1. Wave Equation – Transit of Variable

$u_{tt} = v^2 u_{xx}$ $\qquad$ $u(t \to \pm\infty, t) = 0$

$u(x,0) = g(x)$
$u_t(x) = h(x)$

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} z = x + vt \\ q = x - vt \end{pmatrix} \qquad u(x,t) \to \bar{u}(z,q)$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial}{\partial q}\frac{\partial q}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial q}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial}{\partial q}\frac{\partial q}{\partial t} = v\left[\frac{\partial}{\partial z} - \frac{\partial}{\partial q}\right]$$

$$\partial_{xx} = \left[\frac{\partial}{\partial z} + \frac{\partial}{\partial q}\right]\left[\frac{\partial}{\partial z} + \frac{\partial}{\partial q}\right]$$

$$= \frac{\partial}{\partial z^2} + \frac{\partial}{\partial q} + 2\frac{\partial^2}{\partial zq}$$

$$\partial_{tt} = v^2\left[\frac{\partial}{\partial z} - \frac{\partial}{\partial q}\right]\left[\frac{\partial}{\partial z} - \frac{\partial}{\partial q}\right]$$

$$= v^2\left[\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial q^2} - 2\frac{\partial^2}{\partial zq}\right]$$

Sub into original PDE:

$$v^2\left[\cancel{\frac{\partial^2}{\partial z^2}}+\cancel{\frac{\partial^2}{\partial q^2}}-2\frac{\partial^2}{\partial z q}\right]\bar{u}=v^2\left[\cancel{\frac{\partial^2}{\partial z^2}}+\cancel{\frac{\partial^2}{\partial q^2}}+2\frac{\partial^2}{\partial q z}\right]\bar{u}$$

$$\partial_{zq}\bar{u}=0$$

Generalised Solution:

$$\bar{u}(z,q)=F(z)+G(q)$$

$$u(x,t)=F(x+vt)+G(x-vt)$$

Apply ICs

$$u(x,0)=F+G=f(x)$$

$$u_t(x,0)=vF'-vG'=h(x)$$

$$F'=\frac{f'}{2}+\frac{h}{2v}$$

$$G'=\frac{f'}{2}-\frac{h}{2v}$$

Integrating

$$F(x) - \cancelto{0}{F(\infty)} = \frac{f(x)}{2} - \cancelto{0}{\frac{f(-\infty)}{2}} + \frac{1}{2v}\int_{-\infty}^{x} h(s)\,ds$$

$$G(x) - G(\infty) = \frac{f(x)}{2} - \cancelto{0}{\frac{f(-\infty)}{2}} - \frac{1}{2v}\int_{-\infty}^{x} h(s)\,ds$$

$$u(x,t) = \frac{1}{2}\left[f(x+vt) + f(x-vt)\right]$$ — Average of travelling initial profile

$$+ \frac{1}{2v}\left[\int_{x-ct}^{x+ct} h(s)\,ds\right]$$