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Homework 3 - Topics in Topology

14 March 2023

Please return Tuesday 21 March 2022.

1. Explain how the Kaufmann bracket is an invariant of XC-links. Also explain how the Jones polynomial is the deframing of this invariant of XC-links.

2. Let $A := \mathrm{End}(V)$ be the XC-algebra from Lecture 8 that recovered the Alexander polynomial. The goal of this exercise is to show that if D is an XC-diagram of a link L in $\mathbb{R}^3$, then $Z_A(D) = 0$.
   (a) Show that for an open 1-strand XC-diagram of a long knot K labelled s, we have
   $$Z_A(\mathring{D}^s) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}_s$$
   for some scalars $\lambda, \mu \in \mathbb{Q}(q)$. In fact, argue that $\lambda, \mu \in \mathbb{Q}[q, q^{-1}] \subset \mathbb{Q}(q)$.
   (b) Use exercise 2(b) from the second homework set to show that $\lambda = \mu$, that is,
   $$Z_A(K) = \lambda \cdot \mathrm{Id}_s \in A^{\otimes\{s\}}.$$
   (c) Let $K' := \check{m}_v^{u,u}\check{m}_u^{s,t}(K\check{C}_t)$ be the "closure" of the long knot K. Show that $Z_A(K') = 0$.
   (d) Let D be a XC-diagram of a link in $\mathbb{R}^3$ (with possibly multiple components) and consider the diagram $\mathring{D}^s$ as defined in the lectures whose only open strand is labelled by s. Extend the arguments you used in (a)–(c) to show that $Z_A(\mathring{D}^s) = \lambda \cdot \mathrm{Id}_s$ as well and conclude that $Z_A(D) = 0$. (*Hint.* Recall that $D = \check{m}_v^{u,u}\check{m}_u^{s,t}(\mathring{D}^s\check{C}_t^{\pm})$).

3. Write the long knot 8_{17} shown at the beginning of Lecture 5 as the merging of the disjoint union of eight $\check{X}$'s and some $\check{C}$'s.

   (BONUS) Use the Mathematica code posted on the website to compute the Z_{Dlb} invariant of this knot.

4. (a) Show that there are no non-trivial representations of the fundamental group of the figure eight knot complement into the dihedral group of 6 elements $D_6 = S_3$. (*Hint.* Pick a diagram and show that there is no 3-coloring with three colors.)
   (b) Write down a presentation of the fundamental group of the figure eight knot complement with as few as possible generators as you can find.
   (c) Write expressions for the meridian and the longitude of the knot.

5. Suppose $\mathbb{O}$ and $\mathbb{U}$ are two algebras with two pairings and bases as in the set up of Lecture 10. Simplify the following expression as much as you can for an arbitrary $x \in \mathbb{O}$:
   $$\sum_{i,j} \langle o^i, \tilde{u}^j\rangle \overline{\langle x, u^i\rangle} o^j$$

6. Verify explicitly that $wb = dw + a(s+1)$ in the Sweedler example from Lecture 10 using equation (1) on page 2 of those lecture notes.