The HOMFLY-PT polynomial

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In this presentation we will be introduced to a 2-variable knot polynomial motivated by the, already known, Jones polynomial. Therefore, it is sometimes called the generalized Jones polynomial.

Theorem 1. *There exists a unique link polynomial invariant*

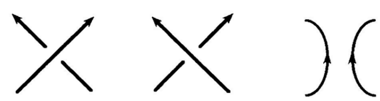
$$P : \mathcal{L} \to \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$$

satisfying

1. $P_{unknot}(x, y) = 1$,
2. *the following skein relation holds:*

$$xP_{L_+}(x, y) + x^{-1}P_{L_-}(x, y) + yP_{L_0}(x, y) = 0$$

where L_+, L_-, L_0 denote three links that are identical except in a neighbourhood of some point where they look like below,



L_+ L_- L_0

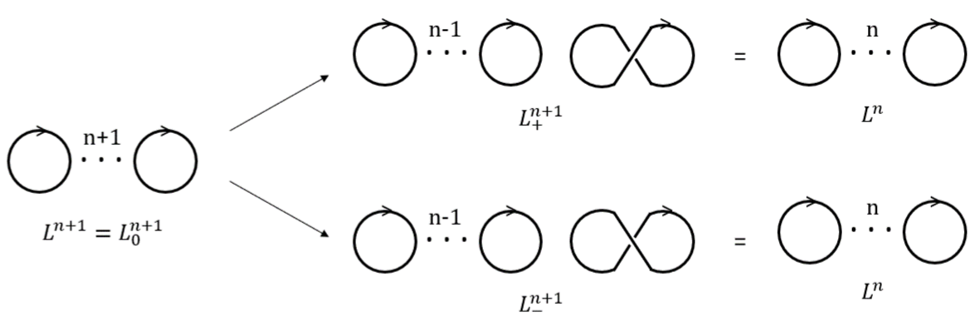
Definition 2. The polynomial of the previous theorem is called the *HOMFLY-PT polynomial.* It is a 2-variable polynomial. The name HOMFLY is an acronym of the initials of the last names of the co-discoverers: Jim Hoste, Adrian Ocneanu, Kenneth Millett, Peter J. Freyd, William B. R. Lickorish, and David N. Yetter. The PT addition recognizes the independent work of Józef H. Przytycki and Paweł Traczyk.

Before we will proof the uniqueness, we will first state and proof the HOMFLY-PT polynomial of the trivial n-component link.

Lemma 3. *If L^n denotes the trivial n-component link, then $P_{L^n}(x, y) = (-(x + x^{-1})y^{-1})^{n-1}$.*

Proof. We will proof this by induction. Assume that $P_{L^n}(x, y) = (-(x + x^{-1})y^{-1})^{n-1}$ is true for n and show that it also holds for $n + 1$. For the base step, we first show that it holds for $n = 1$ (the unknot),

$$P_{\text{unknot}}(x, y) = P_{L^1}(x, y) = (-(x + x^{-1})y^{-1})^0 = 1.$$



Now assume $P_{L^n}(x,y) = (-(x+x^{-1})y^{-1})^{n-1}$ is true. We will show that $P_{L^{n+1}}(x,y) = (-(x+x^{-1})y^{-1})^n$. We apply the skein relation taking $L^{n+1} = L_0^{n+1}$,

$$\begin{aligned} & xP_{L^n}(x,y) + x^{-1}P_{L^n}(x,y) + yP_{L^{n+1}}(x,y) = 0 \\ \iff \qquad & P_{L^{n+1}}(x,y) = -(x+x^{-1})y^{-1} \cdot P_{L^n}(x,y) = (-(x+x^{-1})y^{-1})^n. \end{aligned}$$

Hence we have proven by induction that $P_{L^n}(x,y) = (-(x+x^{-1})y^{-1})^{n-1}$ is the HOMFLY-PT polynomial of the trivial n-component link. □

Let us now proof the uniqueness of Theorem 1.

Proof of uniqueness. Let L be a link, choose a crossing and say it is positive/negative and consider the other two resolutions L_-/L_+ and L_0. L_-/L_+ has the sign of the chosen crossing reversed and L_0 has one fewer crossing. If any of these two links is the trivial unlink, then we stop. Otherwise we choose a different crossing and repeat this step. By lemma 6 of lecture 2 we know that any n-component link can be turned into the trivial n-component link by changing the sign of some of its crossings. This means that this process is finite. A repeated application of the skein relations makes that P_L can be expressed in terms of $P_{\text{unknot}} = 1$ and copies of $P_{L^n} = (-(x+x^{-1})y^{-1})^{n-1}$. Hence the uniqueness. □

The HOMFLY-PT polynomial generalizes two previously discovered polynomials, called the Alexander polynomial and the Jones polynomial. Both can be obtained by appropriate substitutions from HOMFLY-PT polynomial.

Theorem 4. *The HOMFLY-PT polynomial recovers the Jones polynomial (J_L) and the Alexander polynomial (Δ_L):*

1. $J_L(t) = P_L(it^{-1}, i(t^{-1/2} - t^{1/2}))$,
2. $\Delta_L(t) = P_L(i, i(t^{1/2} - t^{-1/2}))$,

where i denotes the imaginary unit.

Proof. (1) The skein relation of the Jones polynomial is:

$$t^{-1}J_{L_+}(t) - tJ_{L_-}(t) = (t^{1/2} - t^{-1/2})J_{L_0}(t).$$

Substituting $x = it^{-1}$ and $y = i(t^{-1/2} - t^{1/2})$ into the skein relation of the HOMFLY-PT polynomial gives:

$$\begin{aligned} it^{-1}P_{L_+} + (it^{-1})^{-1}P_{L_-} + i(t^{-1/2} - t^{1/2})P_{L_0} &= 0, \\ it^{-1}P_{L_+} - itP_{L_-} + i(t^{-1/2} - t^{1/2})P_{L_0} &= 0, \\ t^{-1}P_{L_+} - tP_{L_-} &= (t^{1/2} - t^{-1/2})P_{L_0}. \end{aligned}$$

Hence, $J_L(t) = P_L(it^{-1}, i(t^{-1/2} - t^{1/2}))$.

(2) The skein relation of the Alexander polynomial is:

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{-1/2} - t^{1/2})\Delta_{L_0}(t).$$

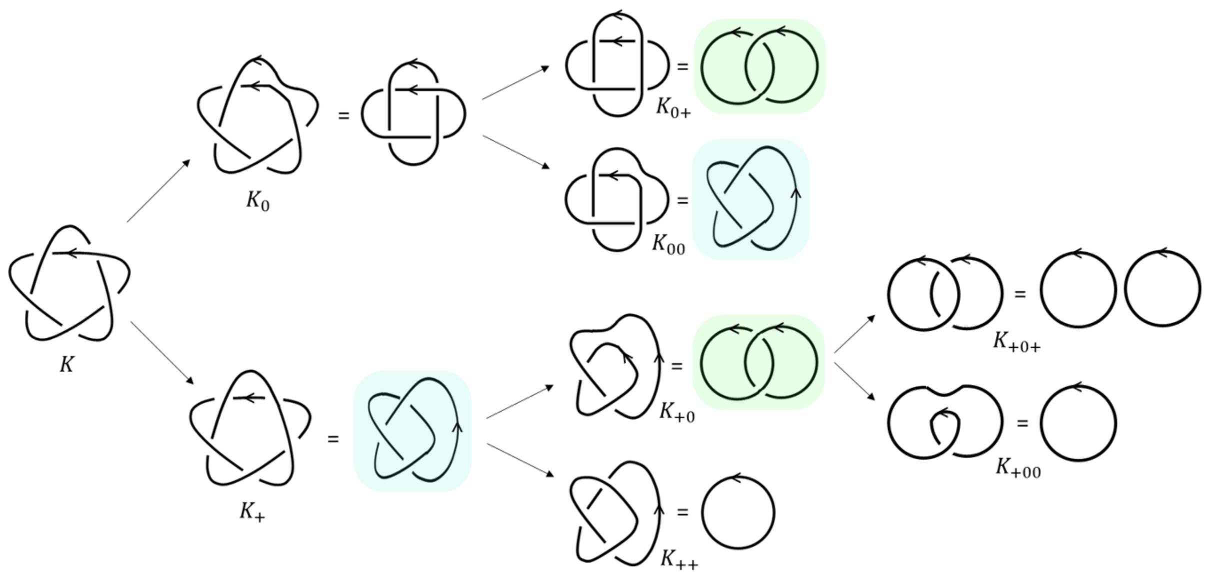
Substituting $x = i$ and $y = i(t^{1/2} - t^{-1/2})$ into the skein relation of the HOMFLY-PT polynomial gives:

$$\begin{aligned} iP_{L_+} + i^{-1}P_{L_-} + i(t^{1/2} - t^{-1/2})P_{L_0} &= 0, \\ iP_{L_+} - iP_{L_-} + i(t^{1/2} - t^{-1/2})P_{L_0} &= 0, \\ P_{L_+} - P_{L_-} &= (t^{-1/2} - t^{1/2})P_{L_0}. \end{aligned}$$

Hence, $\Delta_L(t) = P_L(i, i(t^{1/2} - t^{-1/2}))$. □

Before showing more properties of the HOMFLY-PT polynomial, let us first get more familiar with the statement by giving some examples.

Example 5. Lets compute the HOMFLY-PT polynomial of the cinquefoil knot K by constructing a skein tree for the knot. Below we show this for K:



Note that K_{++} and K_{+00} are the unknot and K_{+0+} is isotopic to the trivial 2-component link. Also note that K_+ and K_{00}, and K_{0+} and K_{+0} are equivalent knots. Using the skein relation, we have

$$\begin{aligned} xP_{K_{+0+}} + x^{-1}P_{K_{+0}} + yP_{K_{+00}} &= 0 \\ x(-(x+x^{-1})y^{-1}) + x^{-1}P_{K_{+0}} + y &= 0 \\ x^{-1}P_{K_{+0}} &= x^2y^{-1} + y^{-1} - y \\ P_{K_{+0}} &= x^3y^{-1} + xy^{-1} - xy \end{aligned}$$

so

$$\begin{aligned} xP_{K_{++}} + x^{-1}P_{K_+} + yP_{K_{+0}} &= 0 \\ x + x^{-1}P_{K_+} + y(x^3y^{-1} + xy^{-1} - xy) &= 0 \\ x^{-1}P_{K_+} &= -2x - x^3 + xy^2 \\ P_{K_+} &= -2x^2 - x^4 + x^2y^2. \end{aligned}$$

On the other hand

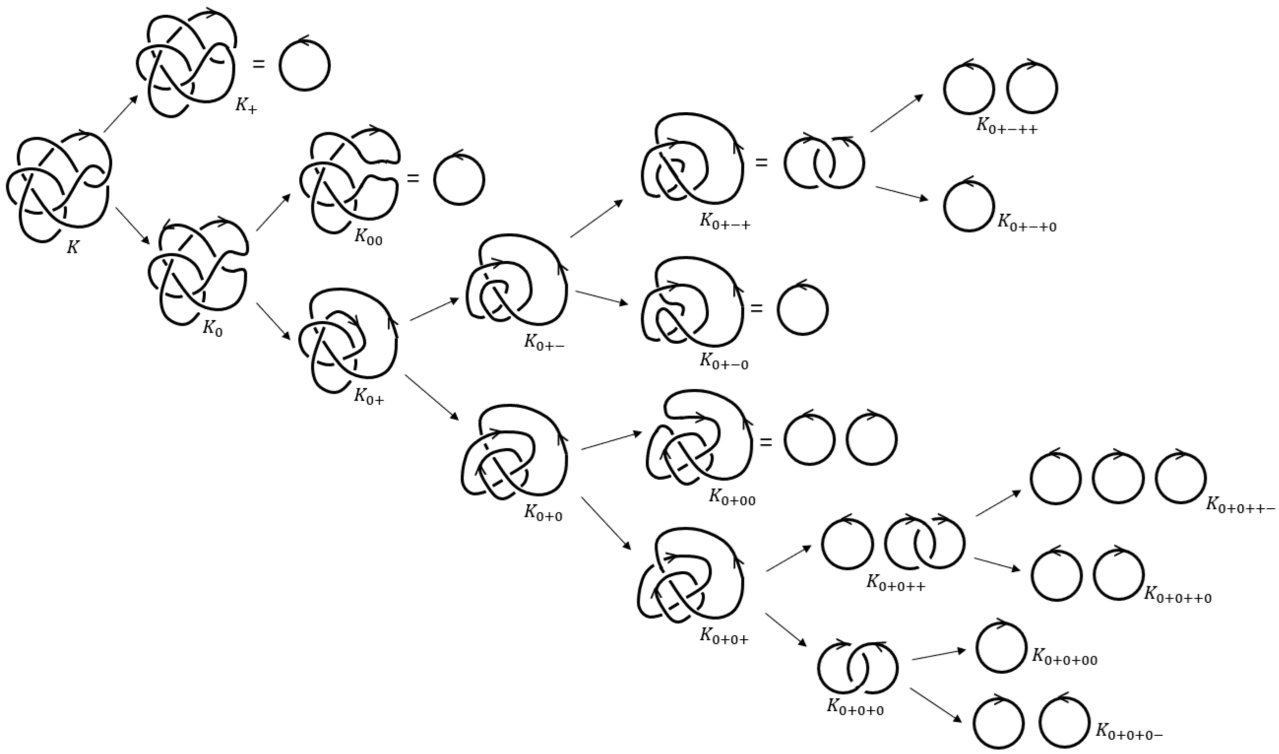
$$\begin{aligned} xP_{K_{0+}} + x^{-1}P_{K_0} + yP_{K_{00}} &= 0 \\ x(x^3y^{-1} + xy^{-1} - xy) + x^{-1}P_{K_0} + y(-2x^2 - x^4 + x^2y^2) &= 0 \\ x^{-1}P_{K_0} &= x^4y - x^4y^{-1} - x^2y^3 + 3x^2y - x^2y^{-1} \\ P_{K_0} &= x^5y - x^5y^{-1} - x^3y^3 + 3x^3y - x^3y^{-1} \end{aligned}$$

and therefore

$$\begin{aligned} xP_{K_+} + x^{-1}P_K + yP_{K_0} &= 0 \\ P_K &= -x^2P_{K_+} - xyP_{K_0} \\ &= -x^2(-2x^2 - x^4 + x^2y^2) - xy(x^5y - x^5y^{-1} - x^3y^3 + 3x^3y - x^3y^{-1}) \\ &= 2x^6 - x^6y^2 + x^4y^4 - 4x^4y^2 + 3x^4. \end{aligned}$$

Although the HOMFLY-PT polynomial can distinguish more knots than the Alexander polynomial and Jones polynomial, it is still not a complete knot invariant. Take for example a look at the following example. It can be seen that even though the cinquefoil and the 10_{132} knot are distinct knots, they have the same HOMFLY-PT polynomial.

Example 6. Lets compute the HOMFLY-PT polynomial of the 10_{132} knot K by constructing a skein tree for the knot. Below we show this for K:



Using the skein relation we get,

$xP_{K_{0+0+0}} + x^{-1}P_{K_{0+0+0-}} + yP_{K_{0+0+00}} = 0 \quad \Rightarrow \quad P_{K_{0+0+0}} = x^{-1}y^{-1} + x^{-3}y^{-1} - x^{-1}y,$

$xP_{K_{0+0++}} + x^{-1}P_{K_{0+0++-}} + yP_{K_{0+0++0}} = 0 \quad \Rightarrow \quad P_{K_{0+0++}} = -y^{-2} - x^{-4}y^{-2} - 2x^{-2}y^{-2} + 1 + x^{-2},$

$xP_{K_{0+0++}} + x^{-1}P_{K_{0+0+}} + yP_{K_{0+0+0}} = 0 \quad \Rightarrow \quad P_{K_{0+0+}} = -x^2y^{-2} + x^{-2}y^{-2} + 2y^{-2} - x^2 - x^{-2} + y^2 - 2,$

$xP_{K_{0+0+}} + x^{-1}P_{K_{0+0}} + yP_{K_{0+00}} = 0 \quad \Rightarrow \quad P_{K_{0+0}} = -x^4y^{-2} - y^{-2} - 2x^2y^{-2} + x^4 + 2 - x^2y^2 + 3x^2,$

$xP_{K_{0+-++}} + x^{-1}P_{K_{0+-+}} + yP_{K_{0+-+0}} = 0 \quad \Rightarrow \quad P_{K_{0+-+}} = x^3y^{-1} + xy^{-1} - xy,$

$xP_{K_{0+}} + x^{-1}P_{K_{0+-}} + yP_{K_{0+0}} = 0 \quad \Rightarrow \quad P_{K_{0+}} = 2x^3y^{-1} - x^{-1}y - 4xy + 3xy^{-1} - x^3y + x^{-1}y^{-1} + xy^3,$

$xP_{K_{0+}} + x^{-1}P_{K_0} + yP_{K_{00}} = 0 \quad \Rightarrow \quad P_{K_0} = -2x^5y^{-1} + 4x^3y - 3x^3y^{-1} + x^5y - xy^{-1} - x^3y^3,$

$xP_{K_+} + x^{-1}P_K + yP_{K_0} = 0 \quad \Rightarrow \quad P_K = 2x^6 - x^6y^2 + x^4y^4 - 4x^4y^2 + 3x^4.$

So the HOMFLY-PT polynomial of the 10_{132} knot is $2x^6 - x^6y^2 + x^4y^4 - 4x^4y^2 + 3x^4$.

Now we will mention some properties of the HOMFLY-PT polynomial.

Proposition 7. *The HOMFLY-PT polynomial satisfies the following properties:*

1. *Let L be a link and denote by $-L$ the orientation reversal of L, then*
$$P_L(x, y) = P_{-L}(x, y).$$

2. Let L be a link and denote by $\bar{L}$ the mirror image of L, then

$$P_L(x,y) = P_{\bar{L}}(x^{-1}, y).$$

3. If L, L' are links, then

$$P_{L\#L'}(x,y) = P_L(x,y)P_{L'}(x,y).$$

4. If L, L' are links, then

$$P_{L\sqcup L'}(x,y) = -\frac{x+x^{-1}}{y}P_L(x,y)P_{L'}(x,y).$$

Proof. (1) Changing the orientation of every component does not change the sign of the crossings. Hence, this allows us to use the same skein tree for L and $-L$.
(2) The mirror image changes every crossing of L to the opposite sign, therefore

$$\begin{aligned} xP_{L_+}(x,y) + x^{-1}P_{L_-}(x,y) + yP_{L_0}(x,y) = 0 &= xP_{\bar{L}_-}(x,y) + x^{-1}P_{\bar{L}_+}(x,y) + yP_{\bar{L}_0}(x,y) \\ &= x^{-1}P_{\bar{L}_-}(x^{-1},y) + xP_{\bar{L}_+}(x^{-1},y) + yP_{\bar{L}_0}(x^{-1},y) \\ &= xP_{\bar{L}_+}(x^{-1},y) + x^{-1}P_{\bar{L}_-}(x^{-1},y) + yP_{\bar{L}_0}(x^{-1},y), \end{aligned}$$

so $P_L(x,y) = P_{\bar{L}}(x^{-1},y)$.
(4) Note that the following relation holds (drawing a skein tree might help)

$$\begin{aligned} &xP_{L\#L'}(x,y) + x^{-1}P_{L\#L'}(x,y) + yP_{L\sqcup L'}(x,y) = 0 \\ &\implies \quad P_{L\sqcup L'}(x,y) = -\frac{x+x^{-1}}{y}P_{L\#L'}(x,y) = -\frac{x+x^{-1}}{y}P_L(x,y)P_{L'}(x,y). \end{aligned}$$

□

We could also use the HOMFLY-PT polynomial to show a relation between the Jones polynomial and the Alexander polynomial.

Proposition 8. *For any link L,*

$$J_L(-1) = \Delta_L(-1).$$

Proof. First note that for any link L, $P_L(x,y) = P_L(-x,-y)$ since from the skein relation we have

$$\begin{aligned} &xP_{L_+}(x,y) + x^{-1}P_{L_-}(x,y) + yP_{L_0}(x,y) = 0, \qquad \text{and} \\ &-xP_{L_+}(-x,-y) - x^{-1}P_{L_-}(-x,-y) - yP_{L_0}(-x,-y) = 0, \\ &\iff \qquad xP_{L_+}(-x,-y) + x^{-1}P_{L_-}(-x,-y) + yP_{L_0}(-x,-y) = 0, \end{aligned}$$

and therefore $P_L(x,y) = P_L(-x,-y)$. From this result and theorem 3 follows,

$$J_L(-1) = P_L(-i, i(1/i - i)) = P_L(-i, 2) = P_L(i, -2) = P_L(i, i(i - 1/i)) = \Delta_L(-1).$$

□

References

[1] *An Introduction to Knot Theory*, WBR Lickorish, (chapter 15 and 16)

[2] *Knots*, Burde Zieschang. (chapter 16)

[3] *A new polynomial invariant of knots and links.* Freyd, P., Yetter, D., Hoste, J., Lickorish, W.R., Millett, K. and Ocneanu, A.