Topics in Topology Presentation

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1 General Overview

The goal of these notes is to give a general overview of Braids and how they relate to the Yang-Baxter Equation. More importantly, it opens a way to compute the Jones Polynomial in a new way. Firstly, we are going to introduce the Braid Group, give its general definition and state the Markov and Alexander theorem. Secondly, we are going to discuss the Yang-Baxter equation and show that for finite dimensional vector spaces V we have $\mathrm{End}(V)\otimes\mathrm{End}(V) \simeq \mathrm{End}(V \otimes V)$. If we represent elements of the spaces $\mathrm{End}(V)$ as matrices the latter gives us a convenient way of manipulating them and doing computations when taking tensor products over such endomorphism spaces. Hereafter we are going to define a braid invariant under the Markov moves and in the end we conclude these notes with some remarks on how to compute the Alexander polynomial using this braid invariant.

2 Braids

First of all we are going to discuss the definition of a braid group B_n and state the Alexander and Markov theorems. We will discuss several different ways to define a braid group and give the Artin representation of the braid group.

A braid is a collection of n strands embedded in $\mathbb{R}^2 \times [0, 1]$ such that its boundary is the set $\{1, \dots n\} \times \{0\} \times \{0, 1\}$ in $\mathbb{R}^2 \times [0, 1]$ and such that no strand has a critical point with respect to the vertical coordinate. Two braids are called isotopic if there are is an isotopy from $\mathbb{R}^2 \times [0, 1]$ preserving its boundary and vertical coordinates.

This can be turned quite easily into a geometrical picture by imagining this n-strands in 3 dimensional space up to isotopy (endpoints of the strand are fixed). The braid group is then the group in n-strands, denoted by B_n where the group operation is defined by gluing of the strands. A picture of a braid is shown below:

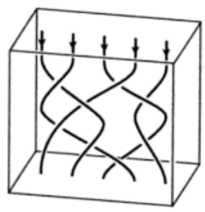

Figure 1: Braid in 5 strands.

Two important theorems about braids are Alexanders theorem and Markovs theorem. Alexander theorem states that every link is isotopic to the closure of a braid. Here with closure we mean connecting the upper and lower part of a braid as shown in the picture below:

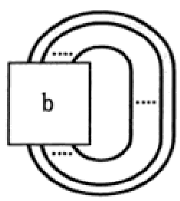

Figure 2: Closure af an arbitrary braid b in B_n

Before discussing Markovs theorem we need first an more convenient way of representing braids.

2.1 Artin Presentation

There are several ways to encode the information of specific braids up to isotopy. However, the Artin presentation of the braid group B_n is particularly usefull and given by the following presentation:

$$B_n = \left\langle \sigma_1, \dots, \sigma_n | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for all } i, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for all } |i - j| > 1 \right\rangle$$

where σ_i and σ_i^{-1} denote the crossings as depicted below.

Figure 3: The crossings σ_i^1 and σ_i^{-1} as in braids.

In therms of Figure 1 this presentation is: $\sigma_1\sigma_3\sigma_2^{-1}\sigma_4^{-1}\sigma_1\sigma_4^{-1}\sigma_2\sigma_3^{-1}$. We can now discuss the Markov theorem. Markov theorem states that the closure of braids A and B are isotopic if and only if B can be obtained from A by either a sequence of the two Markov moves (I and II): (MI) $AA' \longleftrightarrow A'A$ for $A' \in B_n$ and (MII) $A\sigma_n \longleftrightarrow A \longleftrightarrow A\sigma_n^{-1}$ where A in $A\sigma_n$ is regarded as living in the space B_{n+1} by adding the trivial $(n+1)$ strand to A.

Markovs theorem implies the following interesting equality

$$\text{oriented links/isotopy} = \left(\cup_{n=1}^{\infty} B_n/\text{MI and MII moves}\right).$$

This is interesting as we can reduce the study of isotopy classes of links to study of the braids up to Markov moves.

3 Yang Baxter Equation and Vector Space Isomorhpism

In this section wer are going to explore the Yang-Baxter equation and concretely proof the isomorphism $\mathrm{End}(V)\otimes\mathrm{End}(V) \simeq \mathrm{End}(V\otimes V)$. We define an automorphism $R : V\otimes V \to V\otimes V$ as the solution to the Yang-Baxter equation:

$$(R\otimes Id)(Id\otimes R)(R\otimes Id) = (Id\otimes R)(R\otimes Id)(Id\otimes R) \in \mathrm{End}(V^{\otimes 3}).$$

We show that such an R gives rise to a representation $\rho_{R,n} = \rho_n : B_n \to \mathrm{End}(V^{\otimes n})$ given by

$$\rho_n(\sigma_i) = Id^{\otimes i-1}\otimes R\otimes Id^{\otimes n-i-1}.$$

For this we have to show that the identities

$$\rho_n(\sigma_i)\rho_n(\sigma_{i+1})\rho_n(\sigma_i) = \rho_n(\sigma_{i+1})\rho_n(\sigma_i)\rho_n(\sigma_{i+1})$$

and

$$\rho_n(\sigma_i)\rho_n(\sigma_j) = \rho_n(\sigma_j)\rho_n(\sigma_i)$$

hold. The latter we need also $|i-j| > 1$.

The first one is easily seen to hold as:

$$\rho_n(\sigma_i)\rho_n(\sigma_{i+1})\rho_n(\sigma_i) = (Id^{\otimes i-1}\otimes R\otimes Id^{\otimes n-i-1})(Id^{\otimes i}\otimes R\otimes Id^{\otimes n-i-2})(Id^{\otimes i-1}\otimes R\otimes Id^{\otimes n-i-1}) =$$

$$Id^{\otimes i-1}\otimes(R\otimes Id)(Id\otimes R)(R\otimes Id)\otimes Id^{\otimes n-i-2} = Id^{\otimes i-1}\otimes(Id\otimes R)(R\otimes Id)(Id\otimes R)\otimes Id^{\otimes n-i-2} =$$

$$(Id^{\otimes i}\otimes R\otimes Id^{n-i-2})(Id^{\otimes i-1}\otimes R\otimes Id^{\otimes n-i-1})(Id^{\otimes i}\otimes R\otimes Id^{\otimes n-i-2}) = \rho(\sigma_{i+1})\rho(\sigma_i)\rho(\sigma_{i+1}).$$

The second relation is also seen to hold trivially from the definition of the presentation ρ_n. Before proceeding we are going to first discuss a convenient way to work with tensor products over endomorphism spaces.

3.1 Identifying vector spaces

We are also going to show that $f \otimes g \to f \otimes g$ between $\mathrm{End}(V) \otimes \mathrm{End}(V) \simeq \mathrm{End}(V \otimes V)$ is an isomorphism. In the first case we have two elements from $f, g \in \mathrm{End}(V)$ where we see f, g as elements of the vector spaces $\mathrm{End}(V)$ and we take the tensor product of these two elements $f \otimes g \in \mathrm{End}(V) \otimes \mathrm{End}(V)$. This element is mapped to $f \otimes g$ which is an element of the vector space $\mathrm{End}(V \otimes V)$, which consists of endomorphisms over the tensor product $V \otimes V$. Hence the latter one maps elements $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$. We are know going to show the isomorphism between these two spaces.

Note that the map $V^* \otimes V \to \mathrm{End}(V)$ given by $\omega \otimes e \to (v \to \omega(v)e)$ defines a linear ismorphism. Let f, g be two arbitrary elements of $\mathrm{End}(V)$ and let $f \otimes g \in \mathrm{End}(V) \otimes \mathrm{End}(V)$. Note that $V \otimes W \simeq W \otimes V$ which is one of the properties of a tensor product. Hence when we identify f with $\omega \otimes e$ and g with $\omega' \otimes e'$ we see that $f \otimes g \to (\omega \otimes e) \otimes (\omega' \otimes e')$ defines a linear isomorphims between the spaces $\mathrm{End}(V) \otimes \mathrm{End}(V)$. When we use the property on the middle two spaces of this tensor product we obtain: $(\omega \otimes e) \otimes (\omega' \otimes e') \to (\omega \otimes \omega') \otimes (e \otimes e')$. This element lives in the space $V^* \otimes V^* \otimes V \otimes V \simeq (V \otimes V)^* \otimes (V \otimes V)$, as $V^* \otimes V^* \simeq (V \otimes V)^*$ by the map $\omega \otimes \omega' \to \omega \otimes \omega'$.

Therefore we have that $(\omega \otimes \omega') \otimes (e \otimes e') \in (V \otimes V)^* \otimes (V \otimes V) \simeq \mathrm{End}(V \otimes V)$ and hence that the map $f \otimes g \to f \otimes g$ defines a linear isomorphism between $\mathrm{End}(V) \otimes \mathrm{End}(V)$ and $\mathrm{End}(V \otimes V)$.

We can also look at this in terms of matrices. Then we see that a map $A, B \in \mathrm{End}(V)$ are represented by $A(e_i) = \sum_j A_i^j e_j$ and B in a similar manner. Then we see that $A \otimes B \to A \otimes B$, where $(A \otimes B) \in \mathrm{End}(V \otimes V)$ can be seen as $(A \otimes B)(e_i \otimes e_j) = A(e_i) \otimes B(e_j) = \sum_k A_i^k e_k \otimes \sum_l B_j^l e_l = \sum_{k,l} A_i^k B_j^l (e_k \otimes e_l)$. One can define a matrix new $n^2 \times n^2$ matrix $C_{ij}^{kl} = A_i^k B_j^l$.

4 Construction R Matrix

Let V be the 2 dimensional vector space over $\mathbb{Q}(q)$ with fixed basis (e_1, e_2). Let X be as defined as the element that recovered the Jones polynomial as described in lecture 7.

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_i \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}_j + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_i \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}_j + (q - q^{-3}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_j.$$

Note that $X \in \mathrm{End}(V) \otimes \mathrm{End}(V)$ and that we can define a map $\phi : \mathrm{End}(V) \otimes \mathrm{End}(V) \to \mathrm{End}(V \otimes V)$. When we act with this map on X we can use the matrix representation of $\mathrm{End}(V \otimes V)$ derived in the previous paragraph to obtain (this is just the Kronecker product of matrix tensor multiplication)

$$\phi(X) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & q - q^{-3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q^{-1} & q - q^{-3} & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

We can also define a map $P : \mathrm{End}(V \otimes V) \to \mathrm{End}(V \otimes V)$ by $P(e_i \otimes e_j) = e_j \otimes e_i$. P with standard basis for $V \otimes$ consisting of $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$ will have the form:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

When we apply $(P \circ \phi)(X) = P(\phi(X))$ we obtain:

$$R = P(\phi(R)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \circ \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q^{-1} & q - q^{-3} & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & q^{-1} & q - q^{-3} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

We show that R is indeed an R matrix satisfying:

$$(R \otimes Id_V) \circ (Id_V \otimes R) \circ (R \otimes Id_V) = (Id_V \otimes R) \circ (R \otimes Id_V) \circ (Id_V \otimes R) \in \operatorname{End}(V^{\otimes 3}).$$

Note that we have the following identity for $R \otimes Id_V$ and $Id_V \otimes R$

$$R \otimes Id_V = (q) \oplus \begin{pmatrix} q & 0 & 0 \\ 0 & 0 & q^{-1} \\ 0 & q^{-1} & q - q^{-3} \end{pmatrix} \oplus \begin{pmatrix} 0 & q^{-1} & 0 \\ q^{-1} & q - q^{-3} & 0 \\ 0 & 0 & q \end{pmatrix} \oplus (q)$$

and

$$Id_V \otimes R = (q) \oplus \begin{pmatrix} 0 & q^{-1} & 0 \\ q^{-1} & q - q^{-3} & 0 \\ 0 & 0 & q \end{pmatrix} \oplus \begin{pmatrix} q & 0 & 0 \\ 0 & 0 & q^{-1} \\ 0 & q^{-1} & q - q^{-3} \end{pmatrix} \oplus (q).$$

From these identities we see that

$$(R \otimes Id_V) \circ (Id_V \otimes R) \circ (R \otimes Id_V) = (q^3) \oplus \begin{pmatrix} 0 & 0 & q^{-1} \\ 0 & q^{-1} & q - q^{-3} \\ 0 & q - q^{-3} & \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & q^{-1} \\ 0 & q^{-1} & q - q^{-3} \\ q^{-1} & q - q^{-3} & q^3 - q^{-1} \end{pmatrix} \oplus (q^3)$$

and as expected also

$$(Id_V \otimes R) \circ (R \otimes Id_V) \circ (Id_V \otimes R) = (q^3) \oplus \begin{pmatrix} 0 & 0 & q^{-1} \\ 0 & q^{-1} & q - q^{-3} \\ 0 & q - q^{-3} & \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & q^{-1} \\ 0 & q^{-1} & q - q^{-3} \\ q^{-1} & q - q^{-3} & q^3 - q^{-1} \end{pmatrix} \oplus (q^3).$$

5 Markov Trace

In this section we are going to define a braid invariant under the Markov moves. Let $\rho_n : B_n \to \operatorname{End}(V^{\otimes n})$ be the representation for the braid group representation associated to R. Let C be the element $C = \begin{pmatrix} q^2 & 0 \\ 0 & q^{-2} \end{pmatrix}$. Let $C^{\otimes n}$ be the corresponding map $C^{\otimes n} : V^{\otimes n} \to V^{\otimes n}$. In this section we are going to show that $tr(C^{\otimes n} \circ \rho_n(\beta)) \in \mathbb{Q}(q)$ is invariant under the Markov moves. Before proceeding we first discuss some properties and ways to see the trace of an arbitrary matrix.

5.1 General Properties of the Trace

When we have an arbitrary $A \in \mathrm{End}(V)$ where V is n dimensional and thus A can be represented as an $n \times n$ matrix, we have that $tr(A) = \sum_i A_i^i$. Let V_1 and V_2 be vector spaces over $\mathbb{C}$, then we can define tr_i of the endomorphism in $\mathrm{End}(V_i \otimes V_i)$ as

$$tr_1 \;:\; \mathrm{End}(V_1 \otimes V_2) = (V_1 \otimes V_2)^* \otimes (V_1 \otimes V_2) = V_2^* \otimes V_1^* \otimes V_1 \otimes V_2 \to V_2^* \otimes V_2 = \mathrm{End}(V_2),$$

$$tr_2 \;:\; \mathrm{End}(V_1 \otimes V_2) = (V_1 \otimes V_2)^* \otimes (V_1 \otimes V_2) = V_2^* \otimes V_1^* \otimes V_1 \otimes V_2 \to V_1^* \otimes V_1 = \mathrm{End}(V_1)$$

where the arrows depict contractions $V_i^* \otimes V_i \to \mathbb{C}$ for $i = 1, 2$ respectively. Let $\{e_i\}$ and $\{e_i'\}$ depict bases of V_1 and V_2. In the same fashion as section 2 we see that for a linear map $A \in \mathrm{End}(V_1 \otimes V_2)$ we have that $A(e_i \otimes e_j) = \sum_{k,l} A_{ij}^{kl}(e_k \otimes e_l)$. Then the tr_1 and the tr_2 map are represented by

$$(tr_1(A))(e_i') = \sum_{jk} A_{ki}^{kj} e_j'$$

$$(tr_2(A))(e_i) = \sum_{jk} A_{ki}^{kj} e_j.$$

In our case, we have V the two dimensional vector space with basis e_1, e_2. When we have a matrix $A = (A_{ij}^{kl}) \in \mathrm{End}(V \otimes V)$ with respect to the basis $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$ of $V \otimes V$, we see that

$$tr_1(A) = \begin{pmatrix} A_{00}^{00} + A_{10}^{10} & A_{00}^{01} + A_{10}^{11} \\ A_{01}^{00} + A_{11}^{10} & A_{01}^{01} + A_{11}^{11} \end{pmatrix}$$

$$tr_2(A) = \begin{pmatrix} A_{00}^{00} + A_{01}^{01} & A_{00}^{10} + A_{01}^{11} \\ A_{10}^{00} + A_{11}^{01} & A_{10}^{10} + A_{11}^{11} \end{pmatrix}.$$

From this we see that tr_2 might be of help to us in future computations.

5.2 Construction of the Trace Invariant

Recall that $R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & q^{-1} & q - q^{-3} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$ and note that $R^{-1} = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & q^{-1} - q^3 & q & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}$.

Let us compute the following traces:

$$tr_2((Id_V \otimes C) \circ R) = tr_2 \begin{pmatrix} q^3 & 0 & 0 & 0 \\ 0 & 0 & q^{-3} & 0 \\ 0 & q & q^3 - q^{-1} & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix} = q^3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = q^3 Id_V.$$

Similarly we see that

$$tr_2((Id_V \otimes C) \circ R^{-1}) = tr_2 \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q^{-3} - q & q & 0 \\ 0 & q^3 & 0 & 0 \\ 0 & 0 & 0 & q^{-3} \end{pmatrix} = q^{-3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = q^{-3} Id_V.$$

Hence we have that $tr_2((Id_V \otimes C) \circ R^{\pm 1}) = q^{\pm 3} Id_V$. From this we see that

$$tr(C^{\otimes(n+1)} \cdot \rho_{n+1}(\sigma_n^{\pm 1}\beta)) = tr(C^{\otimes(n+1)} \cdot (id_V^{\otimes(n-1)} \otimes R^{\pm 1}) \cdot \rho_{n+1}(\beta)) = q^{\pm 3} tr(C^{\otimes n}\rho_n(\beta)).$$

Hence, we see that tr is invariant under the Markov II move up to a deframing constant. For proving that this identity is also invariant under the Markov I move we note that

$$(C\otimes C)\otimes R = \begin{pmatrix} q^4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-4} \end{pmatrix} \circ \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & q^{-1} & q - q^{-3} & 0 \\ 0 & 0 & 0 & q \end{pmatrix} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & q^{-1} & q - q^{-3} & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \circ \begin{pmatrix} q^4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-4} \end{pmatrix} = R\otimes(C\otimes C).$$

From this we can see that $\rho_n(\beta)$ indeed commutes with $C^{\otimes n}$ ans hence

$$tr(C^{\otimes n}\rho_n(\sigma_n^{-1}\beta\sigma_n^{1})) = tr(C^{\otimes n}\rho_n(\sigma_n^{-1})\rho_n(\beta)\rho_n(\sigma_n^{1})) =$$

$$tr(\rho_n(\sigma_n^{1})C^{\otimes n}\rho_n(\sigma_n^{-1})\rho_n(\beta)) = tr(C^{\otimes n}\rho_n(\sigma_n^{1}\sigma_n^{-1})\rho_n(\beta)) =$$

$$tr(C^{\otimes n}\rho_n(\beta)).$$

Hence we see that this trace is also invariant under the Markov I move and thus Invariant under all the Markov moves up to the deframing.

As an example we are going to compute this Markov invariant trace for the element σ_1^2:

$$tr(C^{\otimes n}\rho_n(\sigma_1^2)) = tr((C^{\otimes 2} \otimes C^{\otimes(n-2)})(R \otimes Id^{\otimes(n-2)})(R \otimes Id^{\otimes(n-2)})) =$$

$$tr((C^{\otimes 2}R^2 \otimes C^{\otimes(n-2)})) = tr(C^{\otimes 2}R^2) \cdot tr(C)^{n-2} = (q^6 + 3q^{-2})(q^2 + q^{-2})^{n-2}$$

5.3 Jones Polynomial and Markov Trace

We are now going to link the Jones Polynomial to the Markov trace. We first note an interesting property of our R matrix which can be computed quite easily:

$$qR - q^{-1}R^{-1} = (q^2 - q^{-2})Id_{V\otimes V}.$$

This shows that the Skein Relation of the Markov trace is the same as the Skein relation of $Z_{\mathrm{End}(V)}$ as in the lecture notes (see theorem 14 of lecture 7 [1]) where $R^{\pm 1}$ is identified with $L^{\pm}$ and $Id_{V\otimes V}$ is identified with L_0. From the previous paragraph we see that deframing property also holds and as in theorem 14 [1].

This together with Alexanders theorem which states that every link is isotopic to the closure of a braid (leaving the writhe invariant) we see that indeed $J_{cl(\beta)} = \frac{1}{q^2+q^{-2}} \hat{tr}(C^{\otimes n}\rho_n(\beta))$ where $\hat{tr}(C^{\otimes n}\rho_n(\beta)) = q^{-3w(\beta)} tr(C^{\otimes n}\rho_n(\beta))$. Hence, we found a new way of computing the Alexander polynomial!

References

[1] Lecture 7: The Jones Polynomial Revisited, dr. J. Becerra and prof. R.I. Van der Veen

[2] Quantum Invariants, a study of knots, 3-manifolds and their sets, T. Ohtsuki. (§2.2)