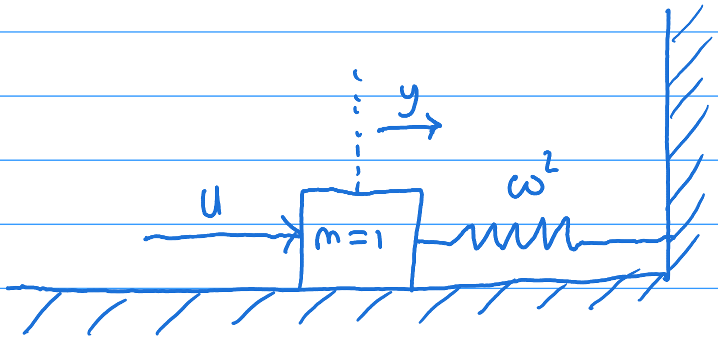
Problem 1:



1. Differential Equation Model:

$$\ddot{y} = u - \omega^2 y$$

Let $x_0 = y$
$x_1 = \dot{y}$

$$\dot{x}_0 = x_1$$
$$\dot{x}_1 = u - \omega^2 x_0$$

$$\dot{\underline{x}} = A\underline{x} + \underline{B}u$$

$$\dot{\underline{x}} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \underline{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

2. $\underline{x}_{k+1} = \hat{A} x_k + \hat{B} u_k$ $\quad\quad \hat{A} = \exp(A\Delta t)$

$$\begin{vmatrix} -\lambda & 1 \\ -\omega & -\lambda \end{vmatrix} = 0 \quad \lambda = \pm i\omega \quad \underline{v} = \begin{pmatrix} \pm \frac{i}{\omega} \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{-i}{\omega} & \frac{i}{\omega} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix} \begin{pmatrix} \frac{-i}{\omega} & \frac{i}{\omega} \\ 1 & 1 \end{pmatrix}^{-1}$$

$$\hat{A} = \exp(A\Delta t) = \begin{pmatrix} \frac{-i}{\omega} & \frac{i}{\omega} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{i\omega\Delta t} & 0 \\ 0 & e^{-i\omega\Delta t} \end{pmatrix} \begin{pmatrix} \frac{-i}{\omega} & \frac{i}{\omega} \\ 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{-i}{\omega} & \frac{i}{\omega} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \cos(\omega\Delta t) + \sin(\omega\Delta t) & 0 \\ 0 & \cos(\omega\Delta t) + \sin(\omega\Delta t) \end{pmatrix} \begin{pmatrix} \frac{-i}{\omega} & \frac{i}{\omega} \\ 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \cos\omega\Delta t & \frac{1}{\omega}\sin(\omega\Delta t) \\ -\omega\sin(\omega\Delta t) & \cos(\omega\Delta t) \end{pmatrix}$$

$$\hat{B} = A^{-1}(A - I) = \begin{pmatrix} \frac{1}{\omega^2}(1 - \cos(\omega\Delta t)) \\ \frac{1}{\omega}\sin(\omega\Delta t) \end{pmatrix}$$

3. $e = (\omega^2 y - F_d)$

$$u = -ke = -k(\omega^2 y - F_d)$$

$$\Rightarrow \dot{y}_1 = y_2$$

$$\dot{y}_2 = -\omega^2 y_1 - k(\omega^2 y_1 - F_d)$$

Equilibrium at $$y_1 = \frac{k F_d}{\omega^2 (1+k)}$$

which gives force $$F = \frac{k F_d}{1+k}$$

Distrete Time:

$$u_{k+1} = -k(\omega^2 y_{1,k} - F_d)$$

$$\underline{x}_{k+1} = \hat{A} x_k + \hat{E} y_{1,k-1} + \hat{G}$$

$$\hat{E} = \begin{pmatrix} 0 \\ -k\omega^2 \end{pmatrix} \qquad \hat{G} = \begin{pmatrix} 0 \\ k F_d \end{pmatrix}$$

Now includes delay: Transform:

$$\underline{z}_k = \begin{pmatrix} z_{1,k} \\ z_{2,k} \\ z_{3,k} \end{pmatrix} = \begin{pmatrix} y_{1,k} \\ y_{2,k} \\ y_{1,k-1} \end{pmatrix}$$

$$\underline{z}_{k+1} = \begin{pmatrix} \hat{A} & \hat{E} \\ (1 \ 0) & 0 \end{pmatrix} \underline{z}_k + \begin{pmatrix} \hat{G} \\ 0 \end{pmatrix}$$