# Control Theory

## Linearisation and linear models

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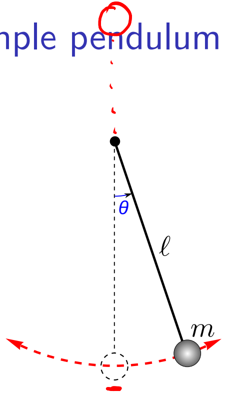
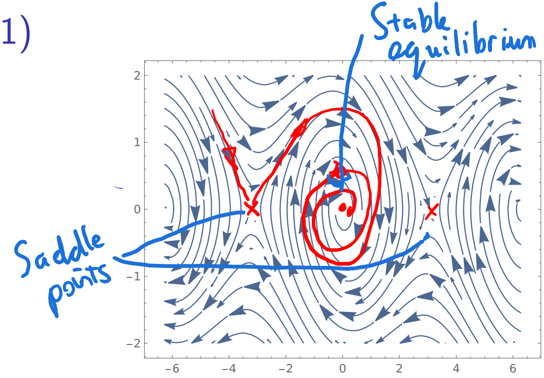
# Simple pendulum (1)




$$\ddot{\theta} + c\dot{\theta} + \sin(\theta) = 0$$




$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\sin x_1 - c x_2 \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = 0$$

Now linearise:

$x_2^* = 0$

$-\sin x_1 = 0 \Rightarrow x_1 = k\pi$

$x_1^* = 0$

# Simple pendulum (2)

$$\begin{pmatrix}\dot x_1\\ \dot x_2\end{pmatrix} = \begin{pmatrix} x_2 \\ -\sin x_1 - c x_2\end{pmatrix} = \underline{f(\underline{x})}, \quad \underline{x}^* = \underline{0}$$

$$x = x^* + \xi, \quad f(x) = \cancel{f(x^*)}^{\to 0} + Df(x^*)\underbrace{(x - x^*)}_{\xi} + \mathcal{O}(\|x-x^*\|^2)$$

$$\dot x = \dot\xi = \underbrace{Df(x^*)}_{A}\, \xi + \cancel{h.o.t} \quad \text{as small } \xi$$

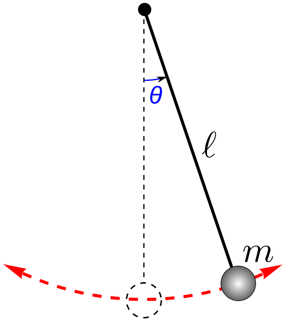
Jacobian — $A$

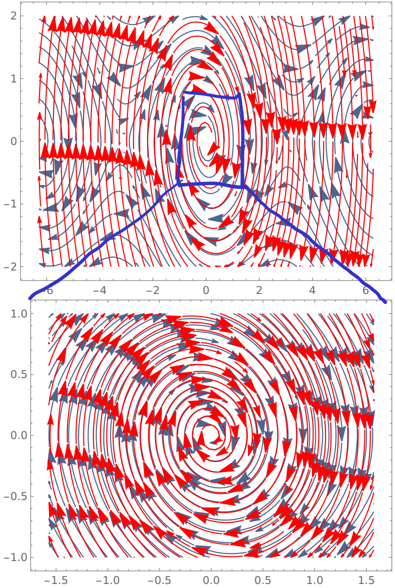
$$\dot\xi = A\xi \qquad \dot\xi = \begin{pmatrix}0 & 1\\ -1 & -c\end{pmatrix}\xi$$

Linearised System

# Simple pendulum (3)

$$\dot{\xi} = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix} \xi$$



# Linear models

$\Delta t$ $\quad x(k\Delta t) \to x((k+1)\Delta t)$

IC $\quad x_k \quad x_{k+1}$

**Continuous time**

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \cancel{\boldsymbol{D}\boldsymbol{u}}$$

**Discrete time**

$$\boldsymbol{x}_{k+1} = \hat{\boldsymbol{A}}\boldsymbol{x}_k + \hat{\boldsymbol{B}}\boldsymbol{u}_k$$
$$\boldsymbol{y}_k = \hat{\boldsymbol{C}}\boldsymbol{x}_k + \hat{\boldsymbol{D}}\boldsymbol{u}_k$$

The general solution is $\quad \dot{x} = At$

$$\boldsymbol{x}(t) = \underbrace{\exp(\boldsymbol{A}t)\boldsymbol{x}(0)}_{\text{Homogeneous solution}} + \underbrace{\int_0^{t} \exp(\boldsymbol{A}(t-\tau))\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau}_{\text{Variation of constants formula}}$$

$t \to (k+1)\Delta t$, $\quad 0 \to k\Delta t$

If $\boldsymbol{u}(t)$ is piecewise-constant, sampling period is $\Delta t$, then we have

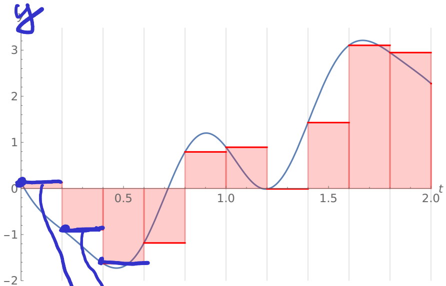
IF $A^{-1}$ exists

$$\hat{\boldsymbol{A}} = \exp(\boldsymbol{A}\Delta t), \quad \hat{\boldsymbol{B}} = \int_0^{\Delta t} \exp(\boldsymbol{A}(\Delta t - \tau))\boldsymbol{B}\mathrm{d}\tau = \boldsymbol{A}^{-1}\left(\hat{\boldsymbol{A}} - \boldsymbol{I}\right)$$

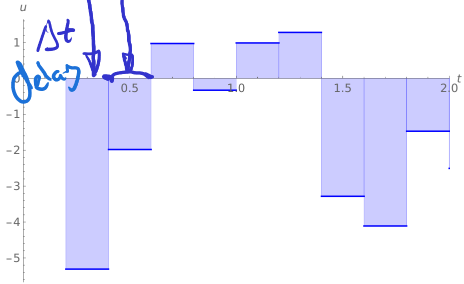
See Applied Linear Algebra videos 16,17,18

# Control with sampling

Output




Control



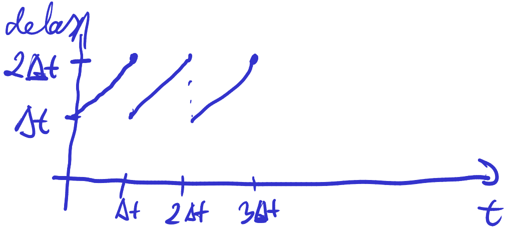

The equations are

$$\begin{bmatrix} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}) \end{bmatrix} = Ax + Bu$$

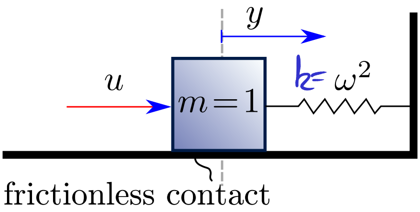
[s]

But the control law is

$$\begin{bmatrix} \boldsymbol{u}(\tau) = \boldsymbol{h}(\underline{\boldsymbol{x}(k\Delta t)}, \cancel{\boldsymbol{x}}u(k\Delta t)), \\ \tau \in [(k+1)\Delta t, (k+2)\Delta t) \end{bmatrix}$$



# Force control




$$u = K(F_d - \omega^2 y)$$

Is this stable

Next week : Stability

# The End