MDM3 Introductory Assignment

A Mathematical Model of a Sphere Falling in Water

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Abstract

This report considers modelling the velocity of sphere falling through water. A simple model using Stokes' law is presented then compared to experimental data. The Reynolds number is then computed to show this problem falls outside the valid domain of Stokes' law. An improved model of drag is proposed using a drag coefficient. This model is also compared to the experimental data and gives good agreement with the terminal velocity reached. Possible further improvements such as the addition of a virtual weight and the inclusion a Basset force are discussed.

1 Introduction

This report seeks to find a mathematical model for a sphere descending in pool of water as shown in Figure 1. The sphere is made from steel and is released from bellow the surface of the water. The model needs to compute the vertical velocity of the sphere as a function of time. The results of the model will be compared to experimental data to validate its performance. The sphere used to generate the experimental data has a mass $m_s = 0.11$g and a radius $R_s = 0.15$cm.

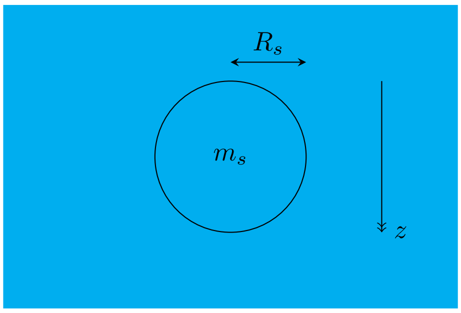

Figure 1: *A sphere of mass m_s and radius R_s descending in water.*

2 Stokes' Law Model

While descending through the fluid, the sphere is acted upon by three forces: the weight of the sphere, the buoyancy and drag caused by the movement through the water. Newton's second law [1] dictates that

$$m_s \ddot{z} = F_{\text{Weight}} - F_{\text{Drag}} - F_{\text{Buoyancy}}. \tag{1}$$

The weight is given by $F_{\text{Weight}} = m_s g$ where g is the acceleration due to gravity. Assuming small Reynolds numbers, the drag force can be modelled Stokes' Law $F_{\text{Drag}} = 6\pi\mu R_s \dot{z} = D\dot{z}$ [2] where μ is the dynamic viscosity of the fluid. The buoyant force is given by Archimedes' principle $F_{\text{Buoyancy}} = \rho_w g V$ [3] where ρ_w is the density of the fluid and $V = (4/3)\pi R_s^3$ is the volume of the sphere. Substituting these terms into Equation 1 and rearranging gives

$$\ddot{z} + \frac{D}{m_s}\dot{z} = g - \frac{\rho_w g V}{m_s}. \tag{2}$$

Since the velocity rather than the position is the quantity of interest substituting $v = \dot{z}$ simplifies Equation 2 into a first order ordinary differential equation (ODE). This ODE can then be solved by multiplying through by the integrating factor $\exp((D/m_s)t)$, integrating and applying the initial condition $v(0) = 0$ as the sphere is released from rest. This gives the solution of the velocity of the sphere as

$$v(t) = \frac{g}{D}(m_s - \rho_w V)\left(1 - \exp\left((-\frac{D}{m}t)\right)\right). \tag{3}$$

mass of sphere, m_S	0.11g
radius of sphere, R_s	0.15cm
acceleration due to gravity, g	$9.8\text{m}\,\text{s}^{-1}$
density of water, ρ_w	$997\text{kg}\,\text{m}^{-3}$
Stokes' drag coefficient, D	$0.0251\text{g}\,\text{s}^{-1}$

Table 1: *The parameter values used to generate results shown in Figure 2.*

As seen in Figure 2 when compared to the experimental data the model has poor agreement with the experimental data. Whilst both the model and the data converge to a terminal velocity, this value is an order of magnitude too high. Once source of error is assuming the drag force can be approximated by Stokes' law. Stokes' law is valid for very small Reynolds numbers, $R_e \ll 1$ [4], however, computing the Reynolds number $R_e = \rho_w U R_s/\mu$ gives $Re \approx 1000$. This means this scenario is well outside the domain where Stokes' law is valid as the drag is no longer dominated by frictional force but by the complex flows in the wake of the sphere.

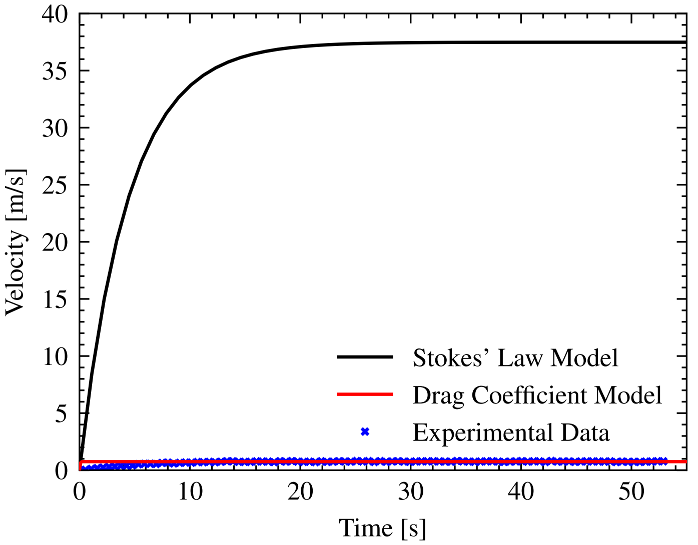

Figure 2: *Comparison of the Stokes' law model, drag coefficient model and the experimental data.*

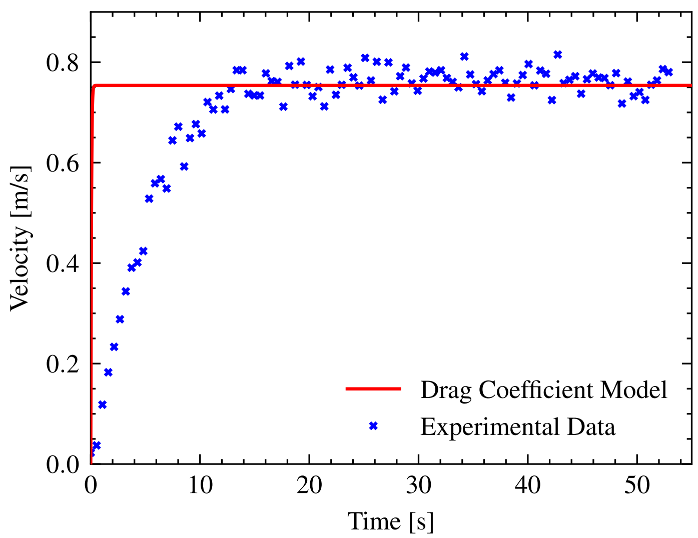

Figure 3: *Comparison of the drag coefficient model and the experimental data.*

3 Drag Coefficient Model

One improvement to Equation 2 is to consider a non-linear model for the drag force $F_{\text{Drag}} = (1/2)\rho_w \dot{z}^2 C_d A$ where C_d is the coefficient of drag and $A = \pi R_s^2$ is the frontal area of the sphere. This results in

$$\ddot{z} + \frac{\rho_w C_d A}{2m_s}\dot{z}^2 = g - \frac{\rho_w g V}{m_s}. \tag{4}$$

Whilst the drag coefficient is a function of the Reynolds and Mach numbers since the range of velocities experienced by the sphere is small C_d is approximated to be constant. A typical value of C_d for a sphere at similar Reynolds numbers to that in the experimental data is $C_d = 0.47$ [5].

Equation 4 was solved numerically using an Adams–Bashforth type scheme provided by SciPy [6]. Figure 3 shows that the model converges to a terminal velocity that is in agreement with the experimental data. However, the sphere is accelerating too quickly relative to the experimental data suggesting the resistive force is too small.

4 Further Improvements

As the sphere accelerates it must move some of the surrounding water. However this added inertia is not accounted for in either model presented as only the inertia of the sphere itself is considered. It is possible to relax this assumption by adding a virtual mass term [4] which accounts for the inertia required to displace water while the sphere is accelerating.

When the sphere accelerate in the water, the development of a boundary layer is delayed relative to the change in velocity. The Basset force accounts for the delay in boundary layer development [7].

These two additional force would increase the resistive force on sphere therefore lessoning the sphere's initial acceleration. This would help to reduce the error between the model and the experimental.

5 Conclusion

In this report two different models have been presented. The first models the drag experienced by the Sphere using Stoke's law. When compared to the experimental data this was shown not to be valid due to the Reynolds number being too high for Stokes' law to be a good model for drag. A second model using a drag coefficient showed much better agreement with the experimental data, with both reaching approximately the same terminal velocity. However the initial acceleration of the sphere is still far greater than the experimental data. This is possibly due the absence of a virtual mass or Basset force which could be included to further improve the model.

References

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