# LECTURE 13: COMPLETING THE INVARIANT $Z_{\mathbb{D}}$

## 1. More on the antipode

Last lecture we argued that the algebras $\mathbb{O}, \mathbb{U}$ that we start with need to be Hopf algebras because of the way the positive and negative crossings relate. Recall that a Hopf algebra $A$ is a bialgebra where we have an additional linear map $\imath : A \longrightarrow A$ that satisfies

$$m_f^{\ell r}\imath_\ell \Delta^s_{\ell r}(a) = 1_f \varepsilon^s(a) = m_f^{\ell r}\imath_r\Delta^s_{\ell r}(a) \tag{1}$$

Our interpretation of the antipode in $XC$-tangles is reversal of the strand so this suggests that the antipode is like the inverse in a group in that $\imath(ab) = \imath(b)\imath(a)$. We will now prove that this is the case in any Hopf algebra. The proof will be purely algebraic and somewhat cumbersome, hopefully illustrating why the use of tangles to do algebra is pleasant.

**Lemma 1.** *For any Hopf algebra $A$ we have*

$$\imath_3 \circ m_3^{12} = m_3^{21} \circ \imath_1\imath_2$$

*Proof.* Recall that the bialgebra axiom says

$$m^{a_1,b_1}_{(ab)_1} m^{a_2,b_2}_{(ab)_2}\Delta^a_{a_1,a_2}\Delta^b_{b_1,b_2} = \Delta^{ab}_{(ab)_1,(ab)_2} m^{a,b}_{ab}$$

Now compose both sides with $m_b^{(ab)_1,(ab)_2,b_3}\imath_{(ab)_1}\imath_{b_3}\Delta^b_{b,b_3}$. The right-hand side will simplify using Equation (1) to

$$m_b^{(ab)_1,(ab)_2,b_3}\imath_{(ab)_1}\Delta^{ab}_{(ab)_1,(ab)_2}m^{a,b}_{ab}\imath_{b_3}\Delta^b_{b,b_3} = m_b^{f,b_3}\big(m_f^{(ab)_1,(ab)_2}\imath_{(ab)_1}\Delta^{ab}_{(ab)_1,(ab)_2}\big)m^{a,b}_{ab}\imath_{b_3}\Delta^b_{b,b_3} =$$

$$m_b^{f,b_3}1_f\varepsilon^{ab}m^{a,b}_{ab}\Delta^b_{b,b_3} = \varepsilon^a\imath_b$$

Likewise, the left-hand side of the bialgebra axiom will also simplify somewhat using the antipode axiom, first using (co)associativity to rearrange terms:

$$m_b^{(ab)_1,(ab)_2,b_3}\imath_{(ab)_1}m^{a_1,b_1}_{(ab)_1}m^{a_2,b_2}_{(ab)_2}\Delta^a_{a_1,a_2}\Delta^b_{b_1,b_2}\imath_{b_3}\Delta^b_{b,b_3} = m_b^{(ab)_1,(ab)_2}\imath_{(ab)_1}m^{a_1,b_1}_{(ab)_1}m^{a_2,b_2}_{(ab)_2}\Delta^a_{a_1,a_2}\big(m^{b_2,b_3}_{b_2}\imath_{b_3}\Delta^{b_2}_{b_2,b_3}\big)\Delta^b_{b_1,b_2}$$

$$= m_b^{(ab)_1,(ab)_2}\imath_{(ab)_1}m^{a_1,b_1}_{(ab)_1}m^{a_2,b_2}_{(ab)_2}\Delta^a_{a_1,a_2}\big(1_{b_2}\varepsilon^{b_2}\big)\Delta^b_{b_1,b_2} = m_b^{(ab)_1,a_2}\imath_{(ab)_1}m^{a_1,b}_{(ab)_1}\Delta^a_{a_1,a_2}$$

Finally composing both sides with $m^{b,a_3}_{ab}\imath_{a_3}\Delta^a_{a,a_3}$ yields the desired result. The right hand side becomes:

$$m^{b,a_3}_{ab}\varepsilon^a\imath_b\imath_{a_3}\Delta^a_{a,a_3} = m^{b,a}_{ab}\imath_b\imath_a$$

while the left-hand side simplifies using the antipode axiom and becomes:

$$m^{b,a_3}_{ab}m_b^{(ab)_1,a_2}\imath_{(ab)_1}m^{a_1,b}_{(ab)_1}\Delta^a_{a_1,a_2}\imath_{a_3}\Delta^a_{a,a_3} = m^{(ab)_1,a_3}_{ab}\imath_{(ab)_1}m^{a_1,b}_{(ab)_1}\big(m^{a_2,a_3}_{a_3}\imath_{a_3}\Delta^{a_2}_{a_2,a_3}\big)\Delta^a_{a_1,a_2} =$$

$$m^{(ab)_1,a_3}_{ab}\imath_{(ab)_1}m^{a_1,b}_{(ab)_1}\big(1_{a_3}\varepsilon^{a_2}\big)\Delta^a_{a_1,a_2} = \imath_{ab}m^{a,b}_{ab}$$

□

The above proof is an extremely painful version of the following argument in a group: First $(ab)^{-1}a = (ab)^{-1}(ab)b^{-1} = b^{-1}$ and multiplying from the right by $a^{-1}$ yields $(ab)^{-1} = b^{-1}a^{-1}$. In the literature one often sees diagrams employing a $Y$-shape to depict the coproduct and an upside down $Y$ for the multiplication to make such proofs more transparent. We will not go into this because usually the tangle diagrams provide an even better graphical notation for our purposes.

## 2. An invariant of 0-rotation $XC$-tangles

**Definition 2. (0-rotation $XC$ tangles)** The total rotation of a strand of an $XC$-diagram is the sum of the rotation numbers of the edges on that strand. We say an $XC$-tangle diagram has 0-rotation if all strands have total rotation 0.

By merging a few $C$'s at the end of a tangle, any tangle diagram can be modified to have 0 rotation. For example Reidemeister $\Omega 2c$ can be modified to become

$$\check{m}_i^{1,3}\check{m}_j^{0,2,4,6}\check{C}_0^{-1}\check{C}_4\check{X}_{1,6}^{-1}\check{X}_{3,2} = \check{1}_i\check{1}_j \quad (\Omega 2c0) \tag{2}$$

We consider 0-rotation $XC$-tangle diagrams up to equivalence by the Reidemeister moves $\Omega 0, \Omega 2b, \Omega 2c0$ and $\Omega 3$ skipping $\Omega 1f$ for now.

To find a version of our invariant $Z_{\mathbb{D}}$ that works for 0-rotation $XC$-tangles even when they have $C$'s we propose to redistribute the $C$'s in pairs $\check{C}, \check{C}^{-1}$ so that we can use the antipode instead of the $C$s themselves. More specifically the basis for our invariant will be the following lemma:

**Lemma 3.** *Any 0-rotation $XC$-tangle diagram can be written as the result of repeatedly applying the operations $\imath^2$, merge and disjoint union to the crossings $\check{X}^{\pm 1}$.*

*Proof.* If the rotation numbers along a strand are $r_1, r_2, \dots r_n$ then we may find integers $s_0, s_1, \dots s_{2n}$ such that $r_i = s_{2i-1} + s_{2i}$ and $s_{2i} = -s_{2i+1}$. This means that we can realize the rotation numbers on the edges by applying an appropriate power of $\imath^2$ to the crossing where the edges meet. □

**Theorem 4.** *Suppose $\mathbb{O}$ and $\mathbb{U}$ are Hopf algebras and $X, X^{-1}$ are built from their bases $o^i, u^i, \tilde{u}^i$ as in the previous lectures.*

*We also constructed a Hopf algebra on $\mathbb{D} = \mathbb{O}\mathbb{U}$ and claim that For any 0-rotation $XC$-tangle diagram $T$ the invariant $Z_{\mathbb{D}}(T) \in \mathbb{D}^{\otimes \mathcal{L}(T)}$ is well-defined, is invariant under the Reidemeister moves $\Omega 2b, \Omega 2c0, \Omega 3$ and is compatible with all $XC$-tangle operations:*

$$Z_{\mathbb{D}}(TT') = Z_{\mathbb{D}}(T)Z_{\mathbb{D}}(T') \quad m_r^{ht}Z_{\mathbb{D}}(T) = Z_{\mathbb{D}}(\check{m}_r^{ht}T) \quad \imath_s Z_{\mathbb{D}}(T) = Z_{\mathbb{D}}(\check{\imath}_s T) \quad \Delta^s_{\ell r}Z_{\mathbb{D}}(T) = Z_{\mathbb{D}}(\check{\Delta}^s_{\ell r}T)$$

*Proof.* It follows from our previous work that $Z_{\mathbb{D}}$ is well-defined. Compatibility with the coproduct, merge and disjoint union are also taken care of. Compatibility with the antipode follows from the fact that in any Hopf algebra the antipode is an algebra anti-morphism, see the lemma above.

Moving on to the Reidemeister moves, these are all equalities between tangles. For all but Reidemeister $\Omega 0$ and $\Omega 2c0$ we already proved invariance. Notice $\Omega 0$ can be restated as $\check{\imath}_1^2\check{\imath}_2^2(\check{X}_{12}) = \check{X}_{12}$. To show this holds for the invariant too (removing the checks) we argue as follows: $(\imath_{\mathbb{D}})_1(\imath_{\mathbb{D}})_2 X_{12} = (\imath_{\mathbb{O}})_1^{-1}(\imath_{\mathbb{U}})_2(X_{12}) = (\imath_{\mathbb{O}})_1^{-1}(X_{12}^{-1}) = X_{12}$. This is because we can write $X_{12}^{-1} = \sum_i o_1^i \tilde{u}_2^i = \sum_i \tilde{o}_1^i u_2^i$ and we claim that $\tilde{o}^i = \imath_{\mathbb{O}}(o^i)$. To prove this we recall that $\imath_{\mathbb{O}} = \imath_{\mathbb{U}}^*$ so $\imath_{\mathbb{O}}(x) = \sum_j \langle \imath_{\mathbb{O}}(x), u^j\rangle o^j = \sum_j \langle x, \imath_{\mathbb{U}}(u^j)\rangle o^j = \sum_j \langle x, \tilde{u}^j\rangle o^j$. Therefore $X_{12}^{-1} = \sum_j o_1^j \tilde{u}_2^j = \sum_{i,j}\langle o^i, \tilde{u}^j\rangle o_1^j u^i = \sum_i \imath_{\mathbb{O}}(o^i)_1 u_2^i$ as required.

Next Reidemeister $\Omega 2c0$ is obtained from $\Omega 2b$ by applying $\check{\imath}$ on one of the strands. Indeed we have (sketch the tangle!)

$$\check{\imath}_1\check{m}_1^{13}\check{m}_2^{24}X_{12}\check{X}_{34}^{-1} = \check{m}_1^{31}\check{m}_2^{24}\check{\imath}_1\check{\imath}_3 X_{12}\check{X}_{34}^{-1} = \check{m}_1^{31}\check{m}_2^{24}\check{\imath}_1^2 X_{12}^{-1}\check{X}_{34}$$

The right hand side is precisely the non-trivial part of $\Omega 2c0$. Since we already showed $Z_{\mathbb{D}}$ is compatible with all the operations we used, invariance under the Reidemeister moves is proven. □

## 3. Adjoining the element $C$

While technically sufficient it may be more satisfying to actually have an element $C$ so that our invariant $Z_{\mathbb{D}}$ of $XC$ tangles just sends $\check{X}$ to $X$ and $\check{C}$ to $C$. While it is not always possible to find a suitable element $C$ in the algebra $\mathbb{D}$ we can always adjoin it formally to get the final form of our knot invariant.

Define the algebra $\mathbb{ID}$ to be the $k$-vector space spanned by the set $\{C^r o^i u^j | r \in \{0,1\}, i, j \in \{1, \dots n\}\}$. We will turn $\mathbb{ID}$ into a Hopf algebra so that $\mathbb{D}$ is a sub-Hopf algebra by the following formulas. Define the element $\gamma = m_1^{1234}\imath_1^{-2}\imath_4^2(X_{21}^{-1}X_{34}) \in \mathbb{D}$. This element is invertible with inverse $\gamma^{-1} = \imath_{\mathbb{D}}(\gamma)$ (exercise, hint: draw a picture). First we define the algebra structure on $\mathbb{ID}$ via the equations

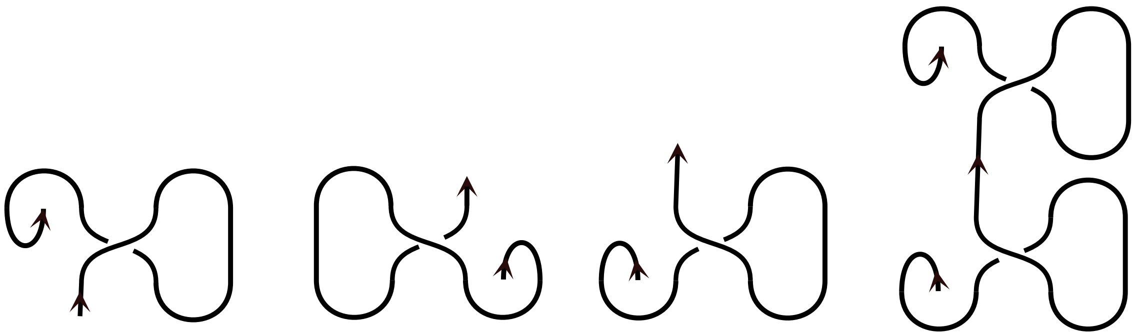
$$C^2 = \gamma \quad xC = C\imath^{-2}(x),\ x \in \mathbb{D} \tag{3}$$

We often speak about $C^{-1}$ and by this we mean $C\gamma^{-1}$. The motivation for the first formula is that the tangle diagram with the same description as we gave for $\gamma$ was shown in the last lecture to be equivalent to $\check{m}_1^{12}\check{C}_1\check{C}_2$. The second equation can be read as $\imath^{-2}(x) = C^{-1}xC$ and that should remind one of the effect of applying the antipode twice on a tangle diagram: we get back to where we started but with some curls.

To check that the proposed algebra is in fact associative we need to verify for all $x, y \in \mathbb{D}$:

$$(CC)x = \gamma x = \imath^{-4}(x)\gamma = C(Cx) \qquad C(xy) = (Cx)y$$

The last equality is a simple consequence of the fact that $\imath$ is an algebra morphism. Proving that $\gamma x = \imath^{-4}(x)\gamma$ holds for all $x \in \mathbb{D}$ is more challenging and is best done using tangle diagrams.



$\check{U}$ $\check{U}^{-1}$ $\check{\imath}(\check{U}^{-1})$ $\check{\gamma}$

FIGURE 1. The tangles $\check{U}, \check{U}^{-1}, \check{\imath}(\check{U}^{-1})$ and $\check{\gamma}$.

**Lemma 5.** *$\gamma x = \imath^{-4}(x)\gamma$ holds for all $x \in \mathbb{D}$ and hence $\mathbb{ID}$ is an associative algebra.*

*Proof.* If we define

$$\check{U}_i = \check{m}_i^{123}\check{X}_{12}\check{C}_3 \qquad \check{U}_i^{-1} = \check{m}_i^{123}\check{X}_{32}^{-1}\check{C}_1^{-1}$$

see Figure 1 then

$$\check{m}_1^{12}U_1^{-1}U_2 = 1_1 \qquad \check{\gamma} = \check{m}^{12}\check{\imath}(\check{U}_1^{-1})\check{U}_2$$

Set $V = \imath(U^{-1})$ so $\gamma = VU$ and it suffices to prove that for all $x \in \mathbb{D}$ we have $xU = U\imath^2(x)$ and $Vx = \imath^{-2}(x)V$.

We will only prove the first equality and leave the one involving $V$ as an exercise for the reader. Since the $o^i$ and $u^j$ generate $\mathbb{D}$ as an algebra it is enough to check that

$$m_1^{12}X_{10}U_2 = m_1^{12}U_1\imath_2^2X_{20} \qquad m_1^{12}X_{01}U_2 = m_1^{12}U_1\imath_2^2X_{02}$$

We prove the first equality by drawing a picture, see Figure 2 and appealing to Theorem 6 to make sure the equality for the tangles shown translates to equality of the implied algebra expressions. Of course we are implicitly using $U = Z_{\mathbb{D}}(\check{U})$ throughout. Notice we needed to add the extra curls in the final picture to emphasize that we are computing the invariant of these diagrams viewed as rotation 0 tangle diagrams. $\square$

To complete the picture we extend $\mathbb{ID}$ to a Hopf algebra by stating the relations that are forced by what we know about tangles and our desire for $Z_{\mathbb{ID}}$ to be compatible with all tangle operations.

$$\Delta(C) = C \otimes C \quad \imath(C) = C^{-1} \quad \varepsilon(C) = 1 \tag{4}$$

The final theorem then is that

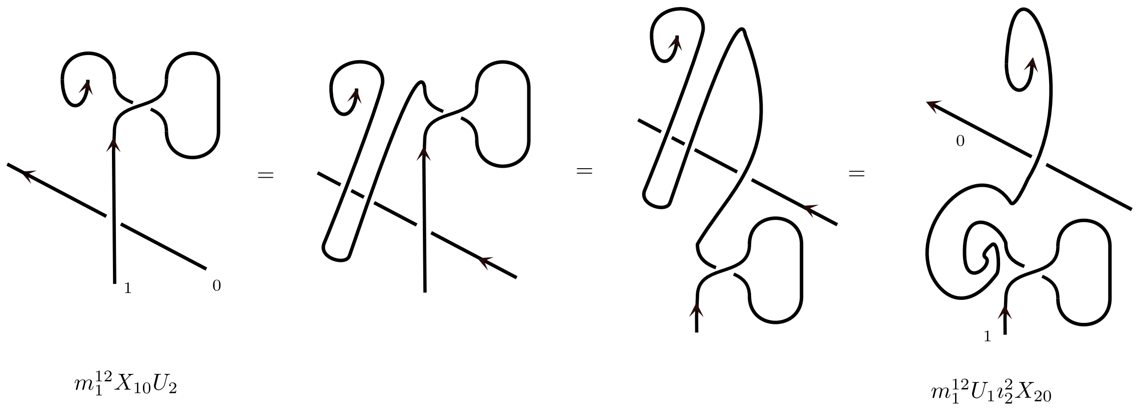



FIGURE 2. The equality $m_1^{12}X_{10}U_2 = m_1^{12}U_1\imath_2^2X_{20}$ $\quad m_1^{12}X_{01}U_2$ follows from the depicted equality of (rotation 0 $XC$) tangles and invariance of $Z_{\mathbb{D}}$.

**Theorem 6.** *Suppose $\mathbb{O}$ and $\mathbb{U}$ are Hopf algebras and construct $X^{\pm}$ as above. Adjoin $C$ to the Hopf algebra $\mathbb{D}$ to get a Hopf algebra $\mathbb{ID}$ as above. For any $XC$-tangle diagram $T$ there exists a well-defined element $Z_{\mathbb{D}}(T) \in \mathbb{D}^{\otimes \mathcal{L}(T)}$ that is invariant under all $XC$-Reidemeister moves and is compatible with all $XC$-tangle operations.*

*In fact, the algebra $\mathbb{ID}$ is characterised uniquely up to isomorphism of Hopf algebras by these requirements.*

*Proof.* Most of the theorem was already checked in the lectures and the serious reader should check to what extent that is really the case. One thing that remains mysterious is the Reidemeister move $\Omega1f$ so let us comment on that. By construction of the algebra $\mathbb{ID}$ and the invariant $Z_{\mathbb{ID}}$ we have

$$Z_{\mathbb{ID}}(\check{\gamma}) = \gamma = C^2 = Z_{\mathbb{ID}}(\check{m}_1^{12}\check{C}_1\check{C}_2) \in \mathbb{ID}$$

Merging both sides of $\check{\gamma} = \check{m}_1^{12}\check{C}_1\check{C}_2$ with $\check{C}^{-1}$ from above and below yields (make a sketch!)

$$\check{m}_1^{12}\check{\alpha}_1\check{\alpha}_2^{-1} = \check{1}_1 \qquad \text{where} \qquad \alpha = \check{m}_1^{123}\check{C}_2^{-1}\check{X}_{13} \qquad \alpha^{-1} = \check{m}_1^{123}\check{C}_2^{-1}\check{X}_{31}^{-1}$$

Since $Z_{\mathbb{ID}}$ is required to be compatible with merging we must conclude that $Z_{\mathbb{ID}}(\check{m}_1^{12}\check{\alpha}_1\check{\alpha}_2^{-1}) = 1$ which is equivalent to $\Omega1f$.

□

Our final theorem emphasizes the inevitability of our constructions. All we had to do was stay true to our tangle diagrams and respect all tangle operations.