First-order ODEs

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`Introduction

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Introduction

Order := order of the highest derivative it contains

eg. $\frac{dy}{dx}$ is first ord, $\frac{d^2y}{dx^2}$ is second order.

Degree := power higest order derivative is raised, after being rationalised to only contain integer powers

eg. $$\frac{d^3y}{dx^3} + x\left(\frac{dy}{dx}\right)^{3/2} + x^2y = 0$$

is order: 3, degree: 2

General solution is most general function $y(x)$ that satisfies the ODE

↳ Contains constants of integration found from the BCs

↳ General Solution to n^{th} Order ODE will contain n constants of integration.

Particular solution is a general solution with BCs applied

General form of solution

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Given a group of functions that each are a solution

$$y = f(x, a_1, a_2, \ldots, a_n)$$

eg. $y = a_1 \sin x + a_2 \cos x$

A group of functions with n parameters satisfies an n'th order ODE is general

First-degree first-order equations

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Only contain $\frac{dy}{dx}$ as a function of x and y

Can be written in the form $\frac{dy}{dx} = F(x,y)$, $A(x,y)dx + B(x,y)dy = 0$

$$\hookrightarrow F(x,y) = -\frac{A(x,y)}{B(x,y)}$$

Separable-variable equations

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$$\frac{dy}{dx} = f(x)g(y)$$

$$\Rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx$$

Example:

$$\frac{dy}{dx} = x + xy$$

$$= (1+y)x$$

$$\Rightarrow \frac{1}{1+y}\,dy = x\,dx$$

$$\Rightarrow \int \frac{1}{1+y}\,dy = \int x\,dx$$

$$\Rightarrow \ln(y+1) = \frac{x^2}{2} + C$$

$$\Rightarrow y = e^{\left(\frac{x^2}{2}+C\right)} - 1$$

$$= Ae^{\frac{x^2}{2}} - 1$$

Exact equations

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Exact ODE is of the form:

$$A(x,y)dx + B(x,y)\,dy = 0 \ \& \ \frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$

Solution: $U(x,y) = \int A(x,y)\,dx + F(y)$

$F(y)$ is found by differentiating with respect to y and equating to $B(x,y)$

Example:

$$x\frac{dy}{dx} + 3x + y = 0$$

$$\Rightarrow \quad x\,dy + (3x+y)\,dx = 0$$

$$A(x,y) = 3x+y \qquad B(x,y) = x$$

$$\frac{\partial A}{\partial y} = 1 \qquad \frac{\partial B}{\partial x} = 1 \quad \Rightarrow \quad A_y = B_x$$

$$U(x,y) = \int 3x+y\,dx + F(y)$$

$$= \frac{3}{2}x^2 + xy + c_1 + F(y)$$

$$\frac{\partial U}{\partial y} = x + \frac{dF}{dy} = B(x,y) = x$$

$$\frac{dF}{dy} = 0 \quad \Rightarrow \quad F(y) = C_2$$

Solution : $\frac{3}{2}x^2 + xy + C = 0$

Inexact equations: Integrating factors

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$$A(x,y)\,dx + B(x,y)\,dy = 0 \quad \text{where} \quad \frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x}$$

is an inexact equation

Can be made exact by multiplying by $\mu(x,y)$

$$\frac{\partial(\mu A)}{\partial \mu} = \frac{\partial(\mu B)}{\partial x}$$

Assume μ is a function of x

then $\mu \frac{\partial A}{\partial y} = \mu \frac{\partial B}{\partial x} + B \frac{d\mu}{dx}$

$$\mu(x) = e^{\int f(x)\,dx} \qquad \mu(y) = e^{\int g(y)\,dy}$$

<u>Example:</u>

$$\frac{dy}{dx} = -\frac{2}{y} - \frac{3y}{2x}$$

$$= \frac{-4x - 3y^2}{2xy}$$

$$\Rightarrow \quad 2xy\,dy + (4x + 3y^2)\,dx = 0$$

$$\frac{\partial}{\partial y}\left(4x + 3y^2\right) = 6y$$

$\therefore$ not exact need an integrating factor

$$\frac{\partial}{\partial x}\left(2xy\right) = 2y$$

$$\frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) = \frac{2}{x}$$

$$\mu(x) = e^{2\int \frac{1}{x}dx} = e^{2\ln x} = x^2$$

$$(4x^3 + 3x^2y^2)\,dx + 2x^3y\,dy = 0$$

$$\Rightarrow 4x^3\,dx + (3x^2y^2\,dx + 2x^3y\,dy) = 0$$

$$\Rightarrow \quad x^4 + y^2x^3 = C$$

Linear equations

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Special case of in exact

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Multiply by integrating factor $\mu(x) = e^{\int P(x)\,dx}$

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x)$$

$$\Rightarrow \quad \frac{d}{dx}\left[\mu(x)y(x)\right] = \mu(x)Q(x) \qquad \text{(by product rule)}$$

$$\Rightarrow \quad \mu(x)y(x) = \int \mu(x)Q(x)\,dx$$

<u>Example:</u>

$$\frac{dy}{dx} + 2xy = 4x$$

$$\mu(x) = e^{\int 2x\,dx} = e^{x^2}$$

$$\Rightarrow \quad e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = 4xe^{x^2} \qquad \left(\text{multiply through by } e^{x^2}\right)$$

$$\Rightarrow \quad \frac{d}{dx}\left[e^{x^2}\right] \quad x^2$$

$$\Rightarrow \quad \frac{d}{dt}\left[e^{x^2} y\right] = 4x e^{x^2}$$

$$e^{x^2} y = \int 4x e^{x^2} dx$$

$$= 2e^{x^2} + C$$

$$y(x) = 2 + \frac{C}{e^{x^2}}$$

Homogeneous equations

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$$\frac{dy}{dx} = \frac{A(x,y)}{B(x,y)} = F\left(\frac{y}{x}\right)$$

Where A and B ~~is~~ homogeneous of degree n if, for any λ, it obeys $g(\lambda x, \lambda y) = \lambda^n g(x,y)$

Solve by substituting $y = vx$

Example:

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$$

Let $y = vx$

$$\Rightarrow \frac{d}{dx}(vx) = v + \tan v$$

$$\Rightarrow \cancel{v} + x\frac{dv}{dx} = \cancel{v} + \tan v$$

$$\int \frac{1}{x}\,dx = \int \cot(v)\,dv$$

$$\ln x = \ln|\sin v|$$

$$x = A\sin v$$

$$= \sin\left(\frac{y}{x}\right)$$

$$y = x\arcsin(Dx)$$

Sturm-Liouville problems

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$$\boxed{[r(x)y']' + [q(x) + \lambda p(x)]y = 0}$$

Sturm-Liouville equation

Example: Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$\Rightarrow [(1-x^2)y']' + \lambda y = 0$$

$$k_1 y(a) + k_2 y'(a) = 0$$

$$l_1 y(b) + l_2 y'(b) = 0$$

Sturm-Liouville Problem