Topics in Topology Lecture 4

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February 17, 2021

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1 Introduction

In this lecture we will cover material from sections 4.4 and 4.6. We will focus on section 4.4 where we define the grid complex $\widetilde{GC}(\mathbb{G})$ of a grid diagram, and prove that $\tilde{\partial}_{\mathbb{O},\mathbb{X}}$ is a differential of degree $(-1, 0)$.

After that we will briefly turn to the contents of section 4.6 where we will mention the existence of two other grid complexes $\widehat{GC}(\mathbb{G})$ and $GC^-(\mathbb{G})$, which are more complicated than $\widetilde{GC}(\mathbb{G})$. This examplifies the richness of structure one can extract from a grid diagram.

We conclude the lecture with a round of questions and exercises. I would like to acknowledge the help of Jorge Becerra for elaborating on some concepts I was not familiar with.

2 Preliminaries

Before we begin, we need to address a few things. We assume that the reader is familiar with the contents of sections 4.1 to 4.3 where the concept of grid states, rectangles and bigrading are introducted. The notation we use will be the same as in the book.

Furthermore we state without proof two equations from section 4.3 that we need, those are property $(M-2)$ from proposition 4.3.1, and the equation from proposition 4.3.3:

$$M_{\mathbb{O}}(\mathbf{x}) - M_{\mathbb{O}}(\mathbf{y}) = 1 - 2\#(r \cap \mathbb{O}) + 2\#(\mathbf{x} \cap \mathrm{Int}(r)) \tag{1a}$$

$$A(\mathbf{x}) - A(\mathbf{y}) = \#(r \cap X) - \#(r \cap \mathbb{O}), \tag{1b}$$

we write $M := M_{\mathbb{O}}$ as in the book, unless stated otherwise.

3 Grid chain complexes

For a grid diagram $\mathbb{G}$ we have a set of grid states $\mathbf{S}(\mathbb{G})$. We may generate an $\mathbb{F}$-vector space $\widetilde{GC}(\mathbb{G}) := \mathbb{F}[\mathbf{S}(\mathbb{G})]$ by taking formal linear combinations of elements of $\mathbf{S}(\mathbb{G})$, where a sum of repeating grid states cancels out, since $x + x = 2x = 0$.

As shown in section 4.3 this vector space has a bigrading, w.r.t., the Maslov and Alexander functions, M and A, respectively, we write:

$$\widetilde{GC}(\mathbb{G}) = \bigoplus_{d,s\in\mathbb{Z}} \widetilde{GC}_d(\mathbb{G}, s),$$

for which $M(\mathbf{x}) = d$ and $A(\mathbf{x}) = s$ for any $\mathbf{x} \in \widetilde{GC}_d(\mathbb{G}, s)$, this means we deal with a bigraded vector space.

From a bigraded vector space we construct chain complexes by defining some homomorphism ∂ called the *differential* (also the *boundary operator*) that maps one grading of the space to another grading as a subset, and for which $\partial^2 = 0$, i.e., the boundary of a boundary is empty. The differential is also sometimes affectionately called the "Master equation", though this term is not reserved only for this equation.

Besides $\widetilde{GC}$, we may generate many bigraded vector spaces over $\mathbf{S}(\mathbb{G})$, we will mention two more as well: $\widehat{GC}$ and GC^-. For each we can find a differential, and in turn each encodes different information and structure. We elaborate on the idea by presenting the chain complex $(\widetilde{GC}, \tilde{\partial}_{\mathbb{O},\mathbb{X}})$ and proving that $\tilde{\partial}_{\mathbb{O},\mathbb{X}}$ is indeed a differential of degree $(-1, 0)$, and only briefly mention the other two.

For a nice initiation into the concept of chain complexes, it is worthwhile to look at the de Rham cohomology, which has a similar flavor but occurs in a different setting.

3.1 The fully blocked chain complex, its homology, and a proof

Definition 1. The **fully blocked grid chain complex** associated to the grid diagram $\mathbb{G}$ is the chain complex $\widetilde{GC}(\mathbb{G})$, whose underlying vector space over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ has a basis corresponding to the set of grid states $\mathbf{S}(\mathbb{G})$, and whose differential is specified by:

$$\tilde{\partial}_{\mathbb{O},\mathbb{X}} = \sum_{\mathbf{y}\in\mathbf{S}(\mathbb{G})} \#\{r \in \mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y}) \,|\, r \cap \mathbb{O} = r \cap \mathbb{X} = \varnothing\} \cdot \mathbf{y}, \tag{2}$$

where $\#\{\cdot\}$ denotes the number of elements in the set modulo 2, and the subscript on $\tilde{\partial}_{\mathbb{O},\mathbb{X}}$ indicates the fact that the map counts rectangles that are disjoint from $\mathbb{O}$ and $\mathbb{X}$.

We need to verify that $\tilde{\partial}_{\mathbb{O},\mathbb{X}}$ is indeed a differential of degree $(-1, 0)$. To do this we need to show it satisfies two criteria:

1. $\mathrm{Im}(\tilde{\partial}_{\mathbb{O},\mathbb{X}}|_{\widetilde{GC}_d(\mathbb{G},s)}) \subseteq \widetilde{GC}_{d-1}(\mathbb{G}, s)$, i.e., the image of $\tilde{\partial}_{\mathbb{O},\mathbb{X}}$ restricted to $\widetilde{GC}_d(\mathbb{G}, s)$ is mapped to $\widetilde{GC}_{d-1}(\mathbb{G}, s)$,
2. and the composition $\tilde{\partial}^2_{\mathbb{O},\mathbb{X}} = 0$, which is the "Boundary of a boundary is empty" criteria.

Proof. (**Proof of degree**) To show Item 1, consider $\mathbf{x} \in \widetilde{GC}_d(\mathbb{G}, s)$, we then need to show that $\tilde{\partial}_{\mathbb{O},\mathbb{X}}(\mathbf{x}) \in \widetilde{GC}_{d-1}(\mathbb{G}, s)$, or even simpler that if the Maslov and Alexander functions on $\mathbf{x}$ evaluate to $M(\mathbf{x}) = d$ and $A(\mathbf{x}) = s$, then the formal sum $\tilde{\partial}_{\mathbb{O},\mathbb{X}}(\mathbf{x})$ in Equation (2) contains only $\mathbf{y} \in \mathbf{S}(\mathbb{G})$ with $M(\mathbf{y}) = d - 1$ and $A(\mathbf{y}) = s$.

For the given $\mathbf{x}$, take an arbitrary $\mathbf{y} \in \mathbf{S}(\mathbb{G})$, then the value of

$$\#\{r \in \mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y}) \,|\, r \cap \mathbb{O} = r \cap \mathbb{X} = \varnothing\},$$

can either be 1 or 0 by definition. We do not consider the case for the value 0, since then the $\mathbf{y}$ vanishes from the sum $\tilde{\partial}_{\mathbb{O},\mathbb{X}}(\mathbf{x})$. If instead the value is 1, we can work with the $r \in \mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})$ that has $r\cap\mathbb{O} = r\cap\mathbb{X} = \varnothing$. We may compute the values $M(\mathbf{y})$ and $A(\mathbf{y})$ using Equation (1a) and (1b) for a $r \in \mathrm{Rect}(\mathbf{x}, \mathbf{y})$ we have

$$\begin{aligned} M(\mathbf{x}) - M(\mathbf{y}) &= 1 - 2\#(r \cap \mathbb{O}) + 2\#(\mathbf{x} \cap \mathrm{Int}(r)) \\ A(\mathbf{x}) - A(\mathbf{y}) &= \#(r \cap \mathbb{X}) - \#(r \cap \mathbb{O}), \end{aligned}$$

given that $r \in \mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})$, we have $\#(\mathbf{x}\cap\mathrm{Int}(r)) = 0$, likewise also $\#(r\cap\mathbb{X}) = \#(r\cap\mathbb{O}) = 0$, since the intersections are empty. Plugging in these values we have:

$$M(\mathbf{x}) - M(\mathbf{y}) = 1, \qquad A(\mathbf{x}) - A(\mathbf{y}) = 0,$$

from which it becomes clear that $M(\mathbf{y}) = d - 1$ and $A(\mathbf{y}) = s$.

(**Proof of differential**) Now we want to prove Item 2. Since $\tilde{\partial}_{\mathbb{O},\mathbb{X}} : \widetilde{GC}(\mathbb{G}) \to \widetilde{GC}(\mathbb{G})$ maps a vector space to a vector space, due to the way we define it, it is automatically linear. In other words, we had defined the image of each basis vector in $\mathbf{S}(\mathbb{G})$, so the image of a sum of basis vectors in $\mathbf{S}(\mathbb{G})$ is a sum of images of those basis vectors. This motivates the first two lines of the following manipulation:

$$\begin{aligned}
\tilde{\partial}^2_{\mathbb{O},\mathbb{X}}(\mathbf{x}) &= \tilde{\partial}_{\mathbb{O},\mathbb{X}}\left(\sum_{\mathbf{y}\in\mathbf{S}(\mathbb{G})} \#\{r \in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y}) \,|\, r\cap\mathbb{O} = r\cap\mathbb{X} = \varnothing\}\cdot\mathbf{y}\right)\\
&= \sum_{\mathbf{y}\in\mathbf{S}(\mathbb{G})} \#\{r \in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y}) \,|\, r\cap\mathbb{O} = r\cap\mathbb{X} = \varnothing\}\cdot\tilde{\partial}_{\mathbb{O},\mathbb{X}}(\mathbf{y})\\
&= \sum_{\mathbf{y}\in\mathbf{S}(\mathbb{G})}\sum_{\mathbf{z}\in\mathbf{S}(\mathbb{G})} \#\{r \in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y}) \,|\, r\cap\mathbb{O} = r\cap\mathbb{X} = \varnothing\}\cdot\\
&\qquad\qquad \#\{r \in \mathrm{Rect}^\circ(\mathbf{y},\mathbf{z}) \,|\, r\cap\mathbb{O} = r\cap\mathbb{X} = \varnothing\}\cdot\mathbf{z}\\
&= \sum_{\mathbf{z}\in\mathbf{S}(\mathbb{G})}\left[\sum_{\mathbf{y}\in\mathbf{S}(\mathbb{G})} \#\{r \in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y}) \,|\, r\cap\mathbb{O} = r\cap\mathbb{X} = \varnothing\}\cdot\right.\\
&\qquad\qquad \left.\#\{r \in \mathrm{Rect}^\circ(\mathbf{y},\mathbf{z}) \,|\, r\cap\mathbb{O} = r\cap\mathbb{X} = \varnothing\}\right]\cdot\mathbf{z},
\end{aligned}$$

we have expressed the composition as a sum of grid states, now we have to determine the value of their coefficients. As before, we ignore even values, since they vanish, and focus on the case with odd values, to this end we pick $\mathbf{z} \in \mathbf{S}(\mathbb{G})$ for which there exists $\mathbf{y} \in \mathbf{S}(\mathbb{G})$ so that we have one of each $r \in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})$ and $r' \in \mathrm{Rect}^\circ(\mathbf{y},\mathbf{z})$ with $r\cap\mathbb{O} = r\cap\mathbb{X} = \varnothing$ and $r'\cap\mathbb{O} = r'\cap\mathbb{X} = \varnothing$. We claim that for each such $\mathbf{y}$ there is exactly one other $\mathbf{y}'$ that produces a pair of rectangles that have the same interior as $r\cup r'$. We provide a sketch elaborating the claim in Figure 1. It is clear now that the coefficient for $\mathbf{z}$ in $\tilde{\partial}^2_{\mathbb{O},\mathbb{X}}(\mathbf{x})$ is always even, since each 1 we find comes with a pair. Since all the coefficients disappear we have $\tilde{\partial}^2_{\mathbb{O},\mathbb{X}} = 0$. ■

We can now construct the homology of groups of $\widetilde{GC}(\mathbb{G})$. Explicitly each grading contributes a group:

$$\widetilde{GH}_d(\mathbb{G},s) = \frac{\mathrm{Ker}(\tilde{\partial}_{\mathbb{O},\mathbb{X}})\cap\widetilde{GC}_d(\mathbb{G},s)}{\mathrm{Im}(\tilde{\partial}_{\mathbb{O},\mathbb{X}})\cap\widetilde{GC}_d(\mathbb{G},s)}$$

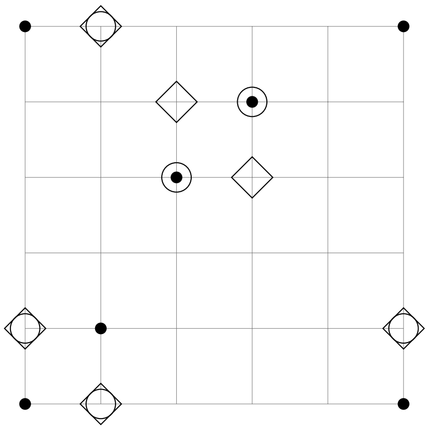
which then defines the following homology:

Definition 2. The **fully blocked grid homology** of $\mathbb{G}$:

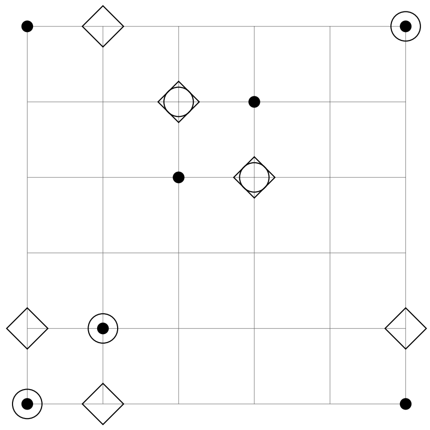
$$\widetilde{GH}(\mathbb{G}) = \bigoplus_{d,s\in\mathbb{Z}} \widetilde{GH}_d(\mathbb{G},s),$$

is the homology of the chain complex $(\widetilde{GC}(\mathbb{G}), \tilde{\partial}_{\mathbb{O},\mathbb{X}})$, thought of as a bigraded vector space.

We end the discussion of the fully blocked grid homology by mentioning an interesting theorem.



(a) A possible arrangement for r and r'



(b) The rectangles but constructed using $\mathbf{y}' \neq \mathbf{y}$

Figure 1: An aid for the proof of the differential, we omit $\mathbb{O}$ and $\mathbb{X}$, and leave some points of the grid states omitted for clarity. The grid states $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in (a) are indicated with black dots, circles, and diamonds respectively, in (b) the new $\mathbf{y}'$ state is given in circles again. Notice in (a) the bottom left rect. goes from $\mathbf{x}$ to $\mathbf{y}$ and the upper right one goes from $\mathbf{y}$ to $\mathbf{z}$, and in (b) $\mathbf{y}, \mathbf{y}'$ differ by two transpositions, and the bottom left is now from $\mathbf{y}$ to $\mathbf{z}$ and the top right is vice versa.

Theorem 1. If $\mathbb{G}$ is a grid diagram with grid number n representing a knot, then the renormalized dimension $\dim_{\mathbb{F}}(\widetilde{GH}(\mathbb{G}))/2^{n-1}$ is an integer valued knot invariant.

In particular this theorem says that by itself the dimension of the homology $\widetilde{GH}(\mathbb{G}))$ is not a knot invariant, instead we scale the dimension by a factor dependent on the grid number of $\mathbb{G}$.

3.2 Other flavours of grid complexes, and their homology

We now construct the **unblocked chain complex** GC^- which we may interpret as n extension of the grid complex $\widetilde{GC}$ by taking the ring $\mathbb{F}[V_1, \ldots, V_n] \supset \mathbb{F}$, where the V_i are formal variables. It is also now convenient to enumerate $\mathbb{O} = \{O_i\}_{i=1}^n$.

Definition 3. The **unblocked grid complex** GC^- is the free module over $\mathcal{R} = \mathbb{F}[V_1, \ldots, V_n]$ generated by $\mathbf{S}(\mathbb{G})$, equiped with the $\mathcal{R}$-module endomorphism whose value on any $\mathbf{x} \in \mathbf{S}(\mathbb{G})$ is given by

$$\partial^-_{\mathbb{X}}(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbf{S}(\mathbb{G})} \sum_{\{r\in\mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})|r\cap\mathbb{X}=\varnothing\}} V_1^{O_1(r)} \ldots V_n^{O_n(r)} \cdot \mathbf{y}.$$

where the *multiplicity* $O_i(r)$ of the rectangle r is an indicator function, if $O_i \in r$, it is 1, otherwise 0.

This grid chain complex is different from the blocked chain complex by allowing the rectangles to contain elements of $\mathbb{O}$. In fact if we take a grid diagram $\mathbb{G}$ that has no rectangle containing elements of $\mathbb{O}$, then the unblocked and blocked complexes coincide. We may extend the Maslov and Alexander functions to this new grid complex.

Finally we mention the simply blocked grid complex.

Definition 4. For some fixed $i = 1, \dots, n$. The quotient complex $GC^-(\mathbb{G})/V_i$ is called the **simply blocked grid complex**, and it is denoted $\widehat{GC}(\mathbb{G})$. The homology of $\widehat{GC}(\mathbb{G})$, viewed as a bigraded chain complex over $\mathbb{F}$, that is, $\widehat{GH}(\mathbb{G})$ is obtained from $\widehat{GC}(\mathbb{G}) = (GC^-/V_i, \partial^-_{\mathbb{X}})$. The resulting complex is independent of the choice of V_i.

4 Exercises

1. The chain complexes $\widetilde{GC}(\mathbb{G})$ and $GC^-(\mathbb{G})$ differ by the way they count rectangles. Do the homologies coincide for a grid diagram $\mathbb{G}$ for which no empty rectangle contains an element of $\mathbb{O}$ or $\mathbb{X}$?
2. Given a grid diagram with $n = 2$ of the unknot, compute explicitly the differential $\tilde{\partial}_{\mathbb{O},\mathbb{X}}(\{1, 2\})$. Try also $\partial^-_{\mathbb{X}}(\{1, 2\})$.
3. Compute $\widetilde{GC}(\mathbb{G})$ where $\mathbb{G}$ is a diagram of the unknot with grid number $n = 2$ or $n = 3$.
4. (Possibly unreasonable) Compute $\widetilde{GH}(\mathbb{G})$ where $\mathbb{G}$ is a diagram for the trefoil knot with grid number $n = 5$. (Use the left handed diagram in figure 3.3 page 45 of the book). We cannot take the simpler $n = 4$ diagram of a Hopf link (which was used in the last week) because chapter 4 restricts to diagrams of knots, not links.