Determining d with a TQFT

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Talk 2 on Khovanov Homology

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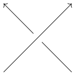
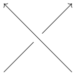
Summary of last week

- q-dimension: $\mathrm{qdim}(\bigoplus_{m\in\mathbb{Z}} W^m) = \sum_{m\in\mathbb{Z}} q^m \dim(W^m)$
- Shift: $W^m\{l\} = W^{m-l}$
- $V = \mathbb{Q}[x] \oplus \mathbb{Q}[1]$, $\deg(x) = -1$, $\deg(1) = 1$
- $\alpha \in \{0,1\}^n$ "bitcode", $n = \#$original crossings of projection
- $n_+ = \#\Bigg($  $\Bigg)$, $n_- = \#\Bigg($  $\Bigg)$
- $r_\alpha = |\alpha| = \#$1's in α, $k_\alpha = \#$ circles in smoothing Γ_α
- $V_\alpha = V^{\otimes k_\alpha}\{r_\alpha + n_+ - 2n_-\}$
- $C^{i,*}(D) = \bigoplus_{\substack{\alpha\in\{0,1\}^n: \\ r_\alpha = i+n_-}} V_\alpha$
- $v \in C_{i,j}(D)$ if $\begin{cases} i = r_\alpha - n_- \\ j = \deg(v) + i + n_+ - n_- \end{cases}$

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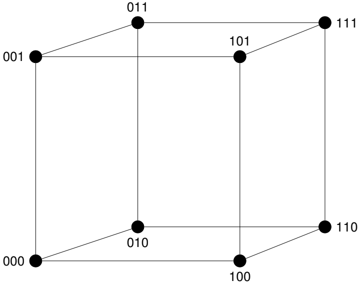
What should *d* do?

- *d* maps on the edges on the *n*-dimensional cube with $\{0,1\}^n$ as vertices



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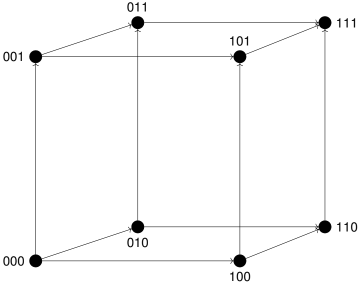
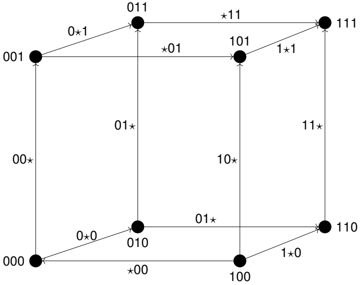


What should d do?

- d maps on the edges on the n-dimensional cube with $\{0, 1\}^n$ as vertices
- d moves "upwards": we go from a "0" to "1".
- We denote the maps by the bitcode and place $\star$ where the change occurs



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- For each edge, the number of circles at the begin and the end differs by 1.
- How can we connect those sets of circles using topological objects?
- Cobordisms!

Cobordisms

Definition (Cobordisms)

The category of $1+1$-cobordisms, denoted by $\mathbf{Cob}_{1+1}$, consists of

- Objects: closed oriented 1-manifolds (i.e. a disjoint union of oriented circles)
- Morphisms: $W : \Gamma \to \Gamma'$ is a oriented 2-manifold such that $\partial W = \Gamma' \sqcup \bar{\Gamma}$, where $\bar{\Gamma}$ is Γ with the reverse orientation

Convention: cobordisms go down the page
Our choice of cobordism along the edges: if the circles do not change, make cylinders. If they do, plug in  or .

How to go back to vector spaces?

- We need something to transforms our cobordisms into vector spaces and linear maps

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- We need something to transforms our cobordisms into vector spaces and linear maps
- This is a TQFT!

TQFTs (part 1)

Definition

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Definition

A (2-dimensional) **TQFT** is

- a functor $T : \mathbf{Cob}_{1+1} \to \mathbf{Vect}_{\mathbb{Q}}$ with
- a natural transformation $\mu_{\Gamma,\Gamma'} : T(\Gamma) \otimes_{\mathbb{Q}} T(\Gamma') \to T(\Gamma \sqcup \Gamma')$ (isomorphism) such that

1. $$\begin{tikzcd} T(\Gamma_1) \otimes T(\Gamma_2) \otimes T(\Gamma_3) \arrow[r, "\mathrm{id} \otimes \mu_{\Gamma_2,\Gamma_3}"] \arrow[d, "\mu_{\Gamma_1,\Gamma_2} \otimes \mathrm{id}"] & T(\Gamma_1) \otimes T(\Gamma_2 \sqcup \Gamma_3) \arrow[d, "\mu_{\Gamma_1,\Gamma_2 \sqcup \Gamma_3}"] \\ T(\Gamma_1 \sqcup \Gamma_2) \otimes T(\Gamma_3) \arrow[r, "\mu_{\Gamma_1 \sqcup \Gamma_2,\Gamma_3}"] & T(\Gamma_1 \sqcup \Gamma_2 \sqcup \Gamma_3) \end{tikzcd}$$ commutes (assoc.)
2. $\mu_{\emptyset,\Gamma} : \mathbb{Q} \otimes_{\mathbb{Q}} T(\Gamma) \xrightarrow{\sim} T(\Gamma)$ and $\mu_{\Gamma,\emptyset} : T(\Gamma) \otimes_{\mathbb{Q}} \mathbb{Q} \xrightarrow{\sim} T(\Gamma)$ are the canonical isomorphisms.

TQFTs (part 2)

Definition

A (2-dimensional) **TQFT** is

- $\ldots$
- $T(\emptyset) = \mathbb{Q}$;
- $T(\Gamma \times [0,1]) = \mathrm{id}_\Gamma$
- If there exists a orientation preserving diffeomorphism $\varphi : V \xrightarrow{\cong} W$ fixing the boundary, then V and W induce the same linear map, that is, $T(V) = T(W)$.

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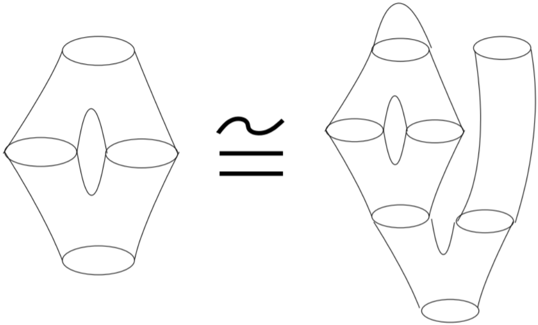
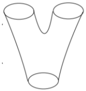
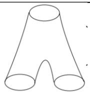
Properties of TQFTs

- Every cobordism Γ gets a vector space V_Γ.
- $\Gamma \sqcup \Gamma' \rightsquigarrow V_\Gamma \otimes V_{\Gamma'}$
- Diffeomorphic cobordisms induce the same linear map



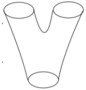
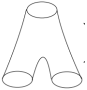
Making algebra easier (part 1)

In a TQFT, the building blocks are circles/cylinders and the following operations:

Coming together	
Branching out	
Starting from scratch	
Ending in scratch	

Making algebra easier (part 2)

This corresponds to linear maps on $V := T(S^1)$:

Coming together		$m : V \otimes V \to V$	multiplication
Branching out		$\Delta : V \to V \otimes V$	comultiplication
Starting from scratch		$\eta : \mathbb{Q} \to V$	unit
Ending in scratch		$\epsilon : V \to \mathbb{Q}$	counit

Frobenius algebra

Definition

A **Frobenius algebra** is a finite-dimensional $\mathbb{Q}$-algebra V with four maps

1. $m : V \otimes V \to V$;
2. $\Delta : V \to V \otimes V$;
3. $\eta : \mathbb{Q} \to V$;
4. $\epsilon : V \to \mathbb{Q}$;

such that $V \to V^*, v \mapsto \epsilon(m(v \otimes \cdot))$ is an isomorphism, and such that $\Delta(v) = \sum_i v_1^i \otimes v_2^i$ if and only if $m(v \otimes w) = \sum_i v_1^i \epsilon(m(v_2^i \otimes w))$.

Frobenius algebras and TQFTs

Theorem

There is an equivalence of categories between the category of (2-dimensional) TQFTs and the category of Frobenius algebras by sending T to $T(S^1)$.

Our Frobenius algebra

Our choice of Frobenius algebra is $V = \mathbb{Q}[x] \oplus \mathbb{Q}[1]$, $\deg(x) = -1$, $\deg(1) = 1$

- $m(1 \otimes 1) = 1$, $m(x \otimes 1) = x = m(1 \otimes x)$, $m(x \otimes x) = 0$;
- $\Delta(1) = 1 \otimes x + x \otimes 1$, $\Delta(x) = x \otimes x$;
- $\eta(1) = 1$;
- $\epsilon(1) = 0$, $\epsilon(x) = 1$.

Call the corresponding TQFT T.

Actually determining d!

Definition

For $v \in V_\alpha \subseteq C^{i,*}(D)$, we set

$$d^i(v) = \sum_{\substack{W \text{ edge of cube} \\ \text{such that } \mathrm{Tail}(W)=\alpha}} \mathrm{sign}(W) d_W(v), \tag{1}$$

where $d_W := T(W) : V_\alpha \to V_\alpha$ is the map from the corresponding TQFT and where $\mathrm{sign}(W) = (-1)^{\#\text{ 1's to the left of } \star \text{ in } \alpha}$.

The start of (co)homology

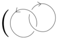
Proposition

$d^2 = 0$.

Proof.

- Look at the faces of the cube $\{0, 1\}^n$.
- Prove that they commute without the signs (use the cobordisms).
- Then put the signs in to make them anti-commutative.

Explicit example

Calculate the (Khovanov) homology of $C^{*,*}($$)$.

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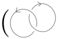
Calculate the (Khovanov) homology of $C^{*,*}($$)$.

$$
\begin{tikzcd}
10 \arrow[r] & 11 \\
00 \arrow[u] \arrow[r] & 01 \arrow[u]
\end{tikzcd}
$$

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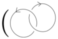
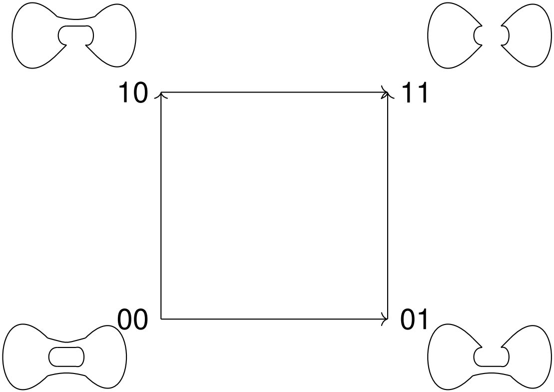
Calculate the (Khovanov) homology of $C^{*,*}($$)$.



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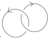
Calculate the (Khovanov) homology of $C^{*,*}($$)$.

$$
\begin{tikzcd}
 & V\{-3\} \arrow[dr, "-\Delta"] & \\
(V \otimes V)\{-4\} \arrow[ur, "m"] \arrow[dr, "m"'] \arrow[rr, phantom, "\oplus"] & & (V \otimes V)\{-2\} \\
 & V\{-3\} \arrow[ur, "\Delta"'] &
\end{tikzcd}
$$

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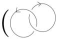
Calculate the (Khovanov) homology of $C^{*,*}($$)$.

Homological degree	*-2*	*-1*	*0*
Cycles	$\{1 \otimes x - x \otimes 1, x \otimes x\}$	$\{(1,1),(x,x)\}$	$\{1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x\}$
Boundaries	-	$\{(1,1),(x,x)\}$	$\{1 \otimes x + x \otimes 1, x \otimes x\}$
Homology	$\{1 \otimes x - x \otimes 1, x \otimes x\}$	-	$\{1 \otimes 1, 1 \otimes x\}$
q-degrees	*-4, -6*		*0, -2*

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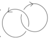
Calculate the (Khovanov) homology of $C^{*,*}($$)$.

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$j \backslash i$	*-2*	*-1*	*0*
0			$\mathbb{Q}$
-1			
-2			$\mathbb{Q}$
-3			
-4	$\mathbb{Q}$		
-5			
-6	$\mathbb{Q}$		