# Knot Theory Seminar

Problem sheet 1

12 March 2020

1. The aim of this exercise is to use the knot group to show that the trefoil $3_1$ is not the unknot (we already showed this by using 3-colourability).

   (a) Show that if $f : A \longrightarrow G$ is a surjective group homomorphism from an abelian group $A$ to a group $G$, then $G$ is also abelian.

   (b) Show that $\pi_1(S^3 - 3_1) \cong \langle a, b | aba = bab \rangle$. (*Hint:* Use the Wirtinger algorithm and later obtain an equivalent presentation).

   (c) Define a surjection $\pi_1(S^3 - 3_1) \longrightarrow S_3$, where $S_3$ denotes the symmetric group of degree 3. Conclude that $3_1$ cannot be equivalent to the unknot.

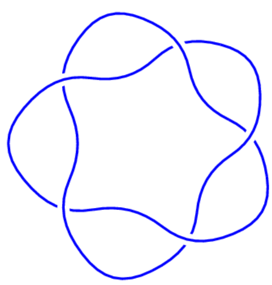
2. Now we will show that the figure-of-eight knot $4_1$ is not the unknot (we were not able to show this by using 3-colourability).

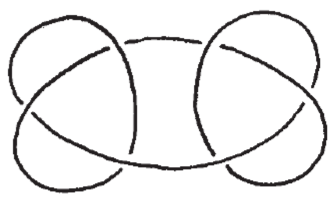
   (a) Let $D_n$ the dihedral group of degree $n$, that is, the group of isometries of the regular $n$-gon ($n > 2$). Show that it is generated by $g \in D_n$, the $2\pi/n$ rotation, and a reflexion $\tau \in D_n$. Show that $D_n$ admits a presentation $D_n \cong \langle g, \tau | g^n = \tau^2 = 1, \tau g \tau = g^{-1} \rangle$.

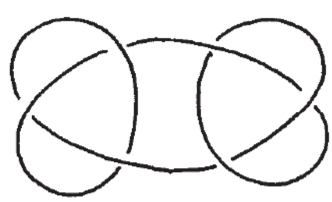
   (b) Show that $\pi_1(S^3 - 4_1) \cong \langle a, b | aba^{-1}bab^{-1} = bab^{-1}a \rangle$.

   (c) Define a surjection $\pi_1(S^3 - 4_1) \longrightarrow D_5$. Conclude that $4_1$ cannot be equivalent to the unknot. (*Hint:* $a \mapsto \tau g^3, b \mapsto \tau g$).

3. Compute the knot group and the Alexander polynomial of the $5_1$ knot, the Square knot and the Granny knot :







4. Show that the degree of the normalized Alexander polynomial of a knot cannot be greater than the number of crossings of any of its diagrams.

5. Let $K, K'$ be knots, let $K\#K'$ its connected sum and denote $\pi_K := \pi_1(S^3 - K)$. Show that
$$\pi_{K\#K'} \cong \pi_K *_{\mathbb{Z}} \pi_{K'}.$$

6. Let $K, K'$ be knots. Show that
$$\Delta_{K\#K'}(t) = \Delta_K(t)\Delta_{K'}(t')$$
(the equality holds in $\mathbb{Z}[t, t^{-1}]/\{\pm t^n\} = \mathbb{Z}[t, t^{-1}]/\mathbb{Z}[t, t^{-1}]^\times$).