Introduction To Khovanov Homology Handout

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We will be following Bar-Natan's paper [2]

0 Notation

We shall understand D and L to be links, K a knot, $\langle - \rangle$ the Kaufman Bracket, and O to be the unknot.

1 Alternative definition of the Kaufman bracket

Definition 1.1. *The bracket $\langle D \rangle' \in \mathbb{Z}[q, q^{-1}]$ is defined via*

1. $\langle \times \rangle' = \langle)(\rangle' - q \langle \asymp \rangle'$
2. $\langle D \amalg O \rangle' = (q + q^{-1}) \langle D \rangle'$
3. $\langle O \rangle' = 1$

Theorem 1.2. *Let n be the number of crossings of D and $A^2 = -q^{-1}$. Then $\langle D \rangle = A^n \langle D \rangle'$*

Proof. Via induction on the number of crossings. When $n = 1$ we see that the two knots arising from 1. must both be equivalent to the unknot. Without loss of generality both are the unknot up to isotopy and hence have bracket equal to 1. So for such a link one has $\langle D \rangle' = 1 - q$, and $A(1 - q) = (-q)^{-1/2} + q^{1/2}$, which we recognise as the corresponding Kauffman bracket.

The same is true for point 2. Namely it is just a multiple of the Kaufman Bracket. It is trivial this works for point 3.

When this relation holds for $n - 1$ crossings, take a link with n crossings and simply expand the bracket along one of its crossings. Keep applying 1 and 2 until one of the crossings disappears and we are left with a link with $n - 1$ crossings, to which we may apply the inductive hypothesis. $\square$

From now on we shall call either bracket the Kauffman bracket depending on what is convenient.

We let $n = n_+ + n_-$ where $n_\pm$ denotes the number of respectively positive or negative crossings in a knot D.

Theorem 1.3. *In this situation, the Jones polynomial is given by*

$$J_L(q) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle'$$

Proof. One sees that $w(L) = n_+ - n_-$

$$\begin{aligned}
(-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle' &= (-q^{-1})^{n/2} (-1)^{n_-} q^{w(L) - n_-} \langle D \rangle \\
&= q^{n_+/2 + n_-/2} (-1)^{n_- - n_+/2 - n_-/2} q^{w(L) - n_-} \langle D \rangle \\
&= (-1)^{-(n_+ - n_-)/2} q^{w(L) + n_+/2 - n_-/2} \langle D \rangle \\
&= (-1)^{-w(D)/2} q^{3w(L)/2} \langle D \rangle
\end{aligned}$$

As $(-1) = (-1)^3$ we do arrive at the formula for the Jones polynomial.

$\square$

We can do a similar construction by instead of saying $\langle O \rangle' = 1$ we define another bracket $\langle - \rangle''$ where the first two parts of the definition are the same, except we say $\langle \emptyset \rangle'' = 1$, and one sees

$$\langle O \rangle' = \langle \emptyset \amalg O \rangle' = (q + q^{-1}) \langle \emptyset \rangle' = q + q^{-1}$$

so we multiply with A^{n+1} instead and we obtain the Jones polynomial via

$$J_L(q) = \frac{(-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle''}{q + q^{-1}}$$

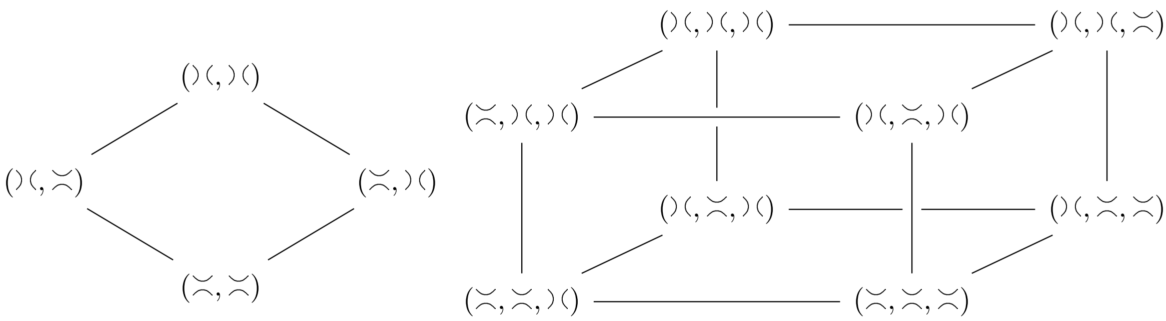
and we call $\hat{J}(q) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle''$.

2 Cube of Complete Smoothings

We will describe a method alternative to Skein-trees to compute the Jones polynomial. In fact we are going to be looking at the end points of the Skein-trees of a knot.

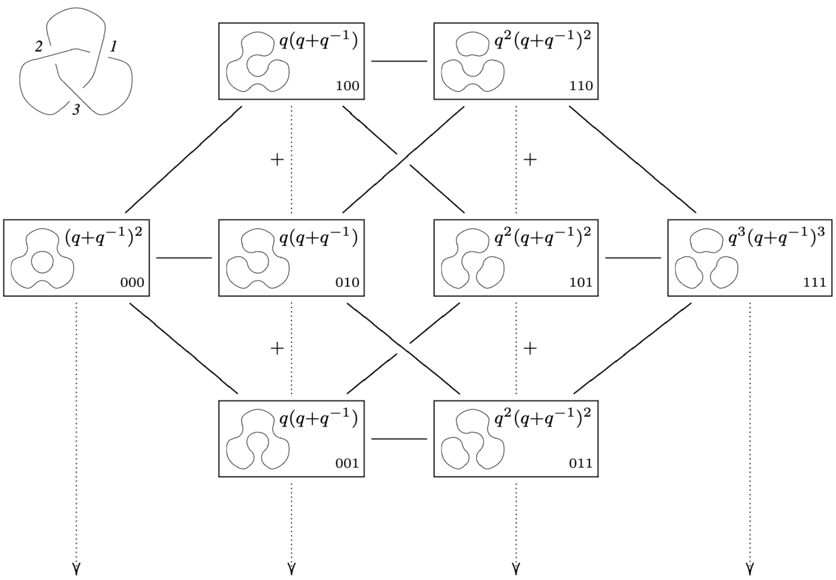
Consider a tangle T with n-crossings. Imagine the vertices of an n-cube $\{0, 1\}^n$ sitting in $\mathbb{R}^n$. We take T and change all its crossings to be either $)($ or $\asymp$ and take the set V to be all graphs obtained by some choice of $)($ and $\asymp$ for every crossing.

For example, we have the Hopf-Link, here we have two crossings, call them a and b, and we can set $(a, b) \in \{)(, \asymp\}^2$, so one might see this as a diagram in this case we form a 2-cube, so a square. being elements of $\{)(, \asymp\}^3$, so we get a cube When instead we take the trefoil, we have three crossings, (a, b, c)



By identifying $)(\sim 1$ and $\asymp \sim 0$ we can turn these tuples of meeting strands into coordinates in $\mathbb{R}^n$ and we call a vertex of this n-cube a smoothing.

So we set r to be the number of times $)($ appears in the smoothing, and we call k the number of disjoint unions of unknots that appear in a smoothing. Then each vertex corresponds to a term $(-1)^r q^r (q + q^{-1})^k$. If we sum over $\{0, 1\}^n$ and multiply with $(-1)^{n_-} q^{n_+ - 2n_-}$ we have gone down all the possible Skein-trees and end up with the Jones polynomial.



$$(q+q^{-1})^2 \quad - \quad 3q(q+q^{-1}) \quad + \quad 3q^2(q+q^{-1})^2 \quad - \quad q^3(q+q^{-1})^3$$

$$= \; q^{-2}+1+q^2-q^6 \xrightarrow[\text{(with } (n_+,n_-)=(3,0))]{\cdot(-1)^{n_-}q^{n_+-2n_-}} q+q^3+q^5-q^9 \xrightarrow{\cdot(q+q^{-1})^{-1}} J(\circledcirc) = q^2+q^6-q^8.$$

Figure 1: Example from [2], this procedure for the trefoil.

3 Graded vector spaces

Definition 3.1. *A grading on a field k is a sequence of subgroups k_n of the additive group of k such that*

$$k = \bigoplus_{n=0}^{\infty} k_n$$

and $k_n k_m \subseteq k_{n+m}$ for all $m, n \in \mathbb{Z}_{\geq 0}$.

Let k be a graded field. A grading on a k-vector space V is a family of subgroups V_n of the additive structure of V such that

$$V = \bigoplus_{n=0}^{\infty} V_n$$

and $k_m V_n \subseteq V_{m+n}$ for all $m, n \in \mathbb{Z}_{\geq 0}$.

Lemma 3.2. *Let V be a graded k-vector space and fix $v \in V$. Then v can be written uniquely as*

$$v = \sum_{n=0}^{\infty} v_n$$

where $v_n \in V_n$ and all but finitely many v_n are zero. We call the v_n the homogeneous components.

For the proof see [1]. We are going to need the notion of a *graded dimension.*

Definition 3.3. *Let V be a graded k-vector space with components $\{V_n\}$. The graded dimension of V is the generating series*

$$q\dim V = \sum_m q^m \dim V_n$$

we also define the degree shift operator $\cdot\{l\}$ as $V\{l\}_m = V_{m-l}$ such that $q\dim V\{l\} = q^l q\dim V$

To proceed we will need to know what a chain complex is [3]

Definition 3.4. *A chain complex is a pair (C, ∂) where C is a collection of (Abelian) groups C_n and ∂ is a collection of homomorphisms $\partial_n : C_{n+1} \to C_n$ such that $\partial_n \circ \partial_{n+1} = 0$ for all n. We also define the nth homology group to be $H_n := \ker \partial_n / \operatorname{Im} \partial_{n+1}$.*

And the final definition we shall require is as follows.

Definition 3.5. *Let $(\bar{C}, \partial)$ be a chain complex. The height shift operation is defined as*

$$\cdot[s] : \bar{C}_n \mapsto \bar{C}_n[s]$$

where we say $C_n = \bar{C}_n[s]$. In particular $C_n = \bar{C}_{n-s}[s]$.

Which allows us to state the final definition of this section

Definition 3.6. *Let (C, ∂) be a chain complex, we define the graded Euler Characteristic as*

$$\chi_q(C) = \sum_n (-1)^n q \dim H_n$$

Now we take a graded vector space with basis (v_+, v_-), where $\deg v_\pm = \pm 1$. Pick a vertex α of the n-cube corresponding to a tangle and define $V_\alpha(L) = V^{\otimes\{k\}}\{|\alpha|\}$, where $|\alpha| = \sum_{i=1}^n \alpha_i$. In this case the polynomial that corresponds to α is $q \dim V_\alpha(L)$. We set

$$[\![L]\!]^r := \bigoplus_{r=|\alpha|} V_\alpha(L)$$

We set $C(L) := ([\![L]\!]^r)_{r=1,2,\ldots}$ to be the C in a chain complex for which we do not yet know ∂. So in the case of the trefoil we have

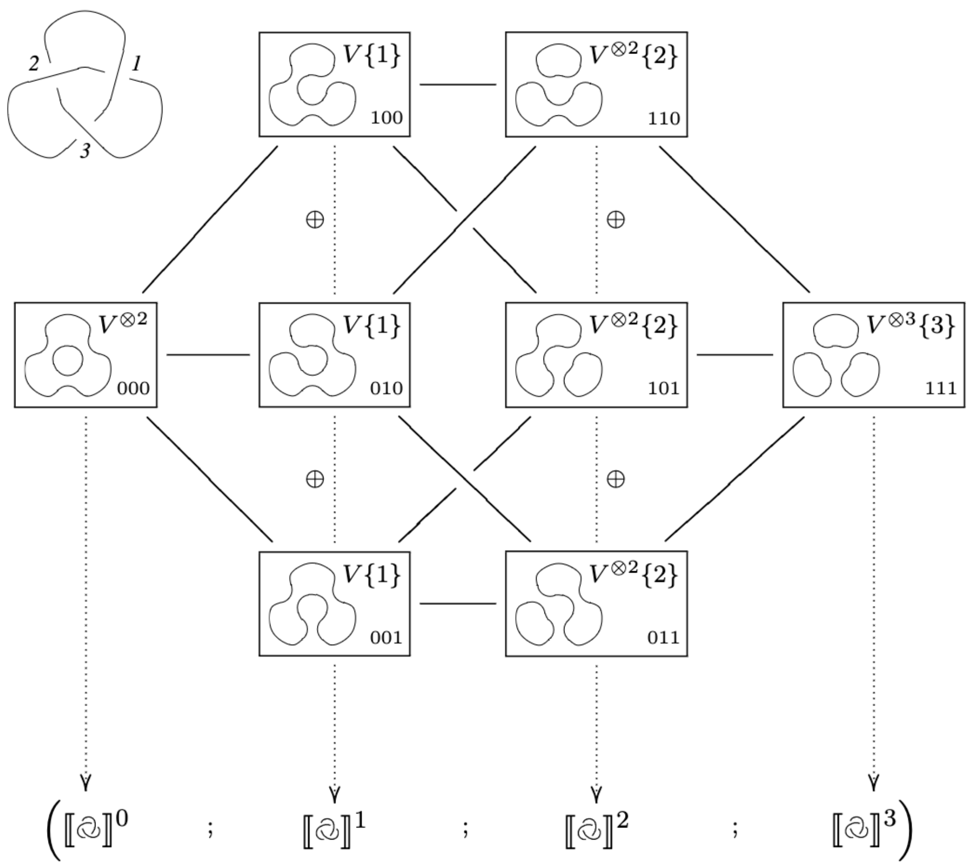


Figure 2: The Chain Complex Corresponding to , from [2]

Theorem 3.7.

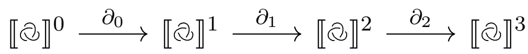
$$\chi_q(C(L)) = \hat{J}(L)$$

Proof. Just compare examples 1 and 2: we see that

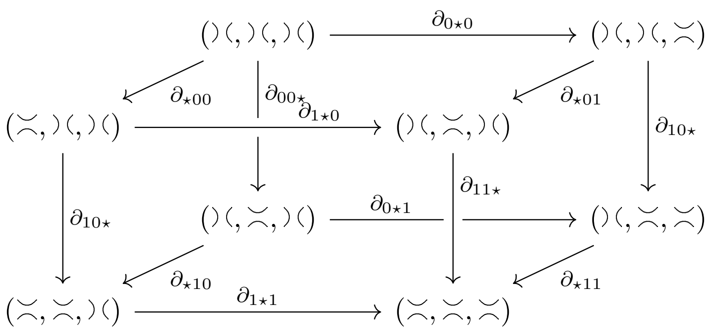
$$\chi_q(C(L)) = \sum_n (-1)^n q \dim H_n$$

we do not yet have H_n, but if we can get it to equal the terms in the sum defining $\hat{J}(L)$ then we are immediately done. □

We shall do a proof by example, namely consider the trefoil knot and the sequence that arises from it:



we can introduce an extra coordinate to get the edges of the cube as $\{0, \star, 1\}^n$, so that the cube becomes



and we define

$$\partial_i = \sum_{\xi \in \{0,\star,1\}^n, |\xi| = i} (-1)^{\xi} \partial_{\xi}$$

where by $(-1)^{\xi}$ we mean

$$(-1)^{\xi} = (-1)^{\sum_{j<i} \xi_j}$$

and i is the location of the $\star$. All that remains is finding ∂_{ξ} which make the cube commute. This construction can be found in [2, Section 3.2].

Now that we have ∂ we can state (but will not prove) the main theorem for this talk, namely.

Theorem 3.8. *(Khovanov, [2, Theorem 2]) The graded dimensions of the homology groups for* $(C(L), \partial)$ *are link invariants.*

References

[1] M.F. Atiyah and I.G. MacDonald. *Introduction To Commutative Algebra.* Addison-Wesley series in mathematics. Avalon Publishing, 1994. ISBN: 9780813345444.

[2] Dror Bar-Natan. "On Khovanov's categorification of the Jones polynomial". In: *Algebraic and Geometric Topology* (2002).

[3] A. Hatcher. *Algebraic Topology.* Algebraic Topology. Cambridge University Press, 2002. ISBN: 9780521795401.