Control Theory

Observability

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Observability: why we need it?

- Controllability checks if the sytem can be brought into any state.
- We need to know the state to check if we reached the desired state!
- However we cannot always measure the full state (cost, technical difficulty, etc)
- Question: can we recover the state from what we can measure?
- If yes, the system is observable.

Observability: definition

Definition

The system

LTI

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$$

$x \in \mathbb{R}^n$
$u \in \mathbb{R}^m$
$y \in \mathbb{R}^p$

with initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ is **observable** if knowing $\boldsymbol{u}$ and $\boldsymbol{y}$ on some interval $[0, t_1]$, $t_1 < \infty$ is sufficient to determine $\boldsymbol{x}_0$.

Observability test

Theorem

The system

LTI

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$$

$x \in \mathbb{R}^n$
$u \in \mathbb{R}^m$
$y \in \mathbb{R}^p$

range $W_o^T = \mathbb{R}^n$
kernel $W_o = \{0\}$

is **observable** *if and only if the matrix*

Observability matrix

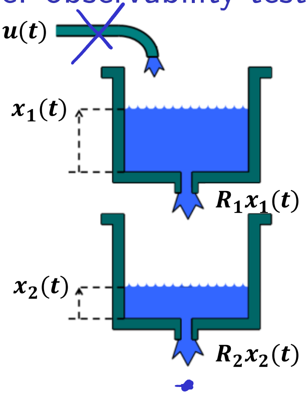
$$\boldsymbol{W}_o = \begin{pmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ \vdots \\ \boldsymbol{C}\boldsymbol{A}^{n-1} \end{pmatrix}$$

has **full rank**.

Notes:

1. Size of $\boldsymbol{W}_o$ is $(n\,p) \times n$
2. If $\boldsymbol{C}$ is a row vector, i.e., $p = 1$, $\boldsymbol{W}_o$ is square, hence $\det(\boldsymbol{W}_o) \neq 0$ means observability

Example: observability test (1)



$$\dot{x} = \begin{pmatrix} -\frac{R_1}{S_1} & 0 \\ \frac{R_1}{S_2} & \frac{R_2}{S_2} \end{pmatrix} x + \begin{pmatrix} \frac{1}{S_1} \\ 0 \end{pmatrix} u$$

① $y = R_1 x_1$

$C = (R_1 \quad 0)$

$$W_o = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} R_1 & 0 \\ -R_1^2 & 0 \end{pmatrix}$$

① not observable

② $y = R_2 x_2$

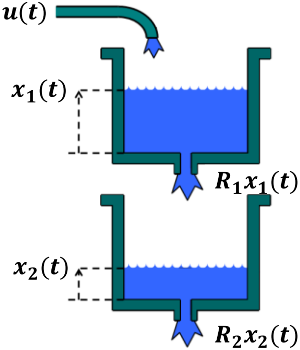
$C = (0 \quad R_2)$

$$W_o = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & R_2 \\ \frac{R_1 R_2}{S_2} & -\frac{R_2^2}{S_2} \end{pmatrix}$$

$$\det W_o = -\frac{R_1 R_2^2}{S_2}$$

$R_1 \neq 0, \; R_2 \neq 0$

Example: observability test (2)



Observability: interpretation (1)

Assume $\boldsymbol{A}$ is complete (semisimple) with eigenvalues

$$\lambda_1, \lambda_2 \dots, \lambda_n$$

and eigenvectors

$$\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$$

define

$$\boldsymbol{T} = [\boldsymbol{v}_1 \ \boldsymbol{v}_2 \ \cdots \ \boldsymbol{v}_n]$$

and

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = \boldsymbol{D} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} \quad \text{and} \quad \boldsymbol{E} = \overline{\boldsymbol{T}}^T \boldsymbol{C}^T = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

$$\overline{D}^T = \overline{D} = \overline{T}^T A^T (\overline{T}^T)^{-1}$$

Transform the matrix

$$(\boldsymbol{W}_o)^T = \left[\boldsymbol{C}^T \ \ \boldsymbol{A}^T\boldsymbol{C}^T \ \cdots \ (\boldsymbol{A}^{n-1})^T\boldsymbol{C}^T\right]$$

into

$$\overline{\boldsymbol{T}}^T(\boldsymbol{W}_o)^T = \overline{\boldsymbol{T}}^T\left[\boldsymbol{C}^T \ \ \boldsymbol{A}^T\boldsymbol{C}^T \ \cdots \ (\boldsymbol{A}^{n-1})^T\boldsymbol{C}^T\right]$$

$$= \Big[E \ \ \underbrace{\overline{T}^T A^T (\overline{T}^T)^{-1}}_{\overline{D}} \underbrace{\overline{T}^T}_{I} \underbrace{C^T}_{E} \ \cdots \ \underbrace{\overline{T}^T A^{T\,n-1} (\overline{T}^T)^{-1}}_{\overline{D}^{n-1}} \underbrace{\overline{T}^T C^T}_{E}\Big]$$

$$= \left[E \ \ \overline{D}E \ \ \cdots \ \ \overline{D}^{n-1}E\right]$$

$$= \underbrace{\begin{bmatrix} e_1 & & 0 \\ & \ddots & \\ 0 & & e_n \end{bmatrix}}_{e_i \neq 0 \ \forall i = 1, \dots n} \underbrace{\begin{bmatrix} 1 & \overline{\lambda}_1 & \cdots & \overline{\lambda}_1^{n-1} \\ \vdots & \overline{\lambda}_2 & & \overline{\lambda}_2^{n-1} \\ 1 & \overline{\lambda}_n & \cdots & \overline{\lambda}_n^{n-1} \end{bmatrix}}$$

$\lambda_i \neq \lambda_j$

$j \neq i$

$\overline{\lambda}_i \neq \overline{\lambda}_j$

Observability: interpretation (2)

Observability: interpretation (3)

Theorem

The system

$$\text{LTI} \quad \begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} \end{cases} \quad \begin{matrix} x \in \mathbb{R}^n \\ u \in \mathbb{R}^n \\ y \in \mathbb{R}^p \end{matrix}$$

is **observable** *if*

1. $\boldsymbol{C}$ *is a row vector*
2. $\boldsymbol{A}$ *is complete (semisimple) with with eigenvalues* $\lambda_1, \lambda_2 \ldots, \lambda_n$, *eigenvectors* $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ *and transformation matrix*

$$\underline{\boldsymbol{T} = [\boldsymbol{v}_1 \ \boldsymbol{v}_2 \ \cdots \ \boldsymbol{v}_n].}$$

3. $\lambda_i \neq \lambda_j$ *for all* $i \neq j$
4. *The vector* $\underline{\boldsymbol{E} = \overline{\boldsymbol{T}}^T \boldsymbol{C}^T}$ *has only non-zero entries*

Example: interpretation of observability

Consider

$$\dot{\boldsymbol{x}} = \overbrace{\begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}}^{A} \boldsymbol{x} + \begin{pmatrix} a \\ b \end{pmatrix} \boldsymbol{u}$$

$$\boldsymbol{y} = \underbrace{(c \;\; d)}_{C} \cdot \boldsymbol{x}$$

$\lambda_1 = -2$ ✓

$\lambda_2 = -1$

For which values of c and d is this observable?

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \bar{T}$

$\bar{T}^T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $\bar{T}^T C^T = \begin{pmatrix} c \\ c+d \end{pmatrix} = E$

$c \neq 0, \quad c+d \neq 0$

The End