Ch 13 Hamiltonian Mechanics

Remember

$$\mathcal{L} = \mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) = T - U$$

n coords $(q_1, \ldots, q_n)$ defines a point in "configuration space"

$2n$ coords $(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$ defines a point in "state space". Specifying initial conditions at $t = t_0$

these determine unique solutions to n differential eqn's:

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad [i = 1, \ldots, n]$$

giving unique equations of motion

$$\underbrace{\frac{\partial \mathcal{L}}{\partial q}}_{\text{generalized force}} = \frac{d}{dt} \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}}_{\text{generalized momentum}}$$

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$ (Canonical momentum or momentum conjugate)

Hamiltonian $\mathcal{H}$ → related to total Energy

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}$$

$$\mathcal{L} \to (q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)$$

$$\mathcal{H} \to (q_1, \dots, q_n, p_1, \dots, p_n)$$

2n coords define a point in "Phase space" (positions & generalized momentum)

Hamiltonians best suited to conservative systems. Similar to Lagrangians.

13.2 Hamilton's Equations for 1D

1 Particle with 1 DoF

$$\mathcal{L} = \mathcal{L}(q, \dot{q}) = T(q, \dot{q}) - U(q)$$

for conservative systen U is only dependant on q

EX: Pendulum

$$\mathcal{L} = \frac{1}{2} m L^2 \dot{\phi}^2 - mgL(1 - \cos\phi$$

Bead on wire

$$\mathcal{L} = \mathcal{L}(x, \dot{x}) = T - U =$$

$$= \frac{1}{2} m[1 + f'(x)^2]\dot{x}^2 - mg f(x)$$

For system with natural coordinates (in 1D)

$$\mathcal{L} = \mathcal{L}(q,\dot{q}) = T - U = \frac{1}{2}[A(q)]\dot{q}^2 - U(q)$$

$$\mathcal{H} = p\dot{q} - \mathcal{L}, \quad p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = A(q)\dot{q}$$

$$p\dot{q} = A(q)\dot{q}^2 = 2T$$

$$\mathcal{H} = p\dot{q} - \mathcal{L} = 2T - (T - U) = T + U$$

So, for a "natural" systen $\mathcal{H}$ is the total energy

Setup $\dot{q}$ in terms of q & P

$$\dot{q} = P/A(q) = \dot{q}(q,P)$$

replace $\dot{q}$ in $\mathcal{H}$

$$\mathcal{H}(q,P) = P\dot{q}(q,P) - \mathcal{L}(q, \dot{q}(q,P))$$

need Hamilton's Equations of motion

find

$$\frac{\partial \mathcal{H}}{\partial q} \quad \& \quad \frac{\partial \mathcal{H}}{\partial P}$$

$$\frac{\partial \mathcal{H}}{\partial q} = P\frac{\partial \dot{q}}{\partial q} - \left[\frac{\partial \mathcal{L}}{\partial q} + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}}}_{P}\frac{\partial \dot{q}}{\partial q}\right]$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = P$$

$$\frac{\partial H}{\partial q} = p\cancel{\frac{\partial \dot{q}}{\partial q}} - \left[\frac{\partial L}{\partial q} + \cancel{p\frac{\partial \dot{q}}{\partial q}}\right]$$

$$\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} = -\frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{q}}}_{p} = -\dot{p}$$

remember

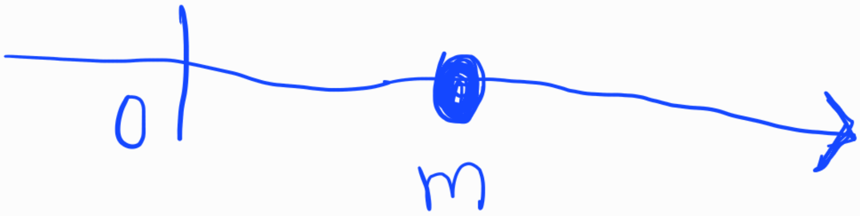
$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$$

$$\frac{\partial H}{\partial p} = \left[\dot{q} + \cancel{p\frac{\partial \dot{q}}{\partial p}}\right] - \cancel{\underbrace{\frac{\partial L}{\partial \dot{q}}}_{p}\frac{\partial \dot{q}}{\partial p}} = \dot{q}$$

$$\dot{q} = \frac{\partial H}{\partial p} \quad \& \quad \dot{p} = -\frac{\partial H}{\partial q} \rightarrow F$$

2 First order differintial Eqns

EX 13.1 Bead on straight wire



Subject to a conservative force with corresponding $U(x)$

Write down Lagrangian & Lagrange eqn's. Find the Hamiltonian & Hamilton Eqn's

$$\mathcal{L}(x,\dot{x}) = T - U = \frac{1}{2}m\dot{x}^2 - U(x)$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \Rightarrow -\frac{dU}{dx} = m\ddot{x}$$

$$F = ma$$

Hamiltonian approach

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \rightarrow \dot{x} = \frac{p}{m}$$

$$\mathcal{H} = p\dot{x} - \mathcal{L} = \frac{p^2}{m} - \left[\frac{p^2}{2m} - U(x)\right]$$

$$= \frac{p^2}{2m} + U = T + U, \quad T = \frac{p^2}{2m}$$

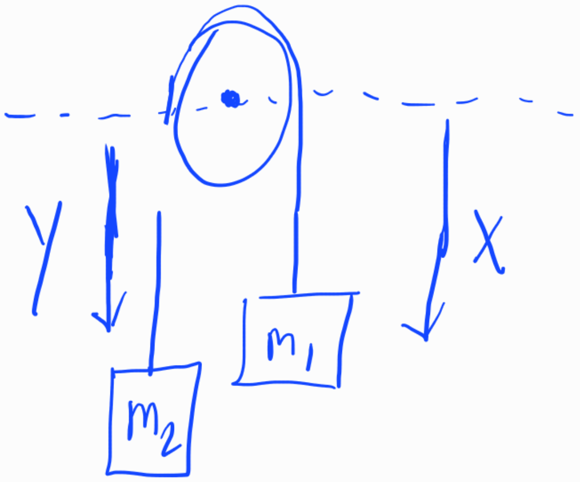
$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m} \quad \text{and} \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -\frac{dU}{dx}$$

$$p = m\dot{x}$$

$$\dot{p} = m\ddot{x} = -\frac{dU}{dx}$$

$$F = ma = \dot{p} = -\nabla U$$

Ex 13.2 Atwood's Machine



using x as the generalized coord.
Setup the hamiltonian formalism

$$\mathcal{L} = T - U, \quad T = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

$$U = -(m_1 - m_2)gx$$

$$\mathcal{H} = p\dot{x} - \mathcal{L} = T + U$$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = (m_1 + m_2)\dot{x}$$

Remember $\mathcal{H}$ is a function of only x & P so,

rewrite $P = (m_1+m_2)\dot{x}$, $\dot{x} = \dfrac{P}{(m_1+m_2)}$

$$\mathcal{H} = \frac{P^2}{(m_1+m_2)} - \frac{1}{2}\frac{\cancel{(m_1+m_2)}P^2}{(m_1+m_2)^{\cancel{2}}} - (m_1-m_2)gx$$

$$= \frac{P^2}{2(m_1+m_2)} - (m_1-m_2)gx$$

or

$$\dot{x} = \frac{P}{m_1+m_2}$$

$$\mathcal{H} = T+U = \frac{1}{2}(m_1+m_2)\dot{x}^2 - (m_1-m_2)gx$$

$$= \frac{P^2}{2(m_1+m_2)} - (m_1-m_2)gx$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m_1+m_2}$$ (restatement of generalized momentum)

$$\dot{p} = \frac{-\partial H}{\partial x} = (m_1 - m_2)g$$

$$\dot{p} = (m_1+m_2)\ddot{x}$$

$$\ddot{x} = \frac{\dot{p}}{m_1+m_2} = \frac{m_1-m_2}{m_1+m_2}g$$

Steps:

1) write down $\mathcal{H}$
- Using $\mathcal{L}, T, U$

2) write down generalized momentum

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

3) Solve for generalized velocity $\dot{q}$

4) Express $\mathcal{H}$ as $\mathcal{H}(q,p)$
- removing $\dot{q}$ dependance

5) Use $\mathcal{H}(q,p)$ in Hamilton Eqn's

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} \quad \text{and} \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}$$

6) Solve for Equations of motion

13.3 Hamilton's Equations in Several Dimensions

New notation for n generalized coordinates

$$\mathbf{q} = (q_1, \dots, q_n)$$

$$\dot{\mathbf{q}} = (\dot{q}_1, \dots, \dot{q}_n)$$

$$\mathbf{p} = (p_1, \dots, p_n)$$

n-dimensional vectors

Use to build up $\mathcal{L}$ & $\mathcal{H}$

Assume Holonomic Constraints

DOF = # generalized coords.

All non constraining forces can be derived from potentials (they can be non conservative if they depend on t)

not necessary for generalized coords to be "natural" (time independant)

$$\mathcal{L} = \mathcal{L}(q, \dot{q}, t) = T - U$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad [i = 1, \ldots, n]$$

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}$$

$$p_i = \frac{\partial \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{q}_i} \quad [i = 1, \dots, n]$$

Need to express $\mathcal{H}$ by $2n$ variables

$\mathbf{q}$ & $\mathbf{p}$

$$\dot{q}_i = \dot{q}_i(q_1, \dots, q_n, p_1, \dots, p_n, t)$$

$$[i = 1, \dots, n]$$

$$\dot{\mathbf{q}} = \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t)$$

$$\mathcal{H} = \mathcal{H}(\mathbf{q}, \mathbf{p}, t) = \sum_{i}^{n} p_i \dot{q}_i(\mathbf{q}, \mathbf{p}, t) - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t))$$

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \qquad [i = 1, \ldots, n]$$

So $2n$ first-order differential Equations for n given DOF

Time dependance of $\mathcal{H}$

$$\frac{d\mathcal{H}}{dt} = \sum_i^n \left[\frac{\partial \mathcal{H}}{\partial q_i} \dot{q}_i + \frac{\partial \mathcal{H}}{\partial p_i} \dot{p}_i \right] + \frac{\partial \mathcal{H}}{\partial t}$$

((Contains $2n+1$ terms))

total derivative

Change in $\mathcal{H}$ due to change in $q, p,$ & t

Partial

Change in $\mathcal{H}$ only due to change of t holding q & p const if $\mathcal{H}$ not dependant on t

$$\frac{\partial \mathcal{H}}{\partial t} = 0$$

Remember : $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$ $\quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$

So

$$\frac{d\mathcal{H}}{dt} = \sum_i^n \left[\cancel{\frac{\partial \mathcal{H}}{\partial q_i}\left(\frac{\partial \mathcal{H}}{\partial p_i}\right)} + \cancel{\frac{\partial \mathcal{H}}{\partial p_i}\left(-\frac{\partial \mathcal{H}}{\partial q_i}\right)} \right] + \frac{\partial \mathcal{H}}{\partial t}$$

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t}$$

if $\mathcal{H}$ is not explicitly dependant on t (t does not show up in expression) than $\mathcal{H}$ is conserved

If the generalized coordinates are "natural" this is also the case. natural = time independant

For these cases the Hamiltonian is just the total energy

$$\mathcal{H} = T + U$$

Ex 13.3 Hamilton's Eqns for a particle in a central force field

Find hamilton's Eqn's for a mass m subject to a conservative central force field with potential energy $U(r)$ using polar coordinates r & ϕ.

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) \quad , \quad \mathcal{H} = T + U$$

need P_r & P_ϕ since U not dependant on $\dot{q}$

$$P_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$$

$$P_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r} \qquad P_\phi = \frac{\partial T}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

$$\dot{r} = \frac{P_r}{m} \quad \& \quad \dot{\phi} = \frac{P_\phi}{mr^2}$$

$$\mathcal{H} = T + U = \frac{1}{2}\cancel{m}\left(\frac{P_r^2}{m^{\cancel{2}}} + \cancel{r^2}\frac{P_\phi}{m^2 r\cancel{^4}{}^2}\right) + U(r)$$

$$= \frac{1}{2m}\left(P_r^2 + \frac{P_\phi}{r^2}\right) + U(r)$$

4 Hamiton Eqn's

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial P_r} = \frac{P_r}{m} \qquad \dot{P}_r = \frac{-\partial \mathcal{H}}{\partial r} = \frac{P_\phi^2}{mr^3} - \frac{dU}{dr}$$

$$\dot{P}_r = m\ddot{r} = \frac{P_\phi^2}{mr^3} - \frac{dU}{dr}$$

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial P_\phi} = \frac{P_\phi}{mr^2} \qquad \dot{P}_\phi = \frac{-\partial \mathcal{H}}{\partial \phi} = 0$$

$$\dot{P}_\phi = mr^2\ddot{\phi} = 0$$, angular momentum is conserved

Hamiltonian Steps

1) Choose suitable generalized coordinates $q_1, \dots, q_n$

2) Write down T & U in terms of q's & $\dot{q}$'s

3) Find Generalized momenta p_i
(now assuming conservative system, U is independant of $\dot{q}$ so $p_i = \frac{\partial T}{\partial \dot{q}}$) General $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$

4) Solve for $\dot{q}$'s in terms of p_i & q_s

5) Write down Hamitonian as a function of p's & q's

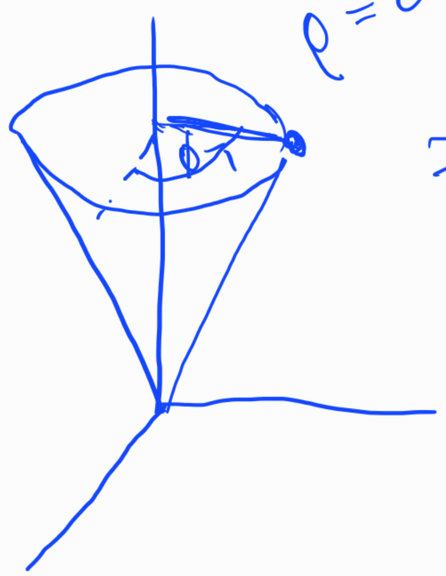
$$\mathcal{H} = T + U \quad \text{or} \quad \mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$$

6) write down Hamilton's Eqn's

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$$

Ex 13.4 Hamilton's Eqn's for a mass on a cone

Set up Hamilton's Eqn's for generalized coordinates z & ϕ for a mass m constrained to move on a vertical cone $\rho = cz$ in a gravitational field g vertically down.
Show that motion is confined between z_{max} & z_{min}. Use this to describe the motion on the cone. Show that for any given value $z_0 > 0$ the solution is a circular path at z_0



$\rho = cz$

$$T = \frac{1}{2} m \left[\dot{\rho}^2 + (\rho \dot{\phi})^2 + \dot{z}^2 \right]$$

$$= \frac{1}{2} m \left[(c^2+1)\dot{z}^2 + (cz\dot{\phi})^2 \right]$$

$$P_z = \frac{\partial T}{\partial \dot{z}} = m(c^2+1)\dot{z}, \quad \dot{z} = \frac{P_z}{m(c^2+1)}$$

$$P_\phi = \frac{\partial T}{\partial \dot{\phi}} = mc^2z^2\dot{\phi}, \quad \dot{\phi} = \frac{P_\phi}{mc^2z^2}$$

$$H = T + U = \frac{1}{2}m\left[(c^2+1)\dot{z}^2 + c^2z^2\dot{\phi}^2\right] + mgz$$

$$= \frac{1}{2}\cancel{m}\left[\cancel{(c^2+1)}\frac{P_z^2}{m^{\cancel{2}}(c^2+1)^{\cancel{2}}} + \cancel{c^2}\cancel{z^2}\frac{P_\phi^2}{m^{\cancel{2}}c^{\cancel{4}2}z^{\cancel{4}2}}\right] + mgz$$

$$= \frac{1}{2m}\left[\frac{P_z^2}{(c^2+1)} + \frac{P_\phi^2}{c^2z^2}\right] + mgz$$

Hamilton's Equation's

$$\dot{z} = \frac{\partial H}{\partial P_z} = \frac{P_z}{m(c^2+1)} \qquad \dot{P}_z = -\frac{\partial H}{\partial z} = \frac{P_\phi^2}{m c^2 z^3} - mg$$

$$\dot{\phi} = \frac{\partial H}{\partial P_\phi} = \frac{P_\phi}{m c^2 z^2} \qquad \dot{P}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

last two show z component of angular momentum is constant

remember

$$H = T + U = E$$

$$E = \frac{1}{2m}\left[\frac{P_z^2}{(c^2+1)} + \frac{P_\phi^2}{c^2 z^2}\right] + mgz$$

$z_{min} > 0 \qquad z_{max} < \infty$

The mass moves around the cone with constant angular momentum P_ϕ as z decreases $\dot{\phi}$ increases and Visa versa oscillating between zmin & z max

For the case that z is constant

$\dot{z} = 0$ & P_z must equal 0 $\rightarrow \dot{P}_z = 0$

$\dot{P}_z = 0$ if $P_\phi = \pm\sqrt{m^2 c^2 g z^3}$

$$\dot{\phi} = \frac{\pm\sqrt{m^2 c^2 g z^3}}{m c^2 z^2} = \frac{\pm\sqrt{g z}}{c z} = \frac{\pm\sqrt{g}}{c\sqrt{z}}$$

Ignorable Coordinates

Remember

If $\mathcal{L}$ is independant of q_i

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{d}{dt} p_i$$

The generalized momentum is conserved.

We say this coordinate is ignorable

If $\mathcal{H}$ is independant of q_i

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} = 0 \qquad p_i \text{ is constant}$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = -\frac{\partial \mathcal{H}}{\partial q_i}$$

Therefore $\mathcal{L}$ is independant if and only if $\mathcal{H}$ is independant or q_i. If q_i is ignorable in the Lagrangian it is ignorable in the hamiltonian

for a 2 DoF system with on ignorable coordinate

$$\mathcal{H}(q_1, P_1, P_2) \to \mathcal{H}(q_1, P_1, K)$$

K is a constant determined by initial coditions

1D Hamiltonian

$$\mathcal{H}(q_1, P_1)$$

So a n DoF system has ignorable coordinate q_i the Hamiltonian looks the same as a $(n-1)$ dimensional problem

This is not the same for the Lagrangian

$\mathcal{L}(q_1, \dot{q}_1, \dot{q}_2)$, if q_2 is ignorable

$P_2 = \text{const} \neq \dot{q}_2$

Ignorable coordinates do not reduce the dimensionalty of a Lagrangian

Lagranges Vs. Hamilton's Eqn's

Number of Eqn's

Lagrangian formalism:

For N- DoF and n GC

nN Second order differential Eqn

Hamitonian Formalism:

2n First order Differential Eqn

n 2nd order

to

2n 1st order is common

Example

$$f(\ddot{q}, \dot{q}, q) = 0$$

define

$$s = \dot{q}$$

see

$$\ddot{q} = \dot{s} \rightarrow f(\dot{s}, s, q) = 0$$

① 2nd $\quad f(\ddot{q}, \dot{q}, q) = 0$

② 1st $\quad s = \dot{q} \qquad f(\dot{s}, s, q) = 0$

This isn't an improvement

But the form is more useful

Rewrite Hamilton's Eqn's to See

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = f(q, p) \quad [i = 1, \dots, n]$$

$$\dot{q} = f(q, p)$$

$$\dot{p} = g(q, p) \quad g_i = -\frac{\partial H}{\partial q_i}$$

Introduce a 2n dimensional vector

$$Z = (q, p) = (q_1, \dots, q_n, p_1, \dots, p_n)$$

Phase-space vector or Phase-Point Z

Z contains all generalized coordinates and their conjugate momenta. Each value of Z identifies a unique point in phase space and a unique set of initial conditions

using

$$\dot{q} = f(q,p) \quad \& \quad \dot{p} = g(q,p)$$

we get a singel Equation of motion (2n components)

$$\dot{Z} = h(Z)$$

where

$$\underline{h} = (f_1, \ldots, f_n, g_1, \ldots, g_n)$$

expressing hamilton's eqn as a first order differential equation of the form

(first derivative of Z) = (function of Z)

Powerful result mathematically that is automatic from the hamiltonian formalism

Lagrange Equationz are flexable in respect to coordinates any

$$q_i = (q_1, \ldots, q_n)$$

can be replaced by

$$Q_i = (Q_1, \ldots, Q_n)$$

Q_i is a function of q_i

So

$$Q_i = Q_i(q_i)$$

Lagrange Equations are invariant under coordinate changes defined in th n-dimensional configuration space $q_A = (q_1, \ldots, q_n)$

Same for Hamiltonian but can also extend to changes in 2n-dimensional Phase space.

$$Q_i = Q_i(q_A, P) \ \& \ P = P(q_A, P)$$

Coordinate changes in which q & P are intermingeled. If they satisfy certain conditions are called

Canonical Transformations

Phase-Space Orbits

The phase-space vector:

$z = (q, p)$ defines a system's position in phase space.

Any point z_0 defines a possible initial condition at time t_0 and Hamilton's equations define a unique phase-space orbit or trajectory, starting from z_0 at t_0

Phase-space has $2n$ dimensions making the visualization of these orbits difficult unless $n=1$

For example an unconstrained Particle with 3 DoF

$2n = 6$ dimensions in Phase-Space

we will look at examples of $n=1$

An Important Property of Phase-Space orbits is that they cannot inersect at any Point.

From Hamiltons equations we can See that for any Point z_0 there Can only be one orbit that Passes through it

If two orbits Pass through the same Point z_0, they must be the same orbit. This is true for all values of time.

*Remember our hamiltonian is time independant, $H(q, P)$

This result limits how orbits trace out and has consequenzes on the analysis of systems such as chaotic motion.

EX 13.5 1-D Harmonic Oscillator

Setup Hamilton's Eqn's for a 1D SHO with mass m and force constant K, describe the possible orbits in phase-space (x, p)

$$T = \frac{1}{2} m \dot{x}^2 \quad , \quad U = \frac{1}{2} K x^2 = \frac{1}{2} m \omega^2 x^2$$

$$\omega = \sqrt{\frac{K}{m}}$$

$$p = \frac{\partial T}{\partial \dot{x}} = m \dot{x} \quad , \quad \dot{x} = \frac{p}{m}$$

$$\mathcal{H} = T + U = \frac{1}{2} \cancel{m} \frac{p^2}{m \cancel{^2}} + \frac{1}{2} m \omega^2 x^2$$

$$= \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -m\omega^2 x$$

$$\ddot{x} = \frac{\dot{p}}{m} = -\omega^2 x$$

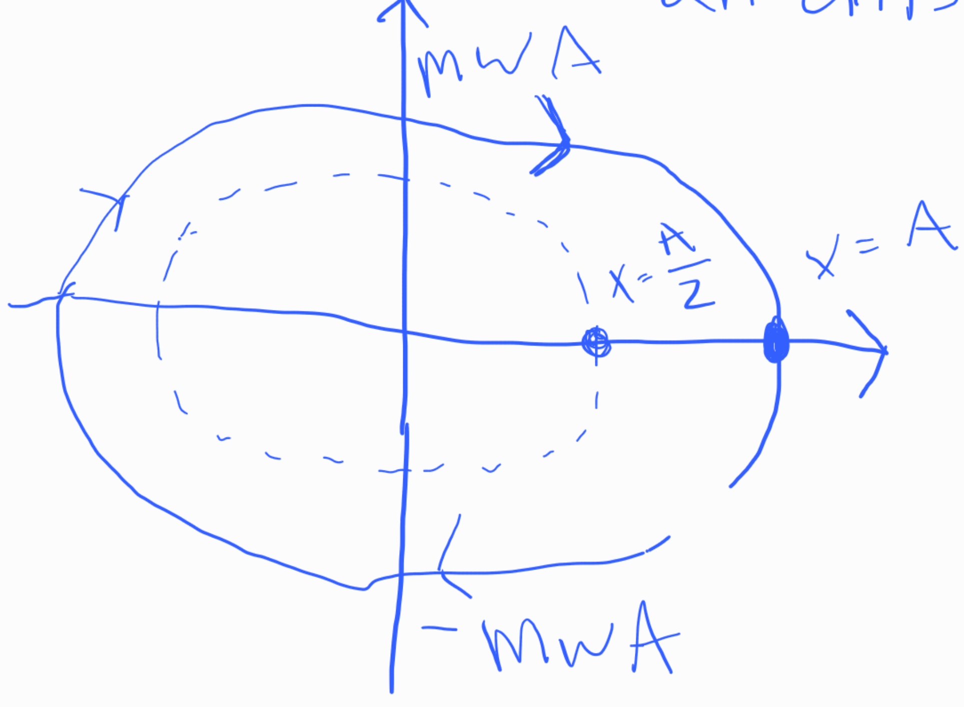
$$x = A\cos(\omega t - \delta)$$

$$p = m\dot{x} = -m\omega \sin(\omega t - \delta)$$

$$z = (x, p) \quad 1D \to 2D \text{ in Phase-space}$$

$$z = \left(A\cos(\omega t - \delta), -m\omega\sin(\omega t - \delta)\right)$$

Parametric form of an ellipse



$$Z_0 = (X = A,\ P = 0)$$

$$\mathcal{H}_0(x = A, P = 0) = \frac{1}{2} m \omega^2 A^2$$

$$\mathcal{H} = \mathcal{H}_0 = \frac{P^2}{2m} + \frac{1}{2} m \omega^2 X^2 = \frac{1}{2} m \omega^2 A^2$$

$$\frac{X^2}{A^2} + \frac{P^2}{(m\omega A)^2} = 1 \quad \text{(Eqn of ellipse)}$$

If we launch a system with 2 different Initial conditions and track their Phase-space orbits, A non chaotic system will have the orbits remain close together and a chaotic system will move apart rapidly and unpredictably

EX 13.6 A Falling body

Set up the Hamitonian formalism for a mass constrained to move in a vertical line, subject to only the force of gravity. Use coordinate x measured downward and it's consugate momentum. Describe the phase-space orbits from time 0 to time t for four different initial conditions.

a) $x_0 = p_0 = 0$ (released at $x=0$ from rest)

b) $x_0 = X$, $p_0 = 0$ (relesed at $x=X$ from rest)

c) $x_0 = X$, $p_0 = P$

d) $x_0 = 0$, $p_0 = P$

$T = \frac{1}{2} m \dot{x}^2 \quad U = -mgx$, (x is measured downwards)

$$\mathcal{H} = T + U = \frac{p^2}{2m} - mgx$$

$\mathcal{H} = \text{const.}$

$$\cancel{m} g x = \frac{p^2}{2\cancel{m}} + \text{const.}, \quad K = \frac{1}{2g}$$

$x = K p^2 + \text{const}$ (Parabola)

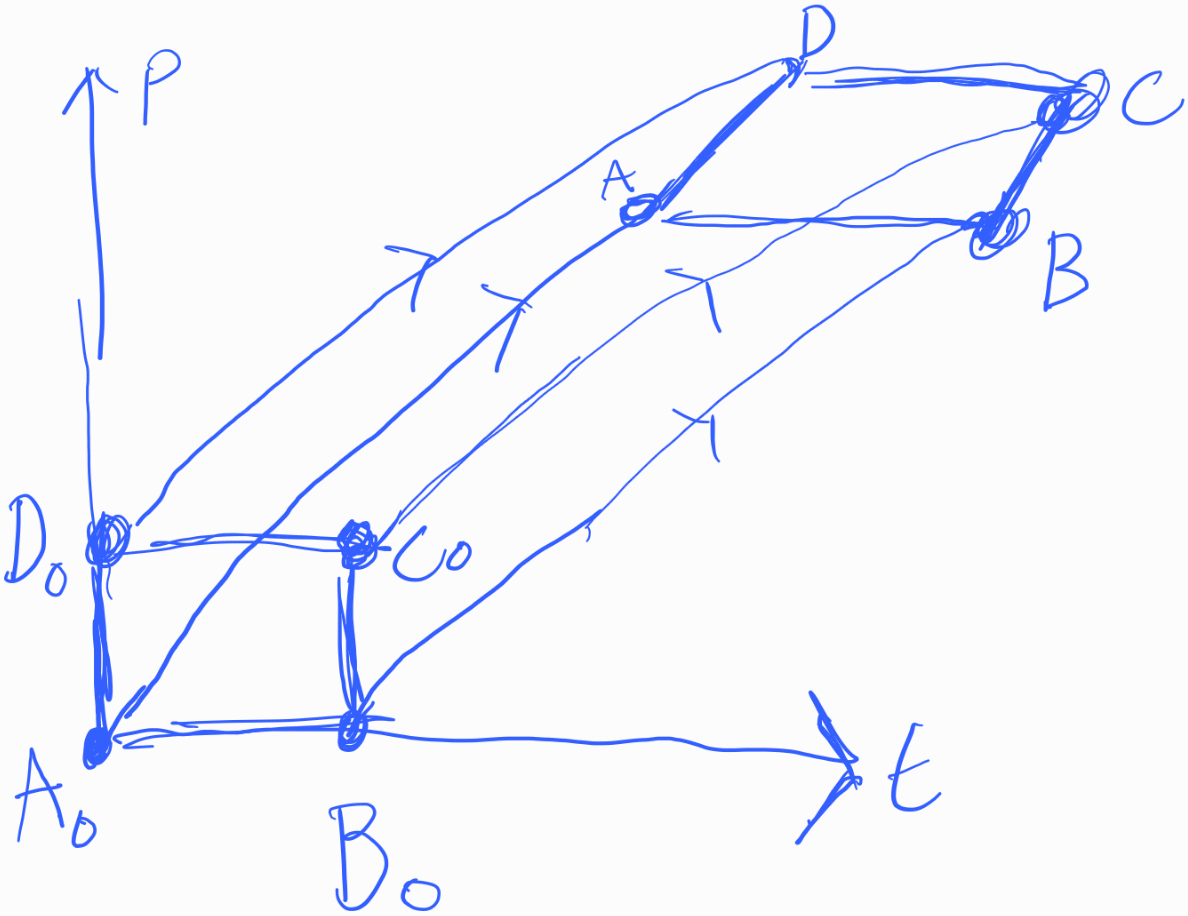
$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot{p} = \frac{-\partial H}{\partial x} = mg$$

Integrate

Integrate

$$p = p_0 + mgt$$

$$x = x_0 + \frac{p_0}{m} t + \frac{1}{2} g t^2$$



Both have same area (Liouville's Theorem)