
Asynchronous lecture 6

- Methods of characteristics: initial value problems II

Methods of characteristics: Example 2

velocity is a function of space

$$\frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = 0$$

$$u(x, 0) = \phi(x)$$

$$a(x, t) \frac{\partial u}{\partial x} + b(x, t) \frac{\partial u}{\partial t} = c(x, t)$$

$$\frac{dx}{ds} = a(x(s), t(s))$$

= x

$$\frac{dt}{ds} = b(x(s), t(s))$$

= 1

$$\frac{du}{ds} = c(x(s), t(s))$$

= 0

Methods of characteristics: Example 2

$$\frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = 0$$

$$u(x, 0) = \phi(x)$$

$$\frac{dx}{ds}(r, s) = x$$

$$\frac{dt}{ds}(r, s) = 1$$

$$\frac{du}{ds}(r, s) = 0$$

$$x(r, 0) = r$$

$$t(r, 0) = 0$$

$$u(r, 0) = \phi(r)$$

Initial value problem



Methods of characteristics: Example 2

$$\frac{dx}{ds}(r, s) = x$$

$$\frac{dt}{ds}(r, s) = 1$$

$$\frac{du}{ds}(r, s) = 0$$

$$x(r, s) = c_2(r)e^s$$

$$t(r, s) = s + c_1(r)$$

$$u(r, s) = c_3(r)$$

Imposing the initial conditions

$$x(r, 0) = r$$

$$t(r, 0) = 0$$

$$u(r, 0) = \phi(r)$$

$$x(r, s) = r e^s$$

$$t(r, s) = s$$

$$u(r, s) = \phi(r)$$

solution

Methods of characteristics: Example 2

$$x(r, s) = r e^s \qquad t(r, s) = s \qquad u(r, s) = \phi(r)$$

Writing r and s in terms of x and t

$$r(x, t) = x e^{-t} \qquad s(x, t) = t$$

$$u(x, t) = u(r(x, t), s(x, t)) = \phi(x e^{-t})$$

Methods of characteristics: Example 2

$$\frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = 0$$

$$u(x, 0) = \phi(x)$$

$$u(x, t) = \phi(xe^{-t})$$

Inserting the solution $u(x, t)$ into the equation one can verify that

$$-xe^{-t}\phi'(xe^{-t}) + x[e^{-t}\phi'(xe^{-t})] = 0$$

Methods of characteristics: Example 3

$$\frac{\partial u}{\partial t} + (x + t) \frac{\partial u}{\partial x} = t$$

Velocity of drift (pointing to $x + t$) *Inhomogeneous term* (pointing to t)

$$u(x, 0) = \phi(x)$$

$$a(x, t) \frac{\partial u}{\partial x} + b(x, t) \frac{\partial u}{\partial t} = c(x, t)$$

$$\frac{dx}{ds} = a(x(s), t(s))$$

$= x + t$

$$\frac{dt}{ds} = b(x(s), t(s))$$

$= 1$

$$\frac{du}{ds} = c(x(s), t(s))$$

$= t$

Methods of characteristics: Example 3

$$\frac{\partial u}{\partial t} + (x + t) \frac{\partial u}{\partial x} = t$$

$$u(x, 0) = \phi(x)$$

$$\frac{dx}{ds}(r, s) = x + t$$

$$\frac{dt}{ds}(r, s) = 1$$

$$\frac{du}{ds}(r, s) = t$$

Methods of characteristics: Example 3

$$\frac{dt}{ds}(r, s) = 1$$



$$t(r, s) = s + c_1(r)$$

$$t(r, 0) = 0$$



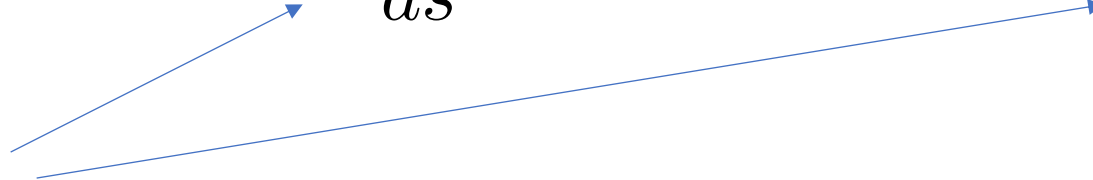
$$t(r, s) = s$$

$$\frac{dx}{ds}(r, s) = x + t$$

$$\frac{dx}{ds}(r, s) = x + s$$

$$\frac{du}{ds}(r, s) = t$$

$$\frac{du}{ds}(r, s) = s$$



Methods of characteristics: Example 3

$$\frac{dx}{ds}(r, s) = x + s$$

Laplace transforming: $\epsilon \tilde{x}(\epsilon) - x(0) = \tilde{x}(\epsilon) + \frac{1}{\epsilon^2}$

$$\tilde{x}(\epsilon) = \frac{x(0)}{\epsilon-1} + \frac{1}{\epsilon^2(\epsilon-1)} = \frac{x(0)}{\epsilon-1} + \frac{1}{\epsilon-1} - \frac{1}{\epsilon} - \frac{1}{\epsilon^2}$$

inverse Laplace transforming: $x(s) = x(0)e^s + e^s - 1 - s$

Methods of characteristics: Example 3

$$\frac{dx}{ds}(r, s) = x + s$$

Laplace transform

$$x(r, s) = [x(r, 0) + 1]e^s - s - 1$$

in earlier notation

$$c_2(r) = [x(r, 0) + 1]$$

using $x(r, 0) = r$

Γ surface of initial cond.

$$x(r, s) = e^s(r + 1) - s - 1$$

Methods of characteristics: Example 3

$$\frac{dt}{ds}(r, s) = 1$$

$$\frac{dx}{ds}(r, s) = x + t$$

$$\frac{du}{ds}(r, s) = t$$

$$t(r, s) = s$$

$$\frac{dx}{ds}(r, s) = x + s$$

$$\frac{du}{ds}(r, s) = s$$

$$t(r, s) = s$$

$$x(r, s) = e^s(r + 1) - s - 1$$

$$\frac{du}{ds}(r, s) = s$$

$$u(r, s) = \frac{s^2}{2} + c_3(r)$$

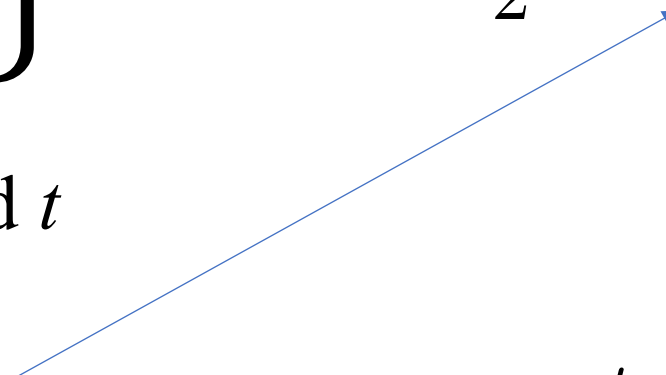
Methods of characteristics: Example 3

$$t(r, s) = s$$

$$x(r, s) = e^s(r + 1) - s - 1$$

$$\left. \begin{array}{l} u(r, s) = \frac{s^2}{2} + c_3(r) \\ u(r, 0) = \phi(r) \end{array} \right\} u(r, s) = \frac{s^2}{2} + \phi(r)$$

Writing r and s in terms of x and t

$$r = (x + s + 1)e^{-s} - 1 \rightarrow r = (x + t + 1)e^{-t} - 1$$


Methods of characteristics: Example 3

$$\frac{\partial u}{\partial t} + (x + t) \frac{\partial u}{\partial x} = t$$

$$u(x, 0) = \phi(x)$$

$$u(x, t) = \phi \left((x + t + 1)e^{-t} - 1 \right) + \frac{t^2}{2}$$

Methods of characteristics: Example 4

$$\frac{\partial u(x, t)}{\partial t} - e^{-x} \frac{\partial u(x, t)}{\partial x} = -u(x, t)$$

$$u(x, 0) = \phi(x)$$

$$\frac{dt}{ds} = 1$$

$$\frac{dx}{ds} = -e^{-x}$$

$$\frac{du}{ds} = -u$$

$$t = s + c_1(r)$$

$$e^x = -s + c_2(r)$$

$$\ln(u) = -s + c_3(r)$$

Methods of characteristics: Example 4

$$\frac{\partial u(x,t)}{\partial t} - e^{-x} \frac{\partial u(x,t)}{\partial x} = -u(x,t)$$

$$u(x,0) = \phi(x)$$

$$t = s + c_1(r)$$

$$t(r,0) = 0$$



$$t = s$$

$$e^x = -s + c_2(r)$$



$$x(r,0) = r$$



$$x = \ln(e^r - s)$$

$$\ln(u) = -s + c_3(r)$$

$$u(r,0) = \phi(r)$$



$$u(r,s) = e^{-s} \phi(r)$$

Methods of characteristics: Example 4

$$\frac{\partial u(x, t)}{\partial t} - e^{-x} \frac{\partial u(x, t)}{\partial x} = -u(x, t)$$

$$u(x, 0) = \phi(x)$$

$$t = s$$

$$x = \ln(e^r - s)$$

$$u(r, s) = e^{-s} \phi(r)$$

$$x = \ln(e^r - t)$$

$$u(r, t) = e^{-t} \phi(r)$$

Writing r and s in terms of x and t

$$r = \ln(e^x + t)$$

$$u(x, t) = e^{-t} \phi(\ln[e^x + t])$$