UNIVERSALITY OF THE BRAID CATEGORY

RUBEN MUD

1. Introduction

Let $\{*\}$ be a singleton set, let G be a group, let F_* denote the free group on the point $*$. An element of F_*, is of the form $\prod_{i=1}^{n} *$, or $\prod_{i=1}^{n} *^{-1}$ or e_* (denoting the neutral element of F_*). Given a homomorphism $\varphi : F_* \longrightarrow M$, then $\varphi(\prod_{i=1}^{n} *) = \prod_{i=1}^{n} \varphi(*)$. So this homomorphism is uniquely determined by the image of the point $*$ under φ. In other words, there is a bijection $\mathrm{Hom}_{\mathbf{Grp}}(F_*, G) \cong \mathrm{Hom}_{\mathbf{Set}}(*, G)$.

This means that F_* has the following universal property: For the inclusion map $\iota : \{*\} \longrightarrow F_*$ given by $\iota(*) = *$. We have that for every function $f : * \longrightarrow G$ there exists a unique group homomorphism $\overline{f} : F_* \longrightarrow G$ such that $\overline{f}\iota = f$. This property is indeed universal, for if H was a group with the same property, then we have uniquely induced group homomorphisms $\bar{i}$ and $\bar{\iota}$ fitting in commutative diagrams:

$$
\begin{tikzcd}
 & H \arrow[d, dotted, "\bar{\iota}"] \\
\{*\} \arrow[ur, "i"] \arrow[r, "\iota"] \arrow[dr, "i"'] & F_* \arrow[d, dotted, "\bar{i}"] \\
 & H
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F_* \arrow[d, dotted, "\bar{i}"] \\
\{*\} \arrow[ur, "\iota"] \arrow[r, "i"] \arrow[dr, "\iota"'] & H \arrow[d, dotted, "\bar{\iota}"] \\
 & F_*
\end{tikzcd}
$$

Notice that the identity homomorphisms id_H and id_{F_*} would also make the outer triangles commute. By uniqueness of the induced homomorphisms, we must find that $\bar{\iota}\bar{i} = \mathrm{id}_{F_*}$ and $\bar{i}\bar{\iota} = \mathrm{id}_H$. Therefore there is a group isomorphism $H \cong F_*$.

The goal of the presentation accompanying this handout is to show that the braid category has a very similar universal property as the one described above. Namely that it is the free strict braided category generated by one object.

---

email: r.n.mud@student.rug.nl

2. Preliminaries

2.1. Definition. *A category* $\mathcal{C}$ *consists of:*

* *A collection of objects* $\mathrm{ob}(\mathcal{C})$
* *For each pair of objects* $A, B \in \mathrm{ob}(C)$ *a set of morphisms* $\mathrm{Hom}_{\mathcal{C}}(A, B)$
* *For each object* $A \in \mathrm{ob}(C)$ *an identity morphism* $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$.

Together with a composition law $\circ : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, B)$ *, mapping a pair of morphisms* (g, f) *to their composite* gf*. Furthermore, composition is associative,* $h(gf) = (hg)f$*, and satisfies the unit axiom,* $\mathrm{id}_B f = f \,\mathrm{id}_A = f$.

2.2. Example. There is a category of sets, is denoted **Set**. Its objects are sets, and its morphisms are functions.

2.3. Example. The one point discrete category is a category with one object $*$. Its morphism set is given by $\mathrm{Hom}(*, *) = \{\mathrm{id}_*\}$.

2.4. Example. There is a category of groups, denoted **Grp**. The objects are groups, and the morphisms are group homomorphisms.

2.5. Example. There is a category of k-vector spaces denoted $\mathbf{Vect}_k$. The objects are k-vector spaces and the morphisms are k-linear mappings.

2.6. Example. For two categories $\mathcal{C}$ and $\mathcal{D}$ there is a product category $\mathcal{C} \times \mathcal{D}$, its objects are given by $\mathrm{ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{ob}(\mathcal{C}) \times \mathrm{ob}(\mathcal{D})$. For two objects, (C_1, D_1) and (C_2, D_2), the associated set of morphisms is given by $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((C_1, D_1), (C_2, D_2)) = \mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \times \mathrm{Hom}_{\mathcal{D}}(D_1, D_2)$. Composition is done component wise. Moreover, $\mathrm{id}_{(C,D)} = (\mathrm{id}_C, \mathrm{id}_D)$.

2.7. Example. A group G can be viewed as a category with one object $*$ and morphism set $\mathrm{Hom}(*, *) = G$

2.8. Definition. *For two categories* $\mathcal{C}$ *and* $\mathcal{D}$*, a functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *consists of a function* $F_0 : \mathrm{ob}(\mathcal{C}) \longrightarrow \mathrm{ob}(\mathcal{D})$ *that is compatible with the composition on* $\mathcal{C}$ *and* $\mathcal{D}$ *in the following sense. For each pair of objects* $A, B \in \mathrm{ob}(\mathcal{C})$*, there is a function* $F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F_0A, F_0B)$ *that satisfies* $F_{A,C}(gf) = F_{B,C}(g)F_{A,B}(f)$ *and* $F_{A,A}(\mathrm{id}_A) = \mathrm{id}_{F_0A}$.

2.9. Example. There is a forgetful functor $U : \mathbf{Grp} \longrightarrow \mathbf{Set}$ that forgets about the group structure. A group G is mapped to its underlying set G. Group homomorphisms are mapped to their underlying functions.

2.10. Example. There is a functor called the free functor $F_{(-)} : \mathbf{Set} \longrightarrow \mathbf{Grp}$ whose action on objects is to map a set S to the free group F_S consisting of all finite words in the alphabet S. The action on morphisms is as follows: given a function $f : S \to T$, we have a rule that assigns each letter of the alphabet S to a letter in the alphabet T. This induces a group homomorphism as follows, take a word $w = \prod_{i=1}^{n} s_{k_i} \in F_S$, then $F_f(\prod_{i=1}^{n} s_{k_i}) := \prod_{i=1}^{n} f(s_{k_i}) \in F_T$. Note that $F_f(e_S) := e_T$, (the empty product is mapped to the empty product). To see that F is a functor, we moreover need that it behaves well under composition. Suppose that we have functions $S \xrightarrow{f} T \xrightarrow{g} U$. Then $F_{gf}(s) = g(f(s)) = F_g(f(s)) = F_g F_f(s)$ for each letter $s \in S$. Moreover, $F_{\mathrm{id}_S}(s) = s$ for all $s \in S$. Thus $F_{\mathrm{id}_S} = \mathrm{id}_{FS}$.

2.11. Example. The tensor product $\otimes_k : \mathbf{Vect}_k \times \mathbf{Vect}_k \longrightarrow \mathbf{Vect}_k$ on k-vector spaces is a functor. In lecture 4 it was shown that for a pair of k-vector spaces (V, W), their tensor product $V \otimes_k W$ exists, so the action on objects is well defined. Moreover it was stated the tensor product maps tuples of k-linear functions (f, g) to a k-linear function $f \otimes_k g$. The statement that $\otimes_k$ is a functor, amounts to saying that $(f_1 \otimes_k g_1)(f_2 \otimes_k g_2) = f_1 f_2 \otimes_k g_1 g_2$ and that $\mathrm{id}_V \otimes_k \mathrm{id}_W = \mathrm{id}_{V \otimes_k W}$.

2.12. Definition. *A monoidal category* $(\mathcal{C}, \otimes, I, a, l, r)$ *consists of:*

* *a category* $\mathcal{C}$,
* *a functor* $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$,
* *an object* I *of* $\mathcal{C}$, *called the tensor unit,*
* *natural isomorphisms;* $a_{XYZ} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$,
  $l_X : I \otimes X \longrightarrow X$, $r_X : X \otimes I \longrightarrow X$

subject to two coherence axioms *expressed by commutativity of the following diagrams:*

$$\begin{tikzcd}
((W \otimes X) \otimes Y) \otimes Z \arrow[r, "a"] \arrow[d, "a\otimes 1"'] & (W \otimes X) \otimes (Y \otimes Z) \arrow[r, "a"] & W \otimes (X \otimes (Y \otimes Z)) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "a"'] & & W \otimes ((X \otimes Y) \otimes Z), \arrow[u, "1\otimes a"']
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "a"] \arrow[dr, "r\otimes 1"'] & & X \otimes (I \otimes Y) \arrow[dl, "1\otimes l"] \\
& X \otimes Y. &
\end{tikzcd}$$

A strict monoidal category is a monoidal category for which a, l, r *are identity morphisms.*

2.13. Example. The category $\mathbf{Vect}_k$ is monoidal, but not strict monoidal. Its tensor unit is given by k.

2.14. Example. Given the one point discrete category $\begin{tikzcd} * \arrow[loop above, "\mathrm{id}_*"] \end{tikzcd}$, we can concatenate finitely many copies of the discrete category to obtain a discrete category $\mathbb{N}$ whose objects consist of all tuples of $*$. For example the n-tuple $(*, *, \ldots, *)$ would be an object in this category. The only morphisms of $\mathbb{N}$ defined as such, are the identity morphisms. Concatenation of tuples provides a strict monoidal functor $\otimes : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ sending the n-tuple and the m-tuple to the $(n+m)$-tuple. The tensor unit is the empty tuple $()$. We say that $\mathbb{N}$ is freely generated by the one point discrete category.

2.15. Example. The power set $\mathcal{P}(S)$ of a set S can be seen as a strict monoidal category. The objects are the subsets of S, there is an arrow $S \longrightarrow T$ precisely whenever $S \subseteq T$. Intersection then provides the monoidal functor $\cap : \mathcal{P}(S) \times \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$ for which S acts as the tensor unit.

2.16. Definition. *For two monoidal categories* $(\mathcal{C}, \otimes, I_{\mathcal{C}}, a, l, r)$ *and* $(\mathcal{D}, \otimes, I_{\mathcal{D}}, a, l, r)$, *a monoidal functor from* $\mathcal{C}$ *to* $\mathcal{D}$ *consists of a functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *on the underlying categories, together with an isomorphisms* $\varphi_0 : I_{\mathcal{D}} \longrightarrow F(I_{\mathcal{C}})$ *and natural isomorphisms* $\varphi_{2,U,V} : F(U) \otimes F(V) \longrightarrow F(U \otimes V)$ *such that the diagrams*

$$\begin{tikzcd}
(F(U) \otimes F(V)) \otimes F(W) \arrow[r, "a"] \arrow[d, "\varphi_2 \otimes \mathrm{id}"'] & F(U) \otimes (F(V) \otimes F(W)) \arrow[d, "\mathrm{id} \otimes \varphi_2"] \\
F(U \otimes V) \otimes F(W) \arrow[d, "\varphi_2"'] & F(U) \otimes F(V \otimes W) \arrow[d, "\varphi_2"] \\
F((U \otimes V) \otimes W) \arrow[r, "Fa"'] & F(U \otimes (V \otimes W)),
\end{tikzcd}$$

and

$$\begin{tikzcd}
I \otimes F(U) \arrow[r, "l"] \arrow[d, "\varphi_0 \otimes \mathrm{id}"'] & F(U) \\
F(I) \otimes F(U) \arrow[r, "\varphi_2"'] & F(I \otimes U), \arrow[u, "F(l)"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
F(U) \otimes I \arrow[r, "r"] \arrow[d, "\mathrm{id} \otimes \varphi_0"'] & F(U) \\
F(U) \otimes F(I) \arrow[r, "\varphi_2"'] & F(U \otimes I), \arrow[u, "F(r)"']
\end{tikzcd}$$

commute for all objects $U, V, W \in \mathrm{ob}(\mathcal{C})$. *The monoidal functor is said to be strict if the isomorphisms* φ_0 *and* φ_2 *are identities.*

2.17. Example. Given a function $f : S \longrightarrow T$, the preimage $f^{-1} : \mathcal{P}(T) \longrightarrow \mathcal{P}(S)$ defined by $f^{-1}(U) = \{s \in S : f(s) \in U\}$, is a strict monoidal functor.

3. Braided Monoidal Categories

3.1. Definition. *For a strict monoidal category* $(\mathcal{C}, \otimes, I)$*, a braiding* c *is a natural isomorphism* $(c_{V,W} : V \otimes W \longrightarrow W \otimes V)_{V,W \in \mathrm{ob}(\mathcal{C})}$ *that satisfies*

$$c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W) \quad \text{and} \quad c_{U\otimes V} = (c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W})$$

A strict monoidal category equipped with a braiding is called a braided monoidal category.

3.2. Theorem. [Thm XIII.1.3, (1)] *Any braiding on a monoidal category satisfies the Yang-Baxter Equation.*

3.3. Definition. *A monoidal functor from a braided monoidal category* $\mathcal{C}$ *to a braided monoidal category* $\mathcal{D}$ *is braided if, for any pair of objects* $V, V' \in \mathrm{ob}(\mathcal{C})$*, the square*

$$\begin{tikzcd} F(V) \otimes F(V') \arrow[r, "\varphi_2"] \arrow[d, "c"'] & F(V \otimes V') \arrow[d, "Fc"] \\ F(V') \otimes F(V) \arrow[r, "\varphi_2"'] & F(V' \otimes V) \end{tikzcd}$$

commutes.

4. The Braid Category

The braid category $\mathcal{B}$ is the category whose objects are given by $\mathrm{ob}(\mathbb{N}) = \{0, 1, 2, \dots\}$, that is, the collection of n-tuples $(*, *, \dots, *)$ described above. For any pair of objects $n, m \in \mathrm{ob}(\mathcal{B})$ there is a morphism set

$$\mathrm{Hom}(n,m) = \begin{cases} \emptyset & \text{if } n \neq m \\ B_n & \text{if } n = m \end{cases}$$

Here B_n denotes the braid group on n strands. There is a functor $\otimes : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B}$ whose action on objects and morphisms is concatenation. In order to put a braiding on the braid category, we have to define isomorphisms $c_{n,m} : n \otimes m \longrightarrow m \otimes n$ for any objects $n, m \in \mathrm{ob}(\mathcal{C})$. This is done as follows: $c_{0,n} = \mathrm{id}_n = c_{n,0}$, and for $n, m > 0$ we set

$$c_{n,m} = (\sigma_m \sigma_{m-1} \cdots \sigma 1)(\sigma_{m+1} \sigma_m \cdots \sigma_2) \cdots (\sigma_{m+n-1} \sigma_{m+n-2} \cdots \sigma_n)$$

where $\sigma_1, \dots, \sigma_{m+n-1}$ denote the generators of B_{m+n}, for example, σ_i is shown in figure 1. The braid $c_{n,m}$ is represented in figure 2.

4.1. Theorem. [XIII.2.1 (1)] *The family of isomorphisms* $(c_{n,m} | n, m \in \mathrm{ob}(\mathbb{N}))$ *defines a braiding on the braid category* $\mathcal{B}$.

Proof. Naturality is stated in theorem XIII.2.1 of (1). To show that c satisfies definition 3.1 we refer to figure 3 and 4. ∎

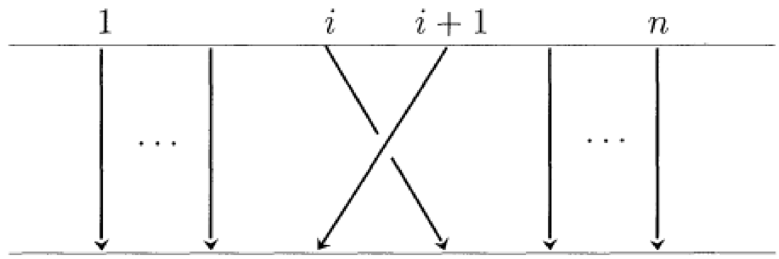

Figure 1: The braid σ_i. taken from: (1)

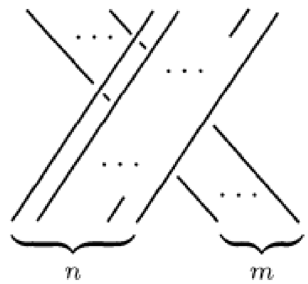

Figure 2: The braid $c_{n,m}$. taken from: (1)

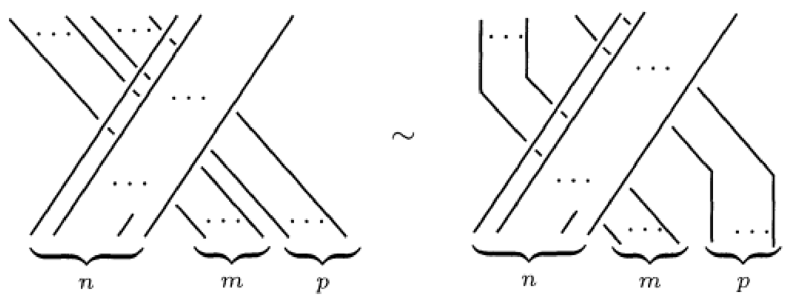

Figure 3: Proof c is a braiding. taken from:(1)

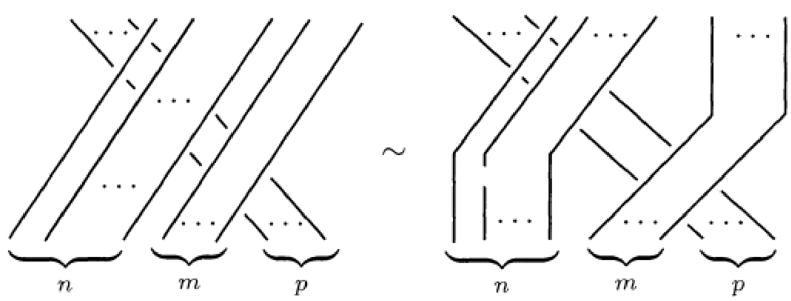

Figure 4: Proof c is a braiding. taken from:(1)

5. Universality of the Braid Category

We will need the following lemma:

5.1. Lemma. [XIII.3.2 (1)] *Let* $(F, \varphi_0, \varphi_2) : \mathcal{C} \longrightarrow \mathcal{D}$ *be a monoidal functor between monoidal categories. If* σ *is a Yang-Baxter Operator on the object* V *in* $\mathcal{C}$*, then*

$$\sigma' = \varphi_{2,V,V}^{-1} F(\sigma) \varphi_{2,V,V}$$

is a Yang-Baxter operator on $F(V)$ *in* $\mathcal{D}$*.*

From this above lemma we deduce that:

5.2. Theorem. *if* $\mathcal{C}$ *is a strict braided monoidal category and* $X \in \mathrm{ob}(\mathcal{C})$*, then there exists a unique braided monoidal functor* $F_X : \mathcal{B} \longrightarrow \mathcal{C}$ *such that* $F_X(*) = X$*.*

Proof. Because F_X is is a strict monoidal functor we know that $F_X(\otimes_{i=1}^n *) = \otimes_{i=1}^n F(*)$ so that setting $F_X(*) = X$ completely determines the action of F_X on the objects of $\mathcal{B}$. By strictness of the functor F_X, we see that the above lemma reduces to $\sigma' = F_X(\sigma)$. This means that the image of Yang-Baxter operators under F_X is again a Yang-Baxter operator. There is only given one such operator on $\mathcal{C}$, namely the one induced by the braiding of $\mathcal{C}$. Asking that F_X is a strict braided monoidal functor is asking that it sends the braiding $\mathcal{B}$ to the braiding on $\mathcal{C}$. There is only one choice for this. ∎

To state explicitly why the above theorem defines a universal property, define the inclusion functor $\iota : * \longrightarrow \mathcal{B}$ to act on objects as $\iota(*) = (*) = 1 \in \mathrm{ob}(\mathcal{B})$. If we are given a functor $G : * \longrightarrow \mathcal{C}$, we have determined an object $G(*) \in \mathrm{ob}(\mathcal{C})$. Therefore there exists a unique strict braided monoidal functor $\widehat{G} : \mathcal{B} \longrightarrow \mathcal{C}$ such that $\widehat{G}(\iota(*)) = G(*)$. Thus $\widehat{G} \circ \iota = G$. This property of the braid category $\mathcal{B}$ is universal in the sense that, if presented with another strict braided monoidal category $\mathcal{B}'$ satisfying the same property, then we would find a pair of uniquely induced strict braided monoidal functors $\widehat{i} : \mathcal{B} \longrightarrow \mathcal{B}'$ and $\widehat{\iota} : \mathcal{B}' \longrightarrow \mathcal{B}$ that act as eachothers inverses.

References

C. Kassel, Quantum Groups. Springer-Verlag (first edition), 1995.

Mac Lane, Categories for the Working Mathematician, Springer Verlag (second edition), 1978.