Chapter 1 - Introduction

November 3, 2022

Theorem 1.1 [2-steps]

If an operation consists of two steps, of which the first can be done in n_1 ways and for each of these the second can be done in n_2 ways, then the whole operation can be done in $n_1 \cdot n_2$ ways.

Theorem 1.2 [k-steps]

If an operation consists of k steps, of which the first can be done in n_1 ways, for each of these the second step can be done in n_2 ways, for each of the first two the third step can be done in n_3 ways, and so forth, then the whole operation can be done in $n_1 \cdot n_2 \cdot \ldots \cdot n_k$ ways.

Theorem 1.3 [# of permutations]

The number of permutations of n distinct objects is $n!$

Theorem 1.4 [# of permutations, r at a time]

The number of permutations of n distinct objects taken r at a time is

$$_nP_r = \frac{n!}{(n-r)!}$$

for $r = 0, 1, 2, \ldots, n$.

Proof

The formula $_nP_r = n(n-1) \cdot \ldots \cdot (n-r+1)$ cannot be used for $r = 0$, but we do have

$$_nP_0 = \frac{n!}{(n-0)!} = 1$$

For $r = 1, 2, \ldots, n$, we have

$$
\begin{aligned}
{}_nP_r &= n(n-1)(n-2)\cdot \ldots \cdot (n-r-1) \\
&= \frac{n(n-1)(n-2)\cdot \ldots \cdot (n-r-1)(n-r)!}{(n-r)!} \\
&= \frac{n!}{(n-r)!}
\end{aligned}
$$

Theorem 1.5 [Circular Permutations]

The number of permutations of n distinct objects arranged in a circle is $(n-1)!$

Theorem 1.6 [Permutations of n objects]

The number of permutations of n objects of which n_1 are of one kind, n_2 are of a second kind, $\ldots, n_k$ are of a kth kind, and $n_1 + n_2 + \cdots + n_k = n$ is

$$\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}$$

Theorem 1.7 [Combination]

The number of combinations of n distinct objects taken r at a time is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

for $r = 0, 1, 2, \ldots, n$

Theorem 1.8 [n objects into k subsets]

The number of ways in which a set of n distinct objects can be partitioned into k subsets with n_1 objects in the first subset, n_2 objects in the second subset, $\ldots$, and n_k objects in the kth subset is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}$$

Proof

Since the n_1 objects going into the first subset can be chosen in $\binom{n}{n_1}$ ways, the n_2 objects going into the second subset can then be chose in $\binom{n-n_1}{n_2}$ ways, the n_3 objects going into the third subset can then be chosen in $\binom{n-n_1-n_2}{n_3}$ ways, and so forth, it follows by Theorem 1.2 that the total number of partitions is

$$
\begin{aligned}
\binom{n}{n_1, n_2, \ldots, n_k} &= \binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdot \ldots \cdot \binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} \\
&= \frac{n!}{n_1! \cdot (n-n_1)!} \cdot \frac{(n-n_1)!}{n_2! \cdot (n-n_1-n_2)!} \cdot \ldots \cdot \frac{(n-n_1-n_2-\cdots-n_{k-1})!}{n_k! \cdot 0!} \\
&= \frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}
\end{aligned}
$$

Theorem 1.9 [Binomial Coefficient]

$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$ for any positive integer n

Theorem 1.10 [Combination of the complimentary set]

For any positive integers n and $r = 0, 1, 2, \ldots, n$,

$$\binom{n}{r} = \binom{n}{n-r}$$

Proof

We might argue that when we select a subset of r objects from a set of n distinct objects, we leave a subset of $n-r$ objects; hence, there are as many ways of selecting r objects as there are ways of leaving (or selecting) $n-r$ objects. To prove the theorem algebraically, we write

$$
\begin{aligned}
\binom{n}{n-r} &= \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} \\
&= \frac{n!}{r!(n-r)!} = \binom{n}{r}
\end{aligned}
$$

Theorem 1.11 [Combination for Pascal's Triangle]

For any positive integer n and $r = 1, 2, \ldots, n-1$,

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

Proof

Substituting $x = 1$ into $(x+y)^n$, let us write $(1+y)^n = (1+y)(1+y)^{n-1} = (1+y)^{n-1} + y(1+y)^{n-1}$ and equate the coefficient of y^r in $(1+y)^n$ with that in $(1+y)^{n-1} + y(1+y)^{n-1}$. Since the coefficient of y^r in $(1+y)^n$ is $\binom{n}{r}$ and the coefficient of y^r in $(1+y)^{n-1}+(1+y)^{n-1}$ is the sum of the coefficient of y^r in $(1+y)^{n-1}$, that is, $\binom{n-1}{r}$, and the coefficient of y^{r-1} in $(1+y)^{n-1}$, that is, $\binom{n-1}{r-1}$, we obtain

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

which completes the proof.

Theorem 1.12 [Sums of combinations]

$$\sum_{r=0}^{k} \binom{m}{k}\binom{n}{k-r} = \binom{m+n}{k}$$

Proof

Using the same technique as in the proof of Theorem 1.11, let us prove this theorem by equating the coefficients of y^k in the expressions on both sides of the equation

$$(1+y)^{m+n} = (1+y)^m(1+y)^n$$

The coefficient of y^k in $(1+y)^{m+n}$ is $\binom{m+n}{k}$, and the coefficient of y^k in

$$(1+y)^m(1+y)^n = [\binom{m}{0}+\binom{m}{1}y+\cdots+\binom{m}{m}y^m]\times[\binom{n}{0}+\binom{n}{1}y+\cdots+\binom{n}{n}y^n]$$

is the sum of the prodcts that we obtain by multiplying the constant term of the first factor by the coefficient of y^k in the second factor, the coefficient of y in the first factor by the coefficient of y^{k-1} in the second factor, ..., and the coefficient of y^k in the first factor by the constant term of the second factor. Thus, the coefficient of y^k in $(1+y)^m(1+y)^n$ is

$$\binom{m}{0}\binom{n}{k}+\binom{m}{1}\binom{n}{k-1}+\binom{m}{2}\binom{n}{n-2}+\cdots+\binom{m}{k}\binom{n}{0} = \sum_{r=0}^{k}\binom{m}{r}\binom{n}{k-r}$$

and this completes the proof.