Control Theory

Stability: Lyapunov's direct method

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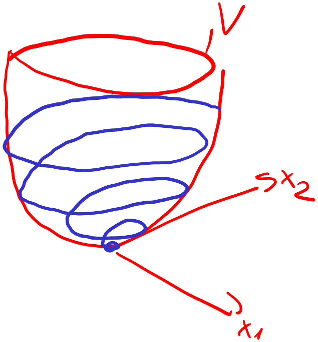
Lyapunov function

- In mechanics an equilibrium is stable if **energy does not increase**
- Create an energy-like function!

Consider an ODE with an equilibrium $\boldsymbol{x}^\star$

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \qquad \boxed{\boldsymbol{f}(\boldsymbol{x}^\star) = \boldsymbol{0}}$$



Definition

A function $V : X \to \mathbb{R}$ is positive definite if:

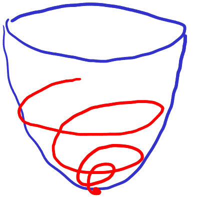
$$V(\boldsymbol{x}^\star) = 0 \quad \text{and}$$
$$V(\boldsymbol{x}) > 0, \quad \text{if } 0 < |\boldsymbol{x} - \boldsymbol{x}^\star| < r$$

for some $r > 0$. Function V is positive semidefinite if we allow $V(\boldsymbol{x}) \geq 0$.

Lie derivative

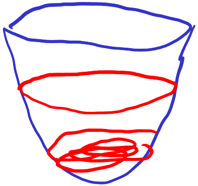
Consider an ODE with an equilibrium $\boldsymbol{x}^\star$

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{f}(\boldsymbol{x}^\star) = \boldsymbol{0}$$



and a positive definite V. The value of V along trajectories is

$$\frac{d}{dt} V(\boldsymbol{x}(t)) = DV(x(t))\,\dot{x}(t)$$



The instantaneous change of V is then

$$\dot{V}(\boldsymbol{x}) = DV(\boldsymbol{x})\dot{\boldsymbol{x}} = DV(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})$$

Theorem Lyapunov

- $\boldsymbol{x}^\star$ *is stable if* $-\dot{V}$ *is positive semidefinite*
- $\boldsymbol{x}^\star$ *is assymptotically stable if* $-\dot{V}$ *is positive definite*

$\dot{V}$ negative definite

Example: Lyapunov's direct method

$$\dot{x}_1 = -x_2 + x_1\left(x_1^2 + x_2^2 - 1\right)$$
$$\dot{x}_2 = x_1 + x_2\left(x_1^2 + x_2^2 - 1\right)$$

Note that $\Re\lambda_i = 0$ and $x_1^2 + x_2^2 = 1$ is a peridic orbit.

Assume $\boxed{V = p_1 x_1^2 + p_2 x_2^2}$

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\lambda_{1,2} = \pm i$

$\dot{V} = 2p_1 x_1 \dot{x}_1 + 2p_2 x_2 \dot{x}_2$

$= 2p_1 x_1 (\cancel{-x_2} + x_1(x_1^2 + x_2^2 - 1))$

$+ 2p_2 x_2 (\cancel{x_1} + x_2(x_1^2 + x_2 - 1))$

$= 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1) < 0$

if $0 < x_1^2 + x_2^2 < 1$

x^* is assymptoticaly stable!

Linear systems

$x_1^2 + x_2^2 < 1$

What is the point: approximate **basin of attraction** of nonlinear systems. Consider

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$

$$V(\boldsymbol{x}) = \boldsymbol{x} \cdot (\boldsymbol{P}\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P}\boldsymbol{x},$$

where $\boldsymbol{P}$ is self-adjoint (real symmetric).

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x}$$

$$= x^T A^T P x + x^T P A x$$

$$= -x^T Q x$$

$$A^T P + PA = -Q$$

$$\begin{bmatrix} \bar{\lambda}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{\lambda}_n \end{bmatrix} \begin{bmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_n \end{bmatrix} + \begin{bmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = -\begin{bmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_n \end{bmatrix}$$

$$\cancel{p_i}\ 2\,\mathrm{Re}\,\lambda_i = -q_i \qquad (p_i = 1)$$

$$q_i = -2\,\mathrm{Re}\,\lambda_i$$

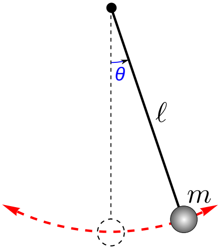
$$J = T^{-1} A T = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$P \leftarrow J$$

$$P = I$$

$$A = Df(x^*) \qquad P = \left(T^{-1}\right)^T I \left(T^{-1}\right)$$

Simple pendulum (1)



$$\ddot{\theta} + c\dot{\theta} + \sin(\theta) = 0$$

Estimate the basin of attraction!

$x_1 = \theta$

$\dot{x}_1 = \dot{\theta}$

$\dot{x}_1 = x_2$

$\dot{x}_2 = -c x_2 - \sin x_1$

$$A = Df(\underset{\shortparallel\atop 0}{x^*}) = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix}$$

$c = \sqrt{3}$

$\lambda_1 = \frac{1}{2}(-\sqrt{3} + i)$, $\lambda_2 = \frac{1}{2}(-\sqrt{3} - i)$

$v_1 = (-\sqrt{3} - i, 2)^T$, $v_2 = (-\sqrt{3} + i, 2)$

$$T = (v_1, v_2) = \begin{pmatrix} -\sqrt{3} - i & -\sqrt{3} + i \\ 2 & 2 \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} i/2 & i/4(\sqrt{3} - i) \\ -i/2 & i/4(-\sqrt{3} - i) \end{pmatrix}$$

$q_i = -2 \operatorname{Re} \lambda_i$ $\qquad \tilde{Q} = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{pmatrix}$

$$P = \overline{(T^{-1})}^T (T^{-1}) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{2} \end{pmatrix}$$

$$V = \frac{1}{2} x_1^2 + \frac{\sqrt{3}}{2} x_1 x_2 + \frac{1}{2} x_2^2$$

$\dot{V} = ?$

Simple pendulum (2)

$$\dot{x}_1 = x_2$$

$$x_2 = -c x_2 - \sin x_1$$

$$V = \frac{1}{2}x_1^2 + \frac{\sqrt{3}}{2}x_1 x_2 + \frac{1}{2}x_2^2$$

$$\dot{V} = x_1\underline{\dot{x}_1} + \frac{\sqrt{3}}{2}\underline{\dot{x}_1} x_2 + \frac{\sqrt{3}}{2}x_1\underline{\dot{x}_2} + x_2\underline{\dot{x}_2}$$

$$\dot{V} = x_1 x_2 + \frac{\sqrt{3}}{2}x_2 x_2 + \frac{\sqrt{3}}{2}(-c x_2 - \sin x_1) + x_2(-c x_2 - \sin x_1)$$

$$\dot{V} = \boxed{0}, -0.5, -1, -1.5$$

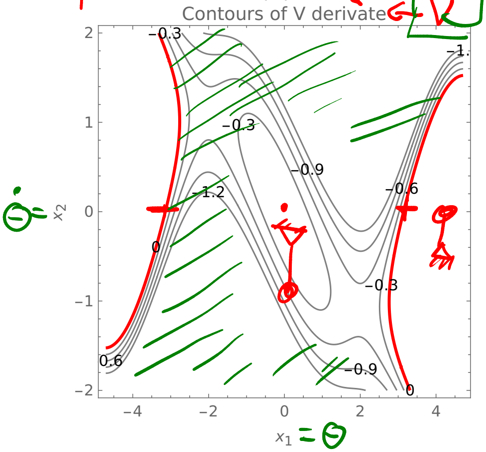
Basin of attraction

Simple pendulum (3)

$$\ddot{\theta} + c\dot{\theta} + \sin(\theta) = 0$$

$\dot{V}$ → not necessary

sufficient condition



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