n-Colourability

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1 Introduction

In the lectures, we already saw the notion of 'three-colourability' come by. Here, we will generalise this idea to $n-$colourability for some $n \in \mathbb{N}$.
As it turns out, $n-$colourability is a knot invariant, and the set of $n-$colourings for a given knot diagram $Col_n(K)$ has the structure of an Abelian group. Moreover, we will give a bijection between $Col_n(K)$ and $HOM_{MR}(\mathcal{W}(K), D_{2n})$ where D_{2n} is the dihedral group defined in the usual way. Finally, we will define the determinant of a knot, and show the main theorem of these lecture notes;

A knot is $n-$ colourable if and only if n is divisible by the determinant of the knot.

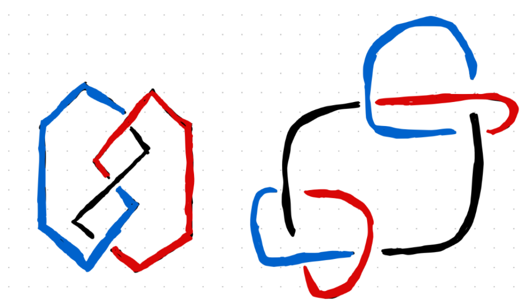
2 Revisiting Three-colourability

Let us start recalling the definition of three-colourability and expanding from there. We say that a diagram of a knot is *three-colourable* if all the edges of the diagram are coloured by an element of the set $\{a, b, c\}$ such that at each crossing either

- one colour meets, or,
- three colours meet.

Moreover, we require that edges that do not pass under other strands are coloured in the same colour.

Example 1. We already saw during the lectures that the trefoil knot is three-colourable. There are many more knots that can be coloured by 3 colours, like the granny knot. These knots are depicted below.



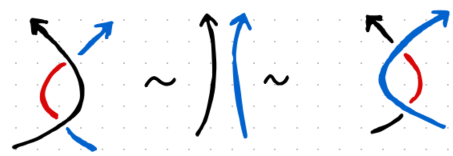
But why are we interested in three-colourability? As with many of the things we discussed in the first few weeks of the course, this is because $3-$coolourability is a knot invariant.

Theorem 1. *Three-colourability is a knot invariant.*

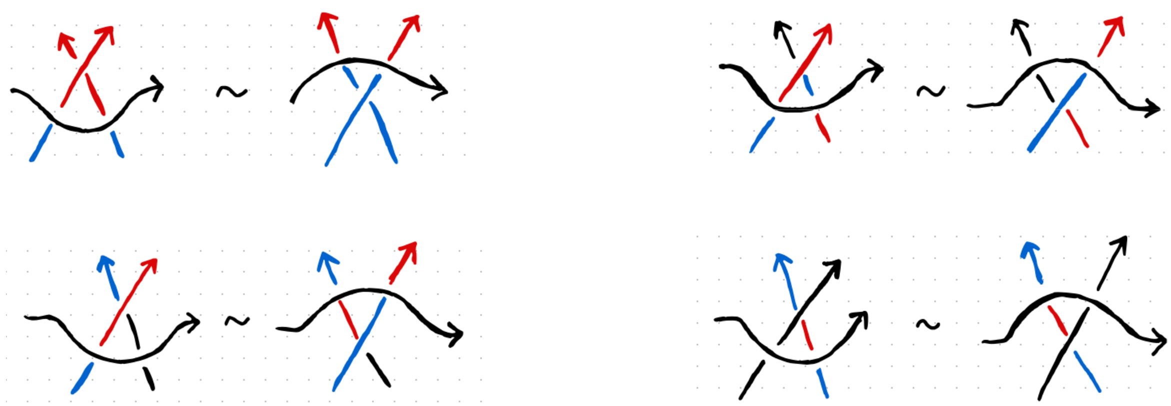
Proof. Let us prove this theorem by investigating what the Reidemeister moves do to a non-trivial colouring of a knot.

Ω1) :

Ω2) :



Ω3) :



We see that that the Reidemeister moves indeed respect the three-colourings, which finishes the proof. □

Remark. *Note that we can always colour knots with the trivial colouring, and that the Reidemeister moves preserve the colours of the strands trivially in this case. The interesting cases are the non-trivial colourings, for which the proof is shown.*

While three-colourability is a knot invariant, it is also quite boring. If we only consider non-trivial colourings into, this invariant only tells us 'yes' or 'no'. We can adapt the invariant to make it more interesting, not by considering *if* a knot is three-colourable, but rather *how many* distinct three colourings there are, and, what kind of structure the set of three-colourings has. This brings us to the following definition.

Definition 1. *We denote the set of distinct three-colourings of a knot by $Col_n(K)$, and we set $col_n(K) = \#Col_n(K)$.*

As the notation suggests, it turns out that $col_3(K)$ is also a knot invariant. To see this, we refer back to the proof of Theorem 1. Note that for each of the Reidemeister moves, once the colours of the incoming strands are fixed, also the colours of the outcoming strands and the strand(s) inbetween are fixed. In particular, this means that we cannot add or remove more three-colourings by performing Reidemeister moves. From this point onward, we will not explicity mention that we are considering a specific diagram of a knot when discussing three-colourability, as it is invariant.

Let us now discuss an interesting property of $col_3(K)$.

Theorem 2. *The number of distinct three-colourings $col_3(K)$ is a power of* 3.

Before we prove this theorem, we will adjust our definition for three-colourability to make the argument more obvious. This adjusted definition is also what we will use for the more general $n-$colourability.

Definition 2. *A knot diagram D is* three-colourable *if every edge is coloured by one of the numbers 0,1,2, in such a way that at each crossing, the sum of the undercrossings is equal to twice the sum of the overcrossing modulo 3.*

Note that this definition agrees with the definition we had for three-colourability before. If we consider any bijection from $\{r, b, g\}$ to $\mathbb{Z}/3\mathbb{Z}$, then it is clear that the restriction in the original definition reduces to Definition 2. Let us now prove Theorem 2.

proof of Theorem 2. Let us consider *any* colouring of our edges, and not just the allowed colourings. Then, it is clear that we can identify these colourings with $(\mathbb{Z}/3\mathbb{Z})^e$, where e is the number of edges in the knot diagram. Note that by definition, the 'good' colourings all have the property that at each crossing, we have that the sum of the colours is $0 \mod 3$. In particular, this means that in $(\mathbb{Z}/3\mathbb{Z})^e$, this characterizes a subgroup. Then, we conclude that $Col_3(K) = 3^n$ for some n [1]. □

3 n-Colourability

Let us now move to $n-$colourability. As mentioned before, the definition we will use is a generalisation of Definition 2.

Definition 3. *A knot diagram D is* n-colourable *if every edge is coloured by one of the numbers $0, 1, ..., n-1$ in such a way that at each crossing, the sum of the undercrossings is equal to twice the sum of the overcrossing modulo n.*

We can prove that $n-$colourability is an knot invariant in the same way as for three-colourability.

Theorem 3. *$n-$Colourability is a knot invariant.*

Proof. We check that $n-$colourability is invariant under the Reidemeister moves.
$\Omega 1)$: Let $a, b \in \mathbb{Z}/n\mathbb{Z}$ be as depicted in the figure below. Then we need the following inequality to hold:

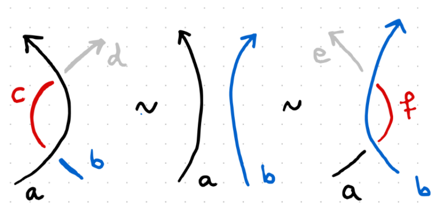
$$2a = a + b \mod n.$$

This implies that $a = b$.

$\Omega 2)$: Let $a, b, c, d, e, f \in \mathbb{Z}/n\mathbb{Z}$ be as depicted in the figure below. Then, we have the following two equalities

$$2a = b + c \mod n,$$
$$2a = c + d \mod n.$$

It follows that $b = d$. In a similar fashion, we can show that $a = e$.



$\Omega 3)$: Let $a, b, c, d, e, f, \tilde{c}, \tilde{e}, \tilde{f} \in \mathbb{Z}/n\mathbb{Z}$ be as depicted below. Then, we have the following 6 equalities

$$\begin{aligned} 2a &= b + c \mod n, & 2a &= \tilde{e} + \tilde{f} \mod n \\ 2a &= d + e \mod n, & 2a &= b + \tilde{c} \mod n, \\ 2c &= e + f \mod n, & 2b &= d + \tilde{e} \mod n. \end{aligned}$$

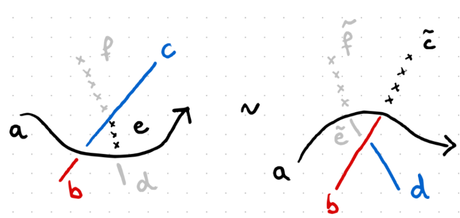
Combining these, we find

$$\begin{aligned} 2a &= b + c \mod n = b + \tilde{c} \mod n \Rightarrow c = \tilde{c} \mod n, \\ e &= 2a - d \mod n, \\ \tilde{e} &= 2b - d \mod n. \end{aligned}$$

Moreover, we see

$$\begin{aligned} f = 2c - e \mod n = 2c - 2a + d \mod n = (4a - 2b) - 2a + d \mod n &= 2a - 2b + d \mod n, \\ \tilde{f} = 2a - \tilde{e} \mod n &= 2a - 2b + d \mod n. \end{aligned}$$

Combining the above, we see that $c = \tilde{c}$ and $f = \tilde{f}$.



We conclude that $n-$ colourability is a knot invariant. □

From this point onward, we will not explicity mention that we are considering a specific diagram of a knot when discussing n-colourability, as it is invariant.

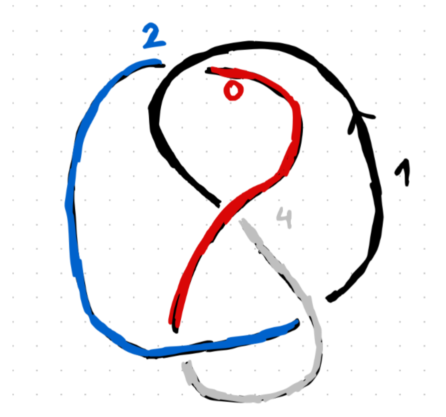
Similarly as before, we can argue that not just the $n-$colourability is a knot invariant, but also the amount of $n-$colourings. We denote the amount of $n-$colourings by $col_n(K)$, and the set of distinct $n-$colourings $Col_n(K)$.

Example 2. Let us consider the figure-of-eight knot. Remember that we showed in homework 3 that the figure-of-eight knot is *not* three-colourable. However, it is five-colourable! To see this, let us take a look at the figure below. We can pick any edge to start at. Let us start at the edge with the arrow on it, and colour it in colour 1. Then, when we traverse the knot and get to the first crossing, we want to find $a, b \in \mathbb{Z}/5\mathbb{Z}$ such that

$$2 \cdot 1 = a + b \mod 5.$$

We see that letting $a = 2$, $b = 0$ works, and we colour the 2 edges accordingly. Then, we get to the second crossing, where we need to find $c \in \mathbb{Z}/5\mathbb{Z}$ such that

$$2 \cdot 0 = 1 + c \mod 5.$$



We deduce that $c = 4$, and colour the remaining edge accordingly. When we traverse to the third crossing, we need that

$$2 \cdot 4 = 2 + 1 \mod 5,$$

which is satisfied as $8 \mod 5 = 3$. Similarly, for the last crossing, we see that twice the overstrand equals the sum of the understrands modulo 5!

Note that we jumped directly to $n = 5$, and that we did not consider $n = 4$. This is because the figure-of-eight knot is also not four-colourable. Showing this is not difficult, and a good exercise to see if you understood the definition of $n-$colourability.

4 Structure of $Col_n(K)$

In the introduction, we mentioned that $Col_n(K)$ can be given a structure of an Abelian group. In order for us to see this, we first need to define what the operation on our group is. We will simply set this to be adding colours strandwise[1], and we denote this operation by $+$.

Theorem 4. *($Col_n(K)$,+) forms an Abelian group.*

Proof. Let us start by checking that $+$ is well-defined. By definition, if we take some colouring $C \in Col_n(K)$, then we know that for the three edges meeting at some arbitrary crossing of K(coloured in e_1, e_2, e_3) we have the relation

$$2e_1 = e_2 + e_3 \mod n.$$

Similarly, for another colouring $C' \in Col_n(K)$, the three edges meeting at the same crossing of K (coloured in f_1, f_2, f_3) satisfy

$$2f_1 = f_2 + f_3 \mod n.$$

Then, we see that $C + C'$ at the same crossing satisfies

$$\begin{aligned} 2(e_1 + f_1) &= 2e_1 + 2f_1 \mod n \\ &= e_2 + e_3 + f_2 + f_3 \mod n \\ &= (e_2 + f_2) + (e_3 + f_3) \mod n. \end{aligned}$$

So the map $+$ is well-defined!

To show that $Col_n(K)$ with $+$ is a group, we need to show

[1]We showed that $Col_n(K)$ is a knot invariant, therefore, we can pick any diagram to represent our knot. Do note that for adding colourings, we need to consider colourings of the same diagram. We can add together two $n-$colourings of distinct diagrams in the sense that we first perform Reidemeister moves (and change colours of strands accordingly) to transform one diagram into the other, and then add colours strandwise.

- associativity,
- existence of an identity element,
- existence of an inverse element.

Note that associativity follows directly from the associativity of $\mathbb{Z}/n\mathbb{Z}$. The identity element is clearly given by the trivial colouring in colour 0, and the inverse element C^{-1} in the group is constructed by changing the colour of each edge from e_i to $-e_i \mod n$. Note that the inverse is well-defined. Indeed, at every crossing of C^{-1}, we have

$$\begin{aligned} 2(-e_1) = -(2e_1) &= -(e_2+e_3) \mod n \\ &= (-e_2)+(-e_3) \mod n. \end{aligned}$$

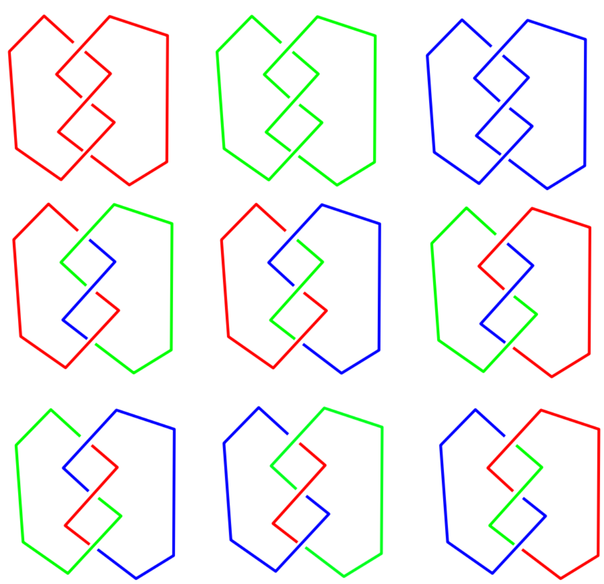
for the edges (coloured in $-e_1$, $-e_2$ and $-e_3$) meeting at the crossing.

Lastly, we note that $Col_n(K)$ is Abelian as $\mathbb{Z}/n\mathbb{Z}$ is Abelian. □

Let us note that we can always find n trivial colourings for a knot K. In particular, these trivial colourings form a subgroup of $Col_n(K)$. When we mod out our colourings by the trivial colourings, we get the *based n-colouring space*, denoted by $Col_n^0(K)$[2]. Now, note that we can write any colouring as a direct sum; namely of a trivial colouring, and of an element from $Col_n^0(K)$

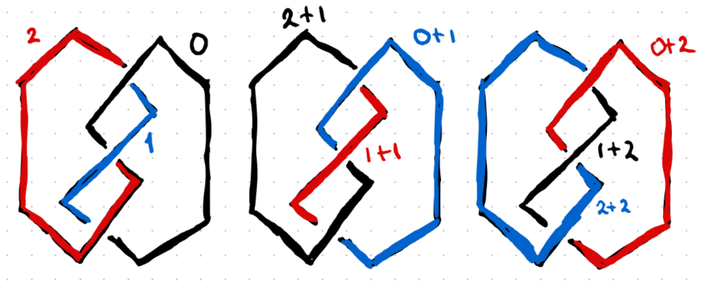
$$Col_n(K) \cong \mathbb{Z}/n\mathbb{Z} \oplus Col_n^0(K).$$

Example 3. Let us see an (easy) example of this. In the lectures, we discussed the trefoil knot, and its 9 colourings. Recall that 3 of these colourings were trivial; the other 6 were not.



To make calculations easier, let us call the colours $0, 1, 2$[2]. Now, let us consider the first nontrivial coloring of the trefoil knot. Then, by adding the (green/black) trivial colouring by the colour 0, we see that it does not change. However, when we add the trivial colourings by colours (blue) 1 and (red) 2, we see that these correspond the the 7th and 9th colourings respectively. Similarly, we can recover the 4th and the 5th colourings from adding trivial colourings to the 8th colouring. Now, we have found 2 elements of $Col_n^0(K)$.

[2]And let green correspond to black in my drawings :)



Note that we are still missing one element of $Col_n^0(K)$. We also notice that we are still missing the trivial colourings. To fix this, we simply add the trivial colouring of colour 0 (green) to $Col_n^0(K)$. Now, we see that

$$Col_n(\text{Trefoil}) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$$

Can we generalise more of what we have seen for three-colourability? The answer is yes! Recall that during the lectures, we discussed a bijection between the distinct three-colourings of a knot, and $HOM_{MR}(\mathcal{W}(K), D_6)$, where the last expression denotes the set of representations $\mathcal{W}(K) \to D_6$ by sending meridians to reflections. In particular, we know that for the trivial colourings, each meridian around every edge e get mapped to $\rho(e) = sc^k$ for some fixed $k \in \{0, 1, 2\}$. The non-trivial colours are a bit trickier. Let us consider some homomorphism between $\mathcal{W}(K)$ and D_6. We claim that such a homomorphisms[3] ϕ_C corresponding to some colouring $C \in Col_n(K)$ is given by

$$\phi_C(e_i) = sc^{k_i},$$

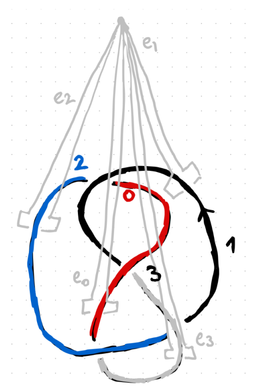
for all edges e_i of the knot[4]. Here, k_i denotes the colour with which we coloured the edge e_i. Note that this way of characterizing the homomorphisms agrees with what we discussed first for the trivial colourings.

We can recreate this same idea for $n-$colourings quite easily. For a given $n-$colouring C, we define a homomorhpism between $\mathcal{W}(K)$ and D_{2n}[5] via

$$\phi_C(e_i) = sc^{k_i},$$

where again, e_i are the edges of the knot, and k_i is the corresponding colour of the edge e_i [1].

Example 4. Let us consider the figure-of-eight knot as an example. We saw earlier that it is $5-$colourable. Let consider the same colouring as before, and see what the meridians are mapped to.



[3] Show that this truly is a homomorphism!

[4] Note that this notation is a bit sloppy. The homomorphism does not send e_i anywhere; it sends the meridian around e_i somewhere.

[5] Remember that D_{2n} is characterized by the generators s (reflection) and c (rotation), with the relations $s^2 = 1$, $c^n = 1$, and $scs = c^{-1}$

We see that

$$\begin{aligned}
\phi_C(e_0) &= sc^0 = s, \\
\phi_C(e_1) &= sc^1 = sc, \\
\phi_C(e_2) &= sc^2, \\
\phi_C(e_3) &= sc^3.
\end{aligned}$$

5 When is a Knot n-Colourable?

We have now seen quite some properties about $n-$colourability, as well as some examples (eg the figure-of-eight knot being $5-$colourable but not three-colourable). But can we find a criterium that tells us whether we can find (non-trivial!) $n-$colourings for a knot, without having to brute-force all the possibilities? For smaller knots, like the figure-of-eight, this is fine, but when our knot becomes more complex, this will clear become more and more difficult. As it turns out, we do have a criterium for $n-$colourability!

Theorem 5. *A knot is $n-$colourable if and only if n is divisible by the determinant of the knot.*

Before we can prove this theorem, we need to define what the determinant of a knot is. To aid with our understanding, we will be using the figure-of-eight knot through these definitions as running example. Let us consider some knot diagram K of our knot. Then, we can define the *pre-colouring matrix* the following way. We label our edges $e_1, ..., e_n$, and at each crossing, we write down the equation

$$e_j + e_k = 2e_l,$$

where e_j and e_k are the understrands, and e_l is the overstrand. Note that, if $n > 1$, we have exactly n crossings. We can now write down a system of n linear equations. If we write all of these linear equations as

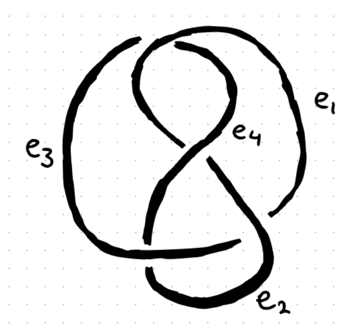
$$e_j + e_k - 2e_l = 0,$$

we see that we can now write this system as

$$\tilde{C}\vec{e} = \vec{0},$$

where $\tilde{C}$ is an $n \times n$ matrix. This $\tilde{C}$ is the pre-colouring matrix[3].

Example 5. Let us compute a pre-colouring matrix of the figure of eight knot.



Note that from this diagram, we get the linear equations

$$\begin{aligned}
e_3 + e_4 &= 2e_1, \\
e_1 + e_3 &= 2e_2, \\
e_2 + e_4 &= 2e_3, \\
e_1 + e_2 &= 2e_4.
\end{aligned}$$

In matrix form, this becomes

$$\begin{bmatrix} -2 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let us move back to the general case. By how the matrix was created, we see that the column vectors add up to 0, i.e. they are linearly dependent. Similarly, we see that the rows are also linearly dependent. As before, we can also predetermine the colour of a specific edge to reduce the system further. Then, we have obtained a $(n-1) \times (n-1)$ linear system. The matrix obtained is called the *colouring matrix* and is denoted by C. Finally, we can define the determinant of a knot.

Definition 4. *The* determinant of a knot *is given by the absolute value of the determinant of a colouring matrix of the knot.*

Example 5 continued. Let us find a colouring matrix of the figure-of-eight knot. When we reduce the dimensions of the matrix, we find

$$C = \begin{bmatrix} -2 & 0 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

Moreover, we can now compute the determinant of the figure-of-eight knot. We see that

$$Det(\text{figure-of-eight}) = \left| Det\left(\begin{bmatrix} -2 & 0 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \right) \right| = |-5| = 5.$$

Theorem 6. *The determinant is a knot invariant.*

Proof. The proof is found by looking at what the Reidemeister moves do to the matrix. For $\Omega 1$ and $\Omega 3$, they do not change the matrix at all, and for $\Omega 2$ it can be shown that the determinant does not change. The full proof can be found in [4]. □

An interesting fact is that we can recover the determinant of a knot from its Alexander polynomial.

Theorem 7. *Let K be a knot, and let $\Delta_K(t)$ be its Alexander polynomial. Then, we have that $Det(K) = |\Delta_K(-1)|$.*

Proof. The proof can be found in [5]. □

Let us now prove Theorem 5.

proof of Theorem 5. ($\Rightarrow$) Let us assume that a knot is $n-$colourable. Then, we see that there must be a nontrivial solution to the linear system

$$C\vec{e} = \vec{0}$$

in $\mathbb{Z}/n\mathbb{Z}$. This can only happen when C is singular, i.e., when $Det(C) = 0 \mod n$. It follows that n divides $Det(C)$.

($\Leftarrow$) Let us assume that for some knot K, n divides $Det(K) = Det(C)$ for some colour matrix C. Then, we know that $Det(C) = 0 \mod n$, so C is singular in $\mathbb{Z}/n\mathbb{Z}$. This implies that there is a non-trivial solution to the system

$$C\vec{e} = \vec{0}$$

in $\mathbb{Z}/n\mathbb{Z}$. It follows that the knot is $n-$colourable. □

References

[1] Józef H. Przytycki. "3-coloring and other elementary invariants of knots". In: ().

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