LECTURE 3: THE ALEXANDER POLYNOMIAL

In the last lecture we constructed a knot polynomial invariant $J : \mathcal{L} \longrightarrow \mathbb{Z}[t^{1/2}, t^{-1/2}]$ called the Jones polynomial, that was characterised by the properties that (1) the value of the unknot is $J_{\text{unknot}}(t) = 1$, (2) a skein relation $t^{-1}J_{L_+}(t) - tJ_{L_-}(t) = (t^{1/2} - t^{-1/2})J_{L_0}(t)$. Following the same scheme, today we will construct another link polynomial invariant more related with how the link is embedded in $\mathbb{R}^3$.

1. Statement and examples

The goal of this lecture is to show the following theorem:

Theorem 1. *There exists a unique link polynomial invariant*

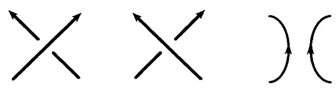
$$\Delta : \mathcal{L} \longrightarrow \mathbb{Z}[t^{1/2}, t^{-1/2}]$$

satisfying

(1) $\Delta_{\text{unknot}}(t) = 1$,
(2) the following skein relation holds:

$$\Delta_{L_+}(t) - \Delta_{L_-} = (t^{-1/2} - t^{1/2})\Delta_{L_0}(t),$$

where L_+, L_-, L_0 *denote three links that are identical except in a neighbourhood of some point where they look like below,*

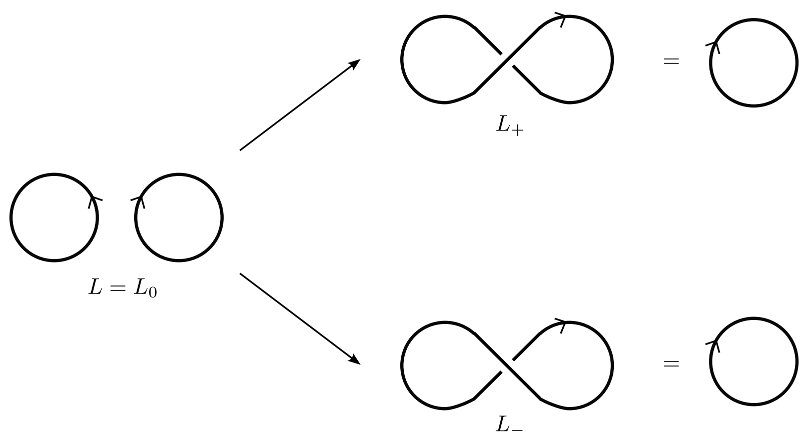


L_+ L_- L_0

Definition 2. The polynomial of the previous theorem is called the *Alexander polynomial.* It is coined after James W. Alexander (1888 – 1971).

As we did last time with the Jones polynomial, we will provide a construction and later check that it satisfies the properties of the theorem. But first of all, let us look at some examples:

Example 3. Let L be the trivial 2-component link. We apply the skein relation taking $L = L_0$. The other links (actually knots) involved L_+, L_0 are trivial by a Reidemeister $\Omega 1$ move, so $\Delta_{L_+} = 1 = \Delta_{L_-}$.

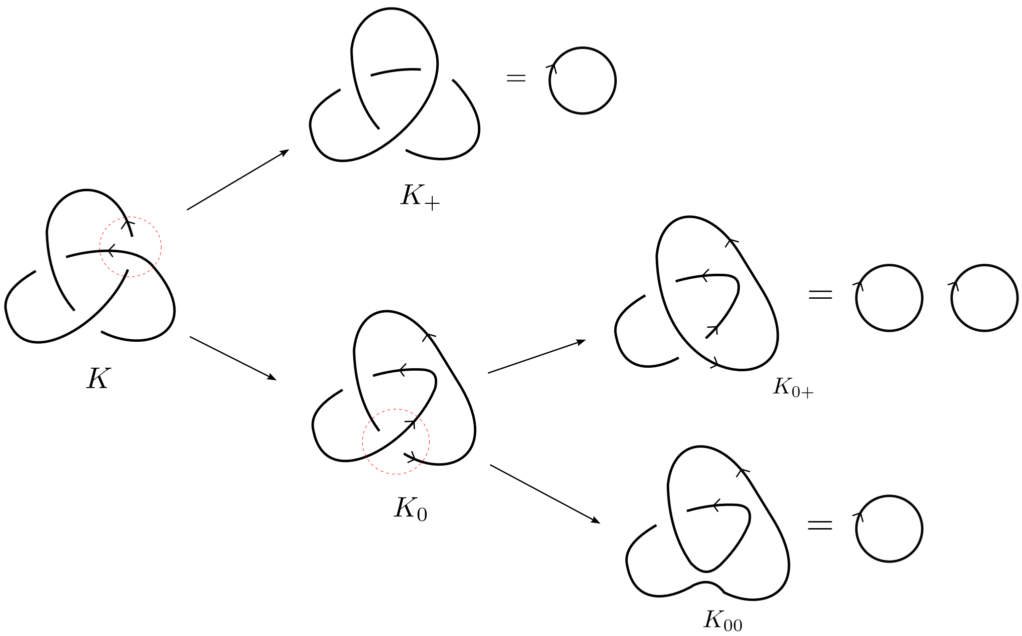


The skein relation then reads

$$0 = 1 - 1 = \Delta_{L_+}(t) - \Delta_{L_-} = (t^{-1/2} - t^{1/2})\Delta_{L_0}(t)$$

so we conclude that $\Delta_L = 0$.

Example 4. Let us now compute the Alexander polynomial of the left-handed trefoil K. Recall from the previous lecture that the strategy consists of constructing a skein tree for the knot. Below we show this for K:



Note both K_+ and K_{00} are the unknot, and K_{0+} is isotopic to the trivial 2-component link, whose Alexander polynomial we just computed. Using the skein relation, we have

$$\Delta_{K_{0+}} - \Delta_{K_0} = (t^{-1/2} - t^{1/2})\Delta_{K_{00}}$$

so $\Delta_{K_0} = -(t^{-1/2} - t^{1/2})$ by the previous example. On the other hand

$$1 - \Delta_K = \Delta_{K_+} - \Delta_K = (t^{-1/2} - t^{1/2})\Delta_{K_0}$$

and therefore

$$\begin{aligned}\Delta_K &= 1 + (t^{-1/2} - t^{1/2})^2 \\ &= 1 + t^{-1} + t - 2 \\ &= t^{-1} - 1 + t.\end{aligned}$$

Exercise 5. Mimicking the above example, show that the Alexander polynomial of the trivial n-component link (that is, n disjoint copies of the unknot) is zero.

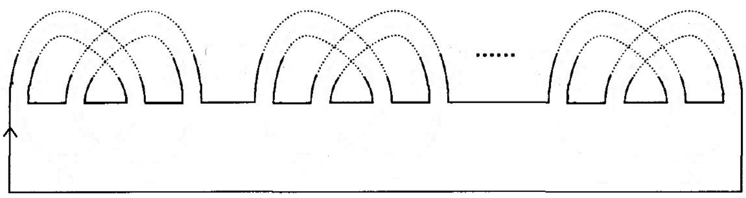
2. Proof of the uniqueness

The proof of the uniqueness follows word by word from the proof of the uniqueness of the Jones polynomial, after the minor change in the skein relation.

3. A construction for the Alexander polynomial

As we mentioned before, the existence is proved by means of providing a construction of the Alexander polynomial.

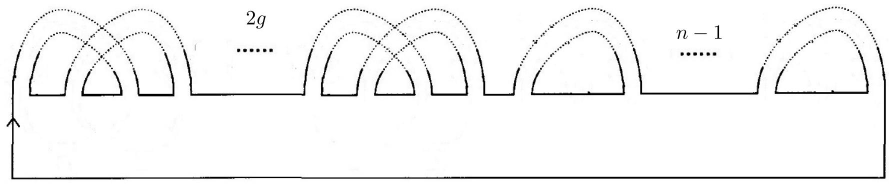
Recall from Lecture 1 that every knot possesses a double-humped Seifert surface,



where the dotted humps mean that all they are possibly knotted and twisted (with full twists). Below we depic an example where four bands are knotted in a non-trivial way:

(1)

For a link, the story is rather similar. It can be easily shown that every n-component link has a Seifert surface of the following form,



(2)

where the dotted lines denote that all these bands are possibly knotted, linked and twisted (with full twists), that we also call *double-humped.* Note that the above surface has n boundary components.

In Lecture 1 we also mentioned that we could easily encode the data of a double-humped Seifert surface by means of a humped framed tangle, which was a collection of arcs encoding the crossings between the bands together with an integer attached to each band encoding the number of full twists.

Definition 6. Let L be an n-component link, and let Σ be a double-humped Seifert surface for L. The humped framed tangle T described above will be called the *humped tangle associated to* Σ. Note this tangle has $2g+n-1$ components.

Exercise 7. Draw the humped tangle associated to (1).

In what follows, let us label the components of such a humped tangle as $T = T_1 \cup \cdots \cup T_{2g+n-1}$.

Definition 8. Let L be a link, let Σ be a double-humped Seifert surface for L, let T be the humped tangle associated to Σ and let the component T_i be labelled with $k_i \in \mathbb{Z}$.

For $i, j = 1, \ldots, 2g+n-1$, $i \neq j$, let p_{ij} (resp. n_{ij}) be the number of positive (resp. negative) crossings in which the overstrand is the i-th component and the understrand is the j-th component.

Then the *Seifert matrix* associated to L is the matrix $V = (v_{ij}) \in \mathcal{M}_{2g+n-1}(\mathbb{Z})$ defined as follows:

$$v_{ij} := \begin{cases} p_{ij} - n_{ij}, & i \neq j \\ k_i, & i = j. \end{cases}$$

This matrix will allow us to define the polynomial invariant we were after:

Definition 9. Let L be a link and let V be a Seifert matrix for L. The *Alexander polynomial* of L is the Laurent polynomial

$$\Delta_L(t) := \det(t^{1/2}V - t^{-1/2}V^T) \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$$

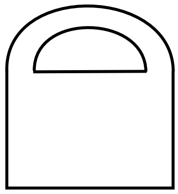
where V^T denotes the transpose of V.

We still have to check that this definition of the Alexander polynomial satisfies the two conditions of Theorem 1 and most importantly that it is indeed an isotopy invariant of L. We will defer this to the end of the lecture.

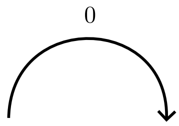
4. Examples and Properties

Let us start with two examples, more concretely, let us redo the two examples that we computed using the skein relation.

Example 10. Once again let L be the trivial 2-component link. A double-humped Seifert surface for L is the following:



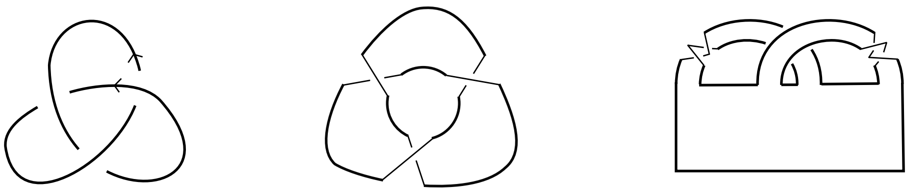
A humped tangle for this Seifert surface is clearly



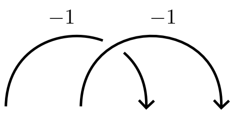
Therefore, $V = 0$ and

$$\Delta_L = \det 0 = 0.$$

Example 11. Let K now be the left-handed trefoil. The first step is to apply the Seifert algorithm to obtain a Seifert surface for K. By isotopying the surface, we can obtain a double-humped Seifert surface for K:



The humped tangle associated to the latter is



The Seifert matrix associated to this is

$$V = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$$

Therefore we compute

$$\begin{aligned} \Delta_K(t) &= \det \left(t^{1/2} \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} - t^{-1/2} \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \right) \\ &= \det \begin{pmatrix} -t^{1/2} + t^{-1/2} & t^{-1/2} \\ -t^{1/2} & -t^{1/2} + t^{-1/2} \end{pmatrix} \\ &= (-t^{1/2} + t^{-1/2})^2 + 1 \\ &= t^{-1} - 1 + t \end{aligned}$$

as we obtained before.

Let us see some general properties of the Alexander polynomial

Proposition 12. *The Alexander polynomial satisfies the following properties:*

(1) If L is a link with an odd number of components (in particular if L is a knot), then

$$\Delta_L(t^{-1}) = \Delta_L(t).$$

and

$$\Delta_L \in \mathbb{Z}[t, t^{-1}].$$

(2) Let L be an n-component link and denote $-L$ and $\overline{L}$ the orientation reversal and mirror image of L, respectively. Then

$$\Delta_L = \Delta_{-L} \qquad , \qquad \Delta_{\overline{L}} = (-1)^{n-1}\Delta_L.$$

(3) If K, K' are knots, then

$$\Delta_{K\#K'} = \Delta_K \cdot \Delta_{K'}.$$

Proof. (1) If L has n components and n is odd, then $r := 2g + n - 1$ is even. Now we simply observe that

$$\begin{aligned}\Delta_L(t^{-1}) &= \det(t^{-1/2}V - t^{1/2}V^T)\\ &= \det(t^{1/2}V^T - t^{-1/2}V)\\ &= \det(t^{1/2}V - t^{-1/2}V^T)\\ &= \Delta_L(t)\end{aligned}$$

and similarly

$$\begin{aligned}\Delta_L(t) &= \det(t^{1/2}V - t^{-1/2}V^T)\\ &= t^{-r/2}\det(tV - V^T) \in \mathbb{Z}[t, t^{-1}].\end{aligned}$$

(2) and (3) Homework. □

5. Proof of the existence

Let us finally check that the polynomial we constructed from a double-humped Seifert surface is the unique link polynomial invariant described in Theorem 1.

Definition 13. Two square matrices A, B (not neccesarily of the same order) are said to be *S-equivalent* if there exists a sequence of square matrices

$$A = A_0, \quad A_1, \quad \dots, \quad A_{m-1}, \quad A_m = B$$

such that for every $i = 0, \dots, m-1$, A_i and A_{i+1} are related by one of the following ways:

(1) $A_{i+1} = \begin{pmatrix} A_i & c & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ or $A_i = \begin{pmatrix} A_{i+1} & c & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$,

(2) $A_{i+1} = \begin{pmatrix} A_i & 0 & 0 \\ r & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ or $A_i = \begin{pmatrix} A_{i+1} & 0 & 0 \\ r & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(3) $A_{i+1} = P^T A_i P$ for some integral square matrix P with $\det P = \pm 1$,

where c and r are a column and row vector, respectively.

We will use the following theorem without proof, as it requires to handle some topological techniques that we will not cover in this course.

Theorem 14. *Any two Seifert matrices of a given link are S-equivalent.*

In particular, the S-equivalence class of any Seifert matrix of a link is an isotopy invariant of the link

We are finally ready to show the existence. First we show that the Alexander polynomial is an isotopy invariant. By the previous theorem, it suffices to show that the quantity $\det(t^{1/2}V - t^{-1/2}V^T)$ does not depend on the three relations in the S-equivalence. If it comes to (3), we have

$$\det(t^{1/2}P^TVP - t^{-1/2}P^TV^TP) = (\det P)^2 \det(t^{1/2}V - t^{-1/2}V^T).$$

Next we show the independence of (1) (for (2) the argument is similar). Let

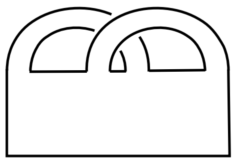
$$V' = \begin{pmatrix} V & c & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

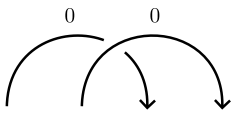
$$(t^{1/2}V' - t^{-1/2}(V')^T) = \begin{pmatrix} t^{1/2}V - t^{-1/2}V^T & t^{1/2}c & 0 \\ -t^{-1/2}c^T & 0 & t^{1/2} \\ 0 & -t^{-1/2} & 0 \end{pmatrix}$$

which is easily seen to have the same determinant as $t^{1/2}V - t^{-1/2}V^T$.

Let us lastly check that Δ_L meets the conditions of Theorem 1. It is easily seen that the following is a double-humped Seifert surface for the unknot,



so that its associated tangle is



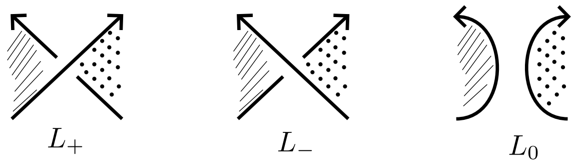
and therefore its Seifert matrix is

$$V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

It follows that

$$\Delta_{\text{unknot}}(t) = \det \begin{pmatrix} 0 & -t^{-1/2} \\ t^{1/2} & 0 \end{pmatrix} = 1.$$

To derive the skein relation, let Σ_0 be a double-humped Seifert surface for a link L_0 as in (2). Consider a point where a neighbourhood of it looks like the figure below for L_0. Here the dotted and the dashed areas represent the two different faces of the Seifert surface.



After isotopying the Seifert surface if neccesary, we can assume that the point is at the rightmost part of the Seifert surface. Now construct Seifert surfaces Σ_+ and Σ_- for L_+ and L_- by replacing the portion of surface indicated for Σ_0 by half-twisted strips, twisted in opposite sense for Σ_+ and Σ_-. The boundaries of these surfaces are links L_+, L_- as the ones indicated in the skein relation.

Let V_0 denote the Seifert matrix associated to Σ_0. Then, L_+ and L_- have Seifert matrices associated to Σ_+ and Σ_- as follows:

$$V_+ = \begin{pmatrix} V_0 & c \\ r & N \end{pmatrix} \quad , \quad V_- = \begin{pmatrix} V_0 & c \\ r & N+1 \end{pmatrix},$$

where c is a column vector, r a row vector and N an integer. Note that these two matrices only differ by adding 1 in the last entry, or equivalently, adding a vector $(0, \dots, 0, 1)^T$ to the last column. Recall from the linear algebra course that if $A = (A_1, \dots, A_n)$ is a square matrix with columns A_i, then

$$\det(A, \dots, A_n + A'_n) = \det(A, \dots, A_n) + \det(A, \dots, A'_n)$$

where A'_n is any column vector. Using this property we compute

$$\begin{aligned}
\Delta_{L_+}(t) - \Delta_{L_-}(t) &= \det(t^{1/2}V_+ - t^{-1/2}V_+^T) - \det(t^{1/2}V_- - t^{-1/2}V_-^T) \\
&= \det\begin{pmatrix} t^{1/2}V_0 - t^{-1/2}V_0^T & t^{1/2}c - t^{-1/2}r^T \\ t^{1/2}r - t^{-1/2}c^T & (t^{1/2} - t^{-1/2})N \end{pmatrix} - \det\begin{pmatrix} t^{1/2}V_0 - t^{-1/2}V_0^T & t^{1/2}c - t^{-1/2}r^T \\ t^{1/2}r - t^{-1/2}c^T & (t^{1/2} - t^{-1/2})(N+1) \end{pmatrix} \\
&= \det(t^{1/2}V_+ - t^{-1/2}V_+^T) - \det(t^{1/2}V_+ - t^{-1/2}V_+^T) - \det\begin{pmatrix} t^{1/2}V_0 - t^{-1/2}V_0^T & 0 \\ t^{1/2}r - t^{-1/2}c^T & (t^{1/2} - t^{-1/2}) \end{pmatrix} \\
&= -(t^{1/2} - t^{-1/2}) \det(t^{1/2}V_0 - t^{-1/2}V_0^T) \\
&= (t^{-1/2} - t^{1/2})\Delta_{L_0}(t),
\end{aligned}$$

which yields the skein relation, as promised.