## Solutions to Problems for Chapter 13

**13.1** ⋆ $\mathcal{L} = \frac{1}{2}m\dot{x}^2$, $p = \partial\mathcal{L}/\partial\dot{x} = m\dot{x}$, and $\mathcal{H} = p\dot{x} - \mathcal{L} = p^2/2m$. The Hamilton equations are

$$\dot{x} = \partial\mathcal{H}/\partial p = p/m \qquad \text{and} \qquad \dot{p} = -\partial\mathcal{H}/\partial x = 0,$$

with solutions $p = p_o = \text{const}$, and $x = x_o + v_o t$, where $v_o = p_o/m$.

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**13.2** ⋆ The Lagrangian is $\mathcal{L} = \frac{1}{2}m\dot{x}^2 + mgx$, so $p = \partial\mathcal{L}/\partial\dot{x} = m\dot{x}$, and $\mathcal{H} = p\dot{x} - \mathcal{L} = p^2/2m - mgx$.The Hamilton equations are

$$\dot{x} = \partial\mathcal{H}/\partial p = p/m \qquad \text{and} \qquad \dot{p} = -\partial\mathcal{H}/\partial x = mg.$$

Combining the two Hamilton equations, we find that $\ddot{x} = g$ as expected.

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**13.3** ⋆ The moment of inertia of a uniform disc is $I = \frac{1}{2}MR^2$, and its kinetic energy is $\frac{1}{2}I\omega^2 = \frac{1}{2}I\dot{x}^2/R^2$. Therefore, $\mathcal{L} = \frac{1}{2}(m_1+m_2+\frac{1}{2}M)\dot{x}^2 + (m_1-m_2)gx$, $p = (m_1+m_2+\frac{1}{2}M)\dot{x}$, and $\mathcal{H} = p\dot{x} - \mathcal{L} = p^2/2(m_1+m_2+\frac{1}{2}M) - (m_1-m_2)gx$. The Hamilton equations are

$$\dot{x} = \partial\mathcal{H}/\partial p = p/(m_1+m_2+\tfrac{1}{2}M) \qquad \text{and} \qquad \dot{p} = -\partial\mathcal{H}/\partial x = (m_1-m_2)g,$$

and the acceleration is $\ddot{x} = g(m_1-m_2)/(m_1+m_2+\frac{1}{2}M)$.

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**13.4** ⋆ The two original coordinates are $x$ and $y$, and the constraint equation is $x + y + \pi R$ $=$ const. Thus the equations for $x$ and $y$ in terms of the generalized coordinate $x$ are $x = x$ (of course) and $y = -x + \text{const}$, both of which are independent of time.

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**13.5** ⋆⋆ Since $\rho = R$ is fixed and $z = c\phi$, there is just one generalized coordinate, which we can choose to be $\phi$. The bead's kinetic energy is $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{z}^2 + R^2\dot{\phi}^2) = \frac{1}{2}m(c^2+R^2)\dot{\phi}^2$, the potential energy is $U = mgz = mgc\phi$, and the generalized momentum is $p = \partial T/\partial\dot{\phi} = m(c^2+R^2)\dot{\phi}$. From these we find the Hamiltonian:

$$\mathcal{H} = T + U = \frac{p^2}{2m(c^2+R^2)} + mgc\phi,$$

and the two Hamilton equations:

$$\dot{\phi} = \frac{\partial\mathcal{H}}{\partial p} = \frac{p}{m(c^2+R^2)} \qquad \text{and} \qquad \dot{p} = -\frac{\partial\mathcal{H}}{\partial\phi} = -mgc.$$

Combining the last two, we can find $\ddot{\phi}$ and thence

$$\ddot{z} = c\ddot{\phi} = -g\frac{c^2}{c^2+R^2} = -g\sin^2\alpha$$

where in the last expression I have introduced tha pitch $\alpha$ of the helix. (The bead rises a height $2\pi c$ in a horizontal run of $2\pi R$, so $\tan\alpha = c/R$ and $\sin\alpha = c/\sqrt{c^2+R^2}$.)

According to Newton, we could argue that the bead's tangential acceleration is $a_{\text{tang}} = g\sin\alpha$ (the well known acceleration down an incline) and its vertical component is $\ddot{z} = -a_{\text{tang}}\sin\alpha = -g\sin^2\alpha$.

Inserting this result into (iv) and solving for $\ddot{x}$ we find that, according to Newton,

$$\ddot{x} = -\frac{gh' + h'h''\dot{x}^2}{1 + h'^2}. \tag{v}$$

Let's now see that we get the same result from the Hamilton equations (iii). From the first of Eqs.(iii) we find

$$\ddot{x} = \frac{d}{dt}\dot{x} = \frac{d}{dt}\frac{p}{m(1+h'^2)} = \frac{\dot{p}}{m(1+h'^2)} - \frac{p}{m}\frac{2h'h''\dot{x}}{(1+h'^2)^2}$$

If we use the second Hamilton equation (iii) to eliminate $\dot{p}$ and the second of Eqs.(ii) to eliminate $p$, this is easily seen to be exactly the same as the Newtonian result (v).

There is a much simpler way to accomplish the same result, though it may seem a cheat at first sight. Hamilton's equations, like Lagrange's from which we derived them, are true with respect to any choice of generalized coordinates. Therefore we can handle the same problem using as our generalized coordinate $s$, the distance measured along the track. If we do this, then the Lagrangian is $\mathcal{L} = \frac{1}{2}m\dot{s}^2 - U(s)$ and the generalized momentum is $p = \partial\mathcal{L}/\partial s = m\dot{s}$. Thus the Hamiltonian is $\mathcal{H} = p^2/(2m) + U(s)$ and the second Hamilton equation is $\dot{p} = -\partial\mathcal{H}/\partial s = -dU/ds$ or $m\ddot{s} = -dU/ds$ in agreement with the Newtonian result Eq.(iv).

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**13.8** $\star$ Since $U = 0$, $\mathcal{L} = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. Therefore, $p_x = \partial\mathcal{L}/\partial\dot{x} = m\dot{x}$, and similarly $p_y = m\dot{y}$ and $p_z = m\dot{z}$, and finally $\mathcal{H} = \mathbf{p}\cdot\dot{\mathbf{r}} - \mathcal{L} = \mathbf{p}^2/2m$. The six Hamilton equations are

$$\dot{x} = \frac{\partial\mathcal{H}}{\partial p_x} = \frac{p_x}{m} \quad \text{and} \quad \dot{p}_x = -\frac{\partial\mathcal{H}}{\partial x} = 0$$

with similar equations for the $y$ and $z$ components. We can combine these into two vector equations $\dot{\mathbf{r}} = \mathbf{p}/m$ and $\dot{\mathbf{p}} = 0$, with the expected solutions $\mathbf{p} = \text{const} = \mathbf{p}_o$ and $\mathbf{r} = \mathbf{r}_o + \mathbf{v}_o t$ where $\mathbf{v}_o = \mathbf{p}_o/m$.

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**13.9** $\star$ The Lagrangian is $\mathcal{L} = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$, and the generalized momentum has components $p_x = \partial\mathcal{L}/\partial\dot{x} = m\dot{x}$ and $p_y = \partial\mathcal{L}/\partial\dot{y} = m\dot{y}$. Therefore, the Hamiltonian is $\mathcal{H} = \mathbf{p}\cdot\dot{\mathbf{r}} - \mathcal{L} = (p_x^2 + p_y^2)/2m + mgy$. The Hamilton equations are

$$\dot{x} = p_x/m, \quad \dot{y} = p_y/m, \quad \dot{p}_x = 0, \quad \text{and} \quad \dot{p}_y = -mg.$$

The first two of these simply reproduce the known relations for $\mathbf{p}$ in terms of $\dot{\mathbf{r}}$. The third says that the $x$ component of $\mathbf{p}$ is constant, and the last that $p_y$ changes at the expected rate $-mg$.

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**13.10** $\star$ The KE is $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$. If we choose the PE to be zero at the origin, $U = -\int_0^{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r} = \frac{1}{2}kx^2 + Ky$. The generalized momenta are given by $\mathbf{p} = m\dot{\mathbf{r}}$ and the Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}(p_x^2 + p_y^2) + \tfrac{1}{2}kx^2 + Ky.$$

The two Hamilton equations for $x$ are

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_x}{m} \quad \text{and} \quad \dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = -kx$$

which combine to give $\ddot{x} = -(k/m)x$. Thus $x$ oscillates in SHM, $x = A\cos(\omega t - \delta)$, with angular frequency $\omega = \sqrt{k/m}$. Meanwhile, the two $y$ equations are

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial p_y} = \frac{p_y}{m} \quad \text{and} \quad \dot{p}_y = -\frac{\partial \mathcal{H}}{\partial y} = -K$$

which combine to give $\ddot{y} = -K/m$. Thus $y$ accelerates in the negative $y$ direction, $y = -\frac{1}{2}(K/m)t^2 + v_{yo}t + y_o$, with constant acceleration $-K/m$.

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**13.11** $\star$ Let's measure $x$ along the tracks (in the direction of travel), $y$ crossways, and $z$ vertically up, all three relative to the car. The ball's velocity relative to the ground is $(V + \dot{x},\ \dot{y},\ \dot{z})$, so the Lagrangian is $\mathcal{L} = T - U = \frac{1}{2}m[(V + \dot{x})^2 + \dot{y}^2 + \dot{z}^2] - mgz$. The generalized momentum has components

$$p_x = \frac{\partial \mathcal{L}}{\partial x} = m(V + \dot{x}), \qquad p_y = m\dot{y}, \qquad \text{and} \qquad p_z = m\dot{z}$$

(Notice that, perhaps unexpectedly, the generalized momentum is the momentum relative to the ground, not to the moving car.) We can solve for $\dot{x} = (p_x - mV)/m$ and then

$$\mathcal{H} = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - \mathcal{L} = \frac{\mathbf{p}^2}{2m} - p_xV + mgz.$$

From this, you can derive the expected equation of motion (as you could check), but our point here is to note that $\mathcal{H}$ is not equal to the energy $T + U$ (neither relative to the car nor relative to the ground), because

$$\begin{aligned}(T+U)_{\text{(rel to car)}} &= \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz \\ &= \frac{\mathbf{p}^2}{2m} - p_xV + \tfrac{1}{2}mV^2 + mgz \neq \mathcal{H}.\end{aligned}$$

and

$$(T+U)_{\text{(rel to ground)}} = \frac{\mathbf{p}^2}{2m} + mgz \neq \mathcal{H}.$$

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**13.12** $\star$ As generalized coordinate I'll use the bead's position $x$ relative to the axis of spin, as measured in the frame of the rod. The bead's PE is zero and its KE (relative to the ground) is $T = \frac{1}{2}m(\dot{x}^2 + x^2\omega^2)$, so $\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + x^2\omega^2)$ and the generalized momentum is $p = \partial\mathcal{L}/\partial\dot{x} = m\dot{x}$. Therefore the Hamiltonian is

$$\mathcal{H} = p\dot{x} - \mathcal{L} = \frac{p^2}{2m} - \tfrac{1}{2}mx^2\omega^2.$$

This is not equal to the energy $T+U$ (neither relative to the rod nor relative to the ground), because

$$(T+U)_{\text{(rel to rod)}} = \tfrac{1}{2}m\dot{x}^2 = \frac{p^2}{2m} \neq \mathcal{H}.$$

and

$$(T+U)_{\text{(rel to ground)}} = \tfrac{1}{2}m(\dot{x}^2 + x^2\omega^2) = \frac{p^2}{2m} + \tfrac{1}{2}mx^2\omega^2 \neq \mathcal{H}.$$

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**13.13** ⋆⋆ The KE is $T = \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{z}^2)$, and the PE is $U = \frac{1}{2}kr^2 = \frac{1}{2}k(R^2+z^2)$. Thus the two generalized momenta are

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = mR^2\dot{\phi} \quad \text{and} \quad p_z = \frac{\partial T}{\partial \dot{z}} = m\dot{z}$$

and the Hamiltonian is

$$\mathcal{H} = T + U = \frac{1}{2m}\left(\frac{p_\phi^2}{R^2} + p_z^2\right) + \frac{1}{2}k(R^2+z^2).$$

The two Hamilton equations for $z$ are

$$\dot{z} = \frac{\partial \mathcal{H}}{\partial p_z} = \frac{p_z}{m} \quad \text{and} \quad \dot{p}_z = -\frac{\partial \mathcal{H}}{\partial z} = -kz.$$

These two combine to give $\ddot{z} = -(k/m)z$, which shows that the motion in the $z$ direction is SHM with frequency $\omega = \sqrt{k/m}$. The two Hamilton equations for $\phi$ are

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mR^2} \quad \text{and} \quad \dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi} = 0.$$

The first of these simply repeats the relation between $\dot{\phi}$ and $p_\phi$. The second tells us that $p_\phi$ (namely, the $z$ component of angular momentum) is constant and hence that the motion around the cylinder proceeds with constant $\dot{\phi}$.

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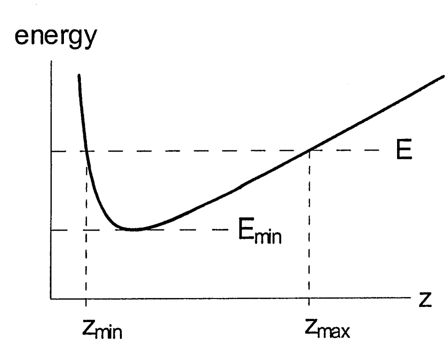
**13.14** ⋆⋆ According to Eq.(13.32), $\dot{z} = p_z/m(c^2+1)$, so $\dot{z}$ can vanish if and only if $p_z = 0$. According to (13.33)

$$\frac{1}{2m}\left[\frac{p_z^2}{(c^2+1)} + \frac{p_\phi^2}{c^2z^2}\right] + mgz = E.$$

Therefore, if $p_z = 0$ it must be that

$$\frac{p_\phi^2}{2mc^2z^2} + mgz = E.$$




To find how many values of $z$ can satisfy this equation consider the graph of the LHS plotted against $z$. When $z \to 0$ or $z \to \infty$ it is clear that the LHS approaches $+\infty$. By differentiating it, you can check that its derivative vanishes exactly once, so the graph of the LHS has the simple cup shape shown. If $E < E_{\text{min}}$,

**13.19** ⋆ The KE is $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ and the PE is $U = U(r) = U\left(\sqrt{x^2+y^2}\right)$. Thus the Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + U\left(\sqrt{x^2+y^2}\right).$$

Clearly neither $\partial\mathcal{H}/\partial x$ nor $\partial\mathcal{H}/\partial y$ is zero (unless $U$ is a constant); that is, neither $x$ nor $y$ is ignorable.

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**13.20** ⋆ **(a)** $U(\mathbf{r}) = -\int \mathbf{F}\cdot d\mathbf{r} = -\mathbf{F}\cdot\mathbf{r}$. Therefore

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} - \mathbf{F}\cdot\mathbf{r} = \frac{p_x^2 + p_y^2}{2m} - F_x x - F_y y.$$

**(b)** If we choose our $x$ axis in the direction of $\mathbf{F}$, then $F_y = 0$ and the coordinate $y$ is ignorable.

**(c)** If neither axis is parallel to $\mathbf{F}$, then neither $F_x$ nor $F_y$ is zero, and neither $\partial\mathcal{H}/\partial x$ nor $\partial\mathcal{H}/\partial y$ is zero, so neither of the coordinates is ignorable.

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**13.21** ⋆⋆ **(a)** The KE is T $= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2)$, and the PE is $U = \frac{1}{2}k(r - l_\text{o})^2$. The generalized momenta are $\mathbf{P} = M\dot{\mathbf{R}}$, $p_r = \mu\dot{r}$, and $p_\phi = \mu r^2\dot{\phi}$, so the Hamiltonian is

$$\mathcal{H} = \frac{1}{2M}(P_x^2 + P_y^2) + \frac{1}{2\mu}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) + \tfrac{1}{2}k(r - l_\text{o})^2.$$

The CM coordinates $X$ and $Y$ are ignorable, because there are no external forces, so that total momentum is conserved. The coordinate $\phi$ is ignorable because the force between the two masses is central, so their angular momentum is conserved. The coordinate $r$ is not ignorable, because there is a radial force.

**(b)** The two Hamilton equations for $X$ are

$$\dot{X} = \frac{\partial\mathcal{H}}{\partial P_x} = \frac{P_x}{M} \quad \text{and} \quad \dot{P}_x = -\frac{\partial\mathcal{H}}{\partial X} = 0$$

with corresponding equations for $Y$. These four equations say that the CM position $\mathbf{R}$ moves like a free particle, with constant velocity. The two equations for $r$ are

$$\dot{r} = \frac{\partial\mathcal{H}}{\partial p_r} = \frac{p_r}{\mu} \quad \text{and} \quad \dot{p}_r = -\frac{\partial\mathcal{H}}{\partial r} = \frac{p_\phi^2}{\mu r^3} - k(r - l_\text{o}),$$

and the two $\phi$ equations are

$$\dot{\phi} = \frac{\partial\mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{\mu r^2} \quad \text{and} \quad \dot{p}_\phi = -\frac{\partial\mathcal{H}}{\partial \phi} = 0$$

which last says that $p_\phi$ is conserved as expected.

**(c)** The two $r$ equations combine to give the familiar radial equation

$$\mu\ddot{r} = \dot{p}_r = \frac{p_\phi^2}{\mu r^3} - k(r - l_\text{o}). \tag{viii}$$

In the special case that $p_\phi = 0$ this shows that $r$ executes SHM about the equilibrium length $r = l_o$. The CM moves with constant velocity while the two masses oscillate toward and away from the CM.

**(d)** If $p_\phi \neq 0$, the radial equation (viii) is no longer linear and cannot be solved in terms of elementary functions. In this case, the two masses still oscillate in and out, but not in SHM, while they orbit around the CM with constant angular momentum.

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**13.22** ⋆⋆ **(a)** According to (13.14) $\mathcal{H}(q,p) = p\dot{q}(q,p) - \mathcal{L}\big(q, \dot{q}(q,p)\big)$, so

$$\frac{\partial \mathcal{H}}{\partial q} = p\frac{\partial \dot{q}}{\partial q} - \left[\frac{\partial \mathcal{L}}{\partial q} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q}\right] = -\frac{\partial \mathcal{L}}{\partial q}$$

where the first and third terms in the middle expression cancel because $p = \partial\mathcal{L}/\partial\dot{q}$.

**(b)** According to (13.24),

$$\mathcal{H} = \mathcal{H}(\mathbf{q},\mathbf{p},t) = \sum_{j=1}^{n} p_j\dot{q}_j(\mathbf{q},\mathbf{p},t) - \mathcal{L}\big(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q},\mathbf{p},t), t\big).$$

so

$$\frac{\partial \mathcal{H}}{\partial q_i} = \sum_j p_j\frac{\partial \dot{q}_j}{\partial q_i} - \left[\frac{\partial \mathcal{L}}{\partial q_i} + \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial q_i}\right] = -\frac{\partial \mathcal{L}}{\partial q_i}$$

where the two sums cancel because $p_j = \partial\mathcal{L}/\partial\dot{q}_j$.

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**13.23** ⋆⋆⋆ **(a)** The gravitational PE is $U_{\text{gr}} = Mgy - mgy - mg(x+y) + \text{const} = -mgx$ if we drop the uninteresting constant. The spring PE is harder. If we let $l_o$ denote the spring's natural, unloaded length, then $k(l_e - l_o) = mg$ and if $x'$ denotes the spring's true extension (from its unloaded length), then $l_o + x' = l_e + x$ so

$$x' = x + (l_e - l_o) = x + \frac{mg}{k}$$

Thus the spring PE is

$$U_{\text{spr}} = \tfrac{1}{2}kx'^2 = \tfrac{1}{2}k\left(x + \frac{mg}{k}\right)^2 = \tfrac{1}{2}kx^2 + mgx + \text{const.}$$

If we add this to $U_{\text{gr}} = -mgx$, the terms in $mgx$ cancel and (dropping another uninteresting constant) we get $U = U_{\text{gr}} + U_{\text{spr}} = \tfrac{1}{2}kx^2$ as claimed.

**(b)** The KE is $T = \tfrac{1}{2}M\dot{y}^2 + \tfrac{1}{2}m\dot{y}^2 + \tfrac{1}{2}m(\dot{x}+\dot{y})^2 = \tfrac{1}{2}m[3\dot{y}^2 + (\dot{x}+\dot{y})^2]$, from which we find the momenta,

$$p_x = \frac{\partial T}{\partial \dot{x}} = m(\dot{x}+\dot{y}) \quad \text{and} \quad p_y = \frac{\partial T}{\partial \dot{y}} = m(\dot{x}+4\dot{y})$$

whence

$$\dot{x}+\dot{y} = \frac{p_x}{m} \quad \text{and} \quad \dot{y} = \frac{1}{3m}(p_y - p_x).$$

From these we can calculate the Hamiltonian,

$$\mathcal{H} = T + U = \frac{1}{2m}\left[\frac{(p_x - p_y)^2}{3} + p_x^2\right] + \frac{1}{2}kx^2.$$

Because this doesn't depend on $y$, the coordinate $y$ is ignorable. This is traceable to the fact that the total mass on each side is the same.

**(c)** The Hamilton equations for $x$ are

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{1}{3m}(4p_x - p_y) \quad \text{and} \quad \dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = -kx \tag{ix}$$

and those for $y$

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial p_y} = \frac{1}{3m}(p_y - p_x) \quad \text{and} \quad \dot{p}_y = -\frac{\partial \mathcal{H}}{\partial y} = 0. \tag{x}$$

The initial conditions are that $x(0) = x_o$, $y(0) = y_o$, and $\dot{x}(0) = \dot{y}(0) = 0$. These imply that $p_x(0) = p_y(0) = 0$, and, because $p_y$ is constant, $p_y = 0$ for all time. Combining the two equations (ix) and setting $p_y = 0$, we find that $\ddot{x} = 4\dot{p}_x/3m = -4kx/3m$. Therefore $x = x_o \cos\omega t$, where $\omega = \sqrt{4k/3m}$. Next, from the first of Eqs.(ix) (with $p_y = 0$) we find that $p_x = \frac{3}{4}m\dot{x} = -\frac{3}{4}m\omega x_o \sin\omega t$ and finally, from the first of Eqs.(x), $\dot{y} = -p_x/3m = \frac{1}{4}\omega x_o \sin\omega t$, so $y = -\frac{1}{4}x_o\cos\omega t + \text{const} = y_o + \frac{1}{4}x_o(1 - \cos\omega t)$.

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**13.24** $\star$ The new variables are defined as $Q = p$ and $P = -q$. So

$$\dot{Q} = \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = -\frac{\partial \mathcal{H}}{\partial(-P)} = \frac{\partial \mathcal{H}}{\partial P}$$

and

$$\dot{P} = -\dot{q} = -\frac{\partial \mathcal{H}}{\partial p} = -\frac{\partial \mathcal{H}}{\partial Q}.$$

These are precisely Hamilton's equations with respect to the new variables.

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**13.25** $\star\star\star$ **(a)** The new variables $Q$ and $P$ are defined so that

$$q = \sqrt{2P}\sin Q \quad \text{and} \quad p = \sqrt{2P}\cos Q \tag{xi}$$

and we are promised that the old variables $q$ and $p$ satisfy Hamilton's equations. So now consider this:

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial Q} &= \frac{\partial \mathcal{H}}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial \mathcal{H}}{\partial p}\frac{\partial p}{\partial Q} = -\dot{p}(\sqrt{2P}\cos Q) - \dot{q}(\sqrt{2P}\sin Q) \\
&= -(\dot{p}p + \dot{q}q) = -\frac{1}{2}\frac{d}{dt}(p^2 + q^2) = -\frac{d}{dt}(P\cos^2 Q + P\sin^2 Q) = -\dot{P}.
\end{aligned}$$

This proves the Hamilton equation for $\dot{P}$. Next

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial P} &= \frac{\partial \mathcal{H}}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial \mathcal{H}}{\partial p}\frac{\partial p}{\partial P} = -\dot{p}\left(\frac{1}{\sqrt{2P}}\sin Q\right) + \dot{q}\left(\frac{1}{\sqrt{2P}}\cos Q\right) \\
&= -\dot{p}\frac{q}{2P} + \dot{q}\frac{p}{2P} = \frac{p^2}{2P}\frac{d}{dt}\left(\frac{q}{p}\right) = \frac{p^2}{2P}\frac{d}{dt}\tan Q = \frac{p^2}{2P}\dot{Q}\sec^2 Q = \dot{Q}.
\end{aligned}$$

Here in moving to the second line I used Eqs.(xi), and for the final equality I used the second of Eqs.(xi). This proves the Hamilton equation for $\dot{Q}$.

**(b)** $\mathcal{H} = (p^2/2m) + \frac{1}{2}kq^2 = \frac{1}{2}(p^2 + q^2)$ because $m = k = 1$.

**(c)** $\mathcal{H} = \frac{1}{2}(p^2 + q^2) = \frac{1}{2}(2P\cos^2 Q + 2P\sin^2 Q) = P$. We see that $Q$ is ignorable and that the new Hamiltonian is just $P$ (so that conservation of $P$ is just conservation of energy).

**(d)** The Hamilton equation for $Q$ reads $\dot{Q} = \partial\mathcal{H}/\partial P = 1$, which implies that $Q = (t - \delta)$ where $\delta$ is an arbitrary constant. Substituting this into the first of Eqs.(xi) and putting $P = E$, we find $q = \sqrt{2E}\sin(t - \delta)$, which correctly describes an SHO with energy $E$ and frequency $\omega = \sqrt{k/m} = 1$.

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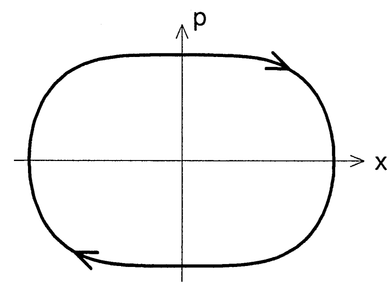
**13.26** ⋆ The potential energy is

$$U = -\int F\,dx = \tfrac{1}{4}kx^4,$$

and the Hamiltonian is

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{4}kx^4 = E.$$

In the two-dimensional phase space, with coordinates $x$ and $p$, this defines the flattened ellipse shown.

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**13.27** ⋆⋆ The position and momentum of an object in free fall are $x = x_o + (p_o/m)t + \frac{1}{2}gt^2$ and $p = p_o + mgt$. Thus the positions in phase space of the eight points in Fig.13.6 are as follows:

| time 0 | | | time $t$ | | |
|---|---|---|---|---|---|
| | $x_o$ | $p_o$ | | $x$ | $p$ |
| $A_o$ | 0 | 0 | $A$ | $\frac{1}{2}gt^2$ | $mgt$ |
| $B_o$ | X | 0 | $B$ | $X + \frac{1}{2}gt^2$ | $mgt$ |
| $C_o$ | X | P | $C$ | $X + (P/m)t + \frac{1}{2}gt^2$ | $P + mgt$ |
| $D_o$ | 0 | P | $D$ | $(P/m)t + \frac{1}{2}gt^2$ | $P + mgt$ |

Inspecting the data in this table, you can see the heights of the two points $D_o$ and $C_o$ above the $x$ axis are both equal to $P$, so the lines $D_oC_o$ and $A_oB_o$ are parallel. Since both of these lines have the same length $(X)$, $A_oB_oC_oD_o$ is a parallelogram. (In fact it is also a rectangle,but this doesn't matter here.) In the same way, the lines $AB$ and $DC$ parallel and of equal length, so $ABCD$ is also a parallogram. Again from the data in the table you can check that both parallograms have bases equal to $X$ and heights equal to $P$. Thus both have area $X \cdot P$. In particular, their areas are the same.

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**13.28** ⋆⋆ **(a)** The potential energy is

$$U = -\int F\,dx = -\tfrac{1}{2}kx^2,$$