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Exercise sheet 7 - Topics in Topology

March 22, 2022

1. In this exercise we make precise the construction of continuous maps $X \cup_\varphi Y \to Z$, where $\varphi : A \subset X \to Y$ and X, Y, Z are topological spaces.

   (a) Let $\sim$ be an equivalence relation on X and consider the quotient topology in $X/\sim$ with the projection $\pi : X \to X/\sim$ (recall: $U \subset X/\sim$ is open if and only if $\pi^{-1}(U) \subset X$ is open).

   Show that if $f : X \to Y$ is a continuous map satisfying that if $x \sim x'$ then $f(x) = f(x')$, there exists a unique continuous map $\bar{f} : X/\sim \to Y$ such that $f = \bar{f} \circ \pi$.

   (b) Consider $X \amalg Y$ the set-theoretic disjoint union of X and Y, and let $i : X \hookrightarrow X \amalg Y$, $j : Y \hookrightarrow X \amalg Y$ be the canonical inclusions. We endow $X \amalg Y$ with a topology defined as follows: $U \subset X \amalg Y$ is open if and only if $i^{-1}(U) \subset X$ and $j^{-1}(U) \subset Y$ are open.

   Show that given continuous maps $f : X \to Z$ and $g : Y \to Z$, there exists a unique continuous map $h = f \amalg g : X \amalg Y \to Z$ such that $f = h \circ i$ and $g = h \circ j$.

   (c) Recall that $X \cup_\varphi Y := (X \amalg Y)/a \sim \varphi(a)$ for $a \in A \subset X$. Write $\bar{i} = \pi \circ i : X \to X \cup_\varphi Y$ and $\bar{j} = \pi \circ j : X \to X \cup_\varphi Y$ for the canonical inclusion-projection maps, where here $\pi : X \amalg Y \to X \cup_\varphi Y$ is the projection to the quotient.

   Show that given continuous maps $f : X \to Z$ and $g : Y \to Z$ such that $f(a) = g(\varphi(a))$ for all $a \in A$, there exists a unique continuous map $h : X \cup_\varphi Y \to Z$ such that $h \circ \bar{i} = f$ and $h \circ \bar{j} = g$.

   (d) Write down commutative diagrams relating the maps for each of the previous exercises.

2. Let X', Y' be additional topological spaces with $\varphi' : A' \subset X' \to Y'$, and let $i' : X' \hookrightarrow X' \amalg Y'$, $j' : Y' \hookrightarrow X' \amalg Y'$ be as before.

   Use the previous exercise to show that a continuous map $f : X \to X'$ and a continuous map $g : Y \to Y'$ such that $i'(f(a)) \sim j'(g(\varphi(a)))$ (in $X' \amalg Y'$) completely determine a continuous map $X \cup_\varphi Y \to X' \cup_{\varphi'} Y'$.

3. Let M be a 3-manifold with boundary $\partial M \cong S^2$. Show that the closed 3 manifold resulting from attaching a 3-disc to M does not depend on the diffeomorphism $f : S^2 \to S^2$ used. More precisely, given any two diffeomorphisms $f, g : S^2 \to S^2$, show that there is a diffeomorphism

$$M \cup_f D^3 \xrightarrow{\cong} M \cup_g D^3.$$

   Hint: Use the Alexander extension lemma.

4. Mimic your argument from the previous exercise to show that a closed 4-manifold M is completely determined by the data of 0,1 and 2-handles. More precisely, if M_2 denotes the union of 0,1 and 2-handles, then $\partial M_2 \cong \#_m S^1 \times S^2$ as 3-handles $\cup$ 4-handle $\cong \natural_m S^1 \times D^3$. Then show that given two diffeomorphisms $f, g : \#_m S^1 \times S^2 \to \#_m S^1 \times S^2$, there is a diffeomorphism

$$M_2 \cup_f (\natural_m S^1 \times D^3) \xrightarrow{\cong} M_2 \cup_g (\natural_m S^1 \times D^3).$$

Hint: Replace the Alexander extension lemma used in the previous exercise by the Laudenbach-Poenaru theorem discussed in the lectures.