Stabilization Invariance of Grid Homology

Sven Bootsma

Rijksuniversiteit Groningen

s.bootsma99@gmail.com

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The Goal of Today

Theorem (Our Main Theorem)

The simply blocked grid homology, that is $\widehat{\mathrm{GH}}$, is invariant under stabilization. More formally, we have the isomorphism $\widehat{\mathrm{GH}}(\mathbb{G}') \cong \widehat{\mathrm{GH}}(\mathbb{G})$ where $\mathbb{G}'$ is obtained from $\mathbb{G}$ by a stabilization move.

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- This turns out to be a corollary of two other results.
- The proof relies heavily on homological and commutative algebra (chain complexes, homology, tensors etc.).
- We will go through these prerequisites.
- Don't be scared.

Prerequisites on Homological and Commutative Algebra

Definition

Let $X = \bigoplus_{d,s \in \mathbb{Z}} X_{d,s}$ and $Y = \bigoplus_{d,s \in \mathbb{Z}} Y_{d,s}$ be two bigraded vector spaces. Their tensor product $X \otimes Y = \bigoplus_{d,s \in \mathbb{Z}} (X \otimes Y)_{d,s}$ is the bigraded vector space with $(X \otimes Y)_{d,s} = \bigoplus_{\substack{d_1+d_2=d \\ s_1+s_2=s}} (X_{d_1,s_1} \otimes Y_{d_2,s_2})$.

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Definition

Let X be a bigraded vector space, and fix integers a and b. The corresponding shift of X, denoted $X[\![a,b]\!]$, is the bigraded vector space that is isomorphic to X as a vector space and given the bigrading $X[\![a,b]\!]_{d,s} = X_{d+a,s+b}$.

Prerequisites on Homological and Commutative Algebra

Lemma (Important!)

Let W be the bigraded vector space of dimension 2 with 1 generator in bigrading $(0,0)$ and 1 generator in bigrading $(-1,-1)$. Let X be any other bigraded vector space, then $X \otimes W \cong X \oplus X[\![1,1]\!]$.

Prerequisites on Homological and Commutative Algebra

$W = W_{(0,0)} \oplus W_{(-1,-1)}$

Lemma (Important!)

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Proof.

Exercise. $\square$

Prerequisites on Homological and Commutative Algebra

Lemma (Also Important!)

Let

$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0,$$

be a short exact sequence of vector spaces, then $V \cong U \oplus W$.

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Lemma (Also Important!)

Let

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be a short exact sequence of vector spaces, then $V \cong U \oplus W$.

Proof.

Exercise. $\square$

- fin. dim ; R-N Thm
- inf. dim ; Every vector space is free

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- Chain complexes are assumed to be known by now.

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- The homology $H(C, \partial)$ of the chain complex (C, ∂) is the quotient module $\operatorname{Ker} \partial / \operatorname{Im} \partial$.
- Let (C, ∂) and (C', ∂') be two chain complexes. A module homomorphism $f\colon C \to C'$ is a chain map if it commutes with the boundary operators, that is, for every $c \in C, f(\partial c) = \partial' f(c)$.

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- These notions are also defined for bigraded chain complexes.

Prerequisites on Homological and Commutative Algebra

Lemma

Let (C, ∂) and (C', ∂') be two chain complexes and let f be a chain map between them. Then f induces a module map $H(f)\colon H(C, \partial) \to H(C', \partial')$ on the homology modules.

Lemma

Let (C, ∂) and (C', ∂') be two chain complexes and let f be a chain map between them. Then f induces a module map $H(f): H(C, \partial) \to H(C', \partial')$ on the homology modules.

Proof.

Exercise. □

- Well-definedness; should look at what happens to cycles ($=\ker(\partial)$) & boundaries ($\mathrm{Im}(\partial)$).

Lemma (The Connecting Homomorphism)

To each short exact sequence of chain complexes

$$0 \longrightarrow (C, \partial) \xrightarrow{f} (C', \partial') \xrightarrow{g} (C'', \partial'') \longrightarrow 0,$$

there is an associated module homomorphism $\delta : H(C'', \partial'') \to H(C, \partial)$, *called the connecting homomorphism, such that*

$$\begin{tikzcd} & H(C,\partial) \arrow[dr, "H(f)"] \arrow[d, phantom, "\circlearrowleft"] & \\ H(C'',\partial'') \arrow[ur, "\delta"] & & H(C',\partial') \arrow[ll, "H(g)"] \end{tikzcd}$$

is an exact triangle.

Prerequisites on Homological and Commutative Algebra

Lemma (The Connecting Homomorphism)

To each short exact sequence of chain complexes

$0 \to H(C) \to \ldots \to \ldots \to 0$

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$$\begin{tikzcd} & H(C,\partial) \arrow[dr, "H(f)"] & \\ H(C'',\partial'') \arrow[ur, "\delta"] & & H(C',\partial') \arrow[ll, "H(g)"] \end{tikzcd}$$

There is no reason for H(f) to be injective

is an exact triangle.

- Note that we do not necessarily obtain a **short** exact sequence.

Proof (Construction only).

- $x \in H(C'', \partial'')$ & a cycle representing x; say c''
- Find an element $c \in C$, $c' \in C'$ so that $g(c') = c''$, $f(c) = \partial'(c)$.
- $\delta(x) := c$

Main Results

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Proposition (Tilde-Homology and Stabilization)

There is an isomorphism of bigraded vector spaces

$\widetilde{GH}(\mathbb{G}) \otimes W$

$$\widetilde{\mathrm{GH}}(\mathbb{G}') \cong \widetilde{\mathrm{GH}}(\mathbb{G}) \oplus \widetilde{\mathrm{GH}}(\mathbb{G})[\![1,1]\!].$$

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Proposition (Tilde-Homology and Hat-Homology)

Let W be the 2-dimensional vector with one generator in bigrading $(0,0)$ and one generator in bigrading $(-1,-1)$. Then there is an isomorphism

$$\widetilde{\mathrm{GH}}(\mathbb{G}) \cong \widehat{\mathrm{GH}}(\mathbb{G}) \otimes W^{\otimes(n-1)},$$

where n is the grid number of $\mathbb{G}$.

Corollary (Our Main Theorem)

The simply blocked grid homology, that is $\widehat{\mathrm{GH}}$, is invariant under stabilization. More formally, we have the isomorphism $\widehat{\mathrm{GH}}(\mathbb{G}') \cong \widehat{\mathrm{GH}}(\mathbb{G})$.

Proof:

- $\widetilde{GH}(\mathbb{G}') \cong \widetilde{GH}(\mathbb{G}) \otimes W$ (1)
- $\widetilde{GH}(\mathbb{G}') \cong \widehat{GH}(\mathbb{G}') \otimes W^{\otimes n}$ (2)
- $\widetilde{GH}(\mathbb{G}) \cong \widehat{GH}(\mathbb{G}) \otimes W^{\otimes (n-1)}$ (3)

$\otimes_{\mathbb{Z}}$

(1)+(3) $\Rightarrow$ $\widetilde{GH}(\mathbb{G}') \cong \widehat{GH}(\mathbb{G}) \otimes W^{\otimes n}$

(2) $\Rightarrow$ $\blacksquare$

Towards a Proof of Tilde-Homology and Stabilization

- Suppose $\mathbb{G}'$ is obtained by a stabilization of type $X{:}SW$.

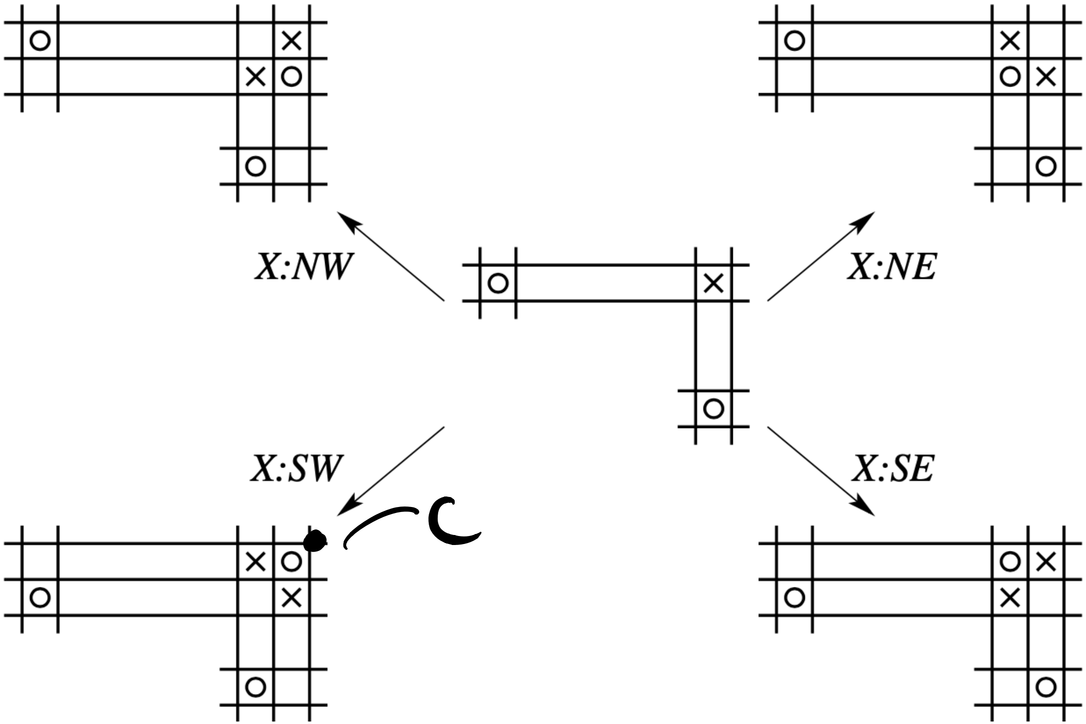

Figure: The four different type of stabilizations (at X)

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- $\mathbb{G}$ is gotten from $\mathbb{G}'$ by destabilizing at the 2×2 square $\begin{array}{c|c} X & O \\ \hline & X \end{array}$.

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- $\mathbb{G}$ is gotten from $\mathbb{G}'$ by destabilizing at the 2×2 square $\begin{array}{c|c} X & O \\ \hline & X \end{array}$.
- We number the O-markings so that O_1 is the newly introduced one and O_2 is the O-marking in the row just below O_1. Moreover, let c denote the intersection point of the newly obtained horizontal and vertical circles and lets label the X-markings $\begin{array}{c|c} X_1 & O_1 \\ \hline & X_2 \end{array}$.

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- Decompose the set of grid states $S(\mathbb{G}')$ of the stabilized diagram $\mathbb{G}'$ as the disjoint union $I(\mathbb{G}') \cup N(\mathbb{G}')$, where $I(\mathbb{G}')$ is the set of grid states $x \in S(\mathbb{G}')$ with $c \in x$.

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- Decompose the set of grid states $S(\mathbb{G}')$ of the stabilized diagram $\mathbb{G}'$ as the disjoint union $I(\mathbb{G}') \cup N(\mathbb{G}')$, where $I(\mathbb{G}')$ is the set of grid states $x \in S(\mathbb{G}')$ with $c \in x$.
- $\widetilde{GC}(\mathbb{G}')$ then decomposes as $\widetilde{GC}(\mathbb{G}') \cong \widetilde{I} \oplus \widetilde{N}$, where $\widetilde{I} = \operatorname{span} I(\mathbb{G}')$ and $\widetilde{N} = \operatorname{span} N(\mathbb{G}')$.

Towards a Proof of Tilde-Homology and Stabilization

- The map $\widetilde{\partial}_{\mathbb{O},\mathbb{X}}$ can be written in matrix form as

$$\widetilde{\partial}_{\mathbb{O},\mathbb{X}} = \begin{bmatrix} \widetilde{\partial}_I^I & 0 \\ \widetilde{\partial}_I^N & \widetilde{\partial}_N^N \end{bmatrix}.$$

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- There is a bijection between $S(\mathbb{G})$ and $I(\mathbb{G}')$. $e: S(\mathbb{G}) \to I(\mathbb{G}')$

$x \mapsto x \cup \{c\}$

Lemma

The map $\widetilde{e} : \widetilde{I} \to \widetilde{GC}(\mathbb{G})$ induced by the correspondence $x \cup \{c\} \mapsto x$ gives an isomorphism between the bigraded chain complexes $(\widetilde{I}, \widetilde{\partial}_I^I)$ and $\widetilde{GC}(\mathbb{G})[\![1, 1]\!]$.

Proof: • e is a bijection between grid states & generated as $\mathbb{F}_2$-v.s's over these grid states $\Rightarrow$ $\widetilde{e}$ iso. of $\mathbb{F}_2$ v.s's

• Empty rectangles disjoint from $\mathbb{O} \cup \mathbb{X}$ $\Leftrightarrow$ empty rectangles disjoint from $\mathbb{O}' \cup \mathbb{X}'$

• $\widetilde{N}$ a subcompl.

• $\widetilde{I}$ quotient complex.

Towards a Proof of Tilde-Homology and Stabilization

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- If $H(\widetilde{I}, \widetilde{\partial}_I^I) \cong \widetilde{\mathrm{GH}}(\mathbb{G})[\![1, 1]\!]$, then $H(\widetilde{N}, \widetilde{\partial}_N^N) \cong \widetilde{\mathrm{GH}}(\mathbb{G})$

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- We need one last lemma before we can give a proof.

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Lemma

The chain map $\widetilde{\partial}_I^N \colon (\widetilde{I}, \widetilde{\partial}_I^I) \to (\widetilde{N}, \widetilde{\partial}_N^N)$ induces the trivial map on homology.

The Proof of Tilde-Homology and Stabilization

Proposition (Tilde-Homology and Stabilization)

There is an isomorphism of bigraded vector spaces

$$\widetilde{\mathrm{GH}}(\mathbb{G}') \cong \widetilde{\mathrm{GH}}(\mathbb{G}) \oplus \widetilde{\mathrm{GH}}(\mathbb{G})[\![1,1]\!].$$

Proof: • $\widetilde{GC}(\mathbb{G}') \cong \tilde{I} \oplus \tilde{N}$, * $\tilde{N}$ is a subcomplex
* $\tilde{I}$ is a quotient complex.

$\Rightarrow 0 \to \tilde{N} \xrightarrow{i} \widetilde{GC}(\mathbb{G}') \xrightarrow{\pi} \tilde{I} \to 0$

$\Rightarrow$

$$\begin{tikzcd}
 & \widetilde{GH}(\mathbb{G}') \arrow[dr, "H(\pi)"] & \\
H(\tilde{N}) \arrow[ur, "H(i)"] & & H(\tilde{I}) \arrow[ll, "\delta = 0"]
\end{tikzcd}$$

$\delta = H(\tilde{\delta}_I^N) = 0$

The Proof of Tilde-Homology and Stabilization

$$\Rightarrow \quad 0 \to H(\tilde{N}) \to \widetilde{GH}(\mathbb{G}') \to H(\tilde{I}) \to 0$$

$$H(\tilde{N}) \cong \widetilde{GH}(\mathbb{G}), \qquad H(\tilde{I}) \cong \widetilde{GH}(\mathbb{G})[\![1,1]\!]$$

$$\Rightarrow \quad \widetilde{GH}(\mathbb{G}') \cong \widetilde{GH}(\mathbb{G}) \oplus \widetilde{GH}(\mathbb{G})[\![1,1]\!]. \qquad \blacksquare$$