Chapter 1

Newton's Laws of Motion

I covered this chapter in 3.5 fifty-minute lectures.

I wrote this chapter on the assumption that most of it would be review for most of our students. Thus, some of the professors using the preliminary version of the book were able to skip this chapter entirely or assign it to be read outside class. Nevertheless, I found that most of my students needed to go through the chapter fairly carefully. Several ideas, such as the concept of an inertial frame, were still quite hazy in the minds of most students, and others, such as the proof that Newton's third law is equivalent to conservation of momentum and the expression (1.47) for acceleration in polar coordinates, were new to many.

Another important role for Chapter 1 is to establish the notations used in the book. I have found that many students are distressingly conservative in the matter of notation. Many students brought up on the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, are surprisingly unwilling to accept any alternatives. Thus, I have tried to push for a little openness to the many other notations that they are bound to meet as they continue in physics. In particular, I have opted mostly for $\hat{\mathbf{x}}, \hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ in simple situations, and $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ when the going gets tougher and there is more need to use summations, such as $\mathbf{a} = \sum_{i=1}^{3} a_i \mathbf{e}_i$.

Without belaboring the point too much, I tried to give fairly rigorous definitions of mass and force, and then to discuss the three laws of Newton. Section 1.6 shows how beautifully simple Newton's second law is in Cartesian coordinates, and 1.7 how distressingly more complicated it is in polar coordinates.

In Chapter 1 an unusually high proportion of the end-of-chapter problems are devoted to refreshing the students' memories about the relevant mathematics. For example, Problems 1.1 through 1.25 can all be seen as a refresher course in vector algebra, and Problems 1.47 and 1.48 introduce cylindrical coordinates.

Solutions to Problems for Chapter 1

1.1 ⋆ $\mathbf{b} + \mathbf{c} = 2\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$, $5\mathbf{b} + 2\mathbf{c} = 7\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 2\hat{\mathbf{z}}$, $\mathbf{b}\cdot\mathbf{c} = 1$, $\mathbf{b} \times \mathbf{c} = \hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}}$.

1.2 ⋆ $\mathbf{b}+\mathbf{c}=(4,4,4)$, $5\mathbf{b}-2\mathbf{c}=(-1,6,13)$, $\mathbf{b}\cdot\mathbf{c}=10$, $\mathbf{b}\times\mathbf{c}=(-4,8,-4)$.

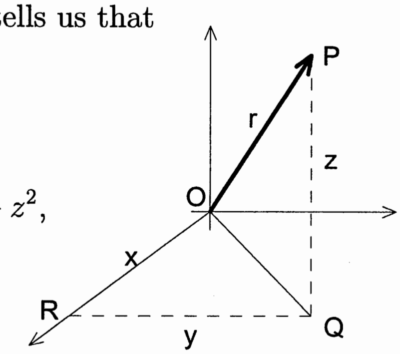
1.3 ⋆ Let P be the point with position vector $\mathbf{r}=(x,y,z)$. Let Q be the projection of P onto the xy plane at $(x,y,0)$, and let R be the projection of Q onto the x axis at $(x,0,0)$. Applied to the right triangle ORQ, Pythagoras' theorem tells us that

$$(OQ)^2=x^2+y^2,$$

and from the right triangle OQP we find

$$r^2=(OQ)^2+z^2=x^2+y^2+z^2,$$

as required.



1.4 ⋆ Since $\mathbf{b}\cdot\mathbf{c}=bc\cos\theta$, it follows that

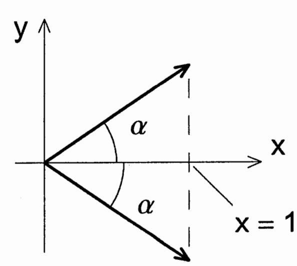
$$\theta=\arccos\left(\frac{\mathbf{b}\cdot\mathbf{c}}{bc}\right)=\arccos\left(\frac{4+4+4}{\sqrt{21}\cdot\sqrt{21}}\right)=55.2^\circ$$

1.5 ⋆ Let P be the corner at $(1,1,1)$ and Q the corner below P at $(1,1,0)$. Then OP is a body diagonal and OQ is a face diagonal, so the angle between OP and OQ is the required angle θ. Thus

$$\overrightarrow{OP}\cdot\overrightarrow{OQ}=\begin{cases}=|\overrightarrow{OP}|\,|\overrightarrow{OQ}|\cos\theta=\sqrt{3}\sqrt{2}\cos\theta\\=(1,1,1)\cdot(1,1,0)=2\end{cases}$$

Equating these two, we find $\cos\theta=\sqrt{2/3}$ and hence $\theta=\arccos\sqrt{2/3}=0.615$ rad or 35.3° .

1.6 ⋆ $\mathbf{b}\cdot\mathbf{c}=1-s^2$, which is zero if and only if $s=\pm1$. The vectors $\mathbf{b}$ and $\mathbf{c}$ make equal angles, α, above and below (or below and above) the x axis. The angle between them can be 90° only if $\alpha=45^\circ$.



1.7 ⋆ If we choose our x axis in the direction of $\mathbf{r}$, then $\mathbf{r}=(r,0,0)$, whereas $\mathbf{s}=(s_x,s_y,s_z)$. As usual $s_x=s\cos\theta$, where θ is the angle between $\mathbf{s}$ and the x axis; with our choice of axes, this means that θ is the angle between $\mathbf{s}$ and $\mathbf{r}$. Thus, according to definition (1.7)

$$\mathbf{r}\cdot\mathbf{s}\text{ [definition (1.7)]}=\sum r_is_i=rs_x+0+0=rs\cos\theta=\mathbf{r}\cdot\mathbf{s}\text{ [definition (1.6)]}$$

1.8 ⋆ **(a)** Starting from the definition (1.7), we see that

$$\mathbf{r}\cdot(\mathbf{u}+\mathbf{v}) = \sum r_i(u_i+v_i) = \sum(r_iu_i + r_iv_i) = \sum r_iu_i + \sum r_iv_i = \mathbf{r}\cdot\mathbf{u} + \mathbf{r}\cdot\mathbf{v}$$

where the second equality follows from the distributive property of ordinary numbers. The third equality is just a rearrangement of the six terms of the sum, and the last is just the definition (1.7) of the two scalar products.

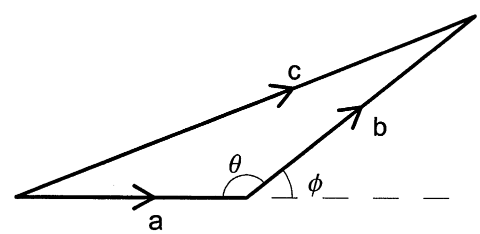
(b) Starting again from (1.7), we find

$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{s}) = \frac{d}{dt}\sum r_is_i = \sum\left(r_i\frac{ds_i}{dt} + \frac{dr_i}{dt}s_i\right) = \mathbf{r}\cdot\frac{d\mathbf{s}}{dt} + \frac{d\mathbf{r}}{dt}\cdot\mathbf{s}.$$

1.9 ⋆ Consider the three vectors **a**, **b**, and $\mathbf{c} = \mathbf{a} + \mathbf{b}$ defined in the figure. The angle ϕ is the angle between **a** and **b**, and the angle θ of the triangle is $\theta = \pi - \phi$. By the given identity,

$$\begin{aligned} c^2 &= (\mathbf{a}+\mathbf{b})^2 = a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b} \\ &= a^2 + b^2 + 2ab\cos\phi = a^2 + b^2 - 2ab\cos\theta, \end{aligned}$$

since $\theta = \pi - \phi$ and hence $\cos\phi = -\cos\theta$.



1.10 ⋆ The particle's polar angle is $\phi = \omega t$, so $x = R\cos(\omega t)$ and $y = R\sin(\omega t)$ or

$$\mathbf{r} = \hat{\mathbf{x}}\,R\cos(\omega t) + \hat{\mathbf{y}}\,R\sin(\omega t).$$

Differentiating, we find that $\dot{\mathbf{r}} = -\hat{\mathbf{x}}\,\omega R\sin(\omega t) + \hat{\mathbf{y}}\,\omega R\cos(\omega t)$ and then

$$\ddot{\mathbf{r}} = -\hat{\mathbf{x}}\,\omega^2 R\cos(\omega t) - \hat{\mathbf{y}}\,\omega^2 R\sin(\omega t) = -\omega^2\mathbf{r} = -\omega^2 R\hat{\mathbf{r}}.$$

That is, the acceleration is antiparallel to the radius vector and has magnitude $a = \omega^2 R = v^2/R$, the well known centripetal acceleration.

1.11 ⋆ The particle's coordinates are $x(t) = b\cos(\omega t), y(t) = c\sin(\omega t)$, and $z(t) = 0$. It remains in the plane $z = 0$ at all times, and, since

$$\frac{x^2}{b^2} + \frac{y^2}{c^2} = \cos^2(\omega t) + \sin^2(\omega t) = 1$$

it moves in an ellipse with semimajor and semiminor axes b and c in the xy plane. It is easy to see that it moves counterclockwise and returns to its starting point in a period $\tau = 2\pi/\omega$.

1.12 ⋆ Note first that the z coordinate increases at the constant rate v_o. Next consider the projection of **r** onto the xy plane, namely, the point $(x, y) = (b\cos\omega t, \sin\omega t)$. By inspection this satisfies $x^2/b^2 + y^2/c^2 = 1$, the equation of an ellipse centered on the origin. It is easy to see that (x, y) moves around this ellipse in a counter-clockwise direction and returns to its starting point in a time $\tau = 2\pi/\omega$. Meanwhile, z is increasing steadily, so the point **r** is moving up the surface of an elliptical cylinder centered on the z axis.

1.22 ⋆⋆ **(a)** According to definition (1.6), $\mathbf{a}\cdot\mathbf{b} = ab\cos(\alpha-\beta)$, but according to (1.7)

$$\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y = ab(\cos\alpha\cos\beta + \sin\alpha\sin\beta).$$

Comparing these two expressions we get the desired result.

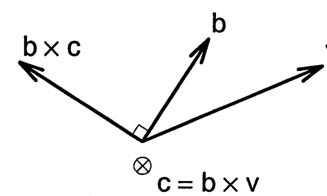
(b) According to the right-hand-rule definition, $\mathbf{a}\times\mathbf{b} = -ab\sin(\alpha-\beta)\hat{\mathbf{z}}$. (The minus sign is easy to check; for example, if $\alpha > \beta$, then the right-hand rule puts $\mathbf{a}\times\mathbf{b}$ in the negative z direction.) On the other hand, the definition (1.9) gives the z component as

$$(\mathbf{a}\times\mathbf{b})_z = a_x b_y - a_y b_x = ab(\cos\alpha\sin\beta - \sin\alpha\cos\beta).$$

Comparing these two expressions we again get the desired result.

1.23 ⋆⋆ The picture shows the plane of $\mathbf{b}$ and $\mathbf{v}$. Because $\mathbf{c} = \mathbf{b}\times\mathbf{v}$, $\mathbf{c}$ is perpendicular to this plane, as indicated. Therefore $\mathbf{b}\times\mathbf{c}$ lies in the plane and is perpendicular to $\mathbf{b}$. This means that $\mathbf{v}$ can be expressed in terms of the two mutually perpendicular vectors $\mathbf{b}$ and $\mathbf{b}\times\mathbf{c}$:

$$\mathbf{v} = \alpha\,\mathbf{b} + \beta\,\mathbf{b}\times\mathbf{c}$$



If we form the dot product of this equation with $\mathbf{b}$ we find that $\mathbf{b}\cdot\mathbf{v} = \alpha\mathbf{b}\cdot\mathbf{b}$. Therefore $\alpha = \mathbf{b}\cdot\mathbf{v}/b^2 = \lambda/b^2$. Similarly, if we form the cross product of the same equation with $\mathbf{b}$, we find that $\mathbf{b}\times\mathbf{v} = \beta\mathbf{b}\times(\mathbf{b}\times\mathbf{c})$. Since $\mathbf{b}\times\mathbf{v} = \mathbf{c}$ and (as you can easily check) $\mathbf{b}\times(\mathbf{b}\times\mathbf{c}) = -b^2\mathbf{c}$, this last implies that $\beta = -1/b^2$, and we conclude that $\mathbf{v} = (\lambda\mathbf{b} - \mathbf{b}\times\mathbf{c})/b^2$.

1.24 ⋆ Integrating the equation $df/f = dt$, we find that $\ln f = t + k$, or $f = Ae^t$, with one arbitrary constant, $A = e^k$.

1.25 ⋆ Integrating the equation $df/f = -3dt$, we find that $\ln f = -3t + k$, or $f = Ae^{-3t}$, with one arbitrary constant, $A = e^k$.

1.26 ⋆⋆ **(a)** Since the puck is frictionless, the net force on it is zero, and, as seen from frame $\mathcal{S}$ the puck heads due north with constant speed v_o, so $x = 0$ and $y = v_o t$, where v_o is the speed with which I kicked the puck.

(b) We must now find how to translate the coordinates (x, y) of any point P, as seen in frame $\mathcal{S}$, to the coordinates (x', y') of the same point P as seen in $\mathcal{S}'$. At time t, the origin of $\mathcal{S}'$ is at $(vt, 0)$ (as seen in $\mathcal{S}$), where v is the speed of $\mathcal{S}'$ relative to $\mathcal{S}$. Therefore $(x, y) = (vt, 0) + (x', y')$; whence $x' = x - vt$ and $y' = y$. This is for any point (x, y). If we substitute the coordinates of the puck from part (a), we find

$$x' = -vt \quad \text{and} \quad y' = v_o t.$$

As seen from frame $\mathcal{S}'$, the puck moves in a straight line toward the north-west quadrant with velocity $(-v, v_o)$. Since the velocity is constant, this motion is consistent with Newton's first law and frame $\mathcal{S}'$ is apparently inertial.

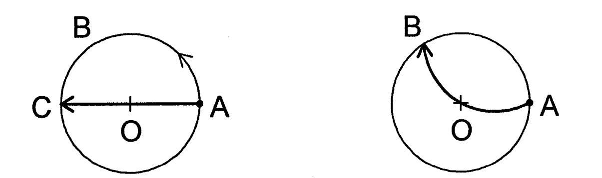
(c) At time t, the origin of frame $\mathcal{S}''$ is at the point $(\frac{1}{2}at^2, 0)$ as seen in $\mathcal{S}$. Therefore, the coordinates of any point (x, y) as seen in $\mathcal{S}$ are $x'' = x - \frac{1}{2}at^2$ and $y'' = y$ as seen in $\mathcal{S}''$. Substituting the coordinates of the puck from part (a), we find

$$x'' = -\tfrac{1}{2}at^2 \qquad \text{and} \qquad y' = v_o t.$$

Therefore, as seen from $\mathcal{S}''$, the puck moves toward the NW quadrant in a parabola, with v''_x steadily increasing (in the negative direction). Since the velocity is not constant, Newton's first law is not valid, and $\mathcal{S}''$ is not an inertial frame.

1.27 ⋆⋆ Since the puck is frictionless, the net force on it is zero, and, as seen from the ground, it travels in a straight line through the center O, as shown in the left picture. It starts from the point A at $t = 0$, travels "due west" with constant speed v_o, and falls onto the ground at point C after a time $T = 2R/v_o$ (where R is the radius of the turntable).

Now imagine an observer sitting on the turntable near A. As seen from the ground, he is traveling north with speed ωR. Therefore, as seen by the observer, the puck's initial velocity has a sideways (southerly) component ωR, in addition to the westerly component v_o; that is, the puck moves initially west *and* south, as shown in the right picture. (The magnitude of the southerly component depends on the table's rate of rotation ω.) As the puck moves in to a smaller radius r, the sideways component ωr gets less, so the puck's path curves to the right. Continuing to curve, its passes through O and eventually reaches the edge of the turntable at point B. The left picture shows the point B of the table at time $t = 0$. The position of B is determined by the following consideration: In the time $T = 2R/v_o$ for the puck to cross the table, point B of the table must move around to point C where we know the puck falls to the ground. Thus the angle BOC is equal to ωT. The faster the table rotates, the larger the angle BOC and the more sharply the puck's path (as seen from the table) is curved.



seen from ground   seen from turntable

1.28 ⋆ When we write out Equations (1.25) and (1.26) for the three particles, we get three equations:

$$\begin{aligned}
\dot{\mathbf{p}}_1 &= (\text{net force on particle 1}) = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_1^{\text{ext}} \\
\dot{\mathbf{p}}_2 &= (\text{net force on particle 2}) = \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_2^{\text{ext}} \\
\dot{\mathbf{p}}_3 &= (\text{net force on particle 3}) = \mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_3^{\text{ext}}\,.
\end{aligned}$$

Adding these three equations, we find for $\dot{\mathbf{P}} = \dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 + \dot{\mathbf{p}}_3$,

$$\dot{\mathbf{P}} = (\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{31} + \mathbf{F}_{32}) + \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_3^{\text{ext}}. \qquad \text{(i)}$$

This corresponds to Equation (1.27). The first six terms on the right can be rearranged to give

$$\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{31} + \mathbf{F}_{32} = (\mathbf{F}_{12} + \mathbf{F}_{21}) + (\mathbf{F}_{13} + \mathbf{F}_{31}) + (\mathbf{F}_{23} + \mathbf{F}_{32}) = 0$$

since each of the three pairs on the right is zero by Newton's third law. Thus Equation (i) for $\dot{\mathbf{P}}$ reduces to

$$\dot{\mathbf{P}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_3^{\text{ext}} = \mathbf{F}^{\text{ext}}$$

which is the required Equation (1.29).

1.29 ⋆ When we write out Equations (1.25) and (1.26) for the four particles, we get four equations:

$$\begin{aligned}
\dot{\mathbf{p}}_1 &= (\text{net force on particle 1}) = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_1^{\text{ext}} \\
\dot{\mathbf{p}}_2 &= (\text{net force on particle 2}) = \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24} + \mathbf{F}_2^{\text{ext}} \\
\dot{\mathbf{p}}_3 &= (\text{net force on particle 3}) = \mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_{34} + \mathbf{F}_3^{\text{ext}} \\
\dot{\mathbf{p}}_4 &= (\text{net force on particle 4}) = \mathbf{F}_{41} + \mathbf{F}_{42} + \mathbf{F}_{43} + \mathbf{F}_4^{\text{ext}}.
\end{aligned}$$

Adding these four equations, we find for $\dot{\mathbf{P}} = \dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 + \dot{\mathbf{p}}_3 + \dot{\mathbf{p}}_4$,

$$\begin{aligned}
\dot{\mathbf{P}} = {} & (\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14}) + (\mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24}) + (\mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_{34}) + (\mathbf{F}_{41} + \mathbf{F}_{42} + \mathbf{F}_{43}) \\
& + (\mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_3^{\text{ext}} + \mathbf{F}_4^{\text{ext}}).
\end{aligned} \tag{ii}$$

This corresponds to Equation (1.27). The twelve terms on the first line of the right side can be rearranged to give

$$(\mathbf{F}_{12} + \mathbf{F}_{21}) + (\mathbf{F}_{13} + \mathbf{F}_{31}) + (\mathbf{F}_{14} + \mathbf{F}_{41}) + (\mathbf{F}_{23} + \mathbf{F}_{32}) + (\mathbf{F}_{24} + \mathbf{F}_{42}) + (\mathbf{F}_{34} + \mathbf{F}_{43}) = 0$$

since each of the six pairs is zero by Newton's third law. Thus Equation (ii) for $\dot{\mathbf{P}}$ reduces to

$$\dot{\mathbf{P}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_3^{\text{ext}} + \mathbf{F}_4^{\text{ext}} = \mathbf{F}^{\text{ext}}$$

which is the required Equation (1.29).

1.30 ⋆ Since mass 2 is at rest, the initial total momentum is just $\mathbf{P}_{\text{in}} = m_1\mathbf{v}$. The final total momentum is $\mathbf{P}_{\text{fin}} = (m_1 + m_2)\mathbf{v}'$. Equating these two and solving for $\mathbf{v}'$, we find that $\mathbf{v}' = \mathbf{v}m_1/(m_1 + m_2)$.

1.31 ⋆ We have to prove for any pair of particles (call them 1 and 2) that $\mathbf{F}_{12} = -\mathbf{F}_{21}$. To do this, suppose that all forces on 1 and 2, except $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$, have been switched off. For example, we could move all bodies with which 1 and 2 interact to a great distance. If the law of conservation of momentum holds, then $\mathbf{p}_1 + \mathbf{p}_2$ is consant, which implies that $\dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 = 0$. Since all other forces have been switched off, the only force on particle 1 is $\mathbf{F}_{12}$ and the only force on 2 is $\mathbf{F}_{21}$. Therefore this last equation implies that $\mathbf{F}_{12} + \mathbf{F}_{21} = 0$, which is the required result. (There is a subtle point here: We have assumed that switching

$$\dot{\mathbf{L}} = \sum \boldsymbol{\Gamma}_\alpha = \boldsymbol{\Gamma}. \tag{iv}$$

That is, the rate of change of $\mathbf{L}$ is the total torque on the system. Unfortunately, this isn't yet enough, since we need to separate the effects of the internal and external forces in accordance with Equation (1.19). This gives

$$\boldsymbol{\Gamma} = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_\alpha = \sum_\alpha \sum_{\beta \neq \alpha} (\mathbf{r}_\alpha \times \mathbf{F}_{\alpha\beta}) + \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_\alpha^{\text{ext}}.$$

Here the final sum is just the net *external* torque $\boldsymbol{\Gamma}^{\text{ext}}$. In the double sum, we can pair off terms, grouping each $\mathbf{F}_{\alpha\beta}$ with $\mathbf{F}_{\beta\alpha} = -\mathbf{F}_{\alpha\beta}$, to give

$$\boldsymbol{\Gamma} = \sum_\alpha \sum_{\beta > \alpha} (\mathbf{r}_\alpha - \mathbf{r}_\beta) \times \mathbf{F}_{\alpha\beta} + \boldsymbol{\Gamma}^{\text{ext}}.$$

Now, the vector $(\mathbf{r}_\alpha - \mathbf{r}_\beta)$ points from particle β to particle α. (This is illustrated in Figure 3.7.) Thus, *provided* the internal forces $\mathbf{F}_{\alpha\beta}$ are *central*, these two vectors are collinear, and their cross product is zero, so that the whole double sum is zero. Therefore, $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{\text{ext}}$ and, according to Equation (iv), $\dot{\mathbf{L}} = \boldsymbol{\Gamma}^{\text{ext}}$. In particular, if there are no external forces, $\mathbf{L}$ is constant.

1.35 ⋆ In the absence of air resistance, the net force on the ball is $\mathbf{F} = m\mathbf{g}$, and with the given choice of axes, $\mathbf{g} = (0, 0, -g)$. Thus Newton's second law, $\mathbf{F} = m\ddot{\mathbf{r}}$, implies that $\ddot{\mathbf{r}} = \mathbf{g}$, or

$$\ddot{x} = 0, \qquad \ddot{y} = 0, \qquad \text{and} \qquad \ddot{z} = -g.$$

The initial velocity has components $v_{ox} = v_o \cos\theta$, $v_{oy} = 0$, and $v_{oz} = v_o \sin\theta$, and we can choose the initial position to be the origin. The first of the above equations can be integrated once to give $\dot{x} = v_{ox}$, and again to give $x(t) = v_{ox}t$. In the same way, the y equation gives $y(t) = 0$, and the z equation gives $z(t) = v_{oz}t - \frac{1}{2}gt^2$. The ball returns to the ground when $z(t) = 0$ which gives $t = 2v_{oz}/g$. Substituting this time into the expression for $x(t)$ gives the range, range $= 2v_{oz}v_{oz}/g$.

1.36 ⋆ **(a)** During the flight the only force on the bundle is its weight, and Newton's second law reads $m\ddot{\mathbf{r}} = \mathbf{F} = m\mathbf{g}$, or $\ddot{\mathbf{r}} = \mathbf{g}$. If we choose the origin at sea level directly below the plane at the moment of launch and measure x in the direction of flight and y vertically up, then the solution is $x = v_o t$, $y = h - \frac{1}{2}gt^2$, and $z = 0$.

(b) The time for the bundle to drop to sea level ($y = 0$) is $t = \sqrt{2h/g}$ and the horizontal distance traveled in this time is $x = v_o t = v_o\sqrt{2h/g}$. With the given numbers this is about 220 m.

(c) If the drop is delayed by a time Δt, the bundle will overshoot by a distance $\Delta x = v_o \Delta t$, so $\Delta t = \Delta x / v_o = 0.2$ sec.

1.37 $\star$ **(a)** The two forces on the puck are its weight $m\mathbf{g}$ and the normal force $\mathbf{N}$ of the incline. If we choose axes with x measured up the slope, y along the outward normal, and z horizontally across the slope, then $\mathbf{N} = (0, N, 0)$ and $\mathbf{g} = (-g\sin\theta, g\cos\theta, 0)$. Thus Newton's second law reads

$$m\ddot{\mathbf{r}} = \mathbf{N} + m\mathbf{g} \quad \text{or} \quad \begin{cases} m\ddot{x} = -mg\sin\theta \\ m\ddot{y} = N - mg\cos\theta \\ m\ddot{z} = 0 \end{cases}$$

Since $\dot{z} = 0$ initially, it remains so and hence $z = 0$ for all t. The normal force adjusts itself so that $\ddot{y} = 0$, and $y = 0$ for all t. Finally, $\ddot{x} = -g\sin\theta$, which can be integrated twice to give $x = v_o t - \frac{1}{2}gt^2\sin\theta$.

(b) Solving for the times when $x = 0$, we find that $t = 0$ (at launch) or $t = 2v_o/(g\sin\theta)$ (the answer of interest).

1.38 $\star$ The two forces on the puck are its weight $m\mathbf{g}$ and the normal force $\mathbf{N}$ of the incline. With the suggested choice of axes, $\mathbf{N} = (0, 0, N)$ and $\mathbf{g} = (0, -g\sin\theta, -g\cos\theta)$. Thus Newton's second law reads

$$m\ddot{\mathbf{r}} = \mathbf{N} + m\mathbf{g} \quad \text{or} \quad \begin{cases} m\ddot{x} = 0 \\ m\ddot{y} = -mg\sin\theta \\ m\ddot{z} = N - mg\cos\theta \end{cases}$$

By integrating the y equation twice, we find that $y = v_{oy}t - \frac{1}{2}gt^2\sin\theta$. Thus the time to return to the line $y = 0$ is $t = 2v_{oy}/(g\sin\theta)$ and the distance from O at that time is $x = v_{ox}t = 2v_{ox}v_{oy}/(g\sin\theta)$.

1.39 $\star\star$ $x = v_o t\cos\theta - \frac{1}{2}gt^2\sin\phi$, $y = v_o t\sin\theta - \frac{1}{2}gt^2\cos\phi$, $z = 0$. When the ball returns to the plane, y is 0, which implies that $t = 2v_o\sin\theta/(g\cos\phi)$. Substituting this time into x and using a couple of trig identities yields the claimed answer for the range R. To find the maximum range, differentiate R with respect to θ and set the derivative equal to zero. This gives $\theta = (\pi - 2\phi)/4$, and substitution into R (plus another trig identity) yields the claimed value of R_{max}.

1.40 $\star\star\star$ **(a)** $x = (v_o\cos\theta)t$, $y = (v_o\sin\theta)t - \frac{1}{2}gt^2$, and $z = 0$

(b) $r^2 = x^2 + y^2 = \frac{1}{4}g^2t^4 - (v_o g\sin\theta)t^3 + v_o^2t^2$, so $d(r^2)/dt = g^2t^3 - 3(v_o g\sin\theta)t^2 + 2v_o^2t$. When t is sufficiently small, r certainly increases with time. It's derivative vanishes if and only if

$$t = \frac{v_o}{2g}\left(3\sin\theta \pm \sqrt{9\sin^2\theta - 8}\right).$$

If θ is small, the argument of the square root is negative, and r always increases. As θ increases, the first value of θ for which the derivative does vanish is given by $\sin\theta = \sqrt{8/9}$; that is, $\theta_{\text{max}} = 70.5°$. Thus for $0 \le \theta < \theta_{\text{max}}$, r always increases. For $\theta_{\text{max}} < \theta \le 90°$, r

increases initially, but then decreases for a while. (This last is particularly clear for the case that $\theta = 90°$.)

1.41 ⋆ The r and ϕ components of Newton's second law are $F_r = m(\ddot{r} - r\dot{\phi}^2)$ and $F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$. Since $r = R$ is constant $\dot{r} = \ddot{r} = 0$, and since $\dot{\phi} = \omega$ is constant $\ddot{\phi} = 0$. Finally the only force is the inward tension of the string, so $F_r = -T$. Thus the r equation becomes $-T = -mR\omega^2$, so $T = m\omega^2 R$, which is just the familiar "centripetal force" mv^2/R, since $\omega = v/R$.

1.42 ⋆ Elementary trigonometry applied to the triangle of Fig.1.10 shows that $x = r\cos\phi$ and $y = r\sin\phi$, and by Pythagoras' theorem $r = \sqrt{x^2 + y^2}$. Clearly $\tan\phi = y/x$, so it is tempting to say that $\phi = \arctan(y/x)$. Unfortunately this isn't quite satisfactory. The difficulty is that the two distinct vectors $\mathbf{r}$ and $-\mathbf{r}$ have the same ratio y/x [since $(-y)/(-x) = y/x$] but their polar angles should differ by π. The simple-minded claim that $\phi = \arctan(y/x)$ can't distinguish these two cases. One way out is to define a function $\arctan(x, y)$ which puts the angle in the right quadrant. For example, if arctan is defined to lie between $-\pi/2$ and $\pi/2$, then we could define

$$\arctan(x, y) = \begin{cases} \arctan(y/x) & \text{if } x > 0 \\ \pi/2 & \text{if } x = 0 \text{ and } y > 0 \\ -\pi/2 & \text{if } x = 0 \text{ and } y < 0 \\ \arctan(y/x) + \pi & \text{if } x < 0 \end{cases}$$

If we define $\phi = \arctan(x, y)$, then every nonzero vector has a well defined polar angle ϕ, any two vectors in different directions have different polar angles in the range $-\pi/2 \le \phi < 3\pi/2$, and (most important) ϕ is always in the correct quadrant.

1.43 ⋆ **(a)** From Figure 1.11(b), you can see that the x and y components of $\hat{\mathbf{r}}$ are $\cos\phi$ and $\sin\phi$. (Remember that $|\hat{\mathbf{r}}| = 1$.) Therefore, $\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$. Similarly, the x and y components of $\hat{\boldsymbol{\phi}}$ are $-\sin\phi$ and $\cos\phi$, so that $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$.

(b) Differentiating these two results with respect to t and using the chain rule, we find (Remember that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are constant.)

$$\frac{d}{dt}\hat{\mathbf{r}} = \dot{\phi}(-\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi) = \dot{\phi}\hat{\boldsymbol{\phi}} \quad \text{and} \quad \frac{d}{dt}\hat{\boldsymbol{\phi}} = \dot{\phi}(-\hat{\mathbf{x}}\cos\phi - \hat{\mathbf{y}}\sin\phi) = -\dot{\phi}\,\hat{\mathbf{r}}.$$

1.44 ⋆ If $\phi = A\sin(\omega t) + B\cos(\omega t)$, then

$$\dot{\phi} = \omega A\cos(\omega t) - \omega B\sin(\omega t) \quad \text{and} \quad \ddot{\phi} = -\omega^2 A\sin(\omega t) - \omega^2 B\cos(\omega t) = -\omega^2\phi$$

and ϕ does indeed satisfy $\ddot{\phi} = -\omega^2\phi$.

1.45 ⋆⋆ Since the magnitude of $\mathbf{v}(t)$ is the same as $\sqrt{\mathbf{v}(t)\cdot\mathbf{v}(t)}$, the magnitude is constant if and only if $\mathbf{v}(t)\cdot\mathbf{v}(t)$ is. Since

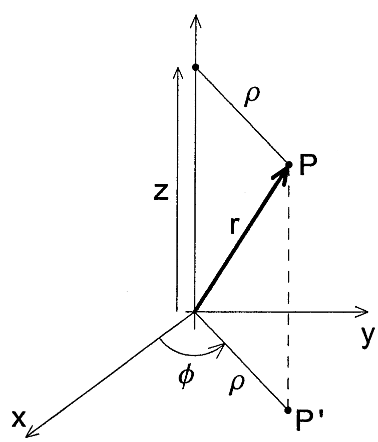
$$\frac{d}{dt}[\mathbf{v}(t)\cdot\mathbf{v}(t)] = 2\mathbf{v}(t)\cdot\dot{\mathbf{v}}(t),$$

this implies that the magnitude of $\mathbf{v}(t)$ is constant if and only if $\mathbf{v}(t)\cdot\dot{\mathbf{v}}(t) = 0$; that is, $\mathbf{v}(t)$ is orthogonal to $\dot{\mathbf{v}}(t)$

1.46 ⋆⋆ **(a)** As seen in the inertial frame $\mathcal{S}$ the puck moves in a straight line with $\phi = 0$ and $r = R - v_o t$

(b) As seen in $\mathcal{S}'$, $r' = r = R - v_o t$ and $\phi' = \phi - \omega t = -\omega t$. This path is sketched in the answer to Problem 1.27. Initially, the puck moves inward with speed v_o but also downward with speed ωR. It curves to its right, passing through the center and continuing to curve to the right until it slides off the turntable.

1.47 ⋆⋆ **(a)** $\rho = \sqrt{x^2 + y^2}$, $\phi = \arctan(y/x)$ (chosen to lie in the right quadrant), and z is the same as in Cartesians. The coordinate ρ is the perpendicular distance from P to the z axis. If we use r for the coordinate ρ, then r is not the same thing as $|\mathbf{r}|$ and $\hat{\mathbf{r}}$ is not the unit vector in the direction of $\mathbf{r}$ [see part (b)].



(b) The unit vector $\hat{\boldsymbol{\rho}}$ points in the direction of increasing ρ (with ϕ and z fixed), that is, directly away from the z axis; $\hat{\boldsymbol{\phi}}$ is tangent to a horizontal circle through P centered on the z axis (counter-clockwise, seen from above); $\hat{\mathbf{z}}$ is parallel to the z axis.

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}.$$

(c) Differentiating this equation we find (Remember that $\hat{\mathbf{z}}$ is constant.)

$$\dot{\mathbf{r}} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\frac{d\hat{\boldsymbol{\rho}}}{dt} + \dot{z}\hat{\mathbf{z}} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}$$

since $d\hat{\boldsymbol{\rho}}/dt = \dot{\phi}\hat{\boldsymbol{\phi}}$ [see (1.42)]. Differentiating again, we find similarly that

$$\ddot{\mathbf{r}} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\mathbf{r}} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\mathbf{z}}.$$

1.48 ⋆⋆ The unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ are the same as in two dimensions (except that what was called r is now called ρ),

$$\hat{\boldsymbol{\rho}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi \qquad \text{and} \qquad \hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$$

and $\hat{\mathbf{z}}$ is the same as in Cartesians. Differentiating with respect to t, we find that

1.17 ★★ **(a)** Let us start with the x component of $\mathbf{r} \times (\mathbf{u} + \mathbf{v})$. From the definition (1.9), we see that

$$[\mathbf{r} \times (\mathbf{u} + \mathbf{v})]_{\mathbf{x}} = r_y(u_z + v_z) - r_z(u_y + v_y) = (r_y u_z - r_z u_y) + (r_y v_z - r_z v_y) = (\mathbf{r} \times \mathbf{u})_x + (\mathbf{r} \times \mathbf{v})_x .$$

Since the y and z components follow in the same way, we conclude that $\mathbf{r} \times (\mathbf{u} + \mathbf{v}) = \mathbf{r} \times \mathbf{u} + \mathbf{r} \times \mathbf{v}$.

(b) Starting again from (1.9), we find for the x component

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{s})_x = \frac{d}{dt}(r_y s_z - r_z s_y) = \left(r_y \frac{ds_z}{dt} - r_z \frac{ds_y}{dt}\right) + \left(\frac{dr_y}{dt} s_z - \frac{dr_z}{dt} s_y\right) = \left(\mathbf{r} \times \frac{d\mathbf{s}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{s}\right)_x .$$

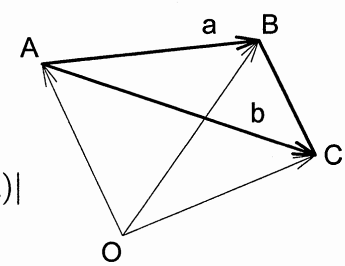
This is the x component of the desired identity. Since the y and z components follow in exactly the same way, our proof is complete.

1.18 ★★ **(a)** $|\mathbf{a} \times \mathbf{b}| = ab \sin\gamma = bh$, where $h = a \sin\gamma$ is the height of the triangle ABC. Therefore $|\mathbf{a} \times \mathbf{b}| = 2$(area of triangle), which is the required first result. The other two follow in the same way.

(b) By part (a), $|\mathbf{c} \times \mathbf{a}| = |\mathbf{b} \times \mathbf{c}|$ or $ca \sin\beta = bc \sin\alpha$, whence $a/\sin\alpha = b/\sin\beta$, as required. The third expression follows in the same way.

1.19 ★★ $\displaystyle \frac{d}{dt}[\mathbf{a}\cdot(\mathbf{v} \times \mathbf{r})] = \frac{d\mathbf{a}}{dt}\cdot(\mathbf{v} \times \mathbf{r}) + \mathbf{a}\cdot\frac{d}{dt}(\mathbf{v} \times \mathbf{r}) = \dot{\mathbf{a}}\cdot(\mathbf{v} \times \mathbf{r}) + \mathbf{a}\cdot(\dot{\mathbf{v}} \times \mathbf{r} + \mathbf{v} \times \dot{\mathbf{r}})$.
The final term $\mathbf{a}\cdot(\mathbf{v} \times \dot{\mathbf{r}})$ is zero because $\dot{\mathbf{r}} = \mathbf{v}$ and $\mathbf{v} \times \mathbf{v} = 0$. The second to last term is $\mathbf{a}\cdot(\mathbf{a} \times \mathbf{r}) = \mathbf{0}$, because $\mathbf{a} \times \mathbf{r}$ is perpendicular to $\mathbf{a}$, so their scalar product is zero. This leaves us with the requested identity.

1.20 ★★ Let two of the sides of the triangle be $\mathbf{a} = \mathbf{B} - \mathbf{A}$ and $\mathbf{b} = \mathbf{C} - \mathbf{A}$ as shown. Then, according to Problem 1.18



$$\begin{aligned} 2(\text{area}) = |\mathbf{a} \times \mathbf{b}| &= |(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A})| \\ &= |(\mathbf{B} \times \mathbf{C}) - (\mathbf{B} \times \mathbf{A}) - (\mathbf{A} \times \mathbf{C}) + (\mathbf{A} \times \mathbf{A})| \\ &= |(\mathbf{B} \times \mathbf{C}) + (\mathbf{C} \times \mathbf{A}) + (\mathbf{A} \times \mathbf{B})| \end{aligned}$$

1.21 ★★ The base of the parallelepiped is a parallelogram with sides $\mathbf{b}$ and $\mathbf{c}$. By Problem 1.18, the area of this base is $|\mathbf{b} \times \mathbf{c}|$. The vector $\mathbf{b} \times \mathbf{c}$ is normal to the base, so if θ is the angle between $\mathbf{a}$ and this normal,

$$|\mathbf{a}\cdot(\mathbf{b} \times \mathbf{c})| = a|\mathbf{b} \times \mathbf{c}|\,|\cos\theta| = (\text{area of base})\,|a\cos\theta| .$$

Now, $|a\cos\theta|$ is the height of the parallelepiped. Therefore $|\mathbf{a}\cdot(\mathbf{b} \times \mathbf{c})|$ is just "(area of base) $\times$ height," which is the volume, as claimed.

1.22 ⋆⋆ **(a)** According to definition (1.6), $\mathbf{a}\cdot\mathbf{b} = ab\cos(\alpha-\beta)$, but according to (1.7)

$$\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y = ab(\cos\alpha\cos\beta + \sin\alpha\sin\beta).$$

Comparing these two expressions we get the desired result.

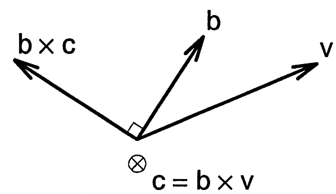
(b) According to the right-hand-rule definition, $\mathbf{a}\times\mathbf{b} = -ab\sin(\alpha-\beta)\hat{\mathbf{z}}$. (The minus sign is easy to check; for example, if $\alpha > \beta$, then the right-hand rule puts $\mathbf{a}\times\mathbf{b}$ in the negative z direction.) On the other hand, the definition (1.9) gives the z component as

$$(\mathbf{a}\times\mathbf{b})_z = a_x b_y - a_y b_x = ab(\cos\alpha\sin\beta - \sin\alpha\cos\beta).$$

Comparing these two expressions we again get the desired result.

1.23 ⋆⋆ The picture shows the plane of $\mathbf{b}$ and $\mathbf{v}$. Because $\mathbf{c} = \mathbf{b}\times\mathbf{v}$, $\mathbf{c}$ is perpendicular to this plane, as indicated. Therefore $\mathbf{b}\times\mathbf{c}$ lies in the plane and is perpendicular to $\mathbf{b}$. This means that $\mathbf{v}$ can be expressed in terms of the two mutually perpendicular vectors $\mathbf{b}$ and $\mathbf{b}\times\mathbf{c}$:

$$\mathbf{v} = \alpha\,\mathbf{b} + \beta\,\mathbf{b}\times\mathbf{c}$$



If we form the dot product of this equation with $\mathbf{b}$ we find that $\mathbf{b}\cdot\mathbf{v} = \alpha\mathbf{b}\cdot\mathbf{b}$. Therefore $\alpha = \mathbf{b}\cdot\mathbf{v}/b^2 = \lambda/b^2$. Similarly, if we form the cross product of the same equation with $\mathbf{b}$, we find that $\mathbf{b}\times\mathbf{v} = \beta\mathbf{b}\times(\mathbf{b}\times\mathbf{c})$. Since $\mathbf{b}\times\mathbf{v} = \mathbf{c}$ and (as you can easily check) $\mathbf{b}\times(\mathbf{b}\times\mathbf{c}) = -b^2\mathbf{c}$, this last implies that $\beta = -1/b^2$, and we conclude that $\mathbf{v} = (\lambda\mathbf{b} - \mathbf{b}\times\mathbf{c})/b^2$.

1.24 ⋆ Integrating the equation $df/f = dt$, we find that $\ln f = t + k$, or $f = Ae^t$, with one arbitrary constant, $A = e^k$.

1.25 ⋆ Integrating the equation $df/f = -3dt$, we find that $\ln f = -3t + k$, or $f = Ae^{-3t}$, with one arbitrary constant, $A = e^k$.

1.26 ⋆⋆ **(a)** Since the puck is frictionless, the net force on it is zero, and, as seen from frame $\mathcal{S}$ the puck heads due north with constant speed v_o, so $x = 0$ and $y = v_o t$, where v_o is the speed with which I kicked the puck.

(b) We must now find how to translate the coordinates (x, y) of any point P, as seen in frame $\mathcal{S}$, to the coordinates (x', y') of the same point P as seen in $\mathcal{S}'$. At time t, the origin of $\mathcal{S}'$ is at $(vt, 0)$ (as seen in $\mathcal{S}$), where v is the speed of $\mathcal{S}'$ relative to $\mathcal{S}$. Therefore $(x, y) = (vt, 0) + (x', y')$; whence $x' = x - vt$ and $y' = y$. This is for any point (x, y). If we substitute the coordinates of the puck from part (a), we find

$$x' = -vt \qquad \text{and} \qquad y' = v_o t.$$

2.14 ★★★ **(a)** With $F = -F_o e^{v/V}$, the second law ($mdv/dt = F$) separates to become $me^{-v/V}dv = -F_o dt$. This is easily integrated (from time 0 to t or v_o to v) and the result solved to give

$$v = -V \ln\left(\frac{F_o t}{mV} + e^{-v_o/V}\right).$$

(b) This is zero when the argument of the log is 1, so that $t = (1 - e^{-v_o/V})mV/F_o$.

(c) Using the result that $\int \ln(x)dx = x\ln(x) - x$, we can evaluate the integral $x = \int v\,dt$ to give

$$x(t) = Vt - \frac{mV^2}{F_o}\left[\left(\frac{F_o t}{mV} + e^{-v_o/V}\right)\ln\left(\frac{F_o t}{mV} + e^{-v_o/V}\right) + \frac{v_o}{V}e^{-v_o/V}\right]$$

and, substituting the time from part (b),

$$x_{\text{max}} = \frac{mV^2}{F_o}\left[1 - e^{-v_o/V}\left(1 + \frac{v_o}{V}\right)\right].$$

2.15 ★ Since the only force is the projectile's weight $m\mathbf{g}$, Newton's second law implies that $\ddot{\mathbf{r}} = \mathbf{g}$ and its two components can be integrated twice to give the well-known results $x = v_{xo}t$ and $y = v_{yo}t - \frac{1}{2}gt^2$ (if we take $x_o = y_o = 0$). At landing, $y = 0$, which gives the time of flight as $t = 2v_{yo}/g$. The range is just the value of x at this time, namely $x = v_{xo}t = 2v_{xo}v_{yo}/g$.

2.16 ★ As usual, $x = (v_o \cos\theta)t$ and $y = (v_o \sin\theta)t - \frac{1}{2}gt^2$. The time to reach the plane of the wall ($x = d$) is $t = d/(v_o\cos\theta)$ and the ball's height at that time is $y = d\tan\theta - \frac{1}{2}gd^2/(v_o\cos\theta)^2$. Notice that this height decreases monotonically as v_o decreases. Thus there is indeed a minimum speed $v_o(\text{min})$ for which the ball clears the wall. Putting $y = h$ and solving for v_o we find that

$$v_o(\text{min}) = \sqrt{\frac{gd^2}{2(d\tan\theta - h)\cos^2\theta}}.$$

If $\tan\theta < h/d$, the argument of the square root is negative and there is no real $v_o(\text{min})$; physically, the ball's initial velocity is aimed below the top of the wall, so the ball cannot possibly clear the wall whatever its speed. With the given numbers, $v_o(\text{min}) = 26.4$ m/s or roughly 50 mi/hr.

2.17 ★ From the first of Equations (2.36) we find that $1 - e^{-t/\tau} = x(t)/(v_{xo}\tau)$ and hence $t = -\tau\ln(1 - x/v_{xo}\tau)$. Substituting these into the second of (2.36), we find

$$y = \frac{v_{yo} + v_{\text{ter}}}{v_{xo}}x + v_{\text{ter}}\tau\ln\left(1 - \frac{x}{v_{xo}\tau}\right)$$

which is (2.37).
