LECTURE 6: XC-ALGEBRAS

1. XC-algebras

An algebraic version of XC-tangle diagrams is obtained by replacing merge by multiplication, and disjoint union by tensor product. This works in the context of any algebra that allows special elements X and C that have the same properties as the tangles with the same names.

Definition 1. **(XC-algebra)**
An XC-algebra $A = (A, m, 1)$ is an algebra together with invertible elements $X \in A^{\otimes 2}$ and $C \in A$ satisfying the following properties:

$$m_i^{1,3,5} m_j^{2,4,6} C_1 C_2 X_{3,4} C_5^{-1} C_6^{-1} = X_{i,j} \tag{$\Omega 0$}$$

$$m_j^{1,2,3} X_{3,1} C_2 = m_j^{1,2,3} X_{1,3} C_2^{-1} \tag{$\Omega 1f$}$$

$$m_s^{1,3} m_r^{2,4} X_{1,2}^{-1} X_{3,4} = 1_r 1_s \tag{$\Omega 2b$}$$

$$m_r^{1,3} m_s^{2,4,6} C_4 X_{1,6}^{-1} X_{3,2} = 1_r C_s \tag{$\Omega 2c$}$$

$$m_i^{1,4} m_j^{2,5} m_k^{3,6} X_{1,2} X_{4,3} X_{5,6} = m_i^{1,4} m_j^{2,5} m_k^{3,6} X_{1,6} X_{2,3} X_{4,5} \tag{$\Omega 3$}$$

Notice that there may be several choices for X and C on the same algebra A so A can be an XC algebra in several ways. For example given an XC-algebra we may multiply X by a scalar ξ and X^{-1} by ξ^{-1} and leave $C^{\pm}$ unchanged to obtain a new XC-algebra.

Many examples of XC-algebras exist, see the next section. However finding them is not easy at all since the equations, especially $\Omega 3$ are non-linear and involve many variables. Later in the course we will show how to construct XC-algebras from a knot theoretical perspective without solving any equations explicitly.

XC-algebras are important because they provide a framework to describe and compute many knot invariants. Indeed, by their very construction XC-algebras give rise to invariants of XC-tangles.

Given an XC-tangle diagram D written as disjoint unions and merges of $\check{X}^{\pm}$ and $\check{C}^{\pm}$ all we need to do is interpret $\check{X}$ and $\check{C}$ as the X and C of a chosen XC-algebra A. In the algebra context the merging of strands is interpreted as multiplication and disjoint union becomes tensor product. This gives an invariant called $Z_A(D) \in A^{\otimes \mathcal{L}(D)}$, where we recall $\mathcal{L}(D)$ is the set of labels of the strands of D.

For example the two strand tangle from the first figure of the previous lecture was written as

$$D = \check{m}_r^{1,6} \check{m}_b^{3,5,7,2,4} (\check{X}_{1,2} \check{X}_{3,4} \check{X}_{5,6} \check{C}_7^{-1})$$

Therefore, removing the checks on every symbol yields a formula for the invariant $Z_A(D) \in A^{\otimes \{b,r\}}$:

$$Z_A(D) = m_r^{1,6} m_b^{3,5,7,2,4} (X_{1,2} X_{3,4} X_{5,6} C_7^{-1})$$

A more precise definition of the invariant follows:

Definition 2. **(XC-tangle invariant)**
Given an XC-algebra A and an XC-tangle diagram D define $Z_A(T) \in A^{\otimes \mathcal{L}(T)}$ by the properties:

$$Z_A(\check{X}_{ou}^{\pm}) = X_{ou}^{\pm} \in A^{\otimes \{o,u\}} \qquad Z_A(\check{C}_s^{\pm}) = C_s^{\pm} \in A^{\otimes \{s\}} \tag{1}$$

$$Z_A(DE) = Z_A(D) Z_A(E) \quad Z_A(\check{m}_n^{h,t}(D)) = m_n^{h,t}(Z_A(D)) \tag{2}$$

Paraphrasing the construction more loosely we can say that $Z_A(D)$ is obtained by placing the components of X on the each positive crossing of the the diagram, and X^{-1} on each negative crossing, placing as many C's as the rotation number on each strand and then multiplying the elements on each strand in order of appearance.

Lemma 3. (Invariance of Z_A)
If T and T' are equivalent XC-tangles then $Z_A(T) = Z_A(T')$.

Proof. By construction of the notion XC-algebra. There is a sequence of XC-Reidemeister moves taking T to T'. Each of the Reidemeister moves is an equality between XC-tangles and each of these equations becomes a true statement in a tensor power of our algebra A. □

Notice that associativity of the algebra and elementary properties of the tensor product make sure that the order in which we merge the strands together does not matter. Otherwise the invariant Z_A would not be well defined.

2. Examples of XC algebras.

In this section we mention three relatively simple examples of XC-algebras that will play an important role in this course. The first is called the Dilbert algebra and its invariant turns out to be special value of the Alexander polynomial. The second algebra is really a family of examples, one for each finite group. Later we will explain how the corresponding invariant counts representations of the fundamental group of the complement of the knot into the given group. The final example is a simple instance of the double construction that will be important later on.

Example 4. (Double group algebra $D(G)$)
For any finite group G and any field k denote by $D(G)$ the vector space spanned by all pairs fg where $g \in G$ and $f : G \longrightarrow k$ is a function. Any function of $f : G \longrightarrow k$ is a linear combination of delta functions $\delta^g : G \longrightarrow k$ defined by $\delta^g(h) = 1$ if $g = h$ and 0 otherwise. This allows us to define an algebra structure on $D(G)$ by setting $(\delta^a g) \cdot (\delta^b h) = \delta^a \delta^{gbg^{-1}} gh$. Here the product gh is taken in the group G. The product of the delta functions is point-wise, for example $\delta^a\delta^b = \delta^a(b)\delta^b$. Also keep in mind that $1 = \sum_{g\in G} \delta^g$ is the unit function.

We claim that $D(G)$ is an XC-algebra with respect to $X^{\pm 1} = \sum_{g\in G} \delta^g \otimes g^{\pm 1}$ and $C^{\pm 1} = 1$. To verify this we check that all equations $\Omega 0, \dots \Omega 3$ of Definition 1.

1. Since $C^{\pm 1} = 1$ equation $\Omega 0$ holds.
2. Next $\Omega 1f$. The left hand side yields $m_i^{1,2,3} X_{3,1} C_2 = m_i^{1,3} X_{3,1} = m_i^{1,3} \sum_{g\in G} \delta_3^g g_1 = \sum_{g\in G} g_i \delta_i^g = \sum_{g\in G} \delta_i^{ggg^{-1}} g_i = \sum_{g\in G} (\delta^g g)_i$. This agrees with the right hand side because: $m_i^{1,2,3} X_{1,3} C_2^{-1} = \sum_{g\in G} m_i^{1,3} \delta_1^g g_3 = \sum_{g\in G} (\delta^g g)_i$.
3. $\Omega 2b$ just verifies that X^{-1} is the multiplicative inverse of X and indeed it is: $m_s^{1,3} m_r^{2,4} X_{1,2}^{-1} X_{3,4} = m_s^{1,3} m_r^{2,4} \sum_{g,h\in G} \delta_1^g g_2^{-1} \delta_3^h h_4 = \sum_{g,h} (\delta^g \delta^h)_s (g^{-1}h)_r = \sum_g \delta_s^g 1_r = 1_r (\sum_g \delta^g)_s = 1_r 1_s$.
4. Next the complicated side of $\Omega 2c$ is $m_i^{1,3} m_j^{2,4} X_{1,4}^{-1} X_{3,2} = m_i^{1,3} m_j^{2,4} \sum_{g,h} \delta_1^g g_4^{-1} \delta_3^h h_2 = \sum_{g,h} (\delta^g \delta^h)_i (hg^{-1})_j = 1_i 1_j$ just like in the previous case.
5. Finally the left hand side of $\Omega 3$ is equal to
$$m_i^{1,4} m_j^{2,5} m_k^{3,6} X_{1,2} X_{4,3} X_{5,6} = m_i^{1,4} m_j^{2,5} m_k^{3,6} \sum_{f,g,h} \delta_1^f f_2 \delta_4^g g_3 \delta_5^h h_6 =$$
$$\sum_{f,g,h} (\delta^f \delta^g)_i (f\delta^h)_j (gh)_k = \sum_{g,h} \delta_i^g (\delta^{ghg^{-1}} g)_j (gh)_k$$
The right hand side is
$$m_i^{1,4} m_j^{2,5} m_k^{3,6} X_{1,6} X_{2,3} X_{4,5} = \sum_{p,q,r} (\delta^p \delta^r)_i (\delta^q r)_j (qp)_k = \sum_{q,r} (\delta^r)_i (\delta^q r)_j (qr)_k$$
Taking $g = r$ and $q = ghg^{-1}$ we see that the two sides agree.

Now that we know $D(G)$ is an XC algebra we gain an invariant $Z_{D(G)}$. Let us investigate its value on the (positive)trefoil (long)knot. Recall that an XC-tangle diagram for this trefoil was given by the formula
$$3_1 = \check{m}_1^{1,2\dots 10} \check{\alpha}_1^{-1} \check{\alpha}_2^{-1} \check{\alpha}_3^{-1} \check{X}_{4,8} \check{X}_{9,5} \check{X}_{6,10} \check{C}_7^{-1}$$
To obtain $Z_{D(G)}(3_1)$ all we need to do is remove the ˇ and interpret the symbols in terms of our algebra. Recall that $\check{\alpha}_i = m_i^{1,2,3} X_{3,1} C_2$ is a positive curl and its inverse is the negative curl $\check{\alpha}_i^{-1} = m_i^{1,2,3} X_{1,3}^{-1} C_2 = \sum_{g\in G} (\delta^g g^{-1})_i$. It is useful to first merge the curls involved as this is

rather simple: $m_j^{12}\alpha_1\alpha_2 = \sum_{g,h}(\delta^g g\delta^h h)_j = \sum_{g,h}(\delta^g\delta^{ghg^{-1}}gh)_j = \sum_g(\delta^g g^2)_j$ because the delta functions force $g = ghg^{-1}$, so $g = h$. The trefoil is thus computed as

$$Z_{D(G)}(3_1) = \sum_{a,b,c,d\in G}(\delta^a a^{-3}\delta^b c\delta^d b\delta^c d)_i$$

Shifting the deltas to the left we get

$$\delta^a a^{-3}\delta^b c\delta^d b\delta^c d = \delta^a\delta^{a^{-3}ba^3}a^{-3}c\delta^d b\delta^c d = \delta^a\delta^{a^{-3}ba^3}\delta^{a^{-3}cdc^{-1}a^3}a^{-3}cb\delta^c d =$$
$$\delta^a\delta^{a^{-3}ba^3}\delta^{a^{-3}cdc^{-1}a^3}\delta^{a^{-3}cbcb^{-1}c^{-1}a^3}a^{-3}cbd =$$
$$\delta^a\delta^b\delta^{a^{-3}cdc^{-1}a^3}\delta^{a^{-3}caca^{-1}c^{-1}a^3}a^{-3}cad = \delta^a\delta^b\delta^{cdc^{-1}}\delta^{caca^{-1}c^{-1}}a^{-3}cac^{-1}ac$$

so we found

$$Z_{D(G)}(3_1) = \sum_{a,c\in G}(\delta^a\delta^{caca^{-1}c^{-1}}a^{-3}cac^{-1}ac)_i$$

We will have more to say about the meaning of this expression after we introduce the fundamental group in a later lecture.

It should be clear from the trefoil computation that the invariant $Z_{D(G)}$ is only interesting if G is non-commutative. In the commutative case all one can hope for is to find something like a linking matrix (exercise!)

Example 5. (Dilbert algebra Dlb**)**
Consider the 4- dimensional algebra Dlb over $k = \mathbb{C}$ generated by $1, d, l, b$ with relations summarized in the multiplication table below, with $a = 1 - l$:

m_{Dlb}	b	l	d	a
b	0	0	l	b
l	b	l	0	0
d	a	d	0	0
a	0	0	d	a

The names of the generators were chosen to make the multiplication rules easy to remember: You only get something non-zero if the two generators you are multiplying match height: for example bl does not match so is 0 but lb does match and is b.

Below we will check explicitly that the Dilbert algebra becomes an XC-algebra when we choose

$$X_{ou} = 1 - 2a_o l_u + 2b_o d_u = X_{ou}^{-1} \qquad C^{\pm} = \pm i(l - a) \tag{3}$$

(1) First we check that C is invertible: $CC^{-1} = (2l-1)^2 = 4l^2 - 4l + 1 = 1$ in the sense First $CC^{-1} = (l-a)^2 = l^2 + a^2 = l + a = 1$.
(2) To check $\Omega 0$ it is useful to study the map $M : Dlb \longrightarrow Dlb$ defined by $M(x) = CxC^{-1}$. We notice that $M(a) = a$ and $M(l) = l$ while $M(b) = (l-a)b(l-a) = -b$ and likewise $M(d) = -d$. The left-hand side of $\Omega 0$ is precisely applying M to both tensor factors so $\Omega 0$ can be rephrased as $(M \otimes M)X = X$ or keeping things in $Dlb^{\otimes\{o,u\}}$ we write $M_o M_u(X_{o,u}) = X_{o,u}$. As with multiplication the subscript z for M_z means that we apply M to the z-th tensor factor of Dlb. To check that this holds we compute $M_o M_u(X_{o,u}) = M_o M_u(1 - 2a_o l_u + 2b_o d_u) = 1 - 2M(a)_o M(l)_u + 2M(b)_o M(d)_u = X_{ou}$
(3) Next we look at $\Omega 1f$. The left hand side is $m_j^{1,2,3}X_{3,1}C_2 = m_j^{j,3}m_j^{1,2}i(1-2a_3l_1+2b_3d_1)(l_2-a_2) = m_j^{j,3}i(l_j-a_j-2a_3l_j+2b_3d_j) = i(l_j-a_j+2a_j) = i(l_j+a_j) = i$. A similar computation on the right hand side yields the same result (exercise!).
(4) Checking that X is an invertible element (with respect to the product on $Dlb^{\otimes 2}$) is the same as $\Omega 2b$: $m_s^{1,3}m_r^{2,4}X_{1,2}^{-1}X_{3,4} = (1 - 2a_r l_s + 2b_r d_s)(1 - 2a_r l_s + 2b_r d_s) = 1 - 4a_r l_s + 4b_r d_s + 4a_r l_s - 4b_r d_s = 1 = 1_r 1_s$ as required.
(5) Similarly we compute the left hand side of $\Omega 2c$ by simply expanding the brackets and multiplying the terms in each monomial in the order indicated: in this case we get 18 terms but most yield 0: $m_r^{1,3}m_s^{2,4,6}C_4X_{1,6}^{-1}X_{3,2} = m_r^{1,3}m_s^{2,4,6}i(l_4-a_4)(1-2a_1l_6+2b_1d_6)(1-2a_3l_2+2b_3d_2) = im_r^{1,3}m_s^{2,4,6}(l_4 - a_4)(1 - 2a_1l_6 + 2b_1d_6 - 2a_3l_2 + 2b_3d_2 + 4a_1l_6a_3l_2 - 4a_1l_6b_3d_2 - 4b_1d_6a_3l_2 + 4b_1d_6b_3d_2) = C_s + i(-2a_rl_s - 2b_rd_s - 2a_rl_s + 2b_rd_s + 4a_rl_s) = C_s = 1_rC_s$.

(6) Finally we check $\Omega 3$ starting with the left-hand side: $m_i^{1,4}m_j^{2,5}m_k^{3,6}X_{1,2}X_{4,3}X_{5,6} = m_j^{2,5}m_k^{3,6}X_{5,6}(m_i^{1,4}(1-2a_1l_2+2b_1d_2)(1-2a_4l_3+2b_4d_3))$ Doing the strands $1,4$ first we find $m_i^{1,4}(1-2a_1l_2+2b_1d_2)(1-2a_4l_3+2b_4d_3) = 1-2a_il_2+2b_id_2-2a_il_3+2b_id_3+4a_il_2l_3-4b_id_2l_3$. Therefore the left-hand side becomes $m_j^{2,5}m_k^{3,6}(1-2a_il_2+2b_id_2-2a_il_3+2b_id_3+4a_il_2l_3-4b_id_2l_3)(1-2a_5l_6+2b_5d_6) = 1-2a_il_j+2b_id_j-2a_il_k+2b_id_k+4a_il_jl_k-4b_id_jl_k-2a_jl_k+4a_ia_jl_k-4b_ia_jd_k+2b_jd_k-4a_ib_jd_k+4b_ia_jd_k$ The right-hand side should give the same result. We start by merging two crossings: $m_j^{2,5}X_{2,3}X_{4,5} = m_j^{2,5}(1-2a_2l_3+2b_2d_3)(1-2a_4l_5+2b_4d_5) = 1-2a_jl_3+2b_jd_3-2a_4l_j+2b_4d_j-4l_3b_4d_j+4d_3b_4l_j$. Bringing in $X_{1,6}$ and merging it with the rest the right-hand side is: $m_i^{1,4}m_k^{3,6}(1-2a_1l_6+2b_1d_6)(1-2a_jl_3+2b_jd_3-2a_4l_j+2b_4d_j-4l_3b_4d_j+4d_3b_4l_j) = 1-2a_jl_k+2b_jd_k-2a_il_j+2b_id_j-4b_id_jl_k+4b_il_jd_k-2a_il_k+4a_ia_jl_k-4a_ib_jd_k+2b_id_k-4b_il_jd_k$. This is indeed equal to the left hand side computed above.

The same trefoil can also be computed in the Dilbert algebra and here the outcome is $Z_{\mathrm{Dlb}}(3_1) = -3$. Recall the diagram of the trefoil we used is the following

$$3_1 = \check{m}_1^{1,2\dots 10}\check{\alpha}_1^{-1}\check{\alpha}_2^{-1}\check{\alpha}_3^{-1}\check{X}_{4,8}\check{X}_{9,5}\check{X}_{6,10}\check{C}_7^{-1}$$

The curls α^{-1} that correct for the framing are easiest to compute first. We already computed $\alpha = i$ while studying $\Omega 1f$ and so $\alpha^{-1} = -i$ and $F = m_1^{1,2,3}\alpha_1^{-1}\alpha_2^{-1}\alpha_3^{-1} = -i^3 = i$. Next let us merge the first two crossings, leaving out the third for now: $Q = m_5^{4,5}m_9^{8,9}X_{4,8}X_{9,5} = m_5^{4,5}m_9^{8,9}(1-2a_4l_8+2b_4d_8)(1-2a_9l_5+2b_9d_5) = 1-2a_5l_9+2b_5d_9+2l_5a_9-2d_5b_9$. Adding in the last crossing yields $B = m_6^{5,6}m_8^{9,10}X_{6,10}Q = m_6^{5,6}m_9^{9,10}(1-2a_5l_9+2b_5d_9+2l_5a_9-2d_5b_9)(1-2a_6l_{10}+2b_6d_{10}) = 1-2a_6l_8+2b_6d_8+2l_6a_8-2d_6b_8-2a_6l_8+2b_6d_8+4a_6l_8-4b_6d_8+4b_6d_8-4a_6l_8 = 1-4a_6l_8+4b_6d_8+2l_6a_8-2d_6b_8$. Putting it all together we find

$$Z_{\mathrm{Dlb}}(3_1) = m_1^{1,7,6,8}FBC_7^{-1} = m_1^{6,7,8} - i(l_7-a_7)i(1-4a_6l_8+4b_6d_8+2l_6a_8-2d_6b_8) =$$
$$l_1 - 2(dlb)_1 - a_1 - 4(bad)_1 = -3(l_1+a_1) = -3$$

Example 6. (Double Sweedler DS)
For later use we mention a more complicated example of an XC-algebra that will later turn out to have a lot of interesting and useful extra structure. It is called the Double Sweedler algebra DS defined as follows
The generators are c,s,w,σ,ω with relations $s^2=\sigma^2=1$ and $w^2=\omega^2=0$ and $c^2=s\sigma$ and c,s,σ commute and $xy+yx=0$ whenever $x\in\{c,\sigma,\omega\}$ and $y\in\{w,\omega\}$ and

$$\omega w - w\omega = \sigma - s$$

This is an XC-algebra with respect to $C=c$ and

$$X_{ij} = \frac{1}{2}\big(1+\sigma_j+s_i(1-\sigma_j)+w_i(\omega_j+\sigma_j\omega_j)+s_iw_i(\omega_j-\sigma_j\omega_j)\big)$$

Here is a multiplication table:

m	c	s	w	σ	ω
c	$s\sigma$	cs	cw	$c\sigma$	$c\omega$
s	cs	1	sw	$s\sigma$	$s\omega$
w	$-cw$	$-sw$	0	σw	$w\omega$
σ	$c\sigma$	$s\sigma$	$-w\sigma$	1	$\sigma\omega$
ω	$-c\omega$	$-s\omega$	$-s+\sigma+w\omega$	$-\sigma\omega$	0

The inverse of C is c^3 and the negative crossing is

$$X_{ij}^{-1} = \frac{1}{2}\big(1+\sigma_j+s_i(1-\sigma_j)-w_i(\omega_j-\sigma_j\omega_j)+s_iw_i(\omega_j+\sigma_j\omega_j)\big)$$

While possible to check the XC-axioms $\Omega 0-\Omega 3$ explicitly either by hand or by computer we will not do so here. Later in the course we will use this example to show how we can make sure that these axioms are true *by construction.* Of course the reader is already welcome to explore the relation with the Dilbert algebra and perhaps find that $Z_{DS}(K)$ equals the Alexander polynomial evaluated at c^2.