LECTURE 5: XC-TANGLES

In this lecture we will introduce a neat way of turning knot diagrams into algebra. To bring out the algebra we will consider a slightly more rigid version of knot diagrams where all crossings are to point upwards as in Figure 2. Instead of studying embedded circles we will study embedded intervals up to isotopy and in diagrams the endpoints of the intervals are also required to point upwards. The benefit of making everything point upwards is that we can assign a rotation number to each edge. This is formalized in the notion of an XC-tangle diagram introduced below. After discussing XC-diagrams in increasingly algebraic terms we get to the point where exactly the same relations between objects make sense in an algebra. In the next lecture we will consider algebras with properties that are the precise counterparts of the Reidemeister moves. Such algebras provide a rich source of knot invariants, including *all* invariants we have seen so far.

1. XC-tangles

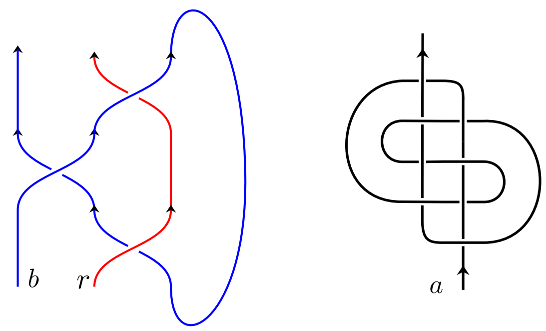

Figure 1. Two XC-diagrams. The diagram on the left hand side has two strands labelled b and r and all of its edges have rotation number 0 except for the right-most edge. The diagram on the right represents the knot 8_{17} from the knot table.

Definition 1. (XC-tangle diagram)
An XC**-tangle** diagram is a directed graph with the following properties.

1. At a vertex either one or four edges meet,
2. in the latter case the four edges meeting the vertex are ordered cyclically such that two adjacent edges enter the vertex and two exit.
3. The vertices where four edges meet are all labelled either $+$ or $-$ and we call them positive[1] and negative crossings. As customary in knot theory are depicted as in Figure 2.
4. Each edge is labelled by an integer called its **rotation number**.
5. A maximal directed path that goes straight at every vertex is called a **strand** of the diagram. Every edge is assumed to be part of a unique strand with distinct start and end points.
6. Every strand carries a unique label. The set of all labels of XC-diagram D is called $\mathcal{L}(D)$.

We like to draw concrete pictures of XC-tangle diagrams in the plane where one can read off the rotation number of each edge by looking at how the tangent vector rotates in the plane. However the actual definition of XC-tangle diagrams is in terms of abstract graph theory, without any reference to the plane or any pictures whatsoever. This makes for cleaner constructions and proofs.

A single strand with rotation number $+1$ and strand label s is denoted by $\check{C}_s$. When the rotation number is -1 we use the notation $\check{C}_s^{-1}$ and when the rotation number is 0 we will use $\check{1}_s$.

[1] I like to check the sign of a crossing using my right hand. Point the thumb in the direction of the over-strand and curl your fingers around the same strand. If the index finger points in the direction of the under-strand your crossing is positive.

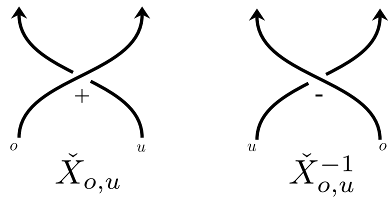

FIGURE 2. The positive crossing $\check{X}$ and the negative crossing $\check{X}^{-1}$, each with over-strand labelled o and under-strand labelled u.

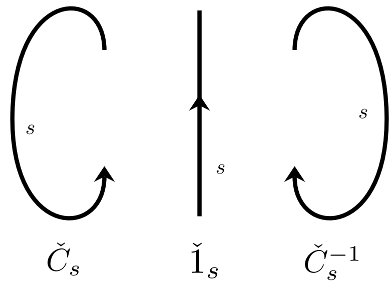

FIGURE 3. The positive and negative full-twisted strand called $\check{C}$ and $\check{C}^{-1}$.

A single positive crossing is denoted $\check{X}_{o,u}$ where o is the label of the overpassing strand and u the the label of the underpassing strand. The negative crossing is labelled $\check{X}^{-1}_{o,u}$, again o refers to the over-strand.

Definition 2. Two XC-tangle diagrams are called (framed Reidemeister equivalent) if they are related by a finite sequence of moves $\Omega 0$, $\Omega 1f$, $\Omega 2b$, $\Omega 2c$ and $\Omega 3$ shown in Figure 4. By an XC-tangle we mean an equivalence class of XC-tangle diagrams.

Please note that as XC-tangle diagrams are officially abstract graphs, there's no such thing as 'planar isotopy'. Also it may seem like some obvious Reidemeister moves are missing, e.g. Reidemeister 2 with the other strand on top. Those can be shown to be consequences of the ones in Figure 4 and the reader can freely use them. Also notice that the usual Reidemeister 1 moves are NOT allowed. Just like in the Kauffman bracket the invariant is really an invariant of thick ribbons or bands that get twisted when we add a curl.

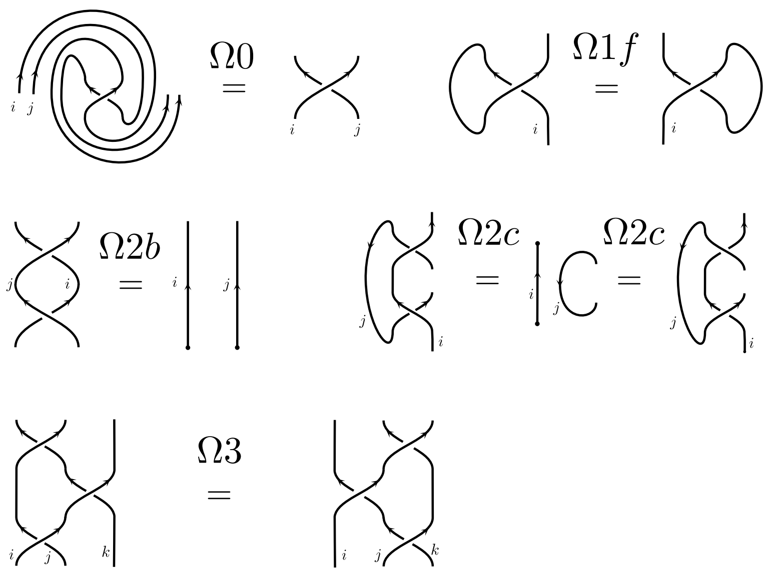

FIGURE 4. The Reidemeister moves for XC-tangle diagrams.

Definition 3. (XC**-Linking matrix)**
The XC-linking matrix of an XC-tangle diagram D is the function $Lk : \mathcal{L}(D)^2 \longrightarrow \mathbb{Z}$ defined by $Lk(o,u) = \sum_r \mathrm{sign}(r)$ where the sum is over all crossings r whose upper strand is o and lower strand is u. The diagonal entry $Lk(i,i)$ is also known as the **writhe** of strand i.

Note how the XC-linking matrix of does not depend on the chosen XC-tangle diagram. Indeed, Reidemeister 0 and 3 preserve the set of crossings while Reidemeister 1f and 2b,c do change the number of crossings but only by cancelling out two crossings of the same type and opposite sign. The XC-linking matrix of the diagrams in Figure 1 are

Lk	b	r
b	1	1
r	1	0

for the left-hand picture and the writhe of the one-strand XC-tangle diagram on the right of the figure is 0.

XC-tangles with a single strand can be related to regular knot diagrams through the notion of a long knot. A long knot is a smoothly embedded $\mathbb{R}$ in $\mathbb{R}^3$ that is assumed to coincide with the z-axis outside a ball so as to prevent infinite knotting. Just like regular knots, long knots can be represented by knot diagrams taken up to isotopy expressed by the usual Reidemeister moves. A long knot can be closed to make a regular knot by connecting the ends outside of the ball where it is knotted. Conversely, a regular knot can be cut open and turned into a long knot in several ways. It is known that two closures of long knots are isotopic if and only if the long knots are isotopic.

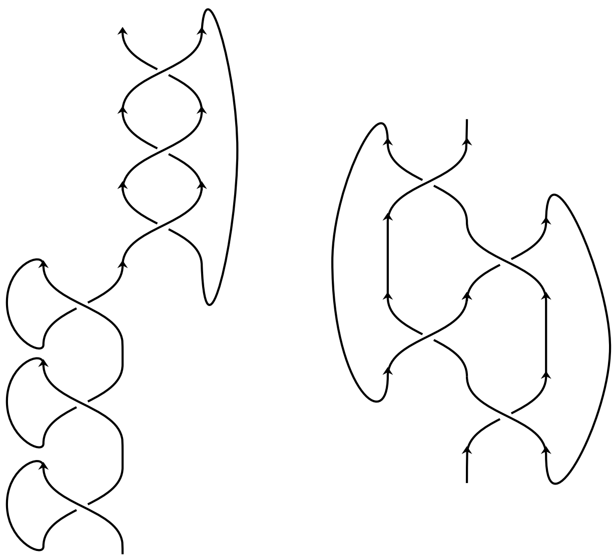

FIGURE 5. 0-writhe long knot diagrams representing the right-hand trefoil 3_1 and the figure eight knot 4_1.

Lemma 4. (Long knots as one-strand XC**-tangles)**
Any long knot diagram can be represented by an XC-tangle diagram with a single strand by cutting it open and rotating all crossings and the end-points so that they point upwards in the plane. To make the XC-tangle canonical we add curls to make its writhe 0.

Two long knot diagrams are equivalent under the usual Reidemeister moves if and only if the XC-tangle diagrams representing them are equivalent in the sense of Definition 2.

The most important operations on XC-tangle diagrams are called disjoint union and merge. In terms of these operations XC-tangle diagrams can be made very explicit without resorting to pictures. Instead of pictures a particular type of algebra emerges.

Definition 5. (Disjoint union and merge)
The **disjoint union** of two XC-tangle diagrams D, E is the disjoint union of the underlying graphs, Notation: DE. It is understood that the labels of the strands are all distinct.

For an XC-tangle diagram D and two distinct strands labelled s,t and any unused label u, define a new XC-tangle diagram $E=\check{m}^{s,t}_u(D)$ called the (s,t,u)-**merge** as follows. E is obtained from D by connecting the endpoint of s to the start of t, removing the resulting vertex and calling the new strand u. The rotation number of the newly created edge is the sum of the rotation numbers of the two edges that it came from.

We will also use the notation $\check{m}^{a,b,c\dots}_k$ to denote the repeated merging of strands a,b,c into a strand called k.

Lemma 6. **(XC-tangle diagram construction)**
All XC-tangle diagrams are obtained from $\check{X}^{\pm}$ and $\check{C}^{\pm}$ by disjoint union and merges.

Proof. Induction on the number of vertices and the rotation number on the edges of the diagram. □

To see this lemma in action let us construct the diagram from Figure 1 (left) using disjoint union and merging. We see three crossings so we start with a disjoint union of three crossings $X_{1,2}X_{3,4}X_{5,6}$. The rotation numbers on all the edges except the rightmost is 0 so we only need one C^{-1} to account for that edge. Taking the lowest crossing to be $\check{X}_{1,2}$ and $\check{X}_{3,4}$ to be in the middle we need to merge $3,5,7,2,4$ to make the blue strand and 1 with 6 to make the red strand. Concretely then we can write this tangle diagram as $\check{m}^{1,6}_r\check{m}^{3,5,7,2}_b(\check{X}_{1,2}\check{X}_{3,4}\check{X}_{5,6}\check{C}_7^{-1})$. See Figure 6 for a step by step explanation.

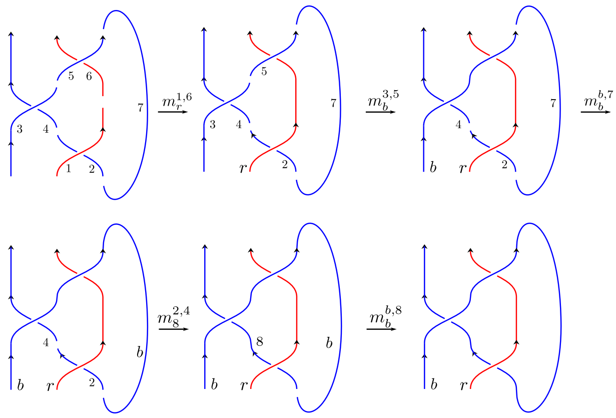

FIGURE 6. Constructing an XC-tangle diagram starting with the disjoint union $\check{X}_{1,2}\check{X}_{3,4}\check{X}_{5,6}\check{C}_7^{-1}$ (top left) and merging strands until only two are left.

Referring to Figure 5 we can write the right-hand trefoil 3_1 and the figure eight knot 4_1 as XC-diagrams with writhe 0 as follows. First define the kink (Reidmeister 1 curl) as $\check{\alpha}_i=\check{m}^{123}_i\check{X}_{13}\check{C}_2^{-1}$ and $\check{\alpha}_i^{-1}=\check{m}^{123}_i\check{X}^{-1}_{13}\check{C}_2$.

$$3_1=\check{m}^{1,2\dots10}_1\check{\alpha}_1^{-1}\check{\alpha}_2^{-1}\check{\alpha}_3^{-1}\check{X}_{4,8}\check{X}_{9,5}\check{X}_{6,10}\check{C}_7^{-1} \tag{1}$$

$$4_1=\check{m}^{1,2\dots10}_1\check{X}^{-1}_{8,1}\check{X}^{-1}_{6,5}\check{X}_{10,3}\check{X}_{9,3}\check{C}_4\check{C}_7^{-1} \tag{2}$$

The XC-Reidemeister moves can be interpreted as equalities between XC-tangle diagrams. It pays off to make these explicit.

$$\check{m}^{1,3,5}_i\check{m}^{2,4,6}_j\check{C}_1\check{C}_2\check{X}_{3,4}\check{C}_5^{-1}\check{C}_6^{-1}=\check{X}_{i,j} \tag{$\check{\Omega}0$}$$

$$\check{m}^{1,2,3}_i\check{X}_{3,1}\check{C}_2=\check{m}^{1,2,3}_i\check{X}_{1,3}\check{C}_2^{-1} \tag{$\check{\Omega}1f$}$$

$$\check{m}^{1,3}_j\check{m}^{2,4}_i\check{X}^{-1}_{1,2}\check{X}_{3,4}=\check{1}_i\check{1}_j \tag{$\check{\Omega}2b$}$$

$$\check{m}^{1,3}_i\check{m}^{2,4}_j\check{X}^{-1}_{1,4}\check{X}_{3,2}=\check{1}_i\check{C}_j \tag{$\check{\Omega}2c$}$$

$$\check{m}^{1,4}_i\check{m}^{2,5}_j\check{m}^{3,6}_k\check{X}_{1,2}\check{X}_{4,3}\check{X}_{5,6}=\check{m}^{1,4}_i\check{m}^{2,5}_j\check{m}^{3,6}_k\check{X}_{1,6}\check{X}_{2,3}\check{X}_{4,5} \tag{$\check{\Omega}3$}$$