Whitehead Doubles

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Topics in Topology

1 Whitehead Doubles

The untwisted double of an arbitrary knot has classical invariants such as the Alexander polynomial and signature identical to those of the unknot. Thus computing values for Whitehead doubles provides an interesting test of any new knot invariant's strength. Moreover, Whitehead doubles can be a useful type of knot to answer questions regarding the Jones polynomial.

Let us first begin by defining what a Whitehead double is. A knot K_P in an unknotted solid torus V_P in $\mathbb{S}^3$ is called a *pattern*. This term comes from the fact that Whitehead doubles are 'build' from these knots.

Let K_C be another knot in $\mathbb{S}^3$, and consider V_C to be a tubular neighbourhood of K_C. Let $h\colon V_P \to V_C$ be a homeomorphism. Then the image of K_P under h is denoted K_W and called a *satellite*. The knot K_C is the *companion* of K_W.

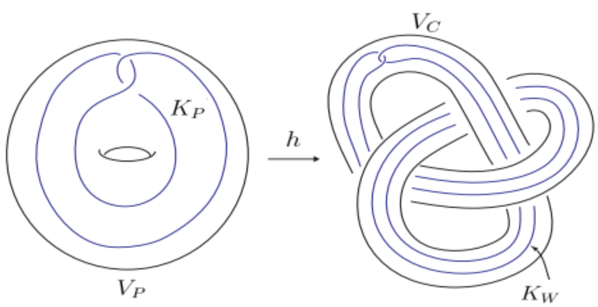

Fig. 1: Construction of the Whitehead double K_W of the trefoil knot.[1]

Definition 1. *The previously constructed K_W knot is called a Whitehead double.*

In Figure 1 the construction of a Whitehead double of a trefoil knot is shown. The left side shows the knot K_P and the untwisted torus V_P. The image under h is visible on the right, where V_C is the distorted torus. K_W is a Whitehead double of a trefoil knot in this figure, as V_C is the tubular neighbourhood of the trefoil knot. Hence, the companion knot K_C of K_W is the trefoil knot.

Remark 1. Note that given K_P and V_P, there are different possibilities for the satellite. For instance, V_P can be twisted around its longitude during its embedding in $\mathbb{S}^3$, resulting in twists in K_W as visible in Figure 2 for a Whitehead double of the trefoil knot.

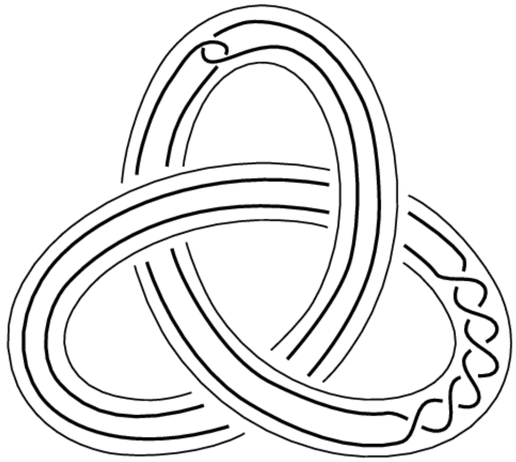

Fig. 2: Whitehead double of the trefoil knot, with twists.

Definition 2. *The sign of the clasp of a knot is denoted by $+$ if the crossings of the clasp are always positive in any orientation, and $-$ if the crossings of the clasp are always negative in any orientation.*

In Figure 3 it is visible that the orientation of the two strands linked at the clasp will always have either $+$ or $-$ sign. The knots are only allowed to make full twists, hence the sign does not depend on the number of twists in a knot.

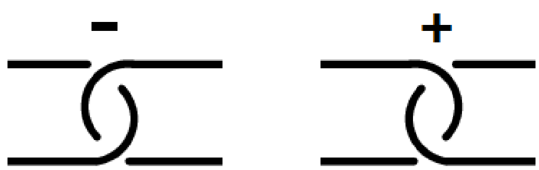

Fig. 3: The sign of two clasps. The left clasp has negative sign, while the other has positive sign.

The image of a twist knot with t full twists and positive sign under h is a positively t-twisted Whitehead double and is denoted by $D_+(K,t)$, where K is the companion knot. In case of a negatively t-twisted Whitehead double, the image is denoted $D_-(K,t)$.

The consideration of sign and the number of twists results in many different types of Whitehead doubles, even if they have the same companion knot. For the rest of this lecture, the Whitehead knot type of interest will be $D_+(K,0)$, an example of which can be found in Figure 4.

2 Seifert Surfaces

In order to construct a Seifert surface for a Whitehead double, observe that the shape of the companion knot does not result in any more crossings in the Whitehead double, and is hence not

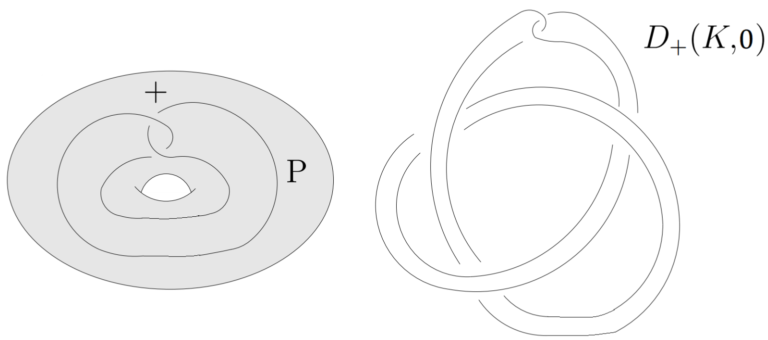

Fig. 4: A Whitehead double $D_+(K,0)$ of the trefoil knot K and the corresponding pattern P [2].

relevant to the Seifert surface. Therefore, in the pictures concerning the Seifert surface, only a K to display the rest of the Whitehead double according to the companion knot K will be drawn, and the clasp and full twists.

In particular, consider the Whitehead double $D_+(K,0)$. Using the construction of a Seifert surface discussed in Lecture 1, the resulting Seifert surface can be found in Figure 5.

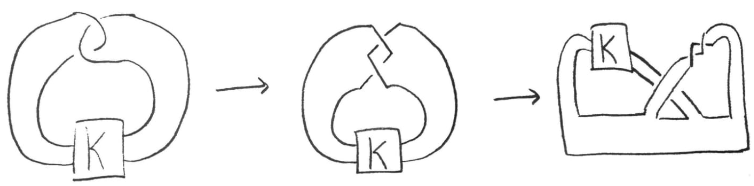

Fig. 5: Construction of the Seifert surface of a Whitehead double.

Calculating the Seifert matrix V according to Lecture 3 then gives the matrix

$$V = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}.$$

The Alexander polynomial

$$\Delta_L(t) = \det(t^{\frac{1}{2}}V - t^{\frac{-1}{2}}V^T)$$

results in

$$\Delta_{D_+(K,0)}(t) = \det\left(\begin{pmatrix} 0 & 0 \\ -t^{\frac{1}{2}} & t^{\frac{1}{2}} \end{pmatrix} - \begin{pmatrix} 0 & -t^{\frac{-1}{2}} \\ 0 & t^{\frac{-1}{2}} \end{pmatrix}\right) = \left|\begin{pmatrix} 0 & t^{\frac{-1}{2}} \\ -t^{\frac{1}{2}} & t^{\frac{1}{2}}t^{\frac{-1}{2}} \end{pmatrix}\right| = 1.$$

Hence, for any companion knot K, the Alexander polynomial of a Whitehead double with no twists is equal to 1.

In fact, if K is the unknot, then the Whitehead double $D_+(K,0)$ is equal to the unknot. This follows from the Reidemeister move $\Omega\,1$ discussed in Lecture 1, as depicted in Figure 6.

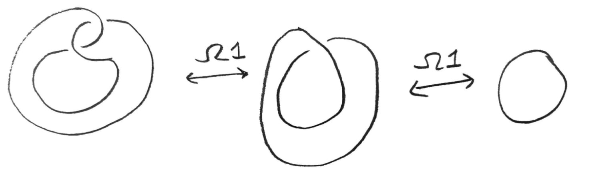

Fig. 6: For K the unknot, $D_+(K,0)$ is also the unknot.

For arbitrary positive Whitehead doubles $D_+(K,q)$ with q twists, the construction of a double-humped Seifert surface can be found in Figure 7. Note that this changes the Siefert matrix to $V = \begin{pmatrix} \pm q & 0 \\ -1 & 1 \end{pmatrix}$, depending on the sign of the q twists. The resulting Alexander polynomial is $\Delta_{D_+(K,q)}(t) = \mp q(-t-t^{-1}) + 1 \mp 2q$. Note that filling in $q = 0$, recovers the Alexander polynomial of $D_+(K,0)$.

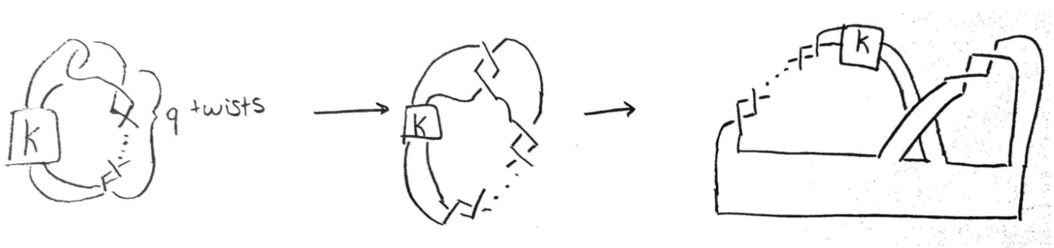

Fig. 7: Construction of a Seifert surface for $D_+(K,q)$.

3 *XC*-tangles

As with the other knots in the past few weeks, we wish to study Whitehead doubles algebraically. To this end, the construction of a Whitehead double for an arbitrary companion knot K will be defined. The construction of a Whitehead double $D_+(K,0)$ defined by

$$\check{m}_1^{3,6,1}\check{m}_1^{2,7,5,8,4,1}\check{\iota}_2\check{\Delta}^1_{2,1}(\check{K}_1\check{X}^-_{3,4}\check{X}^-_{5,6}\check{C}^-_7\check{C}_8)$$

is visible in Figure 8. Note that the orientation of $D_+(K,0)$ matters here. If the orientation would be reversed, the X crossings would change sign. Other than that, nothing would change about the XC-tangle of $D_+(K,0)$. Via this construction, any $D_+(K,0)$ can be expressed in terms of an XC-tangle diagram.

In fact, we can express arbitrarily twisted Whitehead doubles in this manner. The only difference with our previous expression is that a few additional twists need to be added.

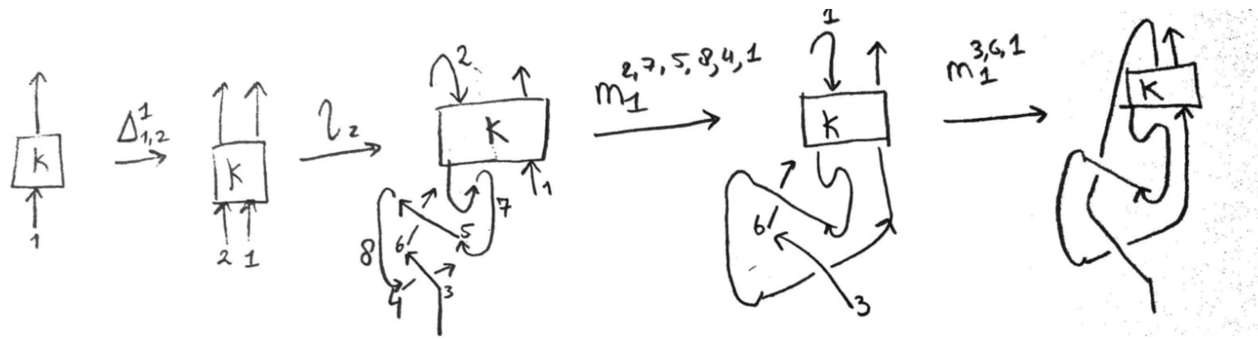

Fig. 8: Constructing the XC-tangle diagram of a Whitehead double $D_+(K,0)$, the boxed K denotes the companion knot.

4 Jones Polynomial

In contrast to the Alexander polynomial, not a lot can be said about the Jones polynomial of Whitehead doubles. As shown in Figure 6, the Whitehead double $D_+(K,0)$ is equal to the unknot when K is the unknot. However, the Jones polynomial of $D_+(K,0)$ in that case is equal to $J_{D_+(K,0)}(t) = t^{1\frac{1}{2}} - t^{\frac{1}{2}} + t^2$, whose construction is shown in Figure 9. Clearly $J_0 = J_- = 1$, and so by the skein relation we find $J_{D_+(K,0)}(t) = t^{1\frac{1}{2}} - t^{\frac{1}{2}} + t^2$. Hence, the Jones polynomial is not a good invariant of Whitehead doubles.

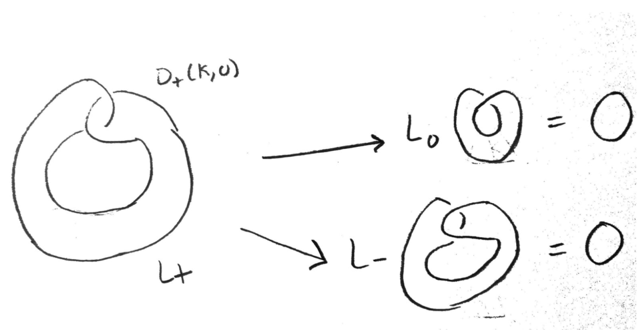

Fig. 9: Finding the Jones polynomial of $D_+(K,0)$, where K is the unknot.

In fact, there has been found no answer to the question whether there exist non-trivial knots with trivial Jones polynomial. It is quite relevant information to know whether an invariant can distinguish the unknot from non-trivial knots, and hence an interesting topic in research.

Whitehead doubles have been crowned candidates to finding a non-trivial knot with trivial Jones polynomial. It has been shown in these lecture notes that only untwisted Whitehead doubles may have trivial Jones polynomial, as their Alexander polynomial has been shown to be trivial. Outside the scope of these lecture notes it has been shown that there exists no untwisted Whitehead double

$D_+(K,0)$ whose companion knot K is a non-trivial knot of 16 or less crossings, with $D_+(K,0)$ having trivial Jones polynomial [3]. Whitehead doubles could be the break-through to this open problem, but they need to be studied quite a bit more thoroughly in order to find a non-trivial Whitehead double with trivial Jones polynomial.

References

1. Gu, S.: Contractible open manifolds which embed in no compact, locally connected and locally 1–connected metric space. Algebraic & Geometric Topology **21**(3), 1327–1350 (2021)
2. Hedden, M.: Knot floer homology of whitehead doubles. Geometry & Topology **11**(4), 2277–2338 (2007)
3. Stoimenow, A.: Non-triviality of the jones polynomial and the crossing numbers of amphicheiral knots. arXiv preprint math/0606255 (2006)