LECTURE 12: ANTIPODE, HOPF ALGEBRAS AND CS

1. Recap

Before moving on to complete the picture let us recap what algebraic structures we proposed so far in our effort to construct an XC-algebra.

We started with two algebras $\mathbb{O}$ and $\mathbb{U}$ of the same dimension and a non-degenerate pairing $\langle,\rangle : \mathbb{O} \longrightarrow \mathbb{U}$ yielding an isomorphism $\mathbb{O} \cong \mathbb{U}^*$. To make this more concrete we had dual bases o^i and u^i so $\langle o^i, u^j\rangle = \delta_{ij}$.

Using the pairing we introduced coproducts on $\mathbb{O}$ and $\mathbb{U}$ called $\Delta_\mathbb{O} : \mathbb{O} \longrightarrow \mathbb{O}\otimes\mathbb{O}$ and $\Delta_\mathbb{U} : \mathbb{U} \longrightarrow \mathbb{U}\otimes\mathbb{U}$ defined by

$$(\Delta_\mathbb{O})^s_{\ell,r}(x) = \sum_{a,b}\langle x, u^a u^b\rangle o^a_\ell o^b_r \qquad (\Delta_\mathbb{U})^s_{\ell,r}(x) = \sum_{a,b}\langle o^a o^b, x\rangle u^a_\ell u^b_r$$

Corresponding to the negative crossing we also introduced a second basis $\tilde{u}^i$ for $\mathbb{U}$ and a second pairing $\overline{\langle\rangle}$ making this basis dual to the o^i basis.

Likewise dual to the units in $\mathbb{O}$ and $\mathbb{U}$ we can introduce co-units $\varepsilon_\mathbb{O} : \mathbb{O} \longrightarrow k$ and $\varepsilon_\mathbb{U} : \mathbb{U} \longrightarrow k$ by setting

$$\varepsilon_\mathbb{O}(x) = \langle x, 1_\mathbb{U}\rangle \qquad \varepsilon_\mathbb{O}(x) = \langle 1_\mathbb{O}, x\rangle$$

Next we introduced the vector space $\mathbb{D}$ spanned by formal products $o^i u^j$ and came up with a multiplication rule for $\mathbb{D}$ by requiring $\mathbb{O}$ and $\mathbb{U}$ to be subalgebras and defining

$$u^p o^q = \sum_{i,j,k,\ell} \overline{\langle o^q, \tilde{u}^i\tilde{u}^j u^\ell\rangle}\langle o^i o^k o^\ell, u^p\rangle o^j u^k \tag{1}$$

With all the definitions in place we can now assert the following:

Theorem 1. *If $\Delta_\mathbb{O}$ is an algebra morphism then both $\mathbb{O}$ and $\mathbb{U}$ become bialgebras referring to the above structures. Moreover the product* (1) *becomes associative so $\mathbb{D}$ is an algebra with unit $1_\mathbb{D} = 1_\mathbb{O}1_\mathbb{U}$. Moreover, $\mathbb{D}$ becomes a bialgebra with respect to the coproduct $\Delta_\mathbb{D}$ and counit $\varepsilon_\mathbb{D}$ defined by*

$$(\Delta_\mathbb{D})^s_{\ell,r}(ou) = (\Delta_\mathbb{O})^s_{r,\ell}(o)(\Delta_\mathbb{U})^s_{\ell,r}(u) \qquad \varepsilon_\mathbb{D}(ou) = \varepsilon_\mathbb{O}(o)\varepsilon_\mathbb{U}(u) \tag{2}$$

The assumption in the above theorem is concretely that $\Delta_\mathbb{O}(xy) = \Delta_\mathbb{O}(x)\Delta_\mathbb{O}(y)$. Equivalently we could've required $\Delta_\mathbb{U}(xy) = \Delta_\mathbb{U}(x)\Delta_\mathbb{U}(y)$ as the algebras are duals. Notice the strange reversal of ℓ and r in the definition of $\Delta_\mathbb{D}$. This, like all the above formulas, is motivated the construction of the tangle invariant $Z_\mathbb{D}$. The philosophy is that the invariant Z should relate all tangle operations to the corresponding algebra operations.

Theorem 2. *Define $Z_\mathbb{D}$, as usual, by*

$$Z_\mathbb{D}(\check{1}) = 1_\mathbb{D} \quad Z_\mathbb{D}(\check{X}_{ij}) = X_{ij} = \sum_p o^p_i u^p_j \quad Z_\mathbb{D}(\check{X}^{-1}_{ij}) = X^{-1}_{ij} = \sum_p o^p_i \tilde{u}^p_j \tag{3}$$

$$Z_\mathbb{D}(\check{m}^{h,t}_r T) = m^{h,t}_r Z_\mathbb{D}(T) \quad Z_\mathbb{D}(DD') = Z_\mathbb{D}(D)Z_\mathbb{D}(D') \tag{4}$$

Then $Z_\mathbb{D}$ is well-defined on all XC-tangle diagrams without C's and it is invariant under all Reidemeister moves that do not involve C's, i.e. $\Omega 2b, \Omega 3$. Moreover, the bialgebra structure on $\mathbb{D}$ satisfies

$$Z_\mathbb{D}(\check{\Delta}^s_{\ell,r}T) = (\Delta_\mathbb{D})^s_{\ell,r}Z_\mathbb{D}(T) \qquad Z_\mathbb{D}(\check{\varepsilon}^s T) = \varepsilon^s_\mathbb{D} Z_\mathbb{D}(T) \tag{5}$$

Both theorems mentioned above were proven in the previous lecture, mostly by choosing the right definitions so as to make them true by construction. As always in mathematics, the serious reader should take up the challenge to work out a detailed proof for herself. Here we will just make two remarks on the invariance under the Reidemeister 3 move $\Omega 3$ and Equation (5).

To check $\Omega 3$ holds, recall form Lecture 10 that by construction of the product (1) we have $Z_\mathbb{D}(L) = Z_\mathbb{D}(R)$ where L and R are the tangles shown in black in the middle of Figure 1. Merging both sides with additional blue crossings as shown yields tangles L' and R' that are precisely the left and right-hand side of the Reidmeister 3 move. On the other hand, $Z_{\mathbb{D}}$ is compatible with merging and disjoint union, showing that $Z_{\mathbb{D}}(L') = Z_{\mathbb{D}}(R')$ so $\Omega 3$ holds.

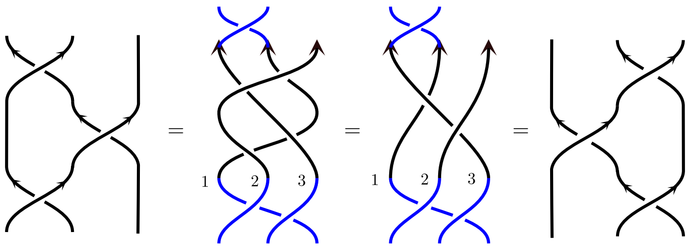

FIGURE 1. Merging the blue crossings onto the two tangle diagrams, L and R from lecture 10 we obtain the two sides of the Reidemeister 3 move $\Omega 3$.

To finish our discussion of Theorems 1 and 2 we comment on Equation (5). Let us check the special case $T = \check{X}_{s,j}$. Sketch a picture to convince yourself that

$$\check{\Delta}^s_{\ell,r}(\check{X}_{s,j}) = \check{m}^{j_1,j_2}_j(\check{X}_{r,j_2}\check{X}_{\ell,j_1})$$

Applying $Z_{\mathbb{D}}$ we just remove the $\check{}$ from the symbols in this formula to get:

$$Z_{\mathbb{D}}(\check{\Delta}^s_{\ell,r}X_{s,j}) = m^{j_1,j_2}_j(X_{r,j_1}X_{\ell,j_2}) = \sum_{a,b} o^a_r o^b_\ell (u^a u^b)_j$$

On the other hand

$$(\Delta_{\mathbb{D}})^s_{\ell,r}Z_{\mathbb{D}}(\check{X}_{s,j}) = \sum_w (\Delta_{\mathbb{D}})^s_{\ell,r}(o^w_s u^w_j) = \sum_{a,b,w} o^a_r o^b_\ell \langle o^w, u^a u^b\rangle u^w_j$$

Bringing the sum over w inside and recalling that $x = \sum_w \langle o^w, x\rangle u^w$ we see that indeed both sides agree. For this to work it was essential to reverse the order of r and ℓ as done in the definition of $\Delta_{\mathbb{D}}$.

Similar checks establish Equation (5) for the other cases where T is a single crossing. Once that is done the Equation will hold in general because by construction $Z_{\mathbb{D}}$ is compatible with disjoint union and merging and $\Delta_{\mathbb{D}}$ is an algebra morphism.

2. Reversing strands

To complete the discussion of XC tangles we need to address the C's and the first step towards them is to ask how the positive and negative crossings are related. One way to turn a positive crossing into a negative crossing is shown on the top right of Figure 2. If we reverse the overpassing strand s of positive crossing $\check{X}_{s,t}$ then it becomes a negative crossing. To make sure the rotation numbers on the edges are preserved we add hooks at the ends of the strand and rotate the picture to make the orientation point upwards near each crossing.

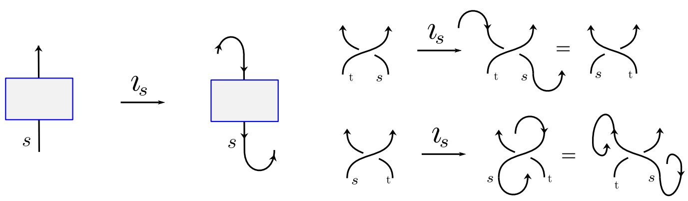

FIGURE 2. Left: Roughly speaking the antipode reverses a strand and places hooks on bottom and top. The effect on a positive crossing is illustrated on the right.

More formally we introduce an operation on XC-tangles by stating what it does on the elementary tangles and explaining how it behaves under merging and disjoint union.

Definition 3. (Strand reversal, antipode)
For an XC-tangle diagram D with strand s define $\check{\imath}_s(D)$ inductively as follows.

$$\check{\imath}_s(\check{X}_{t,s}) = \check{X}^{-1}_{t,s} \quad \check{\imath}_s(\check{X}^{-1}_{s,t}) = \check{X}_{s,t} \quad \check{\imath}_s(\check{C}^{\pm1}_s) = C^{\mp1}_s \tag{6}$$

$$\check{\imath}_s(\check{X}_{s,t}) = \check{m}^{s_1,s_2,s_3}_s(\check{C}^{-1}_{s_1}\check{X}_{s_2,t}\check{C}_{s_3}) \quad \check{\imath}_s(\check{X}^{-1}_{t,s}) = \check{m}^{s_1,s_2,s_3}_s(\check{C}^{-1}_{s_1}\check{X}_{t,s_2}\check{C}_{s_3}) \tag{7}$$

$$\check{\imath}_s(DD') = \check{\imath}_s(D)\check{\imath}_s(D') \quad \check{\imath}_s(\check{m}^{h,t}_s(D)) = \check{m}^{t,h}_s\check{\imath}_t\check{\imath}_h(D) \tag{8}$$

Notice how in the definition applying the antipode to the under-strand of a positive crossing does not exactly yield the negative crossing. Instead we get a conjugation by $\check{C}$'s, see also Figure 2 (bottom right). The appearance of the C's here will allow us to introduce an algebraic version of them in our algebra $\mathbb{D}$.

In Figure 3 the action on of the antipode on a larger tangle is illustrated. The picture in the middle is technically not an XC-tangle diagram because the rotation number of the two edges at the ends of the red strand is not an integer. By a simple planar isotopy rotating the arrows at the crossing upwards this is fixed in line with the above definition.

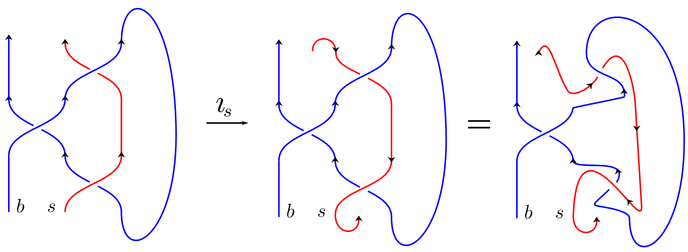

FIGURE 3. Applying the antipode to the red strand s.

The Reidemeister moves for XC tangles imply the following important relation:

$$\check{m}^{\ell r}_x \check{\imath}_l(\Delta^s_{\ell,r}) = \check{m}^{\ell r}_f \check{\imath}_r(\check{\Delta}^s_{\ell r}) = \check{1}_f\check{\varepsilon}^s \tag{9}$$

This relation may look mysterious at first but it just states that if you take any strand and double it back, it comes undone. See Figure 4 for a visual explanation.

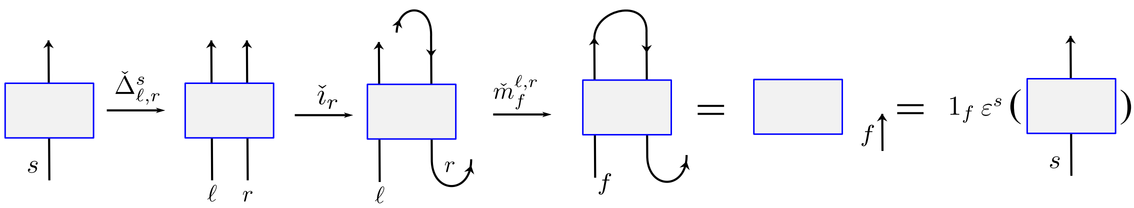

FIGURE 4. The strand s on the left can be knotted and tangled in a very complicated way inside the blue box. Nevertheless it will come undone after applying doubling, antipode and merge as shown.

Notice that applying the antipode twice on the same strand s of tangle T does NOT return the original tangle. Instead we get the tangle conjugated with C. In formulas:

$$\check{\imath}_s(T) = \check{m}^{s_1,s,s_3}_s(C_{s_1}TC_{s_3})$$

Undoing the antipode properly is done by reversing with the opposite hooks. More precisely define

Definition 4. (Strand reversal, inverse antipode)
For an XC-tangle diagram D with strand s define $\check{\imath}^{-1}_s(D)$ inductively as follows.

$$\check{\imath}_s(\check{X}_{s,t}) = \check{X}^{-1}_{s,t} \quad \check{\imath}_s(\check{X}^{-1}_{t,s}) = \check{X}_{t,s} \quad \check{\imath}_s(\check{C}^{\pm1}_s) = C^{\mp1}_s \tag{10}$$

$$\check{\imath}_s(\check{X}_{t,s}) = \check{m}^{s_1,s_2,s_3}_s(\check{C}_{s_1}\check{X}_{t,s_2}\check{C}^{-1}_{s_3}) \quad \check{\imath}_s(\check{X}^{-1}_{s,t}) = \check{m}^{s_1,s_2,s_3}_s(\check{C}_{s_1}\check{X}_{s_2,t}\check{C}^{-1}_{s_3}) \tag{11}$$

$$\check{\imath}_s(DD') = \check{\imath}_s(D)\check{\imath}_s(D') \quad \check{\imath}_s(\check{m}^{h,t}_s(D)) = \check{m}^{t,h}_s\check{\imath}_t\check{\imath}_h(D) \tag{12}$$

Equipped with the antipode and its inverse we can write a tangle formula for the "square" of $\check{C}$, that is a strand with no crossings and rotation number 2 as follows.

$$\check{m}_s^{12}(\check{C}_1\check{C}_2) = \check{m}_s^{1234}\check{\imath}_1^{-2}\check{\imath}_4^{2}(\check{X}_{21}^{-1}\check{X}_{34}) \tag{13}$$

In Figure 5 we show the Reidemeister moves that make Equation 13 hold.

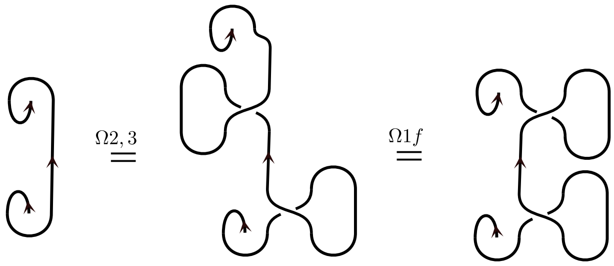

FIGURE 5. An illustration for Equation 13.

We have seen how the antipode relates the positive to the negative crossing, how it contains information about the C's and satisfies an interesting relation with the other tangle operations $\check{\Delta}$ and $\check{m}$. This suggests that finding an algebraic counterpart to the antipode could be the key to completing our knot invariants with C's.

3. Hopf algebras

So far any bialgebra $\mathbb{O}$ yields an invariant $Z_{\mathbb{D}}$ of XC-tangles without C's but most are not compatible with the strand reversal (antipode) operation we introduced in the previous section. The ones that are, are known as Hopf algebras, defined as follows.

Definition 5. (Hopf algebra)
A bialgebra A is called a **Hopf algebra** if there exists a vector space isomorphism $\imath : A \longrightarrow A$ such that $m_1^{12}\imath_1(\Delta) = m_1^{12}\imath_2(\Delta) = 1_1\varepsilon$.

Notice that the final axiom of a Hopf algebra is equivalent to the relation we found between strand doubling and strand reversal, Equation (9).

$$m_x^{\ell r}\imath_\ell(\Delta_{\ell,r}^s) = m_x^{\ell r}\imath_r(\Delta_{\ell r}^s) = 1_x\varepsilon^s \tag{14}$$

We already saw several examples of Hopf algebras. For example the group algebra $k[G] = \{\sum_{g\in G} a_g g : a \in k\}$ with $\Delta(g) = g_1g_2$ and $\varepsilon(g) = 1$ and $\imath(g) = g^{-1}$. Indeed we have $m_1^{12}\imath_1(\Delta)(g) = m_1^{12}\imath_1(g_1g_2) = m_1^{12}g_1^{-1}g_2 = 1_1 = 1_1\varepsilon^1(g_1)$. This example is important in that it suggests that Hopf algebras may be viewed as a generalization of the notion of a group.

Another example is the Sweedler Hopf algebra with $\imath(s) = s, \imath(w) = sw$. Recall that $\Delta(s) = s_1s_2$ and $\Delta(w) = s_2w_1 + w_2$ and $\varepsilon(s) = 1$, $\varepsilon(w) = 0$. Since $sw + ws = 0$ in this algebra we have for example $m_1^{12}\imath_1(\Delta)(w) = m_1^{12}\imath_1(s_2w_1 + w_2) = m_1^{12}(s_2s_1w_1 + w_2) = (sws + w)_1 = 0 = 1_1\varepsilon^1(w_1)$.

One peculiar feature of Hopf algebras is that the antipode is an anti-homomorphism: $\imath(ab) = \imath(b)\imath(a)$. Keeping the group example in mind this statement is similar to the familiar $(ab)^{-1} = b^{-1}a^{-1}$.

Lemma 6. *For any Hopf algebra we have $\imath(ab) = \imath(b)\imath(a)$ or equivalently:*

$$\imath_s \circ m_s^{h,t} = m_s^{t,h} \circ \imath_t \circ \imath_h$$

Proof. The proof is very similar to the case of groups where one argues

$$(ab)^{-1} = (ab)^{-1}(aa^{-1}) = (ab)^{-1}(a(bb^{-1})a^{-1}) = ((ab)^{-1}(ab))b^{-1}a^{-1} = b^{-1}a^{-1}.$$

For a precise proof see next lecture. □

Now that we know what a Hopf algebra is we can ask whether our bialgebras $\mathbb{O}, \mathbb{U}$ and $\mathbb{D}$ from Section 1 are Hopf. If so we would hope to make sure that the relation $Z_{\mathbb{D}}(\check{\imath}_s(T)) = \imath_s(Z_{\mathbb{D}}(T))$ holds for all tangle diagrams T.

The relation $\check{\imath}_s(\check{X}_{t,s}) = \check{X}_{t,s}^{-1}$ suggests the answer is YES. With this in mind we introduce $\imath_{\mathbb{U}} : \mathbb{U} \longrightarrow \mathbb{U}$ by

$$\imath_{\mathbb{U}}(u^i) = \tilde{u}^i$$

This makes sense because $X^{-1} = \sum_i o^i \otimes \tilde{u}^i$ so then we will have $Z_{\mathbb{D}}(\check{\imath}_s(\check{X}_{t,s})) = \imath_s(Z_{\mathbb{D}}(\check{X}_{t,s}))$. Taking the dual using the pairing $\langle , \rangle$ we introduce $\imath_{\mathbb{O}} = \imath_{\mathbb{U}}^*$ so $\imath_{\mathbb{O}}(x) = \sum_i \langle x, \imath(u^i)\rangle o^i$.

Lemma 7. *If we define $\imath_{\mathbb{D}} : \mathbb{D} \longrightarrow \mathbb{D}$ by $\imath_{\mathbb{D}}(ou) = \imath_{\mathbb{U}}(u)\imath_{\mathbb{O}}^{-1}(o)$ then $\mathbb{D}$ becomes a Hopf algebra.*

Proof. First we will show that $\mathbb{U}$ is a Hopf algebra by checking the antipode axioms $m_x^{\ell r}\imath_\ell(\Delta^s_{\ell,r}) = m_x^{\ell r}\imath_r(\Delta^s_{\ell r}) = 1_x\varepsilon^s$ hold in $\mathbb{U}$.

Let us focus on the second equality as the first is similar. We notice that the XC-tangle diagram described by $\check{m}_x^{\ell r}\check{\imath}_r(\check{\Delta}^s_{\ell r}(\check{X}_{t,s}))$ is just a complicated way to write the left-hand side of Reidemeister $\Omega 2b$ (make a sketch to convince yourself!) so it is equal to the tangle $\check{1}_t\check{1}_s$. Since we established that $Z_{\mathbb{D}}(\check{\imath}_s(\check{X}_{t,s})) = \imath_s(Z_{\mathbb{D}}(\check{X}_{t,s}))$ we find that $Z_{\mathbb{D}}(\check{m}_x^{\ell r}\check{\imath}_r(\check{\Delta}^s_{\ell r}(\check{X}_{t,s}))) = Z_{\mathbb{D}}(\check{1}_t\check{1}_s) = 1$. Taking the coefficient of u^i on both sides yields $m_x^{\ell r}\imath_r(\Delta^s_{\ell r}(u^i))$ on the left hand side and $\langle o^i, 1\rangle 1_t = \varepsilon(u^i)1_t$ on the right hand side.

Taking duals (with respect to $\langle , \rangle$) immediately proves that $\mathbb{O} \cong \mathbb{U}^*$ must also be a Hopf algebra with the $\imath_{\mathbb{O}} = \imath_{\mathbb{U}}^*$ as defined above. The proof is now complete because by definition of $\imath_{\mathbb{D}}$ the antipode axioms for $\mathbb{O}$ and $\mathbb{U}$ combine to prove the antipode axioms for $\mathbb{D}$. $\square$