PDEs Exercise Sheet 1

Exercise 1:

1. $u_t + u u_{xx} = \sin(x)$

   Second Order inhomogeneous quasi-linear ✓

2. $(u_t)^2 = u_x + \exp(-x/t)$

   First Order inhomogeneous Non Linear ✓

3. $u_{tt} = u_{xx}$

   ✓ + hyperbolic

   Second Order Homogeneous Linear (Wave Equation)

4. $u_{tt} = u_{xxxx} + t^2 u_x$

   Fourth order Homogeneous Linear ✓

5. $u_{tt} = e^{-t} u_{xx} + u^2$

   Second order Homogeneous Semi-Linear ✓

6. $u_{tt} = -u_{xx} + \sqrt{xt}$

   Second order Nonhomogeneous Linear ✓ + elliptic

7. $u_t = (u^2)_{xx}$

   Second Order Homogeneous Non Linear

   ✗ Quasi-Linear as $(u^2)_{xx} = 2u_x^2 + 2u u_{xx}$

8. $u_t = (u_{xx})^2$

Second Order Homogeneous Non-linear ✓

9. $u_{tt} = -u_{xx} - 2u_{xt}$

Second Order homogeneous Linear + parabolic

10. $(u_t)^2 = u_{xx} + \exp(-x/t)$

Second Order Nonhomogeneous Semi-Linear ✓

Exercise 2

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = \frac{\exp\left(\frac{-x^2}{2\sigma^2}\right)}{\sigma\sqrt{2\pi}}$$

$$u_t(x,0) = \frac{1}{\pi\beta\cosh(x/\beta)}$$

$$u(x,t) = \frac{1}{2}\left[u(x-vt,0) + u(x+vt,0)\right]$$

$$+ \frac{1}{2v}\int_{x-vt}^{x+vt} u_t(\xi,0)\, d\xi$$

$$= \frac{1}{2\sigma\sqrt{2\pi}}\left[\exp\left(\frac{-(x-vt)^2}{2\sigma^2}\right) + \exp\left(\frac{-(x+vt)^2}{2\sigma^2}\right)\right]$$

$$+ \frac{1}{2v}\int_{x-vt}^{x+vt} \frac{1}{T\beta\cosh^2(\xi/\beta)}\, d\xi$$

$$= \frac{1}{2\sigma\sqrt{2\pi}}\left[\exp\left(\frac{-(x-vt)^2}{2\sigma^2}\right) + \exp\left(\frac{-(x+vt)^2}{2\sigma^2}\right)\right]$$

$$+ \frac{1}{2vT}\Big[\tanh(\xi/\beta)\Big]_{x-vt}^{x+vt}$$

$$= \frac{1}{2\sigma\sqrt{2\pi}}\left[\exp\left(\frac{-(x-vt)^2}{2\sigma^2}\right) + \exp\left(\frac{-(x+vt)^2}{2\sigma^2}\right)\right]$$

$$+ \frac{1}{2vT}\left[\tanh\left(\frac{x+vt}{\beta}\right) - \tanh\left(\frac{x-vt}{\beta}\right)\right]$$

First term is two seperate spatial pulses with speed $|v|$

Second term is like a step

Exersize 3

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} = v^2 \frac{\partial^2 u}{\partial x^2}$$

$$\mu_n = \int_{-\infty}^{\infty} (x-c)^n f(x)\, dx$$

First calculate $M(t) = \int_{-\infty}^{\infty} u(x,t)\, dx$

$$\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2} dx + \alpha \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} = v^2 \int_{\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} dx$$

as $u_x(x,t)$ decay steeply for large $|x|$

$$\Rightarrow \frac{d^2 M}{dt^2} + \alpha \frac{dM}{dt} = 0$$

$$\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} dx = \Big[u_x \Big]_{-\infty}^{\infty} = 0$$

$$\lambda^2 + \alpha\lambda = 0$$

$$\lambda(\lambda+\alpha) = 0 \qquad \lambda = 0 \qquad \lambda = -\alpha$$

$$M(t) = A + Be^{-\alpha t}$$

$$M(0) = A = 1 \qquad \left(\text{as } \int_{-\infty}^{\infty} u(x,0) = 1\right)$$

$$M(t) = 1 + Be^{-\alpha t}$$

$$\frac{dM}{dt} = -\alpha B e^{-\alpha t}$$

$$\left.\frac{dM}{dt}\right|_{t=0} = -\alpha B = 0 \quad \left(\text{as } \int_{-\infty}^{\infty} u_t(x,0)\,dx = 0\right)$$

So $M(t) = 1$

First Moment:

$$\int_{-\infty}^{\infty} x\frac{\partial^2 u}{\partial t^2}\,dx + \alpha\int_{-\infty}^{\infty} x\frac{\partial u}{\partial t}\,dx = v^2 \cancel{\int_{-\infty}^{\infty} x\frac{\partial^2 u}{\partial x^2}\,dx} \to 0$$

as odd function decay for large $|x|$

$$\frac{d^2\mu_1}{dt^2} + \alpha\frac{d\mu_1}{dt} = 0$$

$$\mu_1(t) = A + Be^{-\alpha t}$$

Second Moment

$$\int_{-\infty}^{\infty} x^2 \frac{\partial^2 u}{\partial x^2} dx + \alpha \int_{-\infty}^{\infty} x^2 \frac{\partial u}{\partial x} dx = v^2 \int_{-\infty}^{\infty} x^2 \frac{\partial^2 u}{\partial x^2} dx$$

$$\frac{d^2 \mu_2}{dt^2} + \alpha \frac{d\mu_2}{dt} = v^2 \left[x^2 u_x - xu + 1 \right]_{-\infty}^{\infty}$$

$$= v^2$$

$$\mu_2(t) = A + Be^{-\alpha t} + Ct$$

$$msd(t) = \mu_2(t) - \mu_1^2(t)$$

$$= A + Be^{-\alpha t} + Ct - \left(A + Be^{-t}\right)$$

Exercise 4:

$$u_t = G\, u_{xxxx}$$

$$u(x,0) = \delta(x - x_0)$$

$$\int_{-\infty}^{\infty} u(x,t)\, dx = 1 \qquad u(x \to \pm\infty, t) = 0$$

$$\Rightarrow x^n \frac{\partial^m u}{\partial x^m}\bigg|_{x \to \pm\infty} = 0 \quad \text{for } \forall\, m,n \in \mathbb{Z}$$

i. $\mathcal{F}[u_t] = \mathcal{F}[G\, u_{xxxx}]$

$$\hat{u}_t = G(ik)^4 \hat{u}$$
$$= G k^4 \hat{u}$$

$$\hat{u}(t) = A e^{-Gk^4 t}$$

$$\mathcal{F}[u(x,0)] = \mathcal{F}[\delta(x - x_0)]$$

$$u(k,0) = e^{-ikx_0}$$

$$\Rightarrow \hat{u}(t) = e^{-ikx_0}\, e^{-Gk^4 t}$$

ii) $u(x,0) = f(x)$

$$u(x,t) = \mathcal{F}^{-1}\left[\hat{f}(k)\, e^{-6k^4 t}\right]$$

$$= \int_{-\infty}^{\infty} \mathcal{F}^{-1}\left[e^{-6k^4 t}\right](\tau)\, f(x-\tau)\, d\tau$$

iii). First Moment:

$$\int_{-\infty}^{\infty} x\, u_t = 6 \int_{-\infty}^{\infty} x\, u_{xxxx}\, dx$$

$$\frac{d\langle x \rangle}{dt} = 0$$

Second Moment:

$$\int_{-\infty}^{\infty} x^2\, u_t = \int_{-\infty}^{\infty} x^2\, u_{xxxx}\, dx$$

$$\frac{d\langle x^2 \rangle}{dt} = 0$$

Third Moment:

$$\int_{-\infty}^{\infty} x^3 u_t \, dx = G \int_{-\infty}^{\infty} x^3 u_{xxxx} \, dx$$

$$\frac{d\langle x^3 \rangle}{dt} = 0$$

iv. $$\int_{-\infty}^{\infty} x^{2n+1} u_t \, dx = G \int_{-\infty}^{\infty} x^{2n+1} u_{xxxx} \, dx$$

$$\frac{d\langle x^{2n+1} \rangle}{dt} \qquad ?$$

5. $$\frac{\partial u}{\partial t} = \int_0^t \phi(t-s) \frac{\partial^2 u}{\partial x^2} \, ds$$

Diffusion eq: $\phi(t) = D\,\delta(t)$

Wave Equation: $\phi(t) = v^2$