LECTURE 10: WHERE DO XC-ALGEBRAS COME FROM?

The goal of this and the following lectures is to use our intuition from tangle diagrams to construct many examples of XC-algebras. The most important equation that an XC-algebra should satisfy is the $\Omega 3$ or Reidemeister 3 equation. We will start by investigating a simplified case where the tangles are of OU-type. A motivation for studying this class of tangle diagrams is that one of the sides of $\Omega 3$ is OU while the other is not.

Definition 1. **(OU diagrams)**
An open XC-tangle diagram is called over-then-under, abbreviated as OU, when it can be generated by only $X^{\pm}$ and walking along any of the strands we first meet all the overpasses and then all the underpasses.

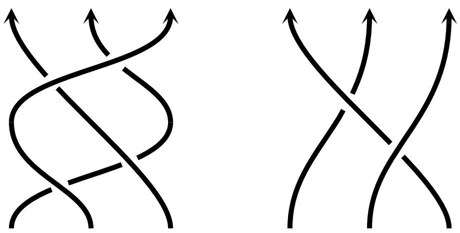

FIGURE 1. Two tangle diagrams, the left is OU, the right is not.

To construct an XC-algebra we start by making an invariant of OU-diagrams that is as nice as possible. For this we will use a pair of algebras $\mathbb{O}$ and $\mathbb{U}$ and an invertible element $X \in \mathbb{O} \otimes \mathbb{U}$. To make full use of our algebras we ask that X is of the form $X = \sum_n o^n \otimes u^n$ for a basis (o^n) of $\mathbb{O}$ and a basis (u^n) of $\mathbb{U}$. Our invariant $Z_{\mathbb{D}}$ will take values in the vector space $\mathbb{D} = \mathbb{O}\mathbb{U}$ spanned by all products $o^i u^j$ where $o^i \in \mathbb{O}$ and $u^j \in \mathbb{U}$ are the basis elements. This implies that for any $o \in \mathbb{O}$ and $u \in \mathbb{U}$, we have $ou \in \mathbb{D}$.

Note that $\mathbb{D}$ is not yet an algebra so we do not know how to multiply two elements unless both are in $\mathbb{O} \subset \mathbb{D}$ or both are in $\mathbb{U} \subset \mathbb{D}$. In that case we can just multiply in $\mathbb{O}$ and/or $\mathbb{U}$ separately. Since $(o \otimes u)(o' \otimes u) = oo' \otimes uu'$ this is enough to make sense of the statement that X should have an inverse. In practice we can just set $X^{-1} = \sum_m o^m \otimes \tilde{u}^m$ for some new basis $\tilde{u}^m$ chosen so that $\sum_{m,n} o^m o^n \otimes \tilde{u}^m u^n = 1 \otimes 1$.

Throughout the lecture we will refer to two running examples to see what is going on:

Example 2. **(Finite group case)**
For any finite group G take $\mathbb{O} = Fun(G) = \{f : G \longrightarrow k\}$, the functions on the group with basis of delta functions $o^g = \delta^g$. Remember that $1 = \sum_g \delta^g$. For $\mathbb{U}$ we choose the group algebra $\mathbb{U} = k[G] = \{\sum a_g g | a_g \in k\}$ with basis G, say $u^g = g$. This means $X = \sum_{g \in G} \delta^g \otimes g$ with inverse $X = \sum_{g \in G} \delta^g \otimes g^{-1}$, so $\tilde{o}^g = o^g$ and $\tilde{u}^g = (u^g)^{-1}$.

Example 3. **Sweedler algebra and its dual**
The algebra $\mathbb{O}$ will be a simplified version of the Dilbert algebra. It is 4-dimensional with basis $o^1 = l, o^2 = a, o^3 = b, o^4 = d$ with multiplication $a^2 = a, l^2 = l, ad = d, ba = b, lb = b, dl = d$ and all other products are 0. This product is simpler than that of $\mathrm{End}(k^2)$ because there db and bd were non-zero. Here is a multiplication table:

$\mathbb{O}$	l	a	b	d
l	l	0	b	0
a	0	a	0	d
b	0	b	0	0
d	d	0	0	0

Notice that we still have $1 = a + l$ as in the Dilbert algebra.

For $\mathbb{U}$ we choose the Sweedler algebra with basis $u^1 = 1, u^2 = s, u^3 = w, u^4 = sw$ and relations $s^2 = 1, w^2 = 0, ws = -sw$. In this case then $X_{ij} = l_i + a_i s_j + b_i w_j + d_i(sw)_j$. To work out what is $X^{-1} = \sum_m o^m \otimes \tilde{u}^m$ we compute $\sum_{m,n} o^m o^n \otimes \tilde{u}^m u^n = l \otimes \tilde{u}^1 u^1 + a \otimes \tilde{u}^2 u^2 + b \otimes (\tilde{u}^1 u^3 + \tilde{u}^3 u^2) + d \otimes (\tilde{u}^4 u^1 + \tilde{u}^2 u^4)$. Remembering that $a + l = 1$ and setting the coefficients of b and d equal to 0 we find $\tilde{u}^1 = 1,\ \tilde{u}^2 = s,\ \tilde{u}^3 = sw,\ \tilde{u}^4 = -w$. So $X_{ij}^{-1} = l_i + a_i s_j + b_i(sw)_j - d_i w_j$.

For any OU-tangle diagram T we define $Z_{\mathbb{D}}(T)$ as usual by the equations $Z_{\mathbb{D}}(\check{X}^{\pm}_{ij}) = X^{\pm}_{ij}$ and $Z_{\mathbb{D}}(ST) = Z_{\mathbb{D}}(S)Z_{\mathbb{D}}(T)$ and $Z_{\mathbb{D}}(\check{m}^{h,t}_r \check{T}) = m^{h,t}_r Z_{\mathbb{D}}(\check{T})$. In the merging equation it is assumed that both T and the result after merging are OU-tangles.

For example $Z_{\mathbb{D}}$ of the left-hand side diagram L in Figure 1 can be computed as indicated in Figure 2. We find

$$Z_{\mathbb{D}}(L) = \sum_{i,j,k,\ell} (\tilde{u}^i \tilde{u}^j u^\ell)_1 (o^i o^k o^\ell)_2 (o^j u^k)_3$$

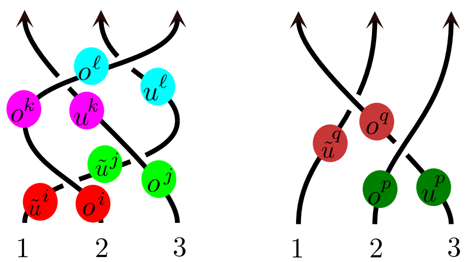

FIGURE 2. Computing the invariants $Z_{\mathbb{D}}(L)$ and $Z_{\mathbb{D}}(R)$ by placing beads on each crossing.

Notice how product terms in $Z_{\mathbb{D}}(T)$ of an OU-tangle diagram always are of the form $ooo..uuuu$.

To extend $Z_{\mathbb{D}}$ to more general tangles such as R in Figure 1 we need a genuine algebra structure on $\mathbb{D}$ so that we can multiply any $o's$ and $u's$ in any order they may appear in the diagram. For example if we follow the same rules to compute $Z_{\mathbb{D}}(R)$ we would find

$$Z_{\mathbb{D}}(R) = \sum_{p,q} \tilde{u}^q_1 o^p_2 (u^p o^q)_3$$

This of course does not yet have a meaning because we do not know what the term $u^p o^q$ in the third tensor factor should mean. On the other hand all the other parts of the expression make sense.

The key idea behind the construction is to notice that L and R are equivalent XC-diagrams by Reidemeister 2 and 3 (check it!). That means that if we ever want to extend $Z_{\mathbb{D}}$ in a way that makes it an invariant under Reidemeister moves we are forced to define it so that $Z_{\mathbb{D}}(R) = Z_{\mathbb{D}}(L)$. Assuming for a moment that this equality holds we see that the unknown product $u^p o^q$ must be equal to the coefficient of $\tilde{u}^q_1 o^p_2$ in the expression of $Z_{\mathbb{D}}(L)$. Since the o^x and the u^y and the $\tilde{u}^z$ all form bases this gives us a formula to define a product on $\mathbb{D}$.

To make these ideas a bit more precise we introduce two pairings $\langle\rangle : \mathbb{O} \otimes \mathbb{U} \longrightarrow k$ and $\overline{\langle\rangle} : \mathbb{O} \otimes \mathbb{U} \longrightarrow k$ defined by setting $\langle o^r, u^s\rangle = \delta_{r,s}$ and $\overline{\langle o^r, \tilde{u}^s\rangle} = \delta_{r,s}$. In other words $\langle x, u^k\rangle$ is the coefficient of o^k when we express x in in the basis $o^1, o^2 \dots$, so $x = \sum_p \langle x, u^p\rangle o^p$ and likewise $y = \sum_j \langle o^j, y\rangle u^j$. In yet other terms o^i is the basis dual to u^i with respect to the pairing $\langle\rangle$. And so this pairing gives a vector space isomorphism $\mathbb{O}^* \cong \mathbb{U}$.

For the tilde basis the same applies: $y = \sum_q \overline{\langle o^q, y\rangle} \tilde{u}^q$. The pairing $\overline{\langle\rangle}$ yields another isomorphism $\mathbb{O}^* \cong \mathbb{U}$ in which the basis dual to o^i is $\tilde{u}^i$.

Using the above notation our definition for the product is then

$$u^p o^q = \sum_{i,j,k,\ell} \overline{\langle o^q, \tilde{u}^i \tilde{u}^j u^\ell\rangle} \langle o^i o^k o^\ell, u^p\rangle o^j u^k \tag{1}$$

Let us see what this product turns out to be in our two running examples.

Finite group example continued.
First in the finite group case we have $\langle \delta^g, h\rangle = \delta^g(h)$ and $\overline{\langle \delta^g, h\rangle} = \delta^g(h^{-1})$. Therefore we get

$$p\delta^q = \sum_{i,j,k,\ell\in G} \overline{\langle \delta^q, i^{-1}j^{-1}\ell\rangle}\langle \delta^i\delta^k\delta^\ell, p\rangle \delta^j k$$

The summands i,k,ℓ must all be equal because of the product of delta functions and then we find $\langle \delta^i, p\rangle =\rangle \delta^i(p)$ so in fact $i=k=\ell=p$. By the same reasoning $q=(p^{-1}j^{-1}p)^{-1}$ because $\overline{\langle \delta^q, i^{-1}j^{-1}\ell\rangle} = \delta^q((p^{-1}j^{-1}i)^{-1})$. It follows that $j=pqp^{-1}$ so the formula for the product is just like in $D(G)$ from the previous lecture:

$$p\delta^q = \delta^{pqp^{-1}}p$$

Now we know why that formula worked so well and it was no coincidence that we found an XC-algebra structure on $D(G)$.

Sweedler example continued.
The notation in the second running example is less uniform but nevertheless we find a product on $\mathbb{D}$ in this case too. For example taking $p=2, q=1$ we compute

$$s\cdot l = u^2o^1 = \sum_{i,j,k,\ell} \overline{\langle o^1, \tilde{u}^i\tilde{u}^j u^\ell\rangle}\langle o^io^ko^\ell, u^2\rangle o^j u^k$$

The pairing $\langle o^io^ko^\ell, u^2\rangle$ can only be non-zero if $o^io^ko^\ell = o^2 = a$ and that only happens when $(i,k,\ell) = (2,2,2)$ corresponding to the product $a = aaa$. In this case the other pairing gives a non-zero result precisely when $j=1$ because then $\tilde{u}^2\tilde{u}^j u^2 = s\tilde{u}^j s = 1$. We conclude that $(i,j,k,\ell) = (2,1,2,2)$ is the only non-zero term in the big sum defining the product sl so we find

$$s\cdot l = o^1u^2 = ls$$

A more exciting product to consider is wd. Taking $p=3, q=4$ we follow the same reasoning to evaluate

$$w\cdot d = u^3o^4 = \sum_{i,j,k,\ell} \overline{\langle o^4, \tilde{u}^i\tilde{u}^j u^\ell\rangle}\langle o^io^ko^\ell, u^3\rangle o^j u^k$$

Which triples (i,k,ℓ) are such that $o^io^ko^\ell = o^3 = b$? We can have $(1,1,3)$ or $(1,3,2)$ or $(3,2,2)$, corresponding to $b = llb = lba = baa$. And for how many of these options will we find a j such that $\tilde{u}^i\tilde{u}^j u^\ell = \tilde{u}^4 = -w$? $(1,1,3)$ works with $j=1$ and the pairing yields -1. $(1,3,2)$ works with $j=3$ because then we get $sws = -w$ and the pairing yields 1. Finally $(3,2,2)$ works with $j=1$ giving 1 as well, there are no other options. Thus the three non-zero terms in the sum are $(i,j,k,\ell) = (1,1,1,3), (1,3,3,2)$ and $(3,1,2,2)$ so we found

$$wd = o^3u^3 + o^1u^2 = bw + l(s-1)$$

Continuing in this fashion we can compute a multiplication table for $\mathbb{D}$

$\mathbb{D}$	l	a	b	d
s	ls	as	$-bs$	$-ds$
w	aw	lw	$dw + a(s+1)$	$bw + l(s-1)$

All products not in the table follow by associativity and the assumption that $\mathbb{O}$ and $\mathbb{U}$ are subalgebras in the obvious way.

Some important questions remain: is the product we found really always associative? And how do we find a suitable value for C? And what is the meaning of pairings of products $\langle o^io^ko^\ell, u^2\rangle$ like the ones appearing in the product formula for $\mathbb{D}$?

If the product is not associative then $Z_{\mathbb{D}}(T)$ is not defined because its value will depend on the order in which we assembled T by merging crossings. These questions will be addressed more fully in the next two lectures but let us make some comments already.

The key to all of these questions will be following interpretation of the dual of the multiplication map. Consider $Z_{\mathbb{D}}(\check{m}_3^{3,4}\check{X}_{14}\check{X}_{23}) = \sum_{a,b} o_1^a o_2^b u_3^b u_3^a = \sum_{a,b} o_1^a o_2^b \sum_r \langle u^b u^a, o^r\rangle u^r =$

$\sum_r \big(\sum_{a,b}\langle u^b u^a, o^r\rangle o_1^a o_2^b\big) u_3^r = \sum_r \Delta(o^r)u^r$, where $\Delta(x) = \sum_{a,b}\langle u^b u^a, x\rangle o_1^a o_2^b$. These computations suggest that Δ has something to do with doubling a strand., see Figure 3 In this case we doubled the over-strand of a positive crossing but one could hope more generally that

$$Z(\text{T with strand s doubled}) = \Delta_s(Z(T))$$

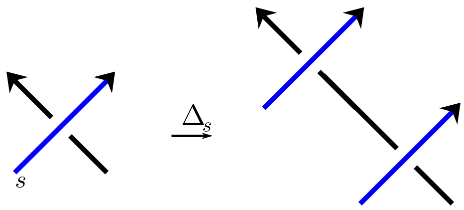

FIGURE 3. Doubling the over-strand s, shown in blue.

To appreciate the problem about associativity let us consider an example where our construction fails: Take $\mathbb{O}$ to be the algebra generated by x with relation $x^2 = 0$ and $\mathbb{U}$ the algebra generated by y with relation $y^2 = 1$. The bases are $o^1 = 1, o^2 = x$ and dual basis $u^1 = y, u^2 = x$ so $X = 1\otimes y + x\otimes 1$. This has inverse $X^{-1} = 1\otimes(-y) + x\otimes 1$, so $\tilde{u}^1 = -y$ and $\tilde{u}^2 = 1$. Now consider

$$yx = u^1 o^2 = \sum_{i,j,k,\ell} \overline{\langle o^2, \tilde{u}^i \tilde{u}^j u^\ell\rangle}\langle o^i o^k o^\ell, u^1\rangle o^j u^k$$

The only way we can have $o^i o^k o^\ell = 1 = o^1$ is $(i,k,\ell) = (1,1,1)$. In this case the second pairing is $\overline{\langle o^2, y\tilde{u}^j y\rangle}$ which gives -1 when $j = 1$ so $yx = -o^1 u^1 = -y$. This product cannot be associative because otherwise $0 = y(xx) = (yx)x = -yx$ and so $x = (yy)x = y(yx) = 0$. However in the four-dimensional vector space $\mathbb{D} = \mathbb{O}\mathbb{U}$ the vector x is a basis element so it cannot be 0.