CHAPTER 4 - MATHEMATICAL EXPECTATION

4.1 Introduction

Mathematical Expectation - Idea arising from games; the product of the amount a player stands to win and the probability that he or she will win

4.2 The Expected Value of a Random Variable

Definition 4.1 (Expected Value)**.** If X is a discrete random variable and $f(x)$ is the value of its probability distribution at x, the **expected value of** X is

$$E(X) = \sum_x x \cdot f(x)$$

Correspondingly, if X is a continuous random variable and $f(x)$ is the value of its probability density at x, the **expected value of** X is

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

Theorem 4.1 (Expected Value w/ function of X)**.** *If X is a discrete random variable and $f(x)$ is the value of its probability distribution at x, the expected value of $g(x)$ is given by*

$$E[g(x)] = \sum_x g(x) \cdot f(x)$$

Correspondingly, if X is a continuous random variable and $f(x)$ is the value of its probability density at x, the expected value of $g(X)$ is given by

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) \cdot f(x) dx$$

Proof. Since a more general proof is beyond the scope of this text, we shall prove this theorem here only for the case where X is discrete and has a finite range. Since $y = g(x)$ does not necessarily define a one-to-one correspondence, suppose that $g(x)$ takes on the value g_i when x takes on the values $x_{i1}, x_{i2}, \ldots, x_{in_i}$. Then, the probability that $g(X)$ will take on the value g_i is

$$P[g(X) = g_i] = \sum_{j=1}^{n_i} f(x_{ij})$$

and if $g(x)$ takes on the values $g_1, g_2, \ldots, g_m$, it follows that

$$\begin{aligned} E[g(X)] &= \sum_{i=1}^{m} g_i \cdot P[g(X) = g_i] \\ &= \sum_{i=1}^{m} g_i \cdot \sum_{j=1}^{n_i} f(x_{ij}) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n_i} g_i \cdot f(x_{ij}) \\ &= \sum_{x} g(x) \cdot f(x) \end{aligned}$$

where the summation extends over all values of X. □

Theorem 4.2 (Expected Value coefficient and sum of constants)**.** *If a and b are constants, then*

$$E(aX + b) = aE(X) + b$$

Proof. Using Theorem 4.1 with $g(X) = aX + b$, we get

$$\begin{aligned} E(aX + b) &= \int_{-\infty}^{\infty} (ax + b) \cdot f(x)dx \\ &= a \int_{-\infty}^{\infty} x \cdot f(x)dx + b \int_{-\infty}^{\infty} f(x)dx \\ &= aE(X) + b \end{aligned}$$

□

Corollary 4.1. *If a is a constant, then*

$$E(aX) = aE(X)$$

Corollary 4.2. *If b is a constant, then*

$$E(b) = b$$

Theorem 4.3 (Expected values for Summation)**.** *If $c_1, c_2, \ldots, c_n$ are constants, then*

$$E\left[\sum_{i=1}^{n} c_i g_i(X)\right] = \sum_{i=1}^{n} c_i E[g_i(X)]$$

Proof. According to Theorem 4.1 with $g(X) = \sum_{i=1}^{n} c_i g_i(X)$, we get

$$\begin{aligned} E\left[\sum_{i=1}^{n} c_i g_i(X)\right] &= \sum_{x}\left[\sum_{i=1}^{n} c_i g_i(x)\right] f(x) \\ &= \sum_{i=1}^{n}\sum_{x} c_i g_i(x) f(x) \\ &= \sum_{i=1}^{n} c_i \sum_{x} g_i(x) f(x) \\ &= \sum_{i=1}^{n} c_i E[g_i(X)] \end{aligned}$$

□

Theorem 4.4 (Expected Value for Joint Probability)**.** *If X and Y are discrete random variables and $f(x, y)$ is the value of their joint probability distribution at (x, y), the expected value of $g(X, Y)$ is*

$$E[g(X, Y)] = \sum_{x}\sum_{y} g(x, y) \cdot f(x, y)$$

Correspondingly, if X and Y are continuous random variables and $f(x, y)$ is the value of their joint probability density at (x, y), the expected value of $g(X, Y)$ is

$$E[g(x, y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f(x, y) dx dy$$

Theorem 4.5 (Expected values for Summation (multiple random variables))**.** *If $c_1, c_2, \ldots, c_n$ are constants, then*

$$E\left[\sum_{i=1}^{n} c_i g_i(X_1, X_2, \ldots, X_k)\right] = \sum_{i=1}^{n} c_i E[g_i(X_1, X_2, \ldots, X_k)]$$

4.3 Moments

Definition 4.2 (Moments About the Origin)**.** The **rth moment about the origin** of a random variable X, denoted by $\mu_r^{'}$, is the expected value of $X^{'}$; symbolically

$$\mu_r^{'} = E(X^r) = \sum_{x} x^r \cdot f(x)$$

for $r = 0, 1, 2, \ldots$ when X is discrete, and

$$\mu_r^{'} = E(X^r) = \int_{-\infty}^{\infty} x^r \cdot f(x) dx$$

when X is continuous.

Definition 4.3 (Mean of a Distribution)**.** $\mu_1^{'}$ is called the **mean** of the distribution of X, or simply the **mean of** X, and it is denoted simply by μ.

Definition 4.4 (Moments about the Mean)**.** The r**th moment about the mean** of a random variable X, denoted by μ_r, is the expected value of $(X-\mu)^r$, symbolically

$$\mu_r = E[(X-\mu)^r] = \sum_x (x-\mu)^r \cdot f(x)$$

for $r = 0, 1, 2, \ldots$ when X is discrete, and

$$\mu_r = E[(X-\mu)^r] = \int_{-\infty}^{\infty} (x-\mu)^r \cdot f(x)dx$$

when X is continuous.

Definition 4.5 (Variance)**.** μ_2 is called the **variance** of the distribution of X, or simply the **variance of** X, and is denoted by σ^2, σ_x^2, $var(X)$ or $V(X)$. The positive square root of the variance, σ, is call the **standard deviation of** X.

Theorem 4.6 (Variance)**.**

$$\sigma^2 = \mu_2^{'} - \mu^2$$

Proof.

$$\begin{aligned}
\sigma^2 &= E[(X-\mu)^2] \\
&= E(X^2 - 2\mu X + \mu^2) \\
&= E(X^2) - 2\mu E(X) + E(\mu^2) \\
&= E(X^2) - 2\mu \cdot \mu + \mu^2 \\
&= \mu_2' - \mu^2
\end{aligned}$$

□

Theorem 4.7 (Coefficients and sums for variance)**.** *If X has the variance σ^2, then*

$$var(aX+b) = a^2\sigma^2$$

Proof. TBD □

4.4 Chebyshev's Theorem

Theorem 4.8 (Chebyshev's Theorem)**.** *If μ and σ are the mean and the standard deviation of a random variable X, then for any positive constant k the probability is* at least $1-\frac{1}{k^2}$ *that X will take on a value within k standard deviationes fo the mean; symbolically,*

$$P(|x-\mu| < k\sigma) \geq 1 - \frac{1}{k^2},\ \sigma \neq 0$$

Proof. According to Definition 4.4 and Definition 4.5, we write

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)dx$$

Then, dividing the integral into three parts as shown in Figure 1, we get

$$\sigma^2 = \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 \cdot f(x)dx + \int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 \cdot f(x)dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 \cdot f(x)dx$$

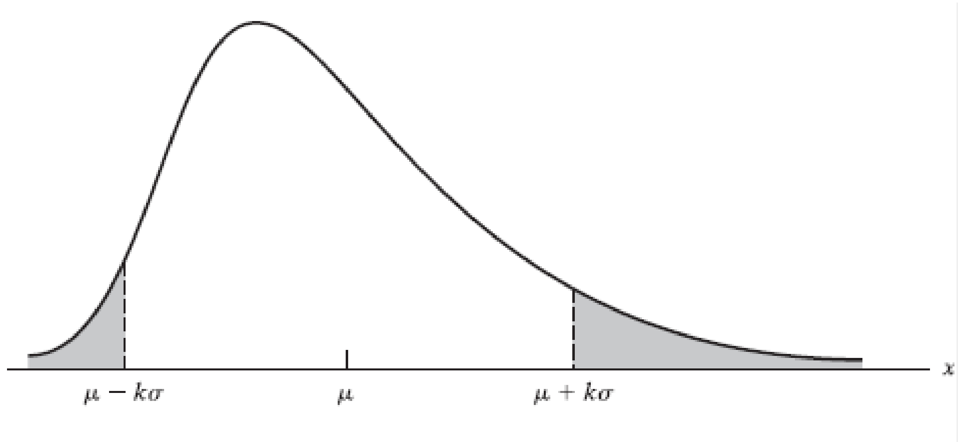

FIGURE 1. Diagram for proof of Chebyshev's Theorem

Since the integrand $(x-\mu)^2 \cdot f(x)$ is nonnegative, we can form the inequality

$$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 \cdot f(x)dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 \cdot f(x)dx$$

by deleting the second integral. Therefore, since $(x-\mu)^2 \geq k^2\sigma^2$ for $x \leq \mu - k\sigma$ or $x \geq \mu + k\sigma$ it follows that

$$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} k^2\sigma^2 \cdot f(x)dx + \int_{\mu+k\sigma}^{\infty} k^2\sigma^2 \cdot f(x)dx$$

and hence that

$$\frac{1}{k^2} \geq \int_{-\infty}^{\mu-k\sigma} f(x)dx + \int_{\mu+k\sigma}^{\infty} f(x)dx$$

provided $\sigma^2 \neq 0$. Since the sum of the two integrals on the right-hand side is the probability that X will take on a value less than or equal to $\mu - k\sigma$ or greater than or equal to $\mu + k\sigma$, we have thus shown that

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

and it follows that

$$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

$\square$

4.5 Moment-Generating Function

Definition 4.6. The **moment generating function** of a random variable X, where it exists, is given by

$$M_X(t) = E(e^{tX}) = \sum_x e^{tX} \cdot f(x)$$

when X is discrete, and

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)dx$$

when X is continuous.

Theorem 4.9 (Derivative of Moment Generating Function)**.**

$$\left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0} = \mu_r^{'}$$

Theorem 4.10. *If a and b are constants, then:*

(1) $M_{X+a}(t) = E[e^{(X+a)t}] = e^{at} \cdot M_X(t)$
(2) $M_{bX}(t) = E(e^{bXt}) = M_X(bt)$
(3) $M_{\frac{X+a}{b}}(t) = E[e^{(\frac{X+a}{b})t}] = e^{\frac{a}{b}t} \cdot M_X(\frac{t}{b})$

Proof. TBD Exercise 4.39 □

4.6 Product Moments

Definition 4.7 (Product Moments About the Origin)**.** The **rth and sth product moment about the origin** of the random variables X and Y, denoted by $\mu_{r,s}^{'}$, is the expected value of X^rY^s; symbolically,

$$\mu_{r,s}^{'} = E(X^rY^s) = \sum_x \sum_y x^r y^s \cdot f(x,y)$$

for $r = 0, 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$ when X and Y are discrete, and

$$\mu_{r,s}^{'} = E(X^rY^s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^r y^s \cdot f(x,y)dxdy$$

when X and Y are continuous.

Definition 4.8. The **rth and sth product moment about the means** of the random variables X and Y, denoted by $\mu_{r,s}$, is the expected value of $(X-\mu_X)^r(Y-\mu_Y)^s$; symbolically,

$$\begin{aligned}\mu_{r,s} &= E[(X-\mu_X)^r(Y-\mu_Y)^s)] \\ &= \sum_x \sum_y (x-\mu_X)^r(y-\mu_Y)^s \cdot f(x,y)\end{aligned}$$

for $r = 0, 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$ when X and Y are discrete, and

$$\begin{aligned}\mu_{r,s} &= E[(X-\mu_X)^r(Y-\mu_Y)^s)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)^r(y-\mu_Y)^s \cdot f(x,y)\end{aligned}$$

when X and Y are continuous.

Definition 4.9 (Covariance)**.** $\mu_{1,1}$ is called the **covaraince** of X and Y, and it is denoted by σ_{XY}, $\text{cov}(X,Y)$, or $\text{C}(X,Y)$

Theorem 4.11.

$$\sigma_{XY} = \mu'_{1,1} - \mu_X\mu_Y$$

Proof. Using the various theorems about expected values, we can write

$$\begin{aligned}
\sigma_{XY} &= E[(X-\mu_X)(Y-\mu_Y)] \\
&= E(XY - X\mu_Y - Y\mu_X - \mu_X\mu_Y) \\
&= E(XY) - \mu_Y E(X) - \mu_X E(Y) + \mu_X\mu_Y \\
&= E(XY) - \mu_Y\mu_X - \mu_X\mu_Y + \mu_X\mu_Y \\
&= \mu'_{1,1} - \mu_X\mu_Y
\end{aligned}$$

□

Theorem 4.12. *If X and Y are independent, then $E(XY) = E(X)\cdot E(Y)$ and $\sigma_{XY} = 0$*

Proof. For the discrete case we have, by definiteion,

$$E(XY) = \sum_x\sum_y xy\cdot f(x,y)$$

Since X and Y are independent, we can write $f(x,y) = g(x)\cdot h(y)$, where $g(x)$ and $h(y)$ are the values of the marginal distributions of X and Y, and we get

$$\begin{aligned}
E(XY) &= \sum_x\sum_y xy\cdot g(x)h(y) \\
&= \left[\sum_x x\cdot g(x)\right]\left[\sum_y y\cdot h(y)\right] \\
&= E(X)\cdot E(Y)
\end{aligned}$$

Hence,

$$\begin{aligned}
\sigma_{XY} &= \mu'_{1,1} - \mu_X\mu_Y \\
&= E(X)\cdot E(Y) - E(X)\cdot E(Y) \\
&= 0
\end{aligned}$$

□

Theorem 4.13. *If $X_1, X_2, \ldots, X_n$ are independent, then*

$$E(X_1, X_2, \ldots, X_n) = E(X_1)\cdot E(X_2)\cdot\ldots\cdot E(X_n)$$

4.7 Moments of Linear Combinations of Random Variables

Theorem 4.14. *If $X_1, X_2, \ldots, X_n$ are random variables and*

$$Y = \sum_{i=1}^{n} a_i X_i$$

where $a_1, a_2, \ldots, a_n$ are constants, then

$$E(Y) = \sum_{i=1}^{n} a_i E(X_i)$$

and

$$var(Y) = \sum_{i=1}^{n} a_i^2 \cdot var(X_i) + 2\sum\sum_{i<j} a_i a_j \cdot cov(X_i X_j)$$

where the double summation extends over all values of i and j, from 1 to n, for which $i < j$.

Proof. From Theorem 4.5 with $g_i(X_1, X_2, \ldots, X_k) = X_i$ for $i = 0, 1, 2, \ldots n$ it follows that immediately that

$$E(Y) = E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i)$$

and this proves the first part of the theorem. To obtain the expression for the variance of Y, let us write μ_i for $E(X_i)$ so that we get

$$\begin{aligned} \mathrm{var}(Y) = E([Y - E(Y)]^2) &= E\left\{\left[\sum_{i=1}^{n} a_i X_i - \sum_{i=1}^{n} a_i E(X_i)\right]^2\right\} \\ &= E\left\{\left[\sum_{i=1}^{n} a_i (X_i - \mu_i)\right]^2\right\} \end{aligned}$$

Then, expanding by means of the multinomial theorem, according to which $(a + b + c + d)^2$, for example, equals $a^2 + b^2 + c^c + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$, and again referring to Theorem 4.5, we get

$$\begin{aligned} \mathrm{var}(Y) &= \sum_{i=1}^{n} a_i^2 E[(X_i - \mu_i)^2] + 2\sum\sum_{i<j} a_i a_j E[(X_i - \mu_i)(X_j - \mu_j)] \\ &= \sum_{i=1}^{n} a_i^2 \cdot \mathrm{var}(X_i) + 2\sum\sum_{i<j} a_i a_j \cdot \mathrm{cov}(X_i, X_j) \end{aligned}$$

Note that we have tacitly made use of the fact that $\mathrm{cov}(X_i, X_j) = \mathrm{cov}(X_j, X_i)$ □

Corollary 4.3. *If the random variables $X_1, X_2, \ldots, X_n$ are independent and $Y = \sum_{i=1}^{n} a_i X_i$, then*

$$var(Y) = \sum_{i=1}^{n} a_i^2 \cdot var(X_i)$$

Theorem 4.15. *If $X_1, X_2, \ldots, X_n$ are random variables and*

$$Y_1 = \sum_{i=1}^{n} a_i X_i \text{ and } Y_2 = \sum_{i=1}^{n} b_i X_i$$

where $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are constants, then

$$cov(Y_1, Y_2) = \sum_{i=1}^{n} a_i b_i \cdot var(X_i) + \sum_{i<j}\sum (a_i b_j + a_j b_i) \cdot cov(X_i, X_j)$$

Corollary 4.4. *If the random variables $X_1, X_2, \ldots, X_n$ are independent, $Y_1 = \sum_{i=1}^{n} a_i X_i$ and $Y_2 = \sum_{i=1}^{n} b_i X_i$*

$$cov(Y_1, Y_2) = \sum_{i=1}^{n} a_i b_i \cdot var(X_i)$$

4.8 Conditional Expectations

Definition 4.10. If X is a discrete random variable, and $f(x|y)$ is the value of the conditional probability distribution of X give $Y = y$ at x, the **conditional expectation of $u(X)$ given $Y = y$** is

$$E[(u(X)|y)] = \sum_{x} u(x) \cdot f(x|y)$$

Correspondingly, if X is a continuous variable and $f(x|y)$ is the value of the conditional probability distribution of X given $Y = y$ at x, the **conditional expectation of $u(X)$ given $Y = y$** is

$$E[(u(X)|y)] = \int_{-\infty}^{\infty} u(x) \cdot f(x|y)dx$$

Conditional Mean

$$\mu_{X|y} = E(X|y)$$

Conditional Variance of X given $Y = y$

$$\begin{aligned}\sigma^2_{X|y} &= E[(X - \mu_{X|y})^2|y] \\ &= E(X^2|y) - \mu^2_{X|y}\end{aligned}$$

4.9 The Theory in Practice

Sample Mean

$$\overline{x} = \sum_{i=1}^{n} \frac{x_i}{n}$$

Median Arrange the observations in ascending order:

- if the number of observations n is odd, then the median is the observation at position $\frac{n+1}{2}$
- if the number of observations n is even, then the median is the average of the two observations at $\frac{n}{2}$ and $\frac{n}{2} + 1$

Sample Standard Deviation

$$s = \sqrt{\frac{\sum_{i=1}^{n}(x-\overline{x})^2}{n-1}}$$

but since this requires finding the mean first the following is equivalent, but doesn't require finding the mean

$$s = \sqrt{\frac{n\sum_{i=1}^{n}x_i^2 - (\sum_{i=1}^{n}x_i)^2}{n(n-1)}}$$