# Oscillations

Any system displaced from a stable equilibrium Position will experience Oscillations.
If the displacement is small the system will undergo simple harmonic motion.

# Hooke's Law

$$F_x(x) = -kx$$

k: Positive number called the Force constant

Since k is positive @ $x=0$ is stable

for $x>0$: Force is negative
for $x<0$: Force is Positive

$F_x(x) = -kx$ is a restoring force

Pushing the system back to equilibrium

$U(x) = \frac{1}{2} K x^2$, $F = -\nabla U$

$U = -\int F(x)\, dx$

Arbitrary 1D conservative system with $U(x)$

$U(x)$ is stable at $x = x_0$ ($x_0 = 0$) origin

expand for value close to $x_0$ (Taylor series around $x_0$)

$$U(x) = U(0) + U'(0)x + \frac{1}{2}U''(0)x^2 + \dots$$

for $x \ll 1$

$U(0)$ is constant and $0$

since $x_0$ is equilibrium point $U'(0) = 0$

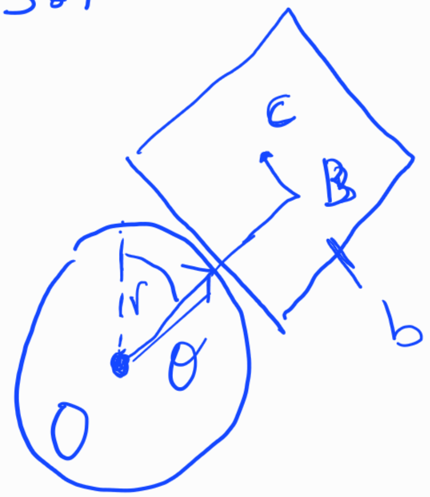
$U''(0) > 0$ (positive) as stable

$$\therefore U(x) = \frac{1}{2}U''(0)x^2 = \frac{1}{2}Kx^2$$

for small displacements *

Ex 5.1

Prove $U(\theta) = \frac{1}{2} k\theta^2$




$$U(\theta) = mg\left[(r+b)\cos\theta + r\theta\sin\theta\right]$$

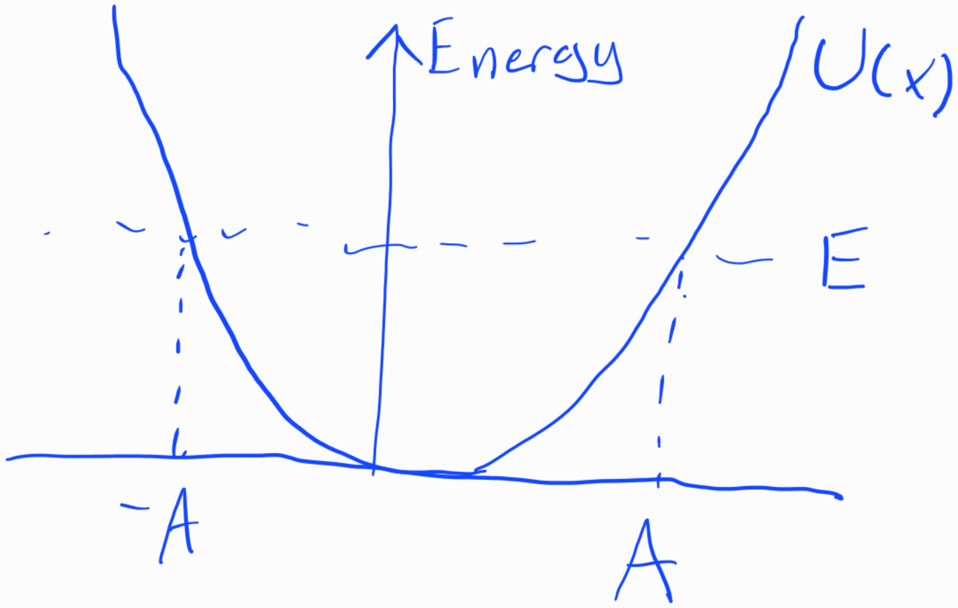
if $\theta$ is small

$$\cos\theta \approx 1 - \frac{\theta^2}{2}, \quad \sin\theta \approx \theta$$

$$U(\theta) \approx mg\left[(r+b)\left(1 - \frac{1}{2}\theta^2\right) + r\theta^2\right]$$

$$= \underbrace{mg(r+b)}_{+C} + \frac{1}{2}\underbrace{mg(r-b)}_{K}\theta^2$$

$$U(\theta) = \frac{1}{2}K\theta^2 + C$$

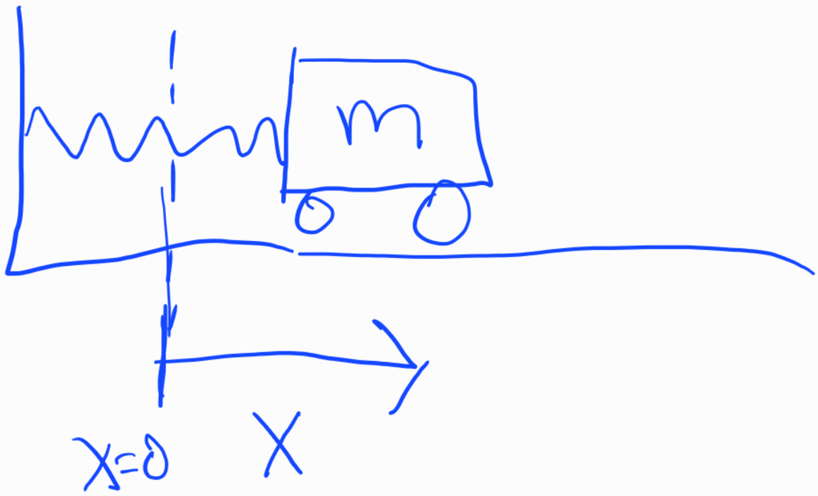



$x = \pm A$, Amplitude

$U(x) = E$, $T = 0$

Particle trapped between turning points

# Simple Harmonic Motion



$$F_x(x) = -kx$$

$$m\ddot{x} = F_x = -kx$$

$$\ddot{x} = -\frac{k}{m}x = -\omega^2 x$$

$$\omega = \sqrt{\frac{k}{m}}$$

(similar to Pendulum $\ddot{\phi} = -\omega^2 \phi$)

$$\ddot{x} = -\frac{k}{m} x$$

$$x(t) = e^{-i\omega t}$$

$$x(t) = e^{+i\omega t}$$

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

$$e^{\pm i\omega t} = \cos(\omega t) \pm i\sin(\omega t)$$

Euler's Formula

$$x(t) = (C_1 + C_2)\cos\omega t + i(C_1 - C_2)\sin\omega t$$

$$= B_1 \cos\omega t + B_2 \sin\omega t$$

$$B_1 = C_1 + C_2 \qquad B_2 = i(C_1 - C_2)$$

$$X(t) = B_1 \cos \omega t + B_2 \sin \omega t$$

is definition of SHM

for $X(t)$ to be real $B_1$ & $B_2$ must be real

@ $t = 0$

$$X(0) = B_1 = X_0$$

$$X'(0) = V(0) = -\omega B_1 \sin \omega t + \omega B_2 \cos \omega t$$

$$= \omega B_2 = V_0, \quad B_2 = \frac{V_0}{\omega}$$

for a pull of $X = X_0$ & release from rest ($V_0 = 0$), $B_2 = 0$

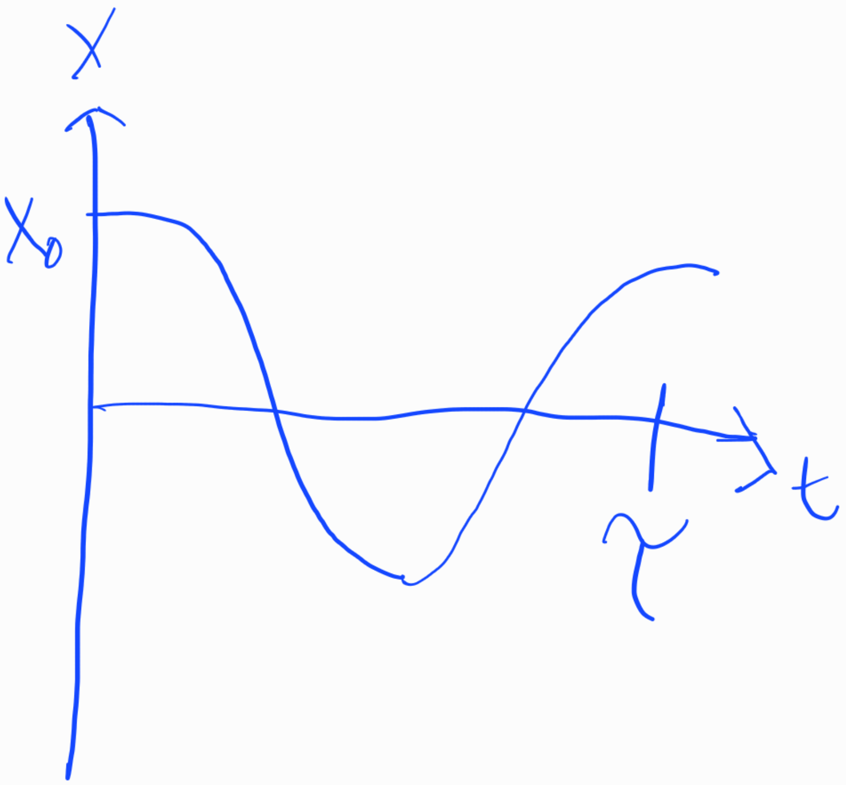
$$X(t) = X_0 \cos \omega t$$

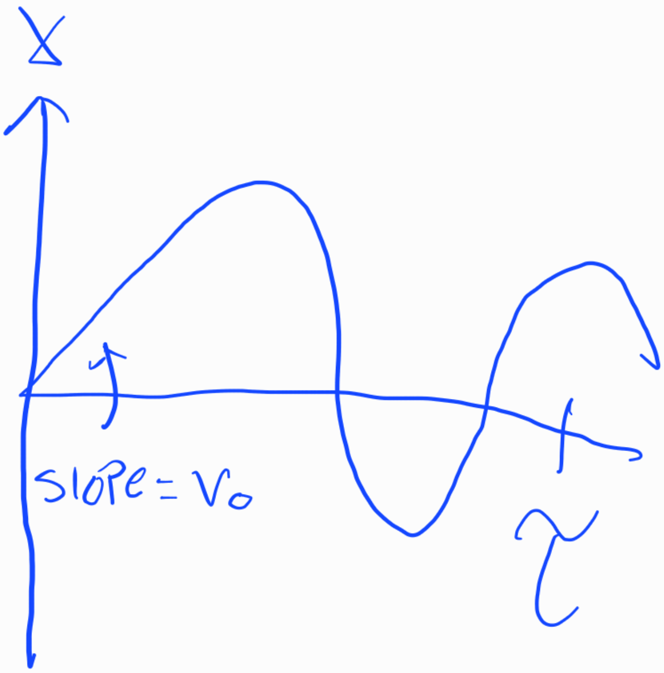
for a push at the origin $x_0 = 0$ & $t = 0$

$$x(t) = \frac{v_0}{\omega} \sin \omega t$$

, repeats when $\omega \tau = 2\pi$

$$\tau = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}$$



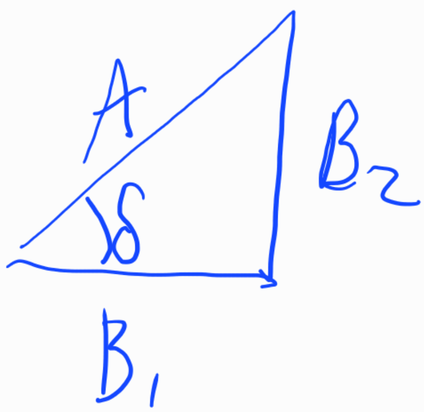



# Phase Shifted Cosine Solution

General solution hard to visualize

$$X(t) = B_1 \cos(\omega t) + B_2 \sin \omega t$$

$$A = \sqrt{B_1^2 + B_2^2}$$




$$\frac{B_1}{A} = \cos\delta, \quad \frac{B_2}{A} = \sin\delta$$

$$X(t) = A\left[\frac{B_1}{A}\cos\omega t + \frac{B_2}{A}\sin\omega t\right]$$

$$= A\left[\cos\delta\cos\omega t + \sin\delta\sin\omega t\right]$$

$$= A\cos(\omega t - \delta)$$

* Difference of angles

$$\cos(x-y) = \cos x\cos y + \sin x \sin y$$

Solution as Real Part of Complex exponential

$$B_1 = C_1 + C_2 \quad \& \quad B_2 = i(C_1 - C_2)$$

$$C_1 = \frac{1}{2}(B_1 - iB_2) \quad \& \quad C_2 = \frac{1}{2}(B_1 + iB_2)$$

$$C_2 = C_1^*$$

$$z = x + iy \qquad z^* = x - iy$$

$$x(t) = C_1 e^{i\omega t} + C_1 e^{-i\omega t}$$

$$z + z^* = (x + iy) + (x - iy) = 2x = 2\,\mathrm{Re}\,z$$

Define $C = 2C_1$

$$x(t) = 2\,\mathrm{Re}\,C_1 e^{i\omega t}$$

$$C = B_1 - iB_2 = Ae^{-i\delta}$$

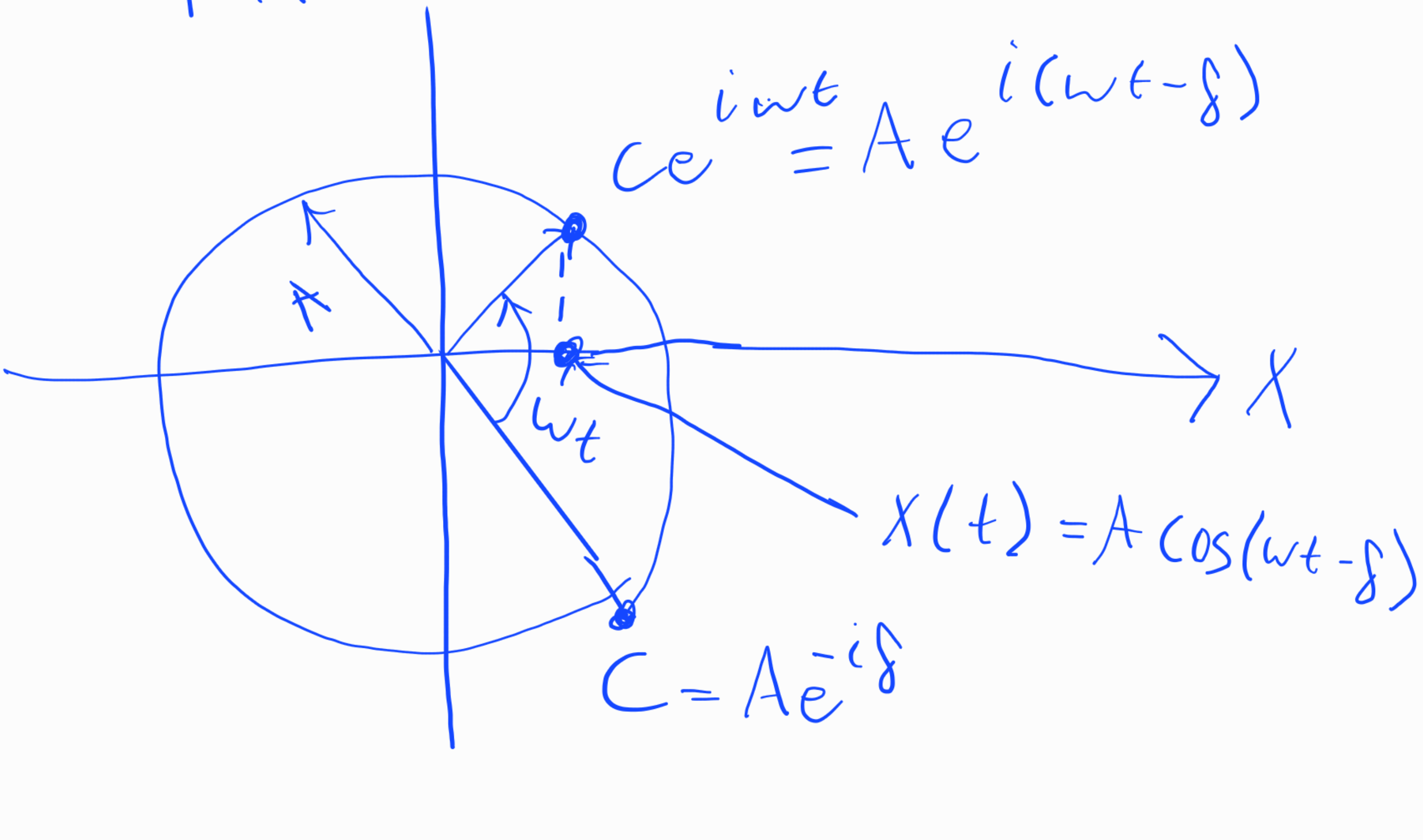
Euler's

$$x(t) = \mathrm{Re}\, Ae^{-i\delta} e^{i\omega t}$$

$$= \mathrm{Re}\, Ae^{i(\omega t - \delta)}$$

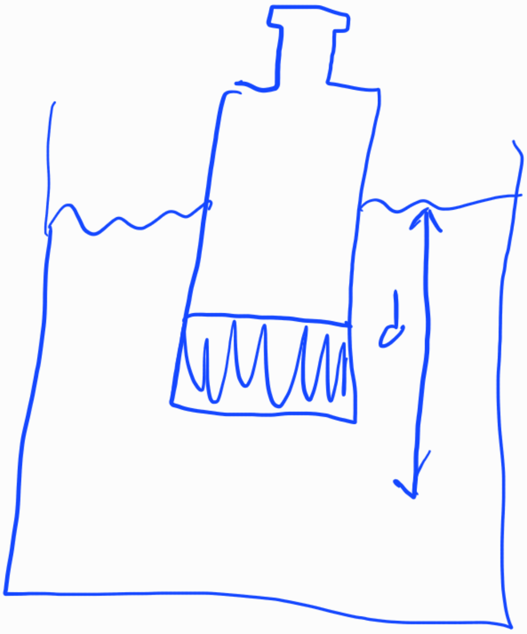
$$= A\cos(\omega t - \delta)$$

$\rho = A$



Ex 5.2

Bottle submerged @ depth $d_0$ if pushed to depth $d$ show it will execute SHM. Find $\omega$ and $\tau$ if $d_0 = 20cm$




Equilibrium

$$mg = \rho g A d_0$$

$$\frac{\rho g A}{m} = \frac{g}{d_0}$$

$$m\ddot{x} = mg - \rho g A(d_0 + x) = \cancel{mg} - \cancel{\rho g A d_0} - \rho g A x$$

$$\ddot{x} = -\frac{\rho g A}{m} x = -\underbrace{\frac{g}{d_0}} x$$

$$\omega = \sqrt{g/d_0}$$

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{d_0}{g}} = 2\pi\sqrt{\frac{0.2m}{9.8m/s^2}} = 0.9 sec$$

Energy

$$U = \frac{1}{2} k x^2 = \frac{1}{2} k A^2 \cos(\omega t - \delta)$$

$$T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m \omega^2 A^2 \sin(\omega t - \delta)$$

$$= \frac{1}{2} k A^2 \sin^2(\omega t - \delta), \quad \omega^2 = \frac{k}{m}$$

U & T oscillate between

$$0 \quad \& \quad \frac{1}{2} k A^2$$

so

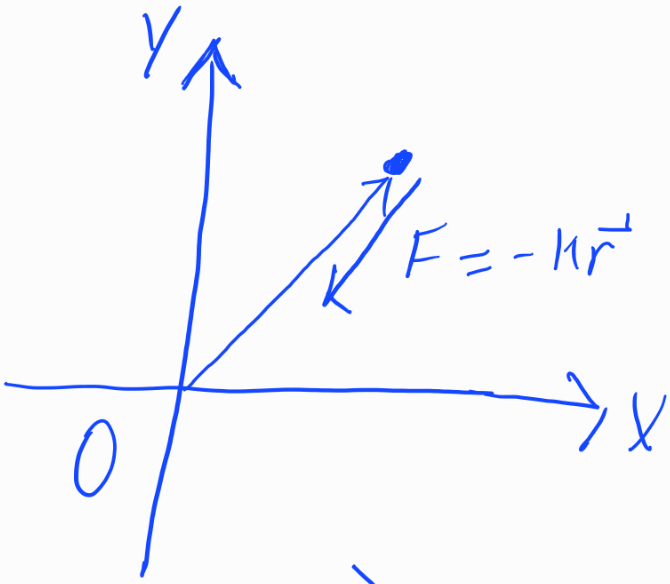
$$E = T + U = \frac{1}{2} k A^2$$

# Two Dimensional Oscillators

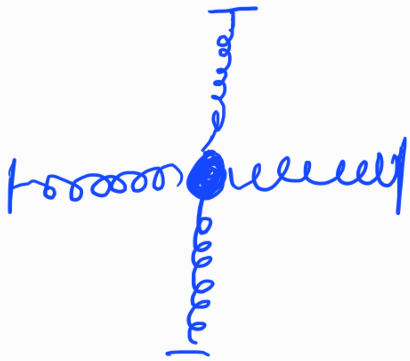
## Isotropic harmonic oscillator

$$\vec{F} = -k\vec{r}$$

$$\ddot{\vec{r}} = \frac{\vec{F}}{m} \begin{cases} \ddot{x} = -\omega^2 x \\ \ddot{y} = -\omega^2 y \end{cases} \qquad \omega = \sqrt{\frac{k}{m}}$$



a)


b)

$$x(t) = A_x \cos(\omega t - \delta_x)$$

$$y(t) = A_y \cos(\omega t - \delta_y)$$

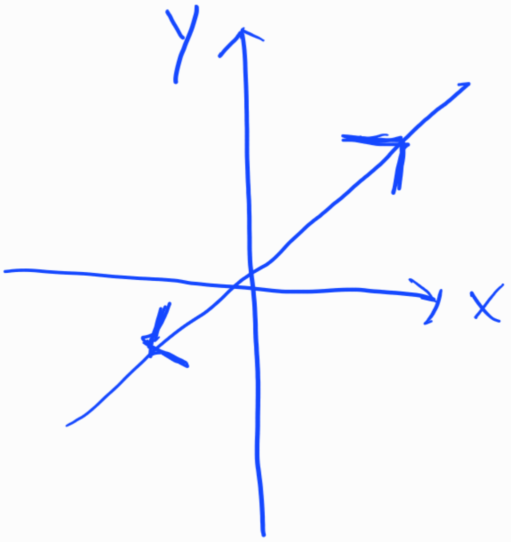
$A_x, A_y, \delta_x, \delta_y$ Determined by Initial Conditions

$$x(t) = A_x \cos(\omega t)$$

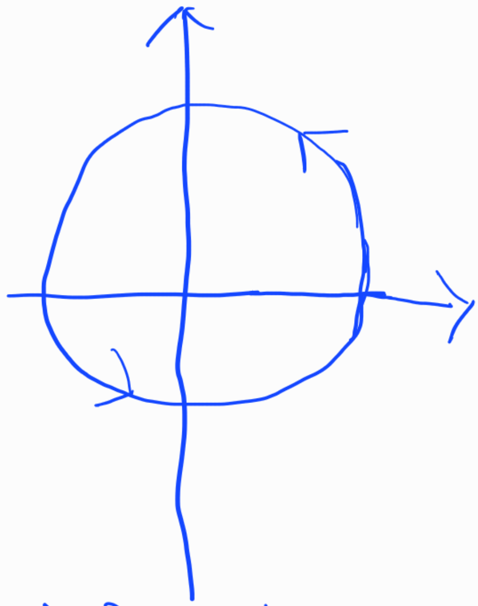
$$y(t) = A_y \cos(\omega t - \delta), \quad \delta = \delta_y - \delta_x$$

If $A_x$ or $A_y$ is zero
motion is only along one axis

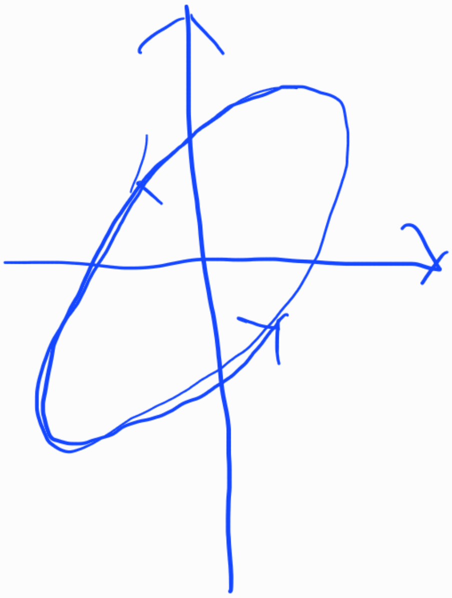
If $A_x$ or $A_y$ is not zero
then motion is dependant on $\delta$




a) $\delta = 0$



b) $\delta = \pi/2$



c) $\delta = \pi/4$

$$F_x = -k_x x, \quad F_y = -k_y y, \quad F_z = -k_z z$$

Force on atom in x-tal of low symmetry

$$\ddot{x} = -\omega_x^2 x \qquad \omega_x = \sqrt{\frac{k_x}{m}}$$

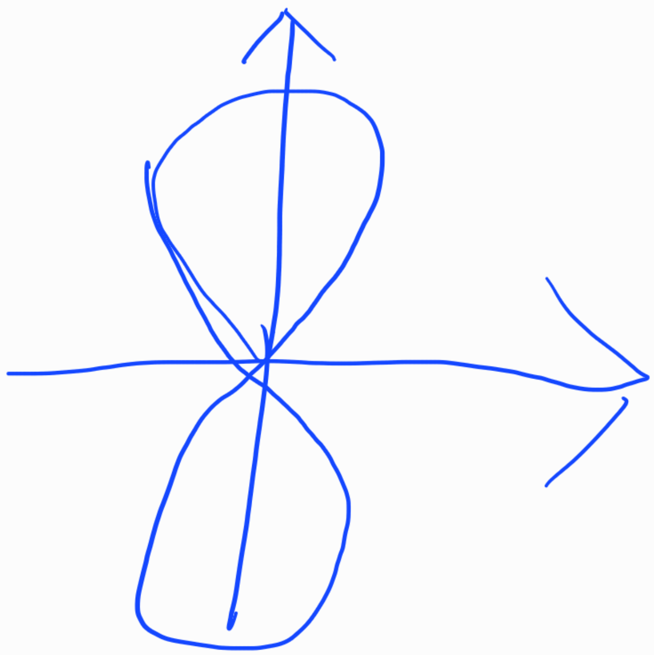
$$\ddot{y} = -\omega_y^2 y \qquad \omega_y = \sqrt{\frac{k_y}{m}}$$

$$x(t) = A_x \cos(\omega_x t)$$

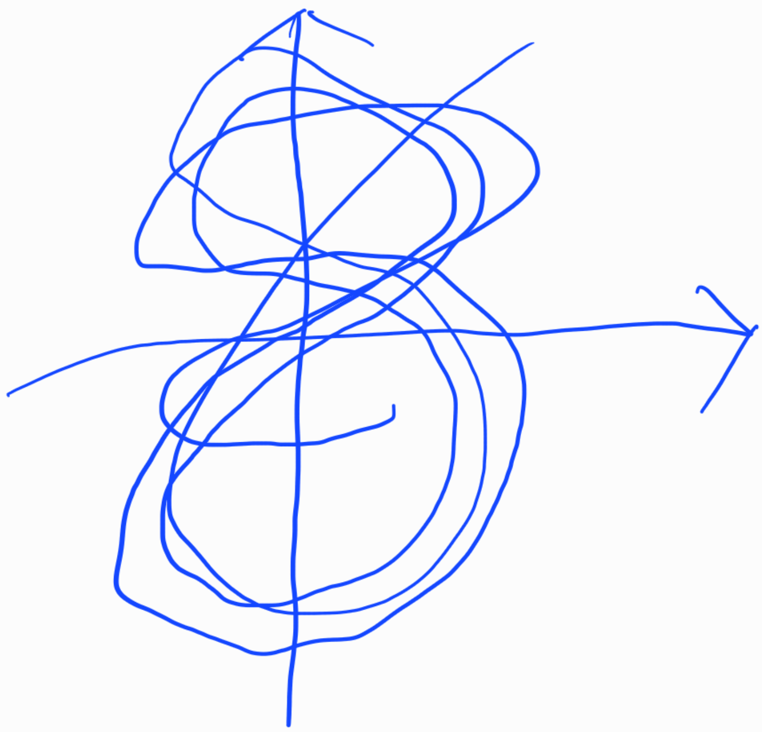
$$y(t) = A_y \cos(\omega_y t - \delta)$$

Lissajous: $\omega_x / \omega_y$

(Lisa-zhoo)



a) $\omega_x = 2\omega_y$



b) $\omega_x = \sqrt{2}\,\omega_y$

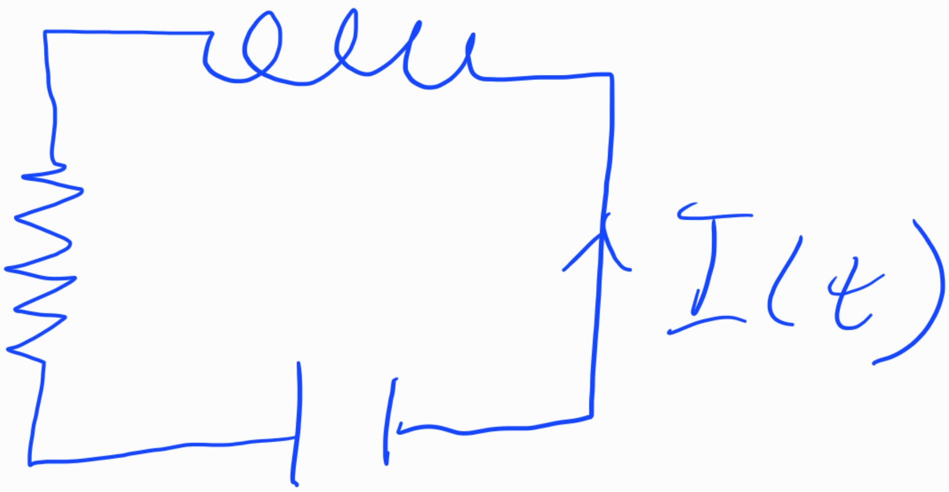
Quasiperiodic

# Damped Oscillations

$$F = \underbrace{-kx}_{\text{Hooks law}} \underbrace{-b\dot{x}}_{\text{drag}}$$

$$m\ddot{x} + b\dot{x} + kx = 0$$

## LRC circuit



$$I(t) = \dot{q}(t)$$

$$\Delta V_L = L\dot{I} = L\ddot{q}$$ (drop across Inductor)

$$\Delta V_{res} = RI = R\dot{q}$$

$$\Delta V_c = q/C$$

Kirchoff's Rule

$$\underbrace{L}_{m}\ddot{q} + \underbrace{R}_{b}\dot{q} + \underbrace{\frac{1}{C}}_{k} q = 0$$

$$\frac{b}{m} = \frac{2\beta}{\tau}$$ (divide by m)

damping constant

$$\omega_0 = \sqrt{\frac{k}{m}}$$ (natural frequency)

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

Second order, linear, homogenous equation

Must be of form

$$C_1 x_1(t) + C_2 x_2(t)$$

$$x(t) = e^{rt}, \quad \dot{x} = re^{rt}$$

$$\ddot{x} = r^2 e^{rt}$$

$$r^2 + 2\beta r + w_0^2 = 0$$

[Auxillary equation]

$$r = -\beta \pm \sqrt{\beta^2 - w_0^2}$$

$$r_1 = -\beta + \sqrt{\beta^2 - w_0^2}$$

$$r_2 = -\beta - \sqrt{\beta^2 - w_0^2}$$

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$= e^{-\beta t}\left(c_1 e^{\sqrt{\beta^2 - \omega_0^2}} + c_2 e^{-\sqrt{\beta^2 - \omega_0^2}}\right)$$

General Solution!

Undamped $\beta = 0$

$$x(t) = c_1 e^{i\omega_0 t} + c_2 e^{-i\omega_0 t}$$

$$= A\cos(\omega_0 t - \delta)$$

# Weak Damping

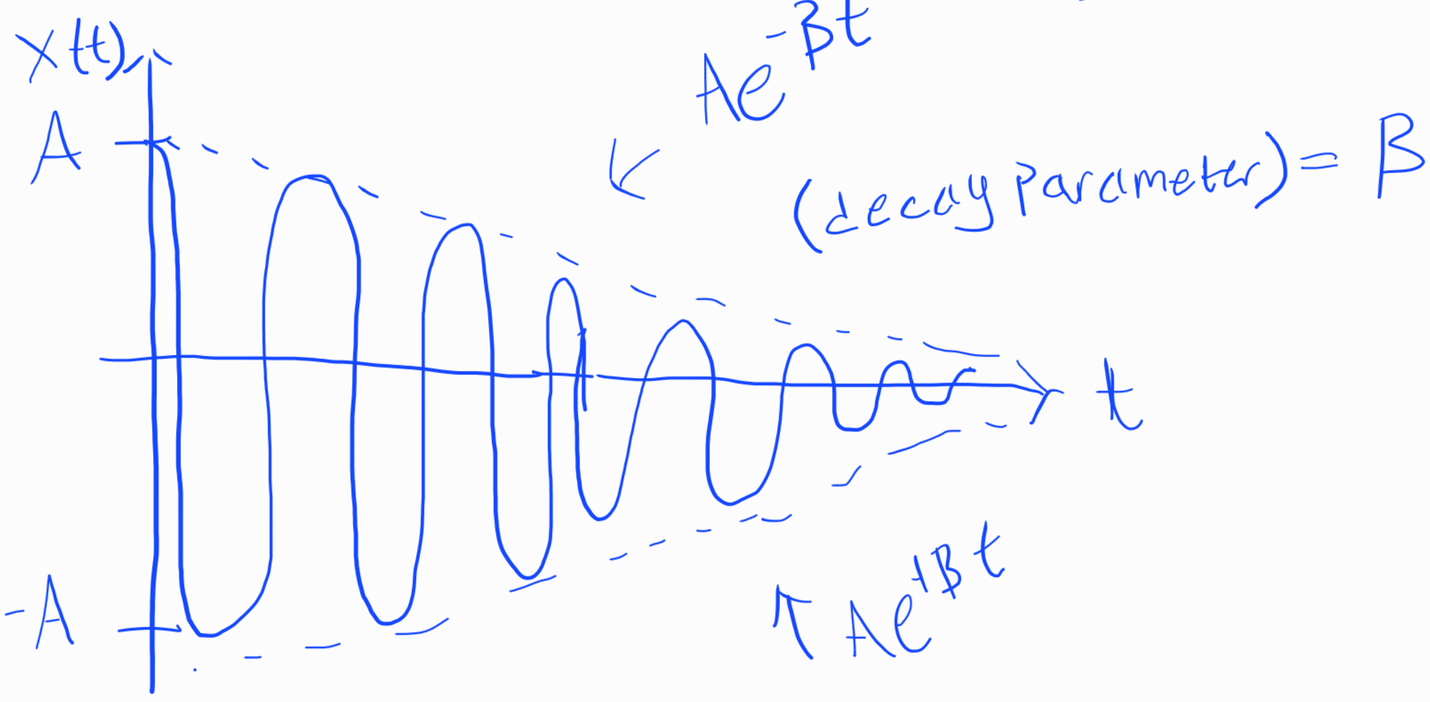
$(\beta \ll \omega_0)$ underdamping

$$\sqrt{\beta^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \beta^2} = i\,\omega_1$$

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2}$$

$$x(t) = e^{-\beta t}\left(e^{i\omega_1 t} + e^{-i\omega_1 t}\right)$$

$$x(t) = A e^{-\beta t}\cos(\omega_1 t - \delta)$$



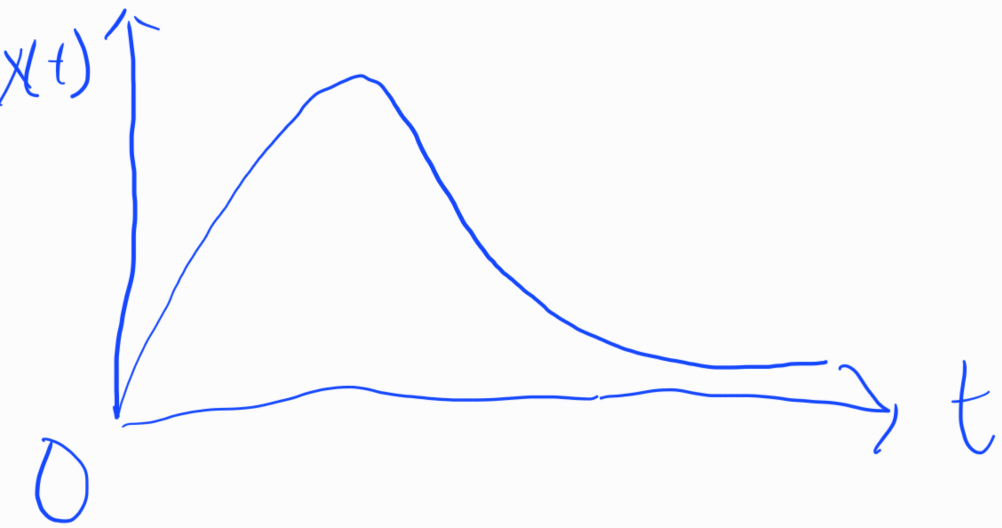
# Strong Damping (overdamping)

$\beta > \omega_0$, $r_1$ & $r_2$ are real

$$x(t) = C_1 e^{-\overbrace{(\beta - \sqrt{\beta^2 - \omega_0^2})}^{r_1} t} + C_2 e^{-\overbrace{(\beta + \sqrt{\beta^2 - \omega_0^2})}^{r_2} t}$$

$r_1 < r_2$ so $Ce^{-r_1 t}$ decreases slower and is the dominating term

$$(\text{decay parameter}) = \beta - \sqrt{\beta^2 - \omega_0^2}$$




* oscillator kicked from origin to maximum and asymptotically decays to zero

# Critical Damping

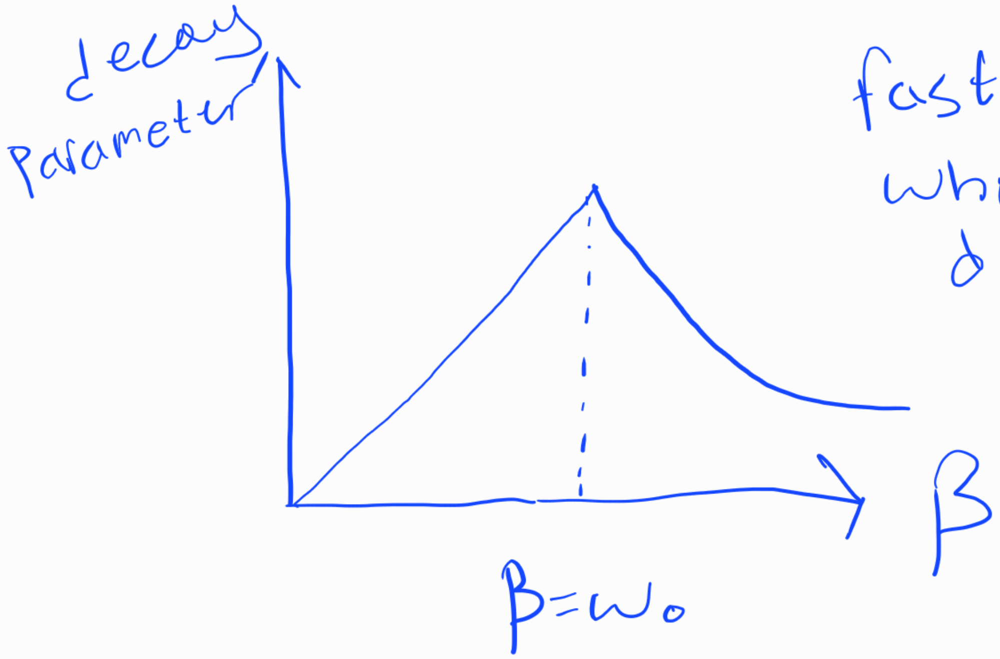
$$\beta \approx \omega_0$$

$$x(t) = e^{-\beta t} \quad \& \quad x(t) = te^{-\beta t}$$

$$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t}$$

$$x(t) = (C_1 + C_2 t) e^{-\beta t}$$

$$(\text{decay parameter}) = \beta$$




fastest decay when critically damped

# Driven Damped Oscillations

$$F = -kx - bv + F(t) = m\ddot{x}$$

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

LRC circuit with EMF $\mathcal{E}(t)$

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = \mathcal{E}(t)$$

$m\ddot{x} + b\dot{x} + kx = F(t)$, divide by $m$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \quad *$$

$$2\beta = \frac{b}{m} \quad \omega_0^2 = \frac{k}{m} \quad f(t) = \frac{F(t)}{m}$$

# Linear Differential Operators

$$D = \frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega_0^2$$

Such that

$$Dx \equiv \ddot{x} + 2\beta \dot{x} + \omega_0^2 x$$

$$Dx = f \text{ , linear}$$

$$D(ax) = aDx$$

$$D(x_1 + x_2) = Dx_1 + Dx_2$$

$$D(ax_1 + bx_2) = aDx_1 + bDx_2$$

# Damped Oscillator

$$Dx = 0 \; , \; Dx_1 = 0 \; , \; Dx_2 = 0$$

$$D(ax_1 + bx_2) = aDx_1 + bDx_2 = 0 + 0 = 0$$

$Dx = 0$ (homogeneous)

involves every term of $x$ or its derivatives exactly once

$Dx = f$ (inhomogeneous)

$f$ does not involve $x$ at all

Particular and Homogeneous solutions:

suppose $X_p(t)$ satisfies -

$$DX_p = f$$

$X_p(t)$ is a particular solution

Consider homogeneous equation:

$$DX_h = 0$$

$X_h$ is a homogeneous solution

we know

$$X_h(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

If $X_p$ exists then $X_p + X_h$ is another solution:

$$D(X_p + X_h) = DX_p + DX_h = f + 0 = f$$

We know $X_p$ is a particular solution, $X_p + X_h$ gives us a larg # of solutions containing all solutions

must find

$$X(t) = X_p(t) + X_h(t)$$

# Complex Solutions for a Sinusodial driving Force

$$f(t) = f_0 \cos(\omega t)$$

$f_0$: Amplitude of driving force divided by mass of oscillator as $f(t) = \frac{F(t)}{m}$

$\omega$: driving frequency

$* \ \omega \neq \omega_0$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

$$\downarrow$$

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 x = f_0 \sin(\omega t)$$

Define

$$z(t) = \underbrace{x(t)}_{\text{Real}} + \underbrace{i\,y(t)}_{\text{imaginary}}$$

multiply y eqn. by $i$ & add to x eqn.

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$$

find solution & take Real part

$$z(t) = C e^{i\omega t}$$

$$(-\omega^2 + 2i\beta\omega + \omega_0^2) C e^{i\omega t} = f_0 e^{i\omega t}$$

$$C = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega}$$

Define

$$C = Ae^{-i\delta}, \quad A \,\&\, \delta \text{ are real}$$

$$A^2 = CC^* = \frac{f_0}{\omega_0^2 - \omega^2 + 2\beta i\omega} \cdot \frac{f_0}{\omega_0^2 - \omega^2 - 2\beta i\omega}$$

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

(Amplitude of Oscillations caused by $f(t)$)

largest $\omega_0 \approx \omega$

$$f_0 e^{i\delta} = A(\omega_0^2 - \omega^2 + 2i\beta\omega)$$

$f_0$ & $A$ are Real

$\therefore$ $\delta$ is the same as the phase angle of the complex number $(\omega_0^2 - \omega^2) + 2i\beta\omega$

$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega}\right)$$

$$z(t) = Ce^{i\omega t} = Ae^{i(\omega t - \delta)}$$

$$x(t) = \mathrm{Re}[z(t)] = A\cos(\omega t - \delta)$$

$A$ & $\delta$ defined above

add it all together

$X_p + X_h$

$$X(t) = A\cos(\omega t - \delta) + \underbrace{C_1 e^{r_1 t} + C_2 e^{r_2 t}}_{\text{transients}}$$

transients: Die out exponentially
Depend on initial conditions.

Long term $A\cos(\omega t - \delta)$ dominates

All of this only applies
to oscillators with linear
restoring & resistive forces
leading to a linear differential
equation

for $\beta < \omega_0$

$$x(t) = \underbrace{A\cos(\omega t - \delta)}_{\text{attractor}} + \underbrace{A_{tr} e^{-\beta t} \cos(\omega_1 t - \delta_{tr})}$$

attractor

arbitrary constants

$(x_0, v_0)$ initial conditions

transients dictate low time value motion.

After some time motion trends to the inhomogeneous or particular solution

# Resonance:

an oscillator driven by a sinusoidal force

$$f(t) = f_0 \cos(\omega t)$$

See's the transient motions die out quickly and the system trends to oscillate sinusoidally @ $\omega$

$$x(t) = A\cos(\omega t - \delta)$$

with

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}, \quad A \propto f_0$$

&

$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega}\right)$$

If $\beta$ is very small

$$A^2 \approx \frac{f_0}{(\omega_0^2 - \omega^2)}$$

for $\omega_0 \gg \omega$ or $\omega_0 \ll \omega$

$A^2$ is small

for $\omega_0 \approx \omega$

$A^2$ is large

Resonance

Resonance occurs when the

$(\omega_0^2-\omega)+4\beta^2\omega^2$ is minimized

for $\omega_0$ varied & $\omega$ fixed

minimum $\rightarrow$ $\omega_0 \approx \omega$

for $\omega$ varied and $\omega_0$ fixed

minimum $\rightarrow$ $\omega=\omega_2=\sqrt{\omega_0^2-2\beta^2}$

# Different $\omega$'s

$\omega_0 = \sqrt{\frac{k}{m}}$ = natural frequency

$\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ = frequency of damped oscillator

$\omega$ = frequency of driving force

$\omega_2 = \sqrt{\omega_0^2 - 2\beta^2}$ = value of $\omega$ at resonance

$$A_{max} \approx \frac{f_0}{2\beta\omega_0}, \quad \omega_0 \approx \omega$$

# Width of the Resonance (Q Factor)

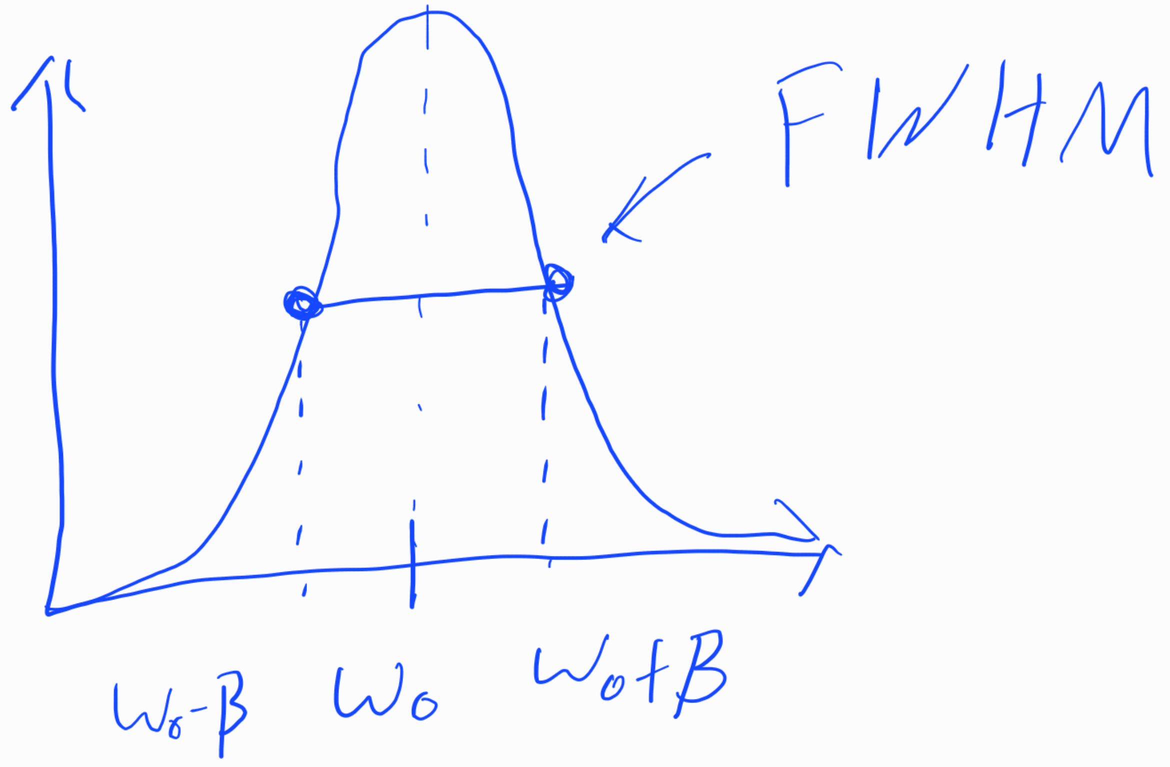
Full width at half maximum (FWHM): Interval between two points where $A^2$ is half it's maximum value

$$\omega \approx \omega_0 \pm \beta$$

$$FWHM = 2\beta$$

Half width at half max (HWHM)

$$HWHM = \beta$$




The Sharpness of the Peak is defined by the ratio:

$$Q = \frac{\omega_0}{2\beta} \left[ \frac{\text{Resonace}}{\text{width}} \right]$$

Q : Quality factor

Large Q : narrow resonance

Small Q : wide Resonance

Remember Oscillations die on the order of $t \sim 1/\beta$

$$\text{Period} = 2\pi/\omega_0 \ , \ \beta \ll \omega_0$$

$$Q = \frac{\omega_0}{2\beta} = \pi \frac{1/\beta}{2\pi/\omega_0}$$

$$= \pi \left[ \frac{\text{decay time}}{\text{Period}} \right]$$

# Phase at Resonance

$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega}\right)$$

Varying $\omega$

for $\omega \ll \omega_0$, $\delta$ small

Oscillations in step with driving force

for $\omega = \omega_0$, $\delta = \pi/2$

Oscillations $90^\circ$ Behind

for $\omega > \omega_0$, $\delta$ approaches $\pi$

Important in atomic & Quantum physics