LECTURE 8: THE ALEXANDER POLYNOMIAL REVISITED

In the last lecture, we proved that for a certain endomorphism XC-algebra $A = \mathrm{End}(V)$ over the field $\mathbb{Q}(q)$, the invariant Z_A restricted to XC-links produced a polynomial invariant that satisfied certain skein relation, and after a process called deframing and some normalisation, we obtained the Jones polynomial as a instance of the invariant Z_A.

Today we will start showing that the Alexander polynomial that we studied in Lecture 3 can also be recovered from another endomorphism XC-algebra using a very similar scheme, although there is an important difference on how the Alexander polynomial will come out. After that, we would like to give some context about how these XC-algebra structures on $\mathrm{End}(V)$ arise in nature.

1. The Alexander polynomial revisited

Recall from Lecture 3 that the Alexander polynomial is a link invariant

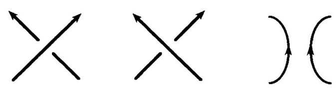
$$\Delta : \mathcal{L} \longrightarrow \mathbb{Z}[t^{1/2}, t^{-1/2}]$$

that is uniquely determined by the following two properties:

(1) $\Delta_{\mathrm{unknot}}(t) = 1$,
(2) the following *skein relation* holds:

$$\Delta_{L_+}(t) - \Delta_{L_-} = (t^{-1/2} - t^{1/2})\Delta_{L_0}(t),$$

where L_+, L_-, L_0 denote three links that are identical except in a neighbourhood of some point where they look like below,



L_+ L_- L_0

Let V be a 2-dimensional vector space over the field $\mathbb{Q}(q)$ with a fixed basis (e_1, e_2), and throughout this section let $A := \mathrm{End}(V) \cong \mathcal{M}_2(\mathbb{Q}(q))$.

Proposition 1. *The elements*

$$X_{ij} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_i \begin{pmatrix} q^{-2} & 0 \\ 0 & q^{-1} \end{pmatrix}_j + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_i \begin{pmatrix} q & 0 \\ 0 & -q^2 \end{pmatrix}_j + (q^{-2} - q^2) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_j \in A^{\otimes\{i,j\}}$$

and

$$C_s := \begin{pmatrix} q^2 & 0 \\ 0 & -q^2 \end{pmatrix}_s \in A^{\otimes\{s\}}$$

define an XC-algebra structure on A.

Proof. We claim that the inverses of these elements are given by

$$X^-_{ij} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_i \begin{pmatrix} q^{2} & 0 \\ 0 & q \end{pmatrix}_j + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_i \begin{pmatrix} q^{-1} & 0 \\ 0 & -q^{-2} \end{pmatrix}_j + (q^{2} - q^{-2}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_j \in A^{\otimes\{i,j\}}$$

and

$$C^-_s := \begin{pmatrix} q^{-2} & 0 \\ 0 & -q^{-2} \end{pmatrix}_s \in A^{\otimes\{s\}}.$$

The proof of this and the rest of the axioms follow from some simple computations as in the previous lecture and they are left to the reader. $\square$

There is a key different with respect to the XC-algebra structure on A that returned the Jones polynomial:

Example 2. The unknot (with writhe 0) as a XC-diagram is simply $O = \check{m}_t^{s,s}(\check{C}_s)$. Now we obtain

$$Z_A(O) = m_t^{s,s}(C_s) = \operatorname{tr}\begin{pmatrix} q^2 & 0 \\ 0 & -q^2 \end{pmatrix} = q^2 - q^2 = 0.$$

For the closed 0-writhe right-handed XC-trefoil 3_1 that was depicted in Lecture 5, a longer but elementary computation shows that $Z_A(3_1) = 0$. So no Alexander here!

This is no incidental: by proceeding as we did for the Jones polynomial during last lecture, we simply get a trivial invariant. This will be part of the homework.

To circumvent this problem, we proceed as follows: let D be a XC-diagram of a link in $\mathbb{R}^3$ and cut one closed component open by removing a copy of $\check{C}^{\pm}$ and pull from the endpoints of the just-created open strand so that its endpoints lie in the uppermost and lowermost part, to obtain a XC-tangle diagram $\mathring{D}^s$ whose only open strand is labelled s. Note that $\check{m}_v^{u,u}\check{m}_u^{s,t}(\mathring{D}^s\check{C}_t^{\pm}) = D$

Proposition 3. *For a XC-diagram $\mathring{D}^s$ as above we have*

$$Z_A(\mathring{D}^s) = \lambda \cdot \mathrm{Id}_s \in A^{\otimes\{s\}}$$

for some $\lambda \in \mathbb{Q}[q, q^{-1}] \subset \mathbb{Q}(q)$.

Proof. Homework. $\square$

So we are forced to obtain a link invariant out of Z_A in a slightly different way, as follows: let D and $\mathring{D}^s$ be as before. Then put

$$Z_A^{\circ}(D) := \lambda = \frac{1}{2} m_t^{s,s}(Z_A(\mathring{D}^s)) \in \mathbb{Q}[q, q^{-1}] \subset \mathbb{Q}(q)$$

where λ is the scalar of Proposition 3.

Lemma 4. *The previous value is a well-defined invariant of D, that is, it is independent of the choice of strand we cut open.*

Proof. Let us write $D = \check{m}_q^{t,t}\check{m}_p^{s,s}\check{m}_t^{t,v}\check{m}_s^{s,u}(E\check{C}_u\check{C}_v^-)$ for some XC-diagram E with two open strands s, t whose endpoints lie in the uppermost and lowermost part (draw a diagram for this!). A similar argument to that one must use in Proposition 3 ensures that

$$Z_A(E) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_s \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}_t + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_s \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}_t + \begin{pmatrix} 0 & \gamma_1 \\ 0 & 0 \end{pmatrix}_s \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_t + \begin{pmatrix} 0 & 0 \\ \gamma_2 & 0 \end{pmatrix}_s \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_t$$

for some scalars $\lambda_i, \mu_i, \gamma_i \in \mathbb{Q}(q)$. Now, applying Z_A to the equality of XC-tangle diagrams

$$\check{m}_2^{2,t}\check{m}_2^{2,3}\check{m}_1^{1,s}\check{m}_1^{1,4}(\check{X}_{1,2}\check{X}_{3,4}E) = \check{m}_2^{2,3}\check{m}_2^{t,2}\check{m}_1^{1,4}\check{m}_1^{s,1}(\check{X}_{1,2}\check{X}_{3,4}E)$$

(draw a picture of this!) and comparing coefficients yields

$$\gamma_2 = q^2\gamma_1 \qquad , \qquad \mu_1 - \lambda_2 = (q - q^{-3})\gamma_2. \tag{1}$$

Likewise, applying Z_A to the equality of XC-diagrams

$$\check{m}_3^{3,6}\check{m}_2^{t,4}\check{m}_1^{s,5}(\check{X}_{4,3}\check{X}_{5,6}E) = \check{m}_3^{3,6}\check{m}_2^{4,t}\check{m}_1^{5,s}(\check{X}_{4,3}\check{X}_{5,6}E)$$

and again comparing coefficients yields

$$\lambda_1 - \lambda_2 = q\gamma_2 = \mu_1 - \mu_2 \qquad , \qquad \lambda_1 - \mu_1 = q^{-1}\gamma_1 = \lambda_2 - \mu_2. \tag{2}$$

Now, note that

$$\mathring{D}^s = \check{m}_q^{t,t}\check{m}_t^{t,v}(E\check{C}_v^-) \qquad , \qquad \mathring{D}^t = \check{m}_p^{s,s}\check{m}_s^{s,u}(E\check{C}_u).$$

Therefore we have

$$c = m_q^{t,t}m_t^{t,v}(Z_A(E)C_v^-) = q^{-2}\begin{pmatrix} \lambda_1 - \lambda_2 & 0 \\ 0 & \mu_1 - \mu_2 \end{pmatrix}_s = q^{-1}\begin{pmatrix} \gamma_2 & 0 \\ 0 & \gamma_2 \end{pmatrix}_s = q\gamma_1\mathrm{Id}_s$$

and similarly

$$Z_A(\mathring{D}^t) = m_p^{s,s}m_s^{s,u}(Z_A(E)C_u) = q^2\begin{pmatrix} \lambda_1 - \mu_1 & 0 \\ 0 & \lambda_2 - \mu_2 \end{pmatrix}_t = q\gamma_1\mathrm{Id}_s,$$

where we have used (1) and (2). Therefore $Z_A(\mathring{\check{D}}^s) = Z_A(\mathring{\check{D}}^t)$ and we conclude. ☐

Theorem 5. *The invariant Z_A° satisfies the following properties:*

(1) $Z_A^\circ(\text{unknot}) = 1$,
(2) Let $L, L_\alpha, L_{\alpha^-}$ be XC-diagrams of links in $\mathbb{R}^3$ that are identical except in a neighbourhood of a point where L looks like $\check{1}$, L_α like the positive kink $\check{\alpha}$ and L_{α^-} like the negative kink $\check{\alpha}^-$. Then
$$Z_A^\circ(L_\alpha) = Z_A^\circ(L) = Z_A^\circ(L_{\alpha^-}).$$
(3) Let L_+, L_-, L_0 denote three XC-diagrams of links in $\mathbb{R}^3$ that are identical except in a neighbourhood of some point where they look like $\check{X}, \check{X}^-$ and $\check{1}\check{1}$, respectively. Then the following skein relation holds:
$$Z_A^\circ(L_+) - Z_A^\circ(L_-) = (q^{-2} - q^2)Z_A^\circ(L_0).$$

Proof. (1) is trivial, (2) follows from the equalities
$$Z_A(\check{\alpha}_i) = Z_A(\check{\alpha}_i^-) = Z_A(\check{1}_i) = \mathrm{Id}_i,$$
which are readily verified, and (3) follows from an argument similar to the one given in the last lecture using the fundamental equality (easily verified)
$$X_{ij} - X_{ji}^{-1} = (q^{-2} - q^2)P_{ij},$$
where
$$P_{ij} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}_i \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}_j + \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}_i \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}_j + \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}_i \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}_j + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}_i \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}_j$$
as in Lecture 7. ☐

In particular, (2) in the previous theorem says that Z_A° is an invariant of links in $\mathbb{R}^3$ as it is invariant under the classical ($\Omega 1$) move, so we do not need to deframe it.

Using the last theorem, the following becomes elementary.

Theorem 6. *Let D be a XC-diagram of a link L in $\mathbb{R}^3$. Then*
$$\Delta_L(t) = Z_A^\circ(D)_{|q^2 = t^{1/2}}$$

2. Producing new XC-algebras

What we would like to do next is to give a context to the XC-algebra structures on $\mathrm{End}(V) \cong \mathcal{M}_2(\mathbb{Q}(q))$ that recover the Jones and Alexander polynomial of links via the invariant Z_A. This algebra (with either of the two XC structures, of course) is four-dimensional and then computations are rather elementary (although some can be involved). Now for an arbitrary XC-algebra A, the computation of $Z_A(D)$ can be potentially difficult, e.g. if A is infinite-dimensional and non-commutative.

How can we deal with this? One possibility is to focus not on the value $Z_A(K)$ but instead on the value $f(Z_A(K)) \in B$ where $f : A \longrightarrow B$ is a certain map, and then studying an alternative way of describing $(f \circ Z_A)(K)$ that does not require to compute the value $Z_A(K)$ first. Since Z_A is a knot invariant, so is $f \circ Z_A$.

In fact, we will focus on a very particular case of this where B is another algebra and $f : A \longrightarrow B$ is an algebra map. The following observation is key:

Lemma 7. *Let A be an XC-algebra over a field k with preferred elements $X \in A \otimes A$ and $C \in A$, and let B be a k-algebra.*

Any k-algebra homomorphism $f : A \longrightarrow B$ induces an XC-algebra structure on B where the preferred elements are given by
$$X' := (f \otimes f)(X) \qquad , \qquad C' := f(C).$$

Proof. This follows directly from the fact that f is an algebra homomorphism. ☐

Therefore, if A is large and difficult to understand, we can try to pass the XC-algebra structure to another algebra B instead that is easier to handle. The price to pay then is that the universal invariant $Z_B(K)$ produced with the new XC-algebra B is potentially weaker. What is sure is that the universal invariant with respect to B is determined by that with respect to A.

Definition 8. Let A, B be XC-algebras preferred elements X, X' and C, C', respectively. An *XC-algebra homomorphism* is an algebra map $f : A \longrightarrow B$ such that $X' = (f \otimes f)(X)$ and $C' = f(C)$.

Corollary 9. *Let A, B be XC-algebras and let $f : A \longrightarrow B$ be a XC-algebra map. Then for XC-tangle T, the value $Z_B(T)$ is determined by $Z_A(T)$,*

$$Z_B(T) = f^{\otimes \mathcal{L}(T)}(Z_A(T)).$$

The main example when we want to apply this is when $B = \mathrm{End}(V)$. In particular, we will see that the XC-algebra structures on $\mathrm{End}(V)$ that recover the Jones and Alexander polynomials are transferred from XC-algebra structures on a more complicated algebra (that we have already seen!).

3. Representations

In this section, we will let A be simply an algebra over some field k.

Definition 10. A *representation* of A or a *(left) A-module* is a pair (V, ρ) where is k-vector space V and ρ is a k-algebra homomorphism

$$\rho : A \longrightarrow \mathrm{End}(V),$$

where $\mathrm{End}(V)$ is viewed as a k-algebra with multiplication given by composition, $f \cdot g := f \circ g$, and unit given by the identity map Id_V.

In other words, a representation of A is a k-vector space V together with a bilinear map

$$A \times V \xrightarrow{\cdot} V$$

such that

$$(ab) \cdot v = a \cdot (b \cdot v) \qquad , \qquad 1 \cdot v = v$$

for all $a, b \in A$, $v \in V$.

Exercise 11. Show that the two phrasings in the definition above are indeed the same via

$$a \cdot v = \rho(a)(v).$$

We will freely use both notations interchangeably.

Examples 12.
(1) For any k-algebra A, then $V = 0$ gives a representation, called the *zero representation.*
(2) If $A = k$, then a k-representation is simply a k-vector space.
(3) If $V = A$, then multiplication in the algebra defines a representation, $\rho(a)b := ab$, called the *regular representation* of A.
(4) Let $k := \mathbb{Q}$ and $A := \mathbb{Q}[\mathbb{Z}/2]$ the rational group algebra of $\mathbb{Z}/2$. If V is any $\mathbb{Q}$-vector space with a choice of basis (e_1, e_2), then

$$\rho(0) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad , \qquad \rho(1) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

extended $\mathbb{Q}$-linearly (that is, $\rho(\lambda \cdot 0 + \mu \cdot 1) := \lambda\rho(0) + \mu\rho(1)$) defines a representation. Indeed, $0 = 1 \cdot 0$[1] is the trivial element of the group algebra so it must be mapped to the identity of V. Since

$$\rho(0) = \rho(1 + 1) = \rho(1)\rho(1) = \rho(1)^2,$$

then the condition for ρ to be a representation is that the matrix defined by $\rho(1)$ squares to the identity, and it is indeed the case.

[1]Watch out! In the expression $1 \cdot 0 \in \mathbb{Q}[\mathbb{Z}/2]$, we have $1 \in \mathbb{Q}$ and $0 \in \mathbb{Z}/2$.

(5) Let us now consider $k = \mathbb{Z}/2$ (this is a field of characteristic 2) and $A := k[\mathbb{Z}/2]$ the group algebra of the group $\mathbb{Z}/2$. Then
$$\rho(0) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \rho(1) := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
defines a representation of A, because $\rho(1)$ squares to the identity mod 2.

Exercise 13. Show that the representation from Examples 12.(5) above is precisely the regular representation of $k[\mathbb{Z}/2]$, where $k = \mathbb{Z}/2$.

Definition 14. Let V be a representation of A. A *subrepresentation* of V is a k-linear subspace $W \subset V$ such that $\rho(a)w \in W$ for all $w \in W$, so that W is a representation of A on its own.

Observe that for any representation V of A, both 0 and V are always subrepresentations.

Definition 15. A representation $V \neq 0$ of A is *irreducible* (or *simple*) if its only subrepresentations are 0 and V.

Example 16. The representation $V = \mathrm{span}_{\mathbb{Q}}(e_1, e_2)$ from Examples 12.(4) is not irreducible because both $W := \mathbb{Q}e_1 \subset V$ and $W' := \mathbb{Q}e_2 \subset V$ are one-dimensional subrepresentations.

The reason we are studying representation theory of algebras is the following.

Corollary 17. *Let A be a XC-algebra, let (V, ρ) be a representation of A and endow $B := \mathrm{End}(V)$ with the XC-algebra structure inherited from A. Then for any XC-tangle T we have*
$$Z_B(T) = \rho^{\otimes \mathcal{L}(T)}(Z_A(T)).$$

4. The $U(\mathfrak{sl}_2)$ example

Recall from Lecture 4 that $U(\mathfrak{sl}_2)$ is the $\mathbb{C}$-algebra generated by the elements X, Y, H and relations
$$[H, X] = 2X \quad , \quad [H, Y] = -2Y \quad , \quad [X, Y] = H$$
where $[u, v] := uv - vu$.

It follows from Lecture 4 that giving a representation
$$\rho : U(\mathfrak{sl}_2) \longrightarrow \mathrm{End}(V)$$
on a vector space V (where we fix a basis) consists on giving matrices $\rho(X), \rho(Y), \rho(H)$ such that
$$[\rho(H), \rho(X)] = 2\rho(X) \quad , \quad [\rho(H), \rho(Y)] = -2\rho(Y) \quad , \quad [\rho(X), \rho(Y)] = \rho(H)$$
where again $[u, v] := uv - vu$.

Proposition 18. *For $n \geq 1$, the matrices*
$$\rho_n(X) = \begin{pmatrix} 0 & n-1 & 0 & \cdots & 0 \\ 0 & 0 & n-2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad , \quad \rho_n(Y) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & n-1 & 0 \end{pmatrix},$$
$$\rho_n(H) = \begin{pmatrix} n-1 & 0 & \cdots & 0 & 0 \\ 0 & n-3 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -n+3 & 0 \\ 0 & 0 & \cdots & 0 & -n+1 \end{pmatrix}$$
define a representation of $U(\mathfrak{sl}_2)$ on an n-dimensional complex vector space V_n.

Proof. It is left to the reader to check that the above matrix relations indeed hold. □

Proposition 19. *Each of the representations V_n above is irreducible for $n \geq 1$.*

Proof. Let $(e_1, \dots, e_n)$ be the fixed basis for V_n. First we make the following claims:

Claim 1: Every finite-dimensional representation W of $U(\mathfrak{sl}_2)$ has a vector $w \in W$ such that w is an eigenvector of H and $w \in \ker X$.

Claim 2: Any vector $w \in V_n$ satisfying the properties of Claim 1 is proportional to e_1.

To see how the proposition follows from the claims, let $V' \subset V_n$ be a subrepresentation. Let $v' \in V'$ satisfy the above-mentioned properties for $W = V'$. Then this element viewed in $W = V$ also satisfies the above properties. Since $e_1 \in V_n$ satisfies these properties, we must have that v' and e_1 are proportional, thus $e_1 \in V'$. But this means that $e_2 = Ye_1$, $e_3 = \frac{1}{2}Y^2 e_1$, etc all belong to V' as well, so $V' = V$.

Let us now prove the Claim 1. Since W is finite-dimensional and we are working over $\mathbb{C}$, the operator H has some eigenvector w, $Hw = \lambda w$. If $w \in \ker X$, we are done; else consider the sequence of vectors $(X^n w)_n$. It is easy to see from the relations defining $U(\mathfrak{sl}_2)$ that

$$H(X^n w) = X^n (H + 2n) w = (\lambda + 2n)(X^n w),$$

so we get a sequence of eigenvectors of H with different eigenvalues. This implies that there must be some n such that $X^n w \neq 0$ and $X^{n+1} w = 0$, as W is finite-dimensional. In this case $X^n w$ is the desired element.

To see Claim 2, such a w must be a scalar multiple of some e_i as it is an eigenvector of H, and in particular of e_1 as none of the others belong to $\ker X$. □

Remark 20. In fact, it is not hard to show (although we will not do it in this course) that

(1) These are the only irreducible representations (up to isomorphism[2]) of $U(\mathfrak{sl}_2)$,
(2) Every finite-dimensional representation of $U(\mathfrak{sl}_2)$ is isomorphic to a direct sum of the irreducible representations V_n.

This gives a full description of all finite-dimensional representations of $U(\mathfrak{sl}_2)$.

5. The $U_q(\mathfrak{sl}_2)$ example

Now recall that $U_q(\mathfrak{sl}_2)$ is the $\mathbb{C}(q)$-algebra generated by X, Y, K, K^{-1} subject to the relations

$$KK^{-1} = 1 = K^{-1}K \quad , \quad KX = q^2 XK \quad , \quad KY = q^{-2} YK \quad , \quad [X, Y] = \frac{K - K^{-1}}{q - q^{-1}}.$$

For any integer n, let us put

$$[n]_q := \frac{q^n - q^{-1}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}.$$

Proposition 21. *For $n \geq 1$ and $\varepsilon = \pm 1$, the matrices*

$$\rho_{\varepsilon,n}(X) = \varepsilon \begin{pmatrix} 0 & [n-1]_q & 0 & \cdots & 0 \\ 0 & 0 & [n-2]_q & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & [1]_q \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad , \quad \rho_{\varepsilon,n}(Y) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ [1]_q & 0 & \cdots & 0 & 0 \\ 0 & [2]_q & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & [n-1]_q & 0 \end{pmatrix},$$

$$\rho_{\varepsilon,n}(K) = \varepsilon \begin{pmatrix} q^{n-1} & 0 & \cdots & 0 & 0 \\ 0 & q^{n-3} & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & q^{-n+3} & 0 \\ 0 & 0 & \cdots & 0 & q^{-n+1} \end{pmatrix}$$

define an n-dimensional representation of $U_q(\mathfrak{sl}_2)$, that we denote $V_{\varepsilon,n}$.

[2]An isomorphism between two representations V_1, V_2 of an algebra A is a linear isomorphism $\varphi : V_1 \longrightarrow V_2$ that preserves the action of A, that is $\varphi(a \cdot v) = a \cdot \varphi(v)$ for any $v \in V_1$

Proof. This is again left to the reader to check. The last relation amounts to showing that

$$[n-2i+1]_q = [n-i]_q[i]_q - [i-1]_q[n-i+1]_q$$

for which the equalities

$$[-n]_q = -[n]_q \qquad , \qquad [n+m]_q = q^n[m]_q + q^{-m}[n_q]$$

are very helpful. □

Remark 22. As in the $U(\mathfrak{sl}_2)$ case, one can show the following:

(1) Each of these representations $V_{\varepsilon,n}$ is irreducible,
(2) These are the only irreducible representations (up to isomorphism) of $U_q(\mathfrak{sl}_2)$,
(3) Every finite-dimensional representation of $U_q(\mathfrak{sl}_2)$ is isomorphic to a direct sum of the irreducible representations $V_{\varepsilon,n}$.

This gives a full description of all finite-dimensional representations of $U_q(\mathfrak{sl}_2)$.

As we hinted before, our interest in studying the representation theory of $U_q(\mathfrak{sl}_2)$ was motivated by the possibility of transferring an XC-algebra structure on $U_q(\mathfrak{sl}_2)$ to $\mathrm{End}(V)$. This XC-structure indeed exists.

Theorem 23 (⚠). *The algebra $U_q(\mathfrak{sl}_2)$ has an XC-algebra structure determined by the elements*

$$X_{ij} = q^{H_iH_j/2}\sum_{n=0}^{\infty} q^{n(n-1)/2}\frac{(q-q^{-1})^{2n}}{[n]_q!}X_i^nY_j^n \qquad , \qquad C_i := K_i$$

where $K = q^H$ and $[n]_q! := [n]_q[n-1]_q\cdots[1]_q$.

Warning 24. The reader should notice that, strictly speaking, the previous statement is not correct, as the expression for X is not a well-defined element in $U_q(\mathfrak{sl}_2)$ (it is given by an infinite sum and $q^{H_iH_j/2}$ is not defined). However it is possible to make it correct by replacing $U_q(\mathfrak{sl}_2)$ by a larger algebra $U_h(\mathfrak{sl}_2)$ where the element X does make sense. The definition of this larger algebra $U_h(\mathfrak{sl}_2)$ requires some technical considerations that we will not cover in this lecture course, and so we will content ourselves with this ill-defined version.

Now let $V := V_{1,2}$ be the 2-dimensional representation of $U_q(\mathfrak{sl}_2)$ for the choice $\varepsilon = +1$, and put $\rho = \rho_{1,2}$. Explicitly,

$$\rho(X) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad , \qquad \rho(Y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad , \qquad \rho(K) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

Theorem 25. *The XC-algebra structure on* $\mathrm{End}(V)$ *induced by the algebra map*

$$\rho : U_q(\mathfrak{sl}_2) \longrightarrow \mathrm{End}(V)$$

is precisely the XC-algebra structure of Proposition 12 from Lecture 7 that recovered the Jones polynomial (after a change of variables $q' := q^2$).

Proof sketch. We recover the element C for $\mathrm{End}(V)$ directly from $\rho(K)$ (after replacing q by q^2). For the element X, using the extension algebra one can show that

$$\rho^{\otimes\{i,j\}}(q^{H_iH_j/2}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_i \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}_j + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_i \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}_j$$

Then we simply note that $\rho(X)^2 = 0 = \rho(Y)^2$, so after applying $\rho^{\otimes\{i,j\}}$ to X_{ij} the sum becomes finite, in fact it only has two summands,

$$\begin{aligned}
\rho^{\otimes\{i,j\}}(X_{ij}) &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_i \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}_j + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_i \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}_j\right] \\
&\times \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_j + (q-q^{-1})\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_j\right] \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_i \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}_j + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_i \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}_j
\end{aligned}$$

$$+ (q^{1/2} - q^{-3/2}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_j$$

so we obtain the same elements X and C as in Lecture 7 after replacing q by q^2. □

Remark 26. Using the rest of irreducible representations $V_{1,n}$ of $U_q(\mathfrak{sl}_2)$, one similarly obtains an XC-algebra structure on $\mathrm{End}(V_{1,n})$ that similarly leads to a collection J_n of link polynomial invariants called the *coloured Jones polynomials*, where $J = J_2$.

It is possible to show, but we will not do it here, that the XC-algebra structure on $\mathrm{End}(V)$ from Proposition 1 that recovers the Alexander polynomial also arises from a 2-dimensional representation of $U_\xi(\mathfrak{sl}_2)$ where $q = \xi = e^{\pi i/2}$ is a root of unity.