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Homework 4 - Topics in Topology

5 April 2023

Please return Thursday April 13 2023 (extended deadline)

1. Fill in the gaps in the proof of Lemma 5 from Lecture 13.

2. In this exercise we revisit the formula for the product of the algebra $\mathbb{D}$ constructed from the pair of Hopf algebras $\mathbb{O}$ and $\mathbb{U}$. Recall the product formula was

$$u^p o^q = \sum_{i,j,k,\ell} \overline{\langle o^q, \tilde{u}^i \tilde{u}^j u^\ell \rangle} \langle o^i o^k o^\ell, u^p \rangle o^j u^k$$

   (a) Prove that for all $x \in \mathbb{O}, y \in \mathbb{U}$ we have $\overline{\langle x, y \rangle} = \langle x, \imath_{\mathbb{U}}^{-1}(y) \rangle$.

   (b) Show that $\tilde{u}^i \tilde{u}^j u^\ell = \imath_{\mathbb{U}}(\imath_{\mathbb{U}}^{-1}(u^\ell) u^j u^i)$ so $\overline{\langle o^q, \tilde{u}^i \tilde{u}^j u^\ell \rangle} = \langle o^q, \imath_{\mathbb{U}}^{-1}(u^\ell) u^j u^i \rangle$.

   (c) If we write $(\Delta_{\mathbb{U}})^1_{123}(y) = \sum y'_1 y''_2 y'''_3$ then show that (using $o^q = x$ and $u^p = y$) $\langle o^i o^k o^\ell, u^p \rangle = \langle o^i, y' \rangle \langle o^k, y'' \rangle \langle o^\ell, y''' \rangle$ and likewise prove that $(\Delta_{\mathbb{O}})^1_{\bar{1}\bar{2}\bar{3}}(x) = \sum x'_{\bar{1}} x''_{\bar{2}} x'''_{\bar{3}}$ implies

$$\langle o^q, \imath_{\mathbb{U}}^{-1}(u^\ell) u^j u^i \rangle = \langle \imath_{\mathbb{O}}^{-1}(x'), u^\ell \rangle \langle x'', u^j \rangle \langle x''', u^i \rangle$$

   (d) We extend the pairing $\langle\rangle$ to unordered tensor products by setting $P_{ij} : \mathbb{O}^{\otimes\{i\}} \times \mathbb{U}^{\otimes\{j\}} \to k$ to be the pairing applied to tensor factors i and j. Combine this notation with your the results from the previous parts to show that

$$\sum_{i,j,k,\ell} \overline{\langle o^q, \tilde{u}^i \tilde{u}^j u^\ell \rangle} \langle o^i o^k o^\ell, u^p \rangle o^j u^k = m_1^{\bar{2},2} P_{\bar{3},1} P_{\bar{1},3} (\imath_{\mathbb{O}}^{-1})_{\bar{1}} (\Delta_{\mathbb{O}})^1_{\bar{1}\bar{2}\bar{3}}(o^q) (\Delta_{\mathbb{U}})^1_{123}(u^p)$$

3. Prove that the element $\alpha_1 = m_1^{123}(X_{31} C_2)$ in $\mathbb{ID}$ has the following properties (hint: sketch the corresponding tangles!):

   (a) α is central: $x\alpha = \alpha x$ for all $x \in \mathbb{ID}$.

   (b) $\imath(\alpha) = \imath^{-1}(\alpha) = \alpha^{\pm 1}$, (you decide if it is $+$ or $-$)

   (c) $\Delta^1_{12}(\alpha) = m_1^{135} m_2^{246} (\alpha_5 \alpha_6 X_{12} X_{43})$

4. Suppose K is a 0-framed, rotation 0, one strand XC-diagram for a (long knot). Define $K^{(2)}$ to be a long knot diagram (it has one strand!) with 0 rotation, 0 writhe that passes through K twice. Write an explicit formula for $Z_{\mathbb{ID}}(K^{(2)})$ in terms of $Z_{\mathbb{ID}}(K)$ and Hopf algebra operations and elements of (tensor powers of) $\mathbb{ID}$.