LECTURE 1: INTRODUCTION TO KNOT THEORY

Knot theory can be described as the mathematical study of the different ways a string can lie in the three-dimensional space. In how many different ways can this occur? As we will see throughout this lecture, a piece of strand can be knotted in very complicated way. Our main tool to study this complexity will be to develop algebraic invariants associated to knots.

Knot theory began to be studied about a hundred years ago from a topological point of view. In this lecture course we aim to study know theory from a relatively modern perspective that arose in the 1980s after the work of the Fields medallists V. Jones, V. Drinfeld and E. Witten (the three of them in 1990). This perspective makes emphasis in the interplay between the topology of the knot and certain algebraic structures that we will study.

1. Knots and links in $\mathbb{R}^3$

Let us start by defining what we precisely mean by a knot.

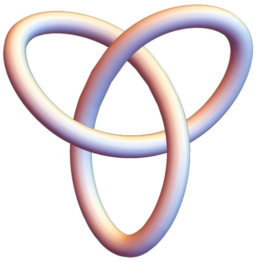
Definition 1. An (oriented) *knot* in $\mathbb{R}^3$ is an oriented smooth embedding[1] $K : S^1 \hookrightarrow \mathbb{R}^3$.

It is also possible to talk about unoriented knots, but in this course we will only talk about the oriented version. By allowing multiple 1-spheres, we obtain what is called a link.

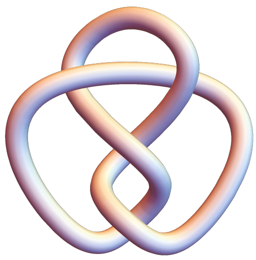
Definition 2. Let $n > 0$ be a positive integer. An (oriented) *n-component link* in $\mathbb{R}^3$ is an oriented smooth embedding $L : \coprod_n S^1 \hookrightarrow \mathbb{R}^3$.

In the previous definition, $\coprod$ denotes the disjoint union of smooth manifolds. A 1-component link is then the same thing as a knot.

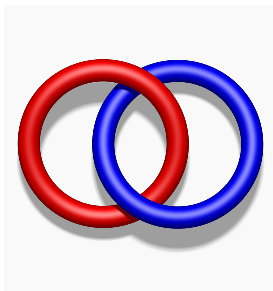
Example 3. Here are some examples of knots and links. For the purpose of illustration, the 1-spheres are depicted thickened, and the choice of orientation is left to the reader.



(A) Left-handed trefoil knot



(B) Figure-of-eight knot



(C) Hopf link

As in every branch of mathematics, we want to define a notion of equality or "sameness". Intuitively, there is an obvious notion of when two knotted strings should be the same. Namely, when one can deform one of the knotted strings by moving the strands and end up with a copy of the other.

This is the mathematical statement of what we have just described.

Definition 4. We say that two knots $K_0, K_1 : S^1 \hookrightarrow \mathbb{R}^3$ are *isotopic* if there exists a smooth map

$$H : S^1 \times [0,1] \longrightarrow \mathbb{R}^3$$

such that

(1) $H(-,0) = K_0$,

[1] Recall that the condition of being an embedding means that K is injective and the differential of K is non-zero at all points. By oriented we simply mean that a choice of orientation in S^1 has been made.

(2) $H(-,1) = K_1$,
(3) $H(-,t)$ is a smooth embedding for all $t \in [0,1]$.

The notion for links is similar.

2. Knot diagrams

According to the previous section, we are interested in studying the set

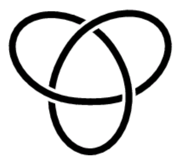
$$\frac{\{\text{ knots in } \mathbb{R}^3 \}}{\text{isotopy}}.$$

However this set, a priori, is rather difficult to study, as its objects are embeddings in the three-dimensional space and there is an equivalence relation involving some smooth maps. It would be much more convenient to keep track only of two-dimensional data rather than three-dimensional, and to have a more combinatorial description of the isotopy relation.

Let $p : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $p(x,y,z) := (x,y)$ be the projection onto the plane xy, and let K be a knot. By slightly isotoping K if needed, we can assume that K is in *general position* with respect to p, that is, that the composite $p \circ K : S^1 \longrightarrow \mathbb{R}^2$ is an immersion with only finitely many transverse double points, ie, an immersion that only self-intersects in a finite amount of points, whose preimage consists only of two points, and the intersection is transversal.

Definition 5. Let K be a knot. A *knot diagram* of K is the projection $D = p \circ K : S^1 \longrightarrow \mathbb{R}^2$ together with the data of "over" or "under" at each double point. Each of these double points is called the *crossings* of the knot diagram.

Here is a knot diagram of the left-handed trefoil:

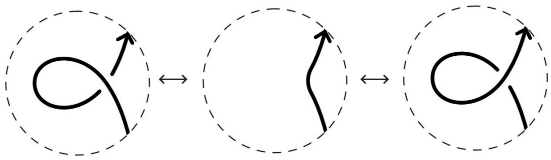


A *link diagram* is defined similarly.

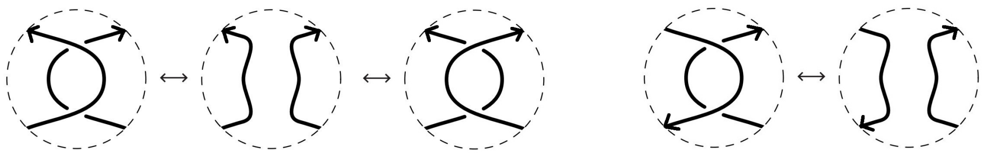
In order to understand isotopy classes of knots via knot diagrams, we have to find in what the isotopy relation translates when studying knots via their diagrams. The answer to this is the content of the following

Theorem 6 (Reidemeister, 1927)**.** *Let K, K' be knots (or in general links) and let D, D' be diagrams of them. Then K and K' are isotopic if and only if D and D' are related by a sequence of Reidemeister moves and planar isotopies, shown below:*

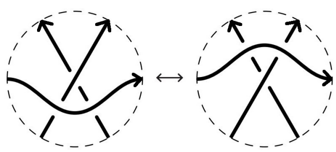
(Ω1)

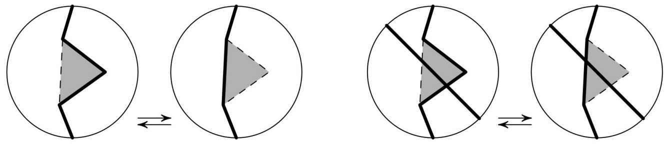


(Ω2)



(Ω3)



(PI) 

In the previous statement, each of the depicted equivalences should be understood as identifying two knots that are identical except in some open neighbourhoods of the knots, where they look as shown.

The proof of this theorem is elementary but rather involved, so we will omit it in this lecture course. The upshot of this is that we can study knots in $\mathbb{R}^3$ up to isotopy by studying their projections on the plane together with some combinatorial moves.

The following is a rephrase of the Reidemeister theorem:

Corollary 7. *There is a bijection*

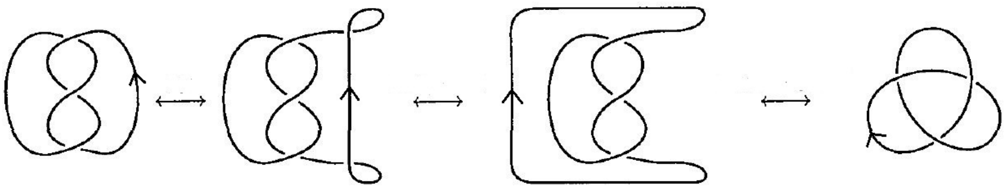
$$\frac{\{\text{ knots in } \mathbb{R}^3 \text{ }\}}{\text{isotopy}} = \frac{\left\{\begin{array}{c}\text{knots diagrams}\\ \text{in } \mathbb{R}^2\end{array}\right\}}{\begin{array}{c}\text{Reidemeister}\\ \text{moves and planar}\\ \text{isotopy}\end{array}}$$

and we will denote any of these bijective sets as $\mathcal{K}$*. Similarly there is a bijection*

$$\frac{\{\text{ links in } \mathbb{R}^3 \text{ }\}}{\text{isotopy}} = \frac{\left\{\begin{array}{c}\text{link diagrams in}\\ \mathbb{R}^2\end{array}\right\}}{\begin{array}{c}\text{Reidemeister}\\ \text{moves and planar}\\ \text{isotopy}\end{array}}$$

and we will denote any of these bijective sets as $\mathcal{L}$*.*

Example 8. The following sequence of Reidemeister moves show two equivalent diagrams for the trefoil:



In the first one we have applied twice $\Omega1$, in the second one we have applied (several times) $\Omega2$ and $\Omega3$ to drag the vertical string to the left, and in the third one we have simply planar isotoped the picture.

3. Knot invariants

If two knots are isotopic, it is rather straightforward to provide a proof by means of a sequence of Reidemeister moves as shown above. However, it is generally not trivial to show to two given knots are not isotopic. The main approach for this is to *construct* a map of sets

$$I : \mathcal{K} \longrightarrow \mathsf{A}$$

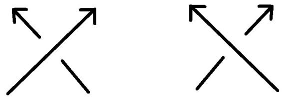
where A is some well-understood set, typically algebraic (e.g. $\mathsf{A} = \mathbb{Z}, \mathbb{Z}[t, t^{-1}], \ldots$). Being a map means that if $K = K'$ then $I(K) = I(K')$, that is, if $I(K) \neq I(K')$ then $K \neq K'$.

Providing a construction for any knot or link invariant requires to check that it is invariant under the Reidemeister moves $(\Omega1) - (\Omega3)$ (so that it gives a well defined map from the quotient).

Let us discuss two of the simplest examples examples of knot invariants.

3.1. **The linking number.** We need an important concept in the first place:

Definition 9. A crossing of a knot or link diagram is called *positive* or *negative* depending on whether it looks like one of these possibilities, respectively:



Let $L = L_1 \cup \cdots \cup L_n$ be an n-component link, and let $D = D_1 \cup \cdots \cup D_n$ be a link diagram of L. The linking number of two components L_i, L_j, $i \neq j$, is the integer

$$lk(L_i, L_j) := \frac{1}{2} \sum_{c} \mathrm{sign}(c)$$

where the sum runs through all crossings c between the components D_i and D_j, and

$$\mathrm{sign}(c) := \begin{cases} +1, & c \text{ is a positive crossing,} \\ -1, & c \text{ is a negative crossing.} \end{cases}$$

Proposition 10. *The linking number is an isotopy invariant.*

Proof. First note that ($\Omega 1$) concerns only one component and then it plays no role in computing the linking number. Similarly the planar isotopy preserves the number of crossings and their signs, so we only have to check ($\Omega 2$) and ($\Omega 3$)

For ($\Omega 2$), note that in each of the neighbourhoods showing two crossings, they have opposite signs, which means that the total contribution cancels out, just as when there is no crossing.

For ($\Omega 3$) there are two possibilities: if the third strand belongs to a different component, then there is only one crossing that contributes and its sign is preserved under this move. If two strands correspond to the same component, then there are two crossings that contribute to the linking number, but likewise their signs are preserved. $\square$

The above proposition says that the linking number gives a well-defined map

$$lk : \mathcal{L}_2 \longrightarrow \mathbb{Z}$$

from the set $\mathcal{L}_2$ of isotopy classes of 2-component links; or more generally

$$lk : \mathcal{L}_{*,*} \longrightarrow \mathbb{Z}$$

from the set $\mathcal{L}_{*,*}$ of isotopy classes of links with a preferred choice of two different components.

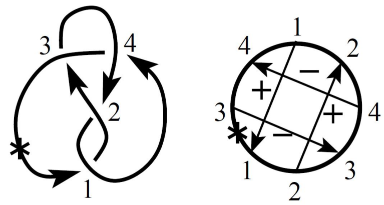
3.2. **The Casson invariant.** Via knot or link diagrams, we have been able to translate topological information about the knot or link in $\mathbb{R}^3$ to the plane, subject to a set of three moves (and planar isotopy). However for some purposes it is important to encode (part of the) information about the knot in a purely combinatorial way, and sometimes this is enough to define knot invariants.

A way of encoding a knot into a purely combinatorial gadget is by means of a Gauss diagram.

Definition 11. Let K be a knot (with a fixed choice of diagram). Choose a basepoint in the knot diagram and number the crossings as they appear when running through the knot along its orientation. Consider an oriented circle with a fixed basepoint and label points along it according to the numbering of the crossings as they appear when traversing the knot. Each number will appear twice in the oriented circle. For each such pair, draw an arrow from the instance corresponding to the overcrossing to the instance corresponding to the undercrossing, and decorate such an arrow with a $+$ or a $-$ depending whether the crossing is positive or negative.

The resulting labelled oriented circle with the decorated arrows is called a *Gauss diagram* for K.

Example 12. Here is a Gauss diagram for the figure-of-eight knot:



Definition 13. Let K be a knot (with a fixed diagram) and let S be its Gauss diagram. The *Casson invariant* of K is the integer

$$v_2(K) := \sum_{S' \subset S} \operatorname{sign}(S')$$

where

$$S' :=$$

and $\operatorname{sign}(S') := \varepsilon_1 \varepsilon_2$.

Proposition 14. *The Casson invariant is an isotopy invariant of knots*

Proof. Homework. □

Example 15. The Casson invariant of the figure-of-eight knot is $v_2(4_1) = -1$.

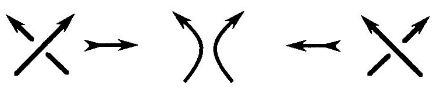
4. Seifert surfaces

Let us back to a more topological perspective of the knot in $\mathbb{R}^3$.

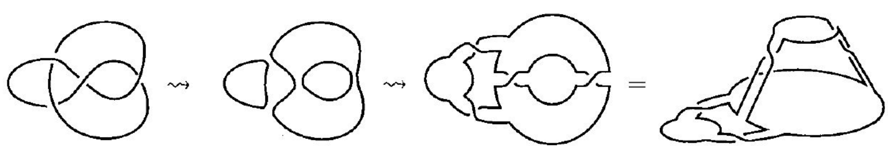
Definition 16. Let K be a knot in $\mathbb{R}^3$. A *Seifert surface* for K is an oriented, connected, compact surface $\Sigma \subset \mathbb{R}^3$ such that $\partial \Sigma = K$.

Lemma 17. *Any knot in $\mathbb{R}^3$ has a Seifert surface.*

Proof. Take any knot diagram of K and replace any positive or negative crossing by two disjoint strands, as shown:



The result of this replacement is a disjoint union of circles. Consider the disjoint discs bounded by these discs, and attach positive or negative half-twisted bands between them for every positive or negative crossing in the original diagram. The resulting surface is a Seifert surface for K. The picture below illustrates the construction for the figure-of-eight knot.



□

The previous proof gives an algorithm to construct *one* possible Seifert surface for a knot (not necessarily the easiest), but a knot can have many different Seifert surfaces. For instance, by glueing a handle (ie, a cylinder $S^1 \times [0,1]$) to any Seifert surface of K, we get a different one of higher genus. The resulting surface should be interpreted as more complicated. We should then be interested in the minimun of the genera among all Seifert surfaces of a given knot. This has a name:

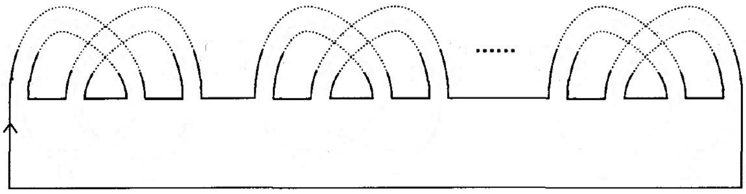
Definition 18. The *genus* of a knot is the non-negative integer

$$g(K) := \min\{g(\Sigma) : \Sigma \text{ is a Seifert surface for } K\}.$$

In general, it is rather hard to compute the genus of a given knot. In the next lecture we will see that the degree of the Alexander polynomial provides a lower bound for the genus of a given knot.

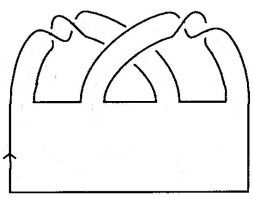
For many purposes, eg to see what is the genus of a given Seifert surface or to compute certain knot invariants, there is a particularly convenient class of Seifert surfaces:

Definition 19. A Seifert surface for a knot K is said to be *double-humped* if it is of the following form,



where the dotted bands are possibly knotted (not only each pair, but all of them) and have full twists.

It can be seen that, for a given knot, any Seifert surface can be isotoped[2] to a such which is double-humped. For instance, the Seifert surface for the figure-of-eight knot constructed in Lemma 17 is isotopic to the following double-humped Seifert surface by making the bottom disc square and stretching the top disc:



From this perspective, all the information about the knot is contained in the humps or bands. More specifically, the knot can be recovered from the cores of the bands

and an extra integer decorating each strand, which keeps track of the number of full twists (positive or negative) that the band should have when *thickening* each strand. We will call such a set of strands a *humped framed tangle*. More about this in the next lecture.

[2]"Isotopy" for an embedded surface means the same than for a knot: it can be smoothly deformed via embeddings.