LECTURE 7: THE JONES POLYNOMIAL REVISITED

In the last lectures we studied the universal invariant of a given XC-tangle T as an element

$$Z_A(T) \in A^{\otimes \mathcal{L}(T)},$$

where $\mathcal{L}(T)$ is the set of labels for the strands of T. An XC-tangle was defined as a graph were only one- and four-valent vertices where allowed, representing an endpoint of a strand or a crossing between two strands, respectively. What this means is that XC-tangles defined like that are representing embeddings of oriented intervals $[0,1]$, whereas we started this lecture course studying knots and links in $\mathbb{R}^3$, which are embeddings of spheres S^1.

To bring the link picture back to the discussion, we have to extend the definition of XC-tangle, allowing closed components.

1. General XC-tangles

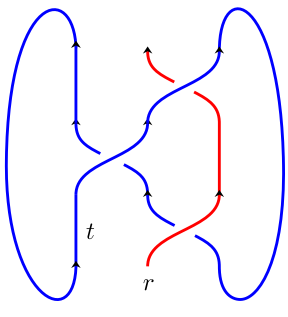
Definition 1. A *(general) XC-tangle diagram* is a directed graph with the following properties:

(i) At a vertex either one, two or four edges meet.
(ii) In the latter case the four edges meeting the vertex are ordered cyclically such that two adjacent edges enter the vertex and two exit.
(iii) The vertices where four edges meet are all labelled either $+$ or $-$ and we call them positive and negative crossings.
(iv) Each edge is labelled by an integer called its *rotation number*.
(v) A maximal directed path that goes straight at every vertex is called a *strand* of the diagram. Every edge is assumed to be part of a unique strand with distinct start and end points.
(vi) Every strand has either (a) no two-valent vertices or (b) one two-valent vertex and no one-valent vertices. If a strand carries a two-valent vertex, it is called *closed strand* (or component). If it does not, then it is called an *open strand*.
(vii) Every strand carries a unique label. The set of labels for the open strands of an XC-diagram D will be denoted by $\mathcal{L}_O(D)$, and the set of labels for the closed strands by $\mathcal{L}_C(D)$.

If D has no closed strands, then we say that D is an *open XC-tangle*. If has no open strands, then D is a *closed XC-tangle* or *XC-link*.

So, the XC-tangle diagrams we studied in the last two lectures were open tangles.

Example 2. The following is a 2-component XC tangle with one open strand labelled r and one closed strand labelled t.



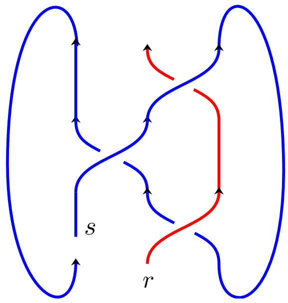
As usual, we view these diagrams up to the Reidemeister moves $(\check{\Omega}0)$, $(\check{\Omega}1f)$, $(\check{\Omega}2b)$, $(\check{\Omega}2c)$, $(\check{\Omega}3)$, where in this pictures we allow the two-valent vertex of a closed strand to appear. On top of these, we must also add the move which consists of sliding one four-valent vertex through a 2-valent vertex (for a closed strand),

As in the case for open strands, every general XC-tangle diagram can be constructed out of the elementary building blocks $\check{X}^{\pm}$ and $\check{C}^{\pm}$ using the disjoint union and merge $\check{m}$ operations. There is however a major difference which is that now we should be able to merge two edges that belong to the same strand. For this we need one extra operation (or rather, an extension of $\check{m}$).

Definition 3. Let D be an XC-tangle diagram and consider an open stand labelled s. Define a new XC-tangle diagram $E = \check{m}^{s,s}_t(D)$, called the (s,s,t)-*merge*, as follows: E is obtained from D by connecting the endpoint of s to the start of s, keeping the resulting vertex (so no new edge is created now), and removing the label s from the set of open strands and adding the label t to the set of closed strands,

$$\mathcal{L}_O(E) := \mathcal{L}_O(D) - \{s\} \qquad , \qquad \mathcal{L}_C(E) := \mathcal{L}_C(D) \amalg \{t\}.$$

For instance, the previous 2-component XC-tangle is the result of applying the operation $\check{m}^{s,s}_t$ to the following one:



Lemma 4. *Any diagram of an n-component link in $\mathbb{R}^3$ can be represented by a closed XC-tangle diagram with n strands.*

The proof of this is similar to that given for knots for which a piece of the strand was removed to make them open.

Warning 5. We want to emphasise that the previous lemma only refers to the diagrams, NOT to diagrams modulo Reidemeister moves (if one wants to be very precise, the lemma refers to diagrams modulo planar isotopy). Indeed for XC-links the Reidemeister move $(\Omega 1)$ from Lecture 1 does *not* hold (there was some other version $(\Omega 1 f)$ for XC-tangles).

2. The invariant Z_A of an endomorphism algebra

Our goal now is to extend the universal invariant Z_A to these general XC-tangles. For an arbitrary XC-algebra A, the invariant Z_A *cannot* be extended to general XC-tangles. This is because for the closed strands there is no a preferred point that determines where we should start multiplying the elements $X^{\pm}$ and $C^{\pm}$, and different choices would lead to different elements in the algebra. However, for a certain class of XC-algebras, this will be properly defined.

Definition 6. A XC-algebra A over a field k is said to be an *endomorphism algebra* (or a *Reshetikhin - Turaev algebra*) if $A = \mathrm{End}(V)$ for some finite-dimensional k-vector space V.

Example 7. The Dilbert algebra from the previous lecture was actually just $End(k^2)$ where we chose to name the elementary matrices as follows: $e_{11} = l$, $e_{12} = b$, $e_{21} = d$ and $e_{22} = a$.

Why should we care about endomorphism algebras? The key difference is as follows: for $A = \mathrm{End}(V)$ a XC-algebra, we can consider the *trace* map,

$$\mathrm{tr} : \mathrm{End}(V) \longrightarrow k.$$

This map can be defined in several equivalent ways:

Exercise 8. Show that the following definitions of the trace $\mathrm{tr} : \mathrm{End}(V) \longrightarrow k$ are equivalent:

(1) If $M = (m_{ij})$ is the matrix expression of an endomorphism $f \in \mathrm{End}(V)$ in some basis $(e_1, \dots, e_n)$ of V, then $\mathrm{tr}(f) = m_{11} + \cdots + m_{nn}$ (and this value is independent of the choice of basis),

(2) The map $V \otimes V^* \xrightarrow{\cong} \mathrm{End}(V)$, $e \otimes \omega \mapsto (v \mapsto \omega(v)e)$ is a linear isomorphism. On the other hand, there is a canonical map $V \otimes V^* \longrightarrow k$ given by contraction, $v \otimes \omega \mapsto \omega(v)$. Then the trace $\mathrm{tr} : \mathrm{End}(V) \longrightarrow k$ is the composition $\mathrm{End}(V) \xrightarrow{\cong} V \otimes V^* \longrightarrow k$.

(3) If $\lambda_1, \dots, \lambda_r$ are the eigenvalues of $f \in \mathrm{End}(V)$ counted with multiplicities, then $\mathrm{tr}(f) = \lambda_1 + \cdots + \lambda_r$.

Exercise 9. Show that if $f, g \in \mathrm{End}(V)$ then

$$\mathrm{tr}(g \circ f) = \mathrm{tr}(f \circ g).$$

Conclude that if $f_1, \ldots, f_n \in \mathrm{End}(V)$ then

$$\mathrm{tr}(f_n \circ \cdots \circ f_1) = \mathrm{tr}(f_{r-1} \circ \cdots \circ f_1 \circ f_n \circ \cdots \circ f_r)$$

for $2 \leq r \leq n$.

Using our unordered tensor powers notation, we will put

$$m_t^{s,s} : A^{\otimes\{s\}} \longrightarrow k^{\otimes\{t\}}$$

for the trace map, $m_t^{s,s} := \mathrm{tr}$.

Let us see now how to extend the definition of the invariant Z_A for general XC-tangles T:

Definition 10. If T is an XC-tangle and $A = \mathrm{End}(V)$ is an XC-algebra of endomorphism type, then $Z_A(T)$ is defined according to the following rules:

(i) $Z_A(\check{X}^{\pm}_{ou}) = X^{\pm}_{ou} \in A^{\otimes\{o,u\}}$,

(ii) $Z_A(\check{C}^{\pm}_{s}) = C^{\pm}_{s} \in A^{\otimes\{s\}}$,

(iii) $Z_A(DE) = Z_A(D)Z_A(E)$,

(iv) $Z_A(\check{m}_n^{h,t}(D)) = m_n^{h,t}(Z_A(D))$,

(v) $Z_A(\check{m}_t^{s,s}(D)) = m_t^{s,s}(Z_A(D))$,

These rules define an element in

$$A^{\otimes\mathcal{L}_O(T)} \otimes k^{\otimes\mathcal{L}_C(T)}.$$

Composing with the isomorphism $W \otimes k \xrightarrow{\cong} W$, $w \otimes \lambda \mapsto \lambda w$ from Lecture 4, we obtain an element

$$Z_A(T) \in A^{\otimes\mathcal{L}_O(T)}$$

(we agree that $Z_A(T) \in k$ if T has only closed strands).

Lemma 11. *The previous element*

$$Z_A(T) \in A^{\otimes\mathcal{L}_O(T)}$$

is a well-defined invariant of general XC-tangles.

Proof. That this value is an invariant follows from the construction as in the last Lecture. So we only need to check that this is well-defined. For a closed strand labelled with s, two different merges closing up the strand that represent the same link will get in the factor k_s values equal the trace of the composite of elements that differ by a cyclic permutation. Exercise 9 concludes that the values will be the same. □

3. The Jones polynomial revisited

Lecture 2 was devoted to the Jones polynomial, a link polynomial invariant

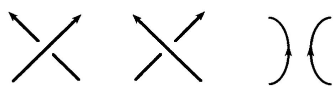
$$J : \mathcal{L} \longrightarrow \mathbb{Z}[t^{1/2}, t^{-1/2}]$$

that was uniquely determined by the properties

(1) $J_{\mathrm{unknot}}(t) = 1$,
(2) the skein relation

$$t^{-1}J_{L_+}(t) - tJ_{L_-}(t) = (t^{1/2} - t^{-1/2})J_{L_0}(t),$$

where L_+, L_-, L_0 denote three links that are identical except in a neighbourhood of some point where they look like below,



L_+ L_- L_0

Our aim now is to recover the Jones polynomial from a particular case of the invariant Z_A we have developed, where A is an endomorphism XC-algebra.

Let V be a 2-dimensional vector space over the field $\mathbb{Q}(q)$ with a fixed basis (e_1, e_2), and let $A := \mathrm{End}(V) \cong \mathcal{M}_2(\mathbb{Q}(q))$.

Proposition 12. *The elements*

$$X_{ij} := \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}_i \begin{pmatrix}q & 0\\ 0 & q^{-1}\end{pmatrix}_j + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}_i \begin{pmatrix}q^{-1} & 0\\ 0 & q\end{pmatrix}_j + (q - q^{-3}) \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}_i \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}_j \in A^{\otimes \{i,j\}}$$

and

$$C_s := \begin{pmatrix}q^2 & 0\\ 0 & q^{-2}\end{pmatrix}_s \in A^{\otimes\{s\}}$$

define an XC-algebra structure on A.

Proof. We start by noting that the elements X_{ij} and C_s are invertible and their inverses of the elements are given by

$$X^-_{ij} := \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}_i \begin{pmatrix}q^{-1} & 0\\ 0 & q\end{pmatrix}_j + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}_i \begin{pmatrix}q & 0\\ 0 & q^{-1}\end{pmatrix}_j + (q^{-1} - q^{3}) \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}_i \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}_j \in A^{\otimes \{i,j\}}$$

and

$$C^-_s := \begin{pmatrix}q^{-2} & 0\\ 0 & q^{2}\end{pmatrix}_s \in A^{\otimes\{s\}}$$

This is clear for C_s and for X_{ij} we simply compute

$$\begin{aligned}(m_j^{j_1j_2} \circ m_i^{i_1i_2})(X_{i_1j_1}X_{i_2j_2}) &= \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}_i \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}_j + (q^{-1} - q^3)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}_i \begin{pmatrix}0 & 0\\ q^{-1} & 0\end{pmatrix}_j \\ &\quad + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}_i \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}_j + (q - q^{-3})\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}_i \begin{pmatrix}0 & 0\\ q & 0\end{pmatrix}_j \\ &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}_i \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}_j + (q^{-2} - q^2 + q^2 - q^{-2})\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}_i \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}_j \\ &= \mathrm{Id}_i\mathrm{Id}_j,\end{aligned}$$

and similarly we get $(m_j^{j_1j_2} \circ m_i^{i_1i_2})(X_{i_2j_2}X_{i_1j_1}) = \mathrm{Id}_i\mathrm{Id}_j$, which amounts to $(\Omega 2b)$. For the axiom $(\Omega 1f)$, a similar computation yields

$$m_\ell^{ikj}(X_{ij}C_k^-) = q^3 \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}_\ell = m_\ell^{jki}(X_{ij}C_k).$$

For $(\Omega 0)$, that is, $m_i^{1,3,5}m_j^{2,4,6}(C_1C_2X_{3,4}C_5^{-1}C_6^{-1}) = X_{i,j}$, one can argue as follows: since C, C^- and the first two summands of X are diagonal matrices, and the product of diagonal matrices is commutative, we only need to check the equality for the third summand of X. This is

$$\begin{aligned}&(q - q^{-3})\left[\begin{pmatrix}q^2 & 0\\ 0 & q^{-2}\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}q^{-2} & 0\\ 0 & q^{2}\end{pmatrix}\right]_i \left[\begin{pmatrix}q^2 & 0\\ 0 & q^{-2}\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}q^{-2} & 0\\ 0 & q^{2}\end{pmatrix}\right]_j \\ &= (q - q^{-3})\begin{pmatrix}0 & q^4\\ 0 & 0\end{pmatrix}_i \begin{pmatrix}0 & 0\\ q^{-4} & 0\end{pmatrix}_j \\ &= (q - q^{-3})\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}_i \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}_j\end{aligned}$$

The axioms $(\Omega 2c)$ and $(\Omega 3)$ are left to the reader. □

This means that we can compute the invariant $Z_A(T)$ for this particular XC-endomorphism algebra A.

Example 13. The unknot (with writhe 0) as a XC-diagram is simply $O = \check{m}_t^{s,s}(\check{C}_s)$. Therefore we obtain

$$Z_A(O) = m_t^{s,s}(C_s) = \operatorname{tr}\begin{pmatrix} q^2 & 0 \\ 0 & q^{-2} \end{pmatrix} = q^2 + q^{-2}.$$

For the closed 0-writhe right-handed XC-trefoil 3_1 that was depicted in Lecture 5 , a longer but elementary computation shows that

$$Z_A(3_1) = -q^{-18} + q^{-10} + q^6 + q^2 = (q^2 + q^{-2})(-q^{-16} + q^{-12} + q^{-4}).$$

The reader might recognise the second factor of the rightmost equation: in Lecture 2 we computed $J_{\bar{3}_1}(t) = -t^{-4} + t^{-3} + t^{-1}$ for the left-handed trefoil, so by the homework we have $J_{3_1}(t) = -t^4 + t^3 + t$ which means that

$$Z_A(3_1) = Z_A(O) \cdot J_{3_1}(q^{-4}).$$

This is of course no coincidence.

In fact, the main step is to show that the invariant Z_A described above satisfies a skein relation very similar to that of the Jones polynomial when restricted to closed XC-tangles:

Theorem 14. *The restriction of Z_A to XC-links satisfies the following properties:*

(1) $Z_A(\text{unknot}) = q^2 + q^{-2}$,

(2) Let $L, L_\alpha, L_{\alpha^-}$ be XC-links that are identical except in a neighbourhood of a point where L looks like $\check{\mathrm{I}}$, L_α like the positive kink $\check{\alpha}$ and L_{α^-} like the negative kink $\check{\alpha}^-$. Then

$$Z_A(L_\alpha) = q^3 Z_A(L) \qquad , \qquad Z_A(L_{\alpha^-}) = q^{-3} Z_A(L).$$

(3) Let L_+, L_-, L_0 denote three XC-links that are identical except in a neighbourhood of some point where they look like $\check{X}, \check{X}^-$ and $\check{\mathrm{I}}\check{\mathrm{I}}$, respectively. Then the following skein relation holds:

$$qZ_A(L_+) - q^{-1} Z_A(L_-) = (q^2 - q^{-2}) Z_A(L_0).$$

Proof. (1) This was already computed in the previous example.

(2) We saw in the proof of Proposition 12 that

$$Z_A(\check{\alpha}_i) = q^3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_i ,$$

which implies that when taking traces $Z_A(L_\alpha) = q^3 Z_A(L)$. Similarly

$$Z_A(\check{\alpha}_i^{-1}) = q^{-3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_i$$

which yields $Z_A(L_{\alpha^{-1}}) = q^{-3} Z_A(L)$.

(3) Let $P_{ij} \in A^{\otimes\{i,j\}}$ be the following element:

$$P_{ij} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_i \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_j + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_j + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_j + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_j .$$

The main point is that we have the following fundamental equality:

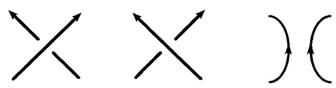
$$qX_{ij} - q^{-1} X_{ji}^- = (q^2 - q^{-2}) P_{ij}. \tag{1}$$

Indeed

$$\begin{aligned} qX_{ij} - q^{-1} X_{ji}^- &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_i \begin{pmatrix} q^2 & 0 \\ 0 & 1 \end{pmatrix}_j + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_i \begin{pmatrix} 1 & 0 \\ 0 & q^2 \end{pmatrix}_j \\ &+ (q^2 - q^{-2}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_j - \begin{pmatrix} q^{-2} & 0 \\ 0 & 1 \end{pmatrix}_i \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_j \\ &- \begin{pmatrix} 1 & 0 \\ 0 & q^{-2} \end{pmatrix}_i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_j - (q^{-2} - q^2) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_j \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_i \left[q^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_j + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_j \right] + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_i \left[q^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_j + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_j \right] \end{aligned}$$

$$
\begin{aligned}
&- \left[q^{-2}\begin{pmatrix}0&0\\0&1\end{pmatrix}_i + \begin{pmatrix}1&0\\0&0\end{pmatrix}_i\right]\begin{pmatrix}0&0\\0&1\end{pmatrix}_j - \left[q^{-2}\begin{pmatrix}0&0\\0&1\end{pmatrix}_i + \begin{pmatrix}1&0\\0&0\end{pmatrix}_i\right]\begin{pmatrix}0&0\\0&1\end{pmatrix}_j \\
&+ (q^2 - q^{-2})\left[\begin{pmatrix}0&1\\0&0\end{pmatrix}_i\begin{pmatrix}0&0\\1&0\end{pmatrix}_j + \begin{pmatrix}0&0\\1&0\end{pmatrix}_i\begin{pmatrix}0&1\\0&0\end{pmatrix}_j\right] \\
&= P_{ij}.
\end{aligned}
$$

Now consider the XC-link L_+ as a composite of merges $\check{X}^{\pm}$ and $\check{C}^{\pm}$ and taking a sequence of self-merges as the last step. Then L_- arises in the same way after replacing one preferred positive crossing $\check{X}^+_{ij}$ by $\check{X}^-_{ji}$. On the other hand, L_0 arises from replacing $\check{X}^+_{ij}$ by $\check{1}_i\check{1}_j$, but every merge map $\check{m}^{is}_t$ needs to be replaced by $\check{m}^{js}_t$, since the top right endpoint of $\check{1}_i\check{1}_j$ is labelled by j and not by i,



$i \quad j \qquad i \quad j \qquad i \quad j$

$L_+ \qquad L_- \qquad L_0$

Therefore, it suffices to show that $Z_A(L_0)$ can be computed with the same merging maps as L_+ but replacing X_{ij} by P_{ij}. For this first we note that the number of components of L_+ and L_0 differ by one. Let us suppose that in L_+ the two stands depicted in the preferred crossing end up in the same component and that the two strands depicted for L_0 end up in different components (the other case is similar), and for simplicity suppose also that these are the only strands in the tangle (as the other strands are not modified). Then if $Z_A(L_0)$ is given by

$$\sum \mathrm{tr}(M)\mathrm{tr}(M')$$

for some matrices

$$M = \begin{pmatrix}m_{11}&m_{12}\\m_{21}&m_{22}\end{pmatrix} \quad , \quad M' = \begin{pmatrix}m'_{11}&m'_{12}\\m'_{21}&m'_{22}\end{pmatrix}$$

then replacing X_{ij} by P_{ij} using the same merging maps as L_+ yields

$$\sum \mathrm{tr}\left(\sum_k Mu^kM'v^k\right)$$

where $P_{ij} = \sum_k u^k_i v^k_j$. Now it is left to the reader to check that each of the four summands of P_{ij} gives one of the four summands of $\mathrm{tr}(M)\mathrm{tr}(M') = m_{11}m'_{11} + m_{11}m'_{22} + m'_{11}m_{22} + m_{22}m'_{22}$. $\square$

The invariant Z_A is an invariant of XC-links, whereas the Jones polynomial is an invariant of ordinary links. Therefore, to obtain the latter from the former, we need first to turn Z_A into an invariant for ordinary links. There is a general procedure, called *deframing*, to turn XC-link invariants into ordinary link invariants, which is contained in the following

Lemma 15 (Deframing)**.** *Let I be an invariant of XC-links with values in a ring k and suppose that*

$$I(L_\alpha) = \lambda I(L) \quad , \quad I(L_{\alpha^{-1}}) = \lambda^{-1}I(L)$$

for some $\lambda \in k$. Then if D is a XC-diagram representing an ordinary link in $\mathbb{R}^3$, then

$$\widetilde{I}(D) := \lambda^{-w(D)}I(D)$$

is an invariant of ordinary links, where $w(D)$ denotes the writhe of D.

Proof. If I is an invariant of XC-links, then I will preserve the Reidemeister moves (Ω2) and (Ω3) for ordinary links, and so will $\widetilde{I}$. But now $\widetilde{I}$ also preserves the Reidemeister (Ω1) move for ordinary links, for

$$\widetilde{I}(D_\alpha) = \lambda^{-w(D_\alpha)}I(D_\alpha) = \lambda^{-(w(D)+1)}\lambda I(D) = \widetilde{I}(D)$$

as claimed (and similarly for the negative kink). $\square$

According to the previous lemma,

$$\widetilde{Z_A}(L) := q^{-3w(D)} Z_A(D),$$

where D is an XC-diagram of a link L in $\mathbb{R}^3$, is an isotopy invariant of ordinary links.

Theorem 16. *For any link L in $\mathbb{R}^3$ we have*

$$J_L(t) = \frac{1}{q^2 + q^{-2}} \widetilde{Z_A}(L)_{|q^2=-t^{-1/2}}.$$

Proof. Clearly it suffices to show that the right-hand side of the previous equality satisfies the two properties that uniquely determine the Jones polynomial. We saw before that $Z_A(O) = q^2 + q^{-2}$ for the 0-writhe diagram so $\widetilde{Z_A}(O) = Z_A(O)$. Therefore

$$\frac{1}{q^2 + q^{-2}} \widetilde{Z_A}(O) = 1.$$

On the other hand, the writhes of L_+, L_- and L_0 are related as

$$w(L_+) - 1 = w(L_0) = w(L_-) + 1.$$

Multiplying the skein relation of Theorem 14 times $q^{-3w(L_0)}$ we obtain

$$qq^{-3(w(L_+)-1)} Z_A(L_+) - q^{-1} q^{-3(w(L_-)+1)} Z_A(L_-) = (q^2 - q^{-2}) \widetilde{Z_A}(L_0),$$

that is,

$$q^4 \widetilde{Z_A}(L_+) - q^{-4} \widetilde{Z_A}(L_-) = (q^2 - q^{-2}) \widetilde{Z_A}(L_0),$$

and setting $q^2 = -t^{-1/2}$ yields

$$t^{-1} \widetilde{Z_A}(L_+) - t \widetilde{Z_A}(L-) = (t^{1/2} - t^{-1/2}) \widetilde{Z_A}(L_0).$$

Dividing both sides by $q^2 + q^{-2}$ we conclude. $\square$