Lagrange's Equations

7.1 Lagrange's Eqns for unconstrained motion

Consider a particle acted on by $\vec{F}(\vec{r})$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{r}^2 = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

$$U = U(\vec{r}) = U(x,y,z)$$

$$\mathcal{L} = T - U$$

$$\mathcal{L} = \mathcal{L}(x,y,z,\dot{x},\dot{y},\dot{z})$$

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x} = P_x$$

Remember $F_x = \dot{P}_x$ (newton's 2nd Law)

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \quad \& \quad \frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}}$$

Hamilton's Principle

$$S = \int_{t_1}^{t_2} \mathcal{L} \, dt$$

The path a particle follows between two points in a time interval $t_1 \to t_2$ is such that the action integral is stationary along the path

We've learned that we can determine a particle's path via 3 ways

1. Newton's 2nd Law $F = ma$
2. 3 Lagrange Eqns
3. Hamilton's Priciple

Hamilton's Principle Proves Lagrange's equations hold for any coordinate system

$$\underbrace{(x, y, z) \to (r, \theta, \phi) \to (\rho, \phi, z)}_{q_1,\, q_2,\, q_3}$$

$$q_i = q_i(r) \quad , i = 1, 2, 3$$

$$\vec{r} = \vec{r}(q_1, q_2, q_3)$$

$$\mathcal{L} = \frac{1}{2} m \dot{r}^2 - U(r)$$

$$\mathcal{L} = \mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)$$

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)$$

The change in variables does nothing to the formalism

Thus

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad , \quad i = 1, 2, 3$$

* for inertial frames

Ex 2.1 1 Particle in 2D

Find $\mathcal{L}$ for a particle moving with a 2D conservative force

$$\mathcal{L} = \mathcal{L}(x, y, \dot{x}, \dot{y}) = T - U$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - U(x, y)$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \Leftrightarrow F_x = m\ddot{x}$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} \Leftrightarrow F_y = m\ddot{y}$$

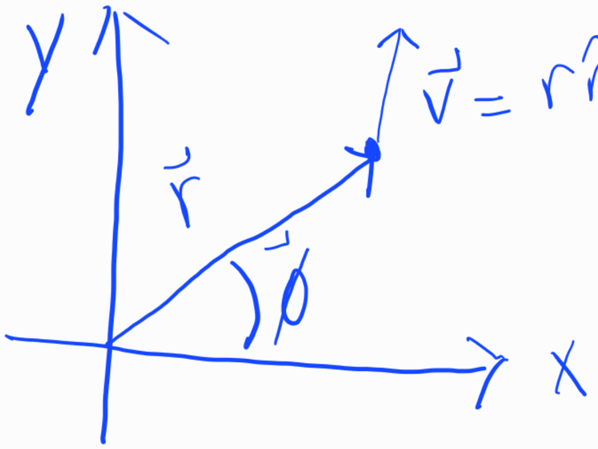
$$\frac{\partial \mathcal{L}}{\partial q_i} \Rightarrow (\text{ith component of generalized force})$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \Rightarrow (\text{ith component of generalized momentum})$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

(generalized force) = (Rate of change of generalized momentum)

Ex 7.2 1 Particle 2D (Polar)



$$\vec{v} = r\hat{r} + r\dot{\phi}\hat{\phi}$$

$$\mathcal{L} = T - U$$

$$\mathcal{L}(r, \phi, \dot{r}, \dot{\phi}) = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r, \phi)$$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$m r\dot{\phi}^2 - \underbrace{\frac{\partial U}{\partial r}}_{F_r} = \frac{d}{dt}(mr) = m\ddot{r}$$

$$\therefore F_r = m\left(\ddot{r} - r\dot{\phi}^2\right)$$

ϕ Eqn:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

$$-\frac{\partial U}{\partial \phi} = \frac{d}{dt}\big(\underbrace{m r^2 \dot{\phi}}_{L}\big)$$

$$\nabla U = \frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\phi}$$

$\downarrow$

$$\Gamma = r F_\phi$$

$$\therefore \; \Gamma = \frac{dL}{dt}$$

$$(\phi \text{ component of generalized force}) = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$= \Gamma \; (\text{torque})$$

(ϕ component of generalized momentum)

$$= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = L \text{ (angular momentum)}$$

if the ith componet of $\frac{\partial \mathcal{L}}{\partial q_i}$ is zero then $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ is constant

and the generalized momentum is conserved

Allows you to quickly identify conservation laws for your system

Can be extended to N Particles

for 2 Particles:

$$\mathcal{L}(\vec{r}_1, \vec{r}_2, \dot{r}_1, \dot{r}_2) = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 - U(\vec{r}_1, \vec{r}_2)$$

$$\vec{F}_1 = -\vec{\nabla}_1 U \quad \& \quad \vec{F}_2 = -\nabla_2 \vec{U}$$

$$F_{1x} = \dot{P}_{1x} \quad F_{1y} = \dot{p}_{1y} \quad F_{1z} = \dot{p}_{1z}$$

$$F_{2x} = \dot{P}_{2x} \quad F_{2y} = \dot{P}_{2y} \quad F_{2z} = \dot{P}_{2z}$$

6 Eqns

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1}, \quad \frac{\partial \mathcal{L}}{\partial y_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}_1}, \quad \frac{\partial \mathcal{L}}{\partial z_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}_1}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_2}, \quad \frac{\partial \mathcal{L}}{\partial y_2} = \frac{\partial \mathcal{L}}{\partial \dot{y}_2}, \quad \frac{\partial \mathcal{L}}{\partial z_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}_2}$$

Each coordinate can become a generalized one q_i

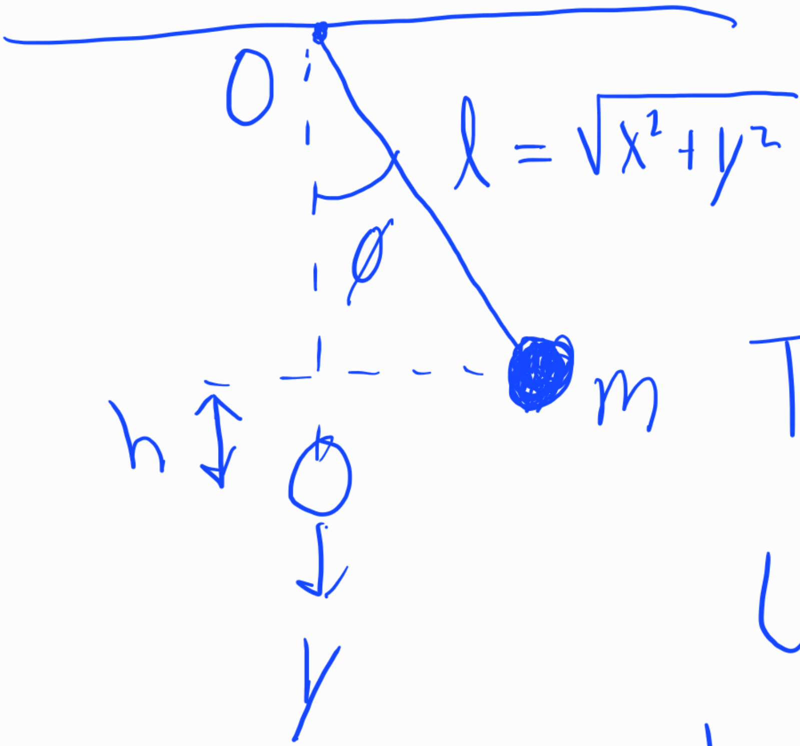
So

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$ ($3N$ Generalized coordinates)

particles

7.2 Constrained Systems

Plane Pendulum



can Express Everything Using ϕ

$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\phi}^2$

$U = mgh$

$h = l - l\cos\phi = l(1-\cos\phi)$

$$L = \frac{1}{2}ml^2\dot{\phi}^2 - mgl(1-\cos\phi)$$

$q \to \phi$

$$\underbrace{\frac{\partial L}{\partial \phi}}_{-mgl\sin\phi} = \frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{\phi}}}_{ml^2\dot{\phi}}$$

$$\underbrace{-mgl\sin\phi}_{\Gamma} = \frac{d}{dt}\left(ml^2\dot{\phi}\right) = \underbrace{ml^2}_{I}\,\underbrace{\ddot{\phi}}_{\alpha}$$

$$\Sigma = I\alpha$$

7.3 Constrained Systems in General

System of N Particles

with positions r_α, $\alpha = 1, 2, \ldots, N$

$q_1, \ldots, q_n$ are a set of generalized coordinates that r_α can be expressed by, along with t.

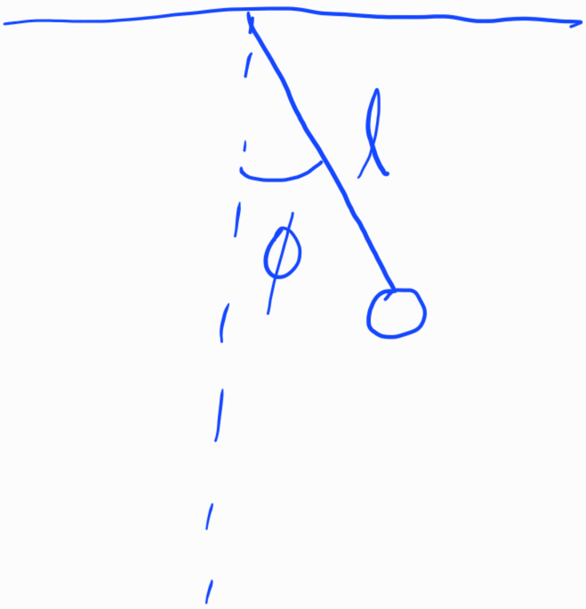
$$r_\alpha = r_\alpha(q_1, \ldots, q_n, t), \quad \alpha = 1, \ldots, N$$

Conversly each q_i can be expressed by r_α & t

$$q_i = q_i(r_1, \ldots, r_N, t) \quad i = 1, \ldots, n$$

The number of generalized Coordinates, n, must be the minimum allowable to Parameterize the system

For a 3D system this is no more than 3N, and even less for a constrained system



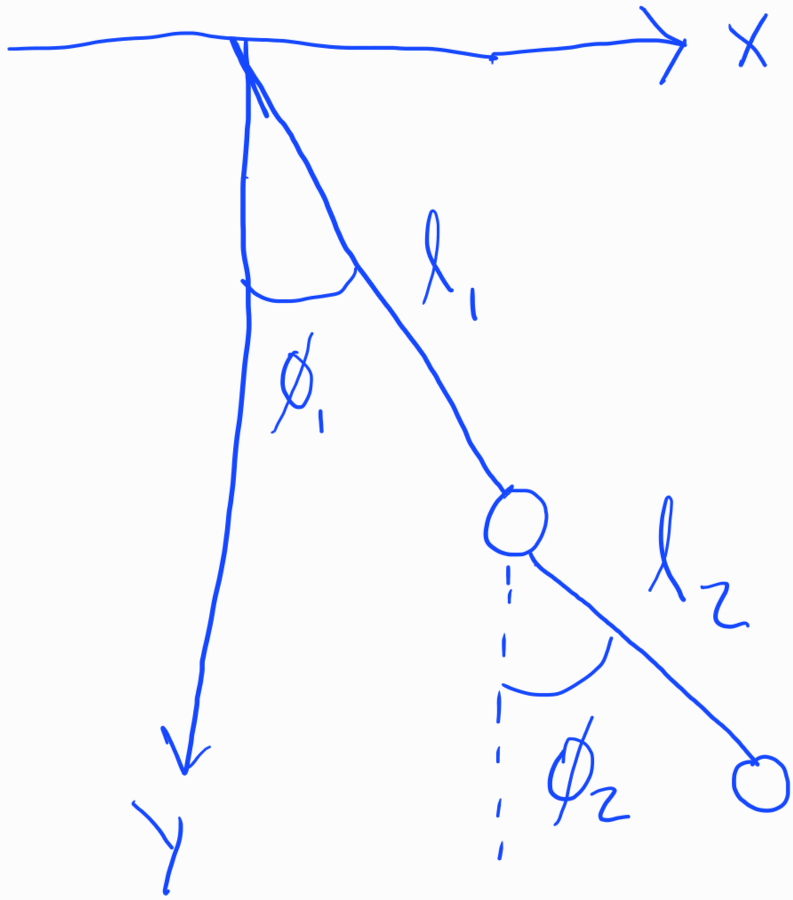
1 Particle

2 coordinates (x, y)

or

$\underline{1}$ generalized coordinate

$$\vec{r} \equiv (x, y) = (l\sin\phi, l\cos\phi)$$



2 Particles

4 Cartesian coord

or

2 generalized coordinates

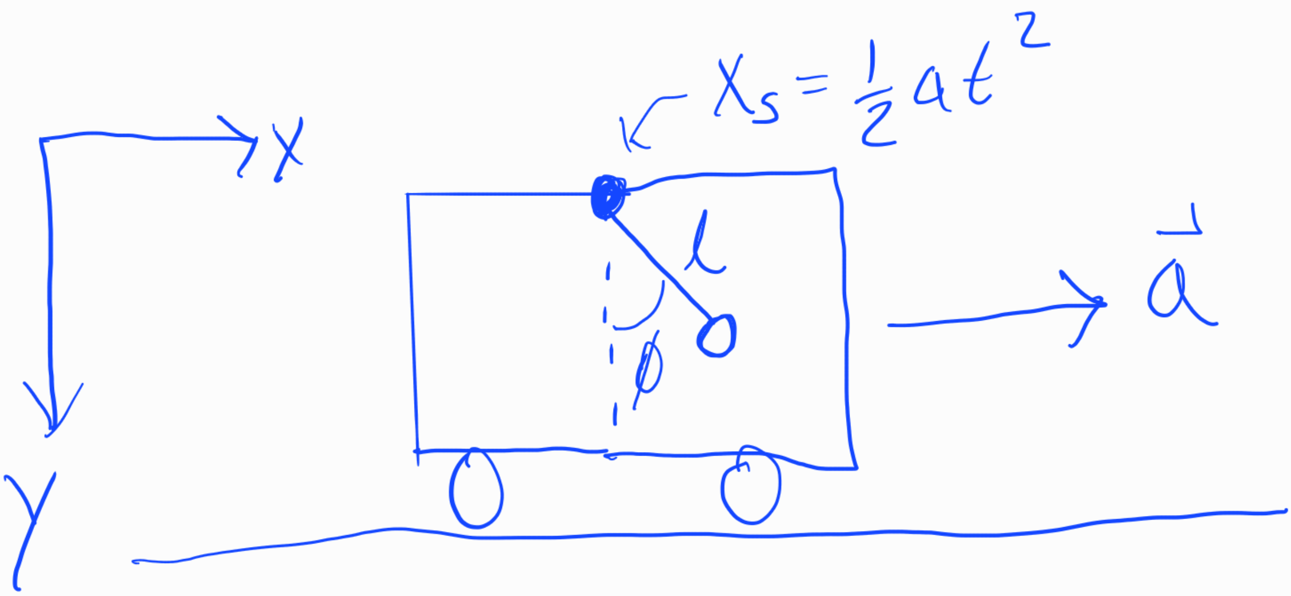
$$\vec{r}_1 = (l_1 \sin\phi_1, l_1 \cos\phi_2) = \vec{r}_1(\phi_1)$$

$$\vec{r}_2 = (l_1 \sin\phi_1 + l_2 \sin\phi_2, l_1 \cos\phi_1 + l_2 \cos\phi_2)$$

$$= \vec{r}_2(\phi_1, \phi_2)$$

Time Independant

Time Dependant Generalized coords



$$\vec{r} = (x,y) = (l\sin\phi + \tfrac{1}{2}at^2,\ l\cos\phi)$$

We call a set of Generalized Coordinates $q_1, \dots, q_N$ "Natural" if the relationship between r_α and q_i does not depend on time,

Degrees of Freedom

The number of degrees of freedom of a system is the number of coordinates that can be varied independantly of one another

Pendulum $\Rightarrow$ 1 DOF

Double Pendulum $\Rightarrow$ 2 DOF

Particle in 3D $\Rightarrow$ 3

Gas of N Particles $\Rightarrow$ 3N

If the number of DOF (in 3D) is less than 3N (2N for 2D) we call the system constrained.

When the number of DOF equals the number of necessary Generalized Coordinates required to describe the system, we call it Holonomic. We will only consider Holonomic systems.

These systems can be described by the n Lagrange eqns.

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}, \quad \mathcal{L} - T - 0$$

7.4 Proof of Lagranges Equations with constraints

Consider a Particle constrained to move on a surface (2 DOF & 2 GC) q_1 & q_2

Two types of forces acting on Particle

1) forces of constraint

$\vec{F}_{cstr} \Rightarrow$ normal, atomic,

2) Non-Constraint Forces

$\vec{F}_{NC} = -\nabla U(\vec{r}, t)$, gravity etc

if conservative independant of t

$$\vec{F} = \overrightarrow{F_{cstr}} + F_{NC}$$

$$\mathcal{L} = T - U_{\pi}$$ only considers F_{NC}

Action integral is stationary at the Right Path

Consider a particle traveling between two points at times t_1 & t_2

right path

$$R(t) = r(t) + \epsilon(t)$$

$\epsilon = 0$ @ t_1 & t_2

Action integral

$$S = \int_{t_1}^{t_2} \mathcal{L}(R, \dot{R}, t)\, dt$$

S is stationary for all $R(t)$ if $R(t) = r(t)$ or $\epsilon(t) = 0$

$$\delta S = S - S_0 = \int \delta \mathcal{L} \, dt$$

$$\delta \mathcal{L} = \mathcal{L}(R, \dot{R}, t) - \mathcal{L}(r, \dot{r}, t)$$

$$\mathcal{L}(r, \dot{r}, t) = T - U = \frac{1}{2} m \dot{r}^2 - U(r, t)$$

$$\mathcal{L}(R, \dot{R}, t) = \frac{1}{2} m (\dot{r} + \dot{\epsilon})^2 - U(r + \epsilon, t)$$

$$\delta \mathcal{L} = \frac{1}{2} m \left[(\dot{r} + \dot{\epsilon})^2 - \dot{r}^2 \right] - \left[U(r + \epsilon, t) - U(r, t) \right]$$

$$= m \dot{r} \cdot \dot{\epsilon} - \epsilon \cdot \nabla U + \cancel{O(\epsilon^2)}$$

ignore higher orders

$$\delta S = \int_{t_1}^{t_2} \delta \mathcal{L} \, dt = \int_{t_1}^{t_2} \left[m \dot{r} \cdot \dot{\epsilon} - \epsilon \cdot \nabla U \right] dt$$

$$\left(f(r + \epsilon) - f(r) \approx \epsilon \cdot \nabla f \right) \text{ * Sec 4.3}$$

$$\delta S = -\int_{t_1}^{t_2} \epsilon \cdot [m\ddot{r} + \nabla U] \, dt$$

$$m\ddot{r} = F_{tot} = F_{cstr} + F_{NC}$$

$$\nabla U = -F_{NC}$$

$$\delta S = -\int_{t_1}^{t_2} \underbrace{\epsilon \cdot F_{cstr}}$$

F_{cstr} is $\perp$ to surface

ϵ is path component along surface so, $\epsilon \cdot F_{cstr} = 0$

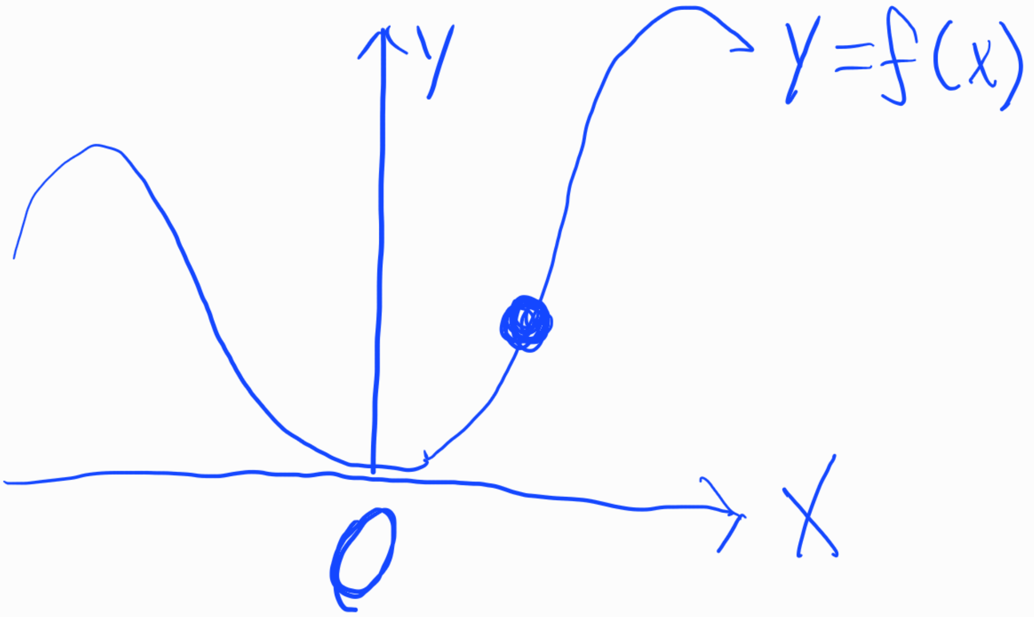
$$\therefore \delta S = 0$$

$$S=\int_{t_1}^{t_2} \mathcal{L}(q_1,q_2,\dot{q}_1,\dot{q}_2,t)\,dt$$

For any holomonic system with n DOF & n GC and with nonconstraint forces derivable from a potential energy, the path followed by the system is determined by the n Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i}=\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i},\ [i=1,\dots,n]$$

Ex 11.1 Bead on a wire



Path of wire is $y=f(x)$

write down T & U for small oscillations around O

$$U = mgy = mgf(x)$$

for small oscillations we can taylor expand f(x)

$$f(x) \approx f(0) + f'(0)x + \frac{1}{2}f''(0)x^2$$

$$f(0) = 0, \quad f'(0) = 0, \quad f''(0) \neq 0$$

So $\quad U(x) = mg\left[\frac{1}{2} f''(0) x^2\right]$

$$T = \frac{1}{2} m \left[\dot{x}^2 + \dot{y}^2\right], \quad \dot{y} = f'(x)\dot{x}$$

$$T = \frac{1}{2} m \left[1 + f'(x)^2\right] \dot{x}^2, \quad f'(x) \approx 0$$

at 0

$$T \approx \frac{1}{2} m \dot{x}^2$$