Asynchronous lecture 7

- Methods of characteristics: Boundary Value Problems I
- Conditions of validity for methods of characteristics
- Lagrange-Charpit equations

Methods of characteristics

Steps to find a solution of the 1st order PDE

Solve three simultaneous ODE

The solution is well defined if we can find $u(x,y)$ instead of $u(r,s)$

One needs to be able to change coordinates $(r, s) \to (x, y)$

unique

Variable change is single-valued if the Jacobian of the transformation is non-singular

$$\frac{\partial(x, y)}{\partial(r, s)} = \frac{\partial x}{\partial r}\frac{\partial y}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial y}{\partial r} \neq 0$$

singular

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix}$$

This requirement is a uniqueness criterion on Γ

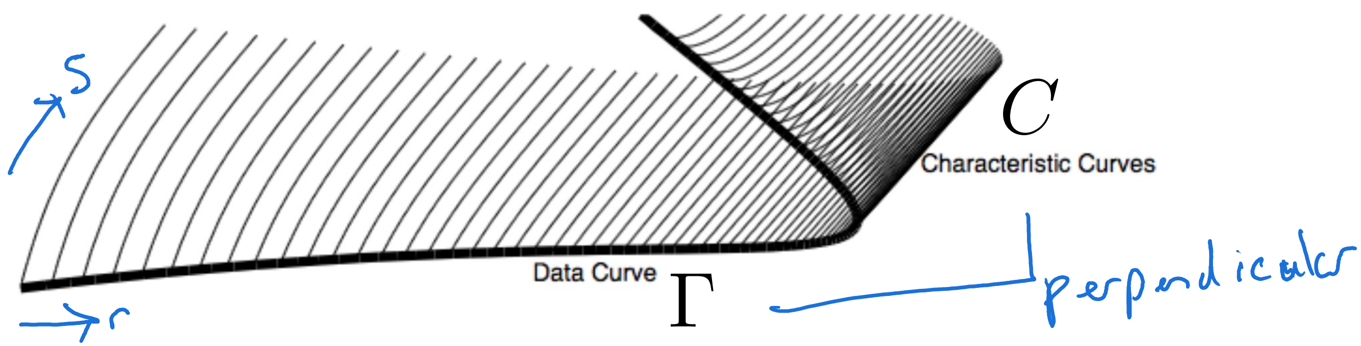
Methods of characteristics

single-valued and non-singular

$$\frac{\partial(x,y)}{\partial(r,s)} = \frac{\partial x}{\partial r}\frac{\partial y}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial y}{\partial r} \neq 0$$

This requirement is a uniqueness criterion on Γ

It is also equivalent to the fact that the characteristics and the initial data curve should be always transversal



One talks about information flowing along characteristics from the boundary

Methods of characteristics

$x(r,s)$ and $t(r,s)$ parametrise the surface C

The vector $\vec{x}(r,s) \times \vec{t}(r,s)$ is orthogonal to the surface C

if $\vec{x}(r,s) \times \vec{t}(r,s) \neq 0$

$$\frac{\partial(x,t)}{\partial(r,s)} = \vec{x}(r,s) \times \vec{t}(r,s)$$

$$\frac{\partial(x,t)}{\partial(r,s)} = \frac{\partial x}{\partial r}\frac{\partial t}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial t}{\partial r}$$

The Jacobian evaluated at $s=0$ as a function of r represents Γ

$\rightarrow \Gamma$ is orthogonal to C

Methods of characteristics

The Jacobian evaluated at $s = 0$ as a function of r represents Γ
$\rightarrow \Gamma$ is orthogonal to C

Non zero Jacobian

When that is not the case, the solution does not exist

or there are conditions of validity for the solution to exist

Methods of characteristics

Let us check if that is true on previous examples

$$\frac{\partial(x,t)}{\partial(r,s)} = \frac{\partial x}{\partial r}\frac{\partial t}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial t}{\partial r}$$

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0$$

$$t(r,s) = s$$

$$x(r,s) = vs + r$$

$$\frac{\partial(x,t)}{\partial(r,s)} = 1$$

Non zero Jacobian

↳ Unique solution

Methods of characteristics

Let us check if that is true on previous examples

$$\frac{\partial(x,t)}{\partial(r,s)} = \frac{\partial x}{\partial r}\frac{\partial t}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial t}{\partial r}$$

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0$$

$$t(r,s) = s$$

$$x(r,s) = r\,e^{s}$$

$$\frac{\partial(x,t)}{\partial(r,s)} = e^{s}$$

Methods of characteristics

Let us check if that is true on previous examples

$$\frac{\partial(x,t)}{\partial(r,s)} = \frac{\partial x}{\partial r}\frac{\partial t}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial t}{\partial r}$$

$$\frac{\partial u}{\partial t} + (x+t)\frac{\partial u}{\partial x} = t$$

$$t(r,s) = s$$

$$x(r,s) = e^s(r+1) - s - 1$$

$$\frac{\partial(x,t)}{\partial(r,s)} = e^s$$

Methods of characteristics

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial u} = c(x,y,u)$$

$$(a(x,y,u), b(x,y,u), c(x,y,u)) \cdot (u_x(x,y), u_y(x,y), -1) = 0$$

The curve $(x(s), y(s), u(s))$ (parametrised by s) that satisfies

$$\frac{dx}{ds} = a(x(s), y(s), u(s)) \qquad \frac{dy}{ds} = b(x(s), y(s), u(s)) \qquad \frac{du}{ds} = c(x(s), y(s), u(s))$$

always lies in the solution surface

To find the solution one proceeds in the same way as studied so far

Methods of characteristics

Another way of solving the PDE

$$(a(x,y,u),b(x,y,u),c(x,y,u))\cdot[u_x,u_y,-1]=0$$

The vector $\bar{A}=(a,b,c)$ is tangent to the solution surface (x,y,u)

The solution (or integral) surface being defined by $f(x,y,u)=u(x,y)-u=0$

The vector $\bar{A}$ determines a direction called the characteristic direction or Monge axis

$$\bar{A}\times d\bar{s}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a & b & c\\ dx & dy & du\end{vmatrix}=0$$

where $d\bar{s}(=dx\vec{i}+dy\vec{j}+du\vec{k})$ is an elemental length along $\bar{A}$

Methods of characteristics

The solution of the PDE can be expressed by the description of the tangent plane in terms of the slope of this surface

Expanding the determinant

$$(b\,du - c\,dy)\vec{i} + (c\,dx - a\,du)\vec{j} + (a\,dy - b\,dx)\vec{k} = 0$$

$$\frac{du}{dx} = \frac{c}{a} \qquad \frac{du}{dy} = \frac{c}{b}$$

equivalently

$$\frac{dx}{dy} = \frac{a}{b}$$

Methods of characteristics

which leads to the so-called Lagrange-Charpit equations

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)}$$

$$\frac{du}{dx} = \frac{c}{a} \qquad \frac{du}{dy} = \frac{c}{b}$$

$$\frac{dx}{dy} = \frac{a}{b}$$

The family of curves defined in (x,y,u) (dependent on u) by the above equation defines the characteristics of the PDE

Methods of characteristics

Solving two of the ODEs gives a two-parameter family of characteristic curves

In (x, y, u) space it can be expressed as

$$F(x, y, u, A, B) = 0 \quad \text{and} \quad G(x, y, u, A, B) = 0$$

with A and B arbitrary constants

specifying B as a function of A, $B = f(A)$

We can write a one-parameter family solution

$$F(x, y, u, A, f(A)) = 0 \quad \text{and} \quad G(x, y, u, A, f(A)) = 0$$

Methods of characteristics

$$F(x, y, u, A, f(A)) = 0 \quad \text{and} \quad G(x, y, u, A, f(A)) = 0$$

The equation of the surface is found implicitly or explicitly by eliminating A between these equations

General approach with Methods of Characteristics

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial u} = c(x,y,u)$$

First write down both the parametrised and the implicit form

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)}$$

$$\frac{dx}{ds} = a(x(s),y(s),u(s)) \qquad \frac{dy}{ds} = b(x(s),y(s),u(s)) \qquad \frac{du}{ds} = c(x(s),y(s),u(s))$$

Integrate what is most convenient