Two-bridge knots

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1 Introduction

In this lecture we will study *two-bridge knots*. Before we can define those, we first need to define a *bridge* and the *bridge number* of a knot. Given a diagram of a knot K, a subarc (a piece of strand between two undercrossings) that includes one or more overcrossings is called a bridge. The number of bridges in a knot diagram is called the bridge number of the diagram. The minimum of the bridge numbers of the diagrams, over all diagrams of K, is called the bridge number of K and denoted by br(K).

Example 1. The most known diagram of the trefoil knot given in Figure 1a has three bridges, as all its subarcs contain one overcrossing. However, the trefoil knot has a different representation in which it has only two bridges, as can be seen in Figure 1b. So the trefoil knot has at most bridge number 2.

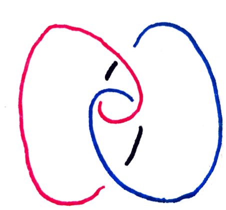
Could it be the case that the trefoil knot has bridge number 1? No, by convention we say that the only knot with bridge number 1 is the unknot. So, in order to show that the trefoil knot has bridge number 2, one only needs to show that the trefoil knot is not the unknot, which we have shown in Lecture 2 by showing its Jones polynomial.

In order to discuss the bridge number br(K) more formally we need to consider a height function. A *height function* on $\mathbb{S}^3$ is a Morse function with exactly two critical points: a maximum and a minimum. On $\mathbb{R}^3$, it is a Morse function with no critical points. More concretely, it is projection onto, say, the z-axis.

From this perspective it makes sense to consider the number of relative maxima of the knot K with respect to a height function. We think of the plane used in a knot diagram as the xy-plane and our height function as projection onto z. Given a diagram of a knot K with m bridges, each subarc of the diagram that is a bridge can be converted into an arc with interior above the plane and exactly one maximum. Subarcs of the diagram that are not bridges can be concatenated and converted into arcs with interior below the plane and exactly one minimum. In this manner we construct a representative of K with exactly m local maxima. As we traverse K, we alternate between traversing arcs above the plane and arcs below the plane. It follows that the representative of K also has exactly m local minima.

Definition 1. *[1] Let K be a knot (or link) in $\mathbb{R}^3$ which meets a plane $E \subset \mathbb{R}^3$ in 2m points such that the arcs of K contained in the upper halfplane have exactly one local maxima and no other critical points and the arcs in the lower halfplane have exactly one local minimum. (K,E) is called an m-bridge presentation of k; the minimal number m possible for a knot K is called its bridge number br(K).*

(a) The trefoil knot with 3 bridges.



(b) The trefoil knot with 2 bridges in blue and red.

Figure 1: The trefoil knot

2 Classification of two-bridge knots

In this presentation we fill focus on 2-bridge knots, so knots with $br(K) = 2$. They are the simplest type of knots that are nontrivial, and they have been classified.

In order to study this classification, we need to represent our knot in a special way. In this section I will discuss two different representations, namely the Conway representation as a rational knot and the Schubert normal form. I will then show how they are related and explain how we can classify two-bridge knots.

2.1 Conway presentation

The Conway presentation of a knot discusses the knot as a rational knot. In order to study these, we first need to study rational tangles.

Rational tangles can be constructed inductively, using two operations. One always starts with the empty tangle, which is just two strands from top-left to top right and bottom left to bottom right. The two operations are called *twisting* and *rotation* and their function is depicted in Figure 2.

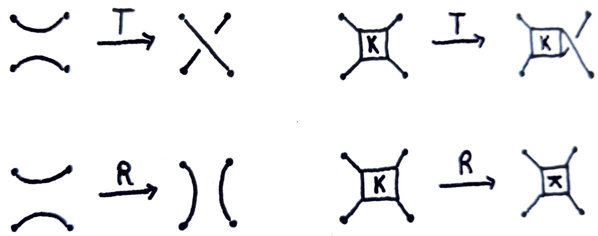


Figure 2: Above: Twisting on an empty and general tangle. Below: Rotation on an empty and general tangle.

Using these operations, we can create a tangle as depicted in Figure 3.

Figure 3: An example of a rational tangle

Now, in the same fashion as we closed a braid to get a knot, we can also close a rational tangle to get a knot. We can close a tangle in two ways, as depicted in Figure 4.

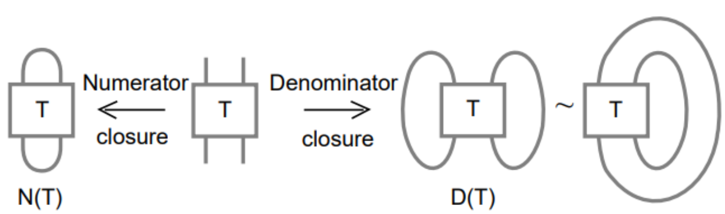


Figure 4: The numerator and denominator closure of a rational tangle [3].

Definition 2. *A knot K is called a rational knot if there exists a rational tangle T such that $K = N(T)$ is the numerator closure of T.*

It is known that any numerator closure of a rational tangle T always produces a rational knot or link, but a denominator closure does not necessarily. [3]

The numerator closure of the rational tangle given in Figure 3 produces the knot 6_2.

The Conway notation of this knot is given by $C(3,1,2)$. More generally, given a rational knot as in Figure 5, it is denoted $C(a_1,\dots,a_n)$.

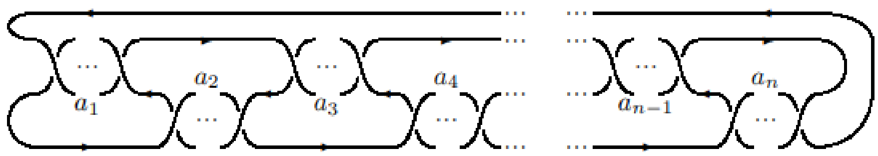


Figure 5: Conway presentation for knots [2].

Any two-bridge link may be expressed with all a_i even. If n is even, then it is a knot and if n is odd, it is a 2-component link. [2]

I've mentioned the term rational knot a few times already, and you might wonder right now, where does this name come from, I haven't seen any rational number yet. Well, we can create a rational number from the Conway notation of a knot using the continued fraction of its elements, as follows

$$\frac{\alpha}{\beta} = a_1 + \cfrac{1}{a_2 + \dots \cfrac{1}{a_{n-1} + \frac{1}{a_n}}}.$$

More on rational tangles can be found in [3].

Coming back to our beloved trefoil knot, recalling Example 8 from Lecture 1, it is easy to see that its Conway presentation is given by $C(3)$, as can be seen in Figure 6. This gives us $\alpha = 3$ and $\beta = 1$.

Figure 6: The trefoil knot in its Conway presentation

2.2 Schubert normal form

Using this notation as a rational knot, we can move naturally to the representation of a knot in its Schubert normal form. Using this presentation, it is more easy to see that these knots are indeed two-bridge knots. Given our fraction $\frac{\alpha}{\beta}$, we can construct the Schubert normal form as follows:

1. Draw two arcs $w_1 = AB$ and $w_2 = CD$, numbered from 0 to $2\alpha - 1$, starting from the inside. These two arcs represent the bridges.
2. From B, start a new knot segment v_1, going first to β on w_2, then to 2β on w_1 and repeat until you reach $\alpha\beta = C$ on w_2.
3. Do the same procedure, creating the segment v_2, now starting at D and ending in A.
4. The knot constructed in this way is denoted $S(\alpha, \beta)$.

For the right-handed trefoil knot, we get the diagram in Figure 7 below.

Theorem 1. *[4] For any two bridge link L, there exist a pair of integers α, β satisfying*

$$\alpha > 0, -\alpha < \beta < \alpha, \gcd(\alpha, \beta) = 1, \beta \text{ odd}$$

such that $L = S(\alpha, \beta)$. If α is odd, we have a knot.

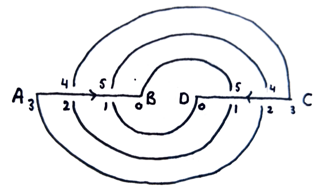


Figure 7: The trefoil knot in its Schubert normal form

So the 2-bridge knots are in fact precisely the rational knots. We can classify them using the following theorem:

Theorem 2. *[1] $S(\alpha, \beta)$ and $S(\alpha', \beta')$ are equivalent as knots if and only if*

$$\alpha = \alpha', \beta^{\pm} \equiv \beta' \mod \alpha.$$

Here β^{-1} denotes the integer with the properties $-\alpha < \beta^{-1} < \alpha$ and $\beta\beta^{-1} = 1 \mod 2\alpha$.

3 Interlude

Before discussing more about two-bridge knots as OU-tangles, I want to shortly discuss a nice theorem about the bridge number of a knot sum.

Theorem 3. $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$.

So, for example, we can find a knot with bridge number 3 by summing two knots with bridge number 2. I haven't looked into doing this, but you can find a list of knots with bridge number 3 by Googling **knots with bridge number 3** and then clicking the second hit.

Using this theorem we can also prove the following

Proposition 4. *Every two-bridge knot is a prime knot.*

Proof. Let $K = K_1 \# K_2$ be a knot with bridge number 2. Then

$$2 = b(K) = b(K_1 \# K_2) = b(K_1) + b(K_2) - 1.$$

Then $b(K_1) + b(K_2) = 3$, so this implies that we must have that $b(K_1) = 1$ and $b(K_2) = 2$ or vice versa, and hence in any decomposition of K as a connected sum, one of the components is the unknot. This is precisely what it means to be a prime knot. $\square$

4 Two-bridge knots as OU-diagrams

To end this lecture, we will compute the Z-invariant of the trefoil knot. In order to do this, I will first show it can be represented as $E = m_r^{a,b}(D)$, where D is an OU-tangle with $\mathcal{L}(D) = \{a, b\}$.

In order to do this, we start with the Schubert normal form of the trefoil knot as in Figure 7, as the pieces of strand from A to C and from C to A are already in OU form. Then we redraw the diagram a little to have a nicer presentation, as in Figure 8.

Now we can rotate it a bit and cut open the strands starting from A and C to end up in the situation as in Figure 9a. This is of course not the result we want yet, as not all crossings have their arrows both pointing up. Hence we need to rotate and redraw some more to eventually end up at a diagram as in Figure 9b.

This example is of course specifically for the trefoil knot, but any two-bridge knot can be written in this form.

Now, writing our two-bridge knot in this special way allows us to calculate the Z-invariant of the knot much faster. As it turns out, every bridge knot has a presentation in which is it alternating, i.e. writing it in terms of o and u, it is $ouou \ldots ou$. However, writing it in the form we discussed above, it is of the form $o \ldots ou \ldots uo \ldots ou \ldots u$, with maybe some C's somewhere.

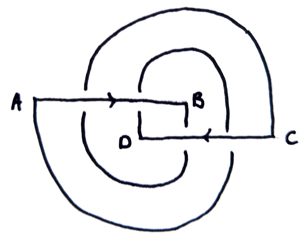
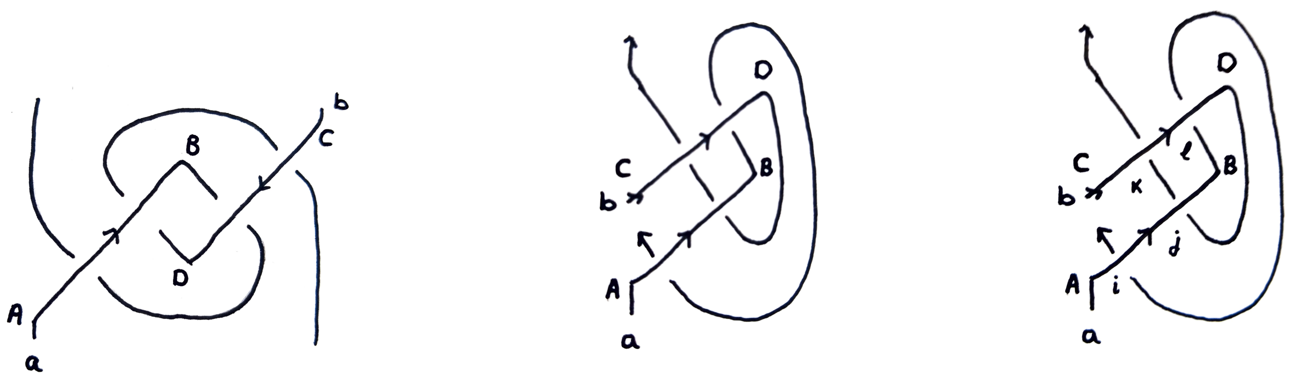


Figure 8: The trefoil knot in its nicer Schubert normal form



(a) The trefoil with two strands.

(b) OU tangle of the trefoil knot

(c) OU tangle of the trefoil knot

Figure 9: Creating an OU-tangle of the trefoil knot

We can easily calculate both the left side and the right side, so we only need to do one hard calculation once, namely in the middle.

So, for the trefoil knot, with i, j, k, l as indicated in Figure 9c, we get

$$\begin{aligned}
Z_{\mathbb{D}}(E) &= Z_{\mathbb{D}}(m_r^{a,b}(D)) \\
&= m_r^{a,b} Z_{\mathbb{D}}(D) \\
&= m_r^{a,b} \sum_{i,j,k,l} (o^i o^j u^l C^{-1} u^i)_a (o^k o^l C^{-1} u^j u^k)_b \\
&= m_r^{a,b} \sum_{i,j,k,l} (o^i o^j u^l i(C) u^i)_a (o^k o^l i(C) u^j u^k)_b \\
&= \sum_{i,j,k,l} (o^i o^j u^l i(C) u^i o^k o^l i(C) u^j u^k)_r
\end{aligned}$$

Then, using the techniques discussed in Lecture 12 and 13 we can work this out further.

References

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[2] David De Wit. The 2-bridge knots of up to 16 crossings. *Journal of Knot Theory and Its Ramifications*, 16(08):997–1019, 2007.

[3] Louis H Kauffman and Sofia Lambropoulou. On the classification of rational knots. *arXiv preprint math/0212011*, 2002.

[4] Akio Kawauchi. *Survey on knot theory*. Springer Science & Business Media, 1996.