Asynchronous lecture 1

- Introduction to PDEs & Classification

Why study partial differential equations?

- The fundamental laws of many physical phenomena, whether in the domain of fluid dynamics, electricity, magnetism, mechanics, optics or heat flow, are described by partial differential equations

- Many other phenomena can be described phenomenologically by partial differential equations

- PDE represents a continuous approximation to computational stochastic models (agent-based simulations)

Classical PDEs

Fundamental laws of many physical phenomena: fluid dynamics, electricity, magnetism, mechanics, optics, heat flow, . . .

Diffusion (heat) equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2}$$

Navier-Stokes (incompressible)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\,\nabla^2\mathbf{u} = -\nabla w + \mathbf{g}$$

Other PDEs

A statistical model of criminal behaviour, Short et al. 2008

$$\frac{\partial B}{\partial t} = \frac{\eta D}{z}\nabla^2 B - \omega B + \varepsilon D \rho A$$
$$\frac{\partial \rho}{\partial t} = \frac{D}{z}\nabla \cdot \left[\nabla \rho - \frac{2\rho}{A}\nabla A\right] - \rho A + \gamma$$

B is attractiveness of an area (to burglars) and ρ is the criminal density

Basic notions

Definition: A *partial differential equation (PDE)* is an equation that

1. has an *unknown function* that depends on at least two variables
2. contains some *partial derivatives* of the unknown function

Independent variables

- t: time coordinate
- x, y, z (or, r, θ, ϕ): space coordinates

Dependent variable $u = u(t, x, \ldots)$ (the unknown function)

Partial derivatives will be (equivalently) written as

$$u_t = \frac{\partial u}{\partial t}, \qquad u_{tt} = \frac{\partial^2 u}{\partial t^2}, \qquad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \qquad \text{etc}$$

Solutions of a PDE

A *solution to a PDE* is any function (in the independent variables) that satisfies the PDE

Find two solutions of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

However, from this family of functions, a single function may be uniquely selected by imposing adequate *initial conditions* and/or *boundary conditions*

A PDE with both initial and boundary conditions constitutes the so-called *initial-boundary-value problem (IBVP)* — most mathematical models of physical phenomena are of this form

Classification of PDEs — order

Different PDEs have different properties; divide them into particular classes depending on certain criteria — there are different solution methods for each class

Order of the PDE: the order of the highest partial derivative in the equation

Examples

$u_t = u_x$	first order
$u_t = u_{xx}$	second order
$u_{xy} = 0$	second order
$u_t + uu_{xxx} = \sin(x)$	third order
$u_{tt} = u_{xxxx}$	fourth order

Classification of PDEs — number of variables

PDEs may be classified by the *number of their independent variables*, that is, the number of variables the unknown function depends on. The dimension is often mentioned as $s + 1$, where s is the number of spatial coordinates.

Examples

$u_t = u_{xx}$	PDE in 2 variables [1+1]; $u = u(t, x)$
$u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$	PDE in 3 variables [2+1]; $u = u(t, r, \theta)$
$u_t = u_{xx} + u_{yy} + u_{zz}$	PDE in 4 variables [3+1]; $u = u(t, x, y, z)$

Classification of PDEs — constant coefficients

A PDE can have *variable coefficients* if at least one of the coefficients (that multiplies the dependent variable) is a function of (some of) the independent variables. If a PDE does not have variable coefficients, then it has *constant coefficients*.

Examples

$$u_{tt} + 5u_{xx} - 3u_{xy} = \cos(x) \qquad \text{constant coefficients}$$

$$u_t + \exp(-t)u_{xx} = 0 \qquad \text{variable coefficients}$$

Classification of PDEs — homogeneity

If all terms contain dependent variables, or its partial derivatives, the PDE is *homogeneous*. Otherwise the PDE is *inhomogeneous*.

Examples

$u_{tt} = u_{xx}$	homogeneous
$u_{tt} = u_{xx} + x^2 \sin(t)$	inhomogeneous

Classification of PDEs — linearity

A PDE is *linear* if the dependent variable and all its derivatives appear in a linear fashion. Otherwise the PDE is *nonlinear*.

Examples

$u_{tt} + \exp(-t)u_{xx} = \sin(t)$	linear
$uu_{xx} + u_t = 0$	nonlinear
$xu_{xx} + yu_{yy} = 0$	linear
$u_x + u_y + u^2 = 0$	nonlinear
$u_x + u_y + x^2 = 0$	linear

Classification of PDEs — linearity

If a PDE is nonlinear, it can be put in one of three subclasses: *semi-linear*, *quasi-linear*, or *fully nonlinear*

- *Semi-linear* if the highest derivatives appear in a linear fashion and the coefficients do not depend on the unknown function or its derivatives. For 2nd order PDEs this corresponds to

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial y^2} = c(x,y,u,u_x,u_y)$$

- *Quasi-linear* is the highest derivatives appear in a linear fashion

$$a(x,y,u,u_x,u_y)\frac{\partial^2 u}{\partial x^2} + b(x,y,u,u_x,u_y)\frac{\partial^2 u}{\partial y^2} = c(x,y,u,u_x,u_y)$$

- *Fully nonlinear* is the highest derivatives appear in a nonlinear fashion

$$u_{xx}u_{yy} = 0$$

Classification of PDEs — 2nd order PDEs

Linear 2nd order PDEs account for most of the physical models in engineering and science. As such there is a further classification of these types of equations

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + \text{lower order terms} = 0$$

The quantity $\alpha = B^2 - 4AC$ is key (think quadratic polynomial formula!)

- $\alpha < 0$ is an *elliptic PDE*, for example, the Laplace equation

$$u_{xx} + u_{yy} = 0$$

- $\alpha = 0$ is a *parabolic PDE*, for example, the heat equation

$$u_t = u_{xx}$$

- $\alpha > 0$ is a *hyperbolic PDE*, for example, the wave equation

$$u_{tt} = u_{xx}$$