The Alexander polynomial:

the topological way

Knot Theory Seminar

28 May 2020

Previously:

K knot, let $\pi_K = \pi_1(X_K) = \pi_1(S^3 - N(K)) \overset{\text{eg. Wirtinger}}{\cong} \dfrac{F(x_1, \dots, x_n)}{\langle r_1, \dots, r_s \rangle}$

If $F_n = F(x_1, \dots, x_n)$, let $\dfrac{\partial}{\partial x_i} : \mathbb{Z}[F_n] \to \mathbb{Z}[F_n]$ Fox derivative

$$\begin{tikzcd} \mathbb{Z}[F_n] \arrow[r, "\mathbb{Z}[\text{quotient}]"] \arrow[rr, bend right, "\rho"'] & \mathbb{Z}[\pi_K] \arrow[r, "\mathbb{Z}[\text{abelianiz}]"] & \mathbb{Z}[\pi_K^{ab}] = \mathbb{Z}[t, t^{-1}] \end{tikzcd}$$

let $A := \left(\rho\left(\dfrac{\partial r_i}{\partial x_j} \right) \right)$ the Alexander matrix

We also defined the <u>elementary ideals</u> (of a $(m \times n)$-matrix) of A as

$$\mathcal{E}_K(A) = (\text{ideal gen by } (n-K)\text{-minors of } A) \subset \mathbb{Z}[t,t^{-1}]$$

$(\mathcal{E}_K(A) = 0$ if $K < n-m$; $\mathcal{E}_K(A) = \mathbb{Z}[t,t^{-1}]$ if $K \geq n)$, and showed that they <u>do not</u> depend on the presentation of π_K, ie it is an inv. of the gp π_K.

• <u>Recall</u>: $\mathbb{Z}[t,t^{-1}]$ is a UFD (so it has gcd's) and $\mathbb{Z}[t,t^{-1}]^{\times} = \{\pm t^{\pm n}\}$.

Definition : The Alexander polynomial of K is the generator of the smallest principal ideal containing $\mathcal{E}_1(A)$, ie,

$$\Delta_K(t) := \gcd(\mathcal{E}_1(A))$$

$$= \gcd((n-1)\times(n-1) \text{ minors of } A)$$

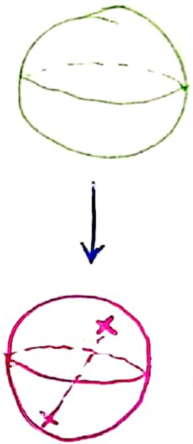
- I will present a more general, alternative notion of Alexander polynomial and a purely topological way to compute it.

<u>Definition</u>: Let X be a space. A <u>covering space</u> of X is a top space E together with a map $p: E \to X$ with the property that every point $x \in X$ has a nbhd U st $p^{-1}(U) \cong \coprod U_i$, $p|_{U_i}: U_i \xrightarrow{\cong} U$.

<u>Examples</u>:

$$t \mapsto e^{2\pi i t}$$

,



$$x \mapsto [x]$$

$$\mathbb{RP}^2 = \mathbb{R}^2 - 0 /_{v \sim \lambda v} \cong S^2 /_{v \sim -v}$$

Definition: Let $p: E \to X$ be a covering map. A covering (or deck) transformation is a homeomorphism $f: E \xrightarrow{\cong} E$ st

$$\begin{tikzcd} E \arrow[rr, "\cong f"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

. $p \circ f = p$.

The set of cov. transformations forms a group (under composition) denoted by $\mathrm{Aut}_X E$.

Example: $\mathrm{Aut}_{S^1} \mathbb{R}$? Observe that if $t \in \mathbb{R}$ and $f \in \mathrm{Aut}_{S^1} \mathbb{R}$,

$$p(t) = p(f(t)) \Leftrightarrow e^{2\pi i t} = e^{2\pi i f(t)} \Leftrightarrow e^{2\pi i (t - f(t))} = 1 \Leftrightarrow t - f(t) = n \in \mathbb{Z}$$

So the family $f_n(t) := t + n$, $n \in \mathbb{Z}$, are the cov. transformations, ie

$$\begin{aligned} \mathrm{Aut}_{S^1} \mathbb{R} &\cong \mathbb{Z} \\ f_n &\longleftrightarrow n \end{aligned}$$

<u>Exercise</u>: $\mathrm{Aut}_{\mathbb{RP}^2} S^2$?

- A remarkable result in covering theory is the following

<u>Theorem</u> (Classification of covering spaces): Let X be a connected simplicial complex, and $x_0 \in X$. Then there is a bijection

$$\left\{ \begin{matrix} \text{subgroups of} \\ \pi_1(X, x_0) \end{matrix} \right\} \quad = \quad \left\{ \begin{matrix} \text{(pointed) connected} \\ \text{covering spaces} \\ (Y, y_0) \to (X, x_0) \end{matrix} \right\}$$

$$H \subseteq \pi_1(X, x_0) \longmapsto ((Y, y_0) \to X, x_0) \quad \text{such that}$$

normal subgp

$$\mathrm{Aut}_X Y \cong \pi_1(X, x_0)/H .$$

*: The theorem holds under far milder hypothesis.

<u>Corollary</u>: Let X be a finite connected simplicial complex. Then there exists a covering space $p: \overline{X} \longrightarrow X$ such that $\mathrm{Aut}_X \overline{X} \cong \mathbb{Z}$.

Pf: Consider

$$H := \mathrm{Ker}\left(\pi_1(X) \xrightarrow{\text{abelian}} \pi_1(X)^{ab} \longrightarrow \frac{\pi_1(X)^{ab}}{\mathrm{Tors}\ \pi_1(X)^{ab}} \cong \begin{matrix} \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ (0,0,\ldots,1,\ldots,0) & \longmapsto & 1 \end{matrix} \right)$$

and apply the theorem. $\square$

- Consider $H_1(\overline{X}) := \pi_1(\overline{X})^{ab}$, an abelian gp. Even more, it is a module over $\mathbb{Z}[t,t^{-1}] = \mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[\mathrm{Aut}_X \overline{X}]$: if $t \in \mathrm{Aut}_X \overline{X}$ is the generator, then

$$t \cdot z := t_*^{ab}(z)$$

Consider a presentation of the $\mathbb{Z}[t,t^{-1}]$-module $H_1(\bar{X})$, i.e. an exact sequence

$$\bigoplus_{n} \mathbb{Z}[t,t^{-1}] \xrightarrow{A} \bigoplus_{m} \mathbb{Z}[t,t^{-1}] \longrightarrow H_1(\bar{X}) \longrightarrow 0$$

which is determined by a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{Z}[t,t^{-1}])$. As before we can consider $\mathcal{E}_k(A)$.

* Definition: Let X be a finite connected simplical complex. The Alexander polynomial of X is

$$\Delta_X(t) := \gcd(\mathcal{E}_0(A)) \in \mathbb{Z}[t,t^{-1}]/\{\pm t^{\pm n}\}.$$

- Both notions agree for knots:

<u>Proposition</u>: $\Delta_K(t) = \Delta_{X_K}(t)$.

(Proof not too hard but it needs tools we have not developed, namely homology).

We would like to compute the RHS of the above equality. First of all, we need to realise $\overline{X}_K$:

<u>Definition</u>: Let K be a knot. A <u>Seifert surface</u> for K is a connected, orientable, compact surface Σ (with boundary) st $\partial\Sigma = K$.

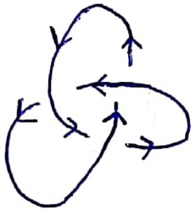
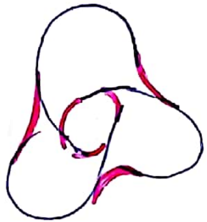
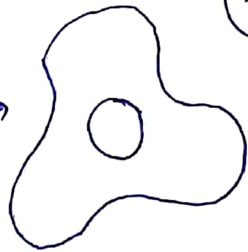
Theorem (Seifert): Any (oriented) knot has a Seifert surface.

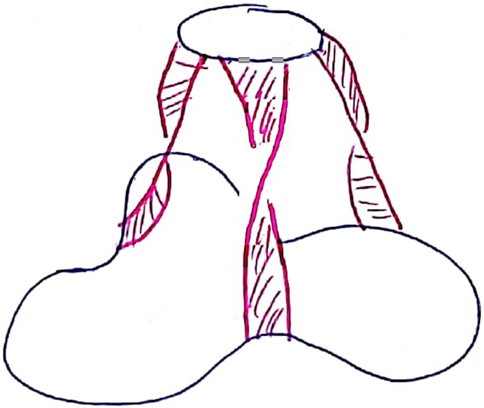
Pf: Let D be a diagram of K. Let D' be the result of modifying D by

D' = disjoint union of circles. Realise those circles as the boundary components of a disjoint union of disks in S^3, and glue half-twisted bands at the crossings.

The resulting surface Σ is a Seifert surface for K. $\square$

Remark: This procedure is called Seifert algorithm.

Example : $3_1 =$  $\rightsquigarrow$  $\rightsquigarrow$ 

$\rightsquigarrow$ 

Σ , $\qquad \partial \Sigma = 3_1$.

⚠ Warning : Surfaces built out of the Seifert algorithm may not be the easiest for any specific use, or the less complex.

Namely, let Σ be a orientable, compact, connected surface with one boundary compont.

Then $\pi_1(\Sigma)^{ab} = H_1(\Sigma) = \mathbb{Z} \oplus \overset{2g}{\cdots} \oplus \mathbb{Z}$, $g :=$ genus of Σ.

Definition : Let K be a knot. The genus of K is

$$g(K) := \min \{ \text{genus}(\Sigma) : \Sigma \text{ Seifert surface for } K \}$$

Remarkable property : $g(K_1 \# K_2) = g(K_1) + g(K_2)$

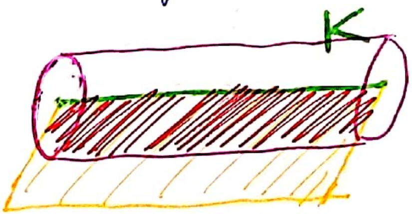
$\rightsquigarrow$ leads to the factorisation of knots as connected sum of prime knots.

Realisation of $\overline{X}_K$

(the infinite cyclic cover of X_K)
$\left(\mathrm{Aut}_{X_K} \overline{X}_K \cong \mathbb{Z}. \right)$

Let $X_K = S^3 - N(K)$ and let Σ be a Seifert surface (any!) for K.

Observe that $X_K \cap \Sigma$ is just Σ with a collar nbhd of $\partial\Sigma$ $X_K \cap N(K)$ removed,



$N(K)$

$X_K \cap N(K)$

Σ

Now let Y_Σ be the result of cutting X_K along $X_K \cap \Sigma$, ie, choose a regular nbhd $U \cong (X_K \cap \Sigma) \times [-1,1]$ of $X_K \cap \Sigma$ and let

$$Y_\Sigma := \overline{X_K - U} \quad . \left(\cong \overline{S^3 - U} \right)$$

The 3-manifold Y_Σ is connected with one boundary component,

$$\partial Y_\Sigma \cong \Sigma_+ \cup A \cup \Sigma_-$$

$\Sigma_\pm$ = copies of Σ ,

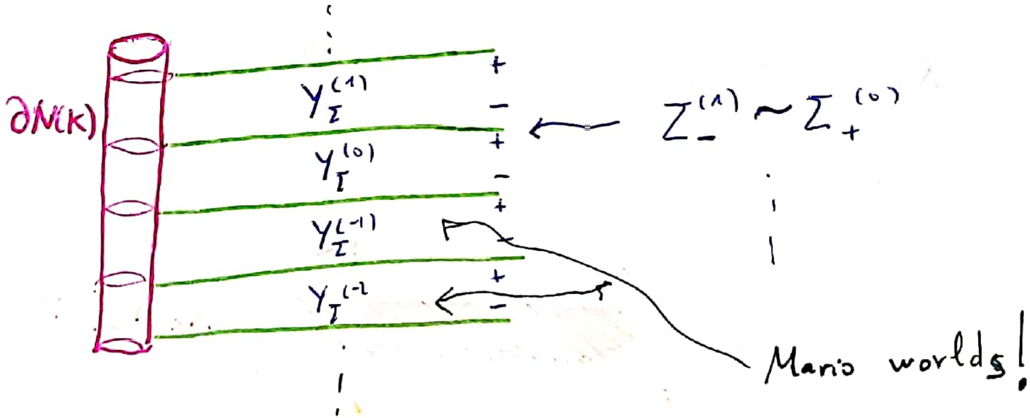
A = annulus

There are inclusion maps $i_\pm : \Sigma_\pm \hookrightarrow Y_\Sigma$. and a reflexion $r : \Sigma_+ \longrightarrow \Sigma_-$

Let $Y_\Sigma^{(i)}$, $i \in \mathbb{Z}$, a copy of Y_Σ , with homeomorphism $h_i : Y_\Sigma \xrightarrow{\cong} Y_\Sigma^{(i)}$

<u>Definition</u>: Let $Y_K^\infty := \bigcup_{i\in\mathbb{Z}} Y_\Sigma^{(i)}$ with boundary identifications

$$h_i(x) \sim h_{i+1}(r(x)) \qquad \forall x \in \Sigma_+ ,\ i \in \mathbb{Z}$$



It turns out that Y_K^∞ is a covering space for X_K with covering map

$$p : Y_K^\infty \longrightarrow X_K , \qquad p|_{Y_K^{(i)}} : Y_K^{(i)} \xrightarrow[\cong]{h_i^{-1}} Y_\Sigma \hookrightarrow X_K$$

• There is an obvious covering transformation

$$t : Y_K^\infty \to Y_K^\infty \,, \qquad t|_{Y_\Sigma^{(i)}} := h_{i+1} \circ h_i^{-1}$$

so t^n is also a covering transf $\forall n \in \mathbb{Z}$ $\longrightarrow$ $\mathrm{Aut}_{X_K} Y_K^\infty \cong \mathbb{Z} = \langle t \rangle$.

By the classification of cov spaces*, $\underline{\underline{Y_K^\infty \cong \overline{X}_K}}$

*<u>Corollary</u>: The manifold Y_K^∞ built does <u>not</u> depend on the choice of Seifert surface Σ used in the construction.

$\pi_1(Y_K^\infty)$ = insanely hard to compute

$H_1(Y_K^\infty) = \pi_1(Y_K^\infty)$ as an ab gp = infinitely many generators

$H_1(Y_K^\infty)$ as a $\mathbb{Z}[t,t^{-1}]$-module = finitely generated!

<u>Recall</u>: Want to compute $\Delta_{X_K}(t) = \gcd(\mathcal{E}_0(A))$ from a presentation

$$\bigoplus_n \mathbb{Z}[t,t^{-1}] \xrightarrow{A} \bigoplus_m \mathbb{Z}[t,t^{-1}] \longrightarrow H_1(Y_K^\infty) \longrightarrow 0.$$

We want to find A using topology!

Lemma: Let Σ be a connected, compact, oriented surface with one boundary component. Let g be the genus of Σ. Then Σ is homeomorphic to the following surface:

and $H_1(\Sigma) \cong \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_{2g}$.

Pf: Copy and paste.

Remark: For Σ a Seifert surface of a knot, it is further true that Σ is isotopic (in S^3) to the above picture, but modifying the bands allowing twists and entanglements between them.

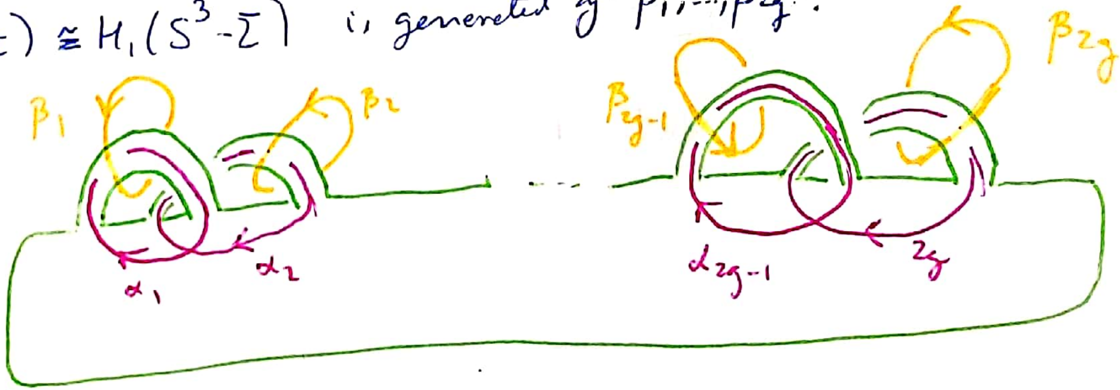
Proposition: Let Σ be a Seifert surface for K. There is a nonsingular $\mathbb{Z}$-bilinear form

$$\langle -,- \rangle : \underset{\substack{\cong \\ S^3-\Sigma}}{H_1(Y_\Sigma)} \times H_1(\Sigma) \to \mathbb{Z}$$

with the property that for oriented simple closed curves $\beta \subset Y_\Sigma$, $\alpha \subset \Sigma$ we have

$$\langle [\beta], [\alpha] \rangle = lk(\beta, \alpha)$$

"Pf": $H_1(Y_\Sigma) \cong H_1(S^3-\Sigma)$ is generated by $\beta_1, \dots, \beta_{2g}$.



$\square$

<u>Definition</u>: Let Σ be a Seifert surface for a knot K, and let $i_- : \Sigma \hookrightarrow Y_\Sigma$ which induces $(i_-)_* : H_1(\Sigma) \longrightarrow H_1(Y_\Sigma)$. The <u>Seifert form</u> associated to Σ is the $\mathbb{Z}$-bilinear form

$$S : H_1(\Sigma) \times H_1(\Sigma) \longrightarrow \mathbb{Z}$$

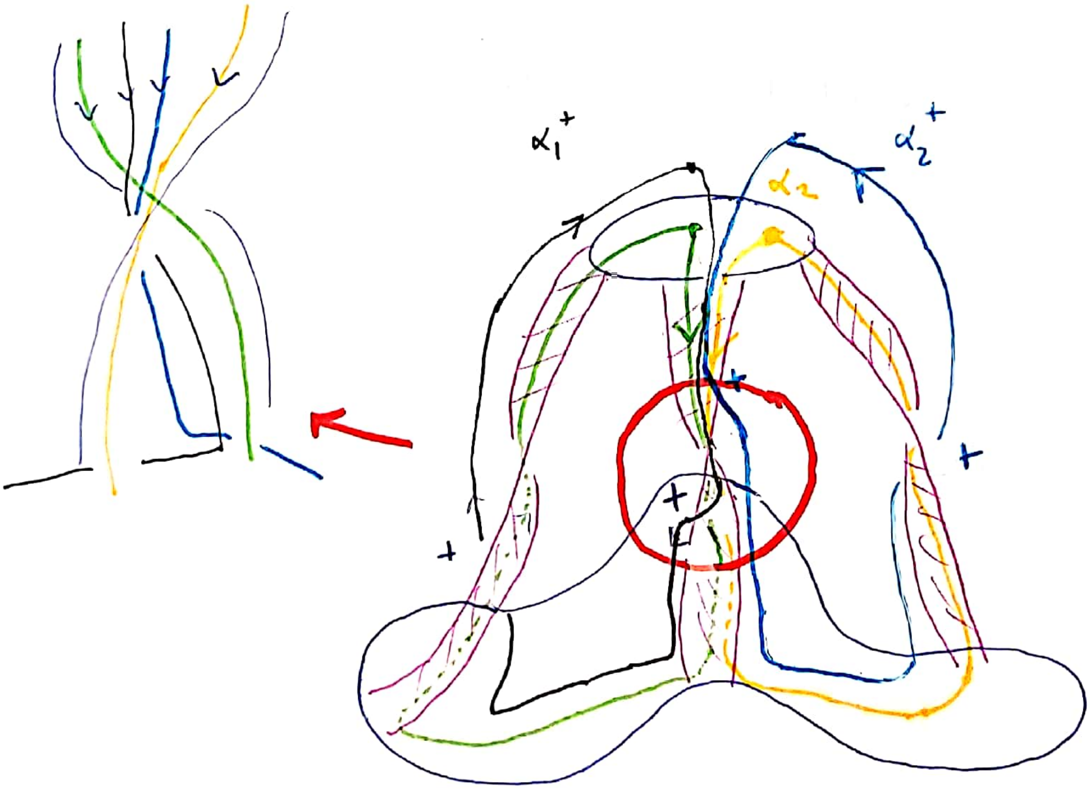
$$(x, y) \longmapsto S(x,y) := \langle (i_-)_*(x), y \rangle .$$

Since $H_1(\Sigma) \cong \mathbb{Z}_{\alpha_1} \oplus \cdots \oplus \mathbb{Z}_{\alpha_{2g}}$, once we choose a basis $\{\alpha_1, \dots, \alpha_{2g}\}$, S is given by a matrix, $V \in M_{2g \times 2g}(\mathbb{Z})$, called the <u>Seifert matrix</u>.

$$V = \left(lk(\alpha_i^-, \alpha_j) \right) = \left(lk(\alpha_i, \alpha_j^+) \right)$$

where $\alpha_p^{\pm} = (i_\pm)(\alpha_p)$.

Example: Seifert matrix for trefoil and the previous Seifert surface:



$$V = \begin{pmatrix} lk(\alpha_1, \alpha_1^+) & lk(\alpha_1, \alpha_2^+) \\ lk(\alpha_2, \alpha_1^+) & lk(\alpha_2, \alpha_2^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

* **Theorem**: Let Σ be a Seifert surface for K and let V be the Seifert matrix associated to any basis of $H_1(\Sigma)$. Then the matrix

$$\boxed{A := tV - V^T}$$

presents the $\mathbb{Z}[t,t^{-1}]$-module $H_1(Y_K^\infty)$, and it is called the Alexander matrix.

In particular,

$$\boxed{\Delta_{X_K}(t) = \det A}$$

Example:

$$\Delta_{X_{3_1}} = \det\left[t\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix} - \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\right] = \det\begin{pmatrix}t-1 & 1\\ -t & t-1\end{pmatrix} = (t-1)^2 + t = t^2 - t + 1$$

<u>Proposition</u> (Properties of Δ_K): Let K be an oriented knot. Then

1) $\Delta_K(t) = \Delta_K(t^{-1})$. More concretely, it can be written as

$$\Delta_K(t) = a_0 + a_1(t+t^{-1}) + \dots + a_n(t^n + t^{-n}).$$

with a_0 odd.

2) $\Delta_K(1) = \pm 1$

3) $\Delta_K(t) = \Delta_{\overline{K}}(t) = \Delta_{-K}(t)$

4) $\Delta_{K_1 \# K_2}(t) = \Delta_{K_1}(t) \cdot \Delta_{K_2}(t)$

$\vdots$

Pf: (of some) 1) If V is a Seifert matrix for K, $V \in M_{2g\times 2g}(\mathbb{Z})$ so

$$\Delta_K(t) = \det(tV - V^T) = \det(tV^T - V) = (-t)^{2g}\det(t^{-1}V - V^T) = \Delta_K(t^{-1})$$

3) If V Seifert surface for K, then $-V$ is a Seifert surface for $\overline{K}$ and V^T is for $-K$.

4) If V_1, V_2 are Seifert matrices for K_1, K_2, then the block matrix

$$\begin{pmatrix} V_1 & \\ & V_2 \end{pmatrix}$$

is a Seifert matrix for $K_1 \# K_2$

⚠ Warning: The equalities take place in $\mathbb{Z}[t,t^{-1}]/\{\pm t^n\}$. !! (up to units) $\square$

To conclude : The Alexander polynomial is ubiquous in knot theory:

- It is the characteristic polinomial of the linear endomorphism

$$t_* : H_1(Y_K^\infty) \otimes \mathbb{Q} \longrightarrow H_1(Y_K^\infty) \otimes K$$

induced by t generator of $\mathrm{Aut}_{X_K} Y_K^\infty$.

- For every finite simplicial complex K there is a topological invariant $\tau(K) \in \mathbb{Z}[t,t^{-1}]$ called Reidemeister torsion. It turns out that $\Delta_K(t) = \tau(X_K)\cdot(t-1)$.

- It appears in many ways using representation theory

⋮