The Conway weight system

Oscar Koster

Knot theory seminar

June 2020

Overview

- Singular knots
- Vassiliev invariants
- Chord diagrams
- Weight systems
- The Conway weight system
- What have I been doing?

My definition of a singular knots

Definition

A **n-singular knot** is a smooth map $S^1 \to \mathbb{R}^3$ that fails to be an injective immersion at n points.

My definition of a singular knots

Definition

A **n-singular knot** is a smooth map $S^1 \to \mathbb{R}^3$ that fails to be an injective immersion at n points.

Intuitively: A knot that intersects itself n times.

My definition of a singular knots

Definition

A **n-singular knot** is a smooth map $S^1 \to \mathbb{R}^3$ that fails to be an injective immersion at n points.

Intuitively: A knot that intersects itself n times.

Notation: 

My definition of a singular knots

Definition

A **n-singular knot** is a smooth map $S^1 \to \mathbb{R}^3$ that fails to be an injective immersion at n points.

Intuitively: A knot that intersects itself n times.

Notation: 

The set of all n-singular knots will be called $\mathcal{K}_n$.

Vassiliev skein relation

A knot invariant $v : \mathcal{K} \to \mathbb{C}$ can be extended to singular knots using the **Vassiliev skein relation:**

$v($  $) = v($  $) - v($  $).$

Vassiliev skein relation

A knot invariant $v : \mathcal{K} \to \mathbb{C}$ can be extended to singular knots using the **Vassiliev skein relation:**

$v($  $) = v($  $) - v($  $).$

For a knot in $\mathcal{K}_n$ we can apply this skein relation recursively to get:

$V^{(n)}($   $\ldots$  $)$

$= V^{(n-1)}($   $\ldots$  $) - V^{(n-1)}($   $\ldots$  $).$

Where moreover we take $V^0 = V(K)$.

Vassiliev invariants

Definition

A knot invariant V is said to be a **Vassiliev invariants of type** n if $V^{(n+1)}\left(\times \ldots \times \right) = 0$.

Vassiliev invariants

Definition

A knot invariant V is said to be a **Vassiliev invariants of type** n if $V^{(n+1)}\Big($  $\ldots$  $\Big) = 0$.

Definition

A finite type invariant is said to be of **order** $n \in \mathbb{Z}_{\geq 0}$ if $V^{(n+1)}($  $\ldots$  $) = 0$ but $V^{(n)}($  $\ldots$  $) \neq 0$

Vassiliev invariants

Definition

A knot invariant V is said to be a **Vassiliev invariants of type** n if $V^{(n+1)}\left(\times \dots \times \right) = 0$.

Definition

A finite type invariant is said to be of **order** $n \in \mathbb{Z}_{\geq 0}$ if $V^{(n+1)}(\times \dots \times) = 0$ but $V^{(n)}(\times \dots \times) \neq 0$

The space of all Vassiliev invariants of type n is denoted by $\mathcal{V}_n$. Notice this is a filtration i.e. $\mathcal{V}_{i-1} \subset \mathcal{V}_i$ for all $i \in \mathbb{Z}_{\geq 0}$.

Example: The Alexander-Conway polynomial

Many well known knot invariants are Vassiliev invariants. For example the n-th coefficient of the Alexander-Conway polynomial.

Example: The Alexander-Conway polynomial

Many well known knot invariants are Vassiliev invariants. For example the *n*-th coefficient of the Alexander-Conway polynomial.

The Alexander-Conway polynomial C can be defined by the Conway skein relation:

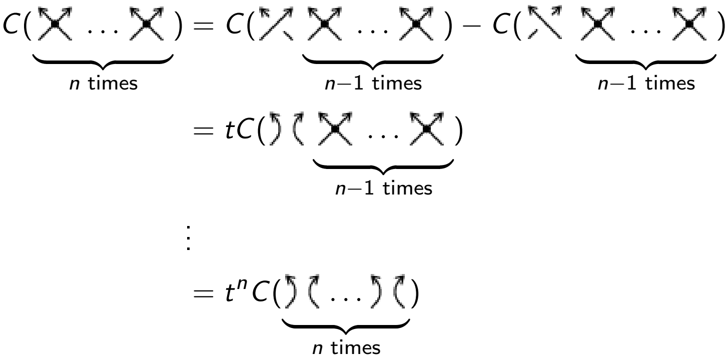
$$C(O) = 1$$

$C($  $) - C($  $) = tC($  $)$

where O denotes the unknot and t is the variable of the polynomial.

Example: The Alexander-Conway polynomial

- Together with the Vassiliev skein relation:



- Consider a knot with more than n double points.
- If a knot has $n+1$ singular points, by the above property $C^{(n+1)}$ must be divisible by t^{n+1}.
- Notice the n-th term of the Alexander-Conway polynomial is of the form $a_n t^n$, which is not divisible by $t^{(n+1)}$. This means the n-th term of the Alexander-Conway polynomial must be 0 on a $n+1$ singular knot.
- Notice that the $n+1$-th term is divisible by $t^{(n+1)}$.
- Therefore the n-th term of the Alexander-Conway polynomial is a Vassiliev invariant of order n.

Why should anyone care?

Why should anyone care?

- Complete knot invariant

Why should anyone care?

- Complete knot invariant
- Conjecture: The set of all Vassiliev invariants forms a complete knot invariant.

Why should anyone care?

- Complete knot invariant
- Conjecture: The set of all Vassiliev invariants forms a complete knot invariant.
- Moreover, the set of all Vassiliev invariants form a module. So for a field $\mathbb{F}$ with characteristic zero, the set of $\mathbb{F}$-valued invariants form a vector space. Therefore, problems can be solved using linear algebra.

Why should anyone care?

- Complete knot invariant
- Conjecture: The set of all Vassiliev invariants forms a complete knot invariant.
- Moreover, the set of all Vassiliev invariants form a module. So for a field $\mathbb{F}$ with characteristic zero, the set of $\mathbb{F}$-valued invariants form a vector space. Therefore, problems can be solved using linear algebra.
- Mostly, because it is fun mathematics!

A few problems that need to be discussed

- How many Vassiliev invariants are of each order?
- Can we make new Vassiliev invariants in an easy way?
- Is there any geometric way in which we can describe these invariants?

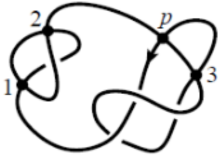
Chord diagrams

A 'recipe' for making chord diagrams.

- Step 1: Pick an orientation for the knot and walk around the knot in the direction of the orientation, parametrise this walk on a circle.
- Step 2: Record the crossings encountered, put labels of the corresponding crossings on the circle.
- Step 3: You always pass both an under- and an overcrossing. Connect the corresponding labels by a chord.

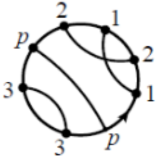
The set of formal linear combinations of chord diagrams with n chords forms a vector space, denoted $\mathcal{C}_n$.

Example of a Chord diagram



K

$\longmapsto$



C

Why do we define chord diagrams?

Theorem

The value of a Vassiliev invariant of degree m on a knot K with m singularities depends only on the chord diagram of K.

Why do we define chord diagrams?

Theorem

The value of a Vassiliev invariant of degree m on a knot K with m singularities depends only on the chord diagram of K.

This means we 'only' have to know chord diagrams to describe Vassiliev invariants!

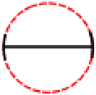
Weight systems

Definition

Given a chord diagram, an **isolated chord** is a chord that does not intersect any other chord of the diagram.

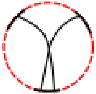
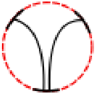
Weight systems: 1T

An element W in $\mathcal{C}^\star$ satisfies the **1T-relation** if W on an isolated chord gives 0, i.e.

$W \Bigg($  $\Bigg) = 0.$

Weight systems: 4T

An element W in $\mathcal{C}^\star$ satisfies the **4T-relation** if the following relation holds

$W\Big($  $\Big) - W\Big($  $\Big) = W\Big($  $\Big) - W\Big($  $\Big)$

Weight systems

Definition

A weight system is an element W in $\mathcal{C}^\star$ such that both the 4T-relation and the 1T-relation hold.

The vector space of all weight systems on $\mathcal{C}_n$ is called $\mathcal{W}_n$.

Kontsevich theorem

Theorem

There is an isomorphism between the space of Vassiliev invariants $\mathcal{V}_n/\mathcal{V}_{n-1}$ *and* $\mathcal{W}_n$.

Kontsevich theorem

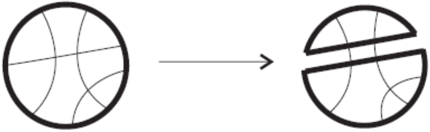
Theorem

There is an isomorphism between the space of Vassiliev invariants $\mathcal{V}_n/\mathcal{V}_{n-1}$ and $\mathcal{W}_n$.

This means that for every Vassiliev invariant there is a weight system and vice versa. So to study Vassiliev invariants we only need to understand weight systems!

Conway weight system

Given a chord diagram D, we turn every chord into a bridge as below. Denote the resulting chord diagram by D'.



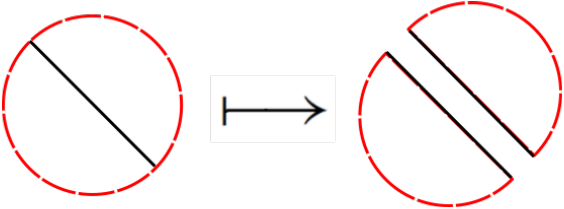
Conway weight system

The Conway weight systems is defined as follows:

$$W_C(D) := \begin{cases} 1 & \text{if } D' \text{ has one connected component} \\ 0 & \text{otherwise.} \end{cases}$$

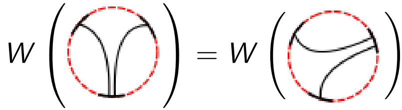
Conway weight system: Why is this a weight system?

- 1T: Suppose we have a diagram D isolated chord. Then changing the isolated chord into a bridge leaves two connected components. Therefore, $W_C(D) = 0$.

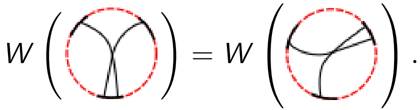


The 2T-relation

- 4T: We need the 2T-relation!
- Set both sides of the 4T relation to zero to get:



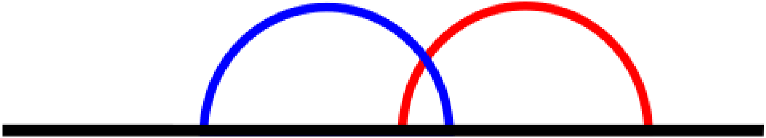
And,



- 2T $\Rightarrow$ 4T

Linear chord diagrams

- Cut open the chord diagrams to get:



Finishing the proof

- Bridge these chords

- The number of connected components does not change.
- So the Conway weight system satisfies 2T.

My bachelor thesis

- Understand Kontsevich theorem and prove part of it.
- Use Kontsevich theorem to find the Vassiliev invariant related to this weight system.
- Show that this invariant is a well-defined invariant.
- Show why this weight systems is so special.
- Count the number of Vassiliev invariants of a certain order.
- Try to say something geometric about these weight systems.
- Relate weight systems to an intersection graph.

Summary

- Definition of Vassiliev invariants
- Vassiliev invariants are conjectured to be a complete invariant.
- Vassiliev invariants can be described by weight systems.