Asynchronous lecture 3

Infinite domains:

- Diffusion/Heat equation
- Drift Equation
- Convection-diffusion equation

Diffusion/Heat equation in 1D (parabolic)

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

Initial-boundary-value problem

Initial conditions: $u(x,0) = f(x)$

Boundary conditions: $u(x \to \pm\infty, t) = 0$

Conserved quantity

$$u(x,t) \geq 0 \quad \longrightarrow \quad \boxed{\int_{-\infty}^{+\infty} dx\, u(x,t) = const.}$$

Solution of the 1D diffusion equation

Fourier transform: $\hat{u}(k,t) = \int_{-\infty}^{+\infty} dx\, e^{-ikx} u(x,t)$

Now an Ordinary Differential Equation (ODE)

from PDE

Transformed equation: $\dfrac{\partial \hat{u}(k,t)}{\partial t} = -Dk^2 \hat{u}(k,t)$

Transformed initial condition: $\hat{u}(k,0) = \int_{-\infty}^{+\infty} dx\, e^{-ikx} \underline{f(x)}$

initial conditions

Solution in Fourier space

$$\hat{u}(k,t) = \hat{u}(k,0)\, e^{-Dk^2 t}$$

Solution in real space

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk e^{ikx}\, \hat{u}(k,0)\, e^{-Dk^2 t}$$

Inverse fourier transform

Solution of the 1D diffusion equation

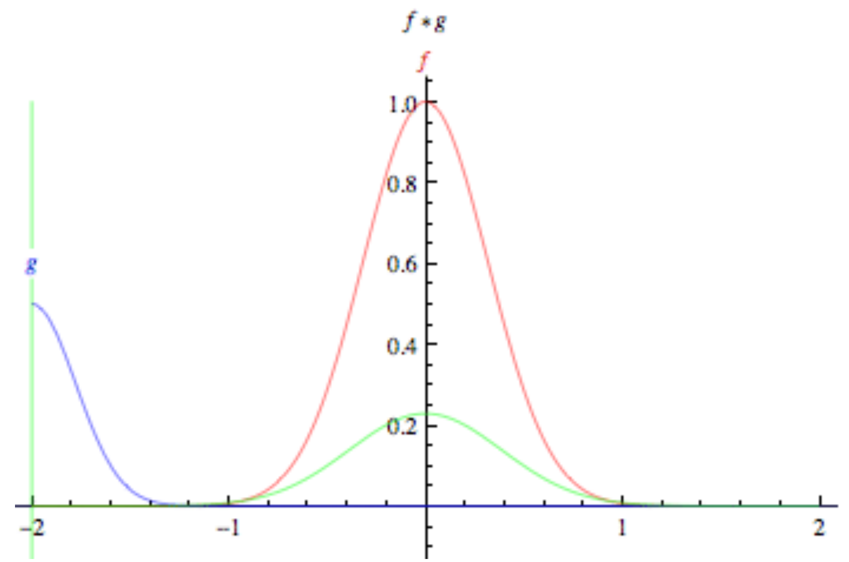
Using convolution theorem

$$\hat{f}(k)\hat{g}(k) = \mathcal{F}\{f(x)\}\mathcal{F}\{g(x)\} = \mathcal{F}\left\{\int_{-\infty}^{+\infty} dy f(x-y)g(y)\right\}$$

Knowing that

guassian

$$\mathcal{F}\left\{\frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4\pi Dt}}\right\} = e^{-Dk^2t}$$

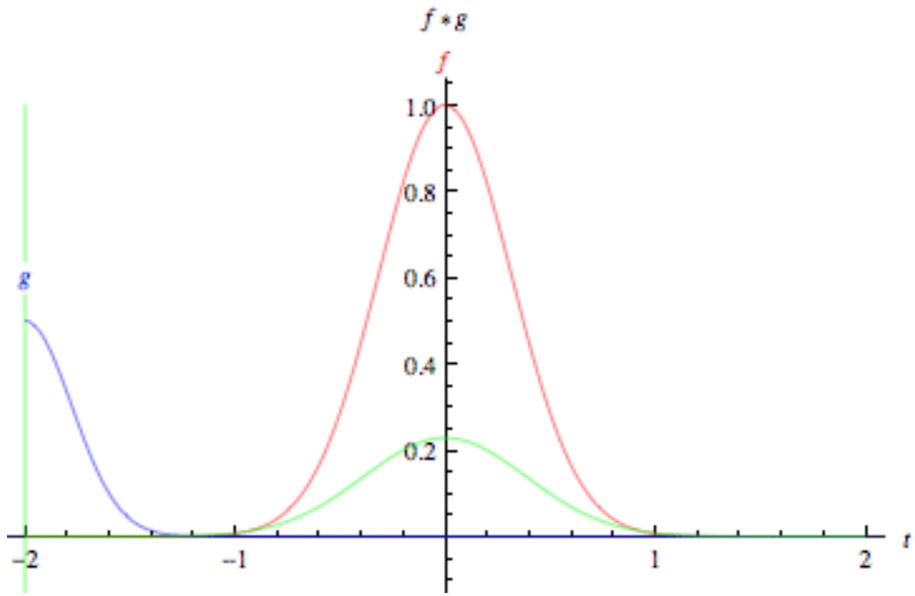


Green -> convolution *f*g*

Applying the initial condition

$$\underbrace{u(x,t) = \int_{-\infty}^{+\infty} dy \frac{e^{-\frac{(x-y)^2}{4Dt}}}{\sqrt{4\pi Dt}} u(y,0)}$$

Green -> convolution *f*g* Convolution integral



$$\int_{-\infty}^{+\infty} dy f(x-y) g(y)$$

amount of overlap of one function *g* as it is shifted over another function *f*

"blends" one function with another

General Solution of the 1D diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

Initial conditions: $u(x,0) = f(x)$

Boundary conditions: $u(x \to \pm\infty, t) = 0$

$$u(x,t) = \int_{-\infty}^{+\infty} dy \frac{e^{-\frac{(x-y)^2}{4Dt}}}{\sqrt{4\pi Dt}}\, u(y,0)$$

Let us try out simple initial conditions

$$u(x,t) = \int_{-\infty}^{+\infty} dy \frac{e^{-\frac{(x-y)^2}{4Dt}}}{\sqrt{4\pi Dt}} u(y,0)$$

located at x_0

Localized initial conditions: $u(x,0) = \delta(x - x_0)$

$\delta(x)$ is a Dirac delta distribution

$$\int_{-\infty}^{+\infty} dx\, \delta(x-x_0) g(x) = g(x_0) \qquad \int_{-\infty}^{+\infty} dx\, \delta(x-x_0) = 1$$

Solution of the 1D diffusion equation with point source

$u(x,0)=\delta(x-x_0)$

$$u(x,t) = \int_{-\infty}^{+\infty} dy \frac{e^{-\frac{(x-y)^2}{4Dt}}}{\sqrt{4\pi Dt}}\, u(y,0)$$

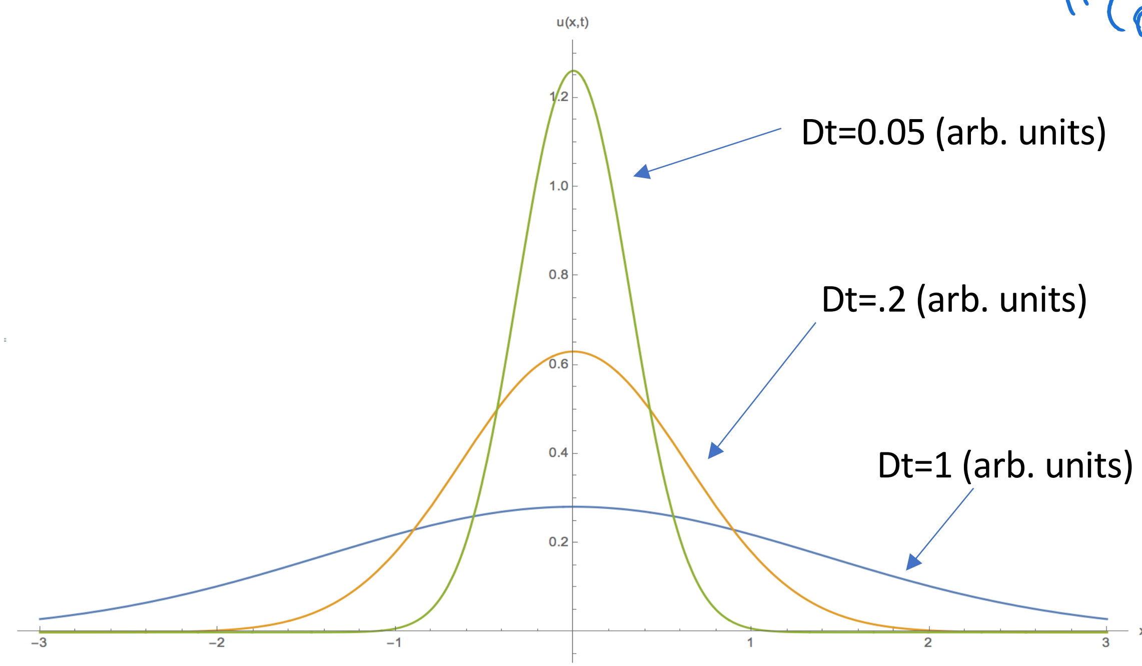
Using Dirac delta as initial condition $\longrightarrow$ $u(x,t) = \dfrac{e^{-\frac{(x-x_0)^2}{4Dt}}}{\sqrt{4\pi Dt}}$

$u(x,t)$ is called a probability distribution function or probability density

Solution of the 1D diffusion equation with point source

$$u(x,t) = \frac{e^{-\frac{(x-x_0)^2}{4Dt}}}{\sqrt{4\pi Dt}}$$

"Concentration diffuses" outwards



The Drift equation

$$\frac{\partial u(x,t)}{\partial t} = -v\frac{\partial u(x,t)}{\partial x}$$

balistic term

Initial conditions $u(x,0) = f(x)$

Boundary conditions: $u(x \to \pm\infty, t) = 0$

Simple solution: initial condition drifting to the right

$$u(x,t) = f(x - vt)$$

No spread! constant movement in one direction

Exercise: show that the above eq. is a solution of the drift equation

$$\frac{\partial u}{\partial t} = -v\frac{\partial u}{\partial x} \qquad u(x,0) = g(x)$$

$$u(x,t) = g(x - vt)$$

$$\frac{\partial u}{\partial t} = -vg'(x-vt)$$

$$\frac{\partial u}{\partial x} = g'(x-vt)$$

$$u(x,0) = g(x - v \cdot 0) = g(x)$$

Diffusion with drift

(Advection-diffusion, convection-diffusion or drift-diffusion equation)

$$\frac{\partial u}{\partial t} = D\underbrace{\frac{\partial^2 u}{\partial x^2}}_{\text{diffusion term}} \underbrace{- v\frac{\partial u}{\partial x}}_{\text{advection term}}$$

$$u(x,0) = f(x)$$

Boundary conditions: $u(x \to \pm\infty, t) = 0$

Solution for Diffusion-drift Equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - v\frac{\partial u}{\partial x}$$

Variable transformation: hop on the drift!

$$(x, t) \to (y, t')$$

$$y = x - vt \qquad t = t'$$

$$u(x - vt, t) = \bar{u}(y, t')$$

Frame of reference moving with the drift

Variable transformation: drifting frame of ref.

Two variable transformation

$$\begin{pmatrix} x \\ t \end{pmatrix} \rightarrow \begin{pmatrix} y \\ t' \end{pmatrix} \longrightarrow \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial x}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial t}$$

$$y(x,t) = x - vt$$

$$t'(x,t) = t$$

$$\longrightarrow \boxed{\frac{\partial y}{\partial x} = 1 \qquad \frac{\partial t'}{\partial x} = 0 \qquad \frac{\partial y}{\partial t} = -v \qquad \frac{\partial t'}{\partial t} = 1}$$

$$\longrightarrow \boxed{\frac{\partial}{\partial x} = \frac{\partial}{\partial y} \qquad \frac{\partial}{\partial t} = -v\frac{\partial}{\partial y} + \frac{\partial}{\partial t'}}$$

Variable transformation

$$\frac{\partial}{\partial t}u(x,t) = D\frac{\partial^2}{\partial x^2}u(x,t) - v\frac{\partial}{\partial x}u(x,t)$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} \qquad \frac{\partial}{\partial t} = -v\frac{\partial}{\partial y} + \frac{\partial}{\partial t'}$$

$$u(x,t) \to \bar{u}(y,t')$$

$$\cancel{-v\frac{\partial}{\partial y}\bar{u}(y,t')} + \frac{\partial}{\partial t'}\bar{u}(y,t') = D\frac{\partial^2}{\partial y^2}\bar{u}(y,t') \cancel{- v\frac{\partial}{\partial y}\bar{u}(y,t')}$$

$$\boxed{\frac{\partial \bar{u}}{\partial t'} = D\frac{\partial^2 \bar{u}}{\partial y^2}}$$

Diffusion Equation

Diffusion with drift

$$\frac{\partial \bar{u}}{\partial t'} = D \frac{\partial^2 \bar{u}}{\partial y^2}$$

Transformed conditions on the new variables

$$u(x,0) = f(x) \rightarrow \bar{u}(y,0) = f(y)$$

$$\text{Boundary conditions: } \bar{u}(y \rightarrow \pm\infty, t') = 0$$

General solution: Green's function convoluted with Int. Cond.

$$\bar{u}(y,t') = \int_{-\infty}^{+\infty} dz\, f(z) \frac{e^{-\frac{(y-z)^2}{4Dt'}}}{\sqrt{4\pi D t'}}$$

z is a dummy variable

Solution for the diffusion-drift Equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - v\frac{\partial u}{\partial x}$$

$$u(x,0) = f(x)$$

Boundary conditions: $u(x \to \pm\infty, t) = 0$

Transform back:

$$\bar{u}(y,t') = u(x - vt, t)$$

Drift

$$u(x,t) = \int_{-\infty}^{+\infty} dz\, f(z)\frac{e^{-\frac{(x-vt-z)^2}{4Dt}}}{\sqrt{4\pi Dt}}$$

The mean value drift in time

Insights: Ballistic vs random (or diffusive) movement

Ballistic movement to the right

$$\frac{\partial u(x,t)}{\partial t} = -v\frac{\partial u(x,t)}{\partial x}$$

Ballistic movement to the left

$$\frac{\partial u(x,t)}{\partial t} = v\frac{\partial u(x,t)}{\partial x}$$

deterministic movement

Diffusive movement

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

Random (solution is a pdf)

The Wave Equation: ballistic movement in two directions

Ballistic movement to the right

$$\frac{\partial u(x,t)}{\partial t} = -v\frac{\partial u(x,t)}{\partial x}$$

Ballistic movement to the left

$$\frac{\partial u(x,t)}{\partial t} = v\frac{\partial u(x,t)}{\partial x}$$

Wave equation

Ballistic movement in both directions

$$\boxed{\frac{\partial^2 u(x,t)}{\partial t^2} = v^2\frac{\partial^2 u(x,t)}{\partial x^2}}$$

The Wave Equation

may be used to describe the travelling of sound waves, electromagnetic waves, water waves, etc.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = v^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

Compared diffusion equation, it represents the (fully deterministic) ballistic movement of a quantity travelling in two opposite direction

Compared with drift equation, the initial condition here may affect the dynamics since the *wave equation* is 2nd order in time

↳ Two initial conditions

The wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = v^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

$$\text{Initial conditions} \begin{cases} u(x,0) = f(x) \\ \dfrac{\partial u(x,0)}{\partial t} = h(x) \end{cases}$$

$$\text{Boundary conditions} \quad u(x \to \pm\infty, t) = 0$$

solution of the wave equation

Variable transformation: $(x,t) \rightarrow (z = x + vt, q = x - vt)$

$$u(x,t) \rightarrow U(z,q)$$

$$\frac{\partial^2 U(z,q)}{\partial z \partial q} = 0$$

Exercise: show that the wave equation simplifies to *the Eq. on the left* using the transformation of variable above.

General solution: $U(z,q) = F(z) + G(q)$

D'alembert Solution

$$u(x,t) = F(x + vt) + G(x - vt)$$

solution of the wave equation: Initial Cond.

$$u(x,t) = F(x+vt) + G(x-vt)$$

$$u(x,0) = f(x) = F(x) + G(x) \rightarrow f'(x) = F'(x) + G'(x)$$

$$\frac{\partial u(x,0)}{\partial t} = h(x) = vF'(x) - vG'(x)$$

$$F'(x) + G'(x) = f(x)$$

$$F'(x) - G'(x) = \frac{h(x)}{v}$$

solving for $F'(x)$ and $G'(x)$ and then integrating

solution of the wave equation

$$u(x,t) = F(x+vt) + G(x-vt)$$

$$F(x) = \frac{f(x)}{2} + \frac{1}{2v}\int_{-\infty}^{x} ds\, h(s)$$

$$G(x) = \frac{f(x)}{2} - \frac{1}{2v}\int_{-\infty}^{x} ds\, h(s)$$

Initial condition

$$u(x,0) = f(x)$$

$$\frac{\partial u(x,0)}{\partial t} = h(x)$$

Exercise: show that the solution is the equation below after integrating *F'* and *G'*

$$\boxed{u(x,t) = \frac{1}{2}[f(x+vt) + f(x-vt)] + \frac{1}{2v}\left[\int_{x-vt}^{x+vt} ds\, h(s)\right]}$$

Two poles with conserved initial shape travelling in both direction f(x) *+ a shape changing function of the initial velocity h(x)*