Asynchronous lecture 4

Dynamics of moments:

- Diffusion/heat Equation
- Advection diffusion Equation
- Wave Equation
- Telegrapher’s equation

Can we understand the dynamics before solving the PDE?

Yes! if we look at the dynamics of the moments of $u(x,t)$

n-th moment of distribution $f(x)$ around c :

Only a function of t

$$\mu_n = \int_{-\infty}^{+\infty} dx\,(x-c)^n f(x)$$

probability distribution

zero-th moment μ_0 is the total probability

fist moment μ_1 is the mean of the distribution

second moment μ_2 is indicative of the width of the distribution

Moments

n-th moment of distribution $f(x)$ around c :

$$\mu_n = \int_{-\infty}^{+\infty} dx\,(x-c)^n f(x)$$

- The moment of a function usually assumes *c = 0*.

- *Second* and *higher* moments, *the central moment* (moments about the mean, with c being the mean) are usually considered rather than the moments about zero -> provides better information about the distribution's shape.

Dynamics of the second moment

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = \delta(x - x_0)$$

Exercise: Integration by parts

$$\frac{d\mu_2(t)}{dt} = \frac{d}{dt}\int_{-\infty}^{+\infty} dx\, x^2 u(x,t) = 2D$$

$$\mu_2(t) = \mu_2(0) + 2Dt = x_0^2 + 2Dt$$

Exercise: using dirac delta initial condition

Dynamics for the second moment

Exercise: calculate directly the time dependence of the second moment from the general solution of a point source (Dirac delta) initial condition, i.e. compute this integral:

$$\mu_2(t) = \int_{-\infty}^{+\infty} dx \, \frac{x^2 e^{-\frac{(x-x_0)^2}{4Dt}}}{\sqrt{4\pi Dt}}$$

$$\text{Hint:} \quad \int_{-\infty}^{+\infty} ds \, e^{-s^2} = \sqrt{\pi}$$

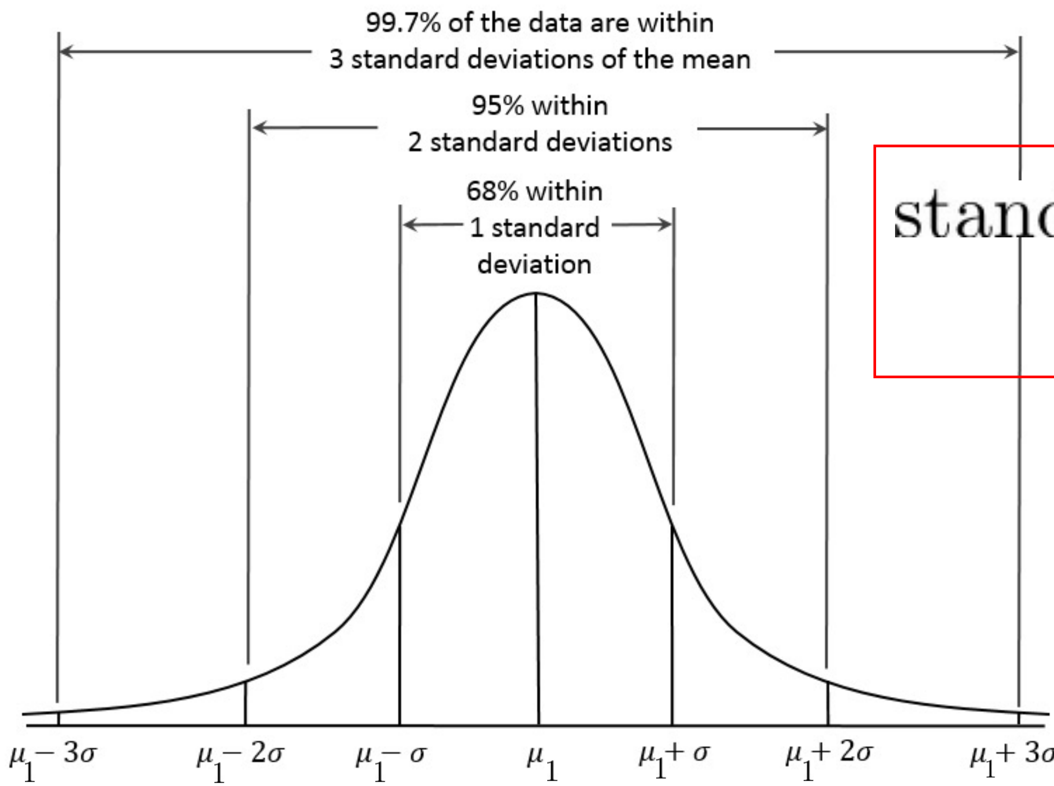
$$\text{Hint:} \quad \int_{-\infty}^{+\infty} ds \, s \, e^{-s^2} = 0$$

Odd function

Recall $x_0 = \mu_1$ the mean of the distribution

Gaussian distribution

$$\text{Gaussian function: } \frac{e^{-(x-\mu_1)^2/(2\mu_2)}}{\sqrt{2\pi\mu_2}} = \frac{e^{-(x-\mu_1)^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}}$$



standard deviation

$$\boxed{\sigma = \sqrt{\mu_2}}$$

Moments dynamics in convection-diffusion

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - v\frac{\partial u}{\partial x}$$

The moment dynamics

$$\frac{d\mu_2(t)}{dt} = 2D + 2v\mu_1(t)$$

Exercise: Integration by parts

$$\frac{d\mu_1(t)}{dt} = v$$

first

fist moment μ_1 is the mean of the distribution

second moment μ_2 is indicative of the width of the distribution

Moments dynamics in convection-diffusion

$$\frac{d\mu_2(t)}{dt} = 2D + 2v\mu_1(t)$$

$$\frac{d\mu_1(t)}{dt} = v \qquad \mu_1(0) = x_0$$

After integration, and using point source δ as initial condition:

$$\mu_1(t) = vt + x_0$$

$$\mu_2(t) = 2Dt + (vt + x_0)^2$$

Mean Square Displacement (msd) dynamics

$$\mu_1(t) = vt$$

$$\mu_2(t) = 2Dt + (vt + x_0)^2$$

How does the initial probability density broaden?

Let us look at the so-called *variance* or *mean square displacement* (msd)

$$\mathrm{msd}(t) = \int_{-\infty}^{+\infty} dx (x - \mu_1(t))^2 \frac{e^{-\frac{(x-vt-x_0)^2}{4Dt}}}{\sqrt{4\pi Dt}}$$

2nd moment around the mean = variance

Mean Square Displacement (msd) dynamics

$$\mathrm{msd}(t) = \int_{-\infty}^{+\infty} dx (x - \mu_1(t))^2 \frac{e^{-\frac{(x - vt - x_0)^2}{4Dt}}}{\sqrt{4\pi Dt}}$$

Exercise: integration of the variance above to get

$$\mathrm{msd}(t) = 2Dt$$

It is the same dynamics of the normal diffusion equation!

Not surprising as we subtracted the *drifting mean* ☺

Dynamics of the 2nd moment: wave Equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = v^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

$$\text{Initial conditions} \begin{cases} u(x,0) = f(x) \\ \dfrac{\partial u(x,0)}{\partial t} = h(x) \end{cases}$$

$$\text{Boundary conditions} \quad u(x \to \pm\infty, t) = 0$$

$$\frac{d^2 \mu_2(t)}{\partial t^2} = 2v^2$$

Exercise: Integration by parts

Dynamics of the 2nd moment: wave Equation

$$\frac{d^2\mu_2(t)}{dt^2} = 2v^2$$

$$\frac{d\mu_2(t)}{dt} = 2v^2 t + \frac{d\mu_2(0)}{dt}$$

$$\frac{d\mu_2(0)}{dt} = \int_{-\infty}^{+\infty} dx\, x^2 \frac{\partial u(x,0)}{\partial t} = \int_{-\infty}^{+\infty} dx\, x^2 h(x)$$

$$\mu_2(t) = v^2 t^2 + \left(\frac{d\mu_2(0)}{dt}\right) t + \mu_2(0)$$

Ballistic movement signature

Much faster than diffusion, quadratic in time!
+ depends on the initial conditions

Exercise: Calculate the dynamics of the 1st moment

The Telegrapher's equation: total probability *M(t)*

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \alpha \frac{\partial u(x,t)}{\partial t} = v^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

Intermediate between wave-like and diffusion-like effects: For short-times behaves like wave equation and longer times diffusion broadening

$$\lim_{\substack{v\to\infty \\ \alpha\to\infty}} \frac{v^2}{\alpha} = D \quad \to \frac{\partial u(x,t)}{\partial t} = D \frac{\partial^2 u(x,t)}{\partial x^2}$$

Diffusion equation

Total probability dynamics

$$M(t) = \int_{-\infty}^{+\infty} dx\, u(x,t)$$

Total *M(t)=M(0) is* conserved for diffusion Equation

$$\frac{\partial^2 M(t)}{\partial t^2} + \alpha \frac{\partial M(t)}{\partial t} = 0$$

The telegrapher's equation: total probability *M(t)*

$$M(t) = \int_{-\infty}^{+\infty} dx\, u(x,t)$$

$$\frac{\partial^2 M(t)}{\partial t^2} + \alpha \frac{\partial M(t)}{\partial t} = 0$$

Exercise: Show that the total moment is:

$$M(t) = M(0) + \dot{M}(0)\frac{1-e^{-\alpha t}}{\alpha}$$

$$M(0) = \int_{-\infty}^{+\infty} dx\, u(x,0) \qquad \dot{M}(0) = \int_{-\infty}^{+\infty} dx\, u_t(x,0)$$

Total probability dynamics depends on time and initial conditions!

Total probability $M(t)$ for diffusion and wave equations

$$M(t) = \int_{-\infty}^{+\infty} dx\, u(x,t) \qquad \dot{M}(0) = \int_{-\infty}^{+\infty} dx\, u_t(x,0)$$

$$\frac{\partial u(x,t)}{\partial t} = v^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad \rightarrow \quad \frac{\partial M(t)}{\partial t} = 0$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = v^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad \rightarrow \quad \frac{\partial^2 M(t)}{\partial t^2} = 0$$

diffusion equation: $M(t) = M(0)$

wave equation: $M(t) = \dot{M}(0)\, t + M(0)$