LECTURE 11: DOUBLING STRANDS AND BIALGEBRAS

1. Operations on XC-tangles

So far we have mainly worked with XC-tangles by piecing them together from the elementary building blocks $X^{\pm}, C^{\pm}$ using merges and disjoint union. There are however many more interesting operations available on XC-tangles. For example we can delete a strand or we can make a parallel copy following a strand. Just like we defined the invariant Z_A by generators and relations we also introduce these two new operations by defining on the building blocks and how they interact with merging and disjoint union.

Definition 1. (Strand deletion)
For an XC-tangle diagram D with strand s define $\check{\varepsilon}^s(D)$ inductively as follows ($\sigma \in \{\pm 1\}$).

$$\check{\varepsilon}^s(\check{X}^\sigma_{s,t}) = 1_t = \check{\varepsilon}^s(\check{X}^\sigma_{t,s}) = \check{\varepsilon}^t(\check{C}^\sigma_t) \tag{1}$$

$$\check{\varepsilon}^s(DD') = \check{\varepsilon}^s(D)\check{\varepsilon}^s(D') \quad \check{\varepsilon}^s(\check{m}^{h,t}_s(D)) = \check{\varepsilon}^t\check{\varepsilon}^h(D) \tag{2}$$

Although we defined this operation on tangle diagrams it should be clear that deleting a strand is compatible with the Reidemeister moves and so is really an operation on tangles.

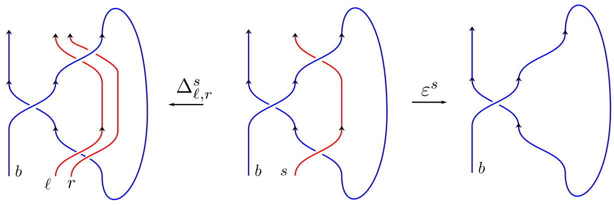

Figure 1. Doubling strand s and deleting strand s of the XC-tangle diagram shown in the middle.

Definition 2. (Strand doubling)
For an XC-tangle diagram D with strand s and $l, r \notin \mathcal{L}(D)$ define $\check{\Delta}^s_{l,r}(D)$ inductively as follows ($\sigma \in \{\pm 1\}$).

$$\check{\Delta}^s_{l,r}(\check{X}^\sigma_{s,t}) = \check{m}^{t_{-1},t_1}_t(\check{X}^\sigma_{r,t_{-\sigma}}\check{X}^\sigma_{l,t_\sigma}) \quad \check{\Delta}^s_{l,r}(\check{X}^\sigma_{t,s}) = \check{m}^{t_{-1},t_1}_t(\check{X}^\sigma_{t_{-\sigma},l}\check{X}^\sigma_{t_\sigma,r}) \quad \check{\Delta}^s_{l,r}(\check{C}^\sigma_s) = \check{C}^\sigma_l\check{C}^\sigma_r \tag{3}$$

$$\check{\Delta}^s_{l,r}(DD') = \check{\Delta}^s_{l,r}(D)\check{\Delta}^s_{l,r}(D') \quad \check{\Delta}^s_{l,r}(\check{m}^{h,t}_s(D)) = \check{m}^{r_1,r_2}_r\check{m}^{l_1,l_2}_l\check{\Delta}^t_{l_2,r_2}\check{\Delta}^h_{l_1,r_1}(D) \tag{4}$$

Just like we sometimes want to do several merges we sometimes like to triple a strand and write $\check{\Delta}^s_{a,b,c} = \check{\Delta}^x_{b,c} \circ \check{\Delta}^s_{a,x}$. As some kind of parallel to associativity we note that the order of tripling does not matter, we have $\check{\Delta}^s_{a,b,c} = \check{\Delta}^x_{a,b} \circ \check{\Delta}^s_{x,c}$.

As with strand deletion, the result of strand doubling does not depend on the particular XC-tangle diagram, only on the tangle (exercise!).

2. Bialgebras

Before moving on we remark that similar structures come up in algebra. Especially in algebras whose dual is also an algebra. Or said differently bialgebras in the sense of

Definition 3. (Bialgebra)
An algebra A with multiplication m is called a **bialgebra** if there are algebra morphisms $\Delta : A \longrightarrow A \otimes A$ (coproduct) and $\varepsilon : A \longrightarrow k$ (counit) that satisfy the following axioms called counit and coassociativity:

$$(\varepsilon \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \varepsilon) \circ \Delta = \mathrm{Id} \quad (\Delta \otimes \mathrm{Id}) \circ \Delta = (\Delta \otimes \mathrm{Id}) \circ \Delta$$

We already saw several examples of bialgebras. For example the group algebra $k[G] = \{\sum_{g\in G} a_g g : a \in k\}$ with $\Delta(g) = g_1 g_2$ and $\varepsilon(g) = 1$.

Another example is the Sweedler algebra with $\Delta(s) = s_1 s_2$ and $\Delta(w) = s_2 w_1 + w_2$ and $\varepsilon(s) = 1$, $\varepsilon(w) = 0$.

As with algebras we like to use subscripts to denote tensor factors and extend Δ and ε to unordered tensor products using the notation $\Delta^s_{l,r} : A^{\otimes\{s\}} \longrightarrow A^{\otimes\{l,r\}}$ and $\varepsilon^s : A^{\otimes\{s\}} \longrightarrow k$. The first takes the coproduct on the elements in tensor factor s and places the result in factors l and r.

Notice that the assertion that ε and Δ are algebra morphisms are precisely the algebraic equivalents of equations (2) and (4). Indeed the equation $\Delta(ab) = \Delta(a)\Delta(b)$ that expresses the fact that Δ is a homomorphism implicitly makes use of the standard algebra structure on $A \otimes A$. When making this explicit the equation becomes

$$\Delta^s_{l,r}(m^{h,t}_s(a_h b_t)) = m^{r_1,r_2}_r m^{l_1,l_2}_l \Delta^t_{l_2,r_2}\Delta^h_{l_1,r_1}(a_h b_t)$$

that looks like Equation (4) as promised. The requirement that coproduct and counit respect the unit is likewise reflected in the above equations on the C's. The coassociativity and counit axioms of bialgebra also clearly reflect what happens when we double and delete strands in a tangle.

Before continuing let us briefly comment on the significance of bialgebras in representation theory. Recall that a linear representation of an algebra is a homomorphism $\rho : A \longrightarrow \mathrm{End}(V)$ for some vector space. If A happens to be a Hopf algebra then linear representations can be combined in useful ways to get more representations. In fact each of the Hopf algebra maps can be interpreted in terms of some operation on representations. First the co-unit coincides with the trivial representation $\varepsilon : A \longrightarrow k \cong \mathrm{End}(k)$. Finally Δ allows the tensor product $V \otimes W$ to become a representation via $\tau : A \longrightarrow \mathrm{End}(V \otimes W)$ defined as $\tau(v \otimes w) = (\rho \otimes \sigma)(\Delta(a))(v \otimes w)$.

Let us connect the concept of a bialgebra to the set up of the previous lecture where we chose algebras $\mathbb{O}$ and $\mathbb{U}$. The element $X = \sum^i_o \otimes u^i \in \mathbb{O} \otimes \mathbb{U}$ defines a non-degenerate[1] pairing $\langle\rangle$ that makes o^i and u^i dual bases. So it sets up an isomorphism $\mathbb{U}^* \cong \mathbb{O}$ and allows us to interpret $m^*_{\mathbb{O}}$ as a coproduct in $\mathbb{U}$ and $m^*_{\mathbb{O}}$ as a coproduct in $\mathbb{O}$. More concretely we define

$$\Delta_{\mathbb{U}}(x) = \sum_{a,b} \langle o^a o^b, x\rangle u^a \otimes u^b \qquad \Delta_{\mathbb{O}}(x) = \sum_{a,b} \langle x, u^a u^b\rangle o^a \otimes o^b$$

The coproduct also appeared implicitly in our product formula see Equation (6) on $\mathbb{D}$ copied below as follows

$$u^p o^q = \sum_{i,j,k,\ell} \overline{\langle o^q, \tilde{u}^i \tilde{u}^j u^\ell\rangle} \langle o^i o^k o^\ell, u^p\rangle o^j u^k \tag{5}$$

The pairing with the triple product is where the coproduct comes in. We have $(\Delta_{\mathbb{U}})^1_{1,2}(x) = \sum_{a,b}\langle o^a o^b, x\rangle u^a_1 u^b_2$ and applying the coproduct again on the second tensor factor we get

$$\Delta^1_{1,2,3} = (\Delta_{\mathbb{U}})^2_{2,3} \circ (\Delta_{\mathbb{U}})^1_{1,2}(x) = \sum_{a,b} \langle o^a o^b, x\rangle u^a_1 (\Delta_{\mathbb{U}})^2_{2,3}(u^b_2) =$$

$$\sum_{a,b} \langle o^a o^b, x\rangle u^a_1 \sum_{c,d} \langle o^c o^d, u^b\rangle u^c_2 u^d_3 = \sum_{a,c,d} \langle o^a \sum_b \langle o^c o^d, u^b\rangle o^b, x\rangle u^a_1 u^c_2 u^d_3 =$$

$$= \sum_{a,c,d} \langle o^a o^c o^d, x\rangle u^a_1 u^c_2 u^d_3$$

What is however NOT guaranteed in our set-up so far is the assurance that the coproducts we defined are algebra maps (i.e. homomorphisms). To make progress we will need to make this assumption.

3. Associativity of the algebra $\mathbb{D}$

Recall from the previous lecture that we chose two algebras $\mathbb{O}, \mathbb{U}$ with elements $X = \sum_i o^i \otimes u^i$, $X^{-1} = \sum_i o^i \otimes \tilde{u}^i$ that correspond to non-degenerate pairings $\langle , \rangle$ and $\overline{\langle , \rangle}$. This data is enough to define an invariant $Z_{\mathbb{D}}$ of OU-tangles, where $\mathbb{D}$ is the vector space spanned by all ordered products $o^i u^j$.

[1] $\langle\rangle$ is non-degenerate if for all x the statement $\forall y \langle x, y\rangle = 0$ implies $x = 0$.

Theorem 4. *If $Z_{\mathbb{D}}$ satisfies $\Delta^s_{l,r} Z_{\mathbb{D}}(T) = Z_{\mathbb{D}}(\check{\Delta}^s_{l,r} T)$ for OU-tangles then the product on $\mathbb{D}$ shown below is associative and $Z_{\mathbb{D}}$ extends to an invariant of all XC-tangles without C's*

$$u^p o^q = \sum_{i,j,k,\ell} \overline{\langle o^q, \tilde{u}^i \tilde{u}^j u^\ell \rangle} \langle o^i o^k o^\ell, u^p \rangle o^j u^k \tag{6}$$

Proof. To prove associativity in $\mathbb{D}$ it suffices to prove that for any a, b, c we have both $u^a(o^b o^c) = (u^a o^b) o^c$ and $u^a(u^b o^c) = (u^a u^b) o^c$. We will focus on proving the first equality as the proof of the second is similar.

The essence of the proof is the equality of the XC-tangles shown in Figure 2 so please take a moment to check that all six XC-tangle diagrams are equivalent.

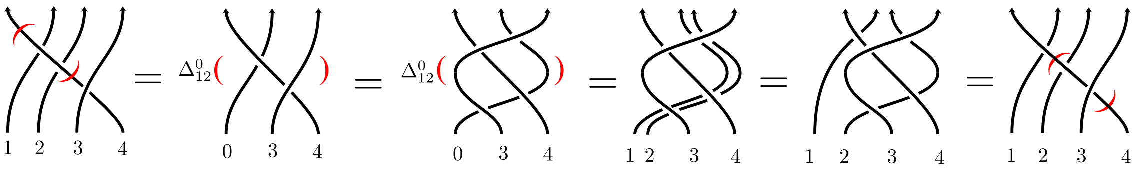

FIGURE 2. The essence of the proof of $u^a(o^b o^c) = (u^a o^b) o^c$.

To see what the figure has to do with associativity we remark that the red brackets in the left- and right-most picture mean that the merge between the brackets happened first. In formulas the left-most picture should be written as $L = \check{m}^{p,s}_4 \check{m}^{q,r}_s (\check{X}^{-1}_{r,1} \check{X}^{-1}_{q,2} \check{X}_{3,p})$. The corresponding invariant is $Z_{\mathbb{D}}(L) = \sum_{a,b,c} \tilde{u}^c_1 \tilde{u}^b_2 o^a_3 (u^a (o^b o^c))_4$, notice the red brackets emphasizing we do not have associativity yet! Taking the $\overline{\langle\rangle}$ pairing with o^c and o^b on the first and second and pairing on the 3rd factor with u^a yields $u^a(o^b o^c)$. Taking the same pairings on the $Z_{\mathbb{D}}$ invariant of the right-most picture likewise yields $(u^a o^b) o^c$ (try it!).

The proof is complete once we notice that the middle pictures in Figure 2 are all OU-tangle diagrams so that in taking their invariant there is no issue of associativity in multiplying the terms. The invariants of all the tangle diagrams shown are equal because we assumed that $Z_{\mathbb{D}}$ is compatible with the strand-doubling/coproduct. $\square$