Knot Groups: Exercises

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Exercise 1: Fundamental group of wedge sum
The wedge sum of two pointed spaces $(X, x_0), (Y, y_0)$ is defined as $X \vee Y = (X \coprod Y)/\{x_0, y_0\}$, identifying $\{x_0, y_0\}$ to a single point p. Suppose that X, Y are path-connected. Suppose also that $\{x_0\}$ is a deformation retract of some neighbourhood in X, and similarly for $\{y_0\}$. Then show that:

$$\pi_1(X \vee Y, p) \cong \pi_1(X, x_0) * \pi_1(Y, y_0).$$

(Hint: use van Kampen's theorem.)

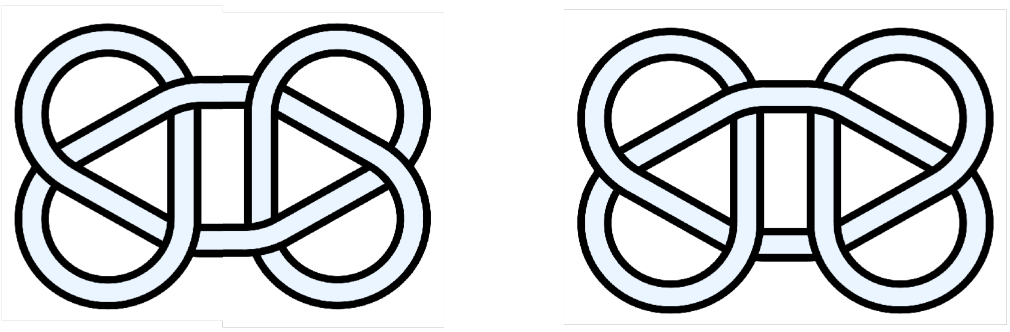

Figure 1: Granny knot (left) and square knot (right).

Exercise 2: Computing knot groups
a) Compute the knot group of the square knot (see figure 1).
b) Compute the knot group of the granny knot (see figure 1).
c) Show that the groups from parts a) and b) are isomorphic. What do you conclude about the knot group as an invariant?

Exercise 3: The Wirtinger presentation In this exercise you'll work through an informal proof of the Wirtinger formula for the knot group. First, recall the statement of the theorem: *Let K be a knot with set of arcs A and set of crossings B. Let W be the free group of $|A|$ generators. Let $N \leq A$ be the normal subgroup generated by elements $r(b)$, where for $b \in B$:*

$$r(b) = \begin{cases} (u(b)+1)o(b)u(b)^{-1}o(b)^{-1} & \text{if } b \text{ right-handed,} \\ o(b)(u(b)+1)o(b)^{-1}u(b)^{-1} & \text{if } b \text{ left-handed.} \end{cases}$$

The knot group G of K is W/N. In other words,

$$G = \langle A \,|\, r(b), b \in B \rangle$$

Choose as a base point in $S^3 \backslash K$ the point x_0 from which a knot diagram of K is viewed. The homotopy class of a loop at x_0 can be specified as a sequence of arcs of K that the loop crosses under, with an inverse $\cdot^{-1}$ if the crossing that the loop makes is left-handed.

a) Argue that any loop at x_0 is homotopic to a concatenation of loops that pass under a single arc of K and then return to x_0.

We have thus shown that each homotopy class of loops at x_0 is represented by an element of W. We now produce the Wirtinger presentation by quotienting out the neccesary relations. There are two cases where an ambient isotopy changes the series of under crossings of a loop:

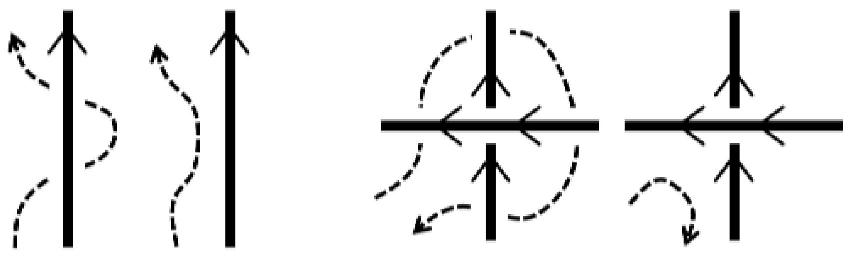

Figure 2: Case 1 (left) and case 2 (right).

b) The first case shows that the loop that first passes under an arc and then back with reverse orientation must be the trivial loop. This is encoded as imposing the relations $aa^{-1} = a^{-1}a = e$ on W. Argue that we don't need to quotient out anything in this case.

c) The other case is regarding the interaction of loops with crossings, and is depicted in figure 2. Write down the relations that we must impose on W to account for this. You must treat left- and right-handed crossings separately!

Note that the subgroup generated by all relations that we now wish to quotient out is precisely N!

d) Considering the Reidemeister theorem for links, convince youself that there is nothing left that we need to quotient out.

We have thus shown that the knot group of K is indeed given by W/N, as required.