CHAPTER 5 - SPECIAL PROBABILITY DISTRIBUTIONS

5.1 Introduction

parameters quantities that are constants for particular distributions, but can take on different values for different members of famiiles of distributions of the same kind

5.2 The Discrete Uniform Distribution

Definition 5.1 (Discrete Uniform Distribution)**.** A random variable X has a **discrete uniform distribution** and it is referred to as a discrete uniform random variable if and only if it probability distribution is given by

$$f(x) = \frac{1}{k} \text{ for } x = x_1, x_2, \ldots, x_k$$

where $x_i \neq x_j$ when $i \neq j$

5.3 The Bernoulli Distribution

Definition 5.2. A random variable X has a **Bernoulli distribution** and it is referred to as a Bernoulli random variable if and only if its probability distribution is given by

$$f(x;\theta) = \theta^x(1-\theta)^{1-x} \text{ for } x = 0, 1$$

Bernoulli Trial an experiment to which a Bernoulli distribution applies; an experiment that has two possible outcomes "success" and "failure"

5.4 The Binomial Distribution

Definition 5.3. A random variable X has a **Binomial distribution** and it is referred to as a binomial random variable if and only if its probability distribution is given by

$$b(x;n,\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x} \text{ for } x = 0, 1, 2, \ldots, n$$

Theorem 5.1 (Binomial Distribution - Complementary)**.**

$$b(x;n,\theta) = b(n-x;n,1-\theta)$$

Proof. TBD Exercise 5.5 □

Theorem 5.2 (Binomial Distribution - Mean and Variance)**.** *The mean and variance of the binomial distribution are*

$$\mu = n\theta \ \ and \ \ \sigma^2 = n\theta(1-\theta)$$

Proof.

$$\begin{aligned}\mu &= \sum_{x=0}^{n} x \cdot \binom{n}{x} \theta^x (1-\theta)^{n-x} \\ &= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} \theta^x (1-\theta)^{n-x}\end{aligned}$$

where we omitted the term corresponding to $x = 0$, which is 0, and canceled the x against the first factor of $x! = x(x-1)$ in the denominator of $\binom{n}{x}$. Then, factoring out the factor n in $n! = n(n-1)!$ and one factor θ, we get

$$\mu = n\theta \sum_{x=1}^{n} x \cdot \binom{n-1}{x-1} \theta^{x-1} (1-\theta)^{n-x}$$

and, letting $y = x - 1$ and $m = n - 1$, this becomes

$$\mu = n\theta \sum_{x=1}^{m} x \cdot \binom{m}{y} \theta^{y} (1-\theta)^{m-y} = n\theta$$

since the last summation is the sum of all the values of a binomial distribution with the parameters m and θ, and hence equal to 1. To find expressions for $\mu_2^{'}$ and σ^2, let us make use of the fact that $E(X^2) = E[X(X-1)] + E(X)$ and first evaluate $E[X(X-1)]$. Duplicating for all practical purposes the steps use before, we thus get

$$\begin{aligned}E[X(X-1)] &= \sum_{x=0}^{n} x(x-1) \binom{n}{x} \theta^x (1-\theta)^{n-x} \\ &= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!} \theta^x (1-\theta)^{n-x} \\ &= n(n-1)\theta^2 \cdot \sum_{x=2}^{n} \binom{n-2}{x-2} \theta^{x-2} (1-\theta)^{n-x}\end{aligned}$$

and, letting $y = x - 2$ and $m = n - 2$, this becomes

$$\begin{aligned}E[X(X-1)] &= n(n-1)\theta^2 \cdot \sum_{x=0}^{m} \binom{m}{y} \theta^{y} (1-\theta)^{m-y} \\ &= n(n-1)\theta^2\end{aligned}$$

Therefore,

$$\mu_2^{'} = E[X(X-1)] + E(X) = n(n-1)\theta^2 + n\theta$$

and, finally

$$\begin{aligned}\sigma^2 &= \mu_2^{'} - \mu^2 \\ &= n(n-1)\theta^2 + n\theta - n^2\theta^2 \\ &= n\theta(1-\theta)\end{aligned}$$

$\square$