Chapter 2 - Probability

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Theorem 2.1 [Sum of Probabilities]

If A is an event in a discrete sample space S, then $P(A)$ equals the sum of the probabilities of the individual outcomes comprising A.

Proof

Let $O_1, O_2, O_3, \ldots$, be the finite or infinite sequence of outcomes that comprise the event A. Thus,

$$A = O_1 \cup O_2 \cup O_3 \cdots$$

and since the individual outcomes, the O's, are mutually exclusive, the third postulate or probability yields

$$P(A) = P(O_1) + P(O_2) + P(O_3) + \cdots$$

Theorem 2.2 [Fraction of Outcomes]

If an experiment can result in any one of N different equally likely outcomes, and if n of these outcomes together constitute event A, then the probability of event A is

$$P(A) = \frac{n}{N}$$

Proof

Let $O_1, O_2, O_3, \ldots, O_N$ represent the individual outcomes in S, each with probability $\frac{1}{N}$. If A is the union of n of these mutually exclusive outcomes, and it does not matter which ones, then

$$\begin{aligned}
P(A) &= P(O_1 \cup O_2 \cup \cdots \cup O_n) \\
&= P(O_1) + P(O_2) + \cdots + P(O_n) \\
&= \underbrace{\frac{1}{N} + \frac{1}{N} + \cdots + \frac{1}{N}}_{n \text{ terms}} \\
&= \frac{n}{N}
\end{aligned}$$

Theorem 2.3 [Complement of a set]

If A and A^{C} are complementary events in a sample space S, then

$$P(A^{\mathsf{C}}) = 1 - P(A)$$

Proof

In the second and third steps of the proof that follows, we make use of the definition of a complement, according to which A and A^{C} are mutually exclusive and $A \cup A^{\mathsf{C}} = S$. Thus, we write

$$\begin{aligned}
1 &= P(S) && \text{(by Postulate 2)} \\
&= P(A \cup A^{\mathsf{C}}) \\
&= P(A) + P(A^{\mathsf{C}}) && \text{(by Postulate 3)}
\end{aligned}$$

and it follows that $P(A^{\mathsf{C}}) = 1 - P(A)$

Theorem 2.4 [Empty set]

$P(\emptyset) = 0$ for any sample space S.

Proof

Since S and $\emptyset$ are mutually exclusive and $S \cup \emptyset = S$ in accordance with the definition of the empty set $\emptyset$, it follows that

$$\begin{aligned}
P(S) &= P(S \cup \emptyset) \\
&= P(S) + P(\emptyset) && \text{(by Postulate 3)}
\end{aligned}$$

and, hence, that $P(\emptyset) = 0$

Theorem 2.5 [My Space is Bigger]

If A and B are events in a sample space S and $A \subset B$, then $P(A) \leq P(B)$.

Proof

Since $A \subset B$, we can write

$$B = A \cup (A^{\mathsf{C}} \cap B)$$

as can easily be verified by means of a Venn diagram. Then, since A and $A^{\mathsf{C}} \cap B$ are mutually exclusive, we get

$$\begin{aligned}
P(B) &= P(A) + P(A^{\mathsf{C}} \cap B) && \text{(by Postulate 3)} \\
&\geq P(A) && \text{(by Postulate 1)}
\end{aligned}$$

Theorem 2.6 [Probability Sandwich]

$0 \leq P(A) \leq 1$ for any event A.

Proof

Using Theorem 2.5 and the fact that $\emptyset \subset A \subset S$ for any event A in S, we have

$$P(\emptyset) \leq P(A) \leq P(S)$$

Then, $P(\emptyset) = 0$ and $P(S) = 1$ leads to the result that

$$0 \leq P(A) \leq 1$$

Theorem 2.7 [Discount Double Check]

If A and B are any two events in a sample space S, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof

Assigning the probabilities a, b and c to the mutually exclusive events $A \cap B$, $A \cap B^{\mathsf{C}}$, and $A^{\mathsf{C}} \cap B$ as shown below in the Venn Diagram in Figure 1. We find that:

$$\begin{aligned} P(A \cup B) &= a + b + c \\ &= (a + b) + (c + a) - a \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

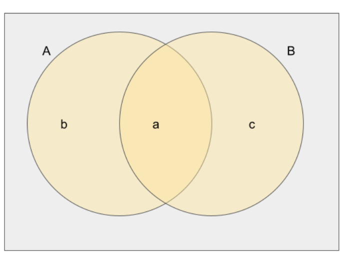

Figure 1: Theorem 2.7 Venn Diagram

Theorem 2.8 [Discount Triple Check]

If A, B, and C are any three events in a sample space S, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Proof

Writing $A \cup B \cup C$ as $A \cup (B \cup C)$ and using the formula of Theorem 2.7 twice, once for $P[A \cup (B \cup C)]$ and once for $P(B \cup C)$, we get

$$\begin{aligned} P(A \cup B \cup C) &= P[A \cup (B \cup C)] \\ &= P(A) + P(B \cup C) - P[A \cap (B \cup C)] \\ &= P(A) + P(B) + P(C) - P(B \cup C) - P[A \cap (B \cup C)] \end{aligned}$$

Then, using the distributive law that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$:

$$\begin{aligned} P[A \cap (B \cup C)] &= P[(A \cap B) \cup (A \cap C)] \\ &= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)] \\ &= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) \end{aligned}$$

and hence that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Theorem 2.9 [Multiplication Rule]

If A and B are any two events in a sample space S and $P(A) \neq 0$, then

$$P(A \cap B) = P(A) \cdot P(B|A)$$

Alternatively if $P(B) \neq 0$, then

$$P(A \cap B) = P(B) \cdot P(A|B)$$

Theorem 2.10 [Intersecting 3]

If A, B, and C are any three events in a sample space S such that $P(A \cap B) \neq 0$, then

$$P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$$

Proof

Writing $A \cap B \cap C$ as $(A \cap B) \cap C$ and using the formula of Theorem 2.9 twice, we get

$$\begin{aligned} P(A \cap B \cap C) &= P[(A \cap B) \cap C] \\ &= P(A \cap B) \cdot P(C|A \cap B) \\ &= P(A) \cdot P(B|A) \cdot P(C|A \cap B) \end{aligned}$$

Theorem 2.11 [Independent Complement]

If A and B are independent, then A and B^{C} are also independent.

Proof

Since $A = (A \cap B) \cup (A \cap B^{\mathsf{C}})$, $A \cap B$ and $A \cap B^{\mathsf{C}}$ are mutually exclusive, and A and B are independent by assumption, we have

$$\begin{aligned} P(A) &= P[(A \cap B) \cup (A \cap B^{\mathsf{C}})] \\ &= P(A \cap B) + P(A \cap B^{\mathsf{C}}) \\ &= P(A) \cdot P(B) + P(A \cap B^{\mathsf{C}}) \end{aligned}$$

It follows that

$$\begin{aligned} P(A \cap B^{\mathsf{C}}) &= P(A) - P(A) \cdot P(B) \\ &= P(A) \cdot [1 - P(B)] \\ &= P(A) \cdot P(B^{\mathsf{C}}) \end{aligned}$$

and hence that A and B^{C} are independent.

Theorem 2.12 [Rule of Total probability or Rule of Elimination]

If the events $B_1, B_2, \ldots, B_k$ constitue a partition of the sample space S and $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$ then for any event A in S

$$P(A) = \sum_{i=1}^{k} P(B_i) \cdot P(A|B_i)$$

Theorem 2.13 [Bayes Theorem]

If $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S and $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$ then for any event A in S such that $P(A) \neq 0$

$$P(B_r|A) = \frac{P(B_r) \cdot P(A|B_r)}{\sum_{i=1}^{k} P(B_i) \cdot P(A|B_i)}$$

for $r = 1, 2, \ldots, k$ In words, the probability that event A was reached via the rth branch of the following tree diagram, given that it was reached via one of its k branches, is the ratio of the probability associated with the rth branch to the sum of the probabilities associated with all k branches of the tree.

Proof

Writing $P(B_r|A) = \frac{P(A \cap B_r)}{P(A)}$ in accordance with the definition of conditional probability, we have only to substitute $P(B_r) \cdot P(A|B_r)$ for $P(A \cap B_r)$ and the formula of Theorem 2.12 for $P(A)$

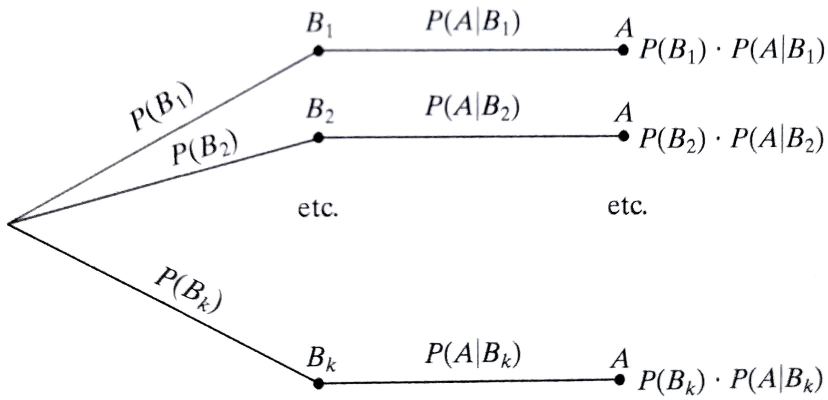

Figure 2: Bayes Theorem Tree Diagram

Theorem 2.14 [Reliability of a Series System]

The reliability of a series system consisting of n independent components is given by

$$R_s = \prod_{i=1}^{n} R_i$$

where R_i is the reliability of the ith component.

Proof

The proof follows immediately by iterating in Definition 2.5 - Independence

Theorem 2.15 [Reliability of a Parallel System]

The reliability of a parallel system consisting of n independent components is given by

$$R_p = 1 - \prod_{i=1}^{n}(1 - R_i)$$

Proof

the proof of this theorem is identical to that of Theorem 2.14, with $(1 - R_i)$ replacing R_i