Observations on Almost OU Tangles, and an Algorithm for Drawing Diagrams

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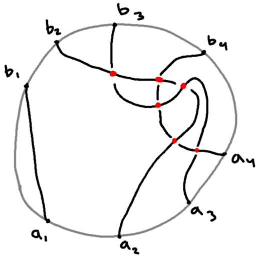
Tangle diagrams, Tame and Wild

A tangle diagram

- is contained in a convex, bounded closed region of $\mathbb{R}^2$,
- has pairs of end points on boundary,
- strands connecting pairs.
- Only two strands can intersect in one point (or once for self-int.),
- strands are homeo. to finite polygonal paths (Tame).

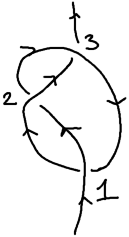
We consider only tangles with 1 strand.



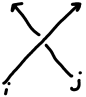
A tame tangle diagram?

Oriented Gauss Code, OU tangle diagrams

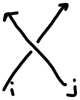
The OGC is a string of symbols O_i^s or U_i^s
with s sign of crossing, i the crossing



A Trefoil tangle diagram



"+"



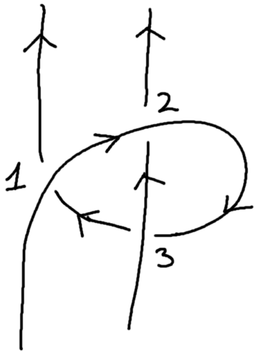
"−"

$$\begin{array}{ccccccccccc}
1 & \to & 2 & \to & 3 & \to & 1 & \to & 2 & \to & 3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
O_1^+ & \to & U_2^+ & \to & O_3^+ & \to & U_1^+ & \to & O_2^+ & \to & U_3^+
\end{array}$$

$$\Rightarrow O_1^+ U_2^+ O_3^+ U_1^+ O_2^+ U_3^+$$

Oriented Gauss Code, OU tangle diagrams

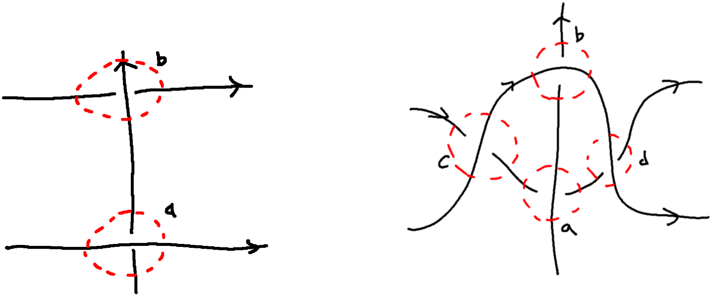
OU tangle diagrams are diagrams with first all Os then all Us
(Non) Examples:



$O_1^+ U_2^+ O_3^+ U_1^+ O_2^+ U_3^+$

$O_1^+ O_2^+ U_3^+ U_1^+, O_3^+ U_2^+$

The Glide move attempts to swap an O and a U in the Gauss code of a tangle diagram

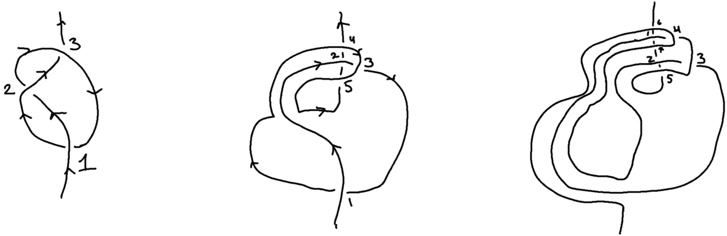


Before applying a Glide   After applying a Glide

Example of the Glide move

The Glide move, OU Algorithm and Acyclic diagrams

The Glide move attempts to swap an O and a U in the Gauss code of a tangle diagram



Example of the Glide move on the Trefoil tangle diagram

The process gets very messy very quick, hence we abstract!

The Glide move, OU Algorithm and Acyclic diagrams

View the Glide move as an operation on the Gauss code.

- Find crossings a, b, for which the GC has $U_a^{\pm} O_b^{\pm}$ or $U_a^{\pm} O_b^{\mp}$
- Replace triples of Os and Us to different triples
- Introduce new crossings c, d

$$U_a^+ O_b^-, O_a^+, U_b^- \mapsto O_a^- U_b^+, O_c^- O_b^+ O_d^+, U_c^- U_a^- U_d^+$$

$$U_a^+ O_b^+, O_a^+, U_b^+ \mapsto O_a^+ U_b^+, O_c^+ O_b^+ O_d^-, U_d^- U_a^+ U_c^+$$

$$U_a^- O_b^-, O_a^-, U_b^- \mapsto O_a^- U_b^-, O_d^+ O_b^- O_c^+, U_c^+ U_a^- U_d^+$$

$$U_a^- O_b^+, O_a^-, U_b^+ \mapsto O_a^+ U_b^-, O_d^+ O_b^- O_c^-, U_d^+ U_a^+ U_c^-,$$

The Glide move, OU Algorithm and Acyclic diagrams

The OU Algorithm is then simply:

1. Read the GC from left to right
2. At the first opportunity, apply the Glide move
3. If the diagram is OU, we are done, if not return to step 1

Remark: the algorithm terminates for only a small fraction of tangle diagrams we consider, specifically the *acyclic* ones.

For the trefoil tangle diagram

$$\text{using } U_a^+ O_b^+, O_a^+, U_b^+ \mapsto O_a^+ U_b^+, O_c^+ O_b^+ O_d^-, U_d^- U_a^+ U_c^+,$$

$$O_1^+ U_2^+ O_3^+ U_1^+ O_2^+ U_3^+ \mapsto O_1^+ O_2^+ U_3^+ O_4^+ O_3^+ O_5^- U_5^- U_2^+ U_4^+$$

Still quite complicated... How about we draw something?

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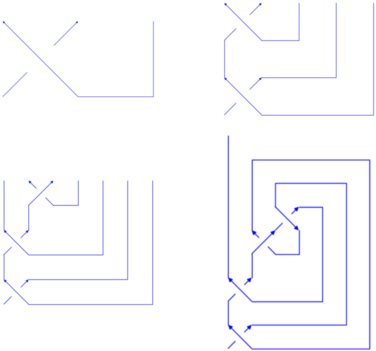
Drawing Algorithm

Top "scanning" method

1. start with one crossing
2. check pairs of strands from the left to connect new crossing
   2.1 first try 2 strands
   2.2 then 1 strand
   2.3 or try to put a "cap"
3. repeat till 1 strand left

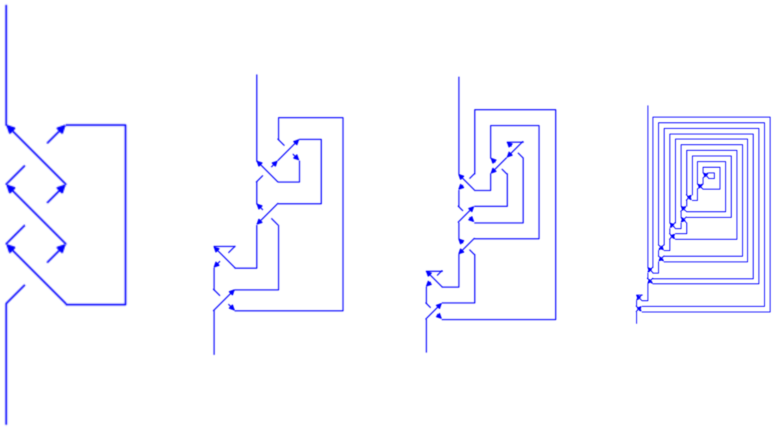


Proof of validity:

- Induction on the number of open strands
- Show that at each iteration the diagram is planar

Drawing Algorithm

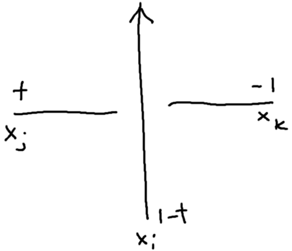
The trefoil after 0, 1, 2, 5 iterations of the OU algorithm.



Still not very illuminating... Perhaps abstraction?

The Alexander Matrix

An Alexander Matrix $\mathcal{A} \in M(n, \mathbb{Z}[t, t^{-1}])$ of a tangle diagram with n crossings summarizes structural information of a tangle



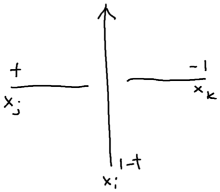
- A column of $\mathcal{A}$ encodes a crossing
- Rows represent connections of over strands
- For some column
    - put $1-t$ in i-th row
    - put t in j-th row
    - put -1 in k-th row

  where x_i, x_j, x_k are over strands

But how do we decide the order of the columns?

The Alexander Matrix

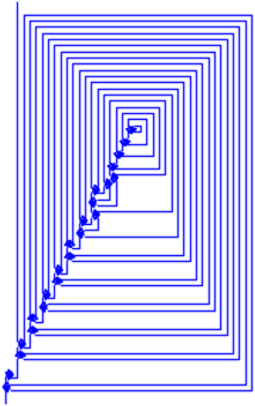
An example with the Trefoil where columns correspond to crossing labels

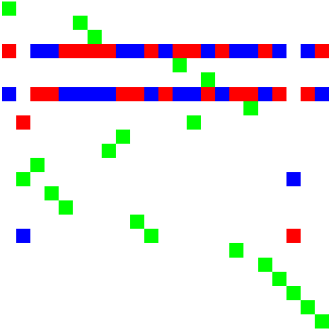


$$\begin{pmatrix} 1-t & -1 & t \\ t & 1-t & -1 \\ -1 & t & 1-t \end{pmatrix}$$

Implications for Alexander Matrix

Notice the drawing algorithm gives an order for the crossings, hence an order for the columns





$t, 1-t, -1 \mapsto R, G, B$ resp.

And this ordering produces a pattern with distinct features

Examples

Movie time!

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Conclusions w.r.t. Alexander Matrix

1. We defined a complex process on tangle diagrams.
2. To visualize we tried to draw the diagrams.
3. Unsatisfied, we made a diagram of a diagram.
4. The drawing algorithm helped make an informed decision

Did this help? In a way no.

In a way yes, since there may exist another algorithm which will order the columns in a different nice way.

More crossings $\Rightarrow$ More column perm.

$\Rightarrow$ Greater interest in picking nice perm.

$\Rightarrow$ Nice perm. may help understand original diagram

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