Categorification of Vassiliev invariants and the Kontsevich integral

Goal. Convince you that (monoidal) category theory plays a crucial role in knot theory.

- Everything I am going to say is classical (90's). [Chord diagram invariants of tangles and graphs, Kassel, Turaev].

- Fix R be a comm ring (w/ unit) and let M be an R-module. We want to study M-valued link invariants, ie maps of sets

$$\left\{\begin{array}{c}\text{isotopy class of}\\ \text{oriented, framed links}\\ \text{in } S^3\end{array}\right\} \longrightarrow M$$

There is an adjunction
$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "{R[-]}"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}_R \arrow[l, bend left, "U"] \end{tikzcd}, \quad \mathrm{Hom}_R(R[X], M) \cong \mathrm{Hom}_{\mathrm{Set}}(X, UM),$$

so such maps of sets correspond uniquely w/ R-module maps

$$R\left[\begin{array}{c}\text{isotopy class}\\ \text{of links}\end{array}\right] =: R\mathcal{L} \longrightarrow M.$$

We run into the same problem as usual: $R\mathcal{L}$ is just too big, we need to make some simplifications! Once we saw the concordance gp, this is another way (more algebraic).

- Recall that a *filtration* of a module N is a sequence of submodules

$$\dots \subseteq \mathcal{F}_m \subseteq \dots \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_0 = N$$

- The *Vassiliev relation*  gives rise to a canonical inclusion

$$\mathrm{res} : \left\{ \begin{array}{c} \text{isotopy class of} \\ \text{singular framed oriented} \\ \text{links in } S^3 \end{array} \right\} \hookrightarrow R\mathcal{L} \qquad \left(\begin{array}{c} \text{Vassiliev} \\ \underline{\text{resolution map}} \end{array} \right)$$

so that any singular link can be viewed as a lin comb of links.

The *Vassiliev - Goussarov filtration* on $R\mathcal{L}$ is given by

$$\mathcal{F}_m^{VG} := \text{submod gen by } \mathrm{res}\begin{pmatrix} \text{sing links} \\ \text{w/} \geq m \text{ sing} \\ \text{crossings} \end{pmatrix}, \qquad m \geq 0$$

Let $R\mathcal{L}_m := R\mathcal{L}/\mathcal{F}_m^{VG}$. The filtration induces a sequence

$$\dots \to R\mathcal{L}_m \twoheadrightarrow \dots \twoheadrightarrow R\mathcal{L}_2 \to R\mathcal{L}_1 \twoheadrightarrow R\mathcal{L}_0 = 0.$$

Definition (let $m \geq 0$). A *Vassiliev - Goussarov invariant* or *finite-type invariant* of degree $\leq m$ w/ values in an R-module M is an R-mod map

$$v : R\mathcal{L} \longrightarrow M$$

st $\mathcal{F}_{m+1}^{VG} \subseteq \operatorname{Ker} v$, i.e. st v factors through $R\mathcal{L}_{m+1}$,

$$\begin{tikzcd} R\mathcal{L} \arrow[r, "v"] \arrow[d] & M \\ R\mathcal{L}_{m+1} \arrow[ur, dashed] & \end{tikzcd}$$

<u>Proposition</u>: Let $\cdots \to N_n \xrightarrow{\varphi_n} \cdots \xrightarrow{\varphi_2} N_1 \xrightarrow{\varphi_1} N_0$ be a sequence of R-mod maps. Then there exists a unique R-module $\lim\limits_i N_i$ (up to iso), called the (inverse) limit of (N_i), together w/ maps $\pi_i : \lim N_i \to N_n$ st $\varphi_i \circ \pi_i = \pi_{i-1}$, which is universal wrt this property, ie

$$\begin{tikzcd}
 & M \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right, "\partial_n"'] \arrow[ddrr, "\partial_1"] \arrow[ddrrr, bend left, "\partial_0"] & & & \\
 & \lim N_i \arrow[dl, "\pi_n"] \arrow[drr, "\pi_1"] \arrow[drrr, "\pi_0"'] & & & \\
\cdots \arrow[r] N_n \arrow[rr] & & \cdots \arrow[r] & N_1 \arrow[r] & N_0
\end{tikzcd}$$

Up to iso, $\lim N_i = \left\{ (x_i) \in \prod_{i=0}^{\infty} N_i : \varphi_n(x_n) = x_{n-1} \right\}$.

• If $(\mathcal{F}_i)$ is a filtration ~~of~~ of an R-mod N, then the limit of the sequence

$$\cdots \to N/\mathcal{F}_n \to \cdots \to N/\mathcal{F}_1 \to N/\mathcal{F}_0$$

is called the <u>completion of N wrt $\mathcal{F}$</u> and denoted $\widehat{N}^{\mathcal{F}}$.

• Given a filtration $(\mathcal{F}_i)$ of N, the <u>associated graded module</u> is $Gr^{\mathcal{F}} N := \bigoplus_{i=0}^{\infty} \mathcal{F}_i/\mathcal{F}_{i+1}$.

<u>Back to links</u>: What we learned is that applying this to the sequence $R\mathcal{L}_n$ we obtain an R-mod $\widehat{R\mathcal{L}}^{vg}$ together w/ a map $R\mathcal{L} \to \widehat{R\mathcal{L}}^{vg}$

<u>Examples</u>: 1) The projection $R\mathcal{L} \to R\mathcal{L}_{m+1}$ is the universal finite type invariant of order m w/ values in an R-module, since by def any other factors through it.

2) $\nabla_L \in \mathbb{Z}[z]$ the Conway polyn of a link L is characterized by

$$\begin{cases} \nabla_{L_+} - \nabla_{L_-} = z\nabla_{L_0} \\ \nabla_{\text{unknot}} = 1 \end{cases}$$

The coeff of z^n is a f t inv of order n. Indeed if a sing link has $n+1$ sing crossings then the skein rel says it is a multple of z^{n+1}.

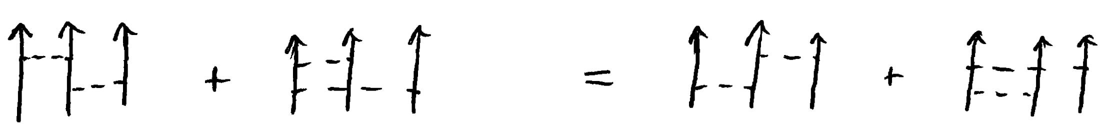
<u>Classic</u>: Give another presentation of $R\mathcal{L}_n$ and $\widehat{R\mathcal{L}}$:

• let $X = \coprod_{\text{finite}} S^1$. A <u>chord diagram</u> on X is the data of

1) an elmt of $\mathbb{Z}/2$ in every component, called the <u>residue</u>

2) a disjoint union of unoriented arcs, called <u>chords</u>, whose endpoints lie in X.

Let $RA_n := R\left[\begin{smallmatrix}\text{homeomorph classes of} \\ \text{chord diag on } X \text{ w/ } n \text{ chords}\end{smallmatrix}\right] \Big/ \text{4T relation}$ $\left(\begin{smallmatrix}\underline{\text{4-term}} \\ \underline{\text{relation}}\end{smallmatrix}\right)$.



where in each of the 4 diagrams of the equation, the residues are required to be the same for the leftmost arrow, also for the middle one and also for the rightmost one.

Theorem (Vassiliev, Kontsevich). If $\mathbb{Q} \subset R$, then there is a R-mod isomorphism

$$\bigoplus_{i=0}^{n} RA_i \xrightarrow{\cong} R\mathcal{L}_{n+1} \qquad (n \geq 0)$$

$$[D] \in A_i \longmapsto \text{res}\begin{pmatrix}\text{singular link having} \\ D \text{ as its chord diagram}\end{pmatrix}$$

(By u.prop of $\oplus$ and $R[-]$, enough to specify what it does on $[D] \in A_i$)

(in the quotient $R\mathcal{L} / \mathcal{F}_{n+1} = R\mathcal{L}_{n+1}$)

(*)

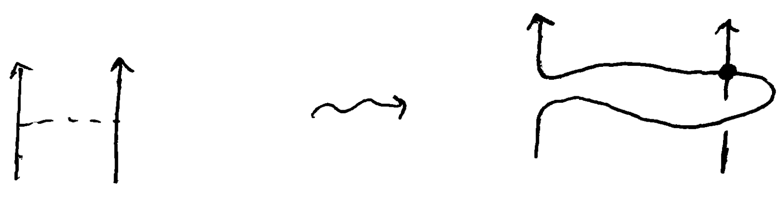
Remarks. 1) By a chord diagram for a framed oriented singular link one means a chord diagram where

(i) # circles = # components

(ii) residue in each circle component = framing mod 2

(iii) there is a chord for evey singular crossing, placed as one parametrises the circles.

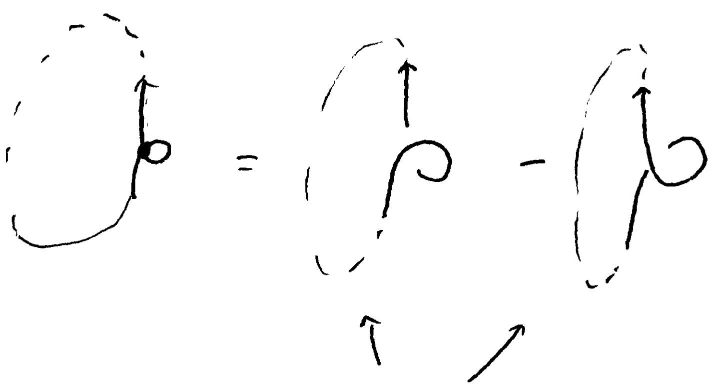
2) A singular link for a given chord diagram can be obtained by contraction of chords,



w/ kinks possibly added to every component so that fr = residue mod 2.

3) The $\mathbb{Z}/2$ residue reflects the fact that flipping crossings preserve the parity of the number of twists, ie to keep track of the framing mod 2. This is the only thing that matters in $R\mathcal{L}_{m+1}$. Indeed for a framed singular link L^s w/ already n double pts,

$R\mathcal{L}_{m+1} \ni o =$



their framing differ by ± 2.

(*) such that the following square commutes:

$$\begin{tikzcd}
\bigoplus_{i=0}^{m} RA_i \arrow[r, "\cong"] \arrow[d, "pr"'] & R\mathcal{L}_{m+1} \arrow[d, "\text{quotient map from filtration.}"] \\
\bigoplus_{i=0}^{m-1} RA_i \arrow[r, "\cong"] & R\mathcal{L}_m
\end{tikzcd}$$

Corollary. If $\mathbb{Q} \subset R$ then there is an isomorphism

$$\bigoplus_{i=0}^{\infty} RA_i \xrightarrow{\cong} Gr^{VS} R\mathcal{L}.$$

Corollary: If $\mathbb{Q} \subset R$, then the previous isomorphisms extend to the inverse limits:

$$\prod_{i=0}^{\infty} RA_i \xrightarrow{\cong} \widehat{R\mathcal{L}} .$$

- Write $\mathcal{V}_n(M) := \mathrm{Hom}_R(R\mathcal{L}_{n+1}, M)$ for the R-mod of finite type invariants of deg n w/ values in an R-mod M. Applying $\mathrm{Hom}_R(-,M)$ to the sequence $(R\mathcal{L}_n)$ induces a sequence

$$0 = \mathcal{V}_{-1}(M) \hookrightarrow \mathcal{V}_0(M) \hookrightarrow \mathcal{V}_1(M) \hookrightarrow \mathcal{V}_2(M) \hookrightarrow \cdots$$

(the arrows are injectives since $R\mathcal{L}_n \to R\mathcal{L}_{n-1}$ are surjectives). Consider $\mathcal{V}(M) := \operatorname{colim}_i \mathcal{V}_i(M)$.

Corollary. If $\mathbb{Q} \subset R$ then $R\mathcal{L}_n$ is free of finite rank.
(Freeness is obtained from the canonical iso $R\mathcal{L}_n \cong \mathbb{Z}\mathcal{L}_n \otimes_{\mathbb{Z}} R$).

- For an R-mod M, write $\mathcal{W}_m(M) := \mathrm{Hom}_R(RA_m, M)$ for the R-mod of <u>weight systems</u> of degree m w/ values in M

Corollary: If $\mathbb{Q} \subset R$ then there is an isomorphism

$$\bigoplus_{i=0}^{n} W_i(M) \xrightarrow{\cong} V_n(M)$$

compatible with the inclusion, i.e. the following square commutes:

$$\begin{array}{ccc} \bigoplus_{i=0}^{n} W_i(M) & \xrightarrow{\cong} & V_n(M) \\ \downarrow & & \downarrow \\ \bigoplus_{i=0}^{n+1} W_i(M) & \xrightarrow{\cong} & V_{n+1}(M) \end{array}$$

Corollary. If $\mathbb{Q} \subset R$ then there is an iso

$$\bigoplus_{i=0}^{\infty} W_i(M) \xrightarrow{\cong} V(M).$$

Corollary. If $\mathbb{Q} \subset R$ then there is an iso

$$W_n(M) \xrightarrow{\cong} V_n(M) / V_{n-1}(M).$$

Note. The inverse of the Vassiliev-Konts. thm is given by the Kontsevich invariant, we will see this at the end of the notes.

Goal. Categorify the previous Vassiliev–Kontsevich thm.

I will start by categorifying the notions of filtration and completion of modules.

- Let R be a comm ring, and consider $\mathcal{C}$ a category enriched over the sym mon cat of Mod_R. That is, $\mathcal{C}$ "is" a (has an underlying) category such that the hom-sets are R-modules and the composite is bilinear.

If $\mathcal{C}$ "is" monoidal, then it is also required that the monoidal product of morphisms is bilinear. We simply say that $\mathcal{C}$ is <u>R-linear</u>.

- If $\mathcal{C}$ is an R-linear (monoidal) category, an <u>ideal</u> $\mathcal{I}$ of $\mathcal{C}$ is a family of linear subspaces $\mathcal{I}(v,w) \subset \mathrm{Hom}_{\mathcal{C}}(v,w)$ for all v, w objects such that whenever f or g is in $\mathcal{I}$, then so is gf (and $f \otimes g$).

If $\mathcal{I}$ is an ideal in $\mathcal{C}$, then it defines a congruence relation so that $\mathcal{C}/\mathcal{I}$ is well def and it is a linear (monoidal) category.

A <u>filtration</u> on $\mathcal{C}$ is a sequence

$$\cdots \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_2 \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{C}$$

of ideals st $\mathcal{F}_n \circ \mathcal{F}_m \subseteq \mathcal{F}_{n+m}$. Then $\mathcal{F}_n \otimes \mathcal{F}_m \subset \mathcal{F}_{n+m}$ follows. $\left(\mathcal{I}^k = \mathcal{I}^{k-1} \cdot \mathcal{I}\right)$

- If $\mathcal{I}$ ideal, the <u>$\mathcal{I}$-adic filtration</u> is $\cdots \subset \mathcal{I}^n \subset \cdots \subset \mathcal{I}^2 \subset \mathcal{I}^1 \subset \mathcal{I}^0 = \mathcal{C}$.

Now let

$$\cdots \to \mathscr{C}_n \xrightarrow{F_n} \cdots \xrightarrow{F_3} \mathscr{C}_2 \xrightarrow{F_2} \mathscr{C}_1 \xrightarrow{F_1} \mathscr{C}_0$$

be a sequence of functors. Its limit $\lim_i \mathscr{C}_i$ can be described as follows:

obj : sequences (V_i), $V_i \in \mathscr{C}_i$ together with isomorphisms $FV_i \xrightarrow{\simeq} V_{i-1}$.

arrows. an arrow $(V_i) \to (W_i)$ is a set of arrows $\varphi_i : V_i \to W_i$

compatible w/ the choice of isomorphism.

If $(\mathscr{C}_i)$ are linear monoidal cat's, so is $\lim \mathscr{C}_i$.

- We can apply this to a category w/ a filtration: the completion of $\mathscr{C}$ wrt $\mathcal{F}$

is

$$\hat{\mathscr{C}}^{\mathcal{F}} := \lim_i \mathscr{C}/\mathcal{F}_i$$

- We also write $\mathrm{Gr}^{\mathcal{F}}\mathscr{C}$ for the graded R-linear monoidal category associated to $\mathcal{F}$: it has same objects as $\mathscr{C}$ and $(\mathrm{Gr}^{\mathcal{F}}\mathscr{C})(V,W) := \bigoplus_{i=0}^{\infty} \mathcal{F}_i(V,W)/\mathcal{F}_{i+1}(V,W)$.
- Now: let $\mathscr{C}$ be an R-linear braided category w/ braiding τ.

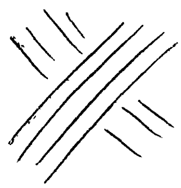
Let $\mathcal{I}$ be the ideal generated by the morphisms

$$\tau_{W,V} \circ \tau_{V,W} - \mathrm{Id}_{V\otimes W} : V \otimes W \to V \otimes W$$

for all $V, W \in \mathscr{C}$, that we call augmentation ideal. The pro-unipotent completion of $\mathscr{C}$ is the limit

$$\hat{\mathscr{C}}^{\mathcal{I}} := \lim_i \mathscr{C}/\mathcal{I}^i.$$

Time to pick up the fruits: we want to apply this to the ~~tangle~~ category $\mathcal{T}$ of framed oriented tangles. Write $R\mathcal{T}$ for its R-linearization, i.e. same obj $\mathcal{T}$ $\mathrm{Hom}_{R\mathcal{T}}(v,w) := R[\mathrm{Hom}_{\mathcal{T}}(v,w)]$, w/ composite enherited from $\mathcal{T}$. Recall that $\mathcal{T}$ (hence $R\mathcal{T}$) is braided where

$\tau_{v,w}$ =  w/ orientations determined by the signs of v and w.

Then we can consider the uni-potent completion $\widehat{R\mathcal{T}}$ of $R\mathcal{T}$ w.r.t. its augmentation ideal. Since in the sequence all functors are id's on objects, $\widehat{R\mathcal{T}}$ can more easily be described as obj = obj of $\mathcal{T}$, arrows = $\varprojlim \mathrm{Hom}_{R\mathcal{T}/\mathcal{I}^n}$.

- The following thm ensures that the previous constructions categorify $R\mathcal{L}_n$ and $\widehat{R\mathcal{L}}$.

Recall that $\mathrm{Hom}_{\mathcal{T}}(\emptyset,\emptyset) = \{$framed oriented links$\}$ and so

$\mathrm{Hom}_{R\mathcal{T}}(\emptyset,\emptyset) = R\mathcal{L}$.

$$\mathrm{Hom}_{\mathrm{Gr}^{\mathcal{I}} R\mathcal{T}}(\emptyset,\emptyset) = \mathrm{Gr}^{vs} R\mathcal{L}.$$

<u>Theorem</u>:

$$\mathrm{Hom}_{R\mathcal{T}/\mathcal{I}^n}(\emptyset,\emptyset) = R\mathcal{L}_n \quad ; \quad \mathrm{Hom}_{\widehat{R\mathcal{T}}}(\emptyset,\emptyset) = \widehat{R\mathcal{L}}.$$

<u>Pf sketch</u>: It sufficies to show that $\mathcal{I}^n(\emptyset,\emptyset)$ is the submodule generated by links w/ n or more singular crossings. One inclusion "$\supset$" is due to the fact that

$$\tau_{+,+} - \tau^{-1}_{+,+} = \tau^{-1}_{+,+}\left(\tau^{2}_{+,+} - \mathrm{Id}_{+^2}\right)$$

For the other inclusion, the axioms of the braiding $\left(\begin{array}{l} \tau_{u,v\otimes w} = (\mathrm{Id}_v \otimes \tau_{u,w})(\tau_{u,v} \otimes \mathrm{Id}_w) \\ \text{and the other} \end{array} \right)$

imply that we can focus on $v, w = +, -$ in the ~~and~~ defining condition for the augmentation ideal. The previous condition also says that a term $\tau_{\pm,\pm}\tau_{\pm\pm} - \mathrm{Id}_{\pm\pm}$ should appear $\geq n$ times, so the singular pt. $\square$.

- With this we have categorified one side of the R-mod iso in the Vassiliev-Kont thm. We will categorify now RA_n.

- A compact, oriented 1-mnfd X $\overset{=(\amalg S^1)\amalg(\amalg I)}{}$ is <u>polarised</u> if $\partial X = \partial_{bot} X \amalg \partial_{top} X$, each part endowed w/ a total order; and ~~also~~ there is a map

$$r = \text{residue} : \pi_0(X) \longrightarrow \mathbb{Z}/2$$

(ie each conn comp gets a 0 or 1). The <u>source</u> $s(X) \in Mon(+,-)$ of X is the elmt obtained by replacing any positive / neg point by $-/+$ following the order. The <u>target</u> $t(X) \in Mon(+-)$ is the same but w/ $+/-$ instead.

- A <u>chord diagram</u> on a polarised 1-mnfd is the data of a disjoint union of unoriented arcs, called chords, whose endpoints lie in $int(X)$.

- Write $RA_n(X) = R[\text{chord diagrams on } X \text{ w/ } n \text{ chords}] / 4T$, and $RA(X) := \bigoplus_n RA_n(X)$.

- Now we will define a category $R\mathcal{A}$:

$\underline{\text{obj}}$: $\mathrm{Mon}(+,-)$ (just as $\mathcal{T}$).

$\underline{\text{arrow}}$: $$\mathrm{Hom}_{R\mathcal{A}}(s,t) := \bigoplus_X RA(X)$$

where X runs through all homeomorphism classes of polarised 1-mfds w/ source s and target t.

$\underline{\text{composite}}$. The closed diagram $D \# D'$ on the 1-mfd $X \cup_{t(X)=s(X')} X'$.
$(D',X') \circ (D,X)$ (extended linearly), and the residue in the glueing is given as follows.
for $c \in \pi_0(X \cup X')$ then

$$r(c) := \sum_{\substack{l \in \pi_0(X) \\ l \subseteq c}} r(l) + \sum_{\substack{l' \in \pi_0(X') \\ l' \subseteq c}} r(l') + n + n' \quad (\mathrm{mod}\, 2)$$

where $n :=$ number of subsets $\{l_1, l_2\} \subseteq \{l \in \pi_0 X : l \subseteq c\}$ $\underline{\underline{\text{not}}}$ admitting an embedding into the square in a way that the order of endpoints is preserved and they don't intersect each other.

$n' :=$ same but for X'

$\underline{\text{id}}$: $\uparrow \cdots \uparrow$.

This is an R-linear category.

Now for every $n \geq 0$ let $\mathcal{I}_n \subset R\mathcal{A}$ be the ideal consisting of lin comb of chord diagrams with $>n$ chords. Write $R\mathcal{A}_n := R\mathcal{A}/\mathcal{I}_n$ for the quotient category. Note that

$$\cdots \subset \mathcal{I}_n(v,w) \subset \cdots \subset \mathcal{I}_2(v,w) \subset \mathcal{I}_1(v,w) \subset \mathcal{I}_0(v,w)$$

and so there is a sequence of functors

$$\to R\mathcal{A}_n \to \cdots \to R\mathcal{A}_2 \to R\mathcal{A}_1 \to R\mathcal{A}_0$$

so we can also consider its limit $\widehat{R\mathcal{A}} := \lim_n R\mathcal{A}_n$.

- The following result, analogous to the one on pg 11, categorifies chord diagrams on $\amalg S^1$:

By def. $\mathrm{Hom}_{R\mathcal{A}}(\emptyset,\emptyset) = \bigoplus_{i=0}^{\infty} RA_i$.

<u>Proposition</u>.

$$\mathrm{Hom}_{R\mathcal{A}_n}(\emptyset,\emptyset) = \bigoplus_{i=0}^{n} RA_i \quad , \quad \mathrm{Hom}_{\widehat{R\mathcal{A}}}(\emptyset,\emptyset) = \prod_{i=0}^{\infty} RA_i .$$

We can finally state the categorification of the Vassiliev-Kontsevich thm.

<u>Theorem</u> (Kassel-Turaev). If $\mathbb{Q} \subset R$, then there exist isomorphisms of categories

$$R\mathcal{A}_n \xrightarrow{\cong} R\mathcal{T}/\mathcal{I}^{n+1}$$

making the following diag commute:

$$\begin{array}{ccc} R\mathcal{A}_n & \xrightarrow{\cong} & R\mathcal{T}/\mathcal{I}^{n+1} \\ \downarrow & & \downarrow \\ R\mathcal{A}_{n-1} & \xrightarrow{\cong} & R\mathcal{T}/\mathcal{I}^{n} \end{array}$$

- Restricting to $\mathrm{Hom}(\emptyset,\emptyset)$ gives the Vassiliev-Kont thm.

<u>Corollary</u>. If $\mathbb{Q} \subset R$ then there is an iso of categories

$$\widehat{R\mathcal{A}} \xrightarrow{\cong} \widehat{R\mathcal{T}}.$$

<u>Corollary</u>. If $\mathbb{Q} \subset R$ then there is an isomorphism of categories

$$R\mathcal{A} \xrightarrow{\cong} \mathrm{Gr}^{\mathcal{I}} R\mathcal{T}$$

Kontsevich invariant

- Let $\mathcal{C}$ be an R-linear strict symmetric monoidal category w/ symmetry τ as before.

An <u>infinitesimal braiding</u> in $\mathcal{C}$ is a natural transformation

$$t_{x,y} : x \otimes y \to x \otimes y$$

such that the two following diagrams commute:

$$\begin{tikzcd} x \otimes y \arrow[r, "\tau_{x,y}"] \arrow[d, "t_{x,y}"'] & y \otimes x \arrow[d, "t_{y,x}"] \\ x \otimes y \arrow[r, "\tau_{x,y}"] & y \otimes x \end{tikzcd} \quad ; \quad \begin{tikzcd} x \otimes y \otimes z \arrow[d, "\mathrm{Id}_x \otimes \tau_{y,z}"'] \arrow[ddd, bend left, "t_{x,y\otimes z} - t_{x,y} \otimes \mathrm{Id}_z"] \\ x \otimes z \otimes y \arrow[d, "t_{x,z} \otimes \mathrm{Id}_y"'] \\ x \otimes z \otimes y \arrow[d, "\mathrm{Id}_x \otimes \tau_{z,y}"'] \\ x \otimes y \otimes z \end{tikzcd}$$

We call such a category a <u>infinitesimal symmetric monoidal category</u>. (Drinfeld, Cartier)

<u>Motivation</u>. For a semisimple complex Lie algebra $\mathfrak{g}$, the category $\mathrm{Mod}^{tf}_{U_h(\mathfrak{g})}$ of topologically free $U_h(\mathfrak{g})$-modules (ie objects are $V[[h]]$ for V a fd $\mathfrak{g}$-module), is braided (in fact ribbon), and the braiding is

$$\sigma_{V,W} = \tilde{\tau}_{V,W} \left(\mathrm{Id}_{V\otimes W} + h \cdot t_{V,W} + \text{terms of higher degree in } h \right)$$

where $\tilde{\tau}_{V,W}$ is an involution and $t_{V,W}$ an endomorphism. Then $t_{V,W}$ forms a natural transformation satisfying the right-hand side diagram.

Example appearing in nature. Let $\mathfrak{g}$ be a semisimple complex Lie algebra, so that its Killing form $\langle x,y\rangle := \mathrm{tr}(\mathrm{ad}(x)\circ\mathrm{ad}(y))$ is a non-degenerate bilinear form. There exists a canonical element $t\in\mathfrak{g}\otimes\mathfrak{g}$ such that if $(e_1,\dots,e_n)$ is any orthonormal basis of $\mathfrak{g}$ wrt the Killing form, then

$$t=\sum_{i=1}^{n} e_i\otimes e_i .$$

(t is called 2-tensor dual to the Killing form). Then for any $\mathfrak{g}$-modules V,W,

$$\begin{aligned} t_{V,W}: V\otimes W &\longrightarrow V\otimes W\\ (v\otimes w) &\longmapsto t\cdot(v\otimes w)\end{aligned}$$

defines an infinitesimal braiding in the sym monoidal cat $\mathrm{Mod}\,\mathfrak{g}$ (although this category is not strict).

- Quantisation of infinitesimal symmetric categories. We first need the notion of Drinfeld associators. The Drinfeld-Kohno algebra is the universal enveloping algebra $U(t_n)$ of the Lie algebra t_n, $n\geq 1$ generated by

$$\{t_{ij}\},\quad i,j=1,\dots,n,\quad i\neq j$$

subject to the relations

$$t_{ij} = t_{ji} \quad , \quad [t_{ij}, t_{ik} + t_{jk}] = 0 \quad \forall i,j,k \text{ distinct} ,$$

$$[t_{ij}, t_{kl}] = 0 \quad \forall i,j,k,l \text{ distinct}.$$

Since the relations are homogeneous by declaring that $\deg(t_{ij}) := 1 \ \forall i,j$, the algebra $U(t_n)$ is actually graded. Denote $\widehat{U(t_n)}$ the degree-completion of $U(t_n)$.

A <u>Drinfeld associator</u> over a field $\mathbb{K}$ of char 0 ($\Leftrightarrow$ contains $\mathbb{Q}$ as subfield) is a formal power series $\varphi \in \mathbb{K}\langle\langle X, Y\rangle\rangle$ in two non-commuting variables X, Y of the form

$$\varphi(X,Y) = \exp\left(\frac{1}{24}[X,Y] + \left(\begin{matrix}\text{infinite sum of iterated commutators in } X,Y \text{ of} \\ \text{length} > 2\end{matrix}\right)\right),$$

$[X,Y] = XY - YX$, which is a solution of the pentagon equation

$$\varphi(t_{12}, t_{23} + t_{24})\, \varphi(t_{13} + t_{23}, t_{34}) = \varphi(t_{23}, t_{34})\, \varphi(t_{12} + t_{13}, t_{24} + t_{34})\, \varphi(t_{12}, t_{23})$$

in the degree-completion $\widehat{U(t_4) \otimes_{\mathbb{Q}} \mathbb{K}}$ (classically, there were two other additional equations called the "hexagon equations", but Furusho showed in 2010 that they are consequences of the pentagon eq).

<u>Remark</u>. Drinfeld constructed an explicit associator w/ real coefficients in the context of the monodromy of a certain system of PDE's called the Knizhnik-Zamolodchikov equations. In a non-constructive way, he also showed the existence of an associator w/ rational coefficients (this was also showed by Bar-Natan).

• <u>Drinfeld-Cartier construction</u> Fix a rational Drinfeld associator $\varphi \in \mathbb{Q}\langle\langle X,Y\rangle\rangle$ once and for all, and let $\mathbb{Q} \subset R$ be a ring. Given an R-linear $\overset{\text{strict}}{\text{infinitesimal}}$ sym monoidal category $\mathcal{C}$, we will construct an $R[\![h]\!]$-linear* braided (non-strict) monoidal category $\mathcal{C}_\varphi[\![h]\!]$. If $\mathcal{C}$ is pivotal (= sovereign = w/ duality), then $\mathcal{C}_\varphi[\![h]\!]$ will be an $R[\![h]\!]$-linear ribbon category.

We define:

• objects of $\mathcal{C}_\varphi[\![h]\!]$ = objects of $\mathcal{C}$

• <u>arrows</u>: $\mathrm{Hom}_{\mathcal{C}_\varphi[\![h]\!]}(v,w) := \underbrace{\mathrm{Hom}_{\mathcal{C}}(v,w)}_{\text{an } R\text{-mod}}[\![h]\!]$ ← a (top free) $R[\![h]\!]$-module.

• <u>composite</u>: $\mathrm{Hom}_{\mathcal{C}}(v_2,v_3)[\![h]\!] \,\hat{\otimes}_{R[\![h]\!]}\, \mathrm{Hom}_{\mathcal{C}}(v_1,v_2)[\![h]\!] \longrightarrow \mathrm{Hom}_{\mathcal{C}}(v_1,v_3)[\![h]\!]$

$$\cong$$

$\left(\mathrm{Hom}_{\mathcal{C}}(v_2,v_3) \otimes_R \mathrm{Hom}_{\mathcal{C}}(v_1,v_2)\right)[\![h]\!]$ induced by composite in $\mathcal{C}$

---

^{1}: More precisely, it is enriched over the cat $\mathrm{Mod}^{tf}_{R[\![h]\!]}$ of top free $R[\![h]\!]$, which is monoidal w/ the completed tensor product $\hat{\otimes}$.

identity: Id_v in $\mathcal{C}_\varphi[[h]]$ = image of Id_v in $\mathcal{C}$ under the canonical map $Hom_{\mathcal{C}}(v,v) \hookrightarrow Hom_{\mathcal{C}_\varphi[[h]]}(v,v)$.

monoidal structure: $\otimes$ on obj same as for $\mathcal{C}$.

on arrows extended $R[[h]]$-linearly as the composite before,

$$
\begin{array}{ccc}
Hom_{\mathcal{C}}(v,w)[[h]] \hat{\otimes}_{R[[h]]} Hom(v',w')[[h]] & \longrightarrow & Hom(v\otimes w, v'\otimes w')[[h]] \\
\| & \nearrow & \\
\left(Hom_{\mathcal{C}}(v,w) \otimes_R Hom_{\mathcal{C}}(v',w')\right)[[h]] & &
\end{array}
$$

This monoidal structure is not-strict: there is a highly non-trivial associativity constraint

$$a_{u,v,w} := \varphi(h\cdot t_{u,v}\otimes Id_w,\ h\cdot Id_u\otimes t_{v,w}) \in Hom_{\mathcal{C}}(u\otimes v\otimes w, u\otimes v\otimes w)[[h]]$$
$$\| $$
$$Hom_{\mathcal{C}_\varphi[[h]]}((u\otimes v)\otimes w,\ u\otimes(v\otimes w)).$$

The monoidal product is strictly left and right unital, since it is the same as in $\mathcal{C}$ for objects.

braiding. let $\tau_{x,y}: x\otimes y \to y\otimes x$ be the symmetry of $\mathcal{C}$. Define the braiding in $\mathcal{C}_\varphi[[h]]$ as

$$\sigma_{x,y} := \tau_{x,y}\circ \exp\left(\frac{1}{2}h\cdot t_{x,y}\right) \in Hom_{\mathcal{C}_\varphi[[h]]}(x\otimes y,\ y\otimes x)$$

This defines a structure of braided monoidal category in $\mathcal{C}_\varphi[\![h]\!]$.

Suppose that $\mathcal{C}$ is pivotal. Then we have the following extra structure on $\mathcal{C}_\varphi[\![h]\!]$:

<u>pivotal structure</u>. duality on objects same as for $\mathcal{C}$, evaluations

$$ev_X : X \otimes X^* \to 1 \quad , \quad \widetilde{ev}_X : X^* \otimes X \to 1$$
$$coev_X : 1 \to X \otimes X^* \quad , \quad \widetilde{coev}_X : 1 \to X^* \otimes X \quad .$$

also the same.

<u>ribbon structure</u>. Given an object X, let $\gamma_X : X \to X$ be defined as

$$X = 1 \otimes X \xrightarrow{coev_X \otimes Id} X \otimes X^* \otimes X \xrightarrow{t_{X,X^*} \otimes Id_X} X \otimes X^* \otimes X \xrightarrow{Id_X \otimes \widetilde{ev}_X} X \otimes 1 = X \xrightarrow{\cdot(-1)} X$$

(associativity / unital constraints are omitted by MacLane coherence).
Then ~~the~~ twist $\theta_X : X \to X$ can be defined as

$$\theta_X := \exp\left(\tfrac{1}{2} h \, \gamma_X\right).$$

- The construction of $\mathcal{C}_\varphi[\![h]\!]$ depends on the choice of Drinfeld associator. However

<u>Theorem</u> (Le, Murakami): Given two Drinfeld associators φ, φ', there is a braided pivotal equivalence of categories

$$\mathcal{C}_\varphi[\![h]\!] \xrightarrow{\simeq} \mathcal{C}_{\varphi'}[\![h]\!] .$$

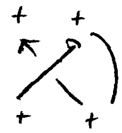
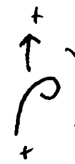
- Hence, we will suppress the subindex φ for $\mathcal{C}[\![h]\!]$.

• Lastly we would like to recall that among the ribbon categories, there is one special, namely the category $\mathcal{T}$ of framed oriented tangles. It is universal among ribbon categories.

<u>Theorem</u>. Let $\mathcal{C}$ be a ribbon category w/ braiding σ, duality $(*, ev, coev)$, and twist θ. Given an object $X \in \mathcal{C}$, there exists a unique strong monoidal functor

$$F_X = F : \mathcal{T} \longrightarrow \mathcal{C}$$

such that $F(+) = X$ and that preserves the ribbon structure, ie $F(-) = X^*$ and

$F($  $) = \sigma_{X,X}$, $F($  $) = \theta_X$,

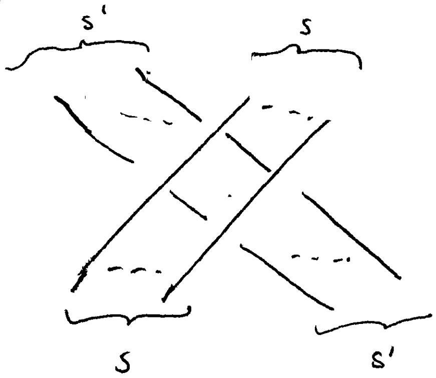
$F($  $) = ev_X$, $F($  $) = coev_X$.

• Let us apply all we have learned to our situation.

Recall the category (R-linear) $R\mathcal{A}$ of chord diagrams on polarised 1-manifolds. We now endow this category w/ the structure of R-linear strict infinitesimal monoidal category w/ duality:

• <u>monoidal product</u>: concatenation at the level of objects and disjoint union at the level of morphisms.

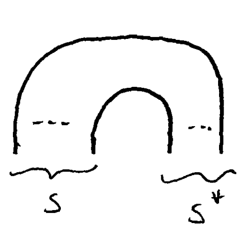
• <u>symmetric braiding</u>: the chordless 1-mnfd (w/ residues 0)

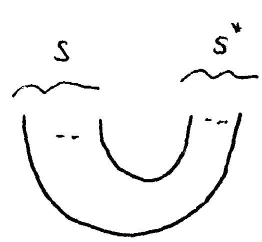


(w/ arrows pointing according to the sequences s, s')

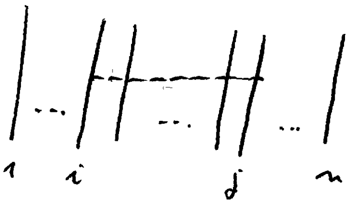
(<u>warning</u>: the above diagram is not any embedding! It's just a pictorial representation saying how the order in source / target of the polarised 1-mnfd should go).

• <u>pivotal structure</u>: $+^* := -$, $-^* := +$ w/

$ev_s =$ 

, $coev_s :=$ 

• <u>infinitesimal braiding</u> : Given an object $s \in R\mathcal{A}$, $s = \varepsilon_1 \cdots \varepsilon_n$ and $i, j = 1, \dots, n$ distinct, let

$$t_s^{i,j} := \varepsilon_i \cdot \varepsilon_j \cdot$$  $: s \to s$

(the orientation of the 1-mf is determined by s). Set

$$t_{s,s'} := \sum_{i=1}^{|s|} \sum_{j=1}^{|s'|} t_{ss'}^{i, |s|+j} \quad : s \otimes s' \longrightarrow s \otimes s'.$$

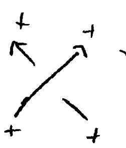
<u>Theorem</u> (Kassel, Turaev). The family $(t_{s,s'})$ define an infinitesimal braiding on $R\mathcal{A}$.

• The category $R\mathcal{A}$ happens to be universal among R-linear strict infinitesimal monoidal categories (w/ duality).

<u>Theorem</u>. Let $\mathcal{C}$ be an R-linear strict inf. monoidal cat (w/ duality). Given $x \in \mathcal{C}$, there exists a unique strong monoidal functor of R-lin categories

$$G_x = G : R\mathcal{A} \longrightarrow \mathcal{C}$$

such that $G(+) = x$ and preserves the structure, ie $\left(F(-) = x^* \text{ and} \right)$

$G($  $) = \sigma_{x,x}$, $\quad G(t_{+,+}) = t_{x,x}$, $\quad G(\uparrow_{res=1}) = \mathrm{Id}_x$,

$$\left(G(ev_+) = ev_x \quad , \quad G(coev_+) = coev_x \right)$$

• We can finally pick up the fruits:

<u>Categorification of weight systems</u> : Recall that by def $Hom_{R\mathcal{A}}(\phi,\phi) = RA = \bigoplus_{n=0}^{\infty} RA_n$.

Also recall that the category $Mod_{\mathfrak{g}}$ of finite-dimensional $\mathfrak{g}$-modules, where $\mathfrak{g}$ is a complex semisimple Lie algebra, is an infinitesimal sym category. By the thm on pg 24, given V a $\mathfrak{g}$-module, we get a functor

$$\mathcal{G} : \mathbb{C}\mathcal{A} \longrightarrow Mod_{\mathfrak{g}}$$

st $\mathcal{G}(+) = V$ and $\mathcal{G}(-) = V^*$. In particular we get a $\mathbb{C}$-linear map restricting to the endomorphisms of ϕ:

$$\begin{array}{ccc} Hom_{\mathbb{C}\mathcal{A}}(\phi,\phi) & \longrightarrow & Hom_{Mod_{\mathfrak{g}}}(\mathbb{C},\mathbb{C}) \\ \| & & \wr\| \\ \bigoplus_{n\geq 0} RA_n & \longrightarrow & \mathbb{C} \end{array}$$

that can be seen to be precisely the weight system $W_{\mathfrak{g},V}$.

<u>Categorification of the Kontsevich invariant</u> Let us take $R=\mathbb{Q}$ here and choose a rational Drinfeld associator $\varphi_{\mathbb{Q}}$. According to the Drinfeld-Cartier construction, we get a ribbon category $\mathbb{Q}\mathcal{A}[[h]]$ out of $\mathbb{Q}\mathcal{A}$ and $\varphi_{\mathbb{Q}}$.

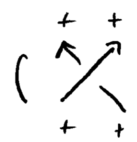
By the universal property of $\mathcal{T}$, we get a strong monoidal functor between ribbon categories

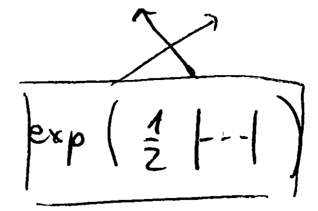
$$Z : \mathcal{T} \longrightarrow \mathbb{Q}\mathcal{A}[\![h]\!] \cong (Gr^{\mathcal{I}}\, \mathbb{Q}\mathcal{T})[\![h]\!]$$

st $Z(+) = +$, ie, Z is the id on objects. For a tangle $T \in \mathcal{T}$ we have

$$Z(T) = \sum_{n \geq 0} Z_n(T) \cdot h^n ,$$

where $Z_n(T)$ is a rational linear combination of chord diagrams w/ n chords sitting in the polarized 1-mfd underlying T. Note that $Z(T)$ <u>does</u> depend on the choice of $\varphi_{\mathbb{Q}}$. For instance, from the description of pg. 20 we get

$$Z\Big($$  $$\Big) = \sigma_{+,+} = \tau_{+,+} \circ \exp\left(\tfrac{1}{2} h \, t_{+,+}\right)$$

$$=$$  .

$$\left(= \sum_{i=0}^{\infty} \frac{h^i}{2^i \, i!} \cdot (\tau_{++} \circ t^i_{+,+}) \right) .$$

In particular for a link L, we have

$$Z(L) = \sum_{n \geq 0} Z_n(L)\, h^n \in \mathrm{Hom}_{\mathbb{Q}\mathcal{A}[\![h]\!]}(\emptyset, \emptyset) = \prod_{n \geq 0} \mathbb{Q}A_n = \widehat{\mathbb{Q}A}$$

with $Z_n(L) \in \mathbb{Q}A_n$. Le-Murakami show that this link invariant is independent of φ.

<u>Proposition</u>. For all $n \geq 0$, the link invariant

$$Z_n : \mathbb{Q}\mathcal{L} \longrightarrow \mathbb{Q}A_n$$

is a finite type invariant (w/ values in the $\mathbb{Q}$-module $\mathbb{Q}A_n$) of degree $\leq n$.

<u>Pf sketch</u>. Extending Z to $\mathbb{Q}\mathcal{T} \longrightarrow \mathbb{Q}\mathcal{A}[\![h]\!]$, one can check that Z sends the augmentation ideal $\mathcal{I}$ of $\mathbb{Q}\mathcal{T}$ to the ideal (h) of $\mathbb{Q}\mathcal{A}[\![h]\!]$, so for any $n \geq 0$ Z gives rise to a functor

$$Z_{\leq n} : \mathbb{Q}\mathcal{T}/\mathcal{I}^{n+1} \longrightarrow \mathbb{Q}\mathcal{A}[\![h]\!]/(h^{n+1})$$

that must take the form $Z_{\leq n}(T) = Z_0(T) + Z_1(T)\cdot h + \cdots + Z_n(T)\cdot h^n$ where $Z_i(T) \in \mathrm{Hom}_{\mathbb{Q}\mathcal{A}}(\text{source}(T), \text{target}(T))$. By the proposition on page 11,

$Z_{\leq n}(K) = Z_0(K) + Z_1(K)\cdot h + \cdots + Z_n(K)\cdot h^n$ is a fin type invariant of order $\leq n$ since it factors through $\mathbb{Q}\mathcal{T}/\mathcal{I}^{n+1}$. Since $Z_{\leq n-1}$ is of deg $\leq n-1$; and $Z_{\leq n} = Z_{\leq n-1} + Z_n \cdot h^n$, Z_n must be of deg $\leq n$ $\square$

• In particular if $\mathbb{Q} \subset R$, one similarly gets a functor

$$Z_{\leq n} : R\mathcal{T}/\mathcal{I}^{n+1} \longrightarrow R\mathcal{A}[[\hbar]]/(\hbar^{n+1})$$

whose image is the category $R\mathcal{A}_n$ (see next page). In this way one shows the Kassel-Turaev thm (this gives the inverse), or the Kontsevich-Vassiliev thm. More precisely, it means that the inverse of the iso of the KV thm is

$$\begin{array}{ccc} R\mathcal{L}_{n+1} & \xrightarrow{\cong} & \bigoplus_{i=0}^{n} RA_i \\ [L] & \longmapsto & (Z_i(L))_{i=0}^{n} \end{array}$$

Given an R-module M, applying $\mathrm{Hom}_R(-,M)$ we get the corollary of pg 8, which now we can write down explicitly

$$\begin{array}{ccc} \bigoplus_{i=0}^{n} \mathcal{W}(M) & \xrightarrow{\cong} & \mathcal{V}_n(M) \\ (w_0,\dots,w_n) & \longmapsto & \left(L \mapsto \sum_{i=0}^{n} w_i(Z_i(L))\right) \end{array}.$$

Or in other words, given v a finite type inv of deg $\leq n$, there exist unique $w_0,\dots,w_n$, $w_i : RA_i \to M$ st $v(L) = \sum_{i=0}^{n} w_i(Z_i(L))$.

• Let me clarify how the functor $\bar{Z}_{\le n} : R\mathcal{T}/\mathcal{I}^{n+1} \xrightarrow{\cong} R\mathcal{A}_n$ arises.

We have the following com diag of functors:

$$\begin{tikzcd}
\mathcal{T} \arrow[d, hook] \arrow[dr, "Z"] & & \\
R\mathcal{T} \arrow[r, "Z"] \arrow[d] & R\mathcal{A}[[h]] \arrow[d] & R\mathcal{A} \arrow[l, "\text{grad}", "\text{(2)}"'] \arrow[d] \\
R\mathcal{T}/\mathcal{I}^{n+1} \arrow[r, dashed, "Z_{\le n}\ \text{(1)}"] \arrow[rr, dashed, bend right, "\cong\ \text{(4)}"'] & R\mathcal{A}[[h]]/(h^{n+1}) & R\mathcal{A}_n \arrow[l, dashed, "\text{(3)}"']
\end{tikzcd}$$

(1) $\mathcal{I}$ augmentation ideal is gen by $\tau_{+,+} - \tau_{+,+}^{-1}$, so

$$Z(\tau_{+,+} - \tau_{+,+}^{-1}) = \sigma_{+,+}\left(\exp\left(\tfrac{1}{2} h\, t_{+,+}\right) - \exp\left(-\tfrac{1}{2} h\, t_{+,+}\right)\right)$$

$$= \sigma_{+,+} \cdot h \cdot t_{+,+} \pmod{h}$$

ie $Z(\mathcal{I}) \subset (h)$ and similarly $Z(\mathcal{I}^{n+1}) \subset (h^{n+1})$

(2) grad is the "grading" functor, $D \mapsto D \cdot h^{\#\text{chords of } D}$

(3) If D has $k > n$ chords then $D h^k = \text{grad}(D) \in (h^{n+1})$ so grad factors; and it is still inj on arrows

(4) Hard [Z, 6.2 - 6.6].

References.

(1) Kassel, Rosso, Turaev - Quantum groups and Knot invariants (ch 8&9)

(2) Kassel, Turaev - Chord diagram invariants of tangles and graphs

(3) Cartier - Construction combinatoire des invariants de Vassiliev-Kontsevich des noeuds.

(4) Habiro, Massuyeau - The Kontsevich integral for bottom tangles in handlebodies.