CHAPTER 5 - SPECIAL PROBABILITY DISTRIBUTIONS

5.1 Introduction

parameters quantities that are constants for particular distributions, but can take on different values for different members of famiiles of distributions of the same kind

5.2 The Discrete Uniform Distribution

Definition 5.1 (Discrete Uniform Distribution)**.** A random variable X has a **discrete uniform distribution** and it is referred to as a discrete uniform random variable if and only if it probability distribution is given by

$$f(x) = \frac{1}{k} \text{ for } x = x_1, x_2, \ldots, x_k$$

where $x_i \neq x_j$ when $i \neq j$.

5.3 The Bernoulli Distribution

Definition 5.2 (Bernoulli Distribution)**.** A random variable X has a **Bernoulli distribution** and it is referred to as a Bernoulli random variable if and only if its probability distribution is given by

$$f(x; \theta) = \theta^x (1 - \theta)^{1-x} \text{ for } x = 0, 1$$

Bernoulli Trial an experiment to which a Bernoulli distribution applies; an experiment that has two possible outcomes "success" and "failure"

5.4 The Binomial Distribution

Definition 5.3 (Binomial Distribution)**.** A random variable X has a **Binomial distribution** and it is referred to as a binomial random variable if and only if its probability distribution is given by

$$b(x; n, \theta) = \binom{n}{x} \theta^x (1 - \theta)^{n-x} \text{ for } x = 0, 1, 2, \ldots, n$$

Theorem 5.1 (Binomial Distribution - Complementary)**.**

$$b(x; n, \theta) = b(n - x; n, 1 - \theta)$$

Proof. TBD Exercise 5.5 $\square$

Theorem 5.2 (Binomial Distribution - Mean and Variance)**.** *The mean and variance of the binomial distribution are*

$$\mu = n\theta \ \text{ and } \ \sigma^2 = n\theta(1-\theta)$$

Proof.

$$\begin{aligned}\mu &= \sum_{x=0}^{n} x \cdot \binom{n}{x}\theta^x(1-\theta)^{n-x} \\ &= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!}\theta^x(1-\theta)^{n-x}\end{aligned}$$

where we omitted the term corresponding to $x = 0$, which is 0, and canceled the x against the first factor of $x! = x(x-1)$ in the denominator of $\binom{n}{x}$. Then, factoring out the factor n in $n! = n(n-1)!$ and one factor θ, we get

$$\mu = n\theta \sum_{x=1}^{n} x \cdot \binom{n-1}{x-1}\theta^{x-1}(1-\theta)^{n-x}$$

and, letting $y = x-1$ and $m = n-1$, this becomes

$$\mu = n\theta \sum_{x=0}^{m} x \cdot \binom{m}{y}\theta^{y}(1-\theta)^{m-y} = n\theta$$

since the last summation is the sum of all the values of a binomial distribution with the parameters m and θ, and hence equal to 1. To find expressions for $\mu_2^{'}$ and σ^2, let us make use of the fact that $E(X^2) = E[X(X-1)] + E(X)$ and first evaluate $E[X(X-1)]$. Duplicating for all practical purposes the steps used before, we thus get

$$\begin{aligned}E[X(X-1)] &= \sum_{x=0}^{n} x(x-1)\binom{n}{x}\theta^x(1-\theta)^{n-x} \\ &= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!}\theta^x(1-\theta)^{n-x} \\ &= n(n-1)\theta^2 \cdot \sum_{x=2}^{n}\binom{n-2}{x-2}\theta^{x-2}(1-\theta)^{n-x}\end{aligned}$$

and, letting $y = x-2$ and $m = n-2$, this becomes

$$\begin{aligned}E[X(X-1)] &= n(n-1)\theta^2 \cdot \sum_{x=0}^{m}\binom{m}{y}\theta^{y}(1-\theta)^{m-y} \\ &= n(n-1)\theta^2\end{aligned}$$

Therefore,

$$\mu_2^{'} = E[X(X-1)] + E(X) = n(n-1)\theta^2 + n\theta$$

and, finally

$$\begin{aligned}\sigma^2 &= \mu_2^{'} - \mu^2 \\ &= n(n-1)\theta^2 + n\theta - n^2\theta^2 \\ &= n\theta(1-\theta)\end{aligned}$$

□

Theorem 5.3 (Binomial w/ parameters). *If X has a binomial distribution with the parameters n and θ and $Y = \frac{X}{n}$, then*

$$E(Y) = \theta \text{ and } \sigma_Y^2 = \frac{\theta(1-\theta)}{n}$$

Theorem 5.4 (Binomial Moment Generation). *The moment-generating function of the binomial distribution is given by*

$$M_X(t) = [1 + \theta(e^t - 1)]^n$$

Proof. By using the Definition 4.6 Moment Generating Functions and Definition 5.3 Binomial Distribution, we get

$$\begin{aligned} M_X(t) &= \sum_{x=0}^{n} e^{xt} \binom{n}{x} \theta^x (1-\theta)^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} (\theta e^t)^x (1-\theta)^{n-x} \end{aligned}$$

and by Theorem 1.9 [Binomial Coefficient] this summation is easily recognized as the binomial expansion of $[\theta e^t + (1-\theta)]^n = [1 + \theta(e^t - 1)]^n$ □

Factorial Moment

$$\mu'_{(r)} = E[X(X-1)(X-2) \cdot \ldots \cdot (X-r+1)]$$

Factorial Moment-Generating Function

$$F_X(t) = E(t^X) = \sum_x t^x \cdot f(x)$$

5.5 The Negative Binomial and Geometric Distributions

Definition 5.4 (Negative Binomial Distribution). A random variable X has a **negative binomial distribution** and it is referred to as a negative binomial random variable if and only if

$$b^*(x; k, \theta) = \binom{x-1}{k-1} \theta^k (1-\theta)^{x-k} \quad \text{for } x = k, k+1, k+2, \ldots$$

binomial waiting-time distributions another name for negative binomial distributions
Pascal distributions another name for negative binomial distributions

Theorem 5.5 (Negative Binomial Distribution w/ a table).

$$b^*(x; k, \theta) = \frac{k}{x} \cdot b(k; x, \theta)$$

Proof. TBD Exercise 5.18 □

Theorem 5.6 (Negative Binomial Distribution - mean and variance). *The mean and the variance of the negative binomial distribution are*

$$\mu = \frac{k}{\theta} \text{ and } \sigma^2 = \frac{k}{\theta}\left(\frac{1}{\theta} - 1\right)$$

Proof. TBD Exercise 5.19 □

Definition 5.5 (Geometric Distribution)**.** A random variable X has a **geometric distribution** and it is referred to as a geometric random varaible if and only if its probability distribution is given by

$$g(x;\theta) = \theta(1-\theta)^{x-1} \ \text{ for } x = 1, 2, 3, \ldots$$

5.6 The Hypergeometric Distribution

Definition 5.6 (Hypergeometric Distribution)**.** A random variable X has a **hypergeometric distribution** and it is referred to as a hypergeometric random variable if and only if its probability distribution is given by

$$h(x;n,N,M) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \ \text{ for } x = 0, 1, 2, \ldots, n \ \ x \leq M \text{ and } n - x \leq N - M$$

Theorem 5.7 (Hypergeometric Distribution - mean and varaince)**.** *The mean and the variance of the hypergeometric distribution are*

$$\mu = \frac{nM}{N} \ \ \textit{and} \ \ \sigma^2 = \frac{nM(N-M)(N-n)}{N^2(N-1)}$$

Proof. To determine the mean, let us directly evaluate the sum

$$\begin{aligned}
\mu &= \sum_{x=0}^{n} x \cdot \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \\
&= \sum_{x=1}^{n} \frac{M!}{(x-1)!(M-x)!} \cdot \frac{\binom{N-M}{n-x}}{\binom{N}{n}}
\end{aligned}$$

where we omitted the term corresponding to $x = 0$, which is 0, and canceled the x against the first factor of $x! = x(x-1)!$ in the denominator of $\binom{M}{x}$. Then factoring out $M/\binom{N}{n}$, we get

$$\mu = \frac{M}{\binom{N}{n}} \cdot \sum_{x=1}^{n} \binom{M-1}{x-1}\binom{N-M}{n-x}$$

and, letting $y = x - 1$ and $m = n - 1$, this becomes

$$\mu = \frac{M}{\binom{N}{n}} \cdot \sum_{x=0}^{m} \binom{M-1}{y}\binom{N-M}{m-y}$$

Finally, using Theorem 1.12 [Sums of combinations] we get

$$\mu = \frac{M}{\binom{N}{n}} \cdot \binom{N-1}{m} = \frac{M}{\binom{N}{n}} \cdot \binom{N-1}{n-1} = \frac{nM}{N}$$

To obtain the formula for σ^2, we proceed as in the proof of Theorem 5.2 [Binomial Distribution - Mean and Variance] by first evaluating $E[X(X-1]$ and then making use of the fact that $E(X^2) = E[X(X-1)] + E(X)$. Leaving it to the reader to show that

$$E[X(X-1)] = \frac{M(M-1)n(n-1)}{N(N-1)}$$

in Exercise 5.27. We thus get

$$\begin{aligned}\sigma^2 &= \frac{M(M-1)n(n-1)}{N(N-1)} + \frac{nM}{N} - \left(\frac{nM}{N}\right)^2 \\ &= \frac{nM(N-M)(N-n)}{N^2(N-1)}\end{aligned}$$

Proof of the following excercise 5.27

$$E[X(X-1)] = \frac{M(M-1)n(n-1)}{N(N-1)}$$

□

5.7 Poisson Distribution

Definition 5.7 (Poisson Distribution)**.** A random variable has a **Poisson distribution** and it is referred to as a Poisson random varaible if and only if its probability distribution is given by

$$p(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \quad \text{for}\, x = 0, 1, 2, \ldots$$

Theorem 5.8 (Poisson Distribution mean and variance)**.** *The mean and variance of the Poisson distribution are given by*

$$\mu = \lambda \;\; \textit{and} \;\; \sigma^2 = \lambda$$

Proof. TBD Exercise 5.33 □

Theorem 5.9 (Poisson Distribution Moment Generating Function)**.**

$$M_X(t) = e^{\lambda(e^t-1)}$$

Proof. Using the definition for Moment-generating Functions and Definition 5.7

$$M_X(t) = \sum_{x=0}^{\infty} e^{xt} \cdot \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

where $\sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$ can be recognized as the Maclaurin's series of e^z with $z = \lambda e^t$. Thus

$$M_X(t) = e^{-\lambda} \cdot e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

□

5.8 Multinomial Distribution

Definition 5.8 (Multinomial Distribution)**.** The random variables $X_1, X_2, \ldots, X_n$ have a **multinomial distribution** and they are referred to as multinomial random variables if and only if their joint probabiltiy distribution is given by

$$f(x_1, x_2, \ldots, x_k; n, \theta_1, \theta_2, \ldots, \theta_k) = \binom{n}{x_1, x_2, \ldots, x_k} \cdot \theta_1^{x_1} \cdot \theta_2^{x_2} \cdot \cdots \cdot \theta_k^{x_k}$$

for $x_i = 0, 1, \ldots, n$ for each i, where $\sum_{i=1}^{k} = n$ and $\sum_{i=1}^{k} \theta_i = 1$.

5.9 Multivariate Hypergeometric Distribution

Definition 5.9 (Multivariate Hypergeometric Distribution)**.** The random variables $X_1, X_2, \ldots, X_k$ have a **multivariate hypergeometric distribution** and they are referred to as multivariate hypergeometric random varaibles if and only if their joint probability distribution is given by

$$f(x_1, x_2, \ldots, x_k; n, M_1, M_2, \ldots, M_k) = \frac{\binom{M_1}{x_1}\binom{M_2}{x_2} \cdot \cdots \cdot \binom{M_k}{x_k}}{\binom{N}{n}}$$

for $x_i = 0, 1, 2, \ldots, n$ and $x_i \leq M_i$ for each i, where $\sum_{i=1}^{k} x_i = n$ and $\sum_{i=1}^{k} M_i = N$.

Theorem 5.10 (Probability of Acceptance)**.** *If n is the size of the sample taken from each large lot and c is the acceptnace number, the **probability of acceptance** is closely approximated by*

$$L(p) = \sum_{k=0}^{c} b(k; n, p) = B(c; n, p)$$

where p is the actual proportion of defectives in the lot.