# Control Theory

## State feedback

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# Feedback control

Consider the system

LTI

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$$

$x \in \mathbb{R}^n$
$u \in \mathbb{R}^m$
$y \in \mathbb{R}^p$

- We have conditions for **controllability** and **observability**
- Q: How to design a controller to make the system behave the way we want?

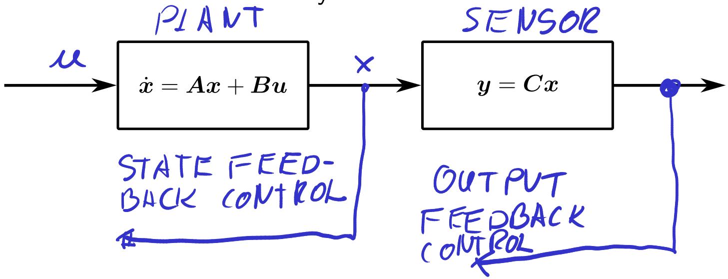
Assume we can observe the whole state space.

The only way is to give feedback.

$$\boxed{\boldsymbol{u} = \varphi(\boldsymbol{x}, \cancel{t})}$$

# Types of feedback

We will look at linear control of linear systems




- This video: **state-feedback controller**
- Next video: **output-feedback controller**

# Pole placement control (1)

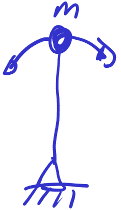
$\frac{1}{P(z)}$ — characteristic polynomial

- Suppose that $\underline{x^\star = \mathbf{0}}$ is an unstable equilibrium of the system

$$\dot{x} = Ax + Bu \qquad (1)$$

for $u = \mathbf{0}$.

$\exists\, i \in \{1 \cdots n\}$ s.t. $\mathrm{Re}\,\lambda_i > 0$




Inverted pendulum

- Now choose

$$\underbrace{u(t)}_{m\times 1} = -\underbrace{K}_{m\times n}\,\underbrace{x(t)}_{n\times 1} \qquad (2)$$

# Pole placement control (2)

- Substitute (2) into (1) to get the closed-loop system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x} \implies$$

$$\implies \dot{x} = (A - BK)x$$

- If we choose matrix $\boldsymbol{K}$ such that:

$$\hat{A} = (A - Bk) \quad \forall i \in \{1 \ldots n\} \, \mathrm{Re}\,\lambda_i < 0$$

then we ensure that the equilibrium $\boldsymbol{x}^\star = \boldsymbol{0}$ is assymptotically stable.

- We will see: if the system is controllable, we can place the eigenvalues (poles) anywhere.

# Pole placement control (3)

Question: How to tune matrix $\boldsymbol{K}$? For example

1. Direct inspection $\hat{A} = (A - BK)$

2. Using Ackermann's formula (MATLAB `place`)

# Inverted pendulum (linearised)

Control variable

$M = u$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$\lambda^2 - \alpha = 0, \quad \lambda_{1,2} = \pm\sqrt{\alpha}$$

UNSTABLE

$$u = -Kx, \text{ where } K = (k_1 \;\; k_2)$$

$$\hat{A} = A - BK = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} (k_1 \;\; k_2)$$

$$= \text{—}''\text{—} - \begin{pmatrix} 0 & 0 \\ k_1 & k_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ \alpha - k_1 & -k_2 \end{pmatrix}$$

$$\lambda^2 - tr(\hat{A})\lambda + det(\hat{A}) = 0$$

$$\lambda^2 + k_2\lambda + (k_1 - \alpha) = 0$$

$$\lambda_{1,2} = \frac{-k_2 \pm \sqrt{k_2^2 - 4(k_1 - \alpha)}}{2}$$

$$\boxed{k_2 > 0}$$

$$k_1 - \alpha > 0 \Rightarrow \boxed{k_1 > \alpha}$$

For negative real part.

# Inverted pendulum (direct inspection) (1)

choosing

$k_1 > \alpha \,, k_2 > 0$

$\Rightarrow x^* = 0$ is assymptotically stable

$\rightarrow$ Do more!

$\lambda_1 = -1 \,, \lambda_2 = -2$ — desired eigen values.

$$p(\lambda) = (\lambda+1)(\lambda+2) = \lambda^2 + 3\lambda + 2$$

$$p(\lambda) = \lambda^2 + k_2 \lambda + (k_1 - \alpha)$$

$k_2 = 3$ $\qquad$ $k_1 - \alpha = 2$

$$k_1 = 2 + \alpha$$

# Ackermann's formula (1)

- Direct inspection cannot be automated, not viable for large systems
- For single-input, single-output (SISO) systems
- Using Ackermann's formula (MATLAB '`place`')

$$\dot{x} = Ax + Bu$$

$$u = -Ku$$

automatic calls

# Ackermann's formula (1)

- We aim to tune the controller

$$u = -\boldsymbol{K}\boldsymbol{x}$$

- such that the characteristic polynomial of $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$ is $\hat{A}$

$$p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_{n-1}\lambda^{n-1} + \lambda^n$$

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \dots + a_{n-1} A^{n-1} + A^n$$

### Theorem

*Row vector $\boldsymbol{K}$ is given by*

Row vector

$$\boldsymbol{K} = (\underbrace{0\ 0\ \cdots\ 0}_{n-1}\ 1)\ \boldsymbol{W}_r^{-1} p(\boldsymbol{A}).$$

Reachability matrix

Wikipedia has a nice proof using the Cayley-Hamilton theorem.

# Inverted pendulum (Ackermann's formula) (1)

Control

$M = u$

$$A = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\lambda_1 = -1 \qquad \lambda_2 = -2$$

$$u = -Kx$$

$$A - BK$$

$$K = [0 \;\; 1]\, W_r^{-1}\, p(A)$$

$$W_r = [B \;\; AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$p(\lambda) = (\lambda + 1)(\lambda + 2) = \lambda^2 + 3\lambda + 2$$

$$= A^2 + 3A + 2I$$

$$= \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ 3\alpha & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \alpha + 2 & 3 \\ 3\alpha & \alpha + 2 \end{pmatrix}$$

# Inverted pendulum (Ackermann's formula) (2)

$$k = \underbrace{(0 \quad 1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{(1 \quad 0)}\begin{pmatrix} \alpha+2 & 3 \\ 3\alpha & \alpha+2 \end{pmatrix}$$

$$= (\alpha+2 \quad 3) \qquad k_1 = \alpha+2$$

$$k_2 = 3$$

$\Rightarrow$ at $x = 0$

assymptotic stability

$\lambda_1 = -1 \quad \lambda_2 = -2$

$$u = -Kx = -(\alpha+2 \quad 3)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -(2+\alpha)x_1 - 3x_2$$

STATE FEEDBACK CONTROL