Control Theory

Stability: basic concepts

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The empirical concept of stability

Stability **of what**?

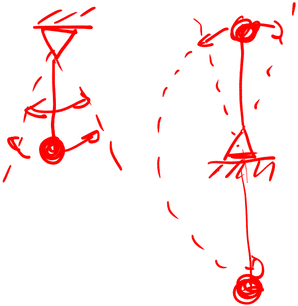
- A **steady state** of the system
- an equilibrium, periodic orbit, quasi-periodic orbit
- Something with recurrence

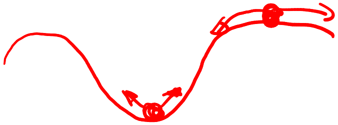
Equilibria

The **definition** should say

- the system remains **near the steady state** for a range of initial conditions
- the system **converges** to the steady state

(See Nonlinear Dynamics for robustness = structural stability) NDC





Stability of an equilibrium

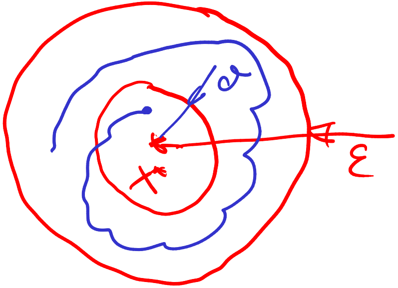
Three definitions

- Stable
- Assymptotically stable
- Unstable

Consider an ODE with an equilibrium $\boldsymbol{x}^\star$

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{f}(\boldsymbol{x}^\star) = \boldsymbol{0}$$

derivatives vanish



Definition

If for all $\epsilon > 0$ there exists $\delta > 0$ such that
if

$$|\boldsymbol{x}_0 - \boldsymbol{x}^\star| < \delta \implies |\boldsymbol{x}(t) - \boldsymbol{x}^\star| < \varepsilon, \; t \geq 0$$

then $\boldsymbol{x}^\star$ is stable.

Stability of an equilibrium: example

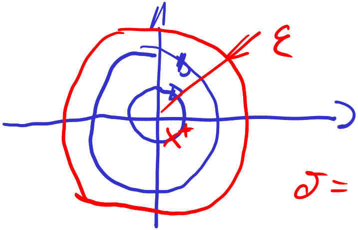
Consider

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$

$x_1 = r\sin(t + \theta_0)$

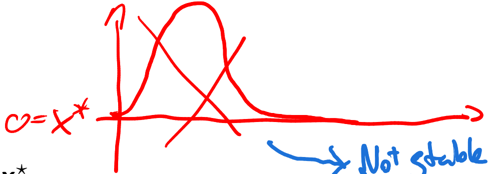
$x_2 = r\cos(t + \theta_0)$

phase shift



$\delta = \varepsilon$

Assymptotic stability



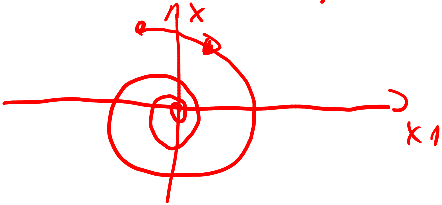
Not stable

Consider an ODE with an equilibrium $\boldsymbol{x}^\star$

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{f}(\boldsymbol{x}^\star) = \boldsymbol{0}$$

$$\dot{x}_1 = x_2 - x_1$$
$$\dot{x}_2 = -x_1 - x_2$$

$$x_1 = r e^{-t} \sin(t + \theta_0)$$
$$x_2 = r e^{-t} \cos(t + \theta_0)$$



Definition

The equilibrium $\boldsymbol{x}^\star$ is **assymptotically stable** if a) it is stable and b)

$$\lim_{t \to \infty} |\boldsymbol{x}(t) - \boldsymbol{x}^\star| = 0$$

Solution converges to equilibrium.

Hartman-Grobman

Consider an ODE with an equilibrium $\boldsymbol{x}^\star$

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{f}(\boldsymbol{x}^\star) = \boldsymbol{0}$$

Let $\boldsymbol{A} = D\boldsymbol{f}(\boldsymbol{x}^\star)$ and λ_i the eigenvalues of $\boldsymbol{A}$.

Jacobian

Theorem

If $\Re\lambda_i \neq 0$, then there exists a homemorphism $\boldsymbol{h} : X \to X$ such that the solutions of

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), & \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y}, & \boldsymbol{y}(0) = \boldsymbol{h}(\boldsymbol{x}_0) \end{cases}$$

are connected by $\boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t))$, $t \in \mathbb{R}$.

homeomorphism