Asynchronous lecture 8

- Methods of characteristics: Boundary Value Problems II
- Lagrange-Charpit method

Methods of characteristics: example 1

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0$$

$$\frac{dx}{dt} = \frac{a(x,t,u)}{b(x,t,u)} = v \qquad \frac{du}{dt} = \frac{c(x,t,u)}{b(x,t,u)} = 0$$

$$x = vt + A \qquad u = B$$

$$B = f(A)$$

$$u(x,t) = f(x - vt)$$

Methods of characteristics: example 2

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

$$\frac{dx}{dy} = \frac{x}{y}$$

$$\frac{du}{dy} = \frac{u}{y}$$

Constant

Constant

$$\ln(x) = \ln(y) + \ln(A)$$

$$\ln(u) = \ln(y) + \ln(B)$$

$$x = Ay$$

$$u = By$$

$$\frac{x}{y} = A \quad \text{and} \quad \frac{u}{y} = B$$

$$B = f(A) \longrightarrow u(x,y) = yf\left(\frac{x}{y}\right)$$

Methods of characteristics: example 2

alternatively we can also solve it this way

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

$$\frac{dy}{dx} = \frac{y}{x} \qquad\qquad \frac{du}{dx} = \frac{u}{x}$$

$$y = xA \qquad\qquad u = xB$$

$$B = \overset{g}{\cancel{f}}(A)$$

$$u(x,y) = xg\left(\frac{x}{y}\right)$$

compare to $u(x,y) = yf\left(\frac{x}{y}\right)$

Methods of characteristics: example 3

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = (x+y)u$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{(x+y)u}$$

$$\frac{dx}{dy} = \frac{x^2}{y^2}$$

$$\frac{1}{x} = \frac{1}{y} + A' \quad \text{or} \quad \frac{xy}{x-y} = A$$

Methods of characteristics: example 3

$$\frac{xy}{(x-y)} = A$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{(x+y)u}$$

$$\text{Subtract } dy = y^2 \frac{du}{(x+y)u} \text{ from } dx = x^2 \frac{du}{(x+y)u}$$

$$dx - dy = (x^2 - y^2)\frac{du}{(x+y)u} = (x-y)\frac{du}{u}$$

Methods of characteristics: example 3

$$\frac{d(x-y)}{(x-y)} = \frac{du}{u}$$

$$\frac{u}{(x-y)} = B \qquad \frac{xy}{(x-y)} = A$$

$$B = f(A)$$

$$u(x,y) = (x-y)f\left(\frac{xy}{(x-y)}\right)$$

Methods of characteristics: example 4

Non-linear example

$$2y\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 2yu^2$$

$$\frac{dx}{2y} = \frac{dy}{u} = \frac{du}{2yu^2}$$

$$\frac{du}{dy} = 2yu \qquad \ln(u) = y^2 + B$$

$$u = Be^{y^2}$$

Methods of characteristics: example 4

$$\frac{dx}{2y} = \frac{dy}{u} = \frac{du}{2yu^2}$$

$$u = Be^{y^2}$$

$$\frac{dx}{dy} = \frac{2y}{u} = \frac{2ye^{-y^2}}{B}$$

$$Bx + e^{-y^2} = A$$

$$B = f(A)$$

$$A = Bx + e^{-y^2} = ue^{-y^2}x + e^{-y^2} = (xu+1)e^{-y^2}$$

$$u(x,y) = e^{y^2} f\left([xu+1]e^{-y^2}\right)$$

Methods of characteristics: example 4

$$2y\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 2yu^2$$

$$u(x,y) = e^{y^2} f\left([xu+1]e^{-y^2}\right)$$

As before, this is an implicit solution of the PDE
for some arbitrary function $f(z)$

General approach with Methods of Characteristics

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial u} = c(x,y,u)$$

First write down both the parametrised and the implicit form

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)}$$

$$\frac{dx}{ds} = a(x(s),y(s),u(s)) \qquad \frac{dy}{ds} = b(x(s),y(s),u(s)) \qquad \frac{du}{ds} = c(x(s),y(s),u(s))$$

Integrate what is most convenient