Similarly, that the collision is elastic implies that $\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2$, whence

$$v_1^2 = v_1'^2 + v_2'^2. \tag{iv}$$

Squaring Equation (iii), we find that

$$v_1^2 = v_1'^2 + v_2'^2 + 2\mathbf{v}_1'\cdot\mathbf{v}_2',$$

and comparing this with Equation (iv), we conclude that $\mathbf{v}_1' \cdot \mathbf{v}_2' = 0$. That is $\mathbf{v}_1'$ is perpendicular to $\mathbf{v}_2'$ (except when one of them is zero, in which case the angle between them is undefined).

3.6 $\star$ Thrust $= -\dot{m}v_{\text{ex}} = (15{,}000 \text{ kg/s}) \times (2500 \text{ m/s}) = 3.75 \times 10^7$ N $\approx$ 4200 tons — more than, though not vastly more than, the vehicle's weight.

3.7 $\star$ Since $v_o = 0$, the final velocity as given by (3.8) is

$$v = v_{\text{ex}} \ln(m_o/m) = 3000 \ln(2) \approx 2100 \text{ m/s}.$$

The thrust is given by (3.7) as

$$\text{thrust} = -\dot{m}v_{\text{ex}} = \frac{m_o - m}{t} v_{\text{ex}} \approx 2.5 \times 10^7 \text{ N}.$$

This is just a little bigger than the initial weight, $m_o g \approx 2.0 \times 10^7$ N.

3.8 $\star$ **(a)** The condition for the rocket to hover is that $-\dot{m}v_{\text{ex}} = mg$. This requires that $-dm/m = g\,dt/v_{\text{ex}}$, which integrates to give $-\ln(m/m_o) = gt/v_{\text{ex}}$. The maximum time occurs when $m = (1-\lambda)m_o$, so $t_{\text{max}} = -\ln(1-\lambda)v_{\text{ex}}/g$.

(b) If $\lambda = 0.1$ and $v_{\text{ex}} = 3000$ m/s, this gives $t_{\text{max}} = 32$ seconds.

3.9 $\star$ From the data of Problem 3.7, $m_o = 2 \times 10^6$ kg, and $\dot{m} \approx -8.33 \times 10^3$ kg/s. The minimum exhaust speed is determined by the condition: thrust $= -\dot{m}v_{\text{ex}} = m_o g$, which gives $v_{\text{ex}} = -m_o g/\dot{m} \approx 2350$ m/s, compared with the actual value of about 3000 m/s.

3.10 $\star$ According to Eq.(3.8), $v = v_{\text{ex}} \ln(m_o/m)$. Therefore, $p = mv = mv_{\text{ex}} \ln(m_o/m)$ and

$$\dot{p} = v_{\text{ex}}[\dot{m}\ln(m_o/m) - m(\dot{m}/m)] = -\dot{m}v_{\text{ex}}[1 - \ln(m_o/m)].$$

Bearing in mind that $\dot{m}$ is negative, we see that $\dot{p}$ is initially positive but drops to 0 when $\ln(m_o/m) = 1$ and is negative thereafter. Therefore, p is maximum when $\ln(m_o/m) = 1$ or $m = m_o/e$.

3.11 ⋆⋆ **(a)** From (3.4) we know that the change in momentum (of rocket plus ejected fuel) in time dt is $dP = m\,dv + dm\,v_{\text{ex}}$. On the other hand we know on general grounds that $\dot{P} = F^{\text{ext}}$, so that $dP = F^{\text{ext}}dt$. Equating these two expressions for dP, we conclude that

$$m\,dv + dm\,v_{\text{ex}} = F^{\text{ext}}dt$$

or, dividing by dt and rearranging,

$$m\dot{v} = -\dot{m}v_{\text{ex}} + F^{\text{ext}}.$$

(b) With $-\dot{m} = k$ (a positive constant) and $F^{\text{ext}} = -mg$, the equation of motion becomes

$$m\dot{v} = kv_{\text{ex}} - mg$$

and since $\dot{m} = -k$ it follows that $m = m_{\text{o}} - kt$. The differential equation separates to give

$$dv = \left(\frac{kv_{\text{ex}}}{m_{\text{o}} - kt} - g\right) dt$$

which can be integrated from time 0 to t (and velocity 0 to v) to give

$$v \;=\; v_{\text{ex}} \ln\left(\frac{m_{\text{o}}}{m_{\text{o}} - kt}\right) - gt \;=\; v_{\text{ex}} \ln\left(\frac{m_{\text{o}}}{m}\right) - gt.$$

(c) Putting in the numbers ($v_{\text{ex}} = 3000$ m/s, $m_{\text{o}}/m = 2$, $g = 9.8$ m/s^2, and $t = 120$ s), we find $v = 900$ m/s. With $g = 0$, the corresponding number is 2100 m/s, so gravity reduces the speed acquired in the first two minutes to a little less than half its weight-free value.

(d) If the thrust kv_{ex} is less than the weight mg, the rocket will just sit on the ground until it has shed enough mass that the thrust *can* overcome the weight. Not a good design!

3.12 ⋆⋆ **(a)** If it uses all the fuel in a single burn, then according to (3.8) the final speed is

$$v = v_{\text{ex}} \ln\left(\frac{m_{\text{o}}}{0.4m_{\text{o}}}\right) = v_{\text{ex}} \ln(2.5) = 0.92v_{\text{ex}}.$$

(b) After the first stage the speed is

$$v_1 = v_{\text{ex}} \ln\left(\frac{m_{\text{o}}}{0.7m_{\text{o}}}\right) = v_{\text{ex}} \ln\left(\frac{1}{0.7}\right)$$

and after the second stage it is

$$v_2 = v_{\text{ex}} \ln\left(\frac{0.6m_{\text{o}}}{0.3m_{\text{o}}}\right) + v_1 = v_{\text{ex}} \ln\left(\frac{0.6}{0.3} \times \frac{1}{0.7}\right) = v_{\text{ex}} \ln(2.86) = 1.05v_{\text{ex}}.$$

3.13 ⋆⋆ We can find the height $y(t)$ by integrating $v(t)$ as found in Problem 3.11:

$$y(t) = \int_0^t v(t')dt' = v_{\text{ex}} \int_0^t [\ln m_{\text{o}} - \ln m(t')]dt' - \int_0^t gt'dt'$$
$$= v_{\text{ex}} t \ln m_{\text{o}} - v_{\text{ex}} \int_0^t \ln m(t')dt' - \tfrac{1}{2}gt^2. \qquad \text{(v)}$$

To do the remaining integral, notice that $m(t') = m_{\text{o}} - kt'$, so that $dt' = -dm'/k$, where m' is short for $m(t')$. Thus the remaining integral in (v) is

$$\int_0^t \ln m' dt' = -\frac{1}{k}\int_{m_{\text{o}}}^{m} \ln m' \, dm' = -\frac{1}{k}\Big[m' \ln m' - m'\Big]_{m_{\text{o}}}^{m} = \frac{1}{k}(m_{\text{o}} \ln m_{\text{o}} - m \ln m) - t,$$

where in the last expression I used the fact that $m_{\text{o}} - m = kt$. Substituting into (v), we get

$$y(t) = v_{\text{ex}}t - \tfrac{1}{2}gt^2 + \frac{v_{\text{ex}}}{k}(kt \ln m_{\text{o}} - m_{\text{o}} \ln m_{\text{o}} + m \ln m) = v_{\text{ex}}t - \tfrac{1}{2}gt^2 - \frac{mv_{\text{ex}}}{k} \ln\left(\frac{m_{\text{o}}}{m}\right)$$

where I have again used that $kt = m_{\text{o}} - m$. Putting in the numbers from Problem 3.11 and $t = 2$ min, we find that $y \approx 40$ km.

3.14 ⋆⋆ The equation of motion (3.29) reads $m\dot{v} = kv_{\text{ex}} - bv$, or $m\, dv/(kv_{\text{ex}} - bv) = dt$. Since $dm/dt = -k$ we can replace dt by $-dm/k$, and the equation of motion becomes $k\, dv/(kv_{\text{ex}} - bv) = -dm/m$. This integrates to give

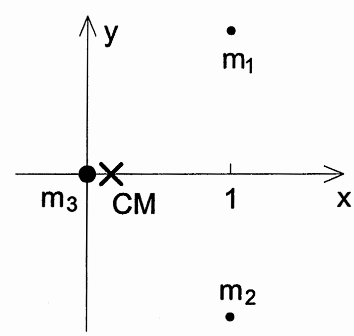
$$\frac{k}{b}\ln\left(\frac{kv_{\text{ex}} - bv}{kv_{\text{ex}}}\right) = \ln\left(\frac{m}{m_{\text{o}}}\right) \quad \text{or} \quad v = \frac{kv_{\text{ex}}}{b}\left[1 - \left(\frac{m}{m_{\text{o}}}\right)^{b/k}\right].$$

3.15 ⋆

$$X = \frac{\sum m_\alpha x_\alpha}{M} = \frac{1+1+0}{12} = \frac{1}{6}$$

$$Y = \frac{\sum m_\alpha y_\alpha}{M} = \frac{1-1+0}{12} = 0$$

$$Z = \frac{\sum m_\alpha z_\alpha}{M} = \frac{0+0+0}{12} = 0$$



Because m_3 is much bigger than m_1 and m_2, the CM is much closer to m_3 than to the other two.

3.16 ⋆ If we put the origin at the sun's center and the x axis through the earth's center, then the CM lies on the x axis at

$$X = \frac{m_{\text{s}}x_{\text{s}} + m_{\text{e}}x_{\text{e}}}{m_{\text{s}} + m_{\text{e}}} = \frac{0 + m_{\text{e}}x_{\text{e}}}{m_{\text{s}} + m_{\text{e}}} = \frac{6.0 \times 10^{24}}{2.0 \times 10^{30}} \times (1.5 \times 10^8 \text{ km}) = 450 \text{ km},$$

which is very, very close to the center of the sun.

3.17 ⋆ If we put the origin at the earth's center and the x axis through the moon's center, then the CM lies on the x axis at

$$X = \frac{m_e x_e + m_m x_m}{m_e + m_m} = \frac{0 + m_m x_m}{m_e + m_m} = \frac{7.4 \times 10^{22}}{(600 + 7.4) \times 10^{22}} \times (3.8 \times 10^5 \text{ km}) = 4600 \text{ km}.$$

Since the earth's radius is 6400 km, the earth-moon CM is comfortably inside the earth

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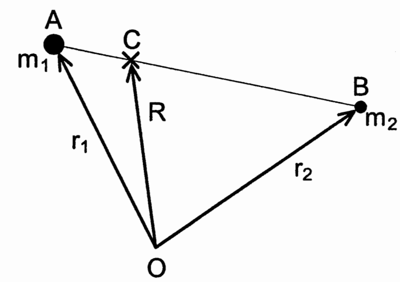
3.18 ⋆⋆ **(a)** The vector pointing from m_1 the CM is



$$\overrightarrow{AC} = \mathbf{R} - \mathbf{r}_1 = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} - \mathbf{r}_1$$
$$= \frac{m_2}{m_1 + m_2}(\mathbf{r}_2 - \mathbf{r}_1) = \frac{m_2}{M}\overrightarrow{AB}$$

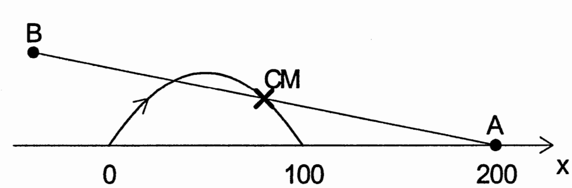
Since the vectors $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are in the same direction, the three points A, B, and C are collinear. Since $AC < AB$, the CM lies between A and B on the line joining them.

(b) By the same argument, $\overrightarrow{BC} = (m_1/M)\overrightarrow{BA}$. Thus the ratio of the two lengths is $AC/BC = m_2/m_1$, as claimed. If $m_1 \gg m_2$, then $AC \ll BC$ and C is very close to A.

3.19 ⋆⋆ **(a)** As long as all pieces are still in the air (so that the only force on them is gravity), (3.11) implies that the CM follows the path of a single particle of mass M, namely, the same parabola.

(b) At the moment when both pieces land ($y = 0$), the CM is also at $y = 0$ and is, therefore, at the target ($x = 100$). Since the CM is half way between the two pieces and the first piece is 100 m beyond the target, the second piece has to be 100 m short of the target, namely back at the launch site ($x = 0$).

(c) Let's call the two halves A and B, and let's imagine the ground removed, so that the pieces can continue to move like projectiles even after they pass the level $y = 0$. At all times, the CM is at the midpoint of the line AB. Suppose that A lands first, at $x_A = 200$. At the same time, B and hence the CM are still above ground, which means that the CM has $X < 100$ (since we know the CM *will* land at $X = 100$). This implies that at the moment when A lands, $x_B < 0$. In other words, B has already passed back over the launch point and will land still further behind it. A similar argument shows that if A lands after B, then B will land at a point $x_B > 0$.



3.20 ⋆⋆ Notice first that the definition (3.9) of the CM can be written as $M\mathbf{R} = \sum m_\alpha \mathbf{r}_\alpha$. Now suppose that we have two bodies made up as follows:

(body 1) = N_1 particles with masses $m_{1\alpha}$ at positions $\mathbf{r}_{1\alpha}$, with $\alpha = 1, \cdots, N_1$,

and

(body 2) = N_2 particles with masses $m_{2\beta}$ at positions $\mathbf{r}_{2\beta}$, with $\beta = 1, \cdots, N_2$.

The total mass of the two-body system is $M = M_1 + M_2 = \sum_\alpha m_{1\alpha} + \sum_\beta m_{2\beta}$ and the CM postion $\mathbf{R}$ of the whole system satisfies

$$M\mathbf{R} = \sum_\alpha m_{1\alpha}\mathbf{r}_{1\alpha} + \sum_\beta m_{2\beta}\mathbf{r}_{1\beta} = M_1\mathbf{R}_1 + M_2\mathbf{R}_2\,,$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are the CM positions of the two separate bodies and the second equality follows from our rewritten definition of the CM applied to each body separately. This is the required result.

3.21 ⋆⋆ Let the disk's mass be M and its mass density (mass/area) be $\sigma = M/A$, where $A = \pi R^2/2$ is its area. The CM position is $\mathbf{R} = \int \sigma\mathbf{r}\,dA/M = \int \mathbf{r}\,dA/A$ where the integral runs over the area of the disk in the plane $z = 0$. Clearly $Z = 0$, and, by symmetry, $X = 0$. Finally

$$Y = \frac{1}{A}\int y\,dA = \frac{2}{\pi R^2}\int_0^R r\,dr \int_0^\pi d\phi(r\sin\phi) = \frac{4}{3\pi}R.$$

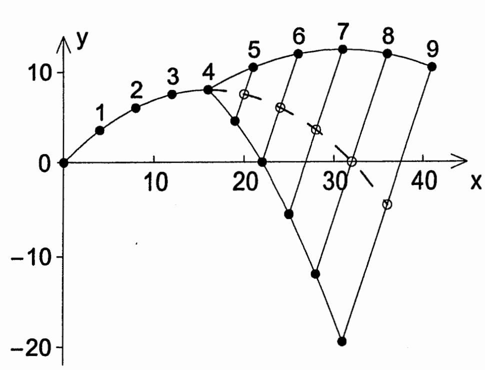
3.22 ⋆⋆ Let the hemisphere's mass be M and its density be $\varrho = M/V$, where $V = 2\pi R^3/3$ is its volume. The CM position is $\mathbf{R} = \int \varrho\mathbf{r}\,dV/M = \int \mathbf{r}\,dV/V$ where the integral runs over the volume of the hemisphere. By symmetry $X = Y = 0$, while

$$Z = \frac{1}{V}\int z\,dV = \frac{3}{2\pi R^3}\int_0^R r^2 dr \int_0^{\pi/2} \sin\theta\,d\theta \int_0^{2\pi} d\phi\, r\cos\theta = \frac{3}{2\pi R^3}\cdot\frac{R^4}{4}\cdot\frac{1}{2}\cdot 2\pi = \frac{3}{8}R$$

3.23 ⋆⋆⋆ **(a)** The orbit is $\mathbf{r}(t) = \mathbf{v}_o t + \frac{1}{2}\mathbf{g}t^2$.

(b) Because the two pieces have equal masses, $m_1 = m_2 = M/2$, conservation of momentum implies that $\mathbf{v}_1 + \mathbf{v}_2 = 2\mathbf{v}$, so that $\mathbf{v}_2 = 2\mathbf{v} - \mathbf{v}_1 = \mathbf{v} - \Delta\mathbf{v}$.

(c) The CM (hollow circles) is at the midpoint of the line joining the two pieces and clearly continues along the original parabola.



3.24 ⋆ If we orient the triangle so that $\mathbf{a}$ is the base, then the height is $h = b\sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Therefore

$$\tfrac{1}{2}|\mathbf{a}\times\mathbf{b}| = \tfrac{1}{2}ab\sin\theta = \tfrac{1}{2}ah = \text{(area of triangle)}.$$

3.25 $\star$ The only force on the particle is the tension of the string, which is necessarily directed toward the hole in the table at O. Therefore the angular momentum $\boldsymbol{\ell}$ about O is constant. When the particle is travelling in a circle of radius r, the vertical component of $\boldsymbol{\ell} = \mathbf{r}\times\mathbf{p}$ is $\ell_z = rp = rmv = rm(r\omega) = mr^2\omega$. Therefore, the quantity $r^2\omega$ is constant and $r^2\omega = r_o^2\omega_o$; whence $\omega = (r_o/r)^2\omega_o$.

3.26 $\star$ **(a)** Since the force is central it has the form $\mathbf{F} = f(\mathbf{r})\hat{\mathbf{r}}$ and the torque on the particle is $\boldsymbol{\Gamma} = \mathbf{r}\times\mathbf{F} = 0$. Therefore the angular momentum $\boldsymbol{\ell}$ is constant.

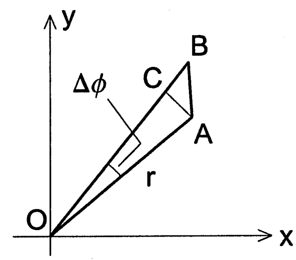
(b) If $\boldsymbol{\ell}_o$ is the particle's initial angular momentum, then conservation of angular momentum implies that $\mathbf{r}\times\mathbf{p} = \boldsymbol{\ell}_o$ at any time and hence that $\mathbf{r}$ is always perpendicular to $\boldsymbol{\ell}_o$. In other words, the position vector $\mathbf{r}$ always lies in the plane through O perpendicular to $\boldsymbol{\ell}_o$.

3.27 $\star\star$ **(a)** Since $\mathbf{r} = r\hat{\mathbf{r}}$ and $\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$, it follows that

$$\boldsymbol{\ell} = \mathbf{r}\times m\dot{\mathbf{r}} = mr^2\dot{\phi}\,\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = mr^2\omega\hat{\mathbf{z}}$$

Therefore, $\ell = mr^2\omega$ as claimed. (Strictly speaking we should put absolute value signs in here if we are to insist that ℓ be positive.)

(b) Suppose that in a time Δt the planet moves from A to B and swings through an angle $\Delta\phi$. The area swept out is the area of the triangle OAB. In the limit of small Δt, this triangle is well approximated by the triangle OAC, with $OC = OA$. This has height r and base $r\Delta\phi$. Therefore the area swept out is $\Delta A \approx \frac{1}{2}r^2\Delta\phi$. Dividing both sides by Δt and letting $\Delta t \to 0$, we conclude that $\dot{A} = \frac{1}{2}r^2\dot{\phi} = \frac{1}{2}r^2\omega$,



as claimed. Comparing this with the result of part (a), we see that $\dot{A} = \ell/2m$ and hence that the conservation of ℓ implies that $\dot{A}$ is constant.

3.28 $\star$ With three particles, Eq.(3.20) reads $\mathbf{L} = (\mathbf{r}_1\times\mathbf{p}_1) + (\mathbf{r}_2\times\mathbf{p}_2) + (\mathbf{r}_3\times\mathbf{p}_3)$, so

$$\dot{\mathbf{L}} = (\dot{\mathbf{r}}_1\times\mathbf{p}_1 + \mathbf{r}_1\times\dot{\mathbf{p}}_1) + (\dot{\mathbf{r}}_2\times\mathbf{p}_2 + \mathbf{r}_2\times\dot{\mathbf{p}}_2) + (\dot{\mathbf{r}}_3\times\mathbf{p}_3 + \mathbf{r}_3\times\dot{\mathbf{p}}_3).$$

The first term in each parenthesis is zero (because each $\dot{\mathbf{r}}$ is parallel to its corresponding $\mathbf{p}$). We can replace $\dot{\mathbf{p}}_1$ by the corresponding net force $\dot{\mathbf{p}}_1 = \mathbf{F}_1 = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_1^{\text{ext}}$ and so on to give

$$\begin{aligned}\dot{\mathbf{L}} &= (\mathbf{r}_1\times\mathbf{F}_1) + (\mathbf{r}_2\times\mathbf{F}_2) + (\mathbf{r}_3\times\mathbf{F}_3)\\ &= \big(\mathbf{r}_1\times[\mathbf{F}_{12}+\mathbf{F}_{13}+\mathbf{F}_1^{\text{ext}}]\big) + \big(\mathbf{r}_2\times[\mathbf{F}_{23}+\mathbf{F}_{21}+\mathbf{F}_2^{\text{ext}}]\big) + \big(\mathbf{r}_3\times[\mathbf{F}_{31}+\mathbf{F}_{32}+\mathbf{F}_3^{\text{ext}}]\big)\end{aligned}$$

Remembering that $\mathbf{F}_{12} = -\mathbf{F}_{21}$ and so on, we can regroup to give

$$\dot{\mathbf{L}} = (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}_{12} + (\mathbf{r}_2 - \mathbf{r}_3) \times \mathbf{F}_{23} + (\mathbf{r}_3 - \mathbf{r}_1) \times \mathbf{F}_{31} + \mathbf{r}_1 \times \mathbf{F}_1^{\text{ext}} + \mathbf{r}_2 \times \mathbf{F}_2^{\text{ext}} + \mathbf{r}_3 \times \mathbf{F}_3^{\text{ext}}.$$

Provided all internal forces are central, each of the first three products here is zero (as discussed in connection with Fig.3.8) and we are left with

$$\dot{\mathbf{L}} = \mathbf{r}_1 \times \mathbf{F}_1^{\text{ext}} + \mathbf{r}_2 \times \mathbf{F}_2^{\text{ext}} + \mathbf{r}_3 \times \mathbf{F}_3^{\text{ext}} = \mathbf{\Gamma}_1^{\text{ext}} + \mathbf{\Gamma}_2^{\text{ext}} + \mathbf{\Gamma}_3^{\text{ext}} = \mathbf{\Gamma}^{\text{ext}}.$$

3.29 ⋆ We are told that the matter accreted by the asteroid is initially at rest. Therefore its initial angular momentum is zero and, by conservation of angular momentum, $I_o\omega_o = I\omega$, where I_o and I are the initial and final moments of inertia of the asteroid. Now,

$$I = \frac{2}{5}MR^2 = \frac{2}{5}\left(\frac{4}{3}\pi\varrho R^3\right)R^2 = \frac{8}{15}\pi\varrho R^5,$$

so, given that ϱ is constant, conservation of angular momentum implies that $R_o^5\omega_o = R^5\omega$, and $\omega = \omega_o(R_o/R)^5$. If $R = 2R_o$, then $\omega = \omega_o/32$.

3.30 ⋆⋆ **(a)** If a particle is a distance ρ from the axis of rotation and the body turns through an angle $d\phi$, then the particle moves a distance $\rho d\phi$ in the tangential (ϕ) direction. Dividing by dt we conclude that the particle's speed is $v = \rho d\phi/dt = \rho\omega$ in the ϕ direction. That is, $\mathbf{v} = \rho\omega\hat{\boldsymbol{\phi}}$.

(b) The particle's position is $\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$, so its angular momentum is $\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p} = (\rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}) \times m\rho\omega\hat{\boldsymbol{\phi}} = m\rho^2\omega\hat{\mathbf{z}} - mz\rho\omega\hat{\boldsymbol{\phi}}$. Therefore its z component is $\ell_z = m\rho^2\omega$.

(c) The total angular momentum has

$$L_z = \sum_{\alpha=1}^{N} \ell_{\alpha z} = \sum_{\alpha=1}^{N} m_\alpha\rho_\alpha^2\omega = I\omega \quad \text{where} \quad I = \sum_{\alpha=1}^{N} m_\alpha\rho_\alpha^2.$$

3.31 ⋆⋆ If we place the disk in the plane $z = 0$ centered on the origin, the sum (3.31) can be written as $I = \sum m_\alpha r_\alpha^2$ (because in the plane $z = 0$ the coordinate ρ is the same as what we usually call r). If the density (mass/area) of the disk is $\sigma = M/\pi R^2$, then, replacing the sum by the appropriate integral, we find that

$$I = \int \sigma r^2 dA = \frac{M}{\pi R^2}\int_0^R r dr \int_0^{2\pi} d\phi\, r^2 = \frac{M}{\pi R^2}\cdot\frac{R^4}{4}\cdot 2\pi = \tfrac{1}{2}MR^2$$

3.32 ⋆⋆ The sum (3.31) becomes the integral $I = \int \varrho\, dV \rho^2$, where $\varrho = M/V = 3M/(4\pi R^3)$ is the density and $\rho = r\sin\theta$ is the distance of a point from the z axis. Therefore

$$I = \frac{3M}{4\pi R^3}\int_0^R r^4 dr \int_0^{\pi} \sin^3\theta\, d\theta \int_0^{2\pi} d\phi = \frac{3M}{4\pi R^3}\cdot\frac{R^5}{5}\cdot\frac{4}{3}\cdot 2\pi = \frac{2}{5}MR^2.$$

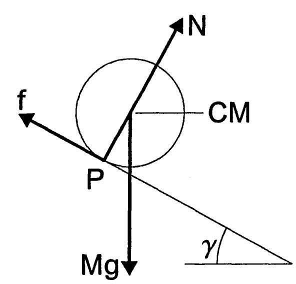
3.33 ⋆⋆ If we place the square in the plane $z = 0$, centered on the origin with its sides parallel to the x and y axes, the sum (3.31) takes the form $I = \sum m_\alpha \rho_\alpha^2 = \sum m_\alpha (x_\alpha^2 + y_\alpha^2) = 2\sum m_\alpha x_\alpha^2$, where the last expression holds because, by symmetry, the two terms of the previous expression are equal. If we denote the density (mass/area) of the square by $\sigma = M/A = M/(4b^2)$, then we can replace the sum by an integral:

$$I = 2\sigma \int x^2 dA = 2\sigma \int_{-b}^{b} x^2 dx \int_{-b}^{b} dy = 2\frac{M}{4b^2} \cdot \frac{2b^3}{3} \cdot 2b = \frac{2}{3}Mb^2$$

3.34 ⋆⋆ The CM moves just like a point mass M, so its height is $Y = v_o t - \frac{1}{2}gt^2$, and the time to return to $Y = 0$ is $t = 2v_o/g$. Since there is no torque about the CM, the angular momentum $L = I\omega$ is constant. Therefore $\omega = \omega_o$ is also constant and the number of complete revolutions in the time t is $n = \omega_o t/2\pi = \omega_o v_o/\pi g$. Therefore, he must arrange that $v_o = n\pi g/\omega_o$ where n is an integer.

3.35 ⋆⋆

(a)



(b) The condition $\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$ applied about P becomes $I_P\dot{\omega} = MgR\sin\gamma$, whence $\dot{v} = \frac{2}{3}g\sin\gamma$.

(c) The same condition applied about the CM gives $I_{\text{cm}}\dot{\omega} = fR$. To eliminate the unknown frictional force f, we must use Newton's second law, $M\dot{v} = Mg\sin\gamma - f$. Eliminating f, we get the same answer as before.

3.36 ⋆⋆ The motion of the CM is the same as in Example 3.4; that is, the momentum of the CM just after the impulse is $\mathbf{P} = \mathbf{F}\Delta t$, so, since $\mathbf{P} = M\dot{\mathbf{R}}$, the initial CM velocity is $\dot{\mathbf{R}} = \mathbf{F}\Delta t/2m$ and, since $\mathbf{P}$ is constant, this velocity remains unchanged. That is, the CM moves with constant speed $F\Delta t/2m$ in the direction of the applied force. The applied torque is $\Gamma^{\text{ext}} = Fb\sin\alpha$, thus the initial angular momentum is $L = Fb\Delta t\sin\alpha$, which also remains constant. Thus the dumbell rotates with constant angular velocity $\omega = L/I = (F\Delta t\sin\alpha)/(2mb)$ about the CM. If $\alpha = 90°$, these results are the same as in the example. If $\alpha = 0$, the CM moves with the same speed but along the x axis, and there is no rotation ($\omega = 0$), just as you would expect.

4.8 ⋆⋆ We'll measure the puck's position by the angle θ it subtends at the sphere's center O (measured down from the top). The puck's PE (defined as zero at the level of O) is $U(\theta) = mgR\cos\theta$, and its total energy is $E = U(0) = mgR$. By conservation of energy,

$$T = \tfrac{1}{2}mv^2 = E - U = mgR(1 - \cos\theta). \tag{i}$$

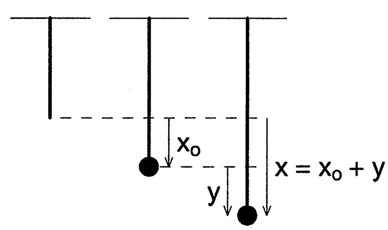
As long as the puck remains in contact with the sphere, the radial component of Newton's second law reads $N - mg\cos\theta = -mv^2/R$, where N denotes the normal force of the sphere on the puck. Substituting from Eq (i) for mv^2 we find

$$N = mg(3\cos\theta - 2).$$

As long as N is positive the puck remains on the sphere. Since the sphere cannot exert a negative normal force, once the predicted value of N becomes negative, the puck must have left the sphere. Therefore it leaves the sphere when $N = 0$ or $\theta = \arccos(2/3) = 48.2°$.

4.9 ⋆⋆ **(a)** $U = -\int_0^x F(x')dx' = k\int_0^x x'dx' = \tfrac{1}{2}kx^2$

(b) The two forces on m are gravity, mg down, and the spring, kx up, and at the new equilibrium x_o they must balance, so $x_o = mg/k$. The total PE is



$$\begin{aligned} U &= \tfrac{1}{2}kx^2 - mgx \\ &= \tfrac{1}{2}k(x_o + y)^2 - mg(x_o + y) \\ &= \tfrac{1}{2}ky^2 + y(kx_o - mg) + \text{const.} \end{aligned}$$

In the last line the second term is zero (the condition for equilibrium) and we can drop the constant, to give $U = \tfrac{1}{2}ky^2$.

4.10 ⋆

function, f	$\partial f/\partial x$	$\partial f/\partial y$	$\partial f/\partial z$
$ax^2 + bxy + cy^2$	$2ax + by$	$bx + 2cy$	0
$\sin(axyz^2)$	$ayz^2\cos(axyz^2)$	$axz^2\cos(axyz^2)$	$2axyz\cos(axyz^2)$
ae^{xy/z^2}	$(ay/z^2)e^{xy/z^2}$	$(ax/z^2)e^{xy/z^2}$	$(-2axy/z^3)e^{xy/z^2}$

4.11 ⋆

function, f	$\partial f/\partial x$	$\partial f/\partial y$	$\partial f/\partial z$
$ay^2 + 2byz + cz^2$	0	$2ay + 2bz$	$2by + 2cz$
$\cos(axy^2z^3)$	$-ay^2z^3\sin(axy^2z^3)$	$-2axyz^3\sin(axy^2z^3)$	$-3axy^2z^2\sin(axy^2z^3)$
$ar = a\sqrt{x^2 + y^2 + z^2}$	ax/r	ay/r	az/r

4.12 ⋆ **(a)** $\nabla f = \hat{\mathbf{x}}\,2x + \hat{\mathbf{z}}\,3z^2$. **(b)** $\nabla f = k\,\hat{\mathbf{y}}$. **(c)** $\nabla f = \hat{\mathbf{r}}$.

4.13 ⋆

function, f	$\partial f/\partial x$	$\partial f/\partial y$	$\partial f/\partial z$	∇f
$\ln(r)$	x/r^2	y/r^2	z/r^2	$\hat{\mathbf{r}}/r$
r^n	nxr^{n-2}	nyr^{n-2}	nzr^{n-2}	$n\,r^{n-1}\,\hat{\mathbf{r}}$
$g(r)$	$g'(r)x/r$	$g'(r)y/r$	$g'(r)z/r$	$g'(r)\,\hat{\mathbf{r}}$

4.14 ⋆ Consider the x component

$$[\nabla(fg)]_x = \frac{\partial}{\partial x}(fg) = f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x} = f(\nabla g)_x + g(\nabla f)_x\,.$$

Since the other two components work the same way, we conclude that $\nabla(fg) = f\nabla g + g\nabla f$.

4.15 ⋆ If $f(\mathbf{r}) = x^2 + 2y^2 + 3z^2$, the gradient is $\nabla f = (2x, 4y, 6z)$, so Equation (4.35) gives the approximation

$$\begin{aligned} df \approx \nabla f\cdot d\mathbf{r} &= (2x, 4y, 6z)\cdot(dx, dy, dz) \\ &= (2, 4, 6)\cdot(0.01, 0.03, 0.05) = 0.02 + 0.12 + 0.30 = 0.4400. \end{aligned}$$

This compares favorably with the exact result

$$df = f(1.01, 1.03, 1.05) - f(1, 1, 1) = 6.4494 - 6.0000 = 0.4494.$$

4.16 ⋆ If $U = k(x^2 + y^2 + z^2)$, then $\mathbf{F} = -\nabla U = -2k(x, y, z) = -2k\mathbf{r}$.

4.17 ⋆ **(a)** The force $\mathbf{F} = q\mathbf{E}_o$ certainly doesn't depend on anything but $\mathbf{r}$ (and doesn't even depend on $\mathbf{r}$ since $\mathbf{E}_o$ is constant). The work integral is $W(1 \to 2) = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = q\mathbf{E}_o\cdot\int_1^2 d\mathbf{r} = q\mathbf{E}_o\cdot(\mathbf{r}_2 - \mathbf{r}_1)$, which is independent of path. Therefore, the force is conservative, and the PE is $U(\mathbf{r}) = -W(0 \to \mathbf{r}) = -q\mathbf{E}_o\cdot\mathbf{r}$.

(b) Since $U = -q(E_{ox}x + E_{oy}y + E_{oz}z)$, $-\nabla U = q(E_{ox}, E_{oy}, E_{oz}z) = q\mathbf{E}_o = \mathbf{F}$.

4.18 ⋆⋆ **(a)** According to (4.35), the change in $f(\mathbf{r})$ resulting from any small displacement $d\mathbf{r}$ is $df = \nabla f\cdot d\mathbf{r}$. If, in particular, we consider any infinitesimal displacement $d\mathbf{r}$ in a surface of constant f, then df will be zero. This implies that $\nabla f\cdot d\mathbf{r} = \mathbf{0}$, that is, ∇f is perpendicular to the surface of constant f.

(b) Consider a displacement $d\mathbf{r} = \epsilon\mathbf{u}$ with fixed magnitude ϵ but variable direction $\mathbf{u}$. Our job is to find the direction of $\mathbf{u}$ for which the corresponding change df is largest. Since $df = \nabla f\cdot d\mathbf{r} = \epsilon\nabla f\cdot\mathbf{u} = \epsilon|\nabla f|\cos\theta$, where θ is the angle between ∇f and $\mathbf{u}$, we see that df is maximum if $\theta = 0$, or ∇f and $\mathbf{u}$ are parallel. That is, the direction of ∇f is the direction in which f increases most rapidly.

4.19 ⋆⋆ **(a)** The equation $x^2 + 4y^2 = K$ defines an ellipse in the xy plane, centered on $x = y = 0$. Since the value of $x^2 + 4y^2$ is unchanged by any variation of z, the surface is an elliptical cylinder, centered on the z axis, with semi-major and semi-minor axes equal to $\sqrt{K}$ and $\sqrt{K}/2$.

(b) We know that the vector ∇f is perpendicular to the surface f = const. In the present case $\nabla f = (2x, 8y, 0) = (2, 8, 0)$ at the point $(1, 1, 1)$. Therefore the unit normal is $\mathbf{n} = (1, 4, 0)/\sqrt{17}$ or $-\mathbf{n}$. The direction of maximum increase is $\mathbf{n}$.

4.20 ⋆

$\mathbf{F} =$	(kx, ky, kz)	(Ax, By^2, Cz^3)	(Ay^2, Bx, Cz)
$\nabla \times \mathbf{F} =$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, B - 2Ay)$

4.21 ⋆ This is exactly the same as Example 4.5 except that $\gamma = kqQ$ is replaced by $\gamma = -GMm$. Thus, if we choose $U = 0$ at $r = \infty$, then $U(r) = \gamma/r = -GMm/r$.

4.22 ⋆ Since $\mathbf{F} = \gamma\hat{\mathbf{r}}/r^2$, the spherical polar components of $\mathbf{F}$ are $F_\theta = F_\phi = 0$ and $F_r = \gamma/r^2$. Since F_r is independent of θ and ϕ, its two derivatives $\partial F_r/\partial\theta$ and $\partial F_r/\partial\phi$ are both zero. Inspection of the expression for $\nabla \times \mathbf{F}$ in spherical polars should convince you that each of the six terms is zero, so $\nabla \times \mathbf{F} = 0$. Since $\mathbf{F}$ certainly depends on $\mathbf{r}$ only, $\mathbf{F}$ is conservative.

4.23 ⋆⋆ All three given forces depend only on $\mathbf{r}$; that is, all satisfy the first condition. It remains to checks their curls.

(a) With $\mathbf{F} = k(x, 2y, 3z)$, we find $\nabla \times \mathbf{F} = (0, 0, 0)$, so $\mathbf{F}$ is conservative. The corresponding PE is $U = -\int \mathbf{F}\cdot d\mathbf{r}' = -k\int(x'dx' + 2y'dy' + 3z'dz') = -k(\frac{1}{2}x^2 + y^2 + \frac{3}{2}z^2)$. Clearly $-\nabla U = k(x, 2y, 3z) = \mathbf{F}$.

(b) With $\mathbf{F} = k(y, x, 0)$, we find $\nabla\times\mathbf{F} = (0, 0, 1-1) = (0, 0, 0)$, so $\mathbf{F}$ is conservative. The corresponding PE is $U = -\int_0^{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r}' = -k\int_0^{\mathbf{r}}(y'dx' + x'dy')$. This integral can be evaluated along any path. One simple choice is to go from the origin out to x on the x axis (this contributes 0 to the integral) and then parallel to the y axis to y (which contributes $-kxy$). Thus $U = -kxy$. Clearly $-\nabla U = k(y, x, 0) = \mathbf{F}$.

(c) With $\mathbf{F} = k(-y, x, 0)$, we find $\nabla\times\mathbf{F} = (0, 0, 1+1) = (0, 0, 2)$, so $\mathbf{F}$ is not conservative.

4.24 ⋆⋆⋆ **(a)** Consider first the force on m due to a short segment dz of the rod at a height z above m. This force has magnitude $dF = Gm\mu\, dz/r^2$ in the direction shown in the left picture, where r is the distance from the element dz to m. To find the total force we must integrate this from $z = -\infty$ to ∞. When we do this, the z components F_z from points z and $-z$ will cancel. Since the component into the page is clearly zero, we have only to worry