LECTURE 9: INTERLUDE ON THE FUNDAMENTAL GROUP

By $F(Y)$ we mean the free group with set of generators Y. A presentation of a group Γ is an isomorphism $\Gamma \cong F(Y)/N(R)$ where Y is some set and $N(R)$ is the smallest normal subgroup of $F(Y)$ generated by some subset $R \subset F(Y)$. The set Y is known as the generators while R is the set of relations. A group presentation is normally denoted by listing the generators and relations between angled brackets with a dash in the middle like this: $\langle Y|R\rangle$. For example a presentation for the cyclic group of order 5 is $\langle\{a\}|\{a^5\}\rangle$ while the dihedral group with ten elements has a presentation like this $\langle\{a,b\}|\{a^5,b^2,baba\}\rangle$. Since the curly braces are cumbersome they are usually dropped. It is also common to write the relation ab^{-1} more informally as $a = b$. The dihedral group presentation then becomes $\langle a,b|a^5 = b^2 = 1, bab = a^{-1}\rangle$.

Without going into the underlying algebraic topology details we will introduce the fundamental group of the complement of a knot (known as the knot group). By the complement of $K \subset S^3$ we mean $S^3 \setminus K$. In general for any topological space B together with a chosen (basepoint) $p \in B$ we can consider the set $\pi_1(B,p)$ of all equivalence classes of continuous paths in B starting and ending at p. This is called the fundamental group where two paths are multiplied by walking first one and then the other. This only works if we consider paths up to homotopy[1] that fixes the base point. We will not use this theory explicitly but instead give a flavour of what kind of groups appear in the case of knots and tangles. It turns out that they have a lot in common with our invariant $Z_{D(G)}$.

In the previous lectures we worked with XC-tangle diagrams but in this lecture we will forget about the C's and the rotation numbers. We call such diagrams simply tangle diagrams. This is natural since the XC-algebra $D(G)$ also had $C^{\pm} = 1$.

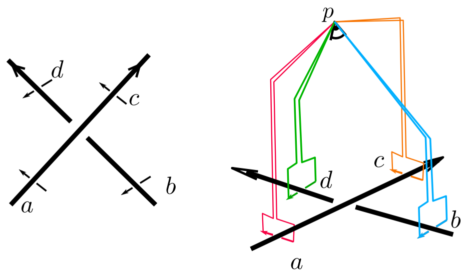

FIGURE 1. The generators of the presentation of the fundamental group near a positive crossing of a knot complement.

Definition 1. (Fundamental group presentation of a tangle)
For any tangle diagram D define $\mathcal{W}(D) = \langle E|R\rangle$ where R consists of two elements per crossing: ac^{-1} and $d^{-1}a^{\mp 1}ba^{\pm 1}$. The labels a, c denote the incoming and outgoing edges of the overpass respectively and b, d the two edges of the underpass and $\pm$ is the sign of the crossing.

The underlying idea here is to choose your eye (or more formally any point far above the plane on which we project the knot) as the basepoint. Every edge of the diagram is turned into a path from your eye, passing under the edge in the orthogonal direction just below and parallel to the projection plane according to the right-hand rule and going back to your eye. It can be shown that such paths generate the whole fundamental group. Studying the four paths close to a crossing we also arrive at the relations we imposed above.

For example if we take a diagram D of the negative trefoil then we find

$$\mathcal{W}(D) = \langle e_1, \dots e_6 | e_6 = e_1, e_2 = e_3, e_4 = e_5, e_2 = e_4e_1e_4^{-1}, e_6 = e_2e_5e_2^{-1}, e_4 = e_6e_3e_6^{-1}\rangle$$

[1] If $\alpha_0, \alpha_1 : [0,1] \longrightarrow B$ are two paths then a homotopy that fixes the basepoint is a continuous map $H : [0,1]^2 \longrightarrow B$ such that $H(i,t) = \alpha_i(t)$ and $H(x,0) = H(x,1) = p$.

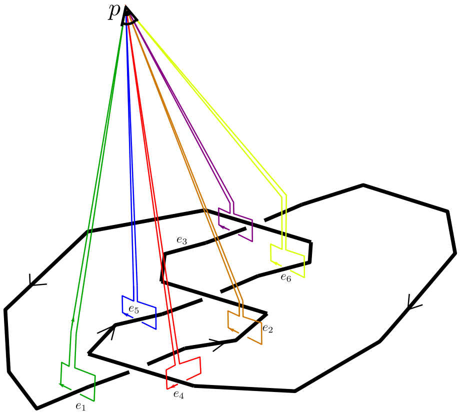

FIGURE 2. The generators of the presentation of the fundamental group of the trefoil knot complement $\mathcal{W}(3_1)$.

This can be simplified by setting $x = e_1$ and $z = e_5$ because then $e_2 = zxz^{-1}$ and the second to last relation yields $x = zxz^{-1}zzx^{-1}z^{-1} = zxzx^{-1}z^{-1}$. The final relation is a consequence of the others so we find $zxz = xzx$ and

$$\mathcal{W}(D) = \langle x, z | zxz = xzx \rangle$$

Working with fundamental groups or infinite groups given by a presentation as the above is in general quite difficult. One way to simplify matters, is to look for representations (i.e. group homomorphisms) into a easier group.

Definition 2. (Set of representations)
For any group Γ and any finite group G define $\mathrm{Hom}(\Gamma, G)$ to be the set of all group homomorphisms $\rho : \Gamma \longrightarrow G$.

For example $\mathrm{Hom}(\mathbb{Z}, G) \cong G$ for any group G since the representation is completely determined by the image of the generator $1 \in \mathbb{Z}$. In general if Γ can be presented with n generators we have $\#\mathrm{Hom}(\Gamma, G) \leq (\#G)^n$.

There is another presentation for the fundamental group of the trefoil knot complement that reads $\mathcal{W}(D) = \langle a, b | a^2 = b^3 \rangle$. It is related to the previous presentation by setting $a = xzx$ and $b = xz$. Any choice of a product of 2-cycles s and a product of 3-cycles t would thus give a representation $\rho : \mathcal{W}(D) \longrightarrow S_n$ with $\rho(a) = s$ and $\rho(b) = t$.

In the simplest non-Abelian case of $G = S_3 = \langle s, c | c^3, s^2, scsc \rangle$ let us have a closer look at the set of representations of a knot group. First there are the representations that send the edges to a power of c but those are not so interesting as (exercise) all edges should get the same element.

There is a bijection between the remaining representations of $\mathrm{Hom}(\mathcal{W}(D), S_3)$ (meaning those that send the edges to elements sc^k) and the set of three-colorings of the knot diagram D. By a three-coloring of the diagram we mean a coloring where all the edges are colored by an element of the set $\{r, g, b\}$ such that at any crossing either three or one colors meet. Also edges that are in the same strand with no underpass in between get the same color. For example Figure 3 shows all possible three-coloring of the trefoil knot diagram.

The idea here is that all the generators in $\mathcal{W}(D)$ are in the same conjugacy class because we assume D is a knot diagram (why?). The conjugacy classes of G are $\{1\}$, $\{c, c^{-1}\}$ and $\{s, sc, sc^{-1}\}$ because the fundamental relation is $scs = c^{-1}$.

The representations where $\rho(e) = sc^k$ for some fixed k are the simplest. These correspond to the colorings where only one color is used. Next if $\rho(e) = sc^k$ for some edge e and some k then all other edges f satisfy $\rho(f) = sc^x$ for some $x \in \{-1, 0, 1\}$. The reader should check that conjugation of sc^k by sc^n yields sc^{-k-n} so that at every crossing we must have ρ assigning three distinct elements sc^x, sc^y, sc^z with $x + y + z = 0 \mod 3$. We leave it as an exercise to finish the argument by showing that any three-coloring of a knot diagram D must either use three colors at all crossings or one color at all crossings.

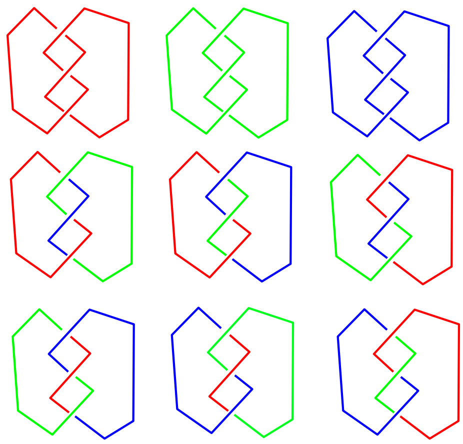

FIGURE 3. The three-colorings of the trefoil knot diagram.

The invariant $Z_{D(G)}(T)$ also counts representations of the fundamental group $\mathcal{W}(T)$ into G but here the count is more sophisticated and uses meridians and longitudes. Before getting into that let us remind ourselves of the invariant of the trefoil knot (diagram with writhe 3):

$$Z_{D(G)}(3_1) = \sum_{x,z\in G} (\delta^x \delta^{zxzx^{-1}z^{-1}} zxz^{-1}xz)_i$$

The number of non-zero terms in the sum is precisely the number of pairs $x, z \in G$ such that $xzx = zxz$ because the delta functions have to agree to get a non-zero term. Since we already showed that $\mathcal{W}(3_1) = \langle x, z | zxz = xzx\rangle$ this shows that the number of non-zero terms in the sum is precisely $\mathrm{Hom}(\mathcal{W}(3_1), G)$. Now let us investigate what the terms actually mean.

Definition 3. (Meridian, longitude, peripheral subgroup)
For a strand s in open XC-diagram D with set of edges E, define its meridian $\mu(s) \in F(E)$ and longitude $\lambda(s) \in F(E)$ as follows. $\mu(s)$ is the first edge belonging to the strand s. The longitude $\lambda(s)$ is the product of $e^{\pm}$ where e is an incoming overpass edge at a $\pm$ crossing where s passes under. The product is taken in order of appearance along s. The subgroup of $\mathcal{W}(D)$ generated by μ and λ is called the peripheral subgroup.

For example in the case of the trefoil diagram used above the longitude and meridian are $\mu = e_1 = x$ and $\lambda = e_5e_1e_3 = zxz^{-1}xz$. Really the longitude is just the path that runs parallel to the strand. When the diagram is closed the longitude is really only defined up to cyclic permutation.

In some sense the fundamental group is all one needs to know about knot theory. Together with its peripheral subgroup it forms a complete invariant, see Waldhausen's classic theorem below.

Theorem 4. (Waldhausen, 1968)
Suppose K, L are two knots. $K = L$ if and only if there is a group isomorphism $\varphi : \mathcal{W}(K) \longrightarrow \mathcal{W}(L)$ that restricts to an isomorphism of their peripheral subgroups.

This theorem is not the last word about knot theory because working with the fundamental group directly is not practical. We now turn to an interpretation of $Z_{D(G)}(T)$ in terms of the fundamental group.

Theorem 5. (Interpretation of $Z_{D(G)}(T)$)
For any open XC-tangle diagram T, consider $\mathcal{W}(T)$ and its longitudes and meridians μ, λ. We have

$$Z_{D(G)}(T) = \sum_{\rho\in\mathrm{Hom}(\mathcal{W}(T),G)} \bigotimes_{s\in\mathcal{L}(T)} \delta^{\rho(\mu(s))}\rho(\lambda(s))$$

The sum runs over all group homomorphisms (representations) ρ.

Proof. Let's call the right-hand side of the formula in the theorem $RH(T)$. We first check that $Z = Z_{D(G)}$ and RH agree on the basic XC-tangles $C^{\pm}, 1$ and X. To finish the proof we then just need to check that they agree also on how the results change under merging and disjoint union.

$\mathcal{W}(C_i^{\pm}) = \mathcal{W}(1_i) = \langle e|\emptyset\rangle$ and $\lambda(i) = 1$ and $\mu_i = f$ because these tangle diagrams consist of a single edge f. Any $g \in G$ yields precisely one representation of $\mathcal{W}(T)$ and all representations $\rho : \mathcal{W} \longrightarrow G$ are like this because they have to send f to $\rho(f) = g$ for some $g \in G$. The right-hand side is thus

$$\sum_{\rho:\langle f\rangle\longrightarrow G} \delta^{\rho(f)}\rho(1) = \sum_{g\in G} \delta^g 1 = 1 = Z(T)$$

because $\sum_{g\in G} \delta^g = 1$ is the unit element in the algebra $D(G)$. For the crossings $X^{\pm 1}_{o,u}$ we similarly compute $\mu(o) = a$, $\mu(u) = b$ and $\lambda(o) = 1$ and $\lambda(u) = a^{\pm 1}$. The presentation of the fundamental group is $\mathcal{W}(X^{\pm 1}_{o,u}) = \langle a, b\rangle$ so again a representation $\rho : \mathcal{W} \longrightarrow G$ is completely determined by any pair $g, h \in G$, setting $\rho(a) = g$ and $\rho(b) = h$. We thus find

$$RH(X^{\pm 1}_{o,u}) = \sum_{g,h} \delta^g_o 1_o \delta^h_u g^{\pm 1}_u = \sum_g \delta^g_o (\sum_h \delta^h_u) g^{\pm 1}_u = \sum_g \delta^g_o g^{\pm 1}_u = Z(X^{\pm 1}_{o,u})$$

Next we consider disjoint union: we note that $\mathcal{W}(DD')$ is obtained from the group presentations of D and D' by throwing together the generators and the relations as they do not interact in any way. In the same way the new meridians and longitudes are simply the union of the ones from D and D'. This means also that any pair of a representation of $\mathcal{W}(D)$ and a representation of $\mathcal{W}(D')$ determines a representation of $\mathcal{W}(DD')$. Therefore we find $RH(DD') = RH(D)RH(D')$ as required.

Finally we investigate $RH(m^{h,t}_r D)$. The meridian $\mu(r)$ is just the first edge of the new strand, that is the first edge of the old strand h so $\mu(r) = \mu(h)$. Also for the same reason $\lambda(r) = \lambda(h)\lambda(t)$. The group presentation changes by identifying the final edge of strand h with the initial edge of t. The initial edge of t is just $\mu(t)$ while the final edge of h is precisely $\lambda(h)^{-1}\mu(h)\lambda(h)$. So we set $\mu(t) = \lambda(h)^{-1}\mu(h)\lambda(h)$ or equivalently $\mu(h) = \lambda(h)\mu(t)\lambda(h)^{-1}$. The set of representations of $\mathcal{W}(m^{h,t}_r D)$ are precisely those $\rho : \mathcal{W}(D) \longrightarrow G$ such that $\rho(\mu(h)) = \rho(\lambda(h)\mu(t)\lambda(h)^{-1})$. It follows that

$$RH(m^{h,t}_r D) = \sum_{\rho\in\mathrm{Hom}(\mathcal{W}(D),G)} \delta_r^{\rho(\mu(h))}\delta_r^{\rho(\lambda(h)\mu(t)\lambda(h)^{-1})}\rho(\lambda(h))_r\rho(\lambda(t))_r \bigotimes_{s\in\mathcal{L}(T)\setminus\{h,t\}} \delta^{\rho(\lambda(s))}\rho(\mu(s))$$

The resulting expression corresponds to the multiplication $m^{h,t}_r Z(D) = Z(m^{h,t}_r D)$ because

$$m^{h,t}_r \delta_h^{\rho(\mu(h))}\rho(\lambda(h))_h \delta_t^{\rho(\mu(t))}\rho(\lambda(t))_t = \delta_r^{\rho(\mu(h))}\delta_r^{\rho(\lambda(h)\mu(t)\lambda(h)^{-1})}\rho(\lambda(h)\lambda(t))_r$$

As we can see the product of the delta functions forces precisely the same relation to hold. This finishes the proof. $\square$