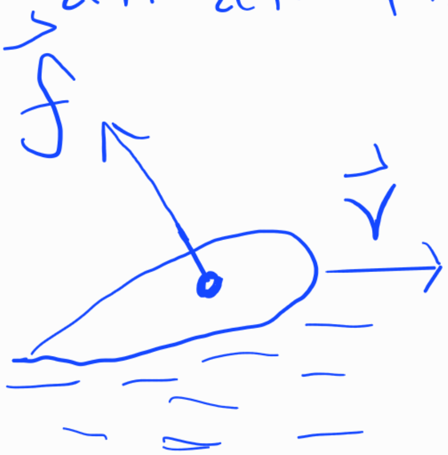
Air Resistance exists!

Drag: resistive force of material an object is moving through.

"f" $\propto$ $\vec{v}$ (velocity)

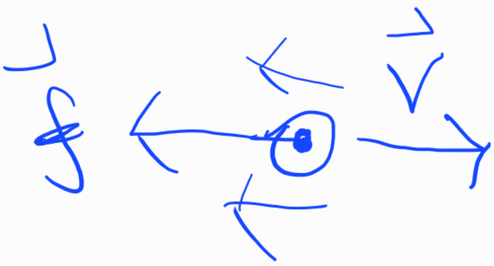
Lift: sideways component of drag on an air plane wing



$f_y \neq 0$

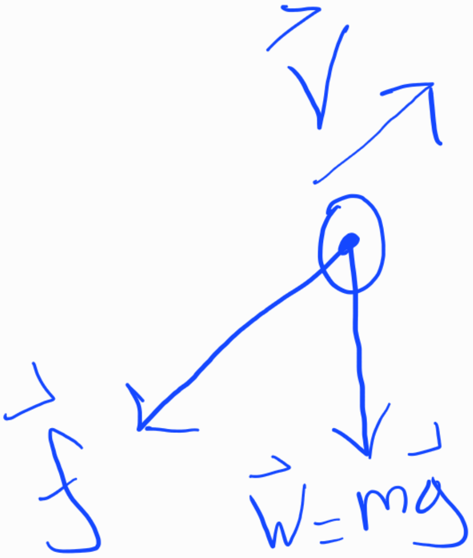
$f_x \neq 0$

Most cases: Drag & Velocity are in opposite directions



$f_y = 0$

$f_x \neq 0$



$$\vec{f} = f(\vec{v})\,\hat{v}$$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

$$f(v) = \underbrace{bv}_{\text{linear}} + \underbrace{cv^2}_{\text{Quadratic}} = f_{lin} + f_{Quad}$$

f_{lin}: Comes from viscosity and linear size of object

f_{Quad}: Due to density of medium and cross sectional Area of object

$b \equiv \beta D \;,\; c \equiv \gamma D^2$

D: diameter of sphere

β & γ depend on medium properties

for a sphere projectile @STP

$\beta \approx 1.6 \times 10^{-4} \frac{N \cdot s}{m^2}$

$\gamma \approx 0.25 \frac{N \cdot s^2}{m^4}$

Ratio of $\frac{f_{quad}}{f_{lin}}$ allows to

decide which can be neglected

$$\frac{f_{quad}}{f_{lin}} \approx \frac{cv^2}{bv^2} = \frac{\gamma}{\beta} Dv$$

$$\approx \left(1.6 \times 10^3 \frac{s}{m^2}\right) Dv$$

Ex 2.1

a) $D = 7cm$ & $v = 5\,m/s$

$\frac{f_{quad}}{f_{lin}} \approx 600$ [baseball], need Quad

$f = -cv^2\hat{v}$

b) $D = 1mm$ & $v = 0.6\,m/s$

$\frac{f_Q}{f_L} \approx 1$ [raindrop], need both

c) $D = 1.5\mu m$, $v = 5 \times 10^{-5}\,m/s$

$\frac{f_Q}{f_L} \approx 10^{-7}$ [milli kandrop], only need Linear

$\vec{f} = -bv\hat{v} = -b\vec{v}$

Reynolds Number:

$$R = \frac{D v \rho}{\eta}, \quad \rho = \text{density}$$
$$\eta = \text{Viscosity}$$

for Large R

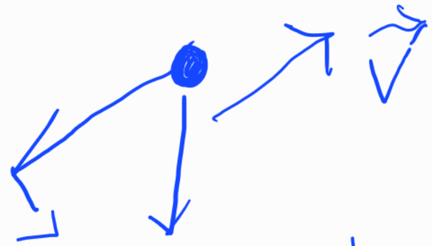
f_{quad} is dominate

for small R

f_{lin} is dominate

Linear Air Resistance

$\vec{F} = m\ddot{\vec{r}}$



$$m\ddot{\vec{r}} = m\vec{g} - b\vec{v}$$

$$m\dot{\vec{v}} = m\vec{g} - b\vec{v}$$ [First order differential Eqn.]

$$\boxed{\begin{aligned} m\dot{v}_x &= -bv \\ m\dot{v}_y &= mg - bv_y \end{aligned}}$$

if $\vec{f} = \vec{f}_{quad}$

$= -cv^2\hat{v}$

$= -cv^2 \dfrac{\vec{v}}{|\vec{v}|} = -cv\vec{v}$

$$\left.\begin{aligned} m\dot{v}_x &= -c\sqrt{v_x^2 + v_y^2}\, v_x \\ m\dot{v}_y &= mg - c\sqrt{v_x^2 + v_y^2}\, v_y \end{aligned}\right\}$$ coupled and complex

For Linear drag

$$\left.\begin{aligned} m\dot{v}_x &= -bv_x \\ m\dot{v}_y &= mg - bv_y \end{aligned}\right\}$$ uncoupled and solved individually

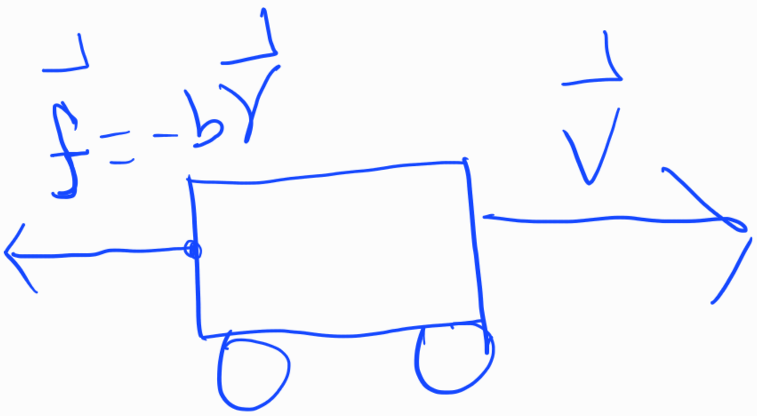
Horizontal motion w/ linear drag:

$m\dot{v}_x = -bv_x$, slows down @ $t=0$, $x=0$, $v_x = v_{0x}$

$$\dot{v}_x = -\frac{b}{m} v_x = -k v_x, \quad k = \frac{b}{m},$$

$$v_x(t) = Ae^{-kt} = v_{x0}e^{-kt} = v_{x0}e^{-t/\tau}$$

$\tau = 1/k = m/b$ [for linear drag]



$$V_x(t) = \dot{x} = V_{x0} e^{-t/\tau}$$

$$x(t) = \int_0^t \dot{x}(t')\,dt = x(t) - x(0)$$

$$x(t) = x(0) + \int_0^t V_{x0} e^{-t'/\tau}\,dt'$$

$$= 0 + \left[-V_{x0}\tau e^{-t'/\tau}\right]_0^t$$

$$= \left[V_{x0}\tau - V_{x0}\tau e^{-t/\tau}\right], \quad x_\infty = V_{x0}\tau$$

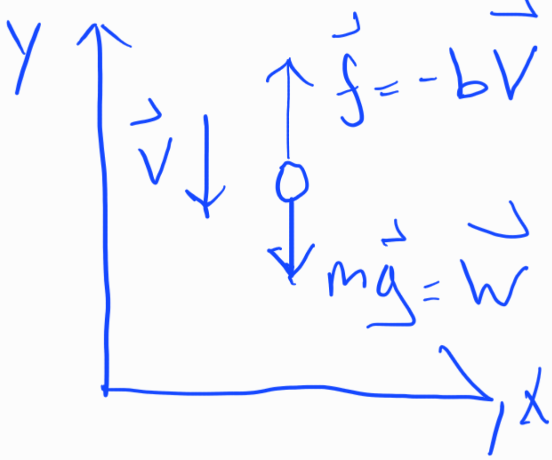
$$\boxed{= x_\infty\left(1 - e^{-t/\tau}\right)}$$

$x(t) \to x_\infty$

as

$t \to \infty$

Vertical Motion with Linear Drag:



$$m\dot{v}_x = -bv_x = 0$$

$$m\dot{v}_y = mg - bv_y$$

$m\dot{v}_y = mg - bv_y$, for small enough v_y eventually

$m\dot{v}_y = 0$, $mg = bv_y$,

$$v_{ter} = v_y = \frac{mg}{b}, \quad [\text{terminal speed}]$$

Ex 2.2

Millikan oil drop

$D = 1.5\,\mu m$ & $\rho = 840\ kg/m^3$

$b = \beta D$, $\beta = 1.6 \times 10^{-4}$

$V_{ter} = \frac{mg}{b}$, $m = \rho V = \rho \frac{\pi}{6} D^3 = \rho \frac{4\pi}{3} r^3$

$$V_{ter} = \frac{\rho \pi D^{\cancel{3}} g}{\beta \cancel{D}} = \frac{\rho \pi D^2 g}{\beta}$$

$= 6.1 \times 10^{-5}\ m/s$

for $D = 0.2\,mm$

$V_{ter} = 1.3\ m/s$

We can rewrite:

$$m\dot{v}_y = mg - bv_y \qquad v_{ter} = \frac{mg}{b}$$

as $m\dot{v}_y = -b(v_y - v_{ter})$

set $u = (v_y - v_{ter})$

$m\dot{u} = -bu$ (v_{ter} is constant)

$u = Ae^{-t/\tau}$, $k = 1/\tau$

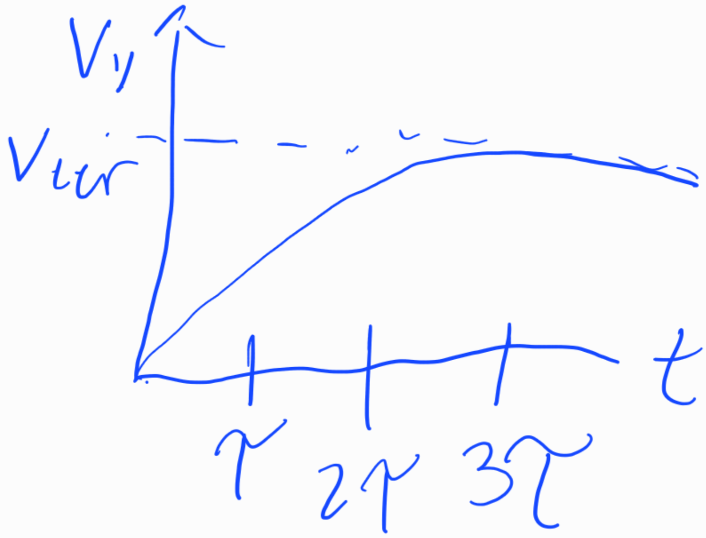
$$v_y - v_{ter} = Ae^{-t/\tau}$$

@ $t=0$, $A = v_{y_0} - v_{ter}$

$$\therefore v_y(t) = v_{ter} + (v_{y_0} - v_{ter})e^{-t/\tau}$$

$$= v_{y_0}e^{-t/\tau} + v_{ter}(1 - e^{-t/\tau})$$

as $t \to \infty$



$V_y(t) \to V_{ter}$

for $V_{y0} = 0$

$$V_y(t) = V_{ter}(1 - e^{-t/\tau})$$

when $t = \tau$

$$V_y = V_{ter}(1 - e^{-1}) = 0.63\, V_{ter}$$

So

t	% V_{ter}
0	0%
τ	63%
2τ	86%
3τ	95%

$\to 3\tau \approx V_{ter}$

Remember:

$$V_y(t) = \dot{y}$$

$$y(t) = \int_0^t V_y(t')\,dt'$$

$$= V_{ter}t + (V_{y0} - V_{ter})\tau\left(1 - e^{-t/\tau}\right)$$

$$\therefore\ y(t) = V_{ter}t + (V_{y0} - V_{ter})\tau\left(1 - e^{-t/\tau}\right)$$

$$x(t) = x_\infty\left(1 - e^{-t/\tau}\right)$$

$$= V_{x0}\tau\left(1 - e^{-t/\tau}\right)$$

Projectile motion in Linear Medium

Trajectory & Range in Linear Media

$$X(t) = V_{xo}\tau\left(1 - e^{-t/\tau}\right)$$

$$Y(t) = (V_{yo} + V_{ter})\tau\left(1 - e^{-t/\tau}\right) - V_{ter}\tau$$

note: $+Y$ direction is up now

by solving first for t & plugging into second

$$Y(t) = \frac{(V_{yo} + V_{ter})}{V_{xo}}X + V_{ter}\tau\ln\left(1 - \frac{X}{V_{xo}\tau}\right)$$

Trajectory Eqn

Range:

$$R_{vac} = \frac{2V_{x0}V_{y0}}{g}$$ from Gen. Phys. [no air resistance]

Range R is X @ Y=0 so,

$$\frac{V_{y0}+V_{ter}}{V_{x0}} R + V_{ter}\tau \ln\left(1 - \frac{R}{V_{x0}\tau}\right) = 0$$

This is a trancendental equation and has no analytical solution but can be solved numerically for a given set of Parameters

Approximate Solution:

f_{air} is usually small

$\therefore V_{ter}$ & τ are large

So $\ln\left(1-\frac{R}{V_{x0}\tau}\right)$ is small

we can expand w/ Talor series

$$\ln(1-\epsilon) = -\left(\epsilon + \frac{1}{2}\epsilon^2 + \frac{1}{3}\epsilon^3 + \dots\right)$$

$$\left[\frac{V_{y0}+V_{ter}}{V_{x0}}\right]R - V_{ter}\tau\left[\frac{R}{V_{x0}\tau} + \frac{1}{2}\left[\frac{R}{V_{x0}\tau}\right]^2 + \frac{1}{3}\left[\frac{R}{V_{x0}\tau}\right]^3\right] = 0$$

Every term has R

so $R=0$ is a solution

$\therefore y=0$ @ $x=0$ but this

is uninteresting

Expanding, cancelling terms and rearranging we get:

$$R = \frac{2V_{x0}V_{y0}}{g} - \underbrace{\frac{2}{3V_{x0}}\tau R^2}_{\text{Very small}}$$

$R < R_{vac}$

First Approximation

$$R \approx \frac{2V_{x0}V_{y0}}{g} = R_{vac}$$

$V_{ter} = \tau g$

if $R \approx R_{vac}$

$$R \approx R_{vac} - \frac{2}{3V_{x0}\tau}(R_{vac})^2$$

$$= R_{vac}\left(1 - \frac{4}{3}\frac{V_{y0}}{V_{ter}}\right)$$

If $\frac{V_{y0}}{V_{ter}} << 1$ $\quad R = R_{vac}$