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# Homework 2 - Topics in Topology

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*Please return Thursday 2 March 2023.*

1. (a) Show that for any pair of knots $K, K' \subset \mathbb{R}^3$, we have $\Delta_{K\#K'}(t) = \Delta_K(t)\Delta_{K'}(t)$.
   (b) Use the skein relation to show that $\Delta_K(1) = 1$.
   (c) Conclude from the previous part that for any Seifert matrix $V$ of a knot we have $\det(V - V^T) = 1$.

2. In this exercise we investigate the universal tangle invariant for the Dilbert algebra Dlb. We will show that $Z_{\mathrm{Dlb}}(K) \in \mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$ for all long knots $K$.
   (a) Recall that the center of an algebra $A$ is $\mathcal{Z}(A) = \{c \in A | \forall a \in A : ca = ac\}$. Determine the center for the Dilbert algebra Dlb.
   (b) Interpreting our long knot $K$ as an 1-strand planar $XC$-tangle labelled 0 argue that if $L := \check{m}_0^{0,1}(K\check{X}_{1,2})$ and $R := \check{m}_0^{1,0}(K\check{X}_{1,2})$ then $L = R$ is an equality of $XC$-tangles (hint: make a sketch!).
   (c) Explain why $Z_{\mathrm{Dlb}}(L) = Z_{\mathrm{Dlb}}(R)$.
   (d) Since $Z(K) \in \mathrm{Dlb}^{\otimes\{0\}}$ we must have $Z_{\mathrm{Dlb}}(K) = w + xd_0 + yl_0 + zb_0$ for some $w, x, y, z \in \mathbb{C}$. Why are they actually in $\mathbb{Z}[i]$?
   (e) Compute $Z_{\mathrm{Dlb}}(L)$ and $Z_{\mathrm{Dlb}}(R)$ explicitly in terms of $w, x, y, z$.
   (f) Use your computation to show that $x = y = z = 0$ and conclude $Z_{\mathrm{Dlb}}(K) = w \in \mathbb{Z}[i]$.
   (g) (BONUS): Improve the result of the previous item in the case of knots with writhe equal to 0: You should get $Z_{\mathrm{Dlb}}(K) \in \mathbb{Z}$.

3. Prove that if $G$ is an Abelian group, then the $XC$-algebra $D(G)$ is commutative. Conclude that $Z_{D(G)}(K) = Z_{D(G)}(\check{1})$ where $K$ is any long knot with writhe 0. Also compute the $Z_{D(G)}$ invariants of the $XC$-tangles $\check{m}_1^{1,4}\check{m}_2^{2,3}(\check{X}_{12}\check{X}_{34})$ and $\check{1}_1\check{1}_2$.

4. Imagine an algebra $A$.
   (a) Show that $m_{\diamondsuit}^{☺,\spadesuit} \circ m_{☺}^{\heartsuit,\clubsuit} = m_{\diamondsuit}^{\heartsuit,☹} \circ m_{☹}^{\clubsuit,\spadesuit}$. Both sides should be interpreted as maps $A^{\otimes\{\heartsuit,\clubsuit,\spadesuit\}} \to A^{\otimes\{\diamondsuit\}}$.
   (b) Prove that $A \otimes A$ is also an algebra with respect to the multiplication $(a \otimes b)(c \otimes d) := ac \otimes bd$.
   (c) What about $A^{\otimes S}$ for a finite set $S$? Can you describe a similar algebra structure on it?
   (d) Is it true that if $B$ is another algebra and $A$ and $B$ are finitely presented then so is $A \otimes B$?

5. Prove that the $XC$-tangle diagrams $\check{m}_1^{1,3,5}\check{m}_2^{2,4,6}(\check{X}_{3,4}^{-1}\check{C}_1\check{C}_2\check{C}_5^{-1}\check{C}_6^{-1})$ and $\check{X}_{1,2}^{-1}$ are related by the $XC$-Reidemeister moves shown in Lecture 5.