5.10 ⋆ If $F = -F_o \sinh \alpha x$, then $U = -\int F\,dx = (F_o/\alpha)\cosh \alpha x$. The only equilibrium position is at $x = 0$ and, for points close to this, Taylor's series gives

$$U(x) \approx (F_o/\alpha)(1 + \tfrac{1}{2}\alpha^2 x^2) = \tfrac{1}{2}kx^2 + \text{const},$$

where $k = \alpha F_o$. The angular frequency of oscillations is $\omega = \sqrt{k/m} = \sqrt{\alpha F_o/m}$.

5.11 ⋆ The given information gives two expressions of the total energy E

$$E = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}kx_1^2 \quad \text{and} \quad E = \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}kx_2^2. \tag{i}$$

Equating these two, we find $m(v_1^2 - v_2^2) = k(x_2^2 - x_1^2)$. This implies that

$$\omega^2 = \frac{k}{m} = \frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}.$$

We know also that $E = \frac{1}{2}kA^2$, and inserting this in the first of Eqs.(i) we conclude that

$$A^2 = \frac{m}{k}v_1^2 + x_1^2 = \frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}v_1^2 + x_1^2 = \frac{x_2^2 v_1^2 - x_1^2 v_2^2}{v_1^2 - v_2^2}.$$

5.12 ⋆⋆ Because $\sin^2(\omega t - \delta)$ oscillates symmetrically between 0 and 1, its average over a cycle is fairly obviously $\frac{1}{2}$. To prove it, write $\sin^2\theta = \frac{1}{2}[1 - \cos 2\theta]$, so that

$$\langle \sin^2(\omega t - \delta) \rangle = \frac{1}{2\tau}\int_0^\tau [1 - \cos 2(\omega t - \delta)]dt = \frac{1}{2} - \frac{1}{2\tau}\big[\sin 2(\omega t - \delta)\big]_0^\tau = \frac{1}{2}$$

where the final square bracket is zero because the sine function is τ-periodic. The corresponding result with a cosine follows in exactly the same way.

We know from (5.16) that $E = \frac{1}{2}kA^2$, and, since $T = \frac{1}{2}kA^2 \sin^2(\omega t - \delta)$, it follows that $\langle T \rangle = \frac{1}{4}kA^2 = \frac{1}{2}E$, and similarly for $\langle U \rangle$.

5.13 ⋆⋆ Since $U(r) = U_o(r/R + \lambda^2 R/r)$, its derivative is $dU/dr = U_o(1/R - \lambda^2 R/r^2)$, which vanishes at $r = \lambda R$ and nowhere else. Clearly $U(r) \to +\infty$ when $r \to 0$ or ∞, so $U(r)$ has a minimum at $r = r_o = \lambda R$. If we write $r = r_o + x$ then

$$\begin{aligned} U = \lambda U_o\left(\frac{r_o + x}{r_o} + \frac{r_o}{r_o + x}\right) &= \lambda U_o\left(1 + \frac{x}{r_o} + \left[1 + \frac{x}{r_o}\right]^{-1}\right) \\ &\approx \lambda U_o\left(1 + \frac{x}{r_o} + 1 - \frac{x}{r_o} + \frac{x^2}{r_o^2}\right) = \lambda U_o\left(2 + \frac{x^2}{\lambda^2 R^2}\right). \end{aligned}$$

where, in the second line, I dropped all terms in $(x/r_o)^3$ and higher. This has the expected form $U = \frac{1}{2}kx^2 + \text{const}$, where $k = 2U_o/(\lambda R^2)$. The angular frequency is $\omega = \sqrt{k/m} = \sqrt{2U_o/(m\lambda R^2)}$.

5.14 ⋆ Given that $\mathbf{F} = -(k_x x, k_y y)$, the PE is

$$U(\mathbf{r}) = -\int_0^{\mathbf{r}} \mathbf{F}(\mathbf{r}')\cdot d\mathbf{r}' = \int_0^{\mathbf{r}} (k_x x' dx' + k_y y' dy') = k_x \int_0^x x' dx' + k_y \int_0^y y' dy' = \tfrac{1}{2}(k_x x^2 + k_y y^2).$$

5.15 ⋆ If we replace the variable t by $t' = t + t_o$, Eq. (5.19) becomes

$$x = A_x \cos(\omega t' - \omega t_o - \delta_x) \quad \text{and} \quad y = A_y \cos(\omega t' - \omega t_o - \delta_y)$$

and if we then choose t_o such that $\omega t_o = -\delta_x$ these become

$$x = A_x \cos(\omega t') \quad \text{and} \quad y = A_y \cos(\omega t' - [\delta_y - \delta_x]).$$

If we rename t' as t and set $\delta_y - \delta_x = \delta$, this is the desired form.

5.16 ⋆ With $\delta = \pi/2$ Eq.(5.20) reads

$$x = A_x \cos(\omega t) \quad \text{and} \quad y = A_y \cos(\omega t - \pi/2) = A_y \sin(\omega t)$$

from which it follows that $x^2/A_x^2 + y^2/A_y^2 = \cos^2(\omega t) + \sin^2(\omega t) = 1$, the equation of an ellipse with semi-major and semi-minor axes A_x and A_y.

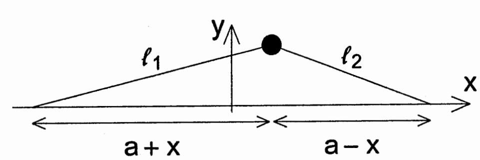
5.17 ⋆⋆ **(a)** Suppose that the ratio of frequencies is rational, that is $\omega_x/\omega_y = p/q$, where p and q are integers. Then let $\tau = 2\pi p/\omega_x = 2\pi q/\omega_y$. Now consider the following

$$x(t+\tau) = A_x \cos[\omega_x(t+\tau)]) = A_x \cos[\omega_x t + 2\pi p] = A_x \cos[\omega_x t] = x(t)$$

where in the second equality I used our definition of τ and in the second the fact that if p is an integer then $\cos(\theta + 2\pi p) = \cos(\theta)$. This shows that $x(t)$ is periodic with period τ. By exactly the same argument, $y(t)$ is also periodic with the same period τ, and we've proved that the whole motion is likewise. What we usually call *the* period of the motion is the value of $\tau = 2\pi p/\omega_x$ with p and q the *smallest* integers for which $\omega_x/\omega_y = p/q$.

(b) Suppose the motion is periodic. Then there is a τ such that $x(t+\tau) = x(t)$ and $y(t+\tau) = y(t)$. Running the previous argument backward, we see that $\omega_x \tau$ must be an integer multiple of 2π, that is $\omega_x \tau = 2\pi p$ for some integer p. Similarly $\omega_y \tau = 2\pi q$ for some integer q. Dividing these two conclusions, we see that $\omega_x/\omega_y = p/q$ and the ratio of frequencies is rational. Therefore, if the ratio is irrational, the motion cannot be periodic.

5.18 ⋆⋆⋆ When the mass is at position (x, y), the lengths of the two springs are l_1 and l_2, where



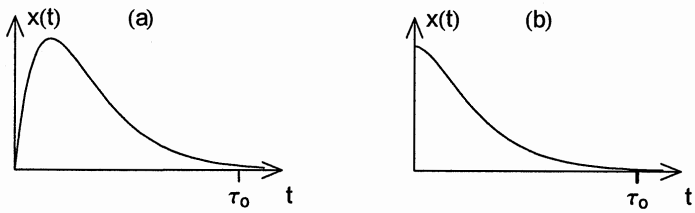
$$l_1 = \sqrt{(a+x)^2 + y^2} = a\left(1 + \frac{2x}{a} + \frac{x^2+y^2}{a^2}\right)^{1/2}$$

$$\approx a\left[1 + \frac{1}{2}\left(\frac{2x}{a} + \frac{x^2+y^2}{a^2}\right) - \frac{1}{8}\left(\frac{2x}{a}\right)^2\right] = a + x + \frac{y^2}{2a}.$$

5.22 ★ **(a)** The general solution for a critically damped oscillator ($\beta = \omega_o$) is given in (5.44) as $x(t) = e^{-\omega_o t}(C_1 + C_2 t)$. Thus

$$x_o = x(0) = C_1 \qquad \text{and} \qquad v_o = \dot{x}(0) = C_2 - \omega_o C_1 \,. \tag{ii}$$

Here $x_o = 0$, so $C_1 = 0$ and $C_2 = v_o$. Therefore, $x(t) = v_o t e^{-\omega_o t}$.



(b) In this case $v_o = 0$ and Eqs.(ii) imply that $C_1 = x_o$ and $C_2 = \omega_o x_o$. Therefore $x(t) = x_o e^{-\omega_o t}(1 + \omega_o t)$. When $t = \tau_o$, the natural period, $x = x_o e^{-2\pi}(1 + 2\pi) = 0.0136 x_o$. The motion is almost 99% damped out.

5.23 ★ Because $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$,

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = \dot{x}(m\ddot{x} + kx) = \dot{x}(-b\dot{x}) = vF_{\text{dmp}}$$

where, for the third equality, I used the differential equation (5.24). Since vF_{dmp} is the rate at which F_{dmp} does work on the oscillator, this is the requested result.

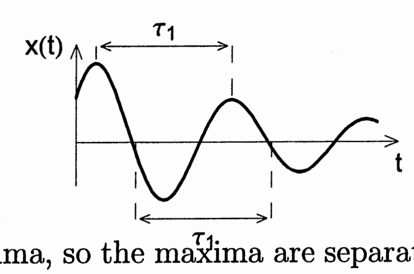
5.24 ★ As long as $\beta < \omega_o$, we can define $\omega_1 = \sqrt{\omega_o^2 - \beta^2}$ and we can use as two independent solutions

$$x_1(t) = e^{-\beta t}\cos(\omega_1 t) \qquad \text{and} \qquad x_2(t) = e^{-\beta t}\sin(\omega_1 t).$$

If $\beta \to \omega_o$, then $\omega_1 \to 0$ and so $x_1(t) \to e^{-\beta t}$. That is, as $\beta \to \omega_o$, our first solution for the case $\beta < \omega_o$ becomes the first solution for the case $\beta = \omega_o$.

Unfortunately, in the same limit, $x_2(t) \to 0$, and our second solution evaporates. However, for our second solution we could equally have used $\tilde{x}_2(t) = e^{-\beta t}\sin(\omega_1 t)/\omega_1$. Since $\sin(kt)/k \to t$ as $k \to 0$, this new second solution does not vanish as $\omega_1 \to 0$; instead, as $\beta \to \omega_o$, the new solution satisfies $\tilde{x}_2(t) \to te^{-\beta t}$ and we obtain the second solution for the case $\beta = \omega_o$.

5.25 ★★ **(a)** Because $x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$, its derivative is $dx/dt = -Ae^{-\beta t}[\beta\cos(\cdots) + \omega_1\sin(\cdots)]$. The maxima and minima of $x(t)$ occur when this derivative vanishes, that is, when $\tan(\omega_1 t - \delta) = -\beta/\omega_1$. Because $\tan\theta$ is π-periodic, the zeroes of dx/dt are equally spaced, with separation π/ω_1. The zeroes of the derivatives correspond alternately to maxima and minima, so the maxima are separated by a time $\tau_1 = 2\pi/\omega_1$.



(b) The zeroes of $x(t)$ occur when $\cos(\omega_1 t - \delta) = 0$. Thus they are regularly spaced with separation of π/ω_1, which equals $\tau_1/2$.

(c) With $\beta = \omega_o/2$, the amplitude shrinks by a factor

$$e^{-\beta\tau_1} = e^{-2\pi\beta/\sqrt{\omega_o{}^2-\beta^2}} = e^{-2\pi/\sqrt{3}} = 0.027$$

(This is much more shrinkage than in the picture, for which β was chosen to be $\omega_o/10$ and the shrinkage factor is about 0.53).

5.26 ⋆⋆ The damping changes the frequency to $\omega_1 = \sqrt{\omega_o^2 - \beta^2}$, which we can solve to give

$$\beta = \omega_o\sqrt{1 - \frac{\omega_1^2}{\omega_o^2}} = \omega_o\sqrt{1 - \frac{\tau_o^2}{\tau_1^2}} = \omega_o\sqrt{1 - 0.998} = 0.0447\omega_o = 0.281\ \mathrm{s}^{-1}$$

After a time $t = 10\tau_1 \approx 10\tau_o$, the amplitude will have changed by a factor of

$$e^{-\beta t} \approx e^{-10\beta\tau_o} = e^{-20\pi\beta/\omega_o} = e^{-20\pi(0.0447)} = 0.060\,.$$

In other words, the amplitude will have diminished by a factor of $1/0.060 = 17$. Clearly the change of amplitude of by a factor of 17 is far more noticeable than the change of period by 0.1%.

5.27 ⋆⋆ The question is: How many times can the function $x(t)$ vanish? If the oscillator is weakly damped ($\beta < \omega_o$), then according to Eq.(5.38) $x(t)$ contains a factor $\cos(\omega t - \delta)$, which vanishes infinitely many times as it oscillates.

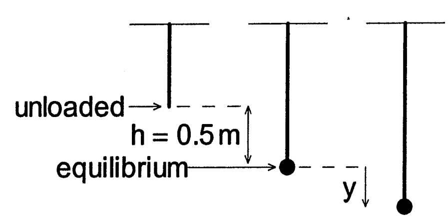
(a) If the oscillator is critically damped ($\beta = \omega_o$), then, according to (5.44), $x(t) = e^{-\beta t}(C_1 + C_2 t)$. This vanishes if and only if $t = -C_1/C_2$; therefore, $x(t)$ vanishes at most once. (It may never vanish — for example, if the motion starts at $t = 0$ and $-C_1/C_2 < 0$.)

(b) If the oscillator is overdamped ($\beta > \omega_o$), then according to (5.40),

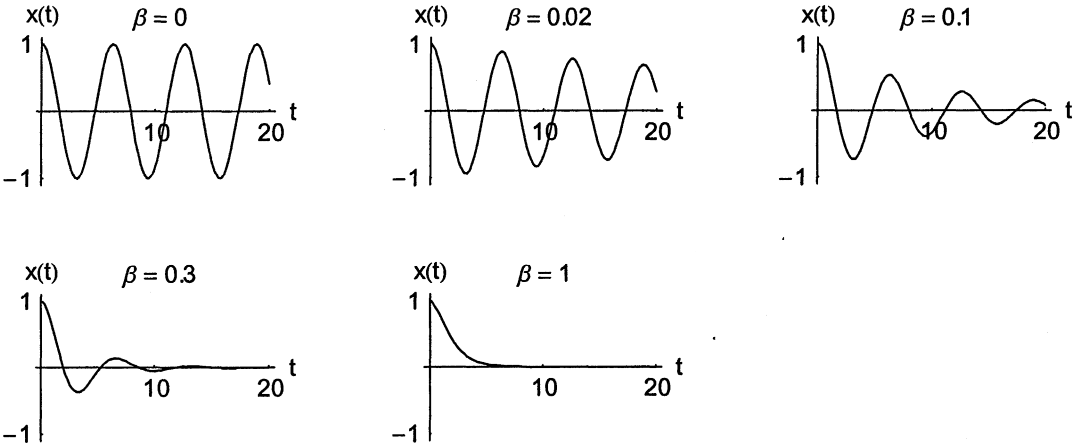
$$x(t) = e^{-\beta t}(C_1 e^{\lambda t} + C_2 e^{-\lambda t}) = e^{-(\beta-\lambda)t}(C_1 + C_2 e^{-2\lambda t})$$

where $\lambda = \sqrt{\beta^2 - \omega_o^2}$. This vanishes if and only if $e^{-2\lambda t} = -C_1/C_2$, and, since $e^{-2\lambda t}$ is a monotonic function (always decreasing), this happens at most once.

5.28 ⋆⋆ The final resting position is the equilibrium position, at a height $h = 0.5$ m below the unloaded position. This height is determined by the condition $kh = mg$, from which we see that $\omega_o{}^2 = k/m = g/h$.



(c)



5.33 ⋆ According to Eq.(5.69), $x(t) = A\cos(\omega t - \delta) + e^{-\beta t}[B_1 \cos(\omega_1 t) + B_2 \sin(\omega_1 t)]$. Setting $t = 0$, we obtain

$$x_o = x(0) = A\cos\delta + B_1 \quad \Longrightarrow \quad B_1 = x_o - A\cos\delta.$$

Similarly, differentiating $x(t)$ and then setting $t = 0$, we find

$$v_o = \dot{x}(0) = A\omega\sin\delta - \beta B_1 + \omega_1 B_2 \quad \Longrightarrow \quad B_2 = \frac{1}{\omega_1}(v_o - A\omega\sin\delta + \beta B_1)$$

as in Eq.(5.70).

5.34 ⋆ We are given that both x_p and x satisfy the same inhomogeneous equation, $Dx_p = f$ and $Dx = f$. Therefore, since D is linear, $D(x - x_p) = Dx - Dx_p = f - f = 0$. That is, the difference $x - x_p = x_h$ is a solution of the homogeneous equation $Dx_h = 0$. Therefore x can always be written as $x = x_p + x_h$ as claimed.

5.35 ⋆⋆ **(a)** $z = x + iy = r\cos\theta + i(r\sin\theta) = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

(b) $zz^* = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$.

(c) $z^* = x - iy = r\cos\theta - i(r\sin\theta) = r[\cos(-\theta) + i\sin(-\theta)] = re^{-i\theta}$

(d) If $z = re^{i\theta}$ and $w = se^{i\phi}$, then $zw = rse^{i(\theta+\phi)}$. Therefore, $(zw)^* = rse^{-i(\theta+\phi)} = (re^{-i\theta})(se^{-i\phi}) = z^*w^*$. If $z = re^{i\theta}$, then $1/z = 1/(re^{i\theta}) = (1/r)e^{-i\theta}$, so $(1/z)^* = (1/r)e^{i\theta}$. Finally, $1/z^* = 1/(re^{-i\theta}) = (1/r)e^{i\theta} = (1/z)^*$.

(e) If $z = \dfrac{a}{b + ic}$, then $|z|^2 = zz^* = \dfrac{a}{b + ic}\left(\dfrac{a}{b + ic}\right)^* = \dfrac{a}{b + ic} \cdot \dfrac{a}{b - ic} = \dfrac{a^2}{b^2 + c^2}$.

5.36 ⋆⋆ From Eqs.(5.64), (5.65), and (5.70) we can calculate the various constants. A and δ are the same as in Example 5.3 (changing the initial condition doesn't affect them), but $B_1 = 0.945$ and $B_2 = 0.0429$. The resulting graph is the solid curve shown. The dashed curve is that of Fig.5.15(b). With different initial conditions, the two curves differ at first, but after a couple of cycles the transients have pretty well died out and the two graphs are indistinguishable.

5.40 ⋆ We can save ourselves a little trouble if we note that A is maximum if and only is A^2 is maximum, which occurs if and only if $1/A^2$ is minimum. In fact, let's consider $(f_o/A)^2 = (\Omega - \omega_o^2)^2 + 4\beta^2\Omega$, where I'll use the variable $\Omega = \omega^2$. To find the minimum of this quantity we have only to differentiate and set the derivative equal to zero:

$$\frac{d}{d\Omega}\left(\frac{f_o^2}{A^2}\right) = 2(\Omega - \omega_o^2) + 4\beta^2,$$

which vanishes when $\Omega = \omega_o^2 - 2\beta^2$, that is, $\omega = \sqrt{\omega_o^2 - 2\beta^2}$. It is easy to check that the second derivative is positive. Therefore $(f_o/A)^2$ is minimum and A is maximum as expected.

5.41 ⋆ Provided β is significantly less that ω_o, the maximum of A^2 comes when $\omega \approx \omega_o$ and, at this point, the denominator of Eq.(5.71) is approximately $4\beta^2\omega_o^2$. Thus A^2 is equal to half its maximum when the denominator is equal to $8\beta^2\omega_o^2$, or when $(\omega^2 - \omega_o^2)^2 = 4\beta^2\omega_o^2$. This simplifies to $(\omega - \omega_o)(\omega + \omega_o) = \pm 2\beta\omega_o$. Since $(\omega + \omega_o) \approx 2\omega_o$, this says that the half maximum occurs at $\omega = \omega_o \pm \beta$.

5.42 ⋆ The period of the pendulum is $\tau = 2\pi\sqrt{l/g} = 10.99$ s. Therefore the quality factor is $Q = \pi(\text{decay time})/\tau = \pi \times (8\text{ h})/(10.99\text{ s}) \approx 8{,}000$.

5.43 ⋆⋆ **(a)** Assuming that the weight of the four men is evenly distributed among the four springs, we can substitute $m = 80$ kg and $x = 2$ cm into the equation $mg = kx$ for any one spring. This gives $k = mg/x = 80 \times 9.8/0.02 \approx 4 \times 10^4$ N/m.

(b) Each axle assembly is attached to two springs, so the effective spring constant of its support is $2k$ and its natural frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{2k}{m}} = \frac{1}{2\pi}\sqrt{\frac{2 \times (4 \times 10^4)}{50}} \approx 6\text{ Hz}.$$

(c) If the distance between bumps is d, then $v = fd = 6 \times 0.8 \approx 5$ m/s or roughly 10 mi/h.

5.44 ⋆⋆ **(a)** Since $x = A\cos(\omega t - \delta)$, the total energy is

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}m\omega^2A^2\cos^2(\omega t - \delta) + \tfrac{1}{2}kA^2\sin^2(\omega t - \delta).$$

Because $\omega \approx \omega_o$, we can replace $k = m\omega_o^2$ by $m\omega^2$, and then, since $\cos^2\theta + \sin^2\theta = 1$, we get $E = \frac{1}{2}m\omega^2A^2$, as claimed.

(b) The rate at which the damping force dissipates energy is $F_{\text{dmp}}v = bv^2 = 2m\beta v^2$. Therefore the energy dissipated in one period is

$$\Delta E_{\text{dis}} = \int_0^\tau 2m\beta v^2 dt = 2m\beta\omega^2A^2\int_0^\tau \sin^2(\omega t - \delta)dt.$$