Calculus of Variations:

For many problems in physics we need to use non-cartesian coordinates

but the expressions for acceleration and Eqn. of motion become complex

$F = ma$ is hard to use

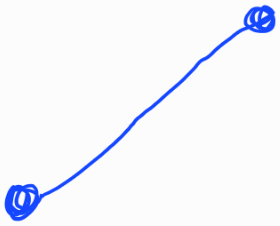
Lagrange's equations are the solution!

these require variational methods

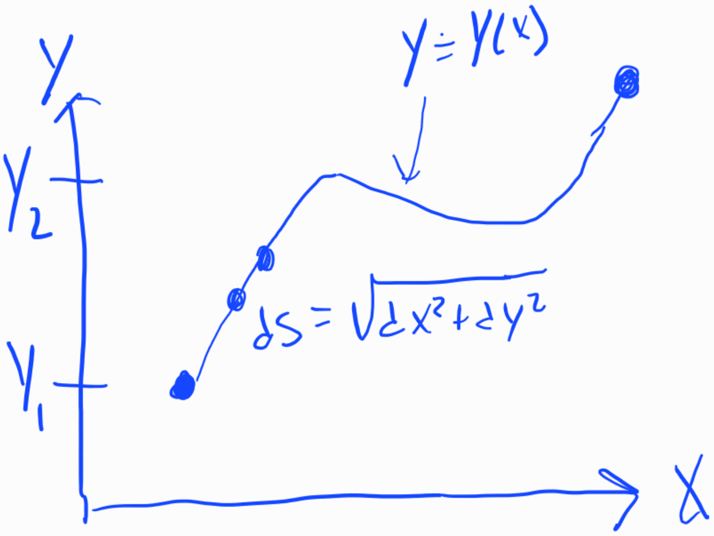
6.1 Two Examples:

Calculus of variations involves finding mins & maxs of an integral quantity

The shortest path between two points



A straight line?



(x_1, y_1)

(x_2, y_2)

$ds = \sqrt{dx^2 + dy^2}$

$dy = \frac{dy}{dx} dx = y'(x) dx$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'(x)^2}\, dx$$

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\, dx$$

must find when minimum

Fermat's Principle

light minimizes its path to minimize time, even when traveling through materials

$$(\text{time of travel}) = \int_1^2 dt = \int_1^2 \frac{ds}{v} = \frac{1}{c}\int_1^2 n\,ds$$

$V = \frac{c}{n}$, if n is const. simplifies to straight line

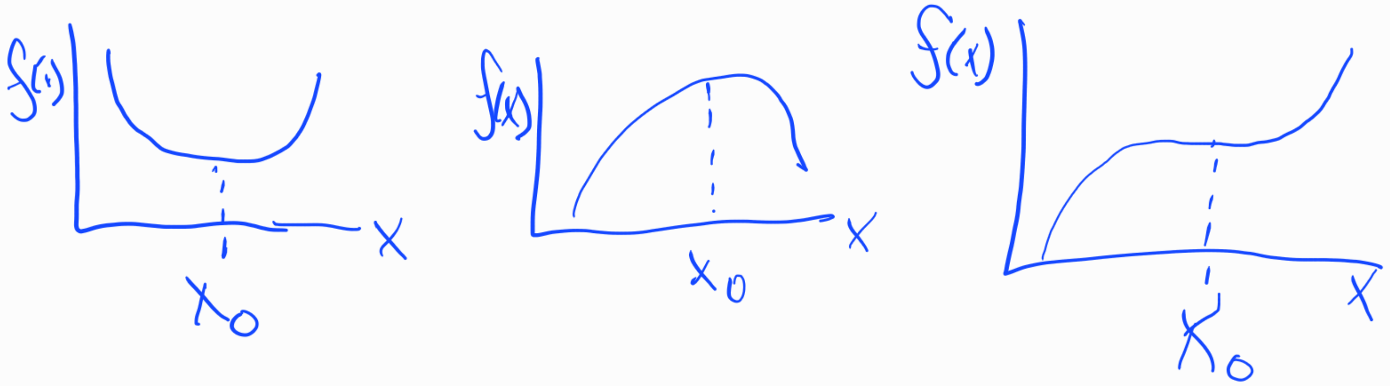
In general
$n = n(x, y)$

$$\int_1^2 n(x,y)\,ds = \int_{x_1}^{x_2} n(x,y)\sqrt{1+y'(x)}\,dx$$

minimize

Sometimes we will want to maximize but

$\frac{df}{dx} = 0$, doesn't always work



The Point x_0 @ which $\frac{df(x)}{dx} = 0$ is a Stationary Point

a small displacement x from x_0 leaves f(x) unchanged

The act of finding a path that does not change for infintesimal changes is the Calculus of Variations

Methods to do so are Variational methods

Principles like Fermats are Variational Principles

Euler-Lagrange Equation:

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]\,dx,$$ Variational Problem

$$y(x) < \begin{matrix}(x_1, y_1)\\(x_2, y_2)\end{matrix} \qquad \begin{cases} y(x_1) = y_1 \\ y(x_2) = y_2 \end{cases}$$

$S \rightarrow$ minimum

$f = [y(x), y'(x), x]$ 3 variables

but a function of only x
(one variable)

Path $\Rightarrow y = y(x)$

Denote correct solution $y = y(x)$

a neighboring curve $y = \bar{Y}(x)$
(larger integral)

write wrong curve as $\bar{Y}(x)$

$$\bar{Y}(x) = y(x) + \eta(x)$$

$\eta \Rightarrow$ the difference between y & $\bar{Y}$

$$\eta(x_1) = \eta(x_2) = 0$$

Infinitely many choices for $\eta(x)$
Such as

$$\eta(x) = (x - x_1)(x_2 - x)$$

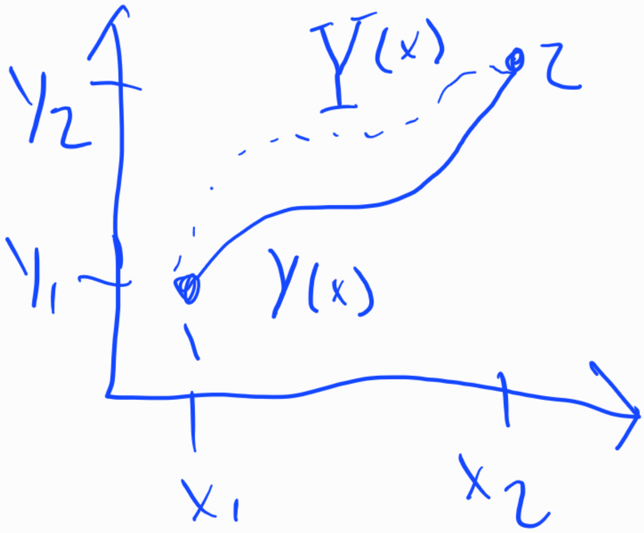
or $\eta(x) = \sin\left[\pi (x - x_1)/(x_2 - x_1)\right]$

redefine with α to make larger than $y(x)$

$$\bar{Y}(x) = y(x) + \alpha\eta(x)$$

$S \to S(\alpha)$

if $\alpha = 0$ $Y \to y$



$S(\alpha)$ is min at $\alpha = 0$

$$\frac{dS}{d\alpha} = 0$$

$$S(\alpha) = \int_{x_1}^{x_2} f(Y, Y', x)\, dx$$

$$= \int_{x_1}^{x_2} f(y + \alpha\eta, y' + \alpha\eta, x)\, dx$$

$$\frac{\partial f(y + \alpha\eta, y' + \alpha\eta', x)}{\partial \alpha} =$$

$$= \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}$$

$$\frac{dS}{d\alpha} = \int_{S_1}^{S_2} \frac{\partial f}{\partial \alpha}\, dx = \int_{x_1}^{x_2} \left(\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} \right) dx = 0$$

$$\int_{x_1}^{x_2} \eta'(x) \frac{\partial f}{\partial y'}\, dx = \left[\eta(x) \frac{\partial f}{\partial y'} \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx} \frac{\partial f}{\partial y'}\, dx$$

$$\int_{x_1}^{x_2} \eta'(x) \frac{\partial f}{\partial y'}\, dx = - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) dx$$

$$\int_{x_1}^{x_2} \eta(x) \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) dx = 0$$

Must satisfy relation regardless of choice of $\eta(x)$

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) = 0$$

6.3 Applications of Euler-Lagrange Equation

Ex 6.1 Shortest path between two points

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1+y^2}\, dx$$

$$f(y, y', x) = (1+y'^2)^{1/2}$$

$$\frac{\partial f}{\partial y} = 0 \quad , \quad \frac{\partial f}{\partial y'} = \frac{y'}{(1+y')^{1/2}}$$

since $\left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) = 0$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 0, \quad \frac{\partial f}{\partial y'} \text{ is constant}$$

$$\frac{\partial f}{\partial y'} = C = \frac{y'}{(1+y'^2)^{1/2}}$$

$$C(1+y'^2)^{1/2} = y'$$

$$C^2(1+y'^2) = y'^2$$

$$C^2 + Cy'^2 = y'^2$$

$$y'^2 - Cy'^2 = C^2$$

$$y'^2(1-C) = C^2$$

$$y'^2 = \frac{C^2}{1-C} = \text{Const}$$

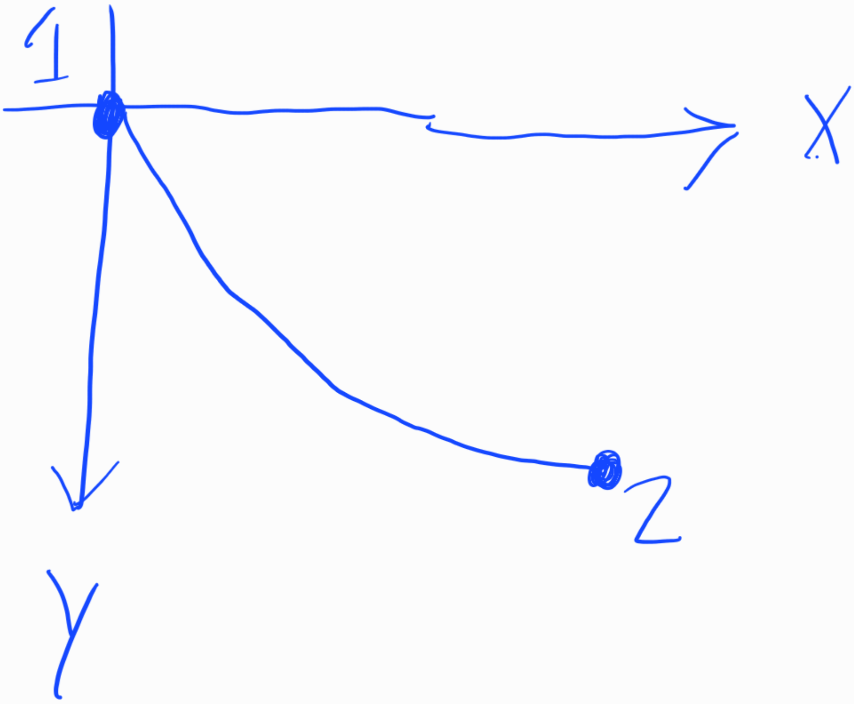
$$m = \sqrt{\frac{C^2}{1-C}}$$

$$y' = m$$

$$y'(x) = m \rightarrow y(x) = \int y'(x)$$

$$y(x) = mx + b$$, Eqn of a line

Ex 6.2 Brachistochrone



$x_1 = y_1 = 0$

$$\text{time}(1 \to 2) = \int_1^2 \frac{ds}{v}$$

$$v = \sqrt{2gy} \to x = x(y)$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{x'(y)^2 + 1}\, dy$$

$$x'(y) = \frac{dx}{dy}$$

$$\text{time}(1\to 2) = \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{x'(y)^2+1}}{\sqrt{y}}\, dy$$

$$f(x, x', y) = \frac{\sqrt{x'(y)+1}}{\sqrt{y}}$$

$$\frac{\partial f}{\partial x} = \frac{d}{dy}\frac{\partial f}{\partial x'}, \quad \frac{\partial f}{\partial x} = 0$$

$$\to \frac{d}{dx}\frac{\partial f}{\partial x'} = 0 \to \frac{\partial f}{\partial x'} = \frac{x'^2}{y(1+x'^2)} = \text{const.}$$

$$\frac{\partial f}{\partial x'} = \text{const} = \frac{1}{2a}, \quad x' = \sqrt{\frac{y}{2a-y}},$$

$$X = \int \sqrt{\frac{Y}{2a-Y}}\, dY$$

Halfangle

$$1-\cos\theta = 2\sin^2\frac{\theta}{2}$$

$$1+\cos\theta = 2\cos^2\frac{\theta}{2}$$

Set $Y = a(1-\cos\theta)$, $dY = a\sin\theta\, d\theta$

$$\sqrt{\frac{a-a\cos\theta}{2a-a+a\cos\theta}}\, dY = \sqrt{\frac{a-a\cos\theta}{a+a\cos\theta}}\, dY = \tan\frac{\theta}{2}\, dY$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}, \quad \tan\theta = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$dY = a\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\tan\frac{\theta}{2}\left(2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = \frac{\sin\frac{\theta}{2}}{\cancel{\cos}\frac{\theta}{2}}\left(2a\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}\right)$$

half angle

$$= 2a\sin^2\frac{\theta}{2}, \quad 2\sin\frac{\theta}{2} = 1-\cos\theta$$

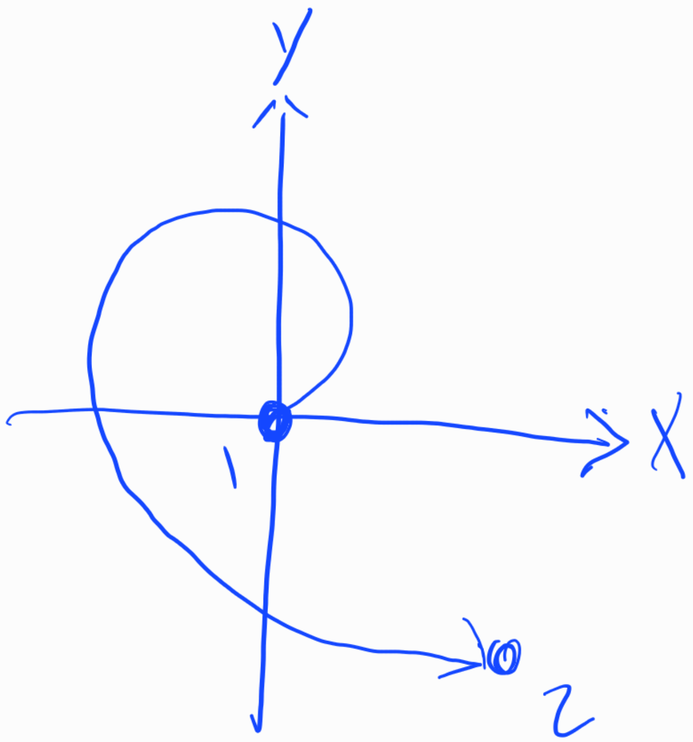
$$2a\sin^2\frac{\theta}{2} = a(1-\cos\theta)$$

$$x = a\int(1-\cos\theta)\,d\theta$$

$$= a(\theta - \sin\theta) + C$$

$$\therefore x = a(\theta - \sin\theta), \quad y = a(1-\cos\theta)$$

6.4 More than two Variables



$$x \neq x(y)$$

$$y \neq y(x)$$

Write x & y in Parametric form for two points

$$x = x(u) \quad \& \quad y = y(u)$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{x'(u)^2 + y'(u)^2}\, du$$

$$x'(u) = \frac{dx}{du}, \quad y'(u) = \frac{dy}{du}$$

$$L = \int_{U_1}^{U_2} \sqrt{x'(u)^2 + y'(u)^2}\, dU$$

must find $x'(u)$ & $y'(u)$ that minimize the integral

$$S = \int_{U_1}^{U_2} f[x(u), y(u), x'(u), y'(u), x]\, dU$$

want to find

$[x(u), y(u)]$ Path between $[x(u_1), y(u_1)]$
$[x(u_2), y(u_2)]$

setup 2 Euler-Lagrange Eqns.

$X = X(u)$ $\qquad Y = Y(u)$

$X = X(u) + \alpha \xi(u)$ $\qquad Y = Y(u) + \beta \eta(u)$

must satisfy when $\alpha = \beta = 0$

$$\frac{\partial S}{\partial \alpha} = 0 \quad \& \quad \frac{\partial S}{\partial \beta} = 0$$

Leads to

$$\frac{\partial f}{\partial x} - \frac{d}{du}\frac{\partial f}{\partial x'} = 0 \,, \quad \frac{\partial f}{\partial x} = \frac{d}{du}\frac{\partial f}{\partial x'}$$

and

$$\frac{\partial f}{\partial y} - \frac{d}{du}\frac{\partial f}{\partial y'} = 0 \,, \quad \frac{\partial f}{\partial y} = \frac{d}{du}\frac{\partial f}{\partial y'}$$

Ex 6.3 Shortest path between two points ... again

$$f(x, x', y, y', u) = \sqrt{x'^2 + y'^2}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \,, \quad \frac{d}{du}\frac{\partial f}{\partial x'} = \frac{d}{du}\frac{\partial f}{\partial y'} = 0$$

$$\therefore \frac{\partial f}{\partial x'} = \frac{\partial f}{\partial y'} = \text{const.} \qquad \frac{\partial f}{\partial x'} = c_1$$

$$\frac{\partial f}{\partial y'} = c_2$$

$$\frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{x'^2+y'^2}} = C_1$$

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{x'^2+y'^2}} = C_2$$

] divide

$$\frac{C_2}{C_1} = \frac{y'}{x'} = \frac{dy}{dx} = \text{const} = m$$

$$\therefore \; y = mx + b$$

Lagrangian:

has independant variable as

t

Dependant variables are denoted

as $q_1, q_2, q_3 \dots q_n$, $n =$ # of coordinates

for $n = 3$ (cartesian)

$$q_1, q_2, q_3 = x, y, z$$

Polar

$$r, \theta, \phi$$

for N Particles

$$x_1, y_1, z_1, \dots x_N, y_N, z_N$$

q is a generalized coordinate

The goal in Lagrangian mechanics is to find how q's vary with time

Lagrange's Eqns take same form in all coordiate sytems making them easier to use than newton's second law

$$S = \int f(\ )\,du$$ is the action integral

The integrand is the Lagrangian

$$\mathcal{L} = \mathcal{L}(q_1, \dot{q}_1, \ldots, q_n, \dot{q}_n, t)$$

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, \dot{q}_1, \ldots, q_n, \dot{q}_n, t)$$

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1}$$

$$\frac{\partial \mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2}$$

$$\frac{\partial \mathcal{L}}{\partial q_n} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_n}$$

n equations

One for each coordinate