Planar Heegaard diagrams

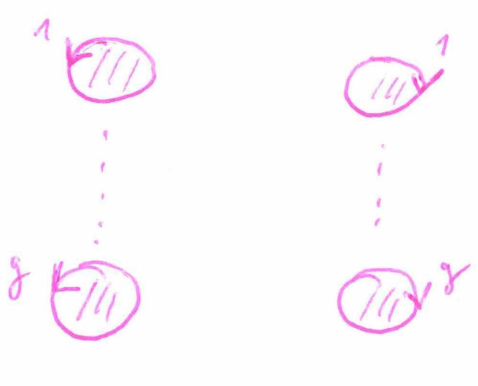
Note

D^3 with two 2-discs with different orientation identified $\cong$ genus 1 handlebody (ie solid torus)

and in general

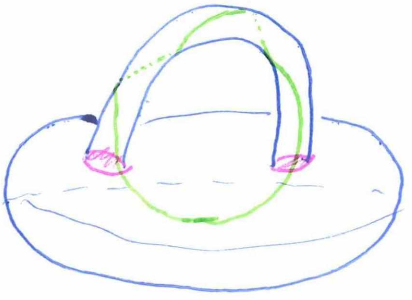
D^3 with $2g$ discs identified in pairs $\cong$ genus g handlebody

Let's look at $\partial D^3 = S^2 = \mathbb{R}^2 \cup \{\infty\}$. Forgetting about the point at ∞, we can draw these discs on the plane $\mathbb{R}^2$



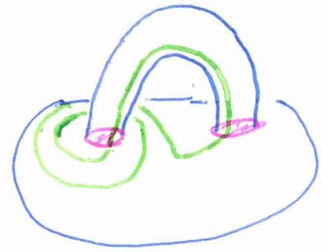
As explained in the lecture, the attaching map $\varphi : \partial D^2 \times D^1 \hookrightarrow \partial H_g$ of a 2-handle is completely determined by $\varphi_0 = \varphi_{|\partial D^2 \times 0} : S^1 \hookrightarrow \Sigma_g$.

Eg:

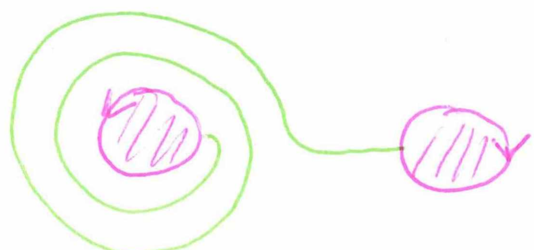


$\cong$

drag the twists down



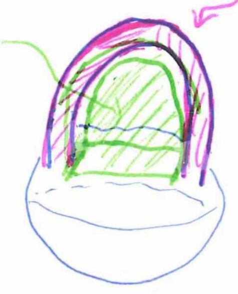
$|||$



Definition. A planar Heegaard diagram consists of $\mathbb{R}^2$ together with the attaching regions $D^2 \amalg D^2$ of 1-handles and the attaching spheres of 2-handles (simple closed curves)

Eg:

$\cong$

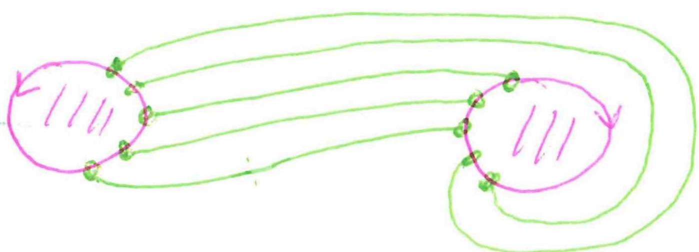


$\cup\, h_3$

$\cong h_0 \cup h_3 \cong S^3.$

Eg: Ruben last week: "$L(p,q)$ can be formed from glueing two solid tori where the meridian of one is glued to a simple closed curve of the boundary of the other that runs through the meridian q times and the longitude p times".

So a planar diagram for $L(5,2)$ looks like



Now, recall that by the Cerf theorem we had a bijection

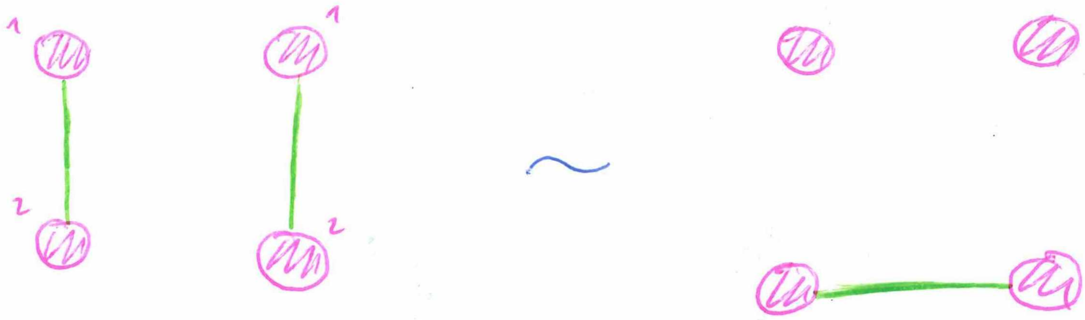
$$\left\{\begin{array}{c}3\text{-manifolds}\\ (\text{up to diffeo}\end{array}\right\} = \{\text{handle attachments}\} \Big/ \begin{array}{l}\text{stabilisation}\\ \text{handle slide.}\end{array}$$

The discussion above says that a 3-dim handle attachment is determined by a planar Heegaard diagram. Now, how do the two relations transform in terms of Heegaard diagrams?

Stabilisation :  ~ $\emptyset$ (empty diagram)

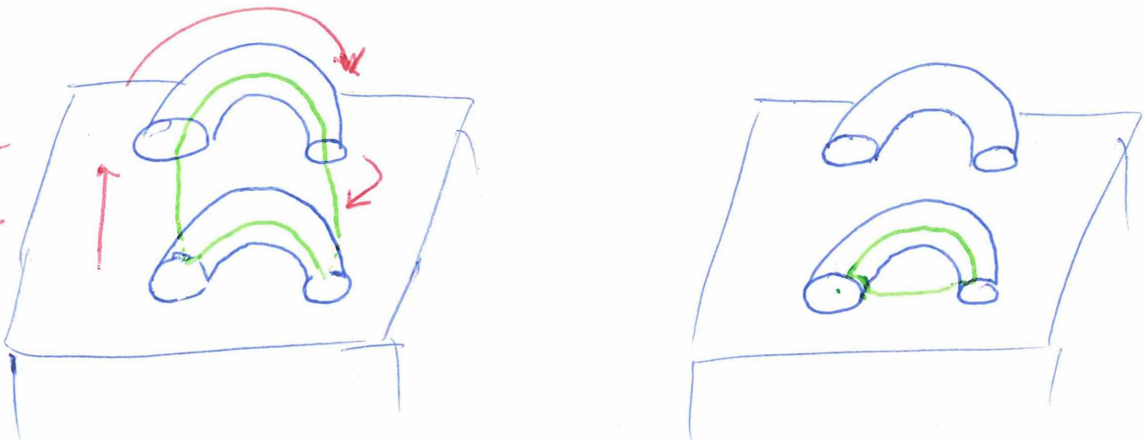
(both represent S^3, see above)

1 - handle slide

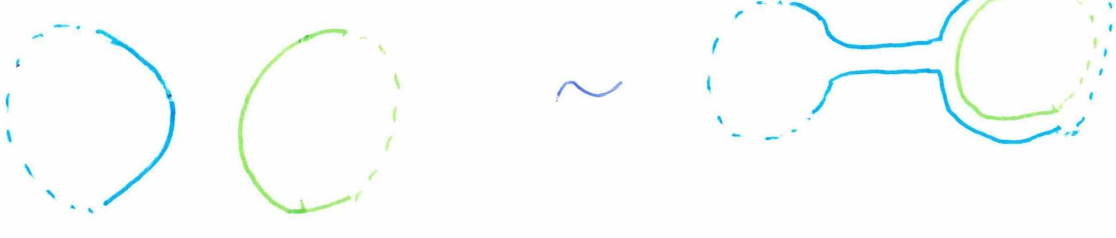


why?

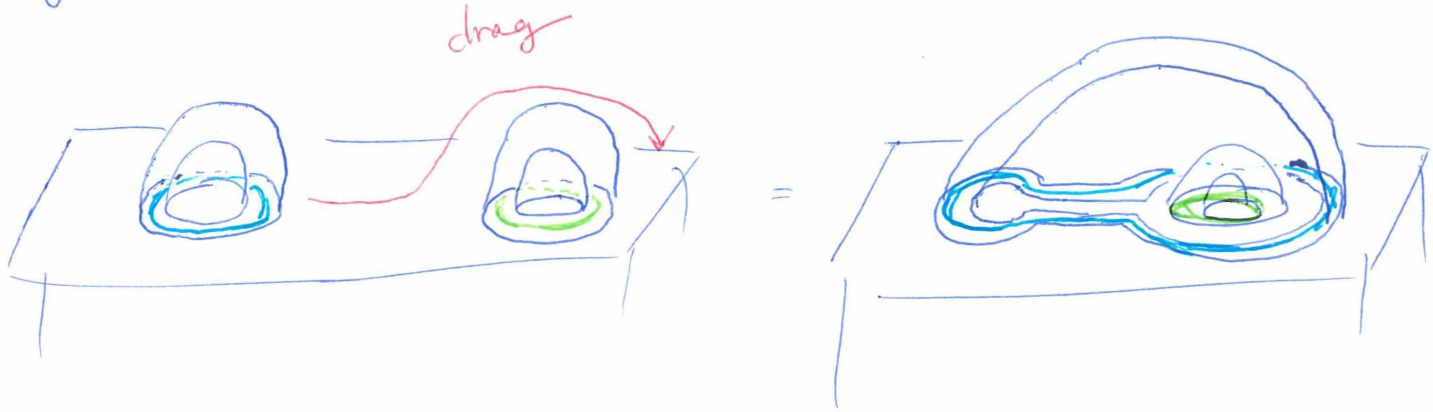
move one handle along the other



2 - handle slide

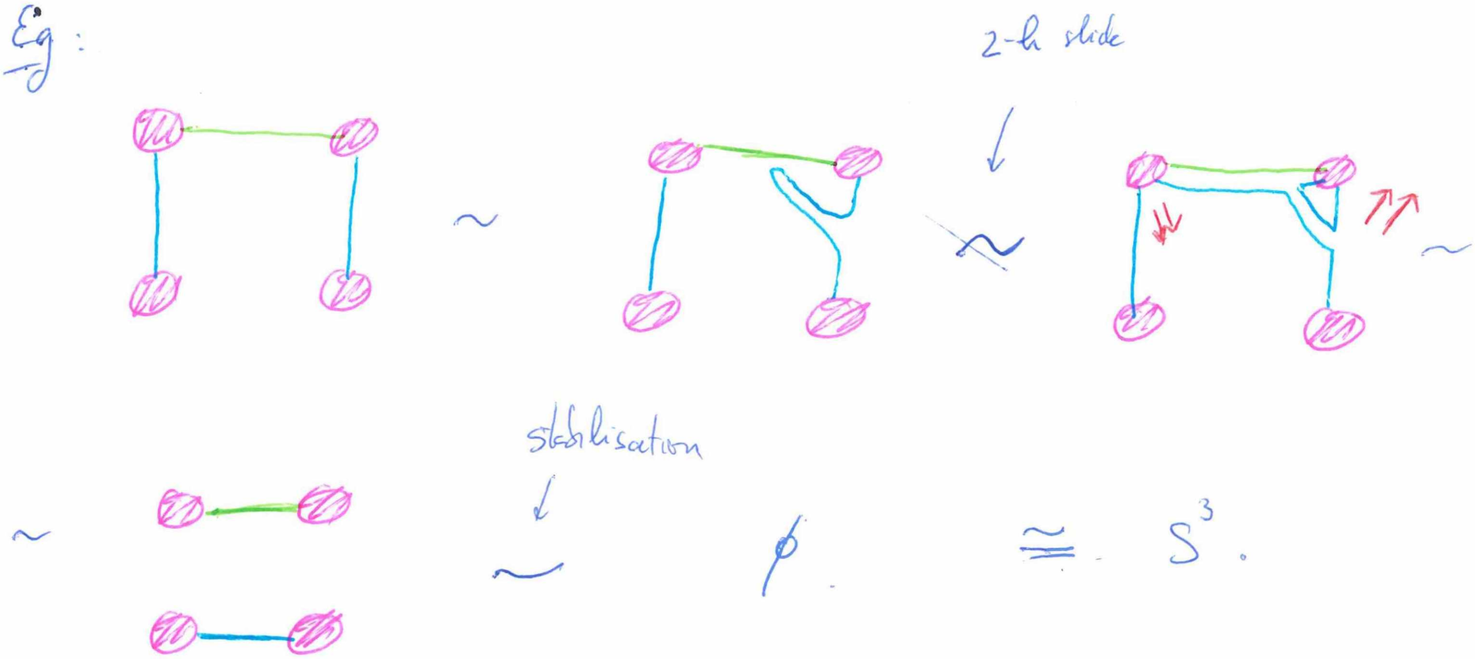


why?



$=$

Eg:



$\sim \emptyset \cong S^3$.

<u>Corollary</u>: There is a bijection

$$\left\{ \begin{array}{c} \text{3-mfds} \\ \text{(up to diffeo)} \end{array} \right\} = \{\text{planar Heegaard diagrams}\} \Big/ \begin{array}{c} \text{planar isotopy} \\ \text{stabilisation} \\ \text{1-h and 2-h slide.} \end{array}$$