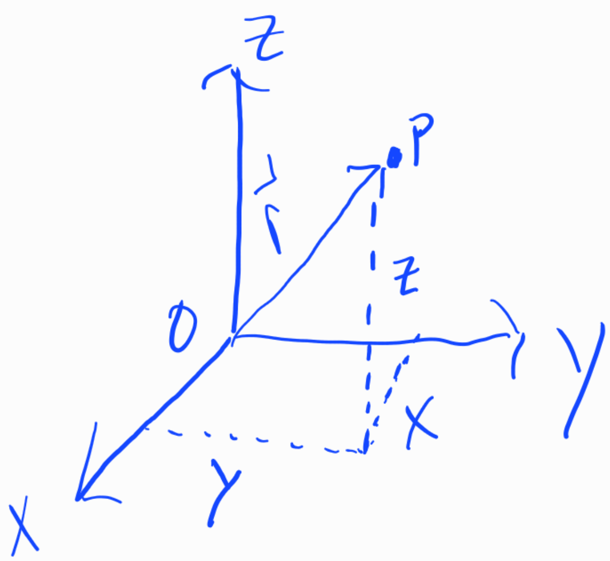
Space:



Cartesian coordinates

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$
$$= x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = (x, y, z)$$

$$\vec{V} = V_x\hat{x} + V_y\hat{y} + V_z\hat{z}$$
$$\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$$

Numerical Notation:

$$\vec{r} = r_1\hat{e}_1 + r_2\hat{e}_2 + r_3\hat{e}_3$$

$r_1 = x \quad r_2 = y \quad r_3 = z$

$\hat{e}_1 = \hat{x} \quad \hat{e}_2 = \hat{y} \quad \hat{e}_3 = \hat{z}$

$\hat{e}$ stands for the German

"eins" or "one"

We can rewrite $\vec{r}$ as:

$$\vec{r} = r_1\hat{e}_1 + r_2\hat{e}_2 + r_3\hat{e}_3 = \sum_{i=1}^{3} r_i\hat{e}_i$$

Vector Operations

$$\vec{r} = (r_1, r_2, r_3) \text{ and } \vec{s} = (s_1, s_2, s_3)$$

$$\vec{r} + \vec{s} = (r_1 + s_1, r_2 + s_2, r_3 + s_3)$$

$$c\vec{r} = (cr_1, cr_2, cr_3)$$

2 types of Vector Products.

Scalar (dot) Product:

$\vec{r} \cdot \vec{s} = rs\cos\theta$, θ is the angle between the vectors

$$= r_1s_1 + r_2s_2 + r_3s_3 = \sum_{n=1}^{3} r_n s_n$$

$$r = |\vec{r}| = \sqrt{\vec{r} \cdot \vec{r}} = \sqrt{r_1^2 + r_2^2 + r_3^2} \rightarrow \vec{r}^{\,2}$$

Vector (cross) Product:

$$\vec{P} = \vec{r} \times \vec{s}$$

$$P_x = r_y s_z - r_z s_y$$

$$P_y = r_z s_x - r_x s_z$$

$$P_z = r_x s_y - r_y s_x$$

$$\vec{r} \times \vec{s} = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ r_x & r_y & r_z \\ s_x & s_y & s_z \end{bmatrix}$$

$$P = |\vec{P}| = rs\sin\theta$$

Time:

$\vec{r}(t)$

$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$

$\vec{a}(t) = \frac{d\vec{v}(t)}{dt}$

$\frac{d\vec{x}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}, \Delta x = x(t+\Delta t) - x(t)$

$\vec{v}$

$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t}, \Delta r = r(t+\Delta t) - r(t)$

Derivative Operations:

$$\frac{d}{dt}(\vec{r}+\vec{s}) = \frac{d\vec{r}}{dt} + \frac{d\vec{s}}{dt}$$

Product rule:

$$\frac{d}{dt}(f\vec{r}) = f\frac{d\vec{r}}{dt} + \frac{df}{dt}\vec{r}$$

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$

$$V = V_x\hat{x} + V_y\hat{y} + V_z\hat{z}$$

$$V_x = \frac{dx}{dt}, \quad V_y = \frac{dy}{dt}, \quad V_z = \frac{dz}{dt}$$

Mass:

Characterizes an object's inertia
the resistance to be accelerated

Force: A Push or Pull.

Newton's 1^{st} & 2^{nd} Laws

Law of Inertia: In the absence of forces, a particle moves with a constant velocity.

$$\text{If } \sum \vec{F} = 0, \vec{a} = 0 \text{ \& } \vec{V} = \text{constant}$$

Second Law:

$$\sum \vec{F} = m\vec{a}$$

$$\vec{a} = \frac{d\vec{V}}{dt} \equiv \dot{V} = \frac{d^2\vec{r}}{dt^2} \equiv \ddot{r}$$

$$\vec{V} = \dot{r} \quad \vec{a} = \dot{v} = \ddot{r}$$

Momentum:

$$\vec{P} = m\vec{v} = m\dot{r}$$

$$\dot{P} = m\dot{v} = m\ddot{r} = \underbrace{m\vec{a}}$$

$$\Sigma\vec{F} = m\vec{a} = \dot{P}$$

Differential Equations:

$$\vec{F} = m\ddot{r}$$

$$\ddot{X}(t) = \frac{F_0}{m} \quad \text{(second order differential equation)}$$

Solution:

Integrate twice!

$$\ddot{x}(t) = \frac{F_0}{m} = a_0$$

$$\dot{x}(t) = \int \ddot{x}(t)\,dt = \frac{F_0}{m} t + C$$

$$\dot{x}(t=0) = \frac{F_0}{m}(0) + C = V_0$$

$$C = V_0$$

$$\boxed{\therefore \dot{x}(t) = V_0 + \frac{F_0}{m} t = V_0 + a_0 t}$$

$$x(t) = \int \left(V_0 + \frac{F_0}{m} t \right) dt$$

$$= V_0 t + \frac{F_0}{2m} t^2 + C$$

$$x(t=0) = x_0 = C$$

a_0

$$\boxed{\therefore x(t) = x_0 + V_0 t + \frac{1}{2} \frac{F_0}{m} t^2}$$

Inertial Frames:

$\vec{F}=0,\ \vec{a}=0$ (Newton's First Law)

Defining S as a frame

S' is an inertial frame relative to S only if it is not rotating or accelerating relative to S

We will focus on inertial frames for the next few lectures but we will see non-inertial frames later in the material!

Newton's 3rd Law:

$$\vec{F}_{12} = -\vec{F}_{21}$$

$$\Sigma \vec{F}_1 = \vec{F}_{12} + \vec{F}_1^{ext}$$

$$\Sigma \vec{F}_2 = \vec{F}_{21} + \vec{F}_2^{ext}$$

$$\dot{\vec{P}}_1 = \vec{F}_1 = \vec{F}_{12} + \vec{F}_1^{ext}$$

$$\dot{\vec{P}}_2 = \vec{F}_2 = \vec{F}_{21} + \vec{F}_2^{ext}$$

$$\vec{P} = \vec{P}_1 + \vec{P}_2 \qquad \vec{F}_{12} = -\vec{F}_{21} \text{ or } \vec{F}_{21} = -\vec{F}_{12}$$

$$\dot{\vec{P}} = \dot{\vec{P}}_1 + \dot{\vec{P}}_2 = \cancelto{0}{\vec{F}_{12}} + \vec{F}_1^{ext} + \cancelto{0}{\vec{F}_{21}} + \vec{F}_2^{ext}$$

$$= \vec{F}_1^{ext} + \vec{F}_2^{ext} \equiv \vec{F}^{ext}$$

$$\therefore \vec{F}^{ext} = 0, \quad \vec{P} = const$$

Equation of Motion (cartesian coord)

$$\vec{F} = m\ddot{\vec{r}}$$

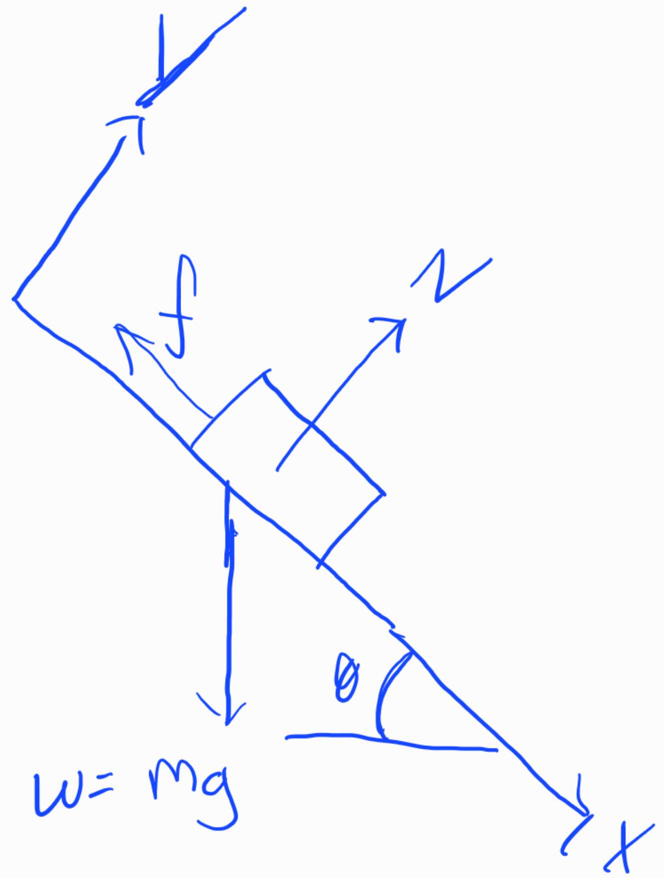
$$\vec{F} = F_x\hat{x} + F_y\hat{y} + F_z\hat{z}$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\ddot{\vec{r}} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$$

$$\vec{F} = \underbrace{m\ddot{x}}_{F_x}\hat{x} + \underbrace{m\ddot{y}}_{F_y}\hat{y} + \underbrace{m\ddot{z}}_{F_z}\hat{z}$$

Ex



m, μ, θ, v_0

$t = ?$

$F_z = m\ddot{z} = 0, \quad \ddot{z} = 0$ for all t

$F_y = N - mg\cos\theta = m\ddot{y} = 0, \quad \ddot{y} = 0$ for all t

$F_y = N - mg\cos\theta = 0$

$N = mg\cos\theta$

$f = \mu N, \quad f = \mu mg\cos\theta$

$\vec{F}_x = w_x - f = m\ddot{x}$

$$\cancel{m}g\sin\theta - \mu \cancel{m} g\cos\theta = \cancel{m}\ddot{x}$$

$$\ddot{x} = g(\sin\theta - \mu\cos\theta)$$

Integrate!

$$\dot{x} = \int g(\sin\theta - \mu\cos\theta)\,dt = g(\sin\theta - \mu\cos\theta)t + \cancel{c}^{\,0}$$

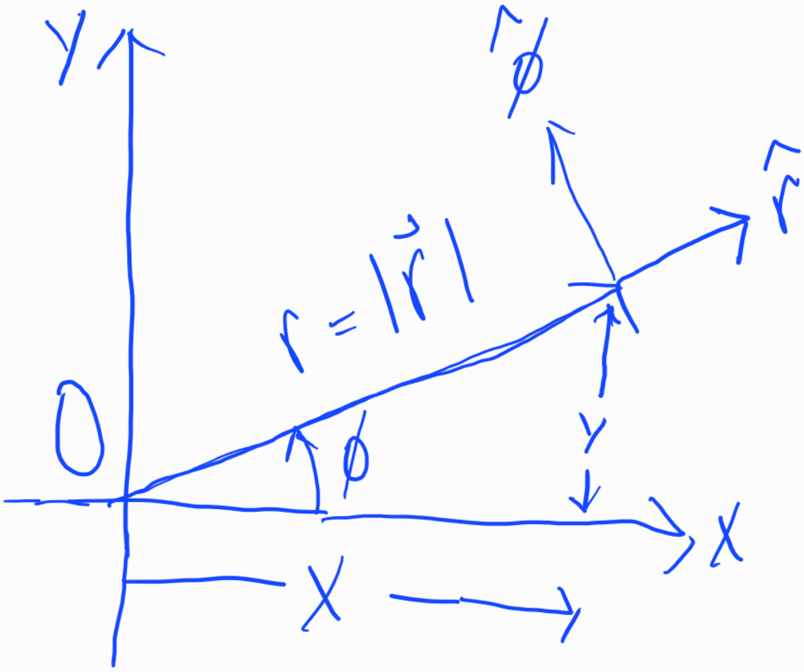
$$\dot{x}(t=0) = 0$$

$$x(t) = \int \dot{x}(t) = \frac{1}{2}g(\sin\theta - \mu\cos\theta)t^2 + \cancel{c}^{\,0}$$

$$x(t=0) = 0$$

$$\therefore x(t) = \frac{1}{2}gt^2(\sin\theta - \mu\cos\theta)$$

Two-Dimension Polar coordinates

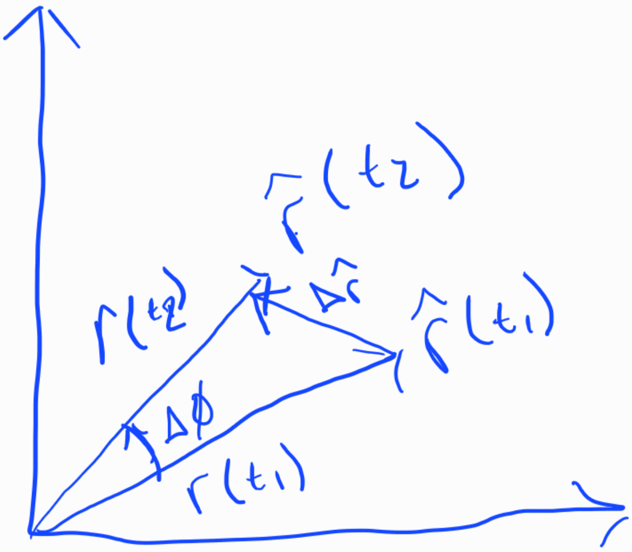


$$x = r\cos\phi \qquad r = \sqrt{x^2 + y^2}$$
$$\longleftrightarrow$$
$$y = r\sin\phi \qquad \phi = \arctan(y/x)$$

$\hat{r}$ is the unit vector along r

$\hat{\phi}$ is the direction of increasing ϕ

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} \quad \text{for any vector } \hat{a} = \frac{\vec{a}}{|\vec{a}|}$$



$$\vec{F} = F_r\hat{r} + F_\phi\hat{\phi}$$

$$\vec{r} = r\hat{r}$$

$$\vec{F} = m\ddot{r}$$

$$\ddot{r}_x = \ddot{x}\ , \ \ddot{r}_y = \ddot{y} \quad \text{but } \Delta\hat{r} \neq 0$$

$$\Delta\hat{r} \simeq \Delta\phi\,\hat{\phi} \simeq \dot{\phi}\,\Delta t\,\hat{\phi}$$

$$\frac{\Delta\hat{r}}{\Delta t} = \dot{\phi}\hat{\phi}$$

$$\dot{\underline{r}} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt} \equiv \vec{V}$$

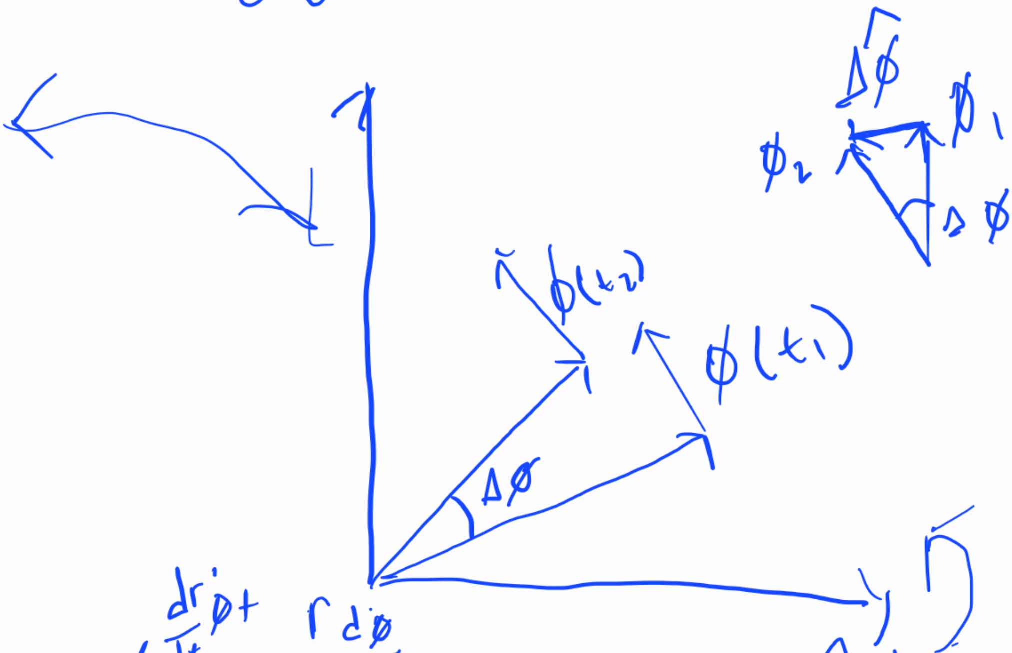
$$\dot{\underline{r}} = \dot{r}\hat{r} + r\dot{\phi}\,\hat{\phi}$$

$V_r = \dot{r}$ & $V_\phi = r\dot{\phi} = r\omega$

ω = angular velocity

$$\vec{a} \equiv \ddot{r} = \frac{d}{dt}\dot{r} = \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi}\right)$$

$$\frac{d\hat{\phi}}{dt} = -\dot{\phi}\hat{r}$$



$$\vec{a} = \Big(\underbrace{\ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt}}_{\frac{d\dot{r}}{dt}\hat{r} + \dot{r}\frac{d\hat{r}}{dt}}\Big) + \Big(\underbrace{(\dot{r}\dot{\phi} + r\ddot{\phi})}_{\frac{d\dot{r}}{dt}\dot{\phi} + r\frac{d\dot{\phi}}{dt}}\hat{\phi} + r\dot{\phi}\frac{d\hat{\phi}}{dt}\Big)$$

$$= (\ddot{r} - r\dot{\phi}^2)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi}$$

If r = Const.

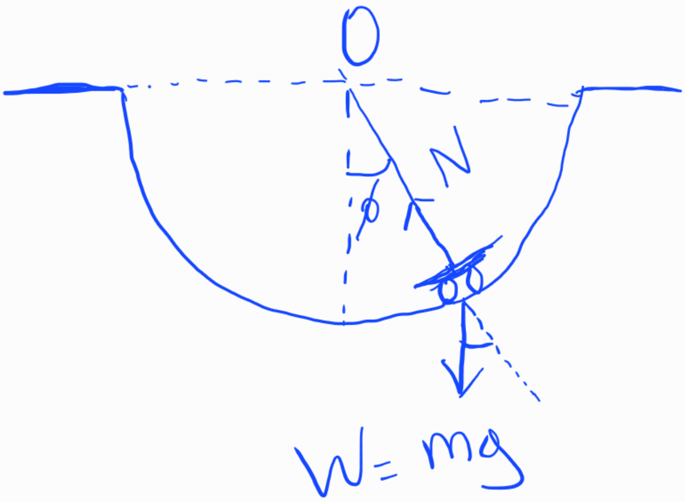
α = angular acceleration

$$\vec{a} = -r\dot{\phi}\hat{r} + r\ddot{\phi}\hat{\phi}$$

$$= -r\omega^2\hat{r} + r\alpha\hat{\phi}$$

$$\vec{F} = m\vec{a} \iff \begin{cases} F_r = m(\ddot{r} - r\dot{\phi}^2) \\ F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) \end{cases}$$

Ex 1.2 Oscillating Skateboard



$r = R = 5\,m$

$\dot{r} = 0$

$\ddot{r} = 0$

$$F_r = -mr\dot{\phi}^2 \quad \& \quad F_\phi = mr\ddot{\phi}$$

$$F_r = mg\cos\phi - N \quad \& \quad F_\phi = -mg\sin\phi$$

$$-\cancel{m}g\sin\phi = \cancel{m}R\ddot{\phi}$$

$$\ddot{\phi} = -\frac{g}{R}\sin\phi, \text{ for } \phi \ll 2\pi$$ (small angle approximation)

$$\sin\phi \approx \phi$$

$$\ddot{\phi} = -\frac{g}{R}\phi, \quad \omega = \sqrt{\frac{g}{R}}$$

$$\ddot{\phi} = -\omega^2\phi$$

$$\phi(t) = A\sin(\omega t) \ \& \ \phi(t) = B\cos(\omega t)$$

$$\phi(t) = A\sin(\omega t) + B\cos(\omega t)$$

general solution

Specific Solution:

$t=0$, $\phi(t=0) = A\cancel{\sin(0)}^{0} + B\cancel{\cos(0)}^{0}$

$B = \phi(t=0) = \phi_0$

from rest

$\dot{\phi}(t=0) = \omega A$, $\dot{\phi}(t=0) = 0$

$0 = \sqrt{\frac{g}{R}} A$, $\therefore A = 0$

$\phi(t) = \phi_0 \cos(\omega t)$, @ $\omega t = 2\pi$ repeat

$\tau = t = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g}}$

$$\boxed{\tau = 4.5\,s}$$