BRAIDS

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Introduction

To make a braid, we take a number of strings and interlace them how we like. In mathematics, braids are objects with some nice properties (for instance, unlike knots and links, they form a group), that have further applications in mathematics and physics. We can view braids as XC-tangles, where no C's are allowed. Formally, we define:

Definition 1. A *braid* in n strands is defined as an embedding of n intervals in $\mathbb{R}^2 \times [0, 1]$, with boundary $\{1, ..., n\} \times \{0\} \times \{0, 1\}$, such that no embedded interval (called a *strand*) has a critical point with respect to the vertical coordinate.

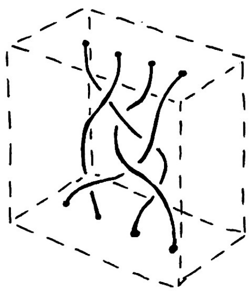

Figure 1. Example of a braid in 4 strands, in $\mathbb{R}^2 \times [0, 1]$. This braid can be constructed using only $X^{\pm}$'s.

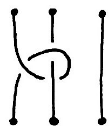

Figure 2. This is *not* a braid. One of the strands has a critical point with respect to the vertical coordinate. We would need a C to construct it.

Remark. The condition that the embedded intervals have no critical point with respect to the vertical coordinate ensures that the braids can only go up, and we can represent the braid as an XC-diagram without C's.

Just like knots, we want to say that two braids are the same if we can smoothly deform one into the other. So really we study equivalence classes: braids modulo isotopy, just as we previously studied knots modulo isotopy.

1. The Braid Group

The equivalence classes of braids form a group in a natural way: define the product of two braids as stacking them. More specifically, take two braids a and b both in n strands, and 'squeeze' them so that a is embedded in $\mathbb{R}^2 \times [1/2, 1]$ and b in $\mathbb{R}^2 \times [0, 1/2]$. We can connect the strands to obtain a new braid, see Figure 3. From Figure 4 it is also clear that this product is associative: $(ab)c$ and $a(bc)$ are related by isotopy and are therefore equivalent.

One can see that the identity element must be the braid consisting of n vertical parallel strands. The inverse element is the mirror image of the braid in the plane $z = 1/2$ after 'squeezing'. A sequence of Reidemeister 2 moves shows that $b^{-1}b$ is isotopic to n parallel strands, see Figure 5 Finally, Figure 4 shows that the group operation is associative.

The group that is obtained this way is called the *braid group* and is denoted B_n.

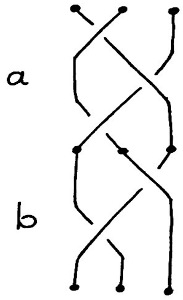

FIGURE 3. The group operation is stacking.

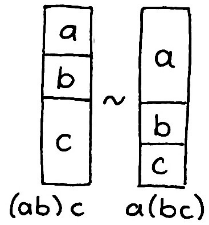

FIGURE 4. By isotophy, $(ab)c$ is equivalent to $a(bc)$.

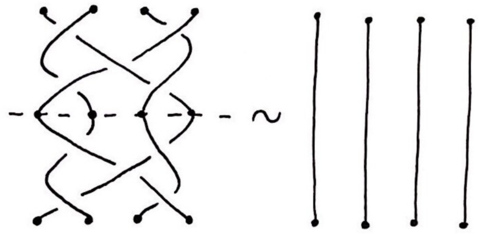

FIGURE 5. The inverse of a braid is its image under reflection. Using Reidemeister 2, we can remove every crossing.

To find a presentation for the braid group, we make a diagram for a class of braids: we project a braid onto the xz-plane similarly as we did for knots, with one extra condition: every crossing occurs at a different height. (Note that every equivalence class has a representative that can be projected this way.) Then a braid will look as in Figure 3. One can see that this is a product of the elements $b_i^{\pm}$, shown in Figure 6.

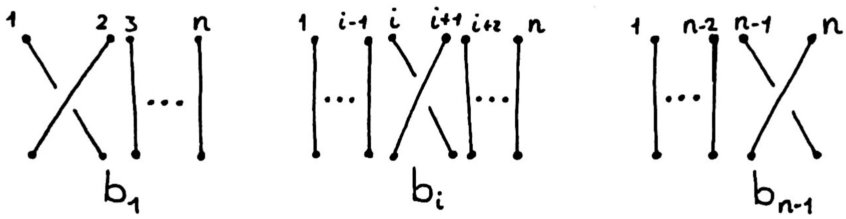

FIGURE 6. The generators $b_1, , , , b_{n-1}$ of B_n. Their inverse is given by their mirror image, which means that $b_i^{-1} = \overline{b_i}$ is obtained by changing the sign of the crossing.

Theorem 2 (Artin's Theorem)**.** The braid group B_n has the following presentation: it is generated by $b_1, ..., b_{n-1}$, satisfying the following relations:

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, \tag{1}$$

$$b_i b_j = b_j b_i \text{ for } |i - j| \geq 2. \tag{2}$$

Proof. We have already seen that B_n is generated by $b_1, ..., b_{n-1}$. Now we check that B_n satisfies relations (1) and (2). Relation (1) looks like Figure 7. We see that this relation is just a consequence of the Reidemeister 3 move. Relation (2) looks like Figure 8. We see that this holds by isotopy.

It remains to show that these relations are sufficient, i.e. that the equivalence of any two equivalent product of $b_i^{\pm}$'s follows from relation (1) and (2). A detailed proof of this can be found in [1]. □

2. The Closure of a Braid

Definition 3. The *closure* of a braid is the link obtained by joining the upper boundaries of the strands to the lower boundaries, as shown in Figure 9.

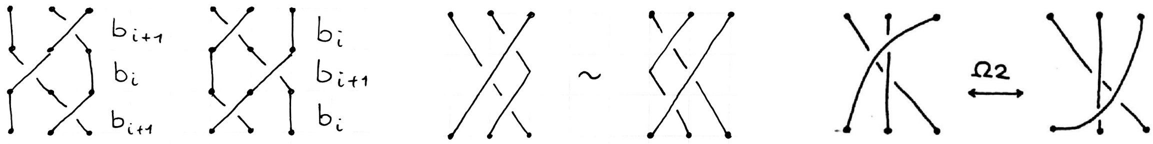

FIGURE 7. This shows that $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ follows from a Reidemeister 3 move. (The $\Omega 2$ is an error and should be $\Omega 3$!)

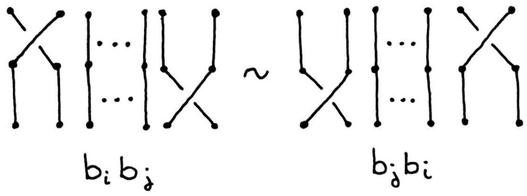

FIGURE 8. This shows that if $|i-j| \geq 2$, we have $b_i b_j = b_j b_i$.

We denote the closure as a map

$$cl : \bigcup_{n=1}^{\infty} B_n \to \mathcal{L}.$$

Example 4. In the language of XC-tangles, we can close a braid in n strands by a number of merges with n $C^{\pm}$'s. Recall that a braid can be represented as an XC-tangle without C's. Then we merge every upper endpoint of the diagram to a C, and merge the endpoints of those C's to the lower endpoints of the diagram. For instance, the diagram in Figure 10 (the same link as in Figure 9, can be written as

$$\check{m}_t^{r_5,t_5}\check{m}_{t_5}^{t_4,9}\check{m}_{t_4}^{t_3,5}\check{m}_{t_3}^{7,4}\check{m}_{t_2}^{t_1,7}\check{m}_{t_1}^{2,3}\check{m}_s^{s_2,s_2}\check{m}_{s_2}^{s_1,8}\check{m}_{s_1}^{1,6}(\check{X}_{1,2}^{-1}\check{X}_{3,4}\check{X}_{5,6}^{-1}\check{C}_7^{-1}\check{C}_8^{-1}\check{C}_9^{-1}).$$

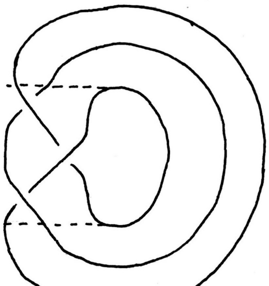

FIGURE 9. The closure of a braid.

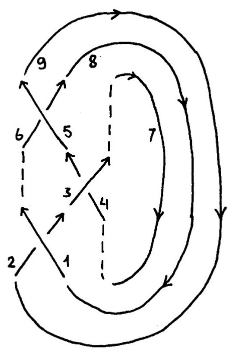

FIGURE 10. The closure of a braid, constructed as an XC-tangle.

In the example above, we end up with two strands, s and t, so the closure of this braid is a link. However, we could easily find a braid whose closure is a knot: take this braid for instance, and delete the middle strand. That raises the question: when is the closure of a braid a knot, and when is it a link?

To answer that question, note that there is a surjective group homoporphism $\sigma : B_n \to S_n$, where $\sigma(b)$ is the permutation of the endpoints following the strands. Formally, for each generator we have $\sigma(b_i) = (i, i+1)$.

Proposition 5. The closure $cl(b)$ of a braid $b \in B_n$ is a knot if and only if $\sigma(b)$ generates the cyclic subgroup of order n in S_n.

The proof is easy to see: a closure produces a knot if all strands end up connected. This happens iff, following a strand, we reach all endpoints and return to the starting point. This occurs whenever the permutation is cyclic and of order n. For example, for the braids in Figure 11, we have $\sigma(a) = (1, 2, 4, 3)$ and $\sigma(b) = (1, 3)(2, 4)$. The closure of a will result in a knot, when the closure of b will result in a 2-component link.

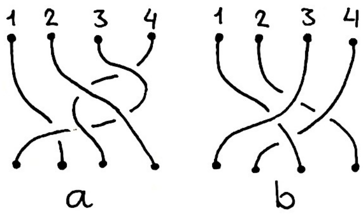

FIGURE 11. Ignoring the crossings, we can view the braid as a permutation of the strands.

Theorem 6 (Alexander's Theorem)**.** The closure map is surjective, i.e. every link is the closure of some braid.

Proof. If we can find a point O such that every edge of the link L, as seen from O, is oriented counterclockwise, we say that K *winds around* the point O.

First consider the case that a link L winds around a point O. We can cut the link along a ray starting from O, and stretch the link into a braid. This is illustrated in Figure 12.

Now we need to show that every link is related to a link that winds around a point via isotopy. To do this, pick some point O, and follow along the link. Whenever an edge is oriented clockwise, throw the strand over the point O, until every edge is oriented counterclockwise. See Figure 13. ∎

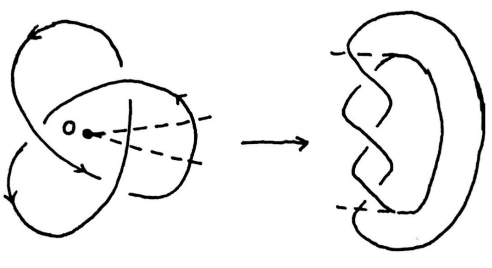

FIGURE 12. Making a braid out of a knot that winds around O.

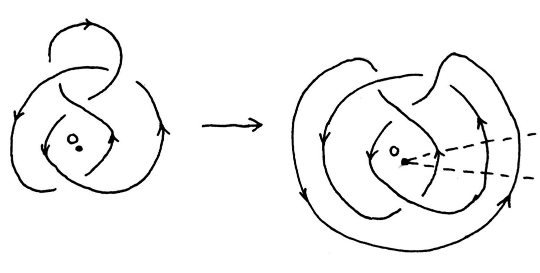

FIGURE 13. Any link is isotopic to a link that winds around a point.

3. Markov Moves

We now know the closure map is surjective. A natural next question is if the closure map is also injective. The answer is no: one can check for instance that $cl(b_1) = cl(b_1^{-1}) = K_{unknot}$. The following theorem describes when two braids have isotopic closures.

Theorem 7 (Markov's Theorem)**.** The closures of two braids are the same link if and only if the braids are related by a finite sequence of Markov moves:

$$\text{MI: } ab \leftrightarrow ba, \quad a, b \in B_n$$
$$\text{MII: } ab_n \leftrightarrow a \leftrightarrow ab_n^{-1}, \quad a \in B_n$$

Here the product ab_n makes sense if we identify a with its image under the inclusion $B_n \to B_{n+1}$, by adding a trivial extra $(n+1)$-th strand.

Proof. We will show only the "if" part of the theorem. Again, this will mostly be a 'proof by picture'. Figure 14 shows that through isotopy, the closure of ab is equivalent to the closure of ba by stretching the strands between them, and sliding the crossings of b around.

Figure 15 shows that adding a strand to a braid b and a crossing just below the braid does not change the closure of a braid. One can see that this takes just a Reidemeister 1 move to change $cl(ab_n)$ to $cl(a)$. □

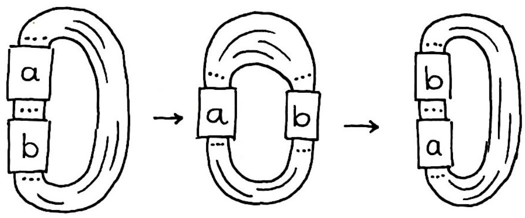

Figure 14. bla.

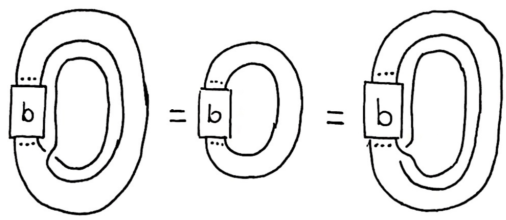

Figure 15. bla.

4. OU-Tangles and Braids

We can find a relation between braids and OU-tangles. Recall that not every tangle can be made into a *OU* tangle diagram. Using *glide moves*, some tangles can be turned into OU-tangles, but this sometimes fails. The cases where this process fails are precisely when the tangle is not *acyclic*. We call a diagram D acyclic when it satisfies the following property: if we draw directed path along strands of D, which may only drop from an upper strand to a lower strand at a crossing (but does not need to change strands at a crossing), it can never be a closed loop. An example of a diagram that is *not* acyclic is shown in Figure 2. This braid cannot be turned into an OU-tangle. More details about this, and a proof, can be found in [2]. It follows that a tangle can be made into an OU-tangle if and only if it is acyclic.

An important property of braids is that they are always acyclic, since we require that there can be no critical points with respect to the vertical coordinate. We can prove the following proposition:

Proposition 8. All OU tangles are equivalent to braids.

References

[1] V. V. Prasolov and A. B. Sosinski, *Knots, links, braids and 3-manifolds: an introduction to the new invariants in low-dimensional topology.* No. 154, American Mathematical Soc., 1997.
[2] D. Bar-Natan, Z. Dancso, and R. van der Veen, "Over then under tangles," 2021.