Asynchronous lecture 2

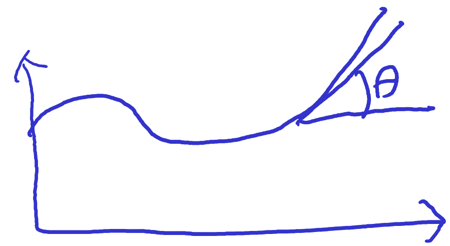
- Diffusion/Heat equation (finite domain)

Diffusion Equation

curvature

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

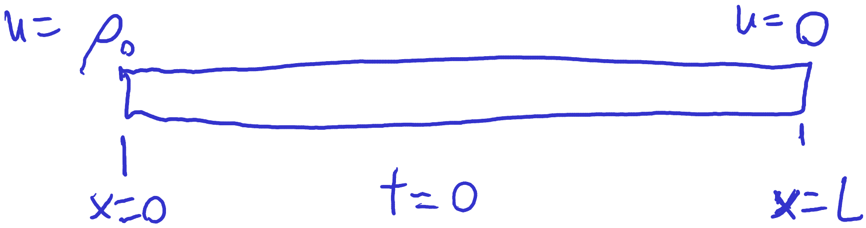
$u(x,t)$



- Probabilistic description of Brownian motion
- Special case of the Fokker-Planck equation
  - Converts a general stochastic differential equation into a PDE on probabilities
- Many uses in physics, finance, time-series analysis, . . .

↳ Temperature

$$\boxed{\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}}$$



$$u(0,0) = \rho_0$$

$$u(L,0) = 0$$

$$u(0,t) = \rho_0 e^{-2t}$$

Seperation of variables:

$$u(x,t) = X(x)T(t)$$

$$\partial_t \left(X(x)T(t)\right) = D\,\partial_{xx}\left(X(x)T(t)\right)$$

$$X \cdot \partial_t T = D\,T \cdot \partial_{xx} X$$

$$\underbrace{\frac{1}{D\,T(t)}\frac{dT}{dt}}_{f(t)} = \underbrace{\frac{1}{X(x)}\frac{d^2X}{dx^2}}_{g(x)} = -c^2$$

constant

$$\frac{1}{T}\frac{dT}{dt} = -c^2 \qquad \frac{dT}{dt} = -c^2\, T$$

$$\frac{D}{X}\frac{d^2X}{dx^2} = -c \qquad \frac{d^2X}{d^2x} = -\frac{c^2}{D}X = 0$$

$$T = T_0\, e^{-c^2 t}$$

$$\frac{d^2X}{dx^2} + \frac{c^2}{D}X = 0 \qquad \text{try } e^{mx}$$

$$\underbrace{m^2 + \frac{c^2}{D} = 0}_{\text{Characteristic equation.}} \Rightarrow m = \pm i\frac{c}{\sqrt{D}}$$

$$\boxed{\lambda \equiv \frac{c}{\sqrt{D}}}$$

$$\boxed{X(x) = A\cos(\lambda x) + B\sin(\lambda x)}$$

$$= \tilde{A}e^{i\lambda x} + \tilde{B}e^{-\lambda x}$$

General Solution:

$$u(x,t) = X(x)\,T(t)$$

$$= \left[A\cos(\lambda x) + B\sin(\lambda x) \right] \cdot e^{-c^2 t}$$

$\uparrow$ Time propergator

$$u(0,0) = \rho_0 = A$$

$$u(L,0) = 0$$

$$= \rho_0 \cos(\lambda L) + B\sin(\lambda L)$$

$$\Rightarrow B = -\rho_0 \cot(\lambda L)$$

$$u(x,t) = \frac{-\rho_0}{\sin \lambda L} \sin(\lambda(x-L))\, e^{-c^2 t}$$

$$u(0,t) = \rho_0 e^{-2t}$$

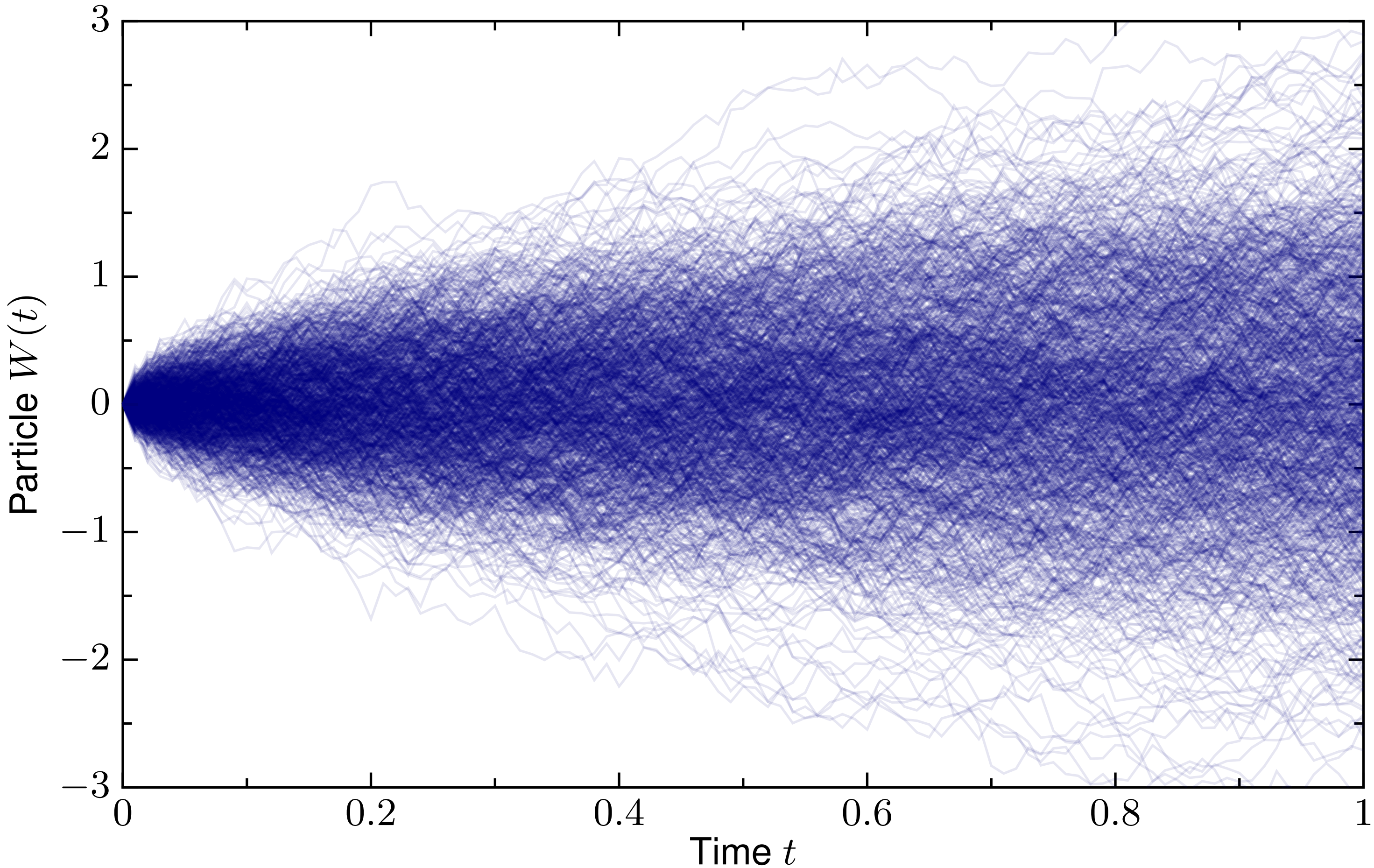
$$= \frac{\rho_0}{\sin(\lambda L)} \sin(\lambda L)\, e^{-c^2 t}$$

$$\Rightarrow c = \pm\sqrt{2}$$

$\sin(-x) = -\sin(x)$ so doesn't matter which branch

$$u(x,t) = \frac{-P_0}{\sin\left(\sqrt{\frac{2}{D}}L\right)} \sin\left[\sqrt{\frac{2}{D}}\,(x-L)\right] e^{-2t}$$

Brownian motion



Solving with finite boundaries

Consider the diffusion equation with finite boundaries and initial data