LECTURE 4: ALGEBRAS OVER A FIELD

In this lecture we give the first step towards the construction of the algebraic knot invariants that we are interested in studying in this course.

Throughout this lecture we let k be a field.

1. Review of tensor products

The tensor product will play a mayor role in this course, so let us start by recalling this operation.

Definition 1. Let V, W be k-vector spaces. The *tensor product* of V and W, if it exists, is a k-vector space $V \otimes_k W$ together with a bilinear map $\pi : V \times W \longrightarrow V \otimes_k W$ with the following universal property: given another k-vector space E and a bilinear map $\phi : V \times W \longrightarrow E$, there exists a unique linear map $\bar{\phi} : V \otimes_k W \longrightarrow E$ such that $\bar{\phi} \circ \pi = \phi$.

In other words, we have the following bijection:

$$\mathrm{Hom}_k(V \otimes_k W, E) \xrightarrow{\cong} \mathrm{Bilin}_k(V, W; E)$$
$$\bar{\phi} \mapsto \bar{\phi} \circ \pi$$

Proposition 2. *The tensor product of two vector spaces exists and it is unique.*

Proof. Uniqueness. If $V \otimes_k W$ and $V \tilde{\otimes}_k W$ satisfy the universal property, with bilinear maps $\pi : V \times W \longrightarrow V \otimes_k W$ and $\tilde{\pi} : V \times W \longrightarrow V \tilde{\otimes}_k W$, the universal property produces a unique map $f : V \otimes_k W \longrightarrow V \tilde{\otimes}_k W$ such that $f \circ \pi = \tilde{\pi}$ and similarly a unique map $g : V \tilde{\otimes}_k W \longrightarrow V \otimes_k W$ such that $g \circ \tilde{\pi} = \pi$. This implies that $\pi = (g \circ f) \circ \pi = \pi$. Since $\pi = \mathrm{Id}_{V \otimes_k W} \circ \pi$, the uniqueness of the universal property concludes that $g \circ f = \mathrm{Id}_{V \otimes_k W}$. Argueing similarly we obtain $f \circ g = \mathrm{Id}_{V \tilde{\otimes}_k W}$, so we conclude that $f : V \otimes_k W \xrightarrow{\cong} V \tilde{\otimes}_k W$ is an isomorphism with inverse g.

Existence. Recall that given a set X, the *free k-vector space generated by X* is the vector space $k[X]$ whose bases is given by the elements of X. Now define

$$V \otimes_k W := \frac{k[V \times W]}{U}$$

where U is the linear subspace generated by the elements

$$(v^1 + v^2, w) - (v^1, w) - (v^2, w) \quad , \quad (v, w^1 + w^2) - (v, w^1) - (v, w^2),$$
$$(\lambda v, w) - \lambda(v, w) \quad , \quad (v, \lambda w) - \lambda(v, w)$$

for all $v, v^1, v^2 \in V, w, w^1, w^2 \in W$, $\lambda \in k$. Note this vector space comes with a canonical map $\pi : V \times W \longrightarrow V \otimes_k W$. Now the universal property of the tensor product follows by the universal property of the quotient vector space. □

Remark 3. Given $(v, w) \in V \times W$, we write $v \otimes w := \pi(v, w)$. Note that since a general element of $k[V \times W]$ is a finite sum $\sum_i^N (v^i, w^i)$, a general element of $V \otimes_k W$ can be written (not uniquely!) as a finite sum $\sum_i^N v^i \otimes w^i$. This also means that the elements of the form $v \otimes w$ span $V \otimes_k W$. In particular, it implies that to define a map $V \otimes_k W \longrightarrow E$ it suffices to specify the map for the elements $v \otimes w$ and check that it is bilinear in each factor. Of course, this is a rephrasing of the universal property given in the definition.

Proposition 4. *The tensor product satisfies the following properties:*

(1) $V \otimes_k W \cong W \otimes_k V$,
(2) $k \otimes_k V \cong V$,
(3) $(V \otimes_k W) \otimes_k E \cong V \otimes_k (W \otimes_k E)$,
(4) $\left(\bigoplus_{i \in I} V_i\right) \otimes_k W \cong \bigoplus_{i \in I}(V_i \otimes_k W)$,

(5) Two linear maps $f : V \longrightarrow W$ and $g : V' \longrightarrow W'$ induces a linear map $f \otimes g : V \otimes_k V' \longrightarrow W \otimes_k W'$.

Notation 5. From now on we will put $\otimes := \otimes_k$ unless otherwise stated.

2. Unordered tensor products

Observe that Proposition 4.(3) is telling us that the tensor product is associative *up to isomorphism*. This means that given k-vector spaces $V_1, \dots, V_n$, any two paranthesisations of $V_1 \otimes \cdots \otimes V_n$ will be isomorphic. We will then denote any of these isomorphic paranthesisations as $V_1 \otimes \cdots \otimes V_n$.

Similarly, Proposition 4.(1) ensures that any two permutation of the indices in $V_1 \otimes \cdots \otimes V_n$ will produce isomorphic vector spaces. For reasons that will become clear later, we are interested in the tensor product of vector spaces where the spaces are not indexed by natural numbers $1, \dots, n$ but instead they are indexed by a finite set S, and the tensor product is taken indexed by S.

Let us be precise about this idea: let S be a finite set (**warning:** no order is assumed in S), and let $(V_s)_{s \in S}$ be a family of k-vector spaces indexed by S. If $n := \#S$, then every bijection

$$\sigma : \{1, \dots, n\} \xrightarrow{\cong} S$$

induces an n-fold tensor product

$$V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)}.$$

Proposition 4.(1) says that for any two such bijections σ, σ', we have an isomorphism

$$V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)} \cong V_{\sigma'(1)} \otimes \cdots \otimes V_{\sigma'(n)}.$$

Definition 6. If S is a finite set and $(V_s)_{s \in S}$ is a family of k-vector spaces indexed by S, then we will denote any of the isomorphic vector spaces from above as

$$\bigotimes_{s \in S} V_s.$$

If $V_s = V$ for all $s \in S$, this vector space will be denoted as $V^{\otimes S}$.

Given any bijection $\sigma : \{1, \dots, n\} \xrightarrow{\cong} S$, we will denote an $v^1 \otimes \cdots \otimes v^n \in V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)}$ viewed in $\bigotimes_{s \in S} V_s$ as

$$v^1_{\sigma(1)} \cdots v^n_{\sigma(n)},$$

where the order here does not matter (as the factor for which every v^i belongs to is already tracked), that is,

$$v^1_{\sigma(1)} \cdots v^n_{\sigma(n)} = v^{p(1)}_{\sigma(p(1))} \cdots v^{p(n)}_{\sigma(p(n))}$$

for any permutation $p \in \Sigma_n$.

Example 7. Let $V = \mathbb{R}^2$, $e^1 := (1, 0)$, $e^2 = (0, 1)$ and let $S := \{\clubsuit, \blacktriangle\}$. Then the following is an element of $V^{\otimes S}$:

$$2e^2_{\clubsuit} e^2_{\blacktriangle} - \frac{1}{33} e^1_{\clubsuit}(3e^1_{\blacktriangle} - 5e^2_{\blacktriangle}) + 9e^1_{\blacktriangle} e^2_{\clubsuit}.$$

3. Algebras over a field

Definition 8. An *algebra* over k is a k-vector space A together with a k-linear maps

$$m : A \otimes A \longrightarrow A,$$

called the *multiplication* map, and an element $1 \in A$, called the *unit*, satisfying the following properties:

(i) (Associativity): $m \circ (m \otimes \mathrm{Id}_A) = m \circ (\mathrm{Id}_A \otimes m)$,

(ii) (Unitality): $m \circ (1 \otimes \mathrm{Id}_A) = \mathrm{Id}_A$ and $m \circ (\mathrm{Id}_A \otimes 1) = \mathrm{Id}_A$, where $1 \otimes \mathrm{Id}_A : A \longrightarrow A \otimes A$ is the map $a \mapsto 1 \otimes a$ (and similarly for $\mathrm{Id}_A \otimes 1$).

If furthermore $P_{A,A} \circ m = m$, where $P_{A,A} : A \otimes A \longrightarrow A \otimes A$, $P_{A,A}(a \otimes b) := b \otimes a$, then we say that A is *commutative*. As usual, we put $m(a \otimes b) =: ab$.

A linear subspace $A' \subset A$ is a *subalgebra* of A if it is closed under multiplication and $1 \in A'$.

A linear map $f : A \longrightarrow B$ between k-algebras is an *algebra homomorphism* if $f \circ m_A = m_B \circ (f \otimes f)$ and $f(1_A) = 1_B$.

Exercise 9. (1) Let $I \subset A$ be a *two-sided ideal* of A, that is, a linear subspace such that $m(I \otimes A) \subset I$ and $m(A \otimes I) \subset I$. Then there exists a unique k-algebra structure on A/I such that the projection $A \longrightarrow A/I$ is a k-algebra homomorphism.

(2) Suppose that $(A, m, 1)$ and $(A', m', 1')$ are k-algebras. Then there is an algebra structure on the tensor product $A \otimes A'$ in which the multiplication map is given by

$$(m \otimes m') \circ (\mathrm{Id}_A \otimes P_{A',A} \otimes \mathrm{Id}_{A'}) : (A \otimes A') \otimes (A \otimes A') \longrightarrow A \otimes A'$$

and the unit is given by $1 \otimes 1' \in A \otimes A'$.

Examples 10. (1) The polynomial algebra $k[x_1, \ldots, x_n]$ in n unknowns is a k-algebra with the usual multiplication of polynomials and the constant polynomial 1 serving as the unit.

(2) The set of square matrices $\mathcal{M}_n(k)$ of order n with coefficients in k is a k-algebra with the usual matrix multiplication and the identity matrix serving as the unit.

(3) For any k-vector space V, its *tensor algebra* is the vector space

$$T(V) := \bigoplus_{n=0}^{\infty} T^n(V)$$

where $T^n(V) := V \otimes \overset{n}{\cdots} \otimes V$ if $n > 0$ and $T^0(V) := k$. This is an algebra over k where the product is induced by the canonical isomorphism

$$T^n(V) \otimes T^k(V) \xrightarrow{\cong} T^{n+k}(V)$$

and the unit is given by $1 \in T^0(V)$.

(4) Let $I(V) \subset T(V)$ denote the two sided ideal generated by $v \otimes v$ for all $v \in V$, that is, the linear subspace generated by elements

$$v^1 \otimes \cdots \otimes v^n \qquad , \qquad n \geq 2,\ v_i \in V,$$

such that $v^i = v^j$ for some $i \neq j$. The quotient

$$\Lambda(V) := T(V)/I(V)$$

is called the *exterior algebra* of V and both its multiplication and unit are inherited from those of $T(V)$.

(5) Let G be a group. The *group algebra* of G over k is the free vector space $k[G]$ with the following multiplication map induced by the multiplication $m : G \times G \longrightarrow G$:

$$k[G] \otimes k[G] \xrightarrow{\cong} k[G \times G] \xrightarrow{k[m]} k[G].$$

Explicitly,

$$\left(\sum_{i=1}^{n} \lambda_i g_i\right)\left(\sum_{j=1}^{s} \mu_j g'_j\right) := \sum_{i=1}^{n}\sum_{j=1}^{s} \lambda_i \mu_j (g_i g'_j),$$

where the last $g_i g'_j$ is the product of g_i and g'_j in G. If $e \in G$ is the unit of G, then the element $1e \in k[G]$ serves as the unit for this algebra.

(6) The algebra $\mathrm{End}(V)$ of endomorphisms of a vector space, with product given by composition, $f \cdot g := f \circ g$, and unit given by the identity map Id_V. By choosing a basis for V, this algebra is isomorphic to that of (2).

Remark 11. The multiplication of an algebra A is a linear map $m : A \otimes A \longrightarrow A$. However we mentioned earlier that we are interested in unordered tensor powers of an algebra, so instead of m as above we really want to use an unordered version

$$m : A^{\otimes S} \longrightarrow A^{\otimes U}$$

where $\#S = 2$ and $\#U = 1$, and for concreteness we put $S = \{s, t\}$ and $U = \{u\}$. In doing so, we must indicate which of the isomorphic copies $A_s \otimes A_t$ or $A_t \otimes A_s$ we are thinking of when writing $A^{\otimes S}$ (here $A = A_s = A_t$). To do so, we simply write $m_u^{s,t}$ or $m_u^{t,s}$ to express whether we refer to $m : A_s \otimes A_t \longrightarrow A_u$ or $m : A_s \otimes A_t \longrightarrow A_u$, respectively.

More precisely,

$$m_u^{s,t} : A^{\otimes S} \longrightarrow A^{\otimes U}$$

is the map sending $a_s b_t = b_t a_s$ to $(ab)_u$, whereas

$$m_u^{t,s} : A^{\otimes S} \longrightarrow A^{\otimes U}$$

is the map sending $a_s b_t = b_t a_s$ to $(ba)_u$.

This notation is especially handy if one wants to multiply only two factors in a large fold tensor product of an algebra. Recall that given two (finite) sets S_1, S_2, its *disjoint union* is the set $S_1 \amalg S_2$ whose elements are[1] those of S_1 and S_2. Observe that

$$A^{\otimes S_1} \otimes A^{\otimes S_2} \cong A^{\otimes (S_1 \amalg S_2)}.$$

Then for a finite set S, we denote by

$$m_z^{x,y} : A^{\otimes (S \amalg \{x,y\})} \longrightarrow A^{\otimes (S \amalg \{z\})}$$

the map

$$\mathrm{Id}_{A^{\otimes S}} \otimes m_z^{x,y} : A^{\otimes S} \otimes A^{\otimes \{x,y\}} \longrightarrow A^{\otimes S} \otimes A^{\otimes \{z\}}.$$

4. Presentation of an algebra

Next we would like to study a general strategy to produce algebras, namely by giving a presentation in terms of generators and relations. In fact, we will see that all algebras arise in this way.

Definition 12. Let S be a set. The *free k-algebra generated by S*, if it exists, is a k-algebra $k\langle S\rangle$ together with a set-theoretic map $i : S \longrightarrow k\langle S\rangle$ satisfying the following universal property: if A is a k-algebra and $f : S \longrightarrow A$ is a set-theoretic map, then there exists a unique k-algebra homomorphism $\hat{f} : k\langle S\rangle \longrightarrow A$ such that $\hat{f} \circ i = f$.

In other words, we have a bijection

$$\mathrm{Hom}_{k-\mathrm{alg}}(k\langle S\rangle, A) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{Set}}(S, A),$$

$$\hat{f} \mapsto \hat{f} \circ i.$$

that is, to define a map from a free k-algebra on a set, we only have to specify the images of the generators.

Proposition 13. *The free k-algebra generated by a set exists and it is unique.*

Proof. The uniqueness is proven as that of Proposition 2, so we will focus here on the existence.

Let S be a set, and write $M(S)$ for the set of all possible words (ie finite sequences of letters) $s_1 \dots s_p$ in the alphabet S, where $s_i \in S$ and $p \geq 0$. For $p = 0$ we mean the empty word $\emptyset$. Put $k\langle S\rangle := k[M(S)]$ for the free vector space generated by $M(S)$. This has the structure of k-algebra where the multiplication induced by concatenation of words and unit given by $1\emptyset$. That this algebra satisfies the universal property follows from the equality

$$f(s_1 \dots s_p) = f(s_1) \cdots f(s_p),$$

where $\hat{f} : k\langle S\rangle \longrightarrow A$ is an algebra map. □

Remark 14. If $S = \{x_1, \dots, x_n\}$, we put $k\langle x_1, \dots, x_n\rangle := k\langle S\rangle$. In particular, we have

$$\mathrm{Hom}_{k-\mathrm{alg}}(k\langle x_1, \dots, x_n\rangle, A) \xrightarrow{\cong} A \times \overset{n}{\cdots} \times A \qquad , \qquad g \mapsto (g(x_1), \dots, g(x_n)).$$

Definition 15. A *presentation* for a k-algebra A is an algebra isomorphism

$$k\langle S\rangle / I \xrightarrow{\cong} A$$

for some set S and some two-sided ideal $I \subset k\langle S\rangle$. If A has a presentation in which S is finite and I is finitely generated, we say that A is *finitely presented.* In such a case, the generators of I are called the *relations.*

[1]The set $\mathbb{R} \amalg \mathbb{R}$ has two *different* 7's, two different 33's, etc and to avoid confusion one can relabel the 7 in the second copy of $\mathbb{R}$ in a different way, but that is a matter of how to label elements. Please be mindful that this is different than union $\cup$ of subsets inside a bigger set.

Remark 16. It follows from above and the universal property of the quotient that for a finitely-presented algebra $A \cong k\langle x_1, \ldots, x_n\rangle/(r_1, \ldots, r_m)$, where $r_i = r_i(x_1, \ldots, x_n)$, we have a bijection

$$\mathrm{Hom}_{k-\mathrm{alg}}(A, B) \xrightarrow{\cong} \{(b_1, \ldots, b_n) \in B^{\times n} : r_i(b_1, \ldots, b_n) = 0\}$$
$$f \mapsto (f(x_1), \ldots, f(x_n))$$

where B is a k-algebra.

Note that every k-algebra A has a presentation, for consider $S := A$ and $I := \ker(k\langle A\rangle \longrightarrow A)$. However, we will be mostly interested in finitely-presented algebras, as they are easier to handle.

Examples 17.
(1) The polynomial algebra $k[x_1, \ldots, x_n]$ from Examples 10.(1) is finitely-presented, for take $\{x_1, \ldots, x_n\}$ as generators and $x_ix_j - x_jx_i$, $1 \leq i, j \leq n$, as relations. It is usual to write the relations as $x_ix_j = x_jx_i$.
(2) The group algebra $k[\mathbb{Z}/2]$ of the cyclic group $\mathbb{Z}/2$ is also finitely-presented, for take a single generator x with the relation $x^2 = 1$.

The mayor advantage of algebra presentations is that one can simply *define* and algebra via generators and relations and, in fact, this will be most of the times the approach we will take.

5. Examples

Let us wrap up this lecture by giving some examples of finitely-presented algebras. The following examples will be key in the forthcoming lectures.

Example 18. Let A be the $\mathbb{C}$-algebra generated by two elements s, w subject to the relations

$$s^2 = 1 \qquad , \qquad w^2 = 0 \qquad , \qquad sw + ws = 0.$$

This algebra is usually called the *Sweedler algebra*.

Example 19. We write $U(\mathfrak{sl}_2(\mathbb{C})) = U(\mathfrak{sl}_2)$ for the $\mathbb{C}$-algebra generated by the elements X, Y, H and relations

$$[H, X] = 2X \qquad , \qquad [H, Y] = -2Y \qquad , \qquad [X, Y] = H$$

where $[u, v] := uv - vu$.

Remark 20. The previous example can be viewed as the "algebraification" of the set of matrices

$$\mathfrak{sl}_2(\mathbb{C}) := \{M \in \mathcal{M}_2(\mathbb{C}) : \mathrm{tr}(M) = 0\},$$

where the trace tr of a matrix is the sum of the diagonal elements. Indeed if

$$X' := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad , \qquad Y' := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad , \qquad H' := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then this elements satisfy the same relations as the generators of $U(\mathfrak{sl}_2)$. In fact, $\mathrm{span}_{\mathbb{C}}\{X, Y, H\} \subset U(\mathfrak{sl}_2)$ is linearly isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ and this isomorphism preserves the brackets. The precise statement here is that "$U(\mathfrak{sl}_2)$ is the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$", but we will not delve into that.

Next we would like to modify the previous example of $U(\mathfrak{sl}_2)$ by introducing an extra parameter q (that physicists call "deformation" or "quantisation"), that will introduce an extra level of complexity in the algebra and will allow us to define more powerful knot invariants.

Definition 21. Let k be a field. The field of *rational functions* in the unknown q with coefficients in k is

$$k(q) := \left\{ \frac{f(q)}{g(q)} : f, g \in k[q],\ g \neq 0 \right\}.$$

Example 22. Let $U_q(\mathfrak{sl}_2(\mathbb{C})) = U_q(\mathfrak{sl}_2)$ be the $\mathbb{C}(q)$-algebra generated by X, Y, K, K^{-1} subject to the relations

$$KK^{-1} = 1 = K^{-1}K \quad , \quad KX = q^2XK \quad , \quad KY = q^{-2}YK \quad , \quad [X, Y] = \frac{K - K^{-1}}{q - q^{-1}}.$$

In the forthcoming lectures we will see that, out of this algebra, we will be able to recover the Jones and Alexander polynomial of links.

The obvious question to ask is: how do $U(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_2)$ relate? The slogan is that "$U(\mathfrak{sl}_2)$ is the limit of $U_q(\mathfrak{sl}_2)$ when $q \to 1$". Formally, this statement does not make sense (what is a limit in an algebra? What happens with the $q - q^{-1}$ denominator?). Let us give a precise statement of this.

Proposition 23. *Let $\widetilde{U}_q(\mathfrak{sl}_2)$ be the $\mathbb{C}(q)$-algebra generated by X, Y, K, K^{-1}, H subject to the relations*

$$KK^{-1} = 1 = K^{-1}K \quad , \quad KX = q^2XK \quad , \quad KY = q^{-2}YK \quad , \quad (q - q^{-1})H = K - K^{-1}$$
$$[X, Y] = H \quad , \quad [H, X] = q(XK + K^{-1}X) \quad , \quad [H, Y] = -q^{-1}(YK + K^{-1}Y).$$

Then the natural $\mathbb{C}(q)$-algebra homomorphism

$$\varphi : U_q(\mathfrak{sl}_2) \xrightarrow{\cong} \widetilde{U}_q(\mathfrak{sl}_2)$$

is a $\mathbb{C}(q)$-algebra isomorphism.

Proof. It is left to the reader to show, using Remark 16, that the map $\psi : \widetilde{U}_q(\mathfrak{sl}_2) \longrightarrow U_q(\mathfrak{sl}_2)$ given by

$$\psi(X) := X \quad , \quad \psi(Y) := Y \quad , \quad \psi(K) := K \quad , \quad \psi(H) := [X, Y]$$

is well-defined and it is the inverse of φ. □

The algebra $\widetilde{U}_q(\mathfrak{sl}_2)$ of the previous proposition is an algebra over $\mathbb{C}(q)$. However, the set of relations also make perfect sense if q is replaced by any complex number $z \in \mathbb{C}$. The same set of generators with the same set of relations (with $q = z$) define a $\mathbb{C}$-algebra $\widetilde{U}_z(\mathfrak{sl}_2)$. We are mostly interested in the case $z = 1$.

Proposition 24. *The natural map*

$$\phi : U(\mathfrak{sl}_2) \xrightarrow{\cong} \widetilde{U}_1(\mathfrak{sl}_2)/(K - 1)$$

is a $\mathbb{C}$-algebra isomorphism.

Proof. We simply note that in the quotient $\widetilde{U}_1(\mathfrak{sl}_2)/(K - 1)$ the first four defining relations for $\widetilde{U}_q(\mathfrak{sl}_2)$ are vacuous, and the remaining three become the defining relations for $U(\mathfrak{sl}_2)$. □