Department of Engineering Mathematics

EMAT20920: Numerical Methods in MATLAB

COURSEWORK ASSESSMENT
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All figures in this report have been saved using `saveFigPDF` function as it automatically resizes the paper to the correct size.

Listing 1: `../src/saveFigPDF.m`

```
function saveFigPDF(fileName)
    %saveFigPDF saves open figure as a PDF file
    %
    %Inputs:
    %   fileName = File name to save figure as
    %Usage:
    %   saveFigPDF("polynomial") -> Saves current figure as polynomial.pdf

    % Get current figure handle
    figureHandle = gcf;
    % Resize paper
    set(figureHandle,'PaperPosition',3*[0 0 6 4]);
    set(figureHandle,'PaperSize',3*[6 4]);
    set(figureHandle,'PaperUnits','centimeters');

    print(fileName,'-dpdf');
end
```

Question 1: Root-finding

(a) To find how many solutions each equation has in the given domain I will rearrange all the equations to be equal to zero and then looks for the zeros of the rearranged equations. As a corollary to the intermediate value theorem, if a function is continuous and changes sign in a bracket then that bracket must contain a zero. So I will plot each of the rearranged equation and I look for appropriate brackets. I will use the `pltFunc` function to plot the functions as it removes values outside a defined limit which prevents MATLAB plotting discontinuous functions as continuous. The limits can then be changed using the property explorer to give a more useful plot.

Listing 2: `../src/q1/pltFunc.m`

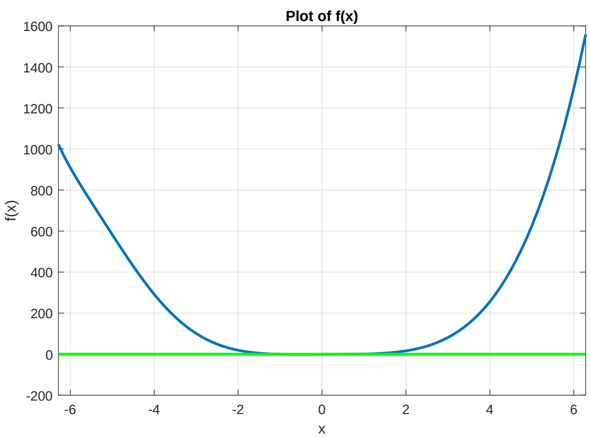
```
function pltFunc(f, domain, discontLim)
    %pltFunc plots function f between values of xLim removing any values
    % that are greater than discontLim to prevent MATLAB plotting
    % discontinuous functions as continous and plots a line of x = 0 to
    % help make any zeros clear
    %
    % Input:
    %   f = function handle to plot
    %   domain = 1x2 vector containing the lower and upper bound of the
    %   domain of f
    %   discountLim = absolute values of the function greater than this are
    %   changed to NaN. Setting to inf will plot all values of the function
    %
```

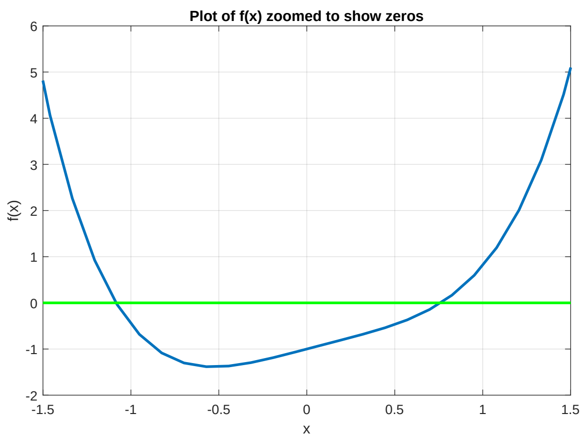
```
    % Usage:
    %   pltFunc(@(x) 1./x, [-10 10], 5) -> Plots 1/x between -10 and 10
    %   changing the values where |1/x|>5 to NaN


    % Check xLim is the correct dimensions
    assert(isequal(size(domain), [1 2]), "domain must be a 1x2 vector")

    %% Generate values to plot
    x = linspace(domain(1), domain(2));
    y = f(x);

    % Remove large values of y to prevent MATLAB plotting discontinuous
    % functions as continuous
    y(abs(y)>discontLim) = NaN;

    %% Plot function and line x = 0
    plot(x, y, [min(x) max(x)], [0 0], "g-", "LineWidth", 2);
    xlabel("x");
    ylabel("f(x)");
    xlim(domain);
    title("Plot of f(x)");
    grid on;
end
```

(i) Rearranging $x^4 = e^{-x}\cos(x)$ gives $f(x) = x^4 - e^{-x}\cos(x)$.

Listing 3: ../src/q1/Q1a_i.m

```
f = @(x) x.^4 - exp(-x).*cos(x);
pltFunc(f, [-2*pi 2*pi], inf);
```



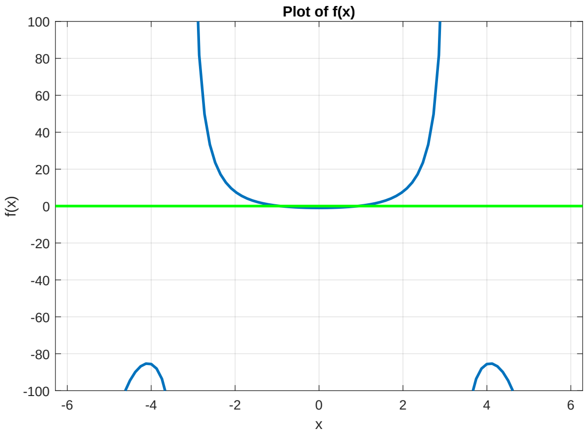


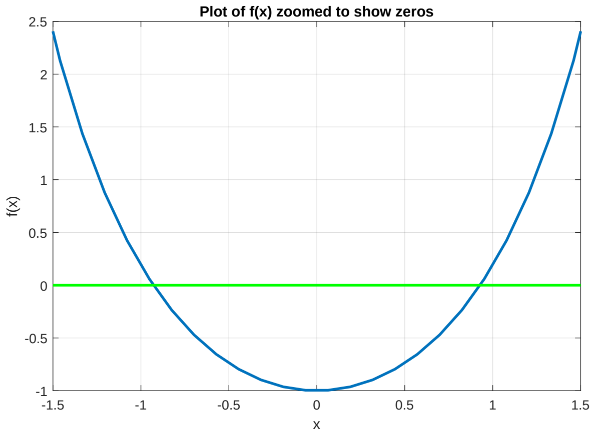
The second zoomed in plot shows there are two zeros in the given domain. The first zero can be bracketed by the interval $[-1.5, -1]$ as $f(-1.5) = 4.7455$ and $f(-1) = -0.4687$ so since the function is continuous and there is a change of sign this bracket must contain a zero. Like wise the second root can be bracketed by the interval $[0.5, 1]$ as $f(0.5) = -0.4698$ and $f(1) = 0.8012$.

(ii) Setting $f(x) = \frac{x^3}{\sin(x)} - 1$.

Listing 4: ../src/q1/Q1a_ii.m

```
f = @(x) (x.^3)./sin(x) - 1;
pltFunc(f, [-2*pi 2*pi], 500);
```



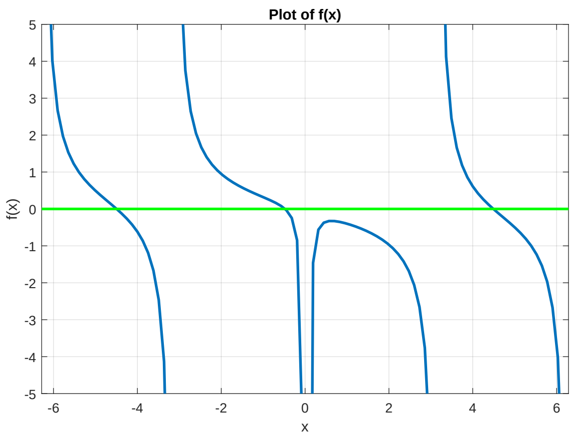


The second plot show there are two roots. The first root can be bracketed by the interval $[-1, -0.5]$ as $f(-1) = 0.1884$ and $f(-0.5) = -0.7393$ and $f(x)$ is continuous in this bracket. Likewise, the second root can be bracketed by the interval $[0.5, 1]$ as $f(0.5) = -0.7393$ and $f(1) = 0.1884$.

(iii) Rearranging $\cot(x) = \frac{25}{25x-1}$ gives $f(x) = \cot(x) - \frac{25}{25x-1}$.

Listing 5: `../src/q1/Q1a_iii.m`

```
f = @(x) cot(x) - 25./(25*x - 1);
pltFunc(f, [-2*pi 2*pi], 30);
```

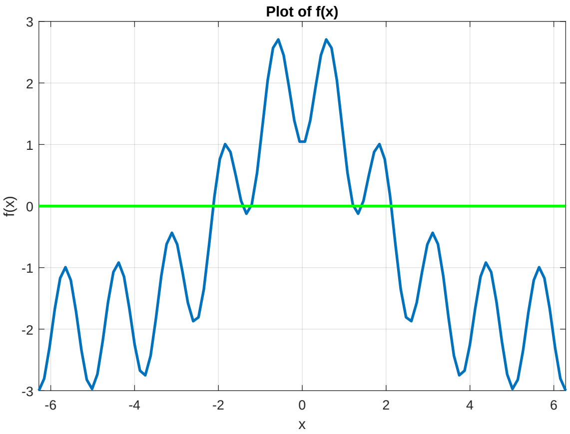


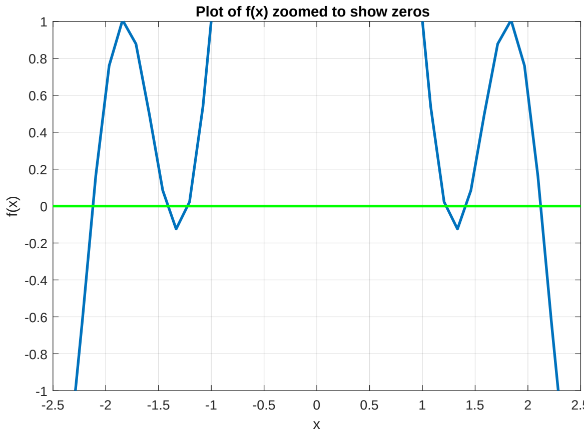
The plot shows that the equation has three solutions. The first can be bracketed by the interval $[-5, -4]$ as $f(-5) = 0.4942$ and $f(-4) = -0.6162$. The second solution can be bracketed by the interval $[-1, -0.1]$ as $f(-1) = 0.3194$ and $f(-0.1) = -2.8238$. The third solution can be bracketed by the interval $[4, 5]$ as $f(4) = 0.6112$ and $f(5) = -0.4974$. $f(x)$ is continuous in each of the bracketing intervals.

(iv) Rearranging $4e^{-x^2/5} = cos(5x) + 2$ gives $f(x) = 4e^{-x^2/5} - cos(5x) - 2$.

Listing 6: `../src/q1/Q1a_iv.m`

```
f = @(x) 4*exp(-x.^2/5) - cos(5*x) - 2;
pltFunc(f, [-2*pi 2*pi], inf);
```





The second plot shows that the equation has 6 solutions. The bracketing intervals are shown in the table below.

$[a, b]$	f(a)	f(b)
$[-2.5, -2]$	-1.8518	0.6364
$[-1.5, -1.25]$	0.2039	-0.0730
$[-1.25, -1]$	-0.0730	0.9913
$[1, 1.25]$	0.9913	-0.0730
$[1.25, 1.5]$	-0.0730	0.2039
$[2, 2.5]$	0.6364	-1.8518

(b) The bisection method is used by calling the the `bisectRoot` function.

Listing 7: `../src/q1/bisectRoot.m`

```
function [sol, i, err] = bisectRoot(f, a, b, tol)
    %bisectRoot Use the bisection method to find roots of the function f
    %   bracketed within the intervals [a, b].
    %
    %Inputs:
    %   f = function handle to function whose root is to be found
    %   a = 1*n array containing all the lower ends of the brackets
    %   where n is the number of roots
    %   b = 1*n array containing all the lower ends of the brackets
    %   where n is the number of roots
    %   tol = absolute error tolerance with which to find the root;
    %   Iteration terminates when the root is known to within +/- tol
    %
    %Outputs:
    %   sol = 1*n array of of roots
    %   i = 1*n array of the number of iterations required to find the nth
    %   root
    %   err =
    %
    %Usage:
    %   [r, i, err] = bisect(@(x) x.^2 - 4, 1, 3, 5e-9) -> returns the
    %   approximation to root of x^2 - 4 = 0 within [1, 3], the number of
    %   iterations required to find the root and the final absolute error

    % check if all intervals are correctly defined
```

```
    assert(isequal(size(a), size(b)),...
        "Must be an equal number of upper and lower bounds");

    % check whether f changes sign
    assert(all(sign(f(a)) ~= sign(f(b))),...
        'f(a) and f(b) should have opposite sign');

    % intialise variables
    % iteration counter
    i = zeros(size(a));
    % current solution estimate
    sol = (a + b)/2;
    % previous solution estimate
    sol_old = Inf;
    % absolute error
    err = Inf;
    withinTol = zeros(size(a));

    % bisection algorithm:
    % at each iteration, find the half-interval that contains a sign change
    % and relabel the endpoints appropriately
    while any(~withinTol)
        i(~withinTol) = i(~withinTol) + 1;
        sol_old = sol;
        mid = (a + b)/2;

        % mid point is a root
        exactRoot = f(mid) == 0;
        sol(exactRoot) = mid(exactRoot);
        err(exactRoot) = 0;
        withinTol(exactRoot) = true;

        % solution is in first half of interval and mid point not a root
        firstHalf = (sign(f(a)) ~= sign(f(mid))) & ~exactRoot;
        b(firstHalf) = mid(firstHalf);

        % solution is in second half of interval and mid point not a root
        secondHalf = (sign(f(a)) == sign(f(mid))) & ~exactRoot;
        a(secondHalf) = mid(secondHalf);

        % update solutions and errors values that aren't within tolerance
        sol(~withinTol) = (a(~withinTol) + b(~withinTol))/2;
        err(~withinTol) = abs(sol(~withinTol) - sol_old(~withinTol));
        withinTol(err < tol) = true;
    end
end
```

(i) Solutions to $f(x) = x^4 - e^{-x}\cos(x) = 0 \quad x \in [-2\pi, 2\pi]$.

Listing 8: `../src/q1/Q1b_i.m`

```
f = @(x) x.^4 - exp(-x).*cos(x);
a = [-1.5 0.5];
b = [-1 1];
[r, i, err] = bisectRoot(f, a, b, [5e-8 5e-9])
```

Note the two different tolerances since one root is an order of magnitude larger so requires one less decimal place of accuracy to be accurate to 8 significant figures.

(ii) Solutions to $f(x) = \frac{x^3}{\sin(x)} - 1 = 0 \quad x \in [-2\pi, 2\pi]$.

$[a, b]$	Root	# Iterations
$[-1.5, -1]$	-1.0843597	23
$[0.5, 1]$	0.76221107	26

Listing 9: ../src/q1/Q1b_ii.m

```
f = @(x) (x.^3)./sin(x) - 1;
a = [-1 0.5];
b = [-0.5 1];
[r, i, err] = bisectRoot(f, a, b, 5e-9)
```

$[a, b]$	Root	# Iterations
$[-1, -0.5]$	-0.92862631	26
$[0.5, 1]$	0.92862631	26

(iii) Solutions to $f(x) = \cot(x) - \frac{25}{25x-1} = 0 \quad x \in [-2\pi, 2\pi]$.

Listing 10: ../src/q1/Q1b_iii.m

```
f = @(x) cot(x) - 25./(25*x - 1);
a = [-5 -1 4];
b = [-4 -0.1 5];
[r, i, err] = bisectRoot(f, a, b, [5e-8 5e-9 5e-8])
```

$[a, b]$	Root	# Iterations
$[-5, -4]$	-4.4953722	24
$[-1, -0.1]$	-0.47773376	27
$[4, 5]$	4.4914097	24

(iv) Solutions to $f(x) = 4e^{-x^2/5} - cos(5x) - 2 = 0 \quad x \in [-2\pi, 2\pi]$.

Listing 11: ../src/q1/Q1b_iv.m

```
f = @(x) 4*exp(-x.^2/5) - cos(5*x) - 2;
a = [-2.5 -1.5 -1.25 1 1.25 2];
b = [-2 -1.25 -1 1.25 1.5 2.5];
[r, i, err] = bisectRoot(f, a, b, 5e-8)
```

$[a, b]$	Root	# Iterations
$[-2.5, -2]$	-2.1222382	23
$[-1.5, -1.25]$	-1.4255432	22
$[-1.25, -1]$	-1.2145933	22
$[1, 1.25]$	1.2145933	22
$[1.25, 1.5]$	1.4255432	22
$[2, 2.5]$	2.1222382	23

(c) The iterative scheme we asked to implement is called Steffensen's method. This is implemented in the `steffensenRoot` function.

Listing 12: ../src/q1/steffensenRoot.m

```
function [r, n, err] = steffensenRoot(f, x0, tol, nMax)
```

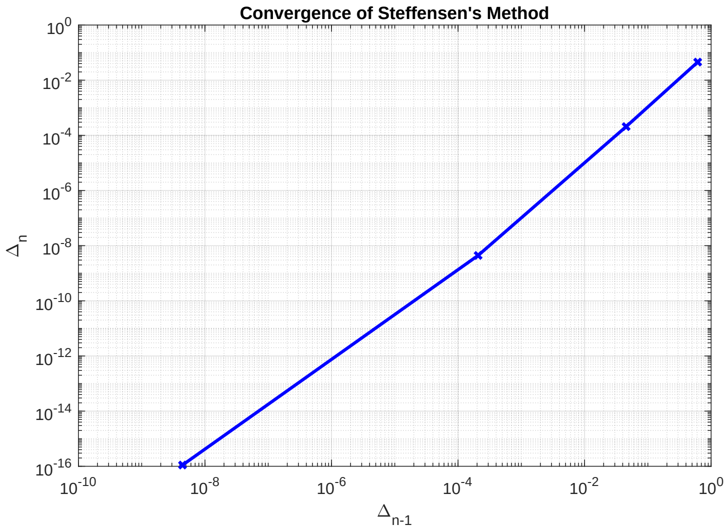
```
    %steffensenRoot uses Steffensen's method to find roots of f(x)
    %   based on an initial guess x0
    %
    %Inputs:
    %   f = function handle to function whose root is to be found
    %   x0 = initial guess of the root to begin iteration at
    %   tol = absolute error tolerance with which to find the root
    %   iteration terminates when the root is known to within +/- tol
    %   nMax = the maximum number of iteration to quit after. Prevents an
    %   infinite loop if the iterations do not converge
    %
    %Outputs:
    %   r = the approximate root of f(x)=0
    %   n = the number of interations
    %   err = 1*n vector of the absolute error after each interation
    %
    %Usage:
    %   [r, n, err] = steffensenRoot(@(x) exp(-x) -x, 0, 5e-9, 50) ->
    %   returns the approximate roo of x^-x -x = 0 after n iterations and
    %   err the absolute error after each iteration

    % set initial guess as first root
    xn = x0;
    %iteration counter
    n = 0;
    % preallocate error array
    err = Inf(1, nMax);

    while all(err > tol) && n < nMax
        n = n + 1;
        xOld = xn;
        % Calculate f(xn) to avoid repeat computation
        fn = f(xn);
        % Calculate next interation
        xn = xn - fn*(f(xn + fn)/fn - 1)^-1;
        err(n) = abs(xn - xOld);
    end

    % remove any unused preallocated element in error array
    err(isinf(err)) = [];

    % check if solution converged
    assert(err(end) < tol, "No convergence")

    r = xn;
end
```

The following uses this function to find the root of $e^{-x} - x = 0$ and calculate the convergence.

Listing 13: `../src/q1/Q1c.m`

```
f = @(x) exp(-x) - x;
[r, n, e] = steffensenRoot(f, 0, 5e-13, 50)
%% Generate plot of convergence
loglog(e(1:end-1),e(2:end), "bx-", "LineWidth", 2);
title("Convergence of Steffensen's Method")
xlabel('\Delta_{n-1}');
ylabel('\Delta_{n}');
grid on;
%% Find order of congerence
polyfit(log(e(1:end-1)), log(e(2:end)), 1)
```

After 5 iterations the root $x = 0.567143290410$ is accurate to 12 decimal places.



The graph shows a straight which shows error $\propto \Delta_{n-1}^q$, where q is the gradient of the line. Using the `MATLAB` function `polyfit` the gradient of the above graph as 1.8 which is close to 2 so Steffensen's method is second order.

(d) The first step in creating a cobweb plot is to implement a fixed point iteration scheme.

Listing 14: `../src/q1/fixedPointRoot.m`

```
function xn = fixedPointRoot(g, x0, nMax)
    % fixedPointRoot Iteration to find solutions of x = g(x)
    %
    %Inputs:
    %   g = function handle to find the solutions of x = g(x)
    %   x0 = first term of the iteration
    %   nMax = the maximum number of iteration to quit after
    %
    %Output:
    %   xn = the iteraterative sequence
    %
    %Usage:
    %   xn = fixedPointRoot(@(x) cos(x), 0.75, 100) -> looks for a
    %   root of the equation x - cos(x) = 0, starting with an inital guess
    %   of 0.75.


    % number of iterations
    n = 0;
    % preallocated sequence array and set initial guess as first term
    xn = NaN(1, nMax);
    xn(1) = x0;
    % set initial error
    err = Inf;

    % iterate x -> g(x)
    while n < nMax
        n = n + 1;
        xn(n + 1) = g(xn(n));
        err = abs(xn(n + 1) - xn(n));
```

```
    end

    % remove any unused elements of the preallocated array
    xn(isnan(xn)) = [];

    fprintf('\nAfter %d steps root is %-20.14g\n', n, xn(end));
    fprintf('Final absolute error is %g\n\n', err);
end
```

Listing 15: `../src/q1/cobwebDiagram.m`

```
function cobwebDiagram(g, x0, nMax, a, b)
    %cobwebDiagram Creates cobweb diagram for x = g(x) in interval [a,b]
    %
    %Inputs:
    %   g = function handle for g(x)
    %   x0 = initial guess to start iteration
    %   nMax = number of iterations to complete
    %   a = lower end of interval [a,b] to plot cobweb diagram over
    %   b = upper end of interval [a,b] to plot cobweb diagram over
    %
    %Usage:
    %   cobwebDiagram(@(x) (x.^5 + 3)/5, 1, 10, 0, 1.5) -> produces a
    %   cobweb diagram of x = (x^5 + 3)/5 based on an initial guess of 10
    %   and 10 iterations. This is shown over the interval [0,1.5].

    %% get fixed point iteration sequence
    xn = fixedPointRoot(g, x0, nMax);

    %% generate cobweb diagram

    % get values for the line y = x and y = g(x)
    x = linspace(a, b);
    y = g(x);
    y(isinf(y)) = NaN;

    % set up figure
    hold on;
    grid on;
    set(gca, "DefaultLineLineWidth", 2);
    title("Cobweb plot for fixed point iteration");
    xlabel("x");
    ylabel("y")
    xlim([a b])
    ylim([min(x(1), y(1)) max(x(end), y(end))]);

    % plot lines y = x and y = g(x)
    plot(x, x, "r-", "DisplayName", "y = x");
    plot(x, y, "k-", "DisplayName", "y = g(x)");
    legend("AutoUpdate", "off");

    % plot the steps
    plot([xn(1) xn(1)], [0 xn(2)],'m-');
    for i=1:length(xn) - 2
        plot([xn(i) xn(i + 1)], [xn(i + 1) xn(i + 1)],'m-');
        plot([xn(i + 1) xn(i + 1)], [xn(i + 1) xn(i + 2)],'m-');
    end
end
```

Question 2: Numerical integration and differentiation

(a) (i) The first expression is Simpson's 3/8 rule and the second is Milne's rule.

Simpson's 3/8 rule can be implemented as follows.

Listing 16: `../src/q2/simpson38.m`

```
function simpQuad = simpson38(f, a, b)
    %simpson38 approximates integral of f(x) over interval [a,b] by using
    %Simpson's 3/8 rule
    %
    %Inputs:
    %   f = function handle of the integrand f(x)
    %   a = lower bound of the interval
    %   b = upper bound of the interval
    %
    %Outputs:
    %   simpQuad = approximate quadrature
    %
    %Usage:
    %   quad = simpson38(@(x) x^2, 0, 0.5) -> returns the approximate
    %   intergal of x^2 in the interval [0, 0.5]

    simpQuad = (b - a)/8 .* (f(a) + 3*f((2*a + b)/3) + 3*f((a + 2*b)/3)...
        + f(b));
end
```

And similarly Milne's rule can be implemented.

Listing 17: `../src/q2/milne.m`

```
function milneQuad = milne(f, a, b)
    %milne approximates integral of f(x) over interval [a,b] by using
    %Milne's rule
    %
    %Inputs:
    %   f = function handle of the integrand f(x)
    %   a = lower bound of the interval
    %   b = upper bound of the interval
    %
    %Outputs:
    %   milneQuad = approximate quadrature
    %
    %Usage:
    %   quad = milne(@(x) x^2, 0, 0.5) -> returns the approximate
    %   intergal of x^2 in the interval [0, 0.5]

    milneQuad = (b - a)/3 .* (2*f((3*a + b)/4) - f((a + b)/2)...
        + 2*f((a + 3*b)/4));
end
```

However to use the composite version the integral must be broken down into smaller intervals. For example breaking the integral into n intervals gives $\int_a^b f(x)dx = \int_a^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^b f(x)dx$ where $x_i = a + i \cdot \frac{b-a}{n}$. Then each of these smaller integrals can be calculated using either of the methods. The `compositeQuad` function breaks down the integral into smaller intervals before using a Newton-Coutes method of choice to approximate the integral.

Listing 18: `../src/q2/compositeQuad.m`

```
function [compQuad, h, err] = compositeQuad(f, i, a, b, tol)
    %compositeQuad amproximate quadrature using a Newton-Coutes method
```

```
    %
    %Inputs:
    %   f = function handle of the integrand
    %   i = function handle of the Newton-Coutes method to use. Must be in
    %   the format i(f, a, b) where f is the integrand and [a,b] is the
    %   interval to integrate over
    %   a = lower bound of the interval
    %   b = upper bound of the interval
    %   tol = desired absolute error tolerance
    %
    %Outputs:
    %   compQuad = vector of the sucessive approximates of the
    %   quadrature where the final entry is the final approximate
    %   h = vector of the step size used at each approximation
    %   err = vector of the absolute error at each approximation
    %
    %Usage:
    %   compositeQuad(@(x) exp(x), @(f, a, b) (b - a)/2 .*(f(a) + f(b)),...
    %   0, 1, 5e-4) -> Estimates the quadrature of e^x in the interval
    %   [0,1] using the trapezium rule to 3 decimal places


    % max number of iterations to prevent infinite loop
    nMax = 25;
    % iteration counter
    n = 1;
    % number of subintervals
    N = 2;
    % preallocate vectors for the error, quadrature and step size
    err = inf(1, nMax);
    compQuad = NaN(1, nMax);
    h = NaN(1, nMax);

    % composite algorithm
    while all(err > tol) && n < nMax
        % generate step size
        h(n) = (b - a)/N;
        % calculate quadrature using given Newton-coutes method
        compQuad(n) = sum(i(f, a + h(n).*[0:N-1], a + h(n).*[1:N]));
        % calculate absolute error
        try
            err(n) = abs(compQuad(n) - compQuad(n - 1));
        catch
            % prevents error when calculating first error term as no
            % previous approximation to compare againsy
            err(n) = inf;
        end
        n = n + 1;
        N = N * 2;
    end
    % removed any used preallocation
    err(isinf(err)) = [];
    compQuad(isnan(compQuad)) = [];
    h(isnan(h)) = [];
end
```

For an example both methods can be used to evaluate $\int_0^5 e^x - x dx$ to 6 decimal places as follows.

Listing 19: ../src/q2/Q2ai_example.m

```
f = @(x) exp(x) - x;
a = 0;
b = 5;
tol = 5e-7;
% using Simpson's 3/8 rule
compositeQuad(f, @simpson38, a, b, tol)
% using Milne's rule
compositeQuad(f, @milne, a, b, tol)
```

Both give the answer to the example as 134.913159.

(ii) test

Question 3: Numerical solution of ODEs

Listing 20: ../src/q3/rhs_projectile.m

(a)
```
function dydt = rhs_projectile(t, y, g, mu)
    %rhs_projectile returns colomn vector of [x,y,u,v]
    dydt = [y(3), y(4), -mu * y(3) * (y(3)^2 + y(4)^2)^0.5,...
        -g - mu * y(4) * ((y(3)^2 + y(4)^2)^0.5)]';
end
```

(b) (i) Forward Euler

Listing 21: ../src/q3/forwardEulerProjectile.m

```
function [t, y] = forwardEulerProjectile(rhs, tSpan, y0, g, mu, n)
    %forwardEulerProjectile solves the projectiles motion using forward
    %euler
    %
    %Inputs:
    %   rhs = function handle for rhs of ode returning a column vector
    %   [x,y,u,v]
    %   tSpan = vector [a,b] which is the time interval to solve ODE over
    %   y0 = 1*4 vector of initial conditions [x0,y0,u0,v0]
    %   g = value of the acceleration due to gravity
    %   mu = value of the drag parameter
    %   n = the number of steps to split integration over
    %
    %Outputs:
    %   t = column vector of solution times
    %   y = matrix of solutions where each row is the values of each of the
    %   varibles at the corisponding value of t
    %Usage:
    %   [t,y]=forwardEulerProjectile(@rhs_projectile, [0 5],...
    %   [0 0 31 21]', 9.81, 2.79e-2, 100) -> Solves the projectile ODE from
    %   t=1 to 5 with 100 steps

    t = linspace(tSpan(1), tSpan(end), n + 1);
    y = zeros(numel(t), numel(y0));
    h = (tSpan(end) - tSpan(1))/n;

    f = @(t,y) rhs(t, y, g, mu);

    % set initial conditions
    y(1,:) = y0';

    for i = 1:n
```

```
    y(i + 1,:) = y(i,:) + h * f(t(i), y(i,:))';
  end

end
```

(ii) 4th-order Runge-Kutta

(iii) `ode45`