PDEs Exersizes Sheet 2

1. $u_t + x u_x = 3x - u$

$\frac{dx}{ds} = x$ ✓ $\qquad x(r,0) = r$

$\frac{dt}{ds} = 1$ ✓ $\qquad t(r,0) = 0$

$\frac{du}{ds} = 3x - u$ ✓ $\qquad u(r,0) = \arctan(r)$

$t(r,s) = s + c_1 \qquad t(0) = c_1 = 0 \Rightarrow \boxed{t(r,s) = s}$ ✓

$\int \frac{1}{x}\,dx = \int ds \Rightarrow x(r,s) = A(r)e^s$

$x(r,0) = A(r) = r$

$\boxed{x(r,s) = re^s}$ ✓

$\frac{du}{ds} + u = 3re^s$ ✓

$u = e^{\lambda s} \Rightarrow \lambda + 1 = 0 \Rightarrow \lambda = -1$

$u(r,s) = Ae^{-s} + Be^s$

To find B: $\quad B\cancel{e^s} + B\cancel{e^s} = 3r\cancel{e^s} \Rightarrow B = \frac{3}{2}r$

$u(r,s) = Ae^{-s} + \frac{3}{2}re^s$

$$u(r,0) = A + \frac{3}{2}r = \arctan(r)$$

$$\Rightarrow A = \arctan(r) - \frac{3}{2}r$$

$$\boxed{u(r,s) = \left(\arctan(r) - \frac{3}{2}r\right)e^{-s} + \frac{3}{2}re^{s}}$$ ✓

From $t = s \Rightarrow x = re^{t} \Rightarrow r = xe^{-t}$

$$x(x,t) = \left(\arctan(xe^{-t}) - \frac{3xe^{-t}}{2}\right)e^{-t} + \frac{3}{2}x$$ ✓

Exercise 2:

$$u_t + 2xtu_x = u \qquad u(x,0) = x$$

$$\frac{dx}{ds} = 2xs \checkmark \qquad x(r,0) = r$$

$$\frac{dt}{ds} = 1 \checkmark \qquad t(r,0) = 0$$

$$\frac{du}{ds} = u \checkmark \qquad u(r,0) = \sin(x)$$

$t(r,s) = s + c$

$t(r,0) = c = 0 \Rightarrow \boxed{t(r,s) = s}$ ✓

$\int \frac{1}{x} dx = \int 2s\,ds$

$\Rightarrow \ln x = s^2 + c$

$x(r,s) = Ae^{s^2}$

$x(r,0) = A = r$

$\Rightarrow x(r,s) = re^{s^2}$ ✓

$u(r,s) = Ae^s$

$u(r,0) = A = r$

$\boxed{u(r,s) = re^s}$

From $s = t \Rightarrow x = re^{t^2} \Rightarrow r = xe^{-t^2}$

$u(x,t) = xe^{-t^2}e^t = xe^{t-t^2}$ ✓

Exercise 3

$$a u_x + b u_y = -cu$$

$$u(x,0) = \sin(x)$$

$\dfrac{dx}{ds} = a$ $\qquad x(r,0) = r$

$\dfrac{dy}{ds} = b$ $\qquad y(r,0) = 0$

$\dfrac{du}{ds} = -cu$ $\qquad u(r,0) = \sin(r)$

$$x(r,s) = as + c$$
$$x(r,0) = c = r$$
$$\Rightarrow \boxed{x(r,s) = as + r}$$

$$y(r,s) = bs + c$$
$$y(r,0) = c = 0 \qquad \Rightarrow \boxed{y(r,s) = bs}$$

$$u(r,s) = Ae^{-cs}$$
$$u(r,0) = A = \sin(r)$$
$$u(r,s) = \sin(r)e^{-cs}$$

Using $s = \frac{y}{b} \Rightarrow x = \frac{ay}{b} + r$

$\Rightarrow r = x - \frac{ay}{b}$

$u(x,y) = \sin\left(x - \frac{ay}{b}\right) e^{-\frac{cy}{b}}$ ✓

Exercise 4

$x u_x - y u_y = u$

$u(x,x) = x^2$

$\frac{dx}{ds} = x$ ✓

$\frac{dy}{ds} = -y$ ✓

$\frac{du}{ds} = u$ ✓ $\qquad u(r,r) = r^2$

$x = c_1(r) e^s \qquad y = c_2(r) e^{-s} \qquad u = c_3(r) e^s$

$xy = g(r) \Rightarrow r = g^{-1}(xy)$

$e^s = \frac{x}{c_1}$ or $e^s = \frac{y}{c_2}$

$$\Rightarrow u(x,y) = \frac{c_3(g^{-1}(xy))\,x}{c_1(g^{-1}(xy))} = x f(xy)$$

$$\text{or} \quad u(x,y) = \frac{c_3(g^{-1}(xy))\,y}{c_2(g^{-1}(xy))} = y\,h(xy)$$

$$u(x,x) = x f(x^2) = x^2$$

$$\Rightarrow f(x^2) = x$$

$$\Rightarrow f(x^2) = \sqrt{x^2}$$

$$u(x,y) = x\sqrt{xy}$$