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Exercise sheet 4 - Topics in Topology

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1. Make sure that for an n-dimensional vector space V, the following two notions of orientation are equivalent:
   - An equivalence class of basis $[\{e_1, \ldots, e_n\}]$, where two bases $\{e_1, \ldots, e_n\}$, $\{e'_1, \ldots, e'_n\}$ are equivalent if the change of basis matrix has positive determinant.
   - An equivalence class of an n-form $[\omega]$, $\omega \in \Lambda^n V$, where two n-forms ω, ω' are equivalent if $\omega' = \lambda\omega$ for $\lambda > 0$.
2. Show that $\partial\mathrm{Mob}_3$ is diffeomorphic to a Klein bottle.
3. Let $r : D^2 \to D^2$ be the reflexion along the x-axis. Let φ be the composite $D^2 \hookrightarrow D^1 \times D^1 \twoheadrightarrow \mathrm{Mob}_2$, where the first map is the inclusion of a disc with radius 0.2. Show explicitly that the maps φ and $r \circ \varphi$ are isotopic.
4. Show that if M_1 is orientable, M_2 non-orientable, $\varphi_1 : D^n \hookrightarrow M_1$ is an orientation-preserving embedding, $\overline{\varphi}_1 : D^n \hookrightarrow M_1$ is an orientation-reversing embedding and $\varphi_2 : D^n \hookrightarrow M_2$, then there is a diffeomorphism
$$(M_1, \varphi_1)\#(M_2, \varphi_2) \cong (M_1, \overline{\varphi}_1)\#(M_2, \varphi_2).$$
   (*Hint:* Consider a reflexion $r : D^n \to D^n$ and compose it with φ_2).
5. Understand the isomophism $H_g \cong \natural_g(D^2 \times S^1)$.
6. Show that for a closed, connected n-manifold M, there is a diffeomorphism $M\#S^n \cong M$.

   Show that for a closed, connected n-manifold M with boundary, there is a diffeomorphism $M\#D^n \cong M$.
7. Show that $\partial(M_1 \natural M_2) \cong \partial M_1 \# \partial M_2$ for closed, connected manifolds with boundary M_1, M_2.
8. Let $\varphi_0 : S^{k-1} \hookrightarrow \partial M$ be an embedding, where M is an n-dimensional manifold. Show that the normal space $N_x(\varphi_0(S^{k-1}))$ in ∂M, $x \in \varphi_0(S^{k-1})$, has dimension $n - k$. Conclude that a trivialisation of the normal bundle $N(\varphi_0(S^{k-1}))$ is a diffeomorphism with $S^{k-1} \times \mathbb{R}^{n-k}$.