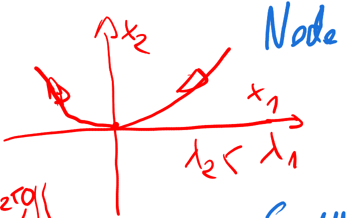
Linear systems

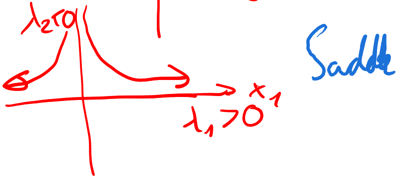
Consider $\boldsymbol{A} \in \mathcal{L}(X)$ with eigenvalues λ_i and the ODE

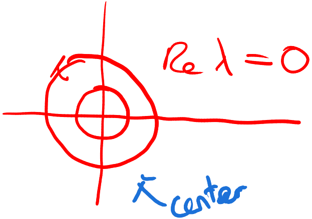
$$\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y}$$

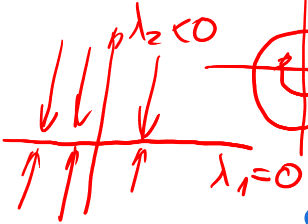
The solution is $\boldsymbol{y}(t) = \exp(\boldsymbol{A}t)\boldsymbol{y}_0$, hence $\boldsymbol{y}^\star = \boldsymbol{0}$ is

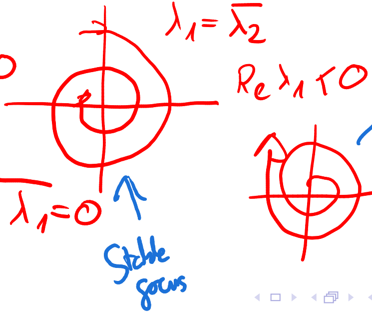
- assyptotically stable if $\Re\lambda_i < 0,\ \forall i$
- stable if $\Re\lambda_i \leq 0,\ \forall i$ (Bounded by a multiple of ICs)
- unstable if $\exists i$ such that $\Re\lambda_i > 0$

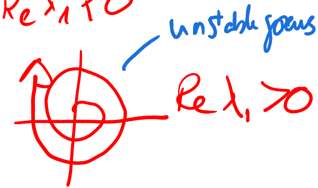












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Form Hartman-Grobman:
Consider $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ and define $\boldsymbol{A} = D\boldsymbol{f}(\boldsymbol{x}^\star)$ and λ_i the eigenvalues of $\boldsymbol{A}$. The equilibrium $\boldsymbol{x}^\star$ is

No equivilance between linear and nonlinear system.

- assymptotically stable if $\Re\lambda_i < 0, \forall i$
- **cannot tell stability** if $\exists i$ s.t. $\Re\lambda_i = 0$ → Lyapunov's direct method
- unstable if $\exists i$ such that $\Re\lambda_i > 0$

BIBO stability

Bounded Input – Bounded Output (BIBO) stability

Consider

$$\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{B}\boldsymbol{u}$$

$\boxed{\|u(t)\| < b}$ $t \in \mathbb{R}$ $\quad \|y(t)\| < c$ $t \in \mathbb{R}$

input is bounded by b

Theorem

If the equilibrium $\boldsymbol{x}^\star = \boldsymbol{0}$ *is assyptotically stable, then our system is BIBO stable.*

Note this applies for nonlinear systems, but only when the bound on $\boldsymbol{u}$ is sufficiently small.

let $\omega = \max_i \mathrm{Re}\,\lambda_i$

Proof.

bounded by largest eigenvalue

Use

$$\le e^{\omega(t-\tau)} \|B\| b$$

$$\le \int_0^t e^{\omega(t-\tau)} \|B\| b \, d\tau = \frac{e^{t\omega} - 1}{\omega} \|B\| b$$

Variation of constants

$$\boldsymbol{x}(t) = \underbrace{\exp(\boldsymbol{A}t)\boldsymbol{x}_0}_{t \to \infty \ \to 0} + \int_0^t \exp(\boldsymbol{A}(t-\tau))\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau$$

and $\int_0^t e^{\omega(t-\tau)}\mathrm{d}\tau = \frac{e^{t\omega}-1}{\omega}$. $\lim_{t \to \infty} = \frac{-1}{\omega}$ □

$$\lim_{t\to\infty} \to -\frac{\|B\| b}{\omega} > 0$$

The End