Asynchronous lecture 10

- Reaction with Drift Equation

Reaction-Drift equation

$$\frac{\partial u(x,t)}{\partial t} + v\,\frac{\partial u(x,t)}{\partial x} = f(u)$$

Concentration dynamics for $u(x,t)$

Two processes at the same time:

spatial spreading

and

growth or decay depending on the sign of $f(u)$

Reaction-Drift equation

$$\frac{\partial u(x,t)}{\partial t} + v\,\frac{\partial u(x,t)}{\partial x} = au - bu^2$$

$$u(x,0) = f(x)$$

Make the transformation $u(x,t) = \dfrac{1}{\phi(x,t)}$

$$-\frac{\phi_t}{\phi^2} - v\,\frac{\phi_x}{\phi^2} = \frac{a}{\phi} - \frac{b}{\phi^2} \Rightarrow$$

$$\boxed{\phi_t + v\phi_x = -a\phi + b}$$

Reaction-Drift equation

$$\phi_t + v\phi_x = -a\phi + b$$

Solution via methods of characteristics ★

$$\begin{cases} \dfrac{dt}{ds} = 1 \text{ with } t(r,0) = 0 \to t(r,s) = s + c_1(r) \to t = s \\ \dfrac{dx}{ds} = v \text{ with } x(r,0) = r \to x(r,s) = vs + c_2(r) \to x = vs + r \\ \dfrac{d\phi}{ds} = -a\phi + b \text{ with } \phi(r,0) = 1/f(r) \to \text{ Laplace transform the } s \text{ variable} \end{cases}$$

$$\epsilon\widetilde{\phi}(\epsilon) - \phi(r,0) = -a\widetilde{\phi}(\epsilon) + \tfrac{b}{\epsilon} \to \widetilde{\phi}(\epsilon) = \tfrac{\phi(r,0)}{\epsilon+a} + \tfrac{b}{\epsilon(\epsilon+a)} = \tfrac{\phi(r,0)}{\epsilon+a} + \tfrac{b}{a}\left(\tfrac{1}{\epsilon} - \tfrac{1}{\epsilon+a}\right)$$

inverse Laplace transforming

$r = x - vt$

$$\phi(r,s) = \tfrac{1}{f(r)}e^{-as} + \tfrac{b}{a}\left(1 - e^{-as}\right) \longrightarrow \boxed{\phi(x,t) = \tfrac{1}{f(x-vt)}e^{-at} + \tfrac{b}{a}\left(1 - e^{-at}\right)}$$

Reaction-Drift equation

$$\frac{\partial u(x,t)}{\partial t} + v\,\frac{\partial u(x,t)}{\partial x} = au - bu^2$$

$$u(x,0) = f(x)$$

$$\text{Make the transformation } u(x,t) = \frac{1}{\phi(x,t)}$$

$$\phi(x,t) = \frac{1}{f(x-vt)}e^{-at} + \frac{b}{a}\left(1 - e^{-at}\right)$$

$$\boxed{u(x,t) = \frac{f(x-vt)}{e^{-at} + \frac{b}{a}\left(1 - e^{-at}\right) f(x-vt)}}$$

Reaction-Drift equation

$$u(x,t) = \frac{f(x-vt)}{e^{-at} + \frac{b}{a}\left(1 - e^{-at}\right) f(x-vt)}$$

What happens as $t \rightarrow \infty$?

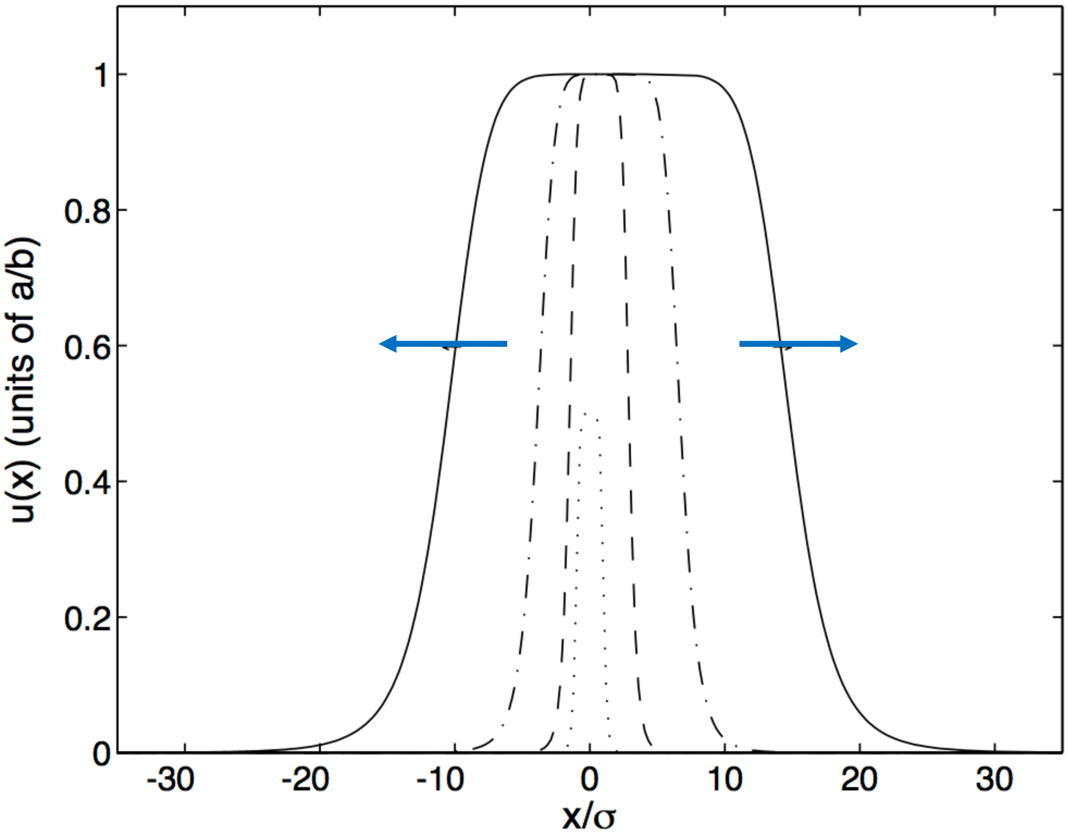
$u -> b/a$

Is this really true ?

No, it depends on the initial condition

More specifically on how the exponential decay in relation to the initial condition

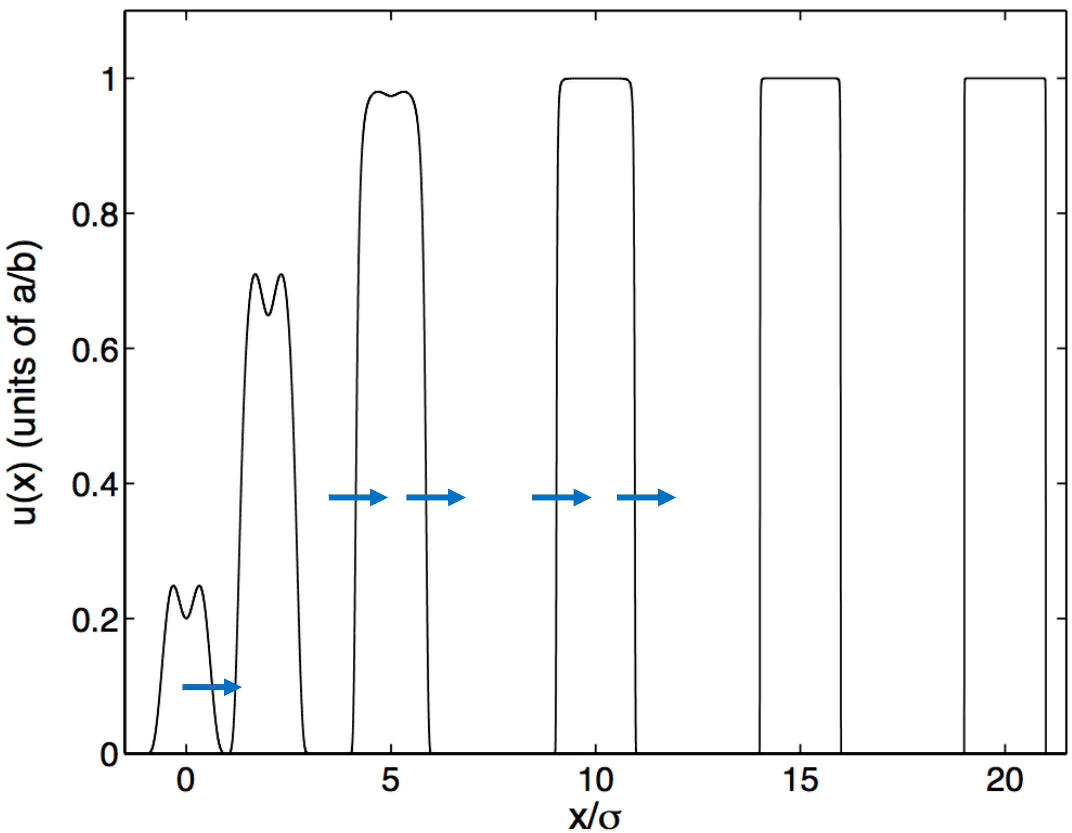
Dynamics for shallow initial conditions



$$u(x,t) = \frac{f(x-vt)}{e^{-at} + \frac{b}{a}\left(1-e^{-at}\right) f(x-vt)}$$

$$f(x) = \frac{1}{2}\frac{1}{1+\left(\frac{x}{\sigma}\right)^8}$$

Dynamics for steep initial conditions

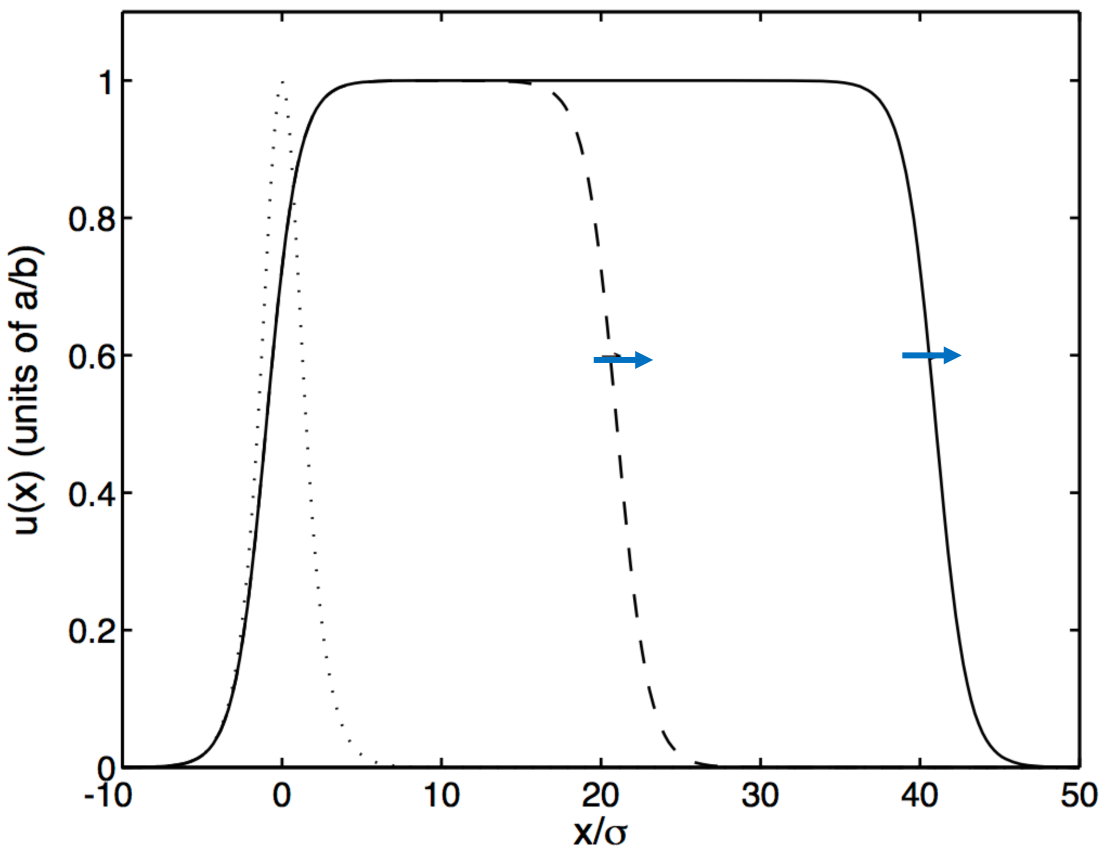


$$u(x,t) = \frac{f(x-vt)}{e^{-at} + \frac{b}{a}\left(1 - e^{-at}\right) f(x-vt)}$$

Compact support initial condition

Dynamics for exponentially decaying initial conditions

Steady front



$$u(x,t) = \frac{f(x-vt)}{e^{-at} + \frac{b}{a}\left(1-e^{-at}\right) f(x-vt)}$$

$$f(x) = \left[\operatorname{sech}\left(\frac{x}{\sigma}\right)\right]^{1/\sqrt{2}}$$

$$\frac{v}{\sigma a} = \sqrt{2}$$

Engineer an initial condition that decays at the same rate as the exponential

Travelling front ansatz

$$\frac{\partial u(x,t)}{\partial t} + v\,\frac{\partial u(x,t)}{\partial x} = au - bu^2$$

$$\text{Ansatz: } u(x,t) = F(x-ct)$$

$$z = x - ct$$

Make the same transformation as before to solve the ODE for F $\quad \frac{1}{\phi(x,t)}$

$$F(z) = \frac{1}{\frac{e^{-\frac{az}{(v-c)}}}{F(0)} + \frac{b}{a}\left(1 - e^{-\frac{az}{(v-c)}}\right)}$$

Travelling front ansatz

$$F(z) = \frac{1}{\frac{e^{-\frac{az}{(v-c)}}}{F(0)} + \frac{b}{a}\left(1 - e^{-\frac{az}{(v-c)}}\right)}$$

$$F(0) = \frac{1}{2}\frac{a}{b}$$ For convenience

$$F(x-ct) = \frac{a}{b}\frac{1}{1 + e^{-\frac{a(x-ct)}{(v-c)}}}$$

$c > v$, right front
$c < v$, left front

Blow-up in finite time

$$u_t + a u_x = u^2 \qquad u(x,0) = \cos(x)$$

$$\frac{dt}{ds}(r,s) = 1 \qquad \frac{dx}{ds}(r,s) = a \qquad \frac{du}{ds}(r,s) = u^2$$

$$t(r,s) = s + c_1(r) \qquad x(r,s) = as + c_2(r) \qquad -u^{-1}(r,s) = s + c_3(r)$$

$$t(r,0) = 0 \qquad x(r,0) = r \qquad u(r,0) = \cos(r)$$

$$t(r,s) = s \qquad x(r,s) = as + r \qquad -\frac{1}{u(r,s)} = s - \frac{1}{\cos(r)}$$

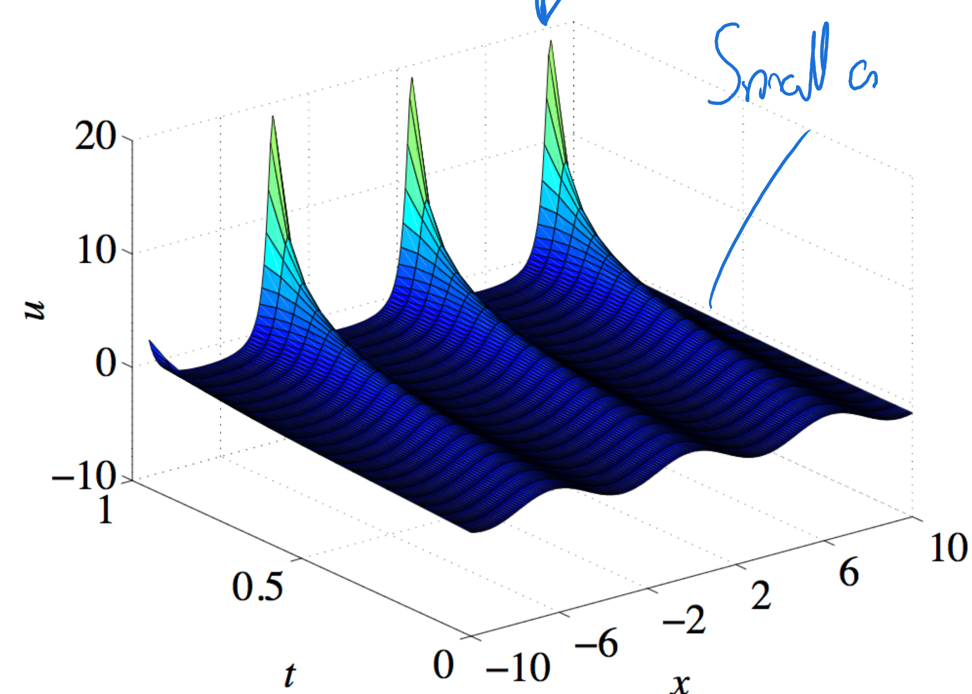
$$r = x - at \qquad \frac{1}{u(x,t)} = \frac{1 - t\cos(x-at)}{\cos(x-at)}$$

Blow-up in finite time

$$u_t + au_x = u^2 \qquad u(x,0) = \cos(x)$$

$$u(x,t) = \frac{\cos(x-at)}{1 - t\cos(x-at)}$$

Blow-up at time t such that $\cos(x-at) = t^{-1}$



When $t=1$ divergencies at $x = a + 2n\pi$

Burger's equation (inviscid)

$$u_t + u\,u_x = 0 \qquad u(x,0) = \phi(x)$$

$$\frac{dt}{ds} = 1 \qquad \frac{dx}{ds} = u \qquad \frac{du}{ds} = 0$$

$$t(r,s) = s \qquad u(r,s) = \phi(r) \qquad x(r,s) = \phi(r)s + r$$

$$t(r,0) = 0 \qquad u(r,0) = \phi(r) \qquad x(r,0) = r$$

$$x = \phi(r)s + r \rightarrow r = x - \phi(r)t = x - u\,t$$

$$u(x,t) = \phi(x - u\,t)$$ implicit solution

Burger's equation (inviscid)

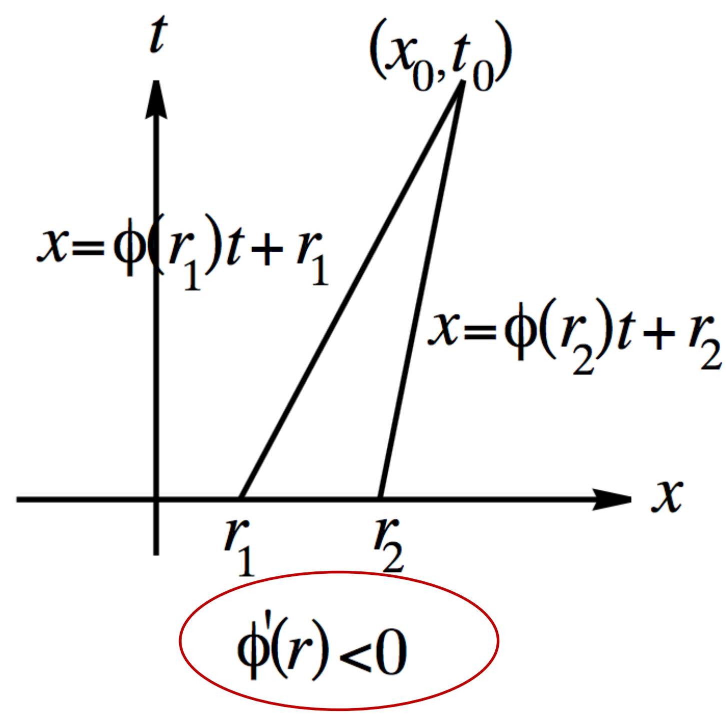
$$u(x,t) = \phi(x - u\,t)$$ implicit solution

Suppose there is $r_1 < r_2$ and $\phi(r_1) > \phi(r_2)$

Projected characteristic curves intersect at some point (x_0, t_0)

But u is constant along characteristics $\left(\frac{du}{ds} = 0\right)$

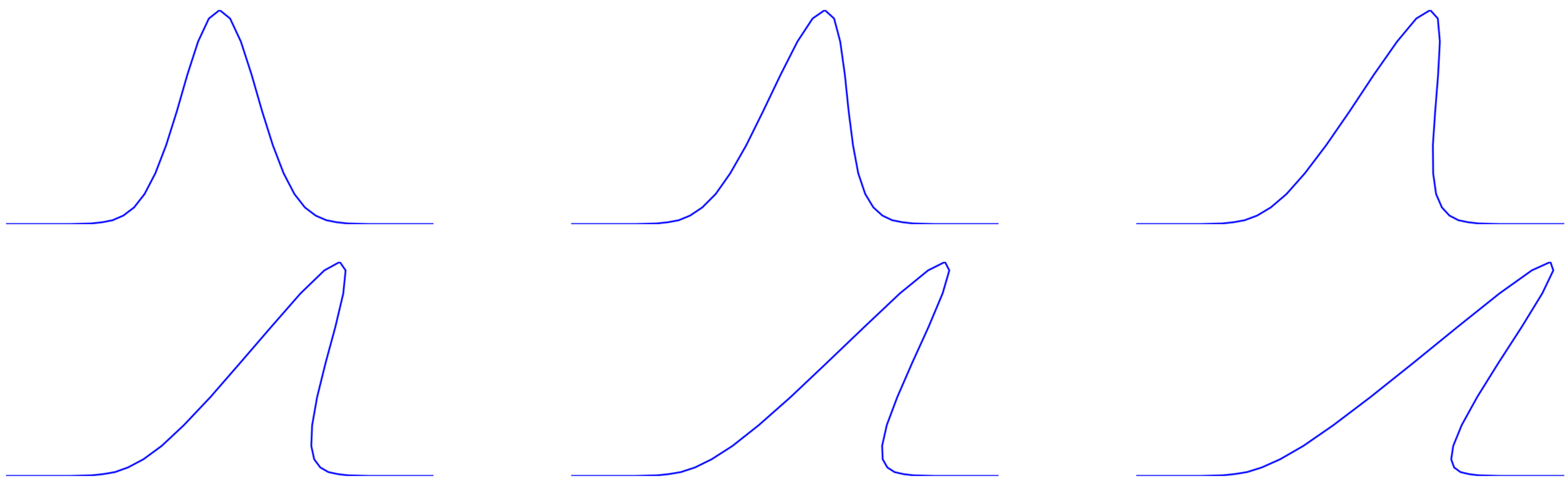
Contradiction!



Burger's equation (inviscid)

$$u_t + u\,u_x = 0 \qquad u(x,0) = \phi(x)$$

$$\phi(x) = e^{-x^2}$$

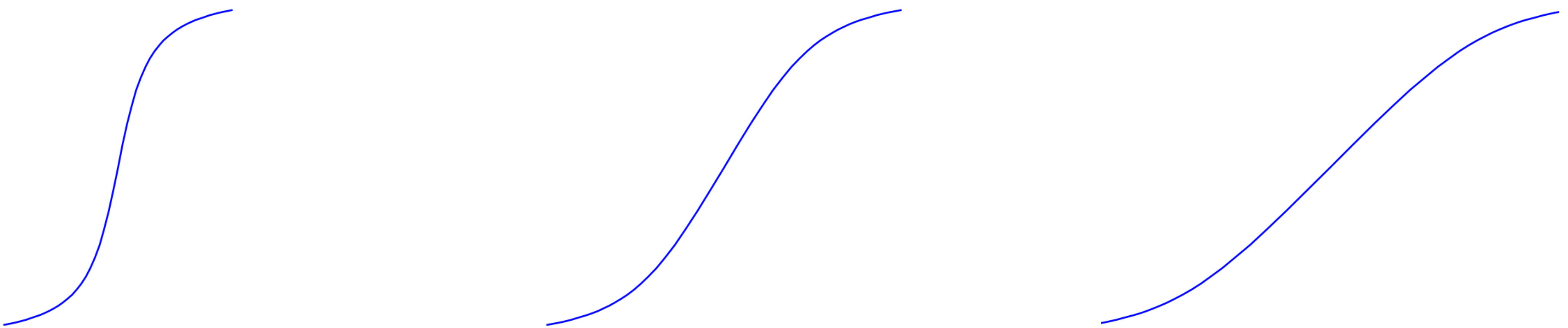


The taller part of the wave will overtake the shorter part of the wave

Burger's equation (inviscid)

$$u_t + u\,u_x = 0 \qquad u(x,0) = \phi(x)$$

$$\phi(x) = \arctan(x)$$



$\phi'(x) \geq 0$ and the projected characteristics do not intersect

$x = \phi(r_1)t + r_1$
$x = \phi(r_2)t + r_2$

do not cross for $r_2 > r_1$