Asynchronous lecture 9

- Methods of characteristics: “Initial value method” on boundary value problems

"IVP method" on BVP: Example 1

$u(x,y)$

$$u_x + u_y = 1 - u \quad \begin{cases} u(x, x + x^2) = f(x) \quad \text{with } y > x & \text{boundary value} \\ u(x, 2x) = f(x) & \text{boundary value} \\ u(x, 0) = f(x) & \text{initial value} \end{cases}$$

$$\frac{dx}{ds} = 1 \qquad \frac{dy}{ds} = 1 \qquad \frac{du}{ds} = 1 - u$$

"IVP method" on BVP: Example 1

$$u_x + u_y = 1 - u \quad \begin{cases} u(x, x+x^2) = f(x) \quad \text{with } y > x & \text{boundary value} \\ u(x, 2x) = f(x) & \text{boundary value} \\ u(x, 0) = f(x) & \text{initial value} \end{cases}$$

$$\frac{dx}{ds} = 1 \qquad \frac{dy}{ds} = 1 \qquad \frac{du}{ds} = 1 - u$$

$$x(r,s) = s + c_1(r) \qquad y(r,s) = s + c_2(r) \qquad u(r,s) = 1 - e^{-s} c_3(r)$$

$$x(r,0) = r \qquad y(r,0) = c_2(r) \qquad u(r,0) = f(r)$$

$c_2(r)$ depends on the problem

$$x(r,s) = s + r \qquad u(r,s) = 1 - e^{-s}\left[1 - f(r)\right]$$

"IVP method" on BVP: Example 1

$$u_x + u_y = 1 - u$$

$$\left.\begin{aligned} u(x, x+x^2) = f(x) \quad \text{with } y > x \\ u(x, 2x) = f(x) \\ u(x, 0) = f(x) \end{aligned}\right\}$$

$$s = 0 \Rightarrow x = r$$

$$y(r, 0) = c_2(r)$$

$$\begin{cases} y(r,0) = r + r^2 \\ y(r,0) = 2r \\ y(r,0) = 0 \end{cases}$$

$$y(r, s) = s + c_2(r)$$

$$\begin{cases} y(r,s) = s + r + r^2 \\ y(r,s) = s + 2r \\ y(r,s) = s \end{cases}$$

$$x(r, s) = s + r$$

$$u(r, s) = 1 - e^{-s}\left[1 - f(r)\right]$$

"IVP method" on BVP: Example 1

$$u(r,s) = 1 - e^{-s}\left[1 - f(r)\right]$$

choose sign such that when $y = x + x^2$ at $s = 0$

so that

$$u(x, x+x^2) = f(x)$$

inverting (r,s) to (x,y)

$$\begin{aligned} x &= s + r \\ y &= s + r + r^2 \end{aligned} \;\rightarrow\; \begin{aligned} s &= x - r \\ y &= x + r^2 \end{aligned} \;\rightarrow\; \begin{aligned} s &= x \mp \sqrt{y-x} \\ r &= \pm\sqrt{y-x} \end{aligned} \;\rightarrow\; \begin{aligned} s &= x - \sqrt{y-x} \\ r &= \sqrt{y-x} \end{aligned}$$

$$\begin{aligned} x &= s + r \\ y &= s + 2r \end{aligned} \;\rightarrow\; \begin{aligned} s &= x - r \\ y &= x + r \end{aligned} \;\rightarrow\; \begin{aligned} s &= 2x - y \\ r &= y - x \end{aligned}$$

$$\begin{aligned} x &= s + r \\ y &= s \end{aligned} \;\rightarrow\; \begin{aligned} s &= x - r \\ y &= s \end{aligned} \;\rightarrow\; \begin{aligned} r &= x - y \\ s &= y \end{aligned}$$

"IVP method" on BVP: Example 1

$$u_x + u_y = 1 - u$$

solution in (r, s) $\longrightarrow$ $u(r, s) = 1 - e^{-s}\left[1 - f(r)\right]$

$u(x, x + x^2) = f(x)$
with $y > x$ $\longrightarrow$ $u(x, y) = 1 - e^{-x+\sqrt{y-x}}\left[1 - f\left(\sqrt{y-x}\right)\right]$

$u(x, 2x) = f(x)$ $\longrightarrow$ $u(x, y) = 1 - e^{y-2x}\left[1 - f\left(y - x\right)\right]$

$u(x, 0) = f(x)$ $\longrightarrow$ $u(x, y) = 1 - e^{-y}\left[1 - f\left(x\right)\right]$ $x - y$

"IVP method" on BVP: Example 2

$$u_x + \frac{u_y}{1+u} = 0$$

$$u(x, g(x)) = f(x)$$

if $g(x) = \text{const.} \rightarrow$ initial value problem

if $g(x) = \text{function of } x \rightarrow$ boundary value problem

we keep $g(x)$ unspecified initially

“IVP method” on BVP: Example 2

$$u_x + \frac{u_y}{1+u} = 0 \qquad u(x, g(x)) = f(x)$$

$$\frac{dx}{ds} = 1 \qquad \frac{du}{ds} = 0 \qquad \frac{dy}{ds} = \frac{1}{1+f(r)}$$

$$x(r,s) = s + c_1(r) \qquad u(r,s) = c_3(r) \qquad y(r,s) = \frac{s}{1+f(r)} + c_2(r)$$

$$x(r,0) = r \qquad u(r,0) = f(r) \qquad y(r,0) = g(r)$$

$$x(r,s) = s + r \qquad u(r,s) = f(r) \qquad y(r,s) = \frac{s}{1+f(r)} + g(r)$$

"IVP method" on BVP: Example 2

$$u(r,s) = f(r)$$

inverting (r,s) to (x,y)

$$x = s + r \qquad \rightarrow \qquad s = x - r$$

$$y = \frac{s}{1+f(r)} + g(r) \qquad \rightarrow \qquad y = \frac{x-r}{1+f(r)} + g(r)$$

$$\downarrow$$

$$[1 + f(r)]\,[y - g(r)] + r - x = 0$$

solve for r once $g(r)$ and $f(r)$ are given $\boxed{u(x, g(x)) = f(x)}$

"IVP method" on BVP: Example 2

$$u(x, g(x)) = f(x) \longrightarrow u(r,s) = f(r)$$

$$[1+f(r)]\,[y-g(r)]+r-x=0$$

example: $g(r) = f(r) = r$ that is boundary value problem $u(x, x) = x$

$$[1+r][y-r]+r-x=0$$

$$r = \frac{y \pm \sqrt{y^2+4(y-x)}}{2} \xrightarrow{u(r,s)=f(r)} u(x,y) = \frac{y \pm \sqrt{y^2+4(y-x)}}{2}$$

to ensure that $u(x, x) = x$ positive sign is the correct solution

$$\boxed{u(x,y) = \frac{y + \sqrt{y^2+4(y-x)}}{2}}$$

"IVP method" on BVP: Example 2

$$u(x, g(x)) = f(x) \longrightarrow u(r,s) = f(r)$$

$$[1 + f(r)]\,[y - g(r)] + r - x = 0$$

example: $g(r) = 0$; $f(r) = r^2$ that is intial value problem $u(x, 0) = x^2$

$$yr^2 + r + y - x = 0$$

$$r = \frac{-1 \pm \sqrt{1+4y(x-y)}}{2y} \xrightarrow{u(r,s)=f(r)} u(x,y) = \frac{\left(-1 \pm \sqrt{1+4y(x-y)}\right)^2}{4y^2}$$

to ensure that $u(x, 0) = x^2$ positive sign is the correct solution

$$u(x,y) = \frac{\left(-1 + \sqrt{1+4y(x-y)}\right)^2}{4y^2}$$

you can verify by doing a Taylor expansion of the square root when $y \to 0$