$$\oint_{\Gamma} \mathbf{F}\cdot d\mathbf{r} = \int \left(\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x}\right) dA = \int (\boldsymbol{\nabla} \times \mathbf{F})\cdot\hat{\mathbf{n}}\, dA.$$

4.26 ⋆ If $U = mgy$, then, whether or not g depends on t, $-\boldsymbol{\nabla}U = -mg\boldsymbol{\nabla}y = -mg(0,1,0) = \mathbf{F}$. But

$$\frac{dE}{dt} = \frac{d}{dt}\left(\tfrac{1}{2}m\mathbf{v}^2 + mgy\right) = m\dot{\mathbf{v}}\cdot\mathbf{v} + (mg\dot{y} + my\dot{g}) = -mgv_y + mgv_y + my\dot{g} = my\dot{g} \neq 0.$$

4.27 ⋆⋆ Stokes's theorem guarantees that the integral (4.48) is path-independent and hence that it defines a unique function $U(\mathbf{r}, t)$. To show that $-\boldsymbol{\nabla}U = \mathbf{F}$, consider the difference between the values of U at two neighboring points $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$:

$$dU = U(\mathbf{r} + d\mathbf{r}, t) - U(\mathbf{r}, t) \tag{iv}$$

$$= -\int_{\mathbf{r}}^{\mathbf{r}+d\mathbf{r}} \mathbf{F}(\mathbf{r}', t)\cdot d\mathbf{r}' = -\mathbf{F}(\mathbf{r}, t)\cdot d\mathbf{r}. \tag{v}$$

But $dU = (\boldsymbol{\nabla}U)\cdot d\mathbf{r}$, and since both of these expressions for dU are valid for all (small) $d\mathbf{r}$, it follows that $\mathbf{F}(\mathbf{r}, t) = -\boldsymbol{\nabla}U(\mathbf{r}, t)$. The argument leading to Eq.(4.19) requires that we look at the change in $E = T + U$ as we follow the particle moving from $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$ as the time advances from t to $t + dt$. The change in T is $dT = \mathbf{F}\cdot d\mathbf{r}$. The change in U is

$$dU = U(\mathbf{r} + d\mathbf{r}, t + dt) - U(\mathbf{r}, t) \tag{vi}$$

and here's the difficulty. This dU is not the same as the dU in Eq.(iv). Here we are concerned with the change dU as the particle moves from $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$ *and* the time changes from t to $t + dt$. In Eq.(iv) dU is the difference between the values at $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$ *at the same time* t. Thus the difference (vi) is not the same as $-\mathbf{F}(\mathbf{r}, t)\cdot d\mathbf{r}$ as in (v); it does not cancel the change in KE when we evaluate $dE = dT + dU$, and mechanical energy is not conserved.

4.28 ⋆⋆ **(a)** Since $E = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$, it follows that $\dot{x}(x) = \sqrt{2/m}\sqrt{E - \frac{1}{2}kx^2}$.

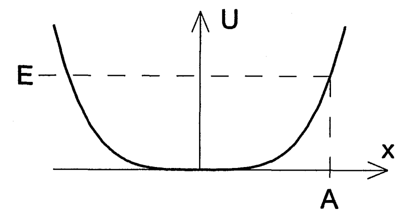
(b) At the end point $x = A$, we know that $T = 0$, so $E = \frac{1}{2}kA^2$. Substituting into the result of part (a), we find $\dot{x}(x) = \omega\sqrt{A^2 - x^2}$, where I have defined $\omega = \sqrt{k/m}$. From (4.58) we find

$$t = \int_0^x \frac{dx'}{\dot{x}(x')} = \frac{1}{\omega}\int_0^x \frac{dx'}{\sqrt{A^2 - x'^2}}.$$

The integral can be evaluated with the substitution $x' = A\sin\theta$ and gives $\arcsin(x/A)$. So $t = (1/\omega)\arcsin(x/A)$.

(c) Solving for x we find $x(t) = A\sin\omega t$. This shows that x is a sinusoidal function of t, which is the definition of simple harmonic motion. In particular, $x(t)$ repeats itself after a time t such that $\omega t = 2\pi$, or $t = 2\pi/\omega = 2\pi\sqrt{m/k}$.

4.29 ⋆⋆ **(a)** The mass moves to the right to the turning point at $x = A$, where $E = U = kA^4$ so that T and hence v are zero. It speeds up as it moves back to O and then moves out to the left until $x = -A$, where $E = U$ again. It then moves back to O, and repeats the whole cycle indefinitely (in the absence of any friction). Notice how this quartic well is much flatter at the bottom than the parabolic harmonic well.



(b) Since $T = E - U = k(A^4 - x^4)$, the velocity is $\dot{x} = \sqrt{2k/m}\sqrt{A^4 - x^4}$ (moving to the right). According to (4.58), the time to move out to $x = A$ is

$$t(0 \to A) = \int_0^A \frac{dx}{\dot{x}} = \sqrt{\frac{m}{2k}} \int_0^A \frac{dx}{\sqrt{A^4 - x^4}}.$$

(c) The period is four times this time. Thus, if we change variables to $u = x/A$,

$$\tau = \frac{1}{A}\sqrt{\frac{8m}{k}} \int_0^1 \frac{du}{\sqrt{1 - u^4}}$$

which is inversely proportional to A as claimed.

(d) Using any suitable software, the integral can be found to be 1.31, and setting $m = k = A = 1$ we find $\tau = 3.71$.

4.30 ⋆ **(a)** As the toy tips, the hemisphere rolls and its center O remains at a fixed height. On the other hand the height of the CM above O changes from $h - R$ to $(h - R)\cos\theta$. Therefore, the PE of the toy is now $U(\theta) = mg[R + (h - R)\cos\theta]$.

(b) Since $dU/d\theta = -mg(h - R)\sin\theta$, which vanishes at $\theta = 0$, we see that the upright position is an equilibrium, as expected. Next, $d^2U/d\theta^2 = -mg(h - R)\cos\theta = mg(R - h)$ at $\theta = 0$. Thus the equilibrium is stable if and only if $R > h$. [If $R = h$, then $U(\theta) = mgR =$ const, and the equilibrium is neutral.]

4.31 ⋆ **(a)** Because the string is inextensible, the height of m_2 below the wheel is $y = k - x$, where k is a constant, and its velocity is $\dot{y} = -\dot{x}$. Thus the total energy is $E = T + U = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 - m_1gx - m_2g(k - x)$ or $E = \frac{1}{2}(m_1 + m_2)\dot{x}^2 - (m_1 - m_2)gx$, if we drop the uninteresting constant $-m_2gk$.

(b) The equation $dE/dt = 0$ yields $(m_1 + m_2)\dot{x}\ddot{x} - (m_1 - m_2)g\dot{x} = 0$, or

$$(m_1 + m_2)\ddot{x} = (m_1 - m_2)g. \tag{vii}$$

Applying the second law to each separate mass, we find

$$m_1\ddot{x} = m_1g - F_\mathrm{T} \quad \text{and} \quad m_2\ddot{x} = F_\mathrm{T} - m_2g$$

where F_T is the tension in the string. Adding these two equations, we can eliminate the tension and we get precisely Eq.(vii).

4.32 ⋆⋆ **(a)** Consider an infinitesimal displacement $d\mathbf{r} = (dx, dy, dz)$ occurring in a time dt. The bead's velocity is $\mathbf{v} = (dx/dt, dy/dt, dz/dt)$, so the speed is

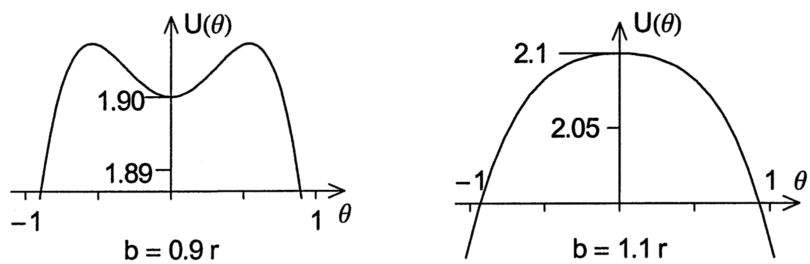
$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{dt} = \frac{ds}{dt} = \dot{s}$$

(b) Differentiating the equation $v^2 = \mathbf{v}^2$, we get $2v\dot{v} = 2\mathbf{v}\cdot\dot{\mathbf{v}} = 2v\hat{\mathbf{v}}\cdot\mathbf{F}/m$. Cancelling the factor of $2v$ and recognising from part (a) that $\dot{v} = \ddot{s}$, we find that $m\ddot{s} = \hat{\mathbf{v}}\cdot\mathbf{F} = F_{\text{tang}}$

(c) The net force on the particle is $\mathbf{F} = \mathbf{N} - \nabla U$. Therefore, $F_{\text{tang}} = -\hat{\mathbf{v}}\cdot\nabla U$ (since $\mathbf{N}$ is perpendicular to $\hat{\mathbf{v}}$). Now, if we imagine a small displacement ds along the wire, $d\mathbf{r} = ds\,\hat{\mathbf{v}}$ (since $\hat{\mathbf{v}}$ is a unit vector tangent to the wire), so $dU = d\mathbf{r}\cdot\nabla U = ds\,\hat{\mathbf{v}}\cdot\nabla U$. Therefore, $F_{\text{tang}} = -\hat{\mathbf{v}}\cdot\nabla U = -dU/ds$.

4.33 ⋆⋆ **(a)** The derivation of $U(\theta)$ is given above Eq.(4.59).

(b)



(c) These two plots do bear out the finding of Example 4.7 that for $b < r$ the equilibrium at $\theta = 0$ is stable, whereas for $b > r$ it is unstable. In addition, they show that for $b < r$ there can be two further equilibrium points (symmetrically placed on either side of $\theta = 0$), both of which are unstable

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4.34 ⋆⋆ **(a)** The distance of the mass m below the support is $l\cos\phi$. Therefore, its height measured up from the equilibrium position is $l - l\cos\phi = l(1 - \cos\phi)$ and its PE is $U = mgl(1 - \cos\phi)$. The total energy is $E = \frac{1}{2}ml^2\dot{\phi}^2 + mgl(1 - \cos\phi)$.

(b) The equation $dE/dt = 0$ reads $ml^2\dot{\phi}\ddot{\phi} + mgl\dot{\phi}\sin\phi = 0$ or $ml^2\ddot{\phi} = -mgl\sin\phi$. That is, $I\alpha = \Gamma$.

(c) Provided ϕ remains small, the equation of motion is well-approximated by $l\ddot{\phi} = -g\phi$, whose solution is $\phi = A\cos(\omega t) + B\sin(\omega t)$, where $\omega = \sqrt{g/l}$. This has period $\tau_o = 2\pi\sqrt{l/g}$.

4.35 ⋆⋆ **(a)** Because the string is inextensible, the height of m_2 below the wheel is $y = k - x$, where k is a constant, and its velocity is $\dot{y} = -\dot{x}$. The angular velocity of the wheel is $\omega = \dot{x}/R$. Thus the total energy is

$$\begin{aligned} E = T + U &= \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}m_2\dot{x}^2 + \tfrac{1}{2}I\omega^2 - m_1 g x - m_2 g(k - x) \\ &= \tfrac{1}{2}(m_1 + m_2 + I/R^2)\dot{x}^2 - (m_1 - m_2)gx, \end{aligned}$$

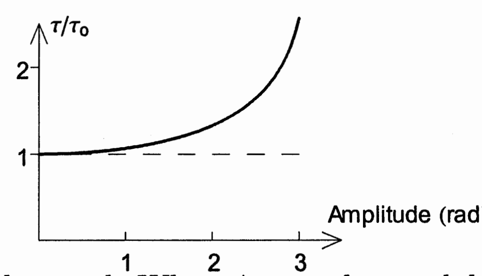
(To get the last expression I used $\sqrt{l/g} = \tau_o/2\pi$.) The time to swing from $\phi = 0$ to Φ is given by the integral $\int d\phi/\dot{\phi}$ taken from 0 to Φ (using the positive square root for $\dot{\phi}$). Since the period is four times this amount, we conclude that

$$\tau = 4\int_0^{\Phi} \frac{d\phi}{\dot{\phi}} = \frac{\tau_o}{\pi}\int_0^{\Phi} \frac{d\phi}{\sqrt{\sin^2\Phi/2 - \sin^2\phi/2}}.$$

If we make the recommended substitutions, $\sin\Phi/2 = A$ and $\sin\phi/2 = Au$, the square root in the denominator becomes $A\sqrt{1-u^2}$ and $d\phi = 2A\,du/\cos\phi/2 = 2A\,du/\sqrt{1-A^2u^2}$. Putting all this together, we find

$$\tau = \tau_o\frac{2}{\pi}\int_0^1 \frac{du}{\sqrt{1-u^2}\sqrt{1-A^2u^2}} = \tau_o\frac{2}{\pi}K(A^2). \tag{x}$$

(b) When the amplitude is small (less than about 0.5 rad), τ is very close to τ_o, as we would expect. When the amplitude is 3 rad (about 170 degrees), τ is about 2.5 times longer than its small-amplitude value. The graph suggests that as the amplitude approaches π, the period may approach infinity, a conclusion that is confirmed by putting $A = 1$ in the integral (x). This is easy to understand: When $\phi = \pi$, the pendulum is actually in unstable equilibrium. Thus the closer it gets to π, the more slowly it moves, causing the period to approach infinity.



4.39 ⋆⋆⋆ **(a)** Same as 4.38(a). **(b)** Ignoring the second square root completely, we get

$$\tau = \tau_o\frac{2}{\pi}\int_0^1 \frac{du}{\sqrt{1-u^2}} = \tau_o$$

as expected. (To show the integral is just $\pi/2$, use the substitution $u = \sin\alpha$.)

(c) If we make the binomial approximation suggested, we get

$$\tau = \tau_o\frac{2}{\pi}\left(\int_0^1 \frac{du}{\sqrt{1-u^2}} + \tfrac{1}{2}A^2\int_0^1 \frac{u^2du}{\sqrt{1-u^2}}\right).$$

The first integral is $\pi/2$ as before, while the same substitution shows the second integral to be $\pi/4$. Recalling that $A = \sin\Phi/2$, we find that $\tau = \tau_o[1 + \frac{1}{4}\sin^2(\Phi/2)]$, as stated. If $\Phi = 45°$, this gives $\tau = 1.037\tau_o$, which represents a 3.7% correction to the small-amplitude approximation (τ_o), and is itself within 0.3% of the exact answer ($1.040\tau_o$).

4.40 ⋆ **(a)** Referring to Fig.4.16, notice that the side OQ has length $OQ = r\sin\theta$, thus

$$x = OQ\cos\phi = r\sin\theta\cos\phi, \quad y = OQ\sin\phi = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

(b) By Pythagoras' theorem, $r = \sqrt{x^2 + y^2 + z^2}$. By elementary trig,

$$\theta = \arccos(z/r) = \arccos(z/\sqrt{x^2 + y^2 + z^2}) \text{ and } \phi = \arctan(y/x).$$

(But note that this leaves an ambiguity of π in ϕ. See the solution to Problem 1.42.)

4.41 $\star$ Since the mass moves in a circle, the radial component of its accelerations is just $-v^2/r$, the centripetal acceleration. By Newton's second law, $m(-v^2/r) = F_r = -dU/dr = -nkr^{n-1}$, from which we see that $mv^2 = nkr^n = nU$, and hence $T = \frac{1}{2}mv^2 = \frac{1}{2}nU$.

4.42 $\star$ Assuming that the spring doesn't bend, its extension is $r - r_o$ and the force which it exerts is $\mathbf{F} = -k(r - r_o)\hat{\mathbf{r}}$. Because this is central and spherically symmetric, it is automatically conservative.

4.43 $\star\star$ **(a)** Since $\hat{\mathbf{r}} = \mathbf{r}/r$ we can write $\mathbf{F} = f(r)\mathbf{r}/r = g(r)\mathbf{r}$, say, where $g(r)$ is simply a convenient new name for $f(r)/r$. Now consider the x component of $\nabla \times \mathbf{F}$:

$$\begin{aligned}(\nabla \times \mathbf{F})_x &= \partial_y F_z - \partial_z F_y \\ &= \partial_y[g(r)z] - \partial_z[g(r)y] = z\,\partial_y g(r) - y\,\partial_z g(r).\end{aligned}$$

Now by the chain rule, $\partial_y g(r) = g'(r)\partial r/\partial y = g'(r)y/r$, and similarly for $\partial_z g(r)$. Therefore

$$(\nabla \times \mathbf{F})_x = zg'(r)y/r - yg'(r)z/r = 0.$$

Since the other two components work the same way, we conclude that $\nabla \times \mathbf{F} = 0$ and $\mathbf{F}(\mathbf{r})$ is conservative.

(b) If $\mathbf{F}(\mathbf{r})$ is central and spherically symmetric, its spherical polar components are $F_r = f(r)$ and $F_\theta = F_\phi = 0$. When we insert these in the given expression for $\nabla \times \mathbf{F}$ in spherical polars, only the two terms involving F_r survive since $F_\theta = F_\phi = 0$, but both of these remaining terms involve derivatives of F_r with respect to θ or ϕ, which are also zero. Therefore $\nabla \times \mathbf{F} = 0$.

4.44 $\star\star$ Since $\mathbf{F} = f(r)\hat{\mathbf{r}}$, the work done going radially out from A to C is $W_{AC} = \int_A^C \mathbf{F}\cdot d\mathbf{r} = \int_{r_A}^{r_B} f(r)dr$. The same argument applies to W_{DB}, so $W_{AC} = W_{DB}$. On the other hand, on the paths CB and AD, $\mathbf{F}$ is perpendicular to $d\mathbf{r}$, so $W_{CB} = W_{AD} = 0$. Therefore

$$W_{ACB} = W_{AC} + W_{CB} = W_{AD} + W_{DB} = W_{ADB}$$

4.45 $\star\star$ That $\mathbf{F}$ is conservative tells us that the amounts of work done along the two paths ACB and ADB are equal, $W_{ACB} = W_{ADB}$. That $\mathbf{F}$ is central implies that no work is done along CB and AD, that is, $W_{CB} = W_{AD} = 0$. Therefore $W_{AC} = W_{DB}$. Since $\mathbf{F} = f(\mathbf{r})\hat{\mathbf{r}}$, $W_{AC} = f(\mathbf{r}_A)dr$ and $W_{DB} = f(\mathbf{r}_D)dr$, and since these two are equal, it follows that $f(\mathbf{r}_A) = f(\mathbf{r}_D)$. Finally, since A and D are any two points at the same distance from O, this says that $f(\mathbf{r})$ depends only on $|\mathbf{r}|$. That is, $f(\mathbf{r}) = f(r)$.

4.46 ⋆ In an elastic collision KE is conserved, so

$$m_1 v_1^2 = m_1 {v_1'}^2 + m_2 {v_2'}^2. \tag{xi}$$

Since momentum is also conserved, $m_1\mathbf{v}_1 = m_1\mathbf{v}_1' + m_2\mathbf{v}_2'$, or, squaring both sides,

$$m_1^2 v_1^2 = m_1^2 {v_1'}^2 + m_2^2 {v_2'}^2 + 2m_1m_2\mathbf{v}_1'\cdot\mathbf{v}_2'. \tag{xii}$$

If we multiply both sides of (xi) by m_1 and subtract from (xii), four terms cancel and we're left with

$$0 = m_2(m_2 - m_1){v_2'}^2 + 2m_1m_2\mathbf{v}_1'\cdot\mathbf{v}_2'.$$

Therefore

$$\mathbf{v}_1'\cdot\mathbf{v}_2' = \frac{m_1 - m_2}{2m_1}{v_2'}^2.$$

If $m_1 > m_2$, $\cos\theta$ has to be positive, so $\theta < \pi/2$. Whereas if $m_1 < m_2$, $\cos\theta$ has to be negative, so $\theta > \pi/2$.

4.47 ⋆ Since the problem is one-dimensional, we can use v to denote velocity (strictly speaking the x component of velocity). Then conservation of KE (a little rearranged) says

$$m_1(v_1^2 - {v_1'}^2) = m_2({v_2'}^2 - v_2^2)$$

and, similarly, conservation of momentum

$$m_1(v_1 - v_1') = m_2(v_2' - v_2).$$

Dividing the first of these by the second, we find

$$v_1 + v_1' = v_2' + v_2 \quad \text{or} \quad v_1 - v_2 = v_2' - v_1'.$$

4.48 ⋆ Let the initial speed of particle 1 be v_1 and the final speed of the composite be v'. Then, conservation of momentum says that $m_1v_1 = (m_1 + m_2)v'$. Therefore the initial and final KEs are $T = \frac{1}{2}m_1v_1^2$ and $T' = \frac{1}{2}(m_1+m_2)v'^2 = \frac{1}{2}m_1^2v_1^2/(m_1+m_2)$, and the fractional loss of KE is

$$\frac{T - T'}{T} = \frac{m_1(m_1+m_2) - m_1^2}{m_1(m_1+m_2)} = \frac{m_2}{m_1+m_2}.$$

If $m_1 \ll m_2$, almost all the initial KE is lost; if $m_2 \ll m_1$, almost none of the initial KE is lost.

4.49 ⋆⋆ Let's define $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, so that $\mathbf{r}$ is a vector pointing from particle 2 to particle 1. The PE is $U = \gamma/r$, the force on particle 1 due to particle 2 is $\mathbf{F}_{12} = (\gamma/r^2)\hat{\mathbf{r}}$, and that on particle 2 due to particle 1 is $\mathbf{F}_{21} = -(\gamma/r^2)\hat{\mathbf{r}}$. We'll start with the x component of $-\nabla_1 U$:

$$(-\nabla_1 U)_x = -\frac{\partial U}{\partial x_1} = \frac{\gamma}{r^2}\frac{\partial r}{\partial x_1} = \frac{\gamma}{r^3}(x_1 - x_2) \tag{xiii}$$

where for the second equality I used the chain rule and for the third I used the following:

$$\frac{\partial r}{\partial x_1} = \frac{\partial}{\partial x_1}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2} = \frac{x_1-x_2}{\sqrt{\cdots}} = \frac{x_1-x_2}{r}. \tag{xiv}$$

Combining (xiii) with the corresponding results for the y and z components, we see that

$$-\nabla_1 U = \frac{\gamma}{r^3}(\mathbf{r}_1-\mathbf{r}_2) = \frac{\gamma}{r^3}\mathbf{r} = \frac{\gamma}{r^2}\hat{\mathbf{r}} = \mathbf{F}_{12}. \tag{xv}$$

If we had evaluated $-\nabla_2 U$, the only difference would have been that in the equivalent of (xiv), we would have had an extra minus sign because $\partial(x_1-x_2)^2/\partial x_2 = -2(x_1-x_2)$. Thus in place of (xv), we would have found $-\nabla_2 U = -(\gamma/r^2)\hat{\mathbf{r}} = \mathbf{F}_{21}$.

4.50 ⋆⋆ By the chain rule

$$\frac{\partial}{\partial x_1}f(x_1-x_2) = f'(x_1-x_2)\frac{\partial}{\partial x_1}(x_1-x_2) = f'(x_1-x_2)$$

where f' denotes the derivative of f with respect to its argument. On the other hand,

$$\frac{\partial}{\partial x_2}f(x_1-x_2) = f'(x_1-x_2)\frac{\partial}{\partial x_2}(x_1-x_2) = -f'(x_1-x_2).$$

This establishes the required result for the one-dimensional function $f(x_1-x_2)$. We can prove the corresponding three-dimensional result, one component at a time. For example, $(\nabla_1 U)_x = \partial U/\partial x_1$, whereas $(\nabla_2 U)_x = \partial U/\partial x_2$, and by the one-dimensional result, the latter is minus the former. (The four extra variables y_1, z_1, y_2, z_2 act as constants and do not affect the result.) Since the y and z components work the same way, this establishes the required three-dimensional result.

4.51 ⋆⋆

$$\begin{aligned} U(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) = U_{12}(\mathbf{r}_1-\mathbf{r}_2) &+ U_{13}(\mathbf{r}_1-\mathbf{r}_3) + U_{14}(\mathbf{r}_1-\mathbf{r}_4) \\ &+ U_{23}(\mathbf{r}_2-\mathbf{r}_3) + U_{24}(\mathbf{r}_2-\mathbf{r}_4) + U_{34}(\mathbf{r}_3-\mathbf{r}_4) \\ &+ U_1^{\text{ext}}(\mathbf{r}_1) + U_2^{\text{ext}}(\mathbf{r}_2) + U_3^{\text{ext}}(\mathbf{r}_3) + U_4^{\text{ext}}(\mathbf{r}_4) \end{aligned}$$

so

$$\begin{aligned} -\nabla_3 U = 0 &+ \mathbf{F}_{31} + 0 \\ &+ \mathbf{F}_{32} + 0 + \mathbf{F}_{34} \\ &+ 0 + 0 + \mathbf{F}_3^{\text{ext}} + 0 = \mathbf{F}_3^{\text{net}}. \end{aligned}$$

4.52 ⋆⋆ **(a)** By the work-KE theorem for each of the four particles, $dT_1 = W_1, dT_2 = W_2, dT_3 = W_3$, and $dT_4 = W_4$. Adding these four equations, we conclude that $dT = W_{\text{tot}}$.

(b) The work done on the four separate particles has the form

Chapter 5

Oscillations

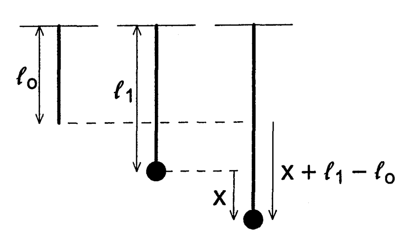
I covered this chapter in 4.5 *fifty-minute lectures.*

The first two sections of this chapter deal with the one-dimensional simple harmonic oscillator, something almost all our students are familiar with. On the other hand, I found many students distressingly weak at the trigonometry behind the various ways to write oscillations — hence the rather ponderous enumeration of the four equivalent forms for an oscillatory solutions in Section 5.2. Section 5.3 is about the two-dimensional oscillator, which is obviously important and comes up from time to time later in the book but plays no role in the rest of the chapter; so you could omit this section.

Section 5.4 is about the damped oscillator (back in one dimension) and 5.5 and 5.6 treat the driven damped oscillator and resonance — topics that every physics major should surely understand and that are essential if you're going to cover Chapter 12 on the chaos of non-linear oscillators. Finally, Sections 5.7–5.9 use Fourier series to solve for the motion of an oscillator that is driven by an arbitrary periodic force. While these three sections could certainly be omitted, my class voted overwhelmingly to include them.

There are several demonstration experiments you can show on driven damped oscillations, including the very simple on described in footnote 14.

Solutions to Problems for Chapter 5



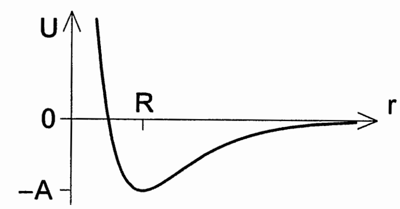
5.1 $\star$ **(a)** When the spring is at the new equilibrium position it is extended by an amount $l_1 - l_o$, so its tension is $k(l_1 - l_o)$. This must balance the weight mg. Therefore, $k(l_1 - l_o) = mg$. When it is stretched a further x, its total extension is $x + l_1 - l_o$ and the tension is $k(x + l_1 - l_o)$ upward. Thus the total downward force on the mass is

$$F = -k(x + l_1 - l_o) + mg = -kx.$$

(b) The total PE is $U = U_{sp} + U_{gr}$ or

$$U = \tfrac{1}{2}k(x + l_1 - l_o)^2 - mgx = \tfrac{1}{2}kx^2 + k(l_1 - l_o)x - mgx + \tfrac{1}{2}k(l_1 - l_o)^2 = \tfrac{1}{2}kx^2 + \text{const}$$

5.2 ⋆ When $r = 0$, $U = A[(e^{R/S} - 1)^2 - 1]$, which is large and positive since $R \gg S$. When $r \to \infty$, U is negative and approaches 0. The smallest possible value of U is when $r = R$ and $U = -A$; that is, the equilibrium separation is $r_o = R$. If we set $r = R + x$ and make a Taylor expansion of the exponential term in U, then

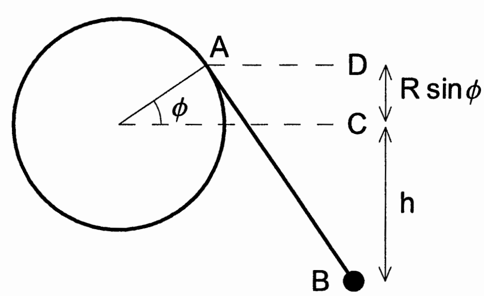


$$U = A\left[\left(\left\{1 - \frac{x}{S} + \cdots\right\} - 1\right)^2 - 1\right] \approx -A + A\left(\frac{x}{S}\right)^2 = \text{const} + \tfrac{1}{2}kx^2$$

where $k = 2A/S^2$.

5.3 ⋆ The height of the mass below the pivot is $l\cos\phi$. Therefore the height above the bottom is $l(1 - \cos\phi)$ and the PE is $U = mgl(1 - \cos\phi)$. If ϕ is small, $\cos\phi \approx 1 - \frac{1}{2}\phi^2$ and $U \approx \frac{1}{2}mgl\phi^2 = \frac{1}{2}k\phi^2$, where $k = mgl$.

5.4 ⋆⋆ The PE is $U = -mgh$ where h is the height of the mass, measured down from the level of the cylinder's center. To find h, note first that as the pendulum swings from equilibrium to angle ϕ, a length $R\phi$ of string unwinds from the cylinder. Thus the length of string away from the cylinder is $AB = (l_o + R\phi)$, and the height BD is $BD = (l_o + R\phi)\cos\phi$. Since the height $CD = R\sin\phi$, we find by subtraction that $h = BD - CD = l_o\cos\phi + R(\phi\cos\phi - \sin\phi)$. Therefore



$$U = -mgh = -mg[l_o\cos\phi + R(\phi\cos\phi - \sin\phi)].$$

If ϕ remains small we can write $\cos\phi \approx 1 - \phi^2/2$ and $\sin\phi \approx \phi$, to give

$$U \approx -mg\left\{l_o - \tfrac{1}{2}l_o\phi^2 + R\left[\phi(1 - \tfrac{1}{2}\phi^2) - \phi\right]\right\} \approx -mgl_o + \tfrac{1}{2}mgl_o\phi^2 = \text{const} + \tfrac{1}{2}k\phi^2$$

where in the third expression I dropped the term in ϕ^3. The constant $k = mgl_o$, which is the same as for a simple pendulum of length l_o. Evidently, wrapping the string around a cylinder makes no difference for small oscillations.

5.5 ⋆ **(a)** In the form (I), $x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$, we can replace the exponentials using Euler's formula $e^{\pm i\omega t} = \cos\omega t \pm i \sin\omega t$ to give

$$x(t) = B_1 \cos\omega t + B_2 \sin\omega t \qquad \text{(II)}$$

where $B_1 = C_1 + C_2$ and $B_2 = i(C_1 - C_2)$.

(b) To get form (III) from form (II), define A and δ to be the hypotenuse and lower angle of a right triangle with base B_1 and height B_2 as in Figure 5.4. Then (II) can be rewritten as in Equation (5.11) to give the form (III).

(c) Since $\cos\theta = \operatorname{Re} e^{i\theta}$, we can rewrite the form (III) as

$$x(t) = \operatorname{Re} A e^{i(\omega t - \delta)} = \operatorname{Re} C e^{i\omega t} \qquad \text{(IV)}$$

where $C = Ae^{-i\delta}$.

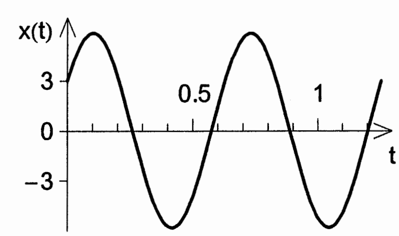
(d) Finally, since $\operatorname{Re} z = \frac{1}{2}(z + z^*)$, we can rewrite (IV) as

$$x(t) = \tfrac{1}{2}(Ce^{i\omega t} + C^* e^{-i\omega t}) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \qquad \text{(I)}$$

where $C_1 = C/2$ and $C_2 = C^*/2$.

5.6 ⋆ The position can be written as $x(t) = A\cos(\omega t - \delta)$, where $A = 2x_o$ and we know that $x(0) = x_o$ and $v(0) < 0$. Putting $t = 0$, we see that $\cos(-\delta) = 0.5$ and hence that $\delta = \pm\pi/3$. The velocity is $v(t) = -A\omega\sin(\omega t - \delta)$, so the condition that $v(0) < 0$ implies that $\sin(\delta) < 0$. Therefore $\delta = -\pi/3$ and $x(t) = 2x_o\cos(\omega t + \pi/3)$.

5.7 ⋆ **(a)** Since $x(t) = B_1\cos\omega t + B_2\sin\omega t$, $x(0) = B_1$ and $v(0) = \omega B_2$. Therefore, $B_1 = x_o$ and $B_2 = v_o/\omega$.



(b) If $k = 50$ N/m and $m = 0.5$ kg, then $\omega = \sqrt{k/m} = 10\ \text{s}^{-1}$. If $x_o = 3.0$ m and $v_o = 50$ m/s, then $B_1 = 3$ m, $B_2 = 5$ m.

(c) If we rewrite $x(t)$ in the form $x(t) = A\cos(\omega t - \delta)$, with $\delta = \arctan(B_2/B_1) = 1.03$ rad, then x first vanishes when $(\omega t - \delta) = \pi/2$ or $t = 0.26$ s, and v first vanishes when $(\omega t - \delta) = 0$ or $t = \delta/\omega = 0.10$ s.

5.8 ⋆ **(a)** $\omega = \sqrt{k/m} = \sqrt{80/0.2} = 20\ \text{s}^{-1}$, $f = \omega/2\pi = 3.2$ Hz, and $\tau = 2\pi/\omega = 0.31$ s.

(b) Since $x_o = 0$, $A\cos(-\delta) = 0$, so $\delta = \pm\pi/2$. Since $v_o = \omega A\sin\delta = 40$ m/s, δ must be positive, $\delta = +\pi/2$, and therefore $A = v_o/\omega = 2$ m.

5.9 ⋆ We are given that $A = 0.2$ m and $v_o = 1.2$ m/s. We know that $E = \frac{1}{2}kA^2 = \frac{1}{2}mv_o^2$. Therefore $m/k = A^2/v_o^2$ and $\tau = 2\pi\sqrt{m/k} = 2\pi A/v_o = 1.05$ s.

5.10 ⋆ If $F = -F_o \sinh \alpha x$, then $U = -\int F\,dx = (F_o/\alpha)\cosh \alpha x$. The only equilibrium position is at $x = 0$ and, for points close to this, Taylor's series gives

$$U(x) \approx (F_o/\alpha)(1 + \tfrac{1}{2}\alpha^2 x^2) = \tfrac{1}{2}kx^2 + \text{const},$$

where $k = \alpha F_o$. The angular frequency of oscillations is $\omega = \sqrt{k/m} = \sqrt{\alpha F_o/m}$.

5.11 ⋆ The given information gives two expressions of the total energy E

$$E = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}kx_1^2 \quad \text{and} \quad E = \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}kx_2^2. \qquad \text{(i)}$$

Equating these two, we find $m(v_1^2 - v_2^2) = k(x_2^2 - x_1^2)$. This implies that

$$\omega^2 = \frac{k}{m} = \frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}.$$

We know also that $E = \tfrac{1}{2}kA^2$, and inserting this in the first of Eqs.(i) we conclude that

$$A^2 = \frac{m}{k}v_1^2 + x_1^2 = \frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}v_1^2 + x_1^2 = \frac{x_2^2 v_1^2 - x_1^2 v_2^2}{v_1^2 - v_2^2}.$$

5.12 ⋆⋆ Because $\sin^2(\omega t - \delta)$ oscillates symmetrically between 0 and 1, its average over a cycle is fairly obviously $\frac{1}{2}$. To prove it, write $\sin^2\theta = \frac{1}{2}[1 - \cos 2\theta]$, so that

$$\langle \sin^2(\omega t - \delta)\rangle = \frac{1}{2\tau}\int_0^\tau [1 - \cos 2(\omega t - \delta)]dt = \frac{1}{2} - \frac{1}{2\tau}\big[\sin 2(\omega t - \delta)\big]_0^\tau = \frac{1}{2}$$

where the final square bracket is zero because the sine function is τ-periodic. The corresponding result with a cosine follows in exactly the same way.

We know from (5.16) that $E = \frac{1}{2}kA^2$, and, since $T = \frac{1}{2}kA^2\sin^2(\omega t - \delta)$, it follows that $\langle T\rangle = \frac{1}{4}kA^2 = \frac{1}{2}E$, and similarly for $\langle U\rangle$.

5.13 ⋆⋆ Since $U(r) = U_o(r/R + \lambda^2 R/r)$, its derivative is $dU/dr = U_o(1/R - \lambda^2 R/r^2)$, which vanishes at $r = \lambda R$ and nowhere else. Clearly $U(r) \to +\infty$ when $r \to 0$ or ∞, so $U(r)$ has a minimum at $r = r_o = \lambda R$. If we write $r = r_o + x$ then

$$U = \lambda U_o\left(\frac{r_o + x}{r_o} + \frac{r_o}{r_o + x}\right) = \lambda U_o\left(1 + \frac{x}{r_o} + \left[1 + \frac{x}{r_o}\right]^{-1}\right)$$
$$\approx \lambda U_o\left(1 + \frac{x}{r_o} + 1 - \frac{x}{r_o} + \frac{x^2}{r_o^2}\right) = \lambda U_o\left(2 + \frac{x^2}{\lambda^2 R^2}\right).$$

where, in the second line, I dropped all terms in $(x/r_o)^3$ and higher. This has the expected form $U = \frac{1}{2}kx^2 + \text{const}$, where $k = 2U_o/(\lambda R^2)$. The angular frequency is $\omega = \sqrt{k/m} = \sqrt{2U_o/(m\lambda R^2)}$.
