Asynchronous lecture 6

- Methods of characteristics: initial value problems II

Methods of characteristics: Example 2

velocity is a function of space

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0$$

$$u(x,0) = \phi(x)$$

$$a(x,t)\frac{\partial u}{\partial x} + b(x,t)\frac{\partial u}{\partial t} = c(x,t)$$

$$\frac{dx}{ds} = a(x(s),t(s)) \qquad \frac{dt}{ds} = b(x(s),t(s)) \qquad \frac{du}{ds} = c(x(s),t(s))$$

$= x \qquad = 1 \qquad = 0$

Methods of characteristics: Example 2

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0$$

$$u(x,0) = \phi(x)$$

$$\frac{dx}{ds}(r,s) = x \qquad \frac{dt}{ds}(r,s) = 1 \qquad \frac{du}{ds}(r,s) = 0$$

$$x(r,0) = r \qquad t(r,0) = 0 \qquad u(r,0) = \phi(r)$$

Initial value problem

Methods of characteristics: Example 2

$$\frac{dx}{ds}(r,s) = x \qquad \frac{dt}{ds}(r,s) = 1 \qquad \frac{du}{ds}(r,s) = 0$$

$$x(r,s) = c_2(r)e^s \qquad t(r,s) = s + c_1(r) \qquad u(r,s) = c_3(r)$$

Imposing the initial conditions

$$x(r,0) = r \qquad t(r,0) = 0 \qquad u(r,0) = \phi(r)$$

$$\boxed{x(r,s) = r\,e^s \qquad t(r,s) = s \qquad u(r,s) = \phi(r)}$$

solution

Methods of characteristics: Example 2

$$x(r, s) = r\, e^{s} \qquad t(r, s) = s \qquad u(r, s) = \phi(r)$$

Writing r and s in terms of x and t

$$r(x, t) = x\, e^{-t} \qquad s(x, t) = t$$

$$u(x, t) = u(r(x, t), s(x, t)) = \phi\left(x\, e^{-t}\right)$$

Methods of characteristics: Example 2

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0$$

$$u(x,0) = \phi(x)$$

$$u(x,t) = \phi(xe^{-t})$$

Inserting the solution *u(x,t)* into the equation one can verify that

$$-xe^{-t}\phi'(xe^{-t}) + x\left[e^{-t}\phi'(xe^{-t})\right] = 0$$

Methods of characteristics: Example 3

Velocity of drift

Inhomogenous term

$$\frac{\partial u}{\partial t} + (x+t)\frac{\partial u}{\partial x} = t$$

$$u(x,0) = \phi(x)$$

$$a(x,t)\frac{\partial u}{\partial x} + b(x,t)\frac{\partial u}{\partial t} = c(x,t)$$

$$\frac{dx}{ds} = a(x(s),t(s)) \qquad \frac{dt}{ds} = b(x(s),t(s)) \qquad \frac{du}{ds} = c(x(s),t(s))$$

$$= x+t \qquad = 1 \qquad = t$$

Methods of characteristics: Example 3

$$\frac{\partial u}{\partial t} + (x+t)\frac{\partial u}{\partial x} = t$$

$$u(x,0) = \phi(x)$$

$$\frac{dx}{ds}(r,s) = x + t \qquad \frac{dt}{ds}(r,s) = 1 \qquad \frac{du}{ds}(r,s) = t$$

Methods of characteristics: Example 3

$$\frac{dt}{ds}(r,s) = 1 \qquad \frac{dx}{ds}(r,s) = x + t \qquad \frac{du}{ds}(r,s) = t$$

$$t(r,s) = s + c_1(r)$$

$$t(r,0) = 0 \qquad \frac{dx}{ds}(r,s) = x + s \qquad \frac{du}{ds}(r,s) = s$$

$$t(r,s) = s$$

Methods of characteristics: Example 3

$$\frac{dx}{ds}(r,s) = x + s$$

Laplace transforming: $\epsilon\widetilde{x}(\epsilon) - x(0) = \widetilde{x}(\epsilon) + \frac{1}{\epsilon^2}$

$$\widetilde{x}(\epsilon) = \frac{x(0)}{\epsilon - 1} + \frac{1}{\epsilon^2(\epsilon-1)} = \frac{x(0)}{\epsilon - 1} + \frac{1}{\epsilon-1} - \frac{1}{\epsilon} - \frac{1}{\epsilon^2}$$

inverse Laplace transforming: $x(s) = x(0)e^s + e^s - 1 - s$

Methods of characteristics: Example 3

$$\frac{dx}{ds}(r,s) = x + s$$

Laplace transform

$$x(r,s) = [x(r,0)+1]e^s - s - 1$$

in earlier notation

$$c_2(r) = [x(r,0)+1]$$

using $x(r,0) = r$

Γ surface of initial cond.

$$x(r,s) = e^s(r+1) - s - 1$$

Methods of characteristics: Example 3

$$\frac{dt}{ds}(r,s) = 1 \qquad \frac{dx}{ds}(r,s) = x + t \qquad \frac{du}{ds}(r,s) = t$$

$$t(r,s) = s \qquad \frac{dx}{ds}(r,s) = x + s \qquad \frac{du}{ds}(r,s) = s$$

$$t(r,s) = s \qquad x(r,s) = e^{s}(r+1) - s - 1 \qquad \frac{du}{ds}(r,s) = s$$

$$u(r,s) = \frac{s^2}{2} + c_3(r)$$

Methods of characteristics: Example 3

$$t(r,s) = s \qquad x(r,s) = e^{s}(r+1) - s - 1$$

$$\left.\begin{aligned} u(r,s) &= \frac{s^2}{2} + c_3(r) \\ u(r,0) &= \phi(r) \end{aligned}\right\} \quad u(r,s) = \frac{s^2}{2} + \phi(r)$$

Writing *r* and *s* in terms of *x* and *t*

$$r = (x+s+1)e^{-s} - 1 \rightarrow r = (x+t+1)e^{-t} - 1$$

Methods of characteristics: Example 3

$$\frac{\partial u}{\partial t} + (x+t)\frac{\partial u}{\partial x} = t$$

$$u(x,0) = \phi(x)$$

$$u(x,t) = \phi\left((x+t+1)e^{-t} - 1\right) + \frac{t^2}{2}$$

Methods of characteristics: Example 4

$$\frac{\partial u(x,t)}{\partial t} - e^{-x}\frac{\partial u(x,t)}{\partial x} = -u(x,t)$$

$$u(x,0) = \phi(x)$$

$$\frac{dt}{ds} = 1 \qquad \frac{dx}{ds} = -e^{-x} \qquad \frac{du}{ds} = -u$$

$$t = s + c_1(r) \qquad e^{x} = -s + c_2(r) \qquad \ln(u) = -s + c_3(r)$$

Methods of characteristics: Example 4

$$\frac{\partial u(x,t)}{\partial t} - e^{-x}\frac{\partial u(x,t)}{\partial x} = -u(x,t)$$

$$u(x,0) = \phi(x)$$

$$t = s + c_1(r)$$

$$t(r,0) = 0$$

$$\downarrow$$

$$t = s$$

$$e^x = -s + c_2(r)$$

$$\downarrow$$

$$x(r,0) = r$$

$$\downarrow$$

$$x = \ln(e^r - s)$$

$$\ln(u) = -s + c_3(r)$$

$$u(r,0) = \phi(r)$$

$$\downarrow$$

$$u(r,s) = e^{-s}\phi(r)$$

Methods of characteristics: Example 4

$$\frac{\partial u(x,t)}{\partial t} - e^{-x}\frac{\partial u(x,t)}{\partial x} = -u(x,t)$$

$$u(x,0) = \phi(x)$$

$$t = s$$

$$x = \ln(e^r - s)$$

$$u(r,s) = e^{-s}\phi(r)$$

$$x = \ln(e^r - t)$$

Writing *r* and *s* in terms of *x* and *t*

$$u(r,t) = e^{-t}\phi(r)$$

$$r = \ln(e^x + t)$$

$$u(x,t) = e^{-t}\phi(\ln[e^x + t])$$