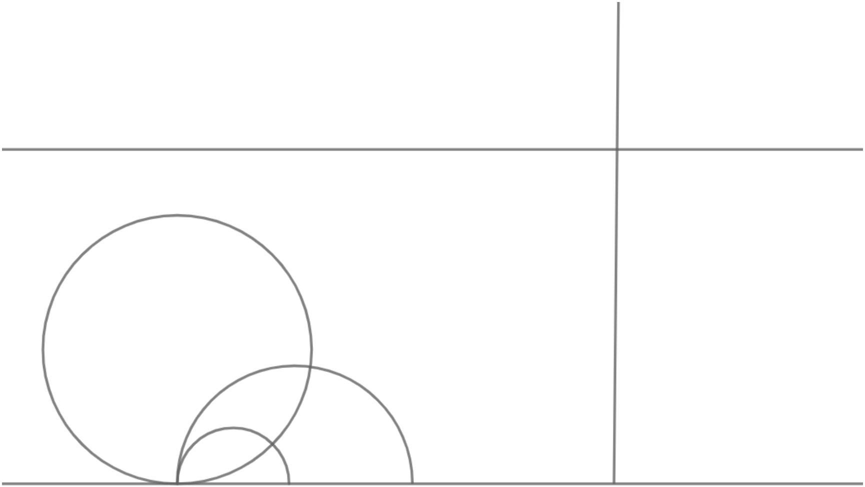


Figure 4: Examples of horocycles

Triangles

Lemma: The area of an ideal triangle is finite.