Existence of a ribbon element

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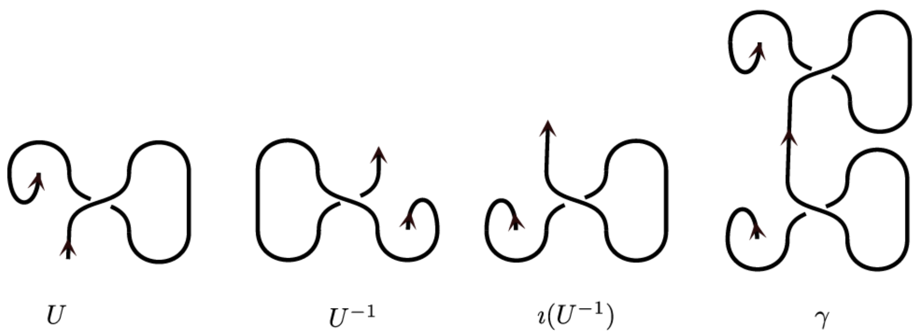
1 Introduction

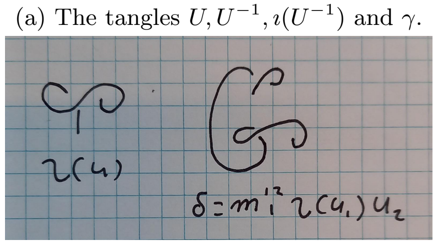
In the last lecture of this course it was claimed that we were satisfactorily finished with the subject of XC tangles. We constructed an XC algebra $\mathbb{D}$ whose universal invariant is well defined for 0-rotation tangles. We can make all tangles 0 rotation by adding some Reidemeister I curls and if we really also want to consider tangles that are not 0-rotation we just adjoin an element C to obtain $\mathbf{I}\mathbb{D}$. The final question that is yet to be answered is whether we always need to adjoin this element or if there are algebras $\mathbb{D}$ that already contain an element with the desired properties.

2 (Quasi-)ribbon elements

Recall the following tangles from lecture 13:



(a) The tangles $U, U^{-1}, \imath(U^{-1})$ and γ.



(b) The tangles $\imath(U)$ and δ.

So $U = m_1^{123} X_{12} C_3, U^{-1} = m_1^{123} X_{32}^{-1} X_1^{-1}$ and $\gamma = m_1^{12}\imath(U_1^1)U_2$. Additionally we define $\delta = U\imath(U)$ or more precisely $\delta = m_1^{12}\imath(U_1)U_2$.

Definition 2.1. *Let $\mathbb{D}$ be the algebra constructed in the lectures. We call an element $\alpha \in \mathbb{D}$ a* ***quasi-ribbon element*** *of $\mathbb{D}$ if the following conditions hold:*

1. $\alpha^2 = \delta$
2. $\imath(\alpha) = \alpha$
3. $\varepsilon(\alpha) = 1$
4. $\Delta(\alpha) = m_1^{135} m_2^{246}(\alpha_5 \alpha_6 X_{12} X_{43})$

A quasi ribbon element is called a ***ribbon element*** *if it is in the centre of $\mathbb{D}$.*

If we compare this definition with exercise 3 of homework 4 we see that the α defined in that exercise is a ribbon element. However this definition of α depends on already having a C.

Definition 2.2. *An element C either in $\mathbb{D}$ or adjoined has the following properties:*

1. $C^2 = \gamma$
2. $xC = C\imath(x)^2$ *for all* $x \in \mathbb{D}$.
3. $\Delta(C) = C_1 C_2$
4. $\imath(C) = C^{-1}$
5. $\varepsilon(C) = 1$

We have two options for obtaining such a C. We could adjoin it to $\mathbb{D}$ to obtain $\mathbf{I}\mathbb{D}$ as was done in lecture 13. In this case C has the desired properties by definition. Alternatively we can search in $\mathbb{D}$ for an element with the desired properties.

We claim that finding such an element α is equivalent to finding an element C which behaves like $\check{C}$ in the XC tangles. Since we would want that a ribbon element α behave in the XC tangles as the Reidemeister I curl we might as well use that as inspiration for how to define C using α. So based on figure 2 we would like to define C as $C := m_1^{123} \imath_1^2 X_{1,2} \alpha_3$.

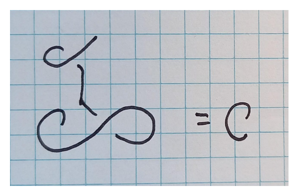


Figure 2: Obtaining C from α

Lemma 2.3. *Finding a ribbon element $\alpha \in \mathbb{D}$ is equivalent to finding an element $C \in \mathbb{D}$.*

Proof. Given a C we claim that $\alpha = m_1^{123} X_{3,1} C_2$ is a ribbon element. Proof: Homework due on Thursday!
Given a ribbon element α we claim that $m_1^{123} \imath_1^2 X_{1,2} \alpha_3$ satisfies the properties of a C. Proof: exercise. □

3 Integrals and co-integrals

Before we can state the theorem of when ribbon elements exists and give an example we need to introduce the concepts of grouplike elements, integrals and co-integrals.

Definition 3.1. *Suppose A is a bialgebra. An element $a \in A$ is called a* ***grouplike element*** *of A if $\Delta(a) = a_1 a_2$. We denote the grouplike elements of A by $G(A)$. The grouplike elements of A^* are denoted by $G(A^*)$.*

Proposition 3.2. *[1]The grouplike elements of A^* are isomorpic to $Alg_k(A,k)$.*

Example 3.3. Let SW denote the Sweedler algebra. To compute the grouplike elements it is enough to check what Δ does to the basis elements. So recall that $\Delta(s) = s_1 s_2$ and $\Delta(w) = w_1 s_2 + w_2$. This means that $G(SW) = (s)$.
By the above proposition the grouplike elements of A^* are the algebra homomorphisms of A to k.

Let f be any algebra homomorphism from A to k. We define such a morphism by defining where it sends s and w and extending. It must be compatible with the relations. We must have $0 = f(0) = f(w^2) = f(w)f(w)$. So $f(w) = 0$. Likewise we must have $1 = f(1) = f(s^2) = f(s)f(s)$. So we can have $f(s) = 1$ or $f(s) = -1$. Note that the first map is a scalar multiple of the second map. So we claim that $G(A^*) = (\eta)$ where $\eta(w) = 0$ and $\eta(s) = -1$. That any $f \in (\eta)$ is compatible with the relations $w^2 = 0$ and $s^2 = 1$ follows from what we did above. We also have $f(sw) = f(s)f(w) = 0 = -f(w)f(s) = f(-ws)$ and the maps are algebra homomorphism by construction.

Remark 3.4. *In Hopf algebras the grouplike elements form a group.*

Definition 3.5. *A* ***(left) integral*** *is an element k of a Hopf algebra H such that $hk = \varepsilon(h)k$ for all $h \in H$. A* ***(right) co-integral*** *is an element λ of $H*$ such that $\mu\lambda =< \mu, 1 > \lambda$.*[1]

Proposition 3.6. *[1] If H is a finite dimensional Hopf algebra then the left integrals and the right co-integrals form one dimensional ideals.*

Example 3.7. We again consider the Sweedler algebra. By the proposition we only need to find one nonzero integral. Recall that $\varepsilon(s) = 1$ and $\varepsilon(w) = 0$ We compute hk and $\varepsilon(h)k$ for $k = w$ and $k = sw$.

h	hw	$\varepsilon(h)w$	h	hsw	$\varepsilon(h)w$
1	w	w	1	sw	sw
s	sw	w	s	w	sw
w	0	0	w	0	0
sw	0	0	sw	0	0

Notice that if we add these columns they are equal so we conclude that $w + sw$ is a left integral.

Before we can compute the co-integrals we introduce a more convenient basis for SW^*. Let $1, \sigma, \omega, \sigma\omega$ be a basis for the dual defined by the pairing given by the following table:

	1	s	w	sw
1	1	1	0	0
σ	1	-1	0	0
ω	0	0	1	1
$\sigma\omega$	0	0	1	-1

This basis has the following properties $\sigma^2 = 1, \omega^2 = 0, \sigma\omega = -\omega\sigma, \Delta(\sigma) = \sigma_1\sigma_2, \Delta(\omega) = \omega_1\sigma_2 + \omega_2, \varepsilon(\sigma) = 1$ and $\varepsilon(\omega) = 0$. In other words $SW^* \cong SW$.
By doing a similar computation as above we find that $\omega - \sigma\omega$ is a right co-integral for SW^*.

Definition 3.8. *Let k be a nonzero left integral of A then there exists a unique $\beta \in G(A^*)$ such that $kx = \beta(x)k$ for all $x \in A$.*
Let λ be a nonzero right co integral of A^* then there is a unique $g \in G(A^{**}) = G(A)$ such that $\pi\lambda =< \pi, g > \lambda$ for all $\pi \in A^*$.
We call β the ***distinguished grouplike element of*** *A^* and g the* ***distinguished grouplike element of*** *A.*

Example 3.9. We look again at the Sweedler algebra and begin with computing the distinguished grouplike element of SW. Take $k = w+sw$ a left integral of SW. We want to find $\beta \in G(A^*)$ such that $kx = \beta(x)k$ for all $x \in A$. It is enough to determine β on the basis of SW. So for example taking $x = 1$ gives $w+sw = \beta(1)(w+sw)$

[1] For those wondering whether this notion of integral is related to the calculus integral it is. See page 181 of [1].

implies that $\beta(1) = 1$. Taking $x = s$ gives $s(w + sw) = sw + w = \beta(s)(w + sw)$ also gives $\beta(s) = 1$. For the last two basis elements we get $\beta(w) = \beta(sw) = 0$. Looking back at the dual basis we see that β behave as 1.

Now take $\lambda = \omega - \sigma\omega$. If we take for π the dual then the equation $\pi\lambda =< \pi, g > \lambda$ determines $< \pi, g >$ which are the coefficients of g with respect to that basis. So taking $\pi = 1$ (the dual 1!) gives $\omega - \sigma\omega =< 1, g > \omega - \sigma\omega$ so $< 1, g >= 1$. Similarly $< \sigma, g >= 1, < \omega, g >= 0, < \sigma\omega = 0 >$. So $g = 1^* + \sigma^*$ which written in the normal basis is s

4 Condition for existence

Before we can formulate the theorem we need a final algebraic definition: the actions of $A*$ on A. We will only give them for the Hopf algebras that we are interested in: $\mathbb{D}$.

Definition 4.1. *Let $\pi \in A^*$ and $a \in A$.*
The left action of $A*$ on A is denoted by $\pi \leftharpoonup a$ as applying p to the second tensor factor of $\Delta(a)$.
The right action is denoted by $a \rightharpoonup \pi$ and is defined by applying p to the first tensor factor of $\Delta(a)$.

Example 4.2. $\varepsilon \rightharpoonup= \varepsilon_2(w_1 s_2 + w_2) = w_1(\varepsilon(s))_2 + \varepsilon(w)_2 = w_1$

Theorem 4.3. *(Kauffman and Radford [2]) Suppose that $\mathbb{O}$ is a finite dimensional Hopf algebra with antipode $\imath$ over a field k. Let g and δ be the distinguished grouplike elements of $\mathbb{O}$ and $\mathbb{U}$, respectively. Then:*

(a) $\mathbb{D}$ has a quasi-ribbon element if and only if there are $l \in G(\mathbb{O})$ and $\beta \in G(\mathbb{U})$ such that $l^2 = g$ and $\beta^2 = \delta$.

(b) $\mathbb{D}$ has a ribbon element if and only if there are l and β as in part (a) such that

$$\imath^2(x) = l(\beta \rightharpoonup x \leftharpoonup \beta^{-1})l^{-1}$$

for all $x \in \mathbb{O}$.

Example 4.4. Let us determine whether it was indeed necessary to adjoin a C to $\mathbb{D}$ defined by the Sweedler algebra or if it in fact has a ribbon element. Recall that for the Sweedler algebra $g = s$ and $\delta = 1$. (Note this is the 1 from the dual basis!) So for $\mathbb{D}$ to have a quasi-ribbon element we must find an $l \in G(SW)$ such that $l^2 = g$. We write $l = a_1 + a_2 s + a_3 w + a_4 sw$. Working out the square gives $l^2 = a_1^2 + a^2 + 2a_1 a_2 s + 2a_1 a_3 w$. This has no solutions in $\mathbb{R}$ but over $\mathbb{C}$ we have for example $a_1 = \frac{1}{2} + \frac{i}{2}, a_2 = \frac{1}{2} - \frac{i}{2}$. However all possible solutions have a nonzero a_1 and are hence not in $G(SW) = (s)$.

Example 4.5. Lets also finish with an example where a ribbon element does exists. We generalise the Sweedler algebra as follows:
Let ζ be a primitive nth root of unity. Let A_n be the algebra generated by a, x subject to the relations

$$a^n = 1, x^n = 0, xa = \zeta ax$$

And has co algebra structure

$$\Delta(a) = a_1 a_2, \Delta(x) = x_1 a_2 + x_2$$

It turns out that this algebra has a unique ribbon element if and only if n is odd. See proposition 7 of [2] for more details.

References

[1] Nastasescu C. Raianu S. Dascalescu, S. Hopf algebra: An introduction. 2000.

[2] Louis H. Kauffman and David E. Radford. A necessary and sufficient condition for a finite-dimensional drinfeld double to be a ribbon hopf algebra. *Journal of Algebra*, 159:98–114, 1993.