Knot Theory seminar lecture 05-03-2020

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In this lecture we deal with a very algebraic topic, no reference to 'knots' is needed. We use the book Introduction to Knot Theory by Crowell and Fox chapter VII (p94-107). All precise definitions and proofs can be found in that book.

1 From group presentations to elementary ideals

Introduction: To identify knots of a certain knot type, we develop different knot invariants. There is always a trade-off between completeness and distinguishablility: Some invariants are easier to compute, so it is easier to distinguish several knots, but are less powerful and not able to completely tell all knots apart.
Last time Wout introduced the invariant of knot groups. But there is no easy way (no general algorithm) to distinguish presentations of groups. Therefore we introduce invariants of group presentations, namely elementary ideals. In the next chapter/hour we further specify this to distinguish presentations of knot groups by Alexander polynomials. Alexander polynomials can be uniquely defined in terms of elementary ideals for tame knot groups, but not for all finitely presented groups. Elementary ideals are defined for any finitely presented group. The invariance of elementary ideals is easier to prove than that of the polynomials. So we obtain the following sequence of knot invariants: Knot type - presentation of knot group - elementary ideals - Alexander polynomial. Below we will explain how to make (a chain of) elementary ideals for a given finitely presented group.
The first step is to define a group ring, that contains the presented group. On that ring we can define a derivation. We can also reduce modulo the relations of the presentation and abelianise the ring. We can form the 'Alexander matrix' with the different derivatives of relations to generators. By looking at the determinants of specific submatrices we construct generators of ideals in the ring. The ideals are contained in one another.

1.1 Group ring

Let $G = (\mathbf{x} : \mathbf{r})$ be a finitely presented multiplicative group with generators $x_1, \ldots, x_n$ and relations $r_1, \ldots, r_m$. Elements of this group are (equalence classes of) words. Multiplication is defined by concatenation of words and the unit element is the empty word.
The group ring is a ring $R = \mathbb{Z}[G]$ that is an extension of a multiplicative group. Every element of the group can be given a weight from a given ring, in our case $\mathbb{Z}$. So elements in $\mathbb{Z}[G]$ are of the form $\sum n_i g_i$, with $n_i \in \mathbb{Z}$ and $g_i \in G$. The multiplication is the multiplication of a multiplicative group, so $\sum n_i g_i \cdot \sum n_j g_j = \sum (n_i n_j)(g_i g_j)$, where $n_i n_j$ is normal multiplication is $\mathbb{Z}$ and $g_i g_j$ is multiplication in G. The addition is defined by addition of the weights, so $\sum n_i g_i + \sum n_j g_i = \sum (n_i + n_j) g_i$ For example: G contains $g_1, g_1 g_2, g_1^{-1} g_2 g_1$ and R contains $2g_1, -5g_2^{-1} + g_1 - 1, 0$.

1.2 Derivation

On the groupring $\mathbb{Z}[G]$ (of a finitely presented group G) we define a derivation D as follows. $D : \mathbb{Z}[G] \to \mathbb{Z}[G]$ such that:
(i) $D(v_1 + v_2) = D(v_1) + D(v_2)$, (so also $D(n) = 0$)
(ii) $D(v_1 v_2) = D(v_1)t(v_2) + v_1 D(v_2)$, where $t : \mathbb{Z}[F] \to \mathbb{Z}[F]$, (for the free group $F = F(x_1, ..., x_n),$) is the

trivialiser that sets every generator equal to one. For example $t(3x_1 + 2x_2x_3^{-1}) = 5$.
Filling in (ii) by $v_1 = g^{-1}, v_2 = g$ gives us $D(g^{-1}g) = D(g^{-1})+g^{-1}D(g)$, thus we obtain $D(g^{-1}) = -g^{-1}D(g)$.

1.3 Alexander matrix

From a group representation, we can derive a matrix. We take the abelianisation $\mathfrak{a}$ of the reduction mod $r_1, \ldots, r_m$ of the derivative of r_i w.r.t. x_j. In a formula: $a_{ij} = \mathfrak{a}\gamma(\frac{\partial r_i}{\partial x_j})$, where γ is the reduction map. For a finitely presented group with n generators and m relations this matrix is $m \times n$.

1.4 Elementary ideals

From a matrix over a (unitary commutative) ring we can define elementary ideals of that ring. Let R be a commutative unitary ring and consider an $m \times n$ matrix A over R. $\forall k \in \mathbb{Z}_{\geq 0}$, the k-th elementary ideal $E_k(A)$ of A is defined as follows:
If $0 < n-k \leq m$, then $E_k(A)$ is the ideal generated by the determinants of all $n-k \times n-k$ submatrices of A.
If $n-k > m$, then $E_k(A) = 0$.
If $n-k \leq 0$, then $E_k(A) = R$.
We have an ascending chain of elementary ideals of A: $E_0(A) \subset E_1(A) \subset \ldots \subset E_n(A) = E_{n+1}(A) = \cdots = R$. (This can easily be seen by realising that every determinant can be expressed/computed in terms of subdeterminants.)
We see that different matrices can lead to the same chain of elementary ideals. We want to look at which matrices lead to the same ideals and regard them as equivalent. As we will see, the next equivalence relation does the job. We define an equivalence relation on matrices: $A \sim A' \iff \exists$ a finite sequence of matrices $A = A_1, \ldots, A_l = A'$ such that A_{i+1} is obtained from A_i or vice versa, by one of the following operations:
(i) Permuting rows or permuting columns.
(ii) Adjoining a row of zeros: $A \to \begin{pmatrix} A \\ 0 \end{pmatrix}$.
(iii) Adding to a row a linear combination of other rows.
(iv) Adding to a column a linear combination of other columns.
(v) Adjoining a new row and column such that the entry in the intersection of the new row and column is 1, and the remaining entries in the new row and column are all 0, $A \to \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.
From the above operations we can derive other equivalence operations (where e is the unit of the ring):
(v') $A \to \begin{pmatrix} A & 0 \\ a & 1 \end{pmatrix}$.
(vi) $\begin{pmatrix} A \\ a \end{pmatrix} \to \begin{pmatrix} A \\ ea \end{pmatrix}$.
(vii) $\begin{pmatrix} A & a \end{pmatrix} \to \begin{pmatrix} A & ea \end{pmatrix}$.
It is a fact that equivalent matrices define the same chain of elementary ideals. For a proof, see Crowell&Fox VII.4.2.

1.5 Invariance of elementary ideals

We want that different presentations of the same group give rise to the same chain of elementary ideals. So different presentations of the same group should lead to equivalent matrices. (The reverse is not possible, there may be different presentations of different groups that also lead to the same elementary ideals.)
It is a theorem (Crowell&Fow VII.4.5) that the above holds: If $(\mathbf{x} : \mathbf{r})$ and $(\mathbf{y} : \mathbf{s})$ are finite group presentations and $f : (\mathbf{x} : \mathbf{r}) \to (\mathbf{y} : \mathbf{s})$ is a presentation equivalence, the the kth elementary ideal of $(\mathbf{x} : \mathbf{r})$ is mapped onto the kth elementary ideal of $(\mathbf{y} : \mathbf{s})$.
A proof of this can be given by using Tietze's theorem and checking invariance under Tietze operations I and II.
An easily derived theorem (Crowell&Fow VII.4.6) is: The elementary ideals are invariants of any finitely presented group G, i.e. any two finite presentations of G have the same chains of ideals.

2 Exercises

Exercise 1. Calculate the elementary ideals corresponding to the trefoil knot using the presentation $< x, y, z \mid yzx^{-1}z^{-1}, xyz^{-1}y^{-1}, zxy^{-1}x^{-1} >$

Exercise 2. Calculate the elementary ideals for the knot group corresponding to some knot (for example the figure-eight knot).

Exercise 3. Calculate the chain of elementary ideals for the free abelian group of rank n, and conclude that, for $n > 1$, the free group of rank n is not abelian.