Exercise Sheet 3

1. $u_x + u_y = 1 - u$

$$dx = dy = \frac{du}{1-u}$$

$$\frac{dx}{dy} = 1 \quad \Rightarrow \quad y = x + A$$

$$\frac{du}{dy} = 1 - u \quad \Rightarrow -\ln|1-u| = y + c$$

$$1 - u = Be^{-y}$$

$$B = f(A)$$

$$u(x,y) = 1 - Be^{-y}$$

$$= 1 - f(A)e^{-y}$$

$$= 1 - f(y-x)e^{-y}$$

$$u(x, x+x^2) = 1 - f(x^2)e^{-x-x^2} = \sin x$$

$$\Rightarrow f(x^2) = (1 - \sin x)e^{x+x^2}$$

$$f(z) = (1 - \sin(\sqrt{z}))e^{\sqrt{z}+z}$$

$$u(x,y) = 1 - (1 - \sin(\sqrt{y-z}))e^{\sqrt{y-x}-x}$$

2. $u_x + 3t^{2/3} u_t = 2$

$\frac{dx}{ds} = 1$ ✓ $\qquad x(r,0) = r$

$x = s + c_1 \qquad x(r,0) = c_1 = r$

$\boxed{x(r,s) = s + r}$

$\frac{dt}{ds} = 3t^{2/3}$ ✓ $\qquad t(r,0) = 1$

$\int t^{-2/3}\, dt = 3\int ds$

$\Rightarrow 3t^{1/3} = 3s + c_2$

$t(r,s) = \left(s + \frac{c_2}{3}\right)^3$

$t(r,0) = \left(\frac{c_2}{3}\right)^3 = 1 \Rightarrow c_2 = 3$

$\boxed{t(r,s) = (s+1)^3}$

$$\frac{du}{ds} = 2 \qquad u(r,0) = 1 + r$$

$$u(r,s) = 2s + c_3$$

$$u(r,0) = \quad c_3 = 1 + r \Rightarrow c_3 = r + 1$$

$$u(r,s) = 2s + r + 1$$

$$s = t^{1/3} - 1 \qquad r = x - s = x - t^{1/3} + 1$$

$$u(x,t) = 2\left(t^{1/3} - 1\right) + x - t^{1/3} + 1 + 1$$

$$= t^{1/3} + x$$

$$dx = \frac{dt}{3t^{2/3}} = \frac{du}{2}$$

$$\int dx = \int \frac{dt}{3t^{2/3}} \Rightarrow \qquad x = t^{1/3} + A$$

$$\frac{du}{dx} = 2 \qquad u = 2x + B$$

$$B = g(A)$$

$$u = 2x + B = 2x + g(A)$$

$$= 2x + g\left(x - t^{1/3}\right)$$

$$u(x,1) = 2x + g(x - 1) = 1 + x$$

$$\Rightarrow \quad g(x-1) = 1 - x$$

$$x - 1 = z \quad \Rightarrow \quad x = z + 1$$

$$g(z) = -z$$

$$u(x,t) = 2x - x + t^{1/3}$$

$$= t^{1/3} + x$$

3. $t\,u_t + (t + u)\,u_x = x - t$

$$\frac{dt}{ds} = t \qquad\qquad u(r,0) = 1$$

$$t(r,s) = Ae^{s}$$

$$t(r,0) = A = 1 \qquad\qquad \boxed{t(r,s) = e^{s}}$$

$$\frac{dx}{ds} = t + u \qquad x(r,0) = r$$

$$= e^s + u$$

$$\frac{du}{ds} = x - t \qquad u(r,0) = 1 + r$$

$$= x - e^s$$

$$\frac{dx}{ds} + \frac{du}{ds} = u + x$$

Let $w = x + u$

$$\frac{dw}{ds} = w \qquad w(r,s) = Ae^s$$

$$\Rightarrow x(r,s) + u(r,s) = Ae^s$$

Let $s = 0$

$$r + 1 + r = A$$

$$x(r,s) + u(r,s) = (1+2r)e^s \qquad (1)$$

$$\frac{dx}{ds} - \frac{du}{ds} = 2e^s + u - x$$

$$= 2e^s - (x - u)$$

Let $v = x - u$

$$\frac{dv}{ds} = 2e^{s} - v$$

$$\frac{dv}{ds} + v = 2e^{s}$$

$V_H = Be^{-s}$ $\qquad$ $V_p = Ce^{s}$

$Ce^{s} + Ce^{s} = 2e^{s} \implies C = 1$

$v(r,s) = Be^{-s} + e^{s}$

$x(r,s) - u(r,s) = Be^{-s} + e^{s}$

Let $s = 1$

$r - (1+r) = B + 1$

$B = -2$

$x(r,s) - u(r,s) = -2e^{-s} + e^{s}$ $\quad (2)$

$(1) + (2)$:

$$2x(r,s) = (1+2r)e^{s} + e^{s} - 2e^{-s}$$

$$= 2(1+r)e^{s} - 2e^{-s}$$

$$x(r,s) = (1+r)e^s - e^{-s}$$

$$u(r,s) = (1+r)e^s - e^{-s} - (-2e^{-s} + e^s)$$

$$= re^s + e^{-s}$$

$$s = \ln(t) \implies x = (1+r)t - \frac{1}{t}$$

$$\frac{xt+1}{t^2} - 1 = r$$

$$u(x,t) = \left(\frac{xt+1}{t^2} - 1\right)t + \frac{1}{t}$$

$$= x + \frac{1}{t} - t + \frac{1}{t}$$

$$= x + \frac{2}{t} - t$$

4. $$dx = -dy = \frac{du}{(x-y)u}$$

Let $q = x+y$ $\quad p = x-y$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial}{\partial p}\frac{\partial p}{\partial x}$$

$$= \frac{\partial}{\partial q} + \frac{\partial}{\partial p}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial}{\partial p}\frac{\partial p}{\partial y}$$

$$= \frac{\partial}{\partial q} - \frac{\partial}{\partial p}$$

Rewrite PDE:

$$2\frac{\partial w}{\partial p} - pw = 0$$

$$\int \frac{2}{w}\,dw = \int p\,dp$$

$$2\ln w = \frac{p^2}{2} + \text{const}$$

$$w = g(q)e^{\frac{p^2}{4}}$$

$$u(x,y) = g(x+y)e^{\frac{(x-y)^2}{4}} = g(x+y)e^{\frac{(x+y)^2}{4} - xy}$$

$$= f(x+y)e^{-xy}$$

5. $$\frac{dx}{u} = dy = du$$

$$\int dx = \int u\, du \Rightarrow x = \frac{u^2}{2} + B$$

$$\int dy = \int du \qquad y = u + A$$

$$B = f(A)$$

$$x - \frac{u^2}{2} = f(y-u)$$

$$x = f(x)$$

$$x - \frac{u^2}{2} = y - u$$

$$u^2 - 2u + 2y - 2x$$

$$u = \frac{2 \pm \sqrt{4 - 8(y-x)}}{2}$$

$$u(x,y) = 1 - \sqrt{1 + 2x - 2y}$$

6. $\frac{dt}{ds} = 1 \qquad \frac{dx}{ds} = u x^2 t \qquad \frac{du}{ds} = 0$

$t(r,0) = 0 \qquad x(r,0) = r \qquad u(r,0) = f(r)$

$t = s \qquad u(r,s) = f(r)$

$$\frac{dx}{ds} = f(r) s x^2$$

$$\int \frac{1}{x^2} dx = f(r) \int s \, ds$$

$$-\frac{1}{x} = f(r) \frac{s^2}{2} + \text{Const}$$

ICs $\quad -\frac{1}{r} = c$

$$-\frac{1}{x} = f(r) \frac{s^2}{2} - \frac{1}{r}$$

$$1 = f(t) \frac{t^2}{2} r + \frac{r}{x}$$

$$r = \frac{1}{f(t)\frac{t^2}{2} + \frac{1}{x}} = \frac{2x}{u t^2 + 2}$$

$$u(x,t) = f\left(\frac{2x}{ut^2+2}\right)$$

7. $$xu_t - tu_x = u$$

$$\frac{dt}{ds} = x \qquad \frac{dx}{ds} = -t \qquad \frac{du}{ds} = u$$

$$t(r,0) = 0 \qquad x(r,0) = r \qquad u(r,0) = \phi(r)$$

$$\boxed{x(r,s) = r\cos s \qquad t(r,s) = r\sin s}$$

$$u(r,s) = g(r)e^s$$

$$u(r,0) = g(r) = \phi(r)$$

$$u(r,s) = \phi(r)e^s$$

$$x^2+t^2 = r^2\cos^2 s + r^2\sin^2 s$$

$$r = \sqrt{x^2+t^2}$$

$$\frac{t}{x} = \frac{r\sin s}{r\cos s} = \tan s \qquad s = \arctan\left(\frac{t}{x}\right)$$

$$u(x,t) = \phi\left(\sqrt{x^2+t^2}\right)\exp\left(\arctan(t/x)\right)$$

8. $u_x + 2u_y + (2x-y)u = 2x^2 + 3xy - 2y^2$
$= (2x-y)(x+2y)$

$x' = x + 2y$
$y' = 2x - y$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial x}$$

$$= \frac{\partial}{\partial x'} + 2\frac{\partial}{\partial y'}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial y}$$

$$= 2\frac{\partial}{\partial x'} - \frac{\partial}{\partial y'}$$

$$\left(\frac{\partial w}{\partial x'} + 2\frac{\partial w}{\partial y'}\right) + 2\left(2\frac{\partial w}{\partial x'} - \frac{\partial w}{\partial y'}\right) + y'w = x'y'$$

$$5\frac{\partial w}{\partial x'} = x'y' - y'w$$

$$\frac{\partial w}{\partial x'} + \frac{y'w}{5} = \frac{x'y'}{5}$$

$$e^{\frac{1}{5}\int y'\,dx} = e^{\frac{y'x'}{5}}$$

$$\frac{\partial}{\partial x}\left(w e^{\frac{y'x'}{5}}\right) = \frac{x'y'}{5} e^{\frac{y'x'}{5}}$$

$$\frac{x'y'}{5} \qquad e^{\frac{y'x'}{5}}$$

$$\frac{y'}{5} \qquad \frac{y'}{5} e^{\frac{y'x'}{5}} \quad +$$

$$\frac{y'^2}{25} e^{\frac{y'x'}{5}} \quad -$$

$$w e^{\frac{y'x'}{5}} = \frac{x'y'^2}{25} e^{\frac{y'x'}{5}} - \frac{y'^3}{125} e^{\frac{y'x'}{5}}$$

$$w = \frac{x'y'^2}{25} - \frac{y'^3}{125} = \left(x' - \frac{y'}{5}\right)\frac{y'^2}{25}$$

$$u(x,t) = \left(x + 2y - \frac{(2x-y)}{5}\right)\frac{(2x-y)^2}{25}$$