Control Theory

Mathematical models

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Newtonian mechanics

Second Law

- Newton's laws incl.

$$m\boldsymbol{a} = \sum_i \boldsymbol{F}_i$$

- Forces in springs

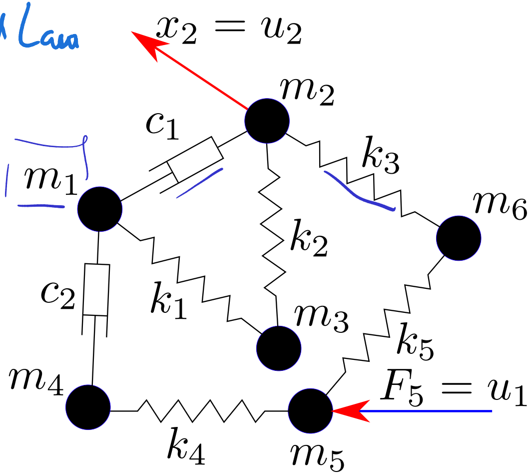
stiffness

$$\boxed{F_s = kx}$$

- Forces in dampers

$$F_d = c\dot{x}$$

damping factor



Lagrangian mechanics

Generalised Coordinates

↳ Uniquely describe position.

- The Lagrangian $L = T - U$ ← potential

  Kinetic energy

- Lagrange equation (2nd kind)

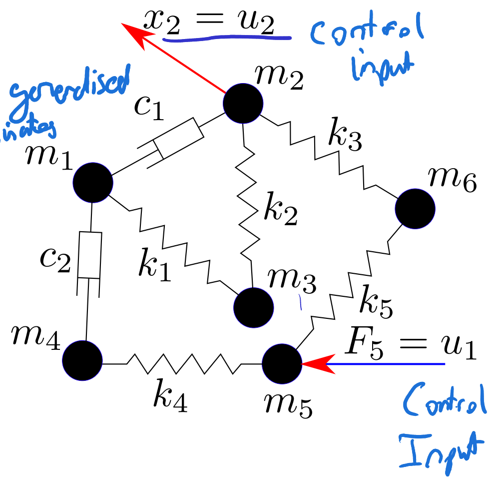
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$$

for each generalised coordinates

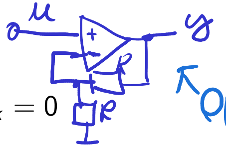
- Generalised forces

$$Q_i = \sum_{j=1}^{\#\text{forces}} \boldsymbol{F}_j \cdot \frac{\partial \boldsymbol{r}_j}{\partial q_i} = \sum_{j=1}^{\#\text{forces}} \boldsymbol{F}_j \cdot \frac{\partial \boldsymbol{v}_j}{\partial \dot{q}_i}$$

- Mass particle $T = \frac{1}{2} m \boldsymbol{v}^2$
- Spring $U = \frac{1}{2} k \boldsymbol{x}^2$
- Damper: use generalised forces OR Rayleigh's dissipation function

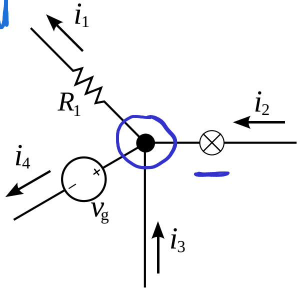


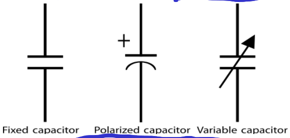
Basic electronics

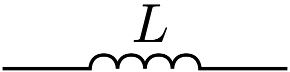


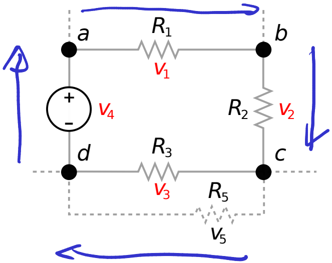
Operational amplifier

- Kirchhoff's current law $\sum_k i_k = 0$
- Kirchhoff's voltage law $\sum_k v_k = 0$
- Resistor $v(t) = R\,i(t)$ Ohm
- Capacitor $i(t) = C\,\dfrac{\mathrm{d}v(t)}{\mathrm{d}t}$ Farad
- Inductor $v(t) = L\,\dfrac{\mathrm{d}i(t)}{\mathrm{d}t}$ Henry
- diode, transistor, etc are nonlinear, see data sheets, eg 1N4148, 2N2222









Model equations: state, input and output

M = Mass Matrix

- Input $\boldsymbol{u} \in \mathbb{R}^p$
  Output $\boldsymbol{y} \in \mathbb{R}^m$
  State $\boldsymbol{x} \in \mathbb{R}^n$
- The equations describing the system

Dynamics $\quad \dot{\boldsymbol{x}} = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}\right)$

Output $\quad \boldsymbol{y} = \boldsymbol{g}\left(\boldsymbol{x}, \boldsymbol{u}\right)$

However mechanical systems tend to have the form $\ddot{q} + M^{-1} f_{nl}(q, \dot{q}, u) = 0$

$$\underbrace{\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{f}_{\mathrm{nl}}\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{u}\right) = \boldsymbol{0},}_{} \quad M^{-1}$$

which we need to re-write.

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} q \\ \dot{q} \end{pmatrix} \Rightarrow \dot{\underline{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -M^{-1} f_{nl}(x_1, x_2, u) \end{pmatrix}$$

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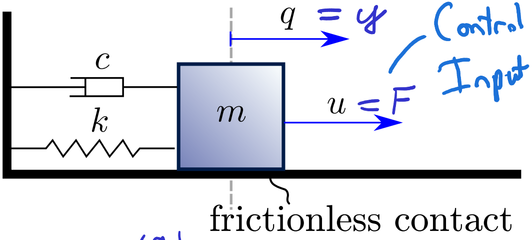
Case study: harmonic oscillator (1)



N2L:

$$m\ddot{q} = \text{force}$$

$$m\ddot{q} = -kq - c\dot{q} + u$$

In mechanics

$$m\ddot{q} + c\dot{q} + kq = u$$

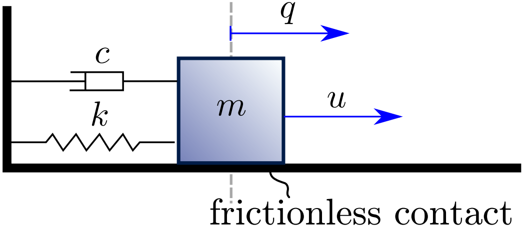
$$\ddot{q} + \frac{c}{m}\dot{q} + \frac{k}{m}q = u$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}$$

Hence

$$\left.\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} x_2 \\ -\frac{c}{m}x_2 - \frac{k}{m}x_1 + u \end{pmatrix} \\ y &= x_1 \end{aligned}\right\} \text{"Plant"}$$

Case study: harmonic oscillator (2)



$$\dot{x} = Ax + Bu$$

$$y = Cx + \cancel{Du}$$

Notice that our equations are linear

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}u \end{pmatrix}$$

$$y = x_1$$

$$\underline{\dot{x}} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}}_{\underline{\underline{A}}\ 2\times 2} \underline{x} + \underbrace{\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}}_{\underline{\underline{B}}\ 2\times 1} u$$

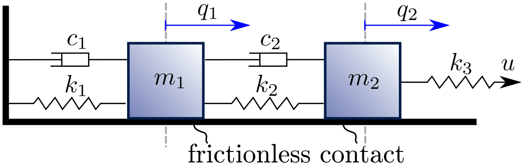
$$y = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\underline{\underline{C}}} x$$

Linear systems

In some cases we can also write the system as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \end{aligned}$$

Assignment



Assume that $y = q_2$, the control input is the displacement of the end of k_3. Derive the mathematical model of the system

The End