Control Theory

Controllability (reachability) part 2

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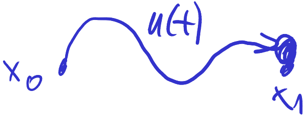
Controllability definition and test

Definition

The system

LTI $$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

with initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ is **controllable** (or **reachable**) if there exists a final time $0 < t_1 < \infty$ and a control input $\boldsymbol{u} : [0, t_1] \to \mathbb{R}^m$, such that for any $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \mathbb{R}^n$ we have $\boldsymbol{x}(t_1) = \boldsymbol{x}_1$.



Theorem

The system

LTI $$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

is **controllable** *(or* **reachable***) if and only if the matrix*

$$\mathbf{W_r} = \begin{bmatrix} \boldsymbol{B} & \underbrace{\boldsymbol{A}\boldsymbol{B} \ \cdots \ \boldsymbol{A}^{n-1}}_{\text{Controllability matrix}}\boldsymbol{B} \end{bmatrix}$$

has **full rank**.

Controllability: interpretation (1)

Assume $\boldsymbol{A}$ is complete (semisimple) with eigenvalues

$$\lambda_1, \lambda_2 \ldots, \lambda_n$$

and eigenvectors

$$\underbrace{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n}$$

define

$$\underbrace{\boldsymbol{T} = [\boldsymbol{v}_1 \ \boldsymbol{v}_2 \ \cdots \ \boldsymbol{v}_n]}$$

and

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix} = \boldsymbol{D} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} \quad \text{and} \quad \boldsymbol{E} = \boldsymbol{T}^{-1}\boldsymbol{B} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

Transform the matrix

$$\mathbf{W_r} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{\mathbf{n}-1}\boldsymbol{B} \end{bmatrix} \qquad x \in \mathbb{R}^n$$

into

$$\boldsymbol{T}^{-1}\mathbf{W_r} = \boldsymbol{T}^{-1} \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{\mathbf{n}-1}\boldsymbol{B} \end{bmatrix}$$

$$= [E \ \ \underbrace{T^{-1}AT}_{D}\underbrace{T^{-1}B}_{E} \ \cdots \ \underbrace{T^{-1}A^{n-1}T}_{D^{n}}\underbrace{T^{-1}B}_{E}]$$

$$= [E \ \ DE \ \cdots \ D^{n-1}E]$$

$$= \begin{bmatrix} e_1 & \lambda_1 e_1 & \cdots & \lambda_1^{n-1} e_1 \\ e_2 & \lambda_2 e_2 & & \\ \vdots & \vdots & & \\ e_n & \lambda_n e_n & & \lambda_n^{n-1} e_n \end{bmatrix}$$

$$= \begin{bmatrix} e_1 & & 0 \\ & e_2 & \\ 0 & & \ddots \ e_n \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & V & \lambda_2^{n-1} \\ \vdots & & & \vdots \\ 1 & \lambda_n & & \lambda_n^{n-1} \end{bmatrix}$$

Vandermode / matrix

$$\det V = \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i)$$

Controllability: interpretation (2)

$$T^{-1}\mathcal{W}_r = \underbrace{\begin{bmatrix} e_1 & & 0 \\ & e_2 & \\ & & \ddots \\ 0 & & e_n \end{bmatrix}}_{e_i \neq 0} \underbrace{\begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & & \\ & & & \\ 1 & \lambda_n & & \lambda_n^{n-1} \end{bmatrix}}_{V} \Rightarrow \det V = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)$$

$\lambda_i \neq \lambda_j \quad i \neq j$

Controllability: interpretation (3)

Theorem

The system

LTI

$$\underbrace{\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}}$$

is **controllable** *(or* **reachable***) if*

1. $\boldsymbol{B}$ *is a vector*
2. $\boldsymbol{A}$ *is complete (semisimple) with with eigenvalues* $\lambda_1, \lambda_2 \ldots, \lambda_n$, *eigenvectors* $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ *and transformation matrix*

$$\underbrace{\boldsymbol{T} = [\boldsymbol{v}_1 \; \boldsymbol{v}_2 \; \cdots \; \boldsymbol{v}_n]}.$$

3. $\lambda_i \neq \lambda_j$ *for all* $i \neq j$ ✓
4. *The vector* $\underbrace{\boldsymbol{T}^{-1}\boldsymbol{B}}$ *has only non-zero entries* ✓

Example: controllability interpretation

Consider

$$\overset{A}{} \qquad \overset{B}{}$$

$$\dot{\boldsymbol{x}} = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} a \\ b \end{pmatrix} \boldsymbol{u}$$

$\lambda_1 = -2$

$\lambda_2 = -1$

For which values of a and b is this controllable?

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$T^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad E = T^{-1} B = \begin{pmatrix} a - b \\ b \end{pmatrix}$

$b \neq 0 \quad a \neq b$

The End