Phase conditions

Solving periodic boundary value problems

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Limit cycle oscillations

A common use of numerical shooting is for finding limit cycle oscillations (periodic orbits)

For example, periodic neural spikes in the Morris-Lecar model

- Also: oscillations in predator-prey models, vibrations in mechanical structures, flutter oscillations in aero-elastic systems, ...

A limit cycle oscillation is an orbit that returns to its starting state after a certain period of time

- (It should also be isolated, i.e., there should be no other periodic orbits infinitesimally close.)

Periodic boundary value problems

A limit cycle oscillation can be found using a periodic boundary value problem

$$\frac{du_1}{dt} = g_1(u_1, u_2), \qquad \frac{du_2}{dt} = g_2(u_1, u_2)$$

with boundary conditions

$$u_1(0) = u_1(T), \qquad u_2(0) = u_2(T)$$

T is the period of the limit cycle

feature of the dynamics

- For an autonomous system (no explicit time dependency) the period is usually unknown!

Now solving for unknowns $[u_1(0), u_2(0), T]$

solving for the unknown period

Root finding problem

As before, boundary conditions give the root finding problem

$$f(u,T) = \begin{bmatrix} u_1(0) - u_1(T) \\ u_2(0) - u_2(T) \end{bmatrix} = 0$$

Problem is that there are two equations but three unknowns!

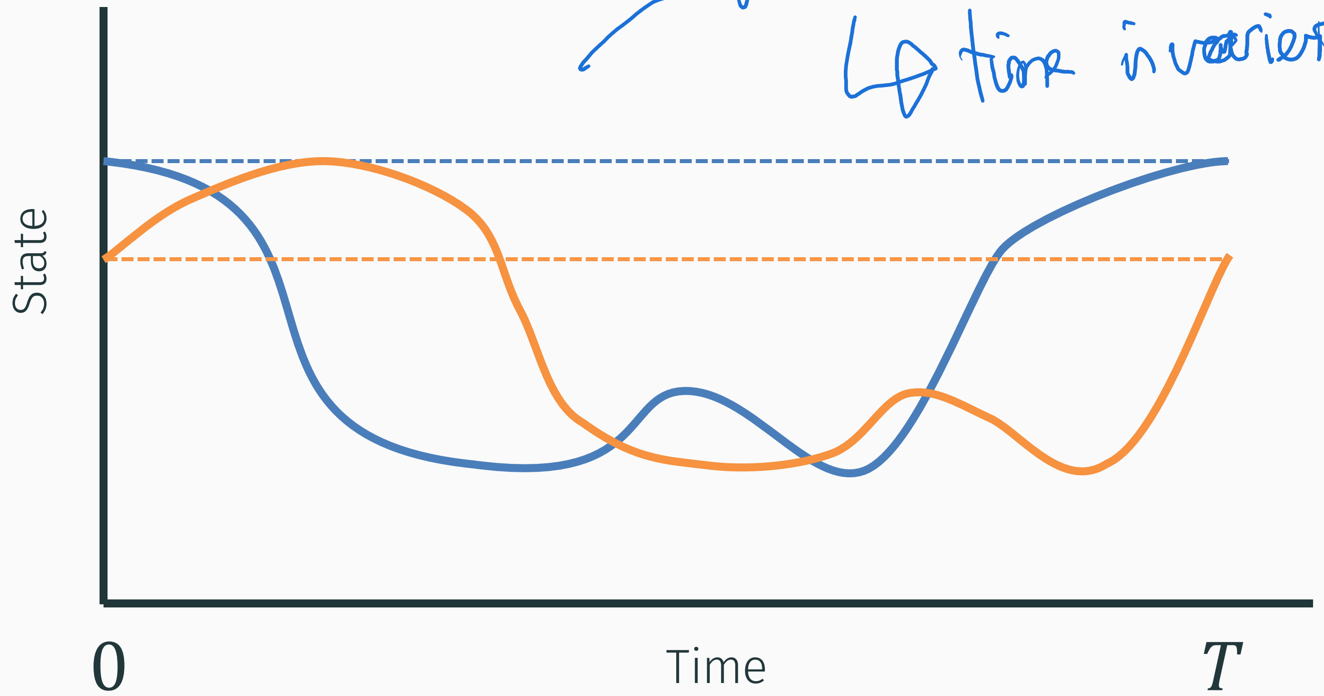
- Generally need an extra equation

What should the extra equation be?

- Consider the family of solutions generated by f

Phase conditions

Still the same limit cycle
↳ time invariant



Phase conditions — fix a variable

Need an equation to fix the translational invariance

Could fix value of $u_1(0) = 1$

$$f(u, T) = \begin{bmatrix} u_1(0) - u_1(T) \\ u_2(0) - u_2(T) \\ u_1(0) - 1 \end{bmatrix} = 0$$

Will work provided u_1 actually passes through that value *but not very robust*

↳ Solution may not pass through 1

Phase conditions — fix a derivative

Could fix value of $\mathrm{d}u_1/dt(0) = 0$

$$f(u,T) = \begin{bmatrix} u_1(0) - u_1(T) \\ u_2(0) - u_2(T) \\ du_1/dt(0) \end{bmatrix} = 0$$

Take the value of $du_1/dt(0)$ from the original ODE

- Should work for almost all limit cycle oscillations
- Will fail if there are inflection points in the limit cycle

Best is to add an integral condition → guaranteed to work

- Harder to implement; not considered in this course

Summary

- Numerical shooting for periodic boundary value problems requires a phase condition
- Phase condition must be incorporated into the root finding problem
  - Don't try to fix the period directly
    - Might give approximately correct results
    - But equally might be completely wrong
- Phase condition can be trivial but the trivial ones can fail at times