Control Theory

Controllability (reachability) part 1

Robert Szalai

Department of Engineering Mathematics
University of Bristol

Controllability: definition (1)

Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

We know the solution

$$\boldsymbol{x}(t) = \exp(\boldsymbol{A}t)\boldsymbol{x}(0) + \int_0^t \exp(\boldsymbol{A}(t-\tau))\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau$$

We know the stability, but

- How does the system react to control input?
- Is it possible to choose $\boldsymbol{u} : [0, t_1] \to \mathbb{R}^m$ such that it brings $\underline{\boldsymbol{x}(0) = \boldsymbol{x}_0}$ to $\underline{\boldsymbol{x}(t_1) = \boldsymbol{x}_1}$ in a finite time $0 < t_1 < \infty$?

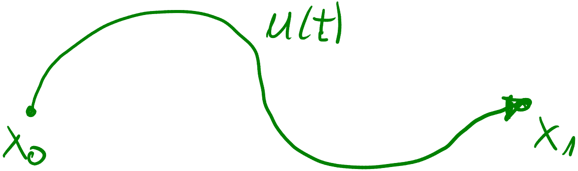
Controllability: definition (2)

Definition

The system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

with initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ is **controllable** (or **reachable**) if there exists a final time $0 < t_1 < \infty$ and a control input $\boldsymbol{u} : [0, t_1] \to \mathbb{R}^m$, such that for any $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \mathbb{R}^n$ we have $\boldsymbol{x}(t_1) = \boldsymbol{x}_1$.



(Counter) example: controllability

Consider the system $x^* = 0$

$a_1, a_2, a_3 > 0$

$x^* = 0$

$$\dot{\boldsymbol{x}} = \underbrace{\begin{pmatrix} -a_1 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & -a_3 \end{pmatrix}}_{A} \boldsymbol{x} + \underbrace{\begin{pmatrix} 1 \\ 1 \\ \cancel{0}\ 2 \end{pmatrix}}_{B} \boldsymbol{u}$$

0 means this not controllable

$$\dot{x}_3 = -a_3 x_3 + 2u$$

Controllability test

Theorem

The system

$$\dot{\mathbf{x}} = \boldsymbol{A}\mathbf{x} + \boldsymbol{B}\mathbf{u}$$

$x \in \mathbb{R}^n$

is **controllable** *(or* **reachable***) if and only if the matrix*

$$\mathbf{W_r} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{\mathbf{n}-\mathbf{1}}\boldsymbol{B} \end{bmatrix}$$

has **full rank**.

range $W_r = \mathbb{R}^n$

column vectors of W_r span $\mathbb{R}^n$

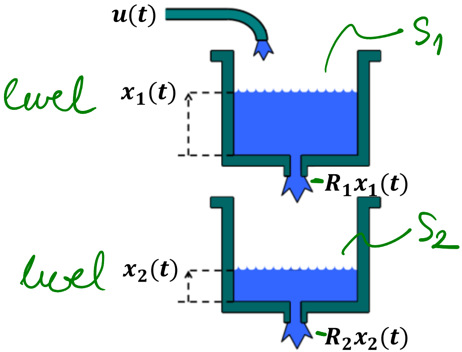
$\ker W_r^T = \{0\}$

Use Gauss eliminat

Notes:

1. Size of $\mathbf{W_r}$ is $n \times (n\,m)$
2. If $\boldsymbol{B}$ is a vector, i.e., $m = 1$, $\mathbf{W_r}$ is square, hence $\det(\mathbf{W_r}) \neq \mathbf{0}$ means controllability.

Example: controllability test (1)



$$\frac{d}{dt}(S_1 x_1) = u - R_1 x_1$$

$$\frac{d}{dt}(S_2 x_2) = R_1 x_1 - R_2 x_2$$

$$\underline{x} = (x_1, x_2)$$

$$\dot{\underline{x}} = \underbrace{\begin{pmatrix} -\frac{R_1}{S_1} & 0 \\ \frac{R_1}{S_2} & -\frac{R_2}{S_2} \end{pmatrix}}_{A} \underline{x} + \underbrace{\begin{pmatrix} \frac{1}{S_1} \\ 0 \end{pmatrix}}_{B} u$$

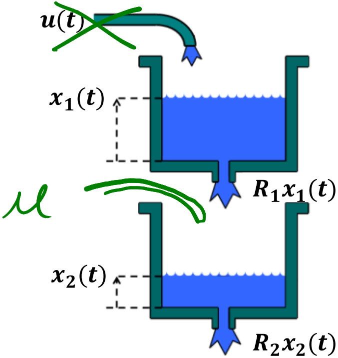
$$W_r = [B \; AB] = \begin{pmatrix} \frac{1}{S_1} & -\frac{R_1}{S_1^2} \\ 0 & \frac{R_1}{S_1 S_2} \end{pmatrix}$$

↑ Reachability matrix.

$$\det W_r = \frac{R_1}{S_1^2 S_2} \neq 0 \text{ if } R_1 \neq 0$$

Example: controllability test (2)

2



$$\dot{x} = \overbrace{\begin{pmatrix} -\frac{R_1}{S_1} & 0 \\ \frac{R_1}{S_2} & -\frac{R_2}{S_2} \end{pmatrix}}^{A} x + \underbrace{\begin{pmatrix} 0 \\ \frac{1}{S_2} \end{pmatrix}}_{B} u$$

$$W_r = [B \quad AB]$$

$$= \begin{pmatrix} 0 & 0 \\ \frac{1}{S_2} & -\frac{R_2}{S_2^2} \end{pmatrix}$$

$\operatorname{rank} W_r = 1$

$\operatorname{range} W_r = \operatorname{span} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

not controllable

The End