Asynchronous lecture 5

- Methods of characteristics: initial value problems I

Methods of characteristics

The method of characteristics is a technique for solving partial differential equations

It applies (mostly) to *first-order PDE* but also to any *hyperbolic PDE*

The method reduces a PDE into a family of ODEs along which the solution can be integrated

Methods of characteristics

First-order linear PDE

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y)$$

Let $u = u(x,y)$ be the solution of the equation

Consider the surface defined implicitly by the equation

$$f(x,y,u) \equiv u(x,y) - u = 0$$

At any point (x,y,u) of the surface $f(x,y,u) = 0$

the gradient vector is $\nabla f = (f_x, f_y, f_u) = (u_x, u_y, -1)$

Normal vector to the surface

Methods of characteristics

First-order linear PDE

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y)$$

Rewrite the equation as

tangent to the surface

$$(a(x,y), b(x,y), c(x,y)) \cdot \overbrace{(u_x(x,y), u_y(x,y), -1)}^{\nabla f} = 0$$

Normal vector to the surface

The vector $((a(x,y), b(x,y), c(x,y))$ is perpendicular to the gradient ∇f

$((a(x,y), b(x,y), c(x,y))$ lies in the tangent plane to the surface $f = 0$

Methods of characteristics

At each point (x, y, u) on the surface $f = 0$
the vector $(a(x,y), b(x,y), c(x,y))$ lies in the tangent plane

tangent to the surface

How do we construct the surface $f = 0$?

Construct a curve C parametrized by s
such that at each point on the curve C

the vector $(a(x(s),y(s)), b(x(s),y(s)), c(x(s),y(s)))$

tangent to the surface

is tangent to the curve

Methods of characteristics

The curve $C = \{(x(s), y(s), u(s)\}$ satisfies the following ODEs s

$$\frac{dx}{ds} = a(x(s), y(s))$$

$$\frac{dy}{ds} = b(x(s), y(s))$$

$$\frac{du}{ds} = c(x(s), y(s))$$

The curve C is called an integral curve for the vector field $(a(x,y), b(x,y), c(x,y))$

Methods of characteristics

These integral curves are known as the characteristic curves of the PDE

$$\frac{dx}{ds} = a(x(s), y(s))$$

$$\frac{dy}{ds} = b(x(s), y(s))$$

$$\frac{du}{ds} = c(x(s), y(s))$$

We have reduced a partial differential equation to a family of ordinary differential equations

Methods of characteristics: example 1

First-order linear PDE

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0$$

We know the solution is

$$u(x,0) = f(x)$$

$$u(x,t) = f(x - vt)$$

Introduce the characteristic equations

$$\frac{dx}{ds} = a(x(s), t(s)) \qquad \frac{dt}{ds} = b(x(s), t(s)) \qquad \frac{du}{ds} = c(x(s), t(s))$$

$$\frac{dx}{ds} = v \qquad \frac{dt}{ds} = 1 \qquad \frac{du}{ds} = 0$$

Methods of characteristics: example 1

Other way to see this

We want to transform the PDE into an ODE along the curves for which

$$\frac{d}{ds}u(x(s),t(s)) = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds}$$

$$\frac{d}{ds}u = v\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

Along the characteristics $u_s = 0$

Methods of characteristics: example 1

Solving the system of equations

$$\frac{dx}{ds} = v \qquad \frac{dt}{ds} = 1 \qquad \frac{du}{ds} = 0$$

$$x(s) = vs + c_1 \qquad t(s) = s + c_2 \qquad u(s) = c_3$$

Eliminating the parameter *s* the characteristic curves are lines given by

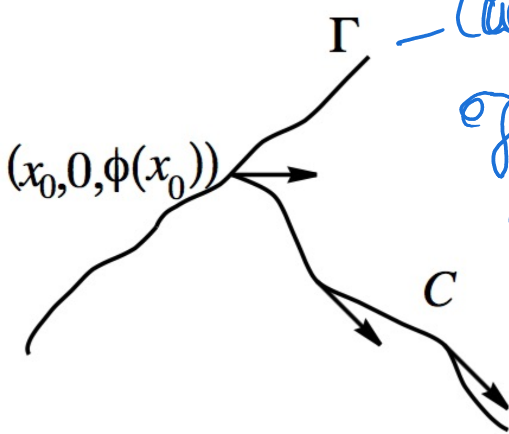
$$x - vt = \text{const.} \qquad u = c_3$$

Methods of characteristics: example 1

Accounting for the initial conditions

$$\frac{dx}{ds} = v \qquad \frac{dt}{ds} = 1 \qquad \frac{du}{ds} = 0$$

$$x(0) = x_0 \qquad t(0) = 0 \qquad u(0) = \phi(x_0) \; / \, f(x)$$



Curve of initial conditions

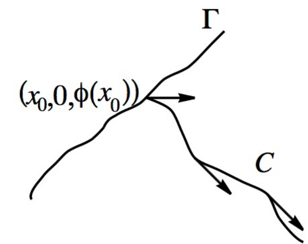
Finding an integral surface for the vector field

$$(v, 1, 0)$$

Family of initial conditions x_0 define several curves C leaving from Γ

Methods of characteristics: example 1

Parametrize the curve on Γ with *r*



Family of initial conditions x_0 parametrized by *r*

$$\frac{dx}{ds}(r,s) = v \qquad \frac{dt}{ds}(r,s) = 1 \qquad \frac{du}{ds}(r,s) = 0$$

$$x(r,0) = r \qquad t(r,0) = 0 \qquad u(r,0) = \phi(r)$$

$$x(r,s) = vs + c_1(r) \qquad x(r,0) = c_1(r) = r$$

$$t(r,s) = s + c_2(r) \qquad t(r,0) = c_2(r) = 0$$

$$u(r,s) = c_3(r) \qquad u(r,0) = c_3(r) = \phi(r)$$

Methods of characteristics: example 1

$$x(r,s) = vs + c_1(r) \qquad x(r,0) = c_1(r) = r$$
$$t(r,s) = s + c_2(r) \qquad t(r,0) = c_2(r) = 0$$
$$u(r,s) = c_3(r) \qquad u(r,0) = c_3(r) = \phi(r)$$

The general solution accounting for the initial condition is

$$x(r,s) = vs + r$$
$$t(r,s) = s$$
$$u(r,s) = \phi(r)$$

Methods of characteristics: example 1

$$x(r,s) = vs + r$$

$$t(r,s) = s$$

$$u(r,s) = \phi(r)$$

Solve for *r* and *s* in terms of *x* and *t*

$$r(x,t) = x - vt$$

$$s(x,t) = t$$

$$u(x,t) = u(r(x,t), s(x,t)) = \phi(x - vt)$$

Methods of characteristics: summary

We can write a 1st order linear PDE as $[a, b, c] \cdot \left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, -1\right] = 0$

Coefficients $\quad \nabla g$

The vector $\left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, -1\right]$ is normal to the solution surface $(x, y, u(x, y))$

Thus the vector $[a, b, c]$ is tangent to the solution surface C at every point

The curve $(x(s), y(s), u(s))$ (parametrised by s) that satisfies

$$\frac{dx}{ds} = a(x(s), y(s)) \quad \frac{dy}{ds} = b(x(s), y(s)) \quad \frac{du}{ds} = c(x(s), y(s)) \qquad (1)$$

always lies in the solution surface

Methods of characteristics: summary

$$\boxed{\frac{dx}{ds} = a(x(s), y(s)) \quad \frac{dy}{ds} = b(x(s), y(s)) \quad \frac{du}{ds} = c(x(s), y(s))} \qquad (1)$$

The solution surface C also passes through the curve of initial data Γ

$$\text{if } \boxed{\text{at } t = 0 \; x = x_0(s), \;\; y = y_0(s), \;\; u = u_0(s)} \qquad (2)$$

Curves that solve (1) and (2) are called characteristics
(and solve the PDE parametrically)

Their projections into the (x, y) plane are called characteristic projections

$$\frac{du}{ds} = c(x(s), y(s))$$

is called the compatibility condition for $u(x, y)$ along the characteristics