Groningen Topology Seminars: Knots

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Knots

Definition 1

A **knot** is an embedding (image of an injective continuous map) of S^1 in $\mathbb{R}^3$.

More precisely, a knot is an equivalence class of such embeddings, under the following equivalence:

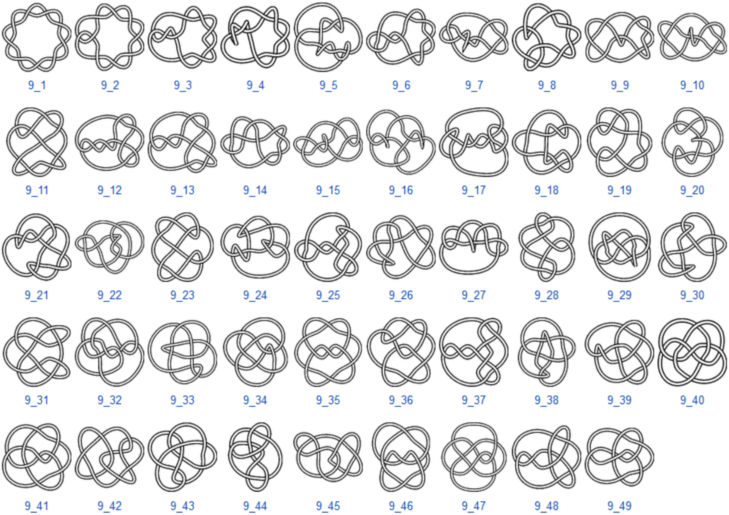
Definition 2

Two knots $a, b : S^1 \to \mathbb{R}^3$ are **equivalent** if there is a continuous function $F : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that

1. $F_0 = \mathrm{id}_{\mathbb{R}^3}$, so that $F_0 \circ a = a$
2. F_t is a homeomorphism for all $t \in [0,1]$
3. $F_1 \circ a = b$

Remark: we only consider so-called 'tame knots'.

Examples



All (prime) knots with 9 crossings; taken from `katlas.org`

Invariants

Central question: how to decide if two knots are equivalent

Definition 3

A **knot invariant** is some quantity or object associated to a knot that is invariant under knot equivalence.

- Two knots with different values of a given knot invariant cannot be equivalent
- A trivial knot invariant of a knot K is the space $\mathbb{R}^3 \backslash K$; for equivalent knots these are homeomorphic (why?)

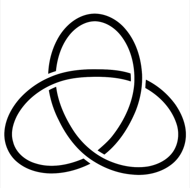
Definition 4

The **knot group** of a knot K is the fundamental group $\pi_1(\mathbb{R}^3 \backslash K)$.

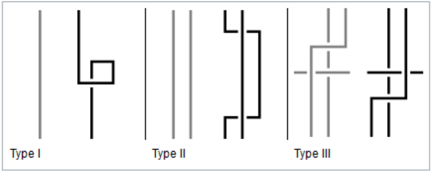
Since $\mathbb{R}^3 \backslash K$ is a knot invariant, clearly the knot group is too.

Knot diagrams

We often draw knots in a plane, specifying at each crossing which strand goes over and which under. By stretching you can always do this such that each intersection contains only 2 strands.



Reidemeister moves

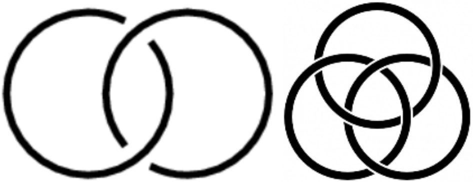


Reidermeister theorem: Two knots are equivalent if and only if their diagrams are related by stretching them and carrying out Reidemeister moves.

Links

Definition 5

A **link** is an embedding of a finite disjoint union $\coprod S^1$ into $\mathbb{R}^3$.



Hopf link and Borromean rings

Link invariants and link groups are defined analogously as for knots; let's compute some of these groups!

Knot groups

Lemma 6

Let K be any knot or link, and let S^3 be the one-point compactification of $\mathbb{R}^3$. Then $\pi_1(\mathbb{R}^3 \backslash K) \cong \pi_1(S^3 \backslash K)$.

Proof.

Write $S^3 \backslash K$ as the union of $\mathbb{R}^3 \backslash K$ and B, where B is the open complement in S^3 of some closed bounded ball containing K. Both B and $B \cap \mathbb{R}^3 \backslash K$ have a trivial fundamental group, namely $B \cap \mathbb{R}^3 \backslash K \cong S^2 \times \mathbb{R}$. Thus van Kampen's theorem gives the desired isomorphism. $\square$

Knot group of the Unknot

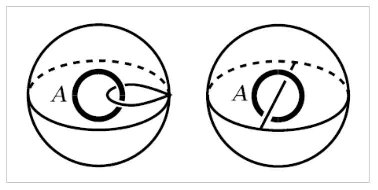
Lemma 7

Let K be the unknot. Then $\pi_1(\mathbb{R}^3 \backslash K) \cong \mathbb{Z}$.

Proof.

The space $\mathbb{R}^3 \backslash K$ retracts onto the wedge sum $S^2 \vee S^1$. Thus using van Kampen's theorem:

$$\pi_1(\mathbb{R}^3 \backslash K) \cong \pi_1(S^2 \vee S^1) \cong \pi_1(S^2) * \pi_1(S^1) \cong \mathbb{Z}.$$



$\square$

Link group of two unlinked circles

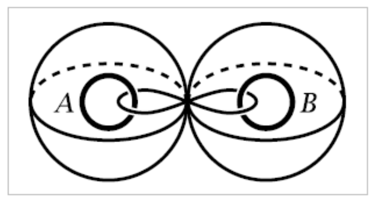
Lemma 8

Let K be the link of two separate circles. Then $\pi_1(\mathbb{R}^3 \backslash K) \cong \mathbb{Z} * \mathbb{Z}$.

Proof.

The space $\mathbb{R}^3 \backslash K$ retracts onto $S^2 \vee S^1 \vee S^2 \vee S^1$. Thus:

$$\pi_1(\mathbb{R}^3 \backslash K) \cong \pi_1(S^2) * \pi_1(S^1) * \pi_1(S^2) * \pi_1(S^1) \cong \mathbb{Z} * \mathbb{Z}.$$



$\square$

Link group of two linked circles

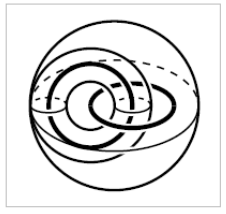
Lemma 9

Let K be the link of two linked circles. Then $\pi_1(\mathbb{R}^3 \backslash K) \cong \mathbb{Z} \times \mathbb{Z}$.

Proof.

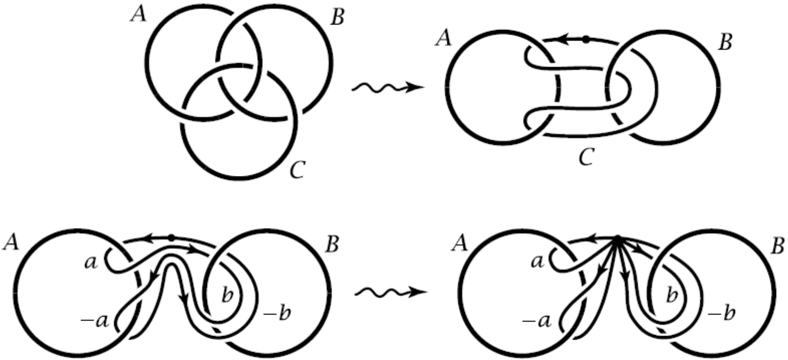
$\mathbb{R}^3 \backslash K$ retracts onto $S^2 \vee (S^1 \times S^1)$ (a bit difficult to see). Hence:

$$\pi_1(\mathbb{R}^3 \backslash K) \cong \pi_1(S^2) * (\pi_1(S^1) \times \pi_1(S^1)) \cong \mathbb{Z} \times \mathbb{Z}.$$



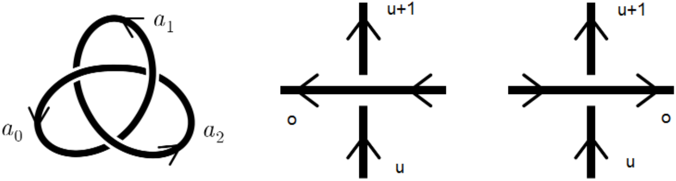
$\square$

Magic Trick



The Wirtinger Presentation

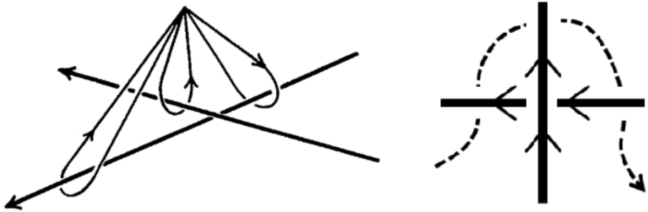
We may partition an oriented knot diagram into its separate arcs, beginning and ending at an 'under' crossing.



Thus at each crossing 3 arcs meet, we distinguish between one 'over' arc o and two 'under' arcs: $u, u+1$.
Here "$u+1$" simply denotes the arc that follows u.
We distinguish between sign of crossings depending on orientation: in the figure the left crossing is left-handed (-) and the right is right-handed $(+)$.

The Wirtinger Presentation

From the basepoint x_0 in $\mathbb{R}^3 \backslash K$ from which a knot K is viewed, we may specify a loop at x_0 by a sequence of passing under arcs, using sign to distinguish between inverse loops.



Clearly, any loop in $\mathbb{R}^3 \backslash K$ is homotopic to a concatenation of loops passing under one arc.
This is pictured for the loop $(u+1)^{-1} \bullet o^{-1} \bullet u$.

The Wirtinger Presentation

Theorem 10

Let K be a knot with set of arcs A and set of crossings B. Let W be the free group of $|A|$ generators. Let $N \leq A$ be the normal subgroup generated by elements $r(b)$, where for $b \in B$:

$$r(b) = \begin{cases} (u(b)+1)o(b)u(b)^{-1}o(b)^{-1} & \text{if } b \text{ right-handed,} \\ o(b)(u(b)+1)o(b)^{-1}u(b)^{-1} & \text{if } b \text{ left-handed.} \end{cases}$$

The knot group of K is W/N.

Example 11

Let K be the trefoil knot. Then

$$\begin{aligned} \pi_1(\mathbb{R}^3 \backslash K) &\cong \langle a_0, a_1, a_2 \mid a_1 a_2 a_0^{-1} a_2^{-1} = a_0 a_1 a_2^{-1} a_1^{-1} = a_2 a_0 a_1^{-1} a_0^{-1} = e \rangle \\ &\cong \langle a_0, a_1 \mid a_0 a_1 a_0 = a_1 a_0 a_1 \rangle \end{aligned}$$

References

- Linov, Larsen. "An Introduction to Knot Theory and the Knot Group." (2014).
- Hatcher, Allen. Algebraic topology, 2005.
- http://katlas.org/wiki/The_Rolfsen_Knot_Table
- Parcly Taxel, https://en.wikipedia.org/wiki/Reidemeister_move