Topics in Topology: Lecture 4
Grid complexes and their homologies

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Introduction

We will define and look at some grid complexes:

$$\mathbb{F} = \mathbb{Z}/2\mathbb{Z} \quad \widetilde{GC} \xrightarrow[\mathbb{F}[V_i,\ldots,V_n]]{\text{extension via}} GC^- \xrightarrow[\text{by } V_i]{\text{quotient}} \widehat{GC}$$

We will prove that $(\widetilde{GC}, \tilde{\partial}_{\mathbb{O},\mathbb{X}})$ is a chain complex.

Then do some exercises!

But First!

We only need two equations for the proof:

part of def.

$$M_{\mathbb{O}}(\mathrm{x}) - M_{\mathbb{O}}(\mathrm{y}) = 1 - 2\#(r \cap \mathbb{O}) + 2\#(\mathrm{x} \cap \mathrm{Int}(r))$$

cor.

$$A(\mathrm{x}) - A(\mathrm{y}) = \#(r \cap X) - \#(r \cap \mathbb{O}),$$

Grid complex GC

$\mathbb{F}[S(\mathbb{G})]$

Take the $\mathbb{F}$-vector space $GC(\mathbb{G}) := \langle S(\mathbb{G}) \rangle$.

$\mathbb{Z}$

$M^{-1}(n) = A \subset GC(\mathbb{G})$

We know the Maslov and Alexander functions grade the space

$$GC(\mathbb{G}) = \bigoplus_{d,s \in \mathbb{Z}} GC_d(\mathbb{G}, s).$$

We obtain a chain complex by choosing a differential.

The map $\tilde{\partial}_{\mathbb{O},\mathbb{X}}$ enters the chat.

$\tilde{\partial}_{\mathbb{O},\mathbb{X}} : GC(\mathbb{G}) \to GC(\mathbb{G})$

$$\tilde{\partial}_{\mathbb{O},\mathbb{X}}(x) = \sum_{y \in S(\mathbb{G})} \#\{r \in \mathrm{Rect}^\circ(x, y) \mid r \cap \mathbb{O} = r \cap \mathbb{X} = \varnothing\} \cdot y,$$

$(d, s) \mapsto (d-1, s)$

Claim: $\tilde{\partial}_{-11-}$ is degree $(-1, 0)$

What we need for the proof

To prove that $\tilde{\partial}_{\mathbb{O},\mathbb{X}}$ is a differential, we need: #( )

Beamer friendly notation! $r \cap X = r \cap Y = \emptyset$

$$\underline{\mathbb{1}_{x,y}} := \mathbb{1}_{1=\#\{r\in \mathrm{Rect}^\circ(x,y) \mid r\cap\mathbb{O}=r\cap\mathbb{X}=\varnothing\}} \Longrightarrow \tilde{\partial}_{\mathbb{O},\mathbb{X}} = \sum_{y\in S(\mathbb{G})} \mathbb{1}_{x,y}\cdot y,$$

$\#\{\cdot\} - e$

and the criteria:

1. $\mathrm{Im}(\tilde{\partial}_{\mathbb{O},\mathbb{X}}|_{\widetilde{GC}_d(\mathbb{G},s)}) \subseteq \widetilde{GC}_{d-1}(\mathbb{G},s)$ $\quad 0 \to A_{n+1} \to A_n \to A_{n-1} \to 0 \ldots$
2. and the composition $\underline{\tilde{\partial}^2_{\mathbb{O},\mathbb{X}} = 0}$

$\tilde{\partial}_{\mathbb{O},\mathbb{X}}$ has degree $(-1, 0)$

Take $\mathrm{x} \in \widetilde{GC}_d(\mathbb{G}, s)$ $\quad$ $M(x)=d$, $A(x)=S$

Then $\tilde{\partial}_{\mathbb{O},\mathbb{X}}(\mathrm{x}) = \sum_{\mathrm{y} \in \mathrm{S}(\mathbb{G})} \mathbb{1}_{\mathrm{x},\mathrm{y}} \cdot \mathrm{y}$ $\leftarrow$ we don't care if zero,

If $\mathbb{1}_{\mathrm{x},\mathrm{y}} = \overline{1}$, we can solve for $M(\mathrm{y})$ and $A(\mathrm{y})$

$$\overset{d}{M_{\mathbb{O}}(\mathrm{x})} - M_{\mathbb{O}}(\mathrm{y}) = 1 - 2\#(r \overset{0}{\cap} \mathbb{O}) + 2\#(\mathrm{x} \overset{0}{\cap} \mathrm{Int}(r)) \Rightarrow M(y) = d-1$$

$$\underset{S}{A(\mathrm{x})} - A(\mathrm{y}) = \#(r \underset{0}{\cap} \mathbb{X}) - \#(r \underset{0}{\cap} \mathbb{O}), \Rightarrow \underline{A(y) = s}$$

Recall $\mathbb{1}_{\mathrm{x},\mathrm{y}} := \mathbb{1}_{1 = \#\{r \in \mathrm{Rect}^\circ(\mathrm{x},\mathrm{y}) \,|\, r \cap \mathbb{O} = r \cap \mathbb{X} = \varnothing\}}$

Hence $\tilde{\partial}_{\mathbb{O},\mathbb{X}}(\mathrm{x}) \in \underline{\widetilde{GC}_{d-1}(\mathbb{G}, s)}$.

Proof that $\tilde{\partial}^2_{\mathbb{O},\mathbb{X}} = 0$

By linearity:

$$\tilde{\partial}(x)$$

$$\|$$

$$\tilde{\partial}^2_{\mathbb{O},\mathbb{X}}(x) = \tilde{\partial}_{\mathbb{O},\mathbb{X}}\left(\sum_{y\in S(\mathbb{G})} \mathbb{1}_{x,y}\cdot y\right) = \sum_{y\in S(\mathbb{G})} \mathbb{1}_{x,y}\cdot \tilde{\partial}_{\mathbb{O},\mathbb{X}}(y)$$

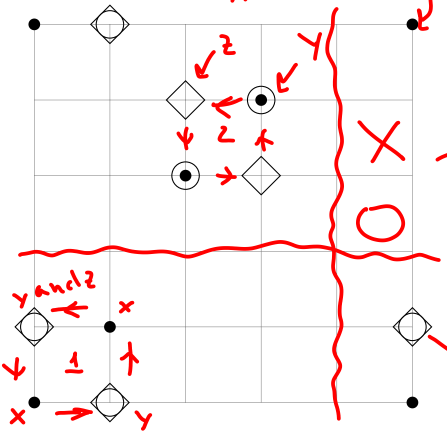
$$= \sum_{y\in S(\mathbb{G})}\left(\mathbb{1}_{x,y}\sum_{z\in S(\mathbb{G})} \mathbb{1}_{y,z}\cdot z\right) = \sum_{z\in S(\mathbb{G})}\left(\overbrace{\underbrace{\sum_{y\in S(\mathbb{G})} \mathbb{1}_{x,y}\mathbb{1}_{y,z}}_{0+0+1+1+0}}^{\forall z \quad =0}\right)\cdot z = 0$$

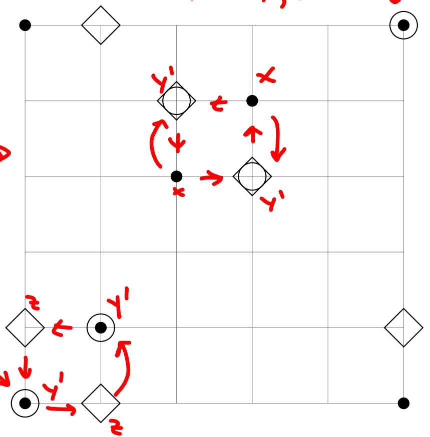
if y $\quad 1_{x,y}1_{y,z} = 1 = 1_{x,y'}1_{y',z}$

then we have another y'

Proof by sketch

$1_{x,y}1_{y,z}=1$ $\qquad$ $1_{x,y'}1_{y',z}=1$ $\quad y \neq y'$





A possible arrangement for r and r', x, y, (y′), z are dot, circ., diam.

The rectangles but constructed using y′ ≠ y

$\sigma_x^{-1}\sigma_y$ and $\sigma_y^{-1}\sigma_z$ both transp. $y \to y'$ (involution with no fixed)

$\sigma_x^{-1}\sigma_{y'}$ $\qquad$ $\sigma_{y'}^{-1}\sigma_z$

Proof that $\tilde{\partial}^2_{\mathbb{O},\mathbb{X}} = 0$

$$\tilde{\partial}^2_{\mathbb{O},\mathbb{X}}(\mathsf{x}) = \sum_{\mathsf{z}\in \mathsf{S}(\mathbb{G})} \left(\underbrace{\sum_{\mathsf{y}\in \mathsf{S}(\mathbb{G})} \mathbb{1}_{\mathsf{x},\mathsf{y}} \mathbb{1}_{\mathsf{y},\mathsf{z}}}_{=0} \right) \cdot \mathsf{z}$$

We see total cancellation, hence $\tilde{\partial}^2_{\mathbb{O},\mathbb{X}} = 0$

blocked grid chain homology

$\tilde\partial$ is a diff. $GC_{d+1}(\mathbb{G},s) \xrightarrow{\tilde\partial} GC_d(\mathbb{G},s) \xrightarrow{\tilde\partial} GC_{d-1}(\mathbb{G},s)$

$\mathrm{Im}(\quad) \subset \partial$

$$\widetilde{GH}_d(\mathbb{G}, s) = \frac{\mathrm{Ker}(\tilde{\partial}_{\mathbb{O},\mathbb{X}}) \cap \widetilde{GC}_d(\mathbb{G}, s)}{\mathrm{Im}(\tilde{\partial}_{\mathbb{O},\mathbb{X}}) \cap \widetilde{GC}_d(\mathbb{G}, s)}$$

which then defines the following homology:

$$\widetilde{GH}(\mathbb{G}) = \underbrace{\bigoplus_{d,s\in\mathbb{Z}} \widetilde{GH}_d(\mathbb{G}, s)}_{},$$

- An interesting theorem: $\dim_{\mathbb{F}}(\widetilde{GH}(\mathbb{G}))/2^{n-1} \in \mathbb{Z}$ is a knot invariant.

grid number

Unblocked grid chain complex

$GC^- = \mathbb{F}[V_1, \ldots, V_n][S(G)]$

Let $\mathbb{F}[V_1, \ldots, V_n]$ be a ring, $\mathbb{O} = \{O_i\}_{i=1}^n$

$\begin{cases} 1 & \text{if } O_i \in r \\ 0 & \text{else} \end{cases}$

$$\partial_{\mathbb{X}}^-(\mathrm{x}) = \sum_{\mathrm{y} \in \mathrm{S}(\mathbb{G})} \sum_{\{r \in \mathrm{Rect}^\circ(\mathrm{x},\mathrm{y}) | r \cap \mathbb{X} = \varnothing\}} V_1^{O_1(r)} \ldots V_n^{O_n(r)} \cdot \mathrm{y}.$$

one to one

is a differential generating the chain complex GC^- and the homology GH^-

In essence allows $r \in \mathrm{R}^\circ(\mathrm{x}, \mathrm{y})$ to have $O_i \in r$

Simply blocked grid chain complex

Consider the quotient GC^-/V_i

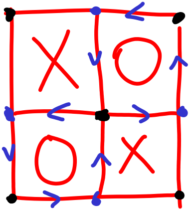
Then $\widehat{GC}(\mathbb{G}) = (GC^-/V_i, \partial_{\mathbb{X}}^-(\mathrm{x}))$ is a chain complex

with homology $\widehat{GH}(\mathbb{G})$.

The resulting homology is independent of the choice V_i. (possibly wrong)

not wrong.

Exercises ?



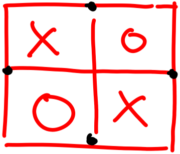
(1,2)

(2,1)

$\#\{ r \in \mathrm{Rect}^\circ(x,y) \mid r \cap \mathbb{O}, \ r \cap \mathbb{X} = \phi \}$

$\phi = \mathrm{Rect}(x,x)$

$$\tilde{\partial}_{\mathbb{O},\mathbb{X}}\big((1,2)\big) = \underline{\quad 0 \quad}\ (1,2) + \underline{\quad 0 \quad}\ (2,1)$$



$$\#\{ r \in \mathrm{Rect}^\circ((1,2),(2,1)) \} = 2$$

but if we take also crit. $r \cap \mathbb{O} = r \cap \mathbb{X} = \phi$

$$\partial^-_{\mathbb{X}}\big((1,2)\big) = 0 \cdot (1,2) + V_1 + V_2 \quad (2,1)$$