# Slice and ribbon knots

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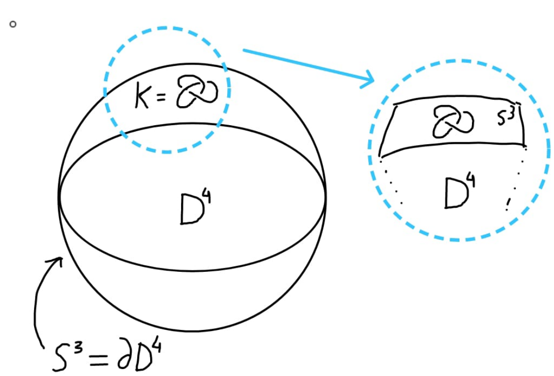
## 1 Slice knots

### 1.1 Generalizing the unknot

So far, our focus was to study knots in $\mathbb{R}^3$ up to isotopy. In particular, we can think of the unknot as an (isotopy) equivalence class of a perfect circle included in our three-dimensional space. Our goal is to obtain an equivalence relation weaker than isotopy by adding, in some sense, an extra dimension. This cannot be done by considering knots in $R^4$ since they would be all isotopic to the unknot.

Before diving into this generalization we need to make a small adjustment to how we think about our ambient space. Instead of $\mathbb{R}^3$ we should now think of a knot to be included in $S^3 = \partial D^4$. Notice that $S^3$ is the 1-point compactification of $\mathbb{R}^3$, i.e. "$S^3 = R^3 \cup \{\infty\}$". Since our knot is, in some sense a copy of $S^1$, it is compact, and this addition of a point at infinity does not interfere with the classical results of knots in $\mathbb{R}^3$.

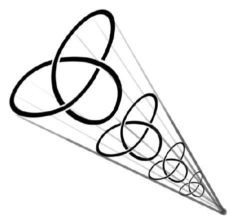
Bellow, we pictorially describe the knot inside $S^3$. Note that we are constrained by our 3-dimensional existence so we have to always visualize $S^2$ instead!




Now, let us note that the usual unknot can be defined to be **the one who bounds a disk $\mathbf{D^2}$**, inside $S^3$. Our generalization now is to allow this disk to also live in the "inside" of the shell $S^3$, i.e. in all $D^4$. Then, we have (almost) defined a slice knot. The precise definition is presented below:

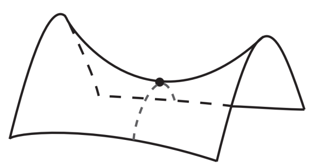
**Definition 1.** A knot $K \subset S^3$ is **slice** if it bounds a locally flat disk $D^2$ embedded into the 4-ball $D^4$.

**Remark 1.** We say that $D^2 \hookrightarrow D^4$ is locally flat if for each point $x \in D^2$ there is a neighbourhood $U$ of $x$ and a neighbourhood $V$ of $q(x)$ such that the pair $(V, q(U))$ can be mapped homeomorphically onto $(\mathrm{int}(D^4), \mathrm{int}(D^2))$. Removing the local flatness condition would make every knot to be slice. This is the case since the cone generated by any knot is a disk in $D^4$ (but local flatness fails at the vertex).

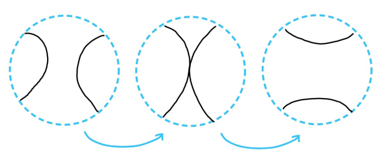


Now let us see what we can do besides isotopy to prove that some knot is slice. Locally we can use the "extra dimension" to obtain different moves.

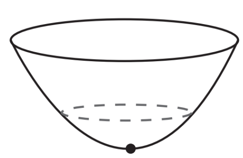
Consider, locally, the following paraboloid:



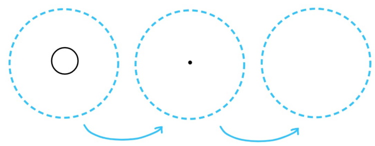
Then the horizontal cross sections of this create the following **movie** when descending on the paraboloid:



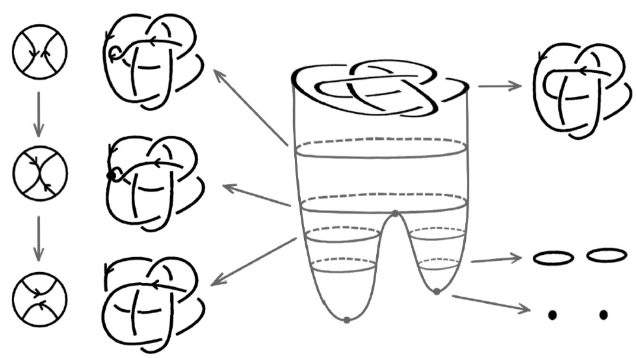
The use of the word movie is justified when we think about the "inward" extra dimension to be the time. Each frame in the movie will be a cross section of our $D^2$. We can do the exact same with the other type of paraboloid:



obtaining the following movie:



**Example 1.** Consider the knot $8_{20}$. The following movie (using the paraboloid moves presented above) proves that the knot is slice by constructing the disc $D^2$ which the knot bounds, in $D^4$.



Now, the natural question to ask is *Are all knots slice?* Fortunately, the answer is *No!*

# 2 Alexander polynomial of slice knots

Now we will deal with the question *How can we prove that a knot is not slice?*. Unsurprisingly, this motivates us to look at the properties of knot invariants for slice knots. In particular, let us study the Alexander polynomial.

**Fact 1.** [1] Any slice knot has a block Seifert matrix $V$ with a zero block matrix in the top left block, i.e. $V = \begin{bmatrix} 0 & P \\ Q & R \end{bmatrix}$.

Using this, we can get some restrictions on the form of the Alexander polynomial for these types of knots:

**Theorem 1.** If the knot $K$ is slice then $\Delta_K(t) = f(t)f(t^{-1})$ for some polynomial $f \in \mathbb{Z}[t]$.

[1] In section 4 we will describe the Seifert surface for some of these knots which allows us to find the Seifert matrix.

**Proof.** By definition, $\Delta_K(t) = \det(t^{1/2}V - t^{-1/2}V^T)$. By the previous fact, we have $V = \begin{bmatrix} 0 & P \\ Q & R \end{bmatrix}$ where $V$ is the Seifert matrix for $K$ and $P, Q, R$ are $g \times g$ matrices. Then we can conclude that

$$\begin{aligned}
\Delta_K(t) &= \det(-(t^{1/2}Q - t^{-1/2}P^T)(t^{1/2}P - t^{-1/2}Q^T)) \\
&= (-1)^g \det(t^{1/2}Q - t^{-1/2}P^T)\det(t^{1/2}P - t^{-1/2}Q^T) \\
&= (-1)^g \det(t^{1/2}Q - t^{-1/2}P^T)\det((t^{1/2}P - t^{-1/2}Q^T)^T) \\
&= (-1)^g \det(t^{1/2}Q - t^{-1/2}P^T)\det(t^{1/2}P^T - t^{-1/2}Q) \\
&= (-1)^g \det(t^{-1/2}(tQ - P^T))\det(-t^{1/2}(t^{-1}(Q - P^T)) \\
&= (-1)^g(-1)^g(t^{1/2})^g(t^{-1/2})^g \det(tQ - P^T)\det(t^{-1}Q - P^T) \\
&= f(t)f(t^{-1})
\end{aligned} \tag{1}$$

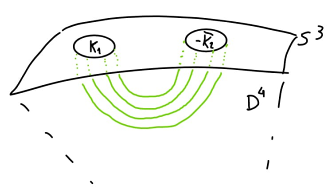
Where $f(t) = \det(tQ - P^T)$.

With this last theorem, we can hope to prove that some knot is not sliced by proving that its Alexander polynomial does not factor that way! For example, let us look at the figure eight knot:

**Example 2.** Recall that the figure eight knot has Alexander polynomial given by $\Delta(t) = -t + 3 - t^{-1}$. Then $\Delta(-1) = 5$. But if $\Delta(t) = f(t)f(t^{-1})$ for some integer polynomial $f$, we would have $\Delta(-1) = f(1)f(1) = f(1)^2$. Since 5 is not a perfect square, we can conclude that the knot is not slice.

# 3 Steps towards concordance

In this section, we will begin the definition of concordance, the weaker equivalence relation mentioned in the beginning. Naively, we can think about concordance as the existence movie that takes place in the interior of $D^4$ that transforms a knot into the mirror on another knot:




**Definition 2.** [2] Two knots $K_1$ and $K_2$ are **concordant** if $K_1 \# -\bar{K}_2$ (where $-\bar{K}$ is the mirror image of $K$) is a slice knot.

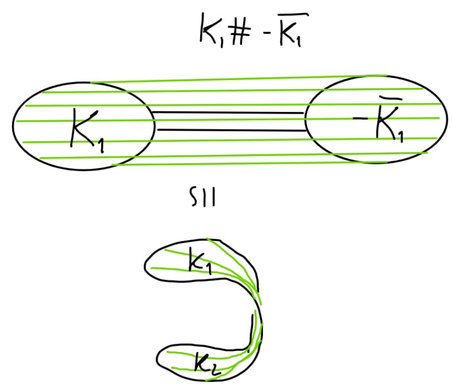
The following result proves that concordance is an equivalence relation.

[2]In some literature, this is presented as a result and not a definition.

**Theorem 2.**

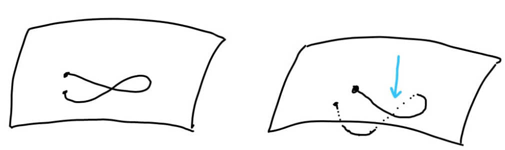
- $K\# - \bar{K}$ is always slice (*Reflexivity*),
- If $K_1\# - \bar{K}_2$ is slice then $K_2\# - \bar{K}_1$ is also slice (*Symmetry*),
- If $K_1\# - \bar{K}_2$ and $K_2\# - \bar{K}_3$ are slice, then $K_1\# - \bar{K}_3$ is also slice (*Transitivity*).

**Proof.** (*Reflexivity*) First, note that there are natural inclusions $K \hookrightarrow K\#-\bar{K}$ and $-\bar{K} \hookrightarrow K\#-\bar{K}$. Now connect every point $x$ on $K \subset K\#-\bar{K}$ to its mirror image in $-\bar{K} \subset K\# - \bar{K}$, using line segments:




Note that, the space constructed this way is always a disk $D^2$, possibly with some self-intersections. Now we shall use the extra dimension to get rid of those self-intersections. In a small open around each intersection (in $S^3$) push one of the self-intersecting pieces to the interior of $D^4$. This way we can obtain the desired disk $D^2$ inside $D^4$.

Below we present a picture describing this procedure in one less dimension, i.e. starting with a self-intersecting $D^1$ in $S^2$ and pushing the intersection through the 3rd dimension.

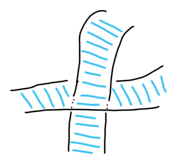


The proof of symmetry and transitivity is not in the scope of this presentation but the proof can be found in the references.

# 4 Ribbon Knots

Ribbon knots are in some sense a simpler kind of slice knots. They have well-behaved properties when looking at the self-intersections of the disk they bound in $S^3$.

**Definition 3.** A knot is ribbon if it bounds a disk that self intersects only with *ribbon singularities*. We define pictorially these types of singularities:



An example of a non-ribbon singularity is the following:

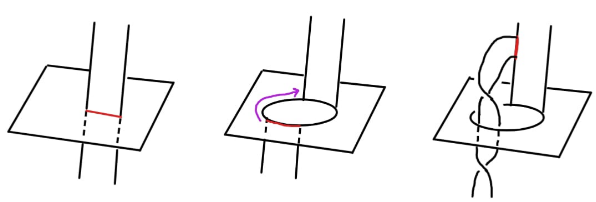
**Theorem 3.** Every ribbon knot is slice.

**Proof.** The proof is simple. Consider the disk that the ribbon knot bounds. By definition, we know how it self-intersects. Again, around each intersection, we can make use of the extra dimension to push a neighborhood into the interior of $D^4$ until there is no more intersection.

The previous result is very straightforward but if you prove or disprove the converse implication you have solved a long-standing open problem!

**Conjecture 1.** (Ralph Fox) Every slice knot is a ribbon knot.

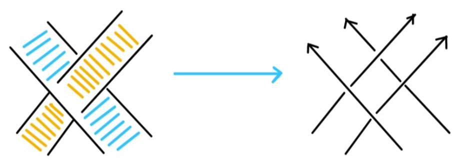
Now, we present the construction of the Seifert surface for a ribbon knot that produces the Seifert matrix with the leftmost block vanishing. Consider one of the self-intersections of the ribbon disk. Then cut at the intersection (and preserve the coherent direction). Then pull the two pieces apart forming a hole. Finally, slide one edge of the cut ribbon to the side of the other. The picture below should make this construction more clear.



Using *ribbon diagrams* to study the first homology group of the surface it is proven in [4] that fact 1 holds for ribbon knots.

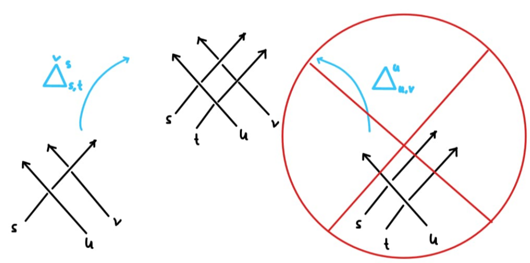
## 4.1 A quick note on XC-tangles for ribbons

Since we know that the self-intersections of a ribbon disk are of the specific form stated previously we can describe a neighborhood of the knot around each self-intersection as the natural XC-tangle:



Then we can notice the following: in that neighborhood, the ribbon disk looks like two different intersecting ribbons segments. We will see how we can pair each of the XC strands correctly to obtain exactly these two ribbons. Moreover, by analyzing how we can obtain the XC tangle using the doubling of strands we can identify the "inner" and "outer" strands (and this is useful since we want to push down the inner one into the 4−ball. Note that the discussion below is local (in particular, the global XC-tangle of each knot has only one strand)

Start by drawing only 3 of the 4 strands. Then we want to double one of them to get back our original XC tangle:




This shows that our procedure works only when we double the strand that passes under and over. This way we have found that the correct pairing of strands $\{s, t, u, v\}$ is always $\{(s, t), (u, v)\}$, where we doubled $s$ to obtain the two strands $(s, t)$. Actually, this also gives the information that the inner ribbon on the self-intersection is the one with the pair $(s, t)$, and that is the one that should be pushed down into the $4^{\text{th}}$ dimension!

Also, intuitively it seems that any ribbon knot could be constructed by some smart doubling of strands. So we may conjecture that any ribbon knot, in its XC-tangle presentation, can be obtained by doubling a strand and merging the two strands obtained. Below we show an example of a ribbon knot. The colors yellow and red represent the two strands (one obtained from the other by doubling) and they are merged in the purple points. Note that it is still not clear how to do this procedure since we have to avoid the forbidden cases presented in the image above (also there are needed two strange mergings of the same strands, to obtain the actual knot and not the long knot).

# References


[1] Slice Knots: Knot Theory in the 4th Dimension, P Teichner

[2] An introduction to knot theory, WBR Lickorish

[3] Notes by Jorge Becerra

[4] Alexander polynomial of ribbon knots, Sheng Bai, https://arxiv.org/abs/2103.07128