followed in Section 7.5 by lots of worked examples. Section 7.6 has a brief discussion of ignorable coordinates, and 7.7 is a summary of the chapter so far.

One of the controversial aspects of Lagrangian mechanics concerns what to do next. My own view is that, having swallowed this huge and unfamiliar pill, our students should be given a chance to digest it by studying its many applications in well known, interesting, and reasonably straightforward problems — planetary motion, rigid-body rotation, normal modes of coupled oscillators, and so on — and this is what I do in Chapters 8 through 11. However, there are also many theoretical developments that call out to be pursued. Three of these, the relation between invariance principles and conservation laws, the Lagrangian for a charge in a magnetic field, and the method of Lagrange multipliers, I decided to treat as optional sections at the end of this chapter. If you feel that any of these are pressingly important, you could include them now, though my preference would be to come back to them later. The most important and obvious theoretical sequel to Lagrangian mechanics is Hamiltonian mechanics, and several colleagues argued that Hamiltonian mechanics should be the subject of Chapter 8. Obviously I did not agree. When our students are still reeling from the challenge of mastering Lagrange is, I believe, the worst moment to hit them with Hamilton, and I postponed introduction of the latter to Chapter 13. Nevertheless, Hamiltonian mechanics is a tremendously important part of modern physics[1] and I designed Chapter 13 to be read at any time after Chapter 7. If your students are doing well and feeling strong, you could even jump to Chapter 13 immediately after Chapter 7.

This chapter offers several opportunities to bring out demonstration experiments to give your students a break from theory. Few students have seen an Atwood machine (Example 7.3) and even fewer know what it was designed for (to measure g). You can quite easily make demonstrations of Examples 7.5 and 7.6, and of several of the problems at the end of the chapter (for example, the "yoyo" of Problem 7.14). Even more than usual, it's important that your students do lots of problems for themselves. Many of the "$\star$" problems are intentionally very simple and are solvable by Newtonian mechanics. I think you should assign several of these just to get your students thinking in Lagrangian mode.

Solutions to Problems for Chapter 7

7.1 $\star$ The kinetic and potential energies are

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \qquad \text{and} \qquad U = mgz.$$

Therefore

$$\mathcal{L} = T - U = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

The three Lagrange equations (7.7) are

[1]Though I can't agree with the view that a good understanding of classical Hamiltonian mechanics is a prerequisite for quantum mechanics.

$$0 = m\ddot{x}, \qquad 0 = m\ddot{y}, \qquad \text{and} \qquad -mg = m\ddot{z},$$

which are just the three components of the equation $\mathbf{F} = m\mathbf{a}$ for a projectile with $\mathbf{F} = m\mathbf{g}$.

7.2 ⋆ With $F = -kx$, the PE is $U = \frac{1}{2}kx^2$ and the Lagrangian is $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$. The Lagrange equation is $-kx = m\ddot{x}$ and its solution is $x = A\cos(\omega t - \delta)$ where $\omega = \sqrt{k/m}$ and A and δ are arbitrary constants.

7.3 ⋆ $\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}k(x^2 + y^2)$ and the two Lagrange equations are $-kx = m\ddot{x}$ and $-ky = m\ddot{y}$. In the general solution, x and y oscillate with the same angular frequency $\omega = \sqrt{k/m}$ and the point (x, y) moves around an ellipse.

7.4 ⋆ The kinetic and potential energies are

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) \qquad \text{and} \qquad U = mgh = -mgy\sin\alpha.$$

Therefore

$$\mathcal{L} = T - U = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy\sin\alpha$$

The two Lagrange equations (7.7) are

$$0 = m\ddot{x} \qquad \text{and} \qquad mg\sin\alpha = m\ddot{y}$$

which imply that the acceleration across the slope is zero while that down the slope is $g\sin\alpha$, as expected.

7.5 ⋆ If we make a small displacement from $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$, we know from (4.35) that the change in $f(\mathbf{r})$ is $df = \nabla f \cdot d\mathbf{r}$. In two-dimensional polars, $d\mathbf{r} = (dr, r\,d\theta)$. Therefore,

$$df = (\nabla f)_r\,dr + (\nabla f)_\theta\,r\,d\theta.$$

On the other hand, we know from two-variable calculus that

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta.$$

Both of these are valid for arbitrary (small) values of dr and $d\theta$, so we can compare coefficients to give

$$(\nabla f)_r = \frac{\partial f}{\partial r} \qquad \text{and} \qquad (\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}.$$

7.6 ⋆ **(a)** Newton's second law gives the two vector equations (or six scalar equations)

$$\mathbf{F}_1 = -\boldsymbol{\nabla}_1 U = m_1\ddot{\mathbf{r}}_1 \quad \text{and} \quad \mathbf{F}_2 = -\boldsymbol{\nabla}_2 U = m_2\ddot{\mathbf{r}}_2\,. \tag{ii}$$

(b) The Lagrangian is

$$\mathcal{L} = T - U = \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) - U(x_1, \cdots, z_2).$$

There are six Lagrange equations, one for each of the coordinates x_1, y_1, z_1, x_2, y_2, and z_2:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \quad \Longrightarrow \quad -\partial U/\partial x_1 = m_1\ddot{x}_1$$

$$\cdots\cdots\cdots$$

$$\frac{\partial \mathcal{L}}{\partial z_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}_2} \quad \Longrightarrow \quad -\partial U/\partial z_2 = m_2\ddot{z}_2$$

which you will recognize as precisely the six components of Newton's second law as in Eq.(ii).

7.7 ⋆ **(a)** Newton's second law gives the N vector equations (or $3N$ scalar equations)

$$\mathbf{F}_\alpha = -\boldsymbol{\nabla}_\alpha U = m_\alpha\ddot{\mathbf{r}}_\alpha \quad [\alpha = 1, \cdots, N]. \tag{iii}$$

(b) The Lagrangian is

$$\mathcal{L}(\mathbf{r}_1, \cdots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \cdots, \dot{\mathbf{r}}_N) = T - U = \sum_1^N \tfrac{1}{2}m_\alpha\dot{\mathbf{r}}_\alpha^2 - U(\mathbf{r}_1, \cdots, \mathbf{r}_N)$$

and the $3N$ Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \quad \Longrightarrow \quad -\partial U/\partial x_1 = m_1\ddot{x}_1$$

and so on. These are just the $3N$ components of the N Newtonian equations (iii).

7.8 ⋆⋆ **(a)** $\mathcal{L} = T - U = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}kx^2$.

(b) Solving for x_1 and x_2 in terms of X and x, we find

$$x_1 = X + \tfrac{1}{2}x + \tfrac{1}{2}l \quad \text{and} \quad x_2 = X - \tfrac{1}{2}x - \tfrac{1}{2}l.$$

Differentiating these, and substituting into $\mathcal{L}$, we find

$$\mathcal{L} = \tfrac{1}{2}m[(\dot{X} + \tfrac{1}{2}\dot{x})^2 + (\dot{X} - \tfrac{1}{2}\dot{x})^2] - \tfrac{1}{2}kx^2 = m\dot{X}^2 + \tfrac{1}{4}m\dot{x}^2 - \tfrac{1}{2}kx^2.$$

The two Lagrange equations are

$$\text{X eqn:} \quad \frac{\partial \mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} \quad \text{or} \quad 0 = 2m\ddot{X}$$

and

$$\text{x eqn:} \quad \frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \quad \text{or} \quad -kx = \tfrac{1}{2}m\ddot{x}.$$

(c) The X equation implies that $\dot{X}(t) = \text{const} = V_o$ and hence that $X(t) = V_o t + X_o$; that is, the CM moves like a free particle, which we could have anticipated, since there are no external forces. The x equation has the general solution $x(t) = A\cos(\omega t - \delta)$; that is, the two masses oscillate in and out, relative to each other, with frequency $\omega = \sqrt{2k/m}$. The factor of $2k$ inside the square root, can be understood in several ways; for example, the spring is compressed (or stretched) by twice the amount that either separate mass moves. Thus the force on either mass is as if the spring had force constant $2k$.

7.9 ⋆ The requested equations are just the standard equations for two-dimensional polar coordinates, with the radius fixed at that of the hoop, $r = R$, namely $x = R\cos\phi$, $y = R\sin\phi$ and, in the other direction, $\phi = \arctan(y/x)$ with ϕ chosen to lie in the right quadrant.

7.10 ⋆ $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = \rho/\tan\alpha$, and, in the other direction, $\rho = \sqrt{x^2+y^2}$ (or $\rho = z\tan\alpha$) and $\phi = \arctan(y/x)$ with ϕ chosen to lie in the right quadrant.

7.11 ⋆ $x = x_s + l\sin\phi = A\cos(\omega t) + l\sin\phi$, and $y = y_s + l\cos\phi = l\cos\phi$. In the other direction, $\phi = \arctan[(x - A\cos\omega t)/y]$.

7.12 ⋆ If we define $\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 - U(x)$, then $\partial\mathcal{L}/\partial x = -\partial U/\partial x = F$ and $(d/dt)(\partial\mathcal{L}/\partial\dot{x}) = m\ddot{x}$. Substituting into Newton's second law $F + F_{\text{fric}} = m\ddot{x}$, we find that

$$\frac{\partial\mathcal{L}}{\partial x} + F_{\text{fric}} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}}\,.$$

7.13 ⋆⋆ The Lagrangian is $\mathcal{L} = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - U(\mathbf{r}_1, \mathbf{r}_2, t)$. Suppose that in the actual motion the two particles follow the "right" path $\mathbf{r}_1 = \mathbf{r}_1(t)$ and $\mathbf{r}_2 = \mathbf{r}_2(t)$, and we'll compare the action along this path with that on the "wrong" one $\mathbf{R}_1(t) = \mathbf{r}_1(t) + \boldsymbol{\epsilon}_1(t)$ and $\mathbf{R}_2(t) = \mathbf{r}_2(t) + \boldsymbol{\epsilon}_2(t)$. The difference between the Lagrangians on these two paths is

$$\begin{aligned}\delta\mathcal{L} &= \mathcal{L}(\mathbf{R}_1, \mathbf{R}_2, \dot{\mathbf{R}}_1, \dot{\mathbf{R}}_2, t) - \mathcal{L}(\mathbf{r}_1, \mathbf{r}_2, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, t)\\ &= \tfrac{1}{2}m_1[(\dot{\mathbf{r}}_1 + \dot{\boldsymbol{\epsilon}}_1)^2 - \dot{\mathbf{r}}_1^2] + \tfrac{1}{2}m_2[(\dot{\mathbf{r}}_2 + \dot{\boldsymbol{\epsilon}}_2)^2 - \dot{\mathbf{r}}_2^2] - [U(\mathbf{r}_1 + \boldsymbol{\epsilon}_1, \mathbf{r}_2 + \boldsymbol{\epsilon}_2, t) - U(\mathbf{r}_1, \mathbf{r}_2, t)]\\ &\approx m_1\dot{\mathbf{r}}_1\cdot\dot{\boldsymbol{\epsilon}}_1 + m_2\dot{\mathbf{r}}_2\cdot\dot{\boldsymbol{\epsilon}}_2 - \boldsymbol{\epsilon}_1\cdot\nabla_1 U - \boldsymbol{\epsilon}_2\cdot\nabla_2 U.\end{aligned}$$

Therefore, the difference between the two action integrals is

$$\begin{aligned}\delta S = \textstyle\int \delta\mathcal{L}\,dt &= -\textstyle\int[\boldsymbol{\epsilon}_1\cdot(m_1\ddot{\mathbf{r}}_1 + \nabla_1 U) + \boldsymbol{\epsilon}_2\cdot(m_2\ddot{\mathbf{r}}_2 + \nabla_2 U)]dt\\ &= -\textstyle\int[\boldsymbol{\epsilon}_1\cdot\mathbf{F}_1^{\text{cstr}} + \boldsymbol{\epsilon}_2\cdot\mathbf{F}_2^{\text{cstr}}]dt.\end{aligned}$$

(In the first line I used integration by parts to move a time derivative from each $\dot{\boldsymbol{\epsilon}}$ to the corresponding $\dot{\mathbf{r}}$, and in the second I used Newton's second law.) The integral on the second line is the work done by the constraint forces in the displacement from $\mathbf{r}_1$ to $\mathbf{r}_1 + \boldsymbol{\epsilon}_1$ and from $\mathbf{r}_2$ to $\mathbf{r}_2 + \boldsymbol{\epsilon}_2$. Provided this displacement is consistent with the constraints, this work is zero. Thus we have proved that the action integral is stationary for any displacement of path consistent with the constraints. If we now introduce generalized coordinates $q_1, \cdots, q_n$, then *any* variation of $q_1, \cdots, q_n$ is consistent with the constraints. Therefore the action integral $S = \int \mathcal{L}(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t)dt$ is stationary for any variations of $q_1, \cdots, q_n$, and the correct path must satisfy the n Euler-Lagrange equations (7.52).

7.14 ⋆ Recalling the $I = \frac{1}{2}mR^2$ and that $\omega = \dot{x}/R$, we see that the kinetic energy is $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 = \frac{3}{4}m\dot{x}^2$. Therefore, the Lagrangian is $\mathcal{L} = \frac{3}{4}m\dot{x}^2 + mgx$, the Lagrange equation is $mg = 3m\ddot{x}/2$, and $\ddot{x} = 2g/3$.

7.15 ⋆ Since both masses have the same speed $\dot{x}$, the total KE is $T = \frac{1}{2}(m_1 + m_2)\dot{x}^2$, whereas the PE is due to the second mass alone, $U = -m_2gx$. Therefore,

$$\mathcal{L} = T - U = \tfrac{1}{2}(m_1 + m_2)\dot{x}^2 + m_2gx.$$

The Lagrange equation is $m_2g = (m_1 + m_2)\ddot{x}$, from which it follows $\ddot{x} = gm_2/(m_1 + m_2)$.

7.16 ⋆ Since $\omega = v/R$, the cylinder's KE is $T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(m + I/R^2)\dot{x}^2$. The PE is $U = -mgx\sin\alpha$, so the Lagrangian is $\mathcal{L} = \frac{1}{2}(m + I/R^2)\dot{x}^2 + mgx\sin\alpha$ and the Lagrange equation is $mg\sin\alpha = (m + I/R^2)\ddot{x}$. Therefore $\ddot{x} = (mg\sin\alpha)/(m + I/R^2)$.

7.17 ⋆ The KE of rotation of the pulley is $\frac{1}{2}I\omega^2 = \frac{1}{2}I\dot{x}^2/R^2$ since $\omega = \dot{x}/R$. Therefore, the total KE is $T = \frac{1}{2}(m_1 + m_2 + I/R^2)\dot{x}^2$, while the PE is $U = -(m_1 - m_2)gx$ as before. Thus the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}(m_1 + m_2 + I/R^2)\dot{x}^2 + (m_1 - m_2)gx$$

and the Lagrange equation is

$$(m_1 - m_2)g = (m_1 + m_2 + I/R^2)\ddot{x}.$$

That is, $\ddot{x} = (m_1 - m_2)g/(m_1 + m_2 + I/R^2)$.

7.18 ⋆ We'll use as our generalized coordinate the height x of m below the axis of the cylinder. The angular velocity of the cylinder is $\omega = \dot{x}/R$, so the total KE is $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(m+I/R^2)\dot{x}^2$, while the PE is $U = -mgx$. Therefore, $\mathcal{L} = T - U = \frac{1}{2}(m+I/R^2)\dot{x}^2 + mgx$ and the Lagrange equation is $mg = (m+I/R^2)\ddot{x}$, which implies that $\ddot{x} = mg/(m+I/R^2)$.

$(l\dot{\phi}\cos\phi, at + l\dot{\phi}\sin\phi)$. The bob's height above the ground is $y = \frac{1}{2}at^2 - l\cos\phi$. You can now write down the KE and PE and (after a little algebra) the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}mv^2 - mgy = \tfrac{1}{2}m\left(a^2t^2 + 2atl\dot{\phi}\sin\phi + l^2\dot{\phi}^2\right) - mg\left(\tfrac{1}{2}at^2 - l\cos\phi\right).$$

The Lagrange equation is

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} \quad \Longrightarrow \quad matl\dot{\phi}\cos\phi - mgl\sin\phi &= \frac{d}{dt}(ml^2\dot{\phi} + matl\sin\phi)\\ &= ml^2\ddot{\phi} + matl\dot{\phi}\cos\phi + mal\sin\phi.\end{aligned}$$

Making a couple of cancelations and rearranging, we arrive at the equation $l\ddot{\phi} = -(g+a)\sin\phi$, which is the equation for a normal (non-accelerating) pendulum, except that g has been replaced by $(g + a)$.

7.23 ⋆ The small cart's KE is $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x} + \dot{X})^2 = \frac{1}{2}m(\dot{x} - A\omega\sin\omega t)^2$, and $U = \frac{1}{2}kx^2$. Thus $\partial\mathcal{L}/\partial\dot{x} = m(\dot{x} - A\omega\sin\omega t)$ and Lagrange's equation reads

$$-kx = m\ddot{x} - mA\omega^2\cos\omega t \qquad \text{or} \qquad \ddot{x} + \omega_o^2 x = B\cos\omega t$$

where I have replaced k/m by ω_o^2 and renamed $A\omega^2$ as B.

7.24 ⋆ The Lagrangian for the Atwood machine is given by Eq.(7.54) as $\mathcal{L} = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx$. Therefore

$$(\text{generalized force}) = \frac{\partial\mathcal{L}}{\partial x} = (m_1 - m_2)g \;\;\&\;\; (\text{generalized momentum}) = \frac{\partial\mathcal{L}}{\partial\dot{x}} = (m_1 + m_2)\dot{x}.$$

The "effective weight" is $(m_1 - m_2)g$ and the "effective mass" is $(m_1 + m_2)$.

7.25 ⋆ If $\mathbf{F} = kr^n\hat{\mathbf{r}}$, then $U(\mathbf{r}) = -\int_{\mathbf{r}_o}^{\mathbf{r}} \mathbf{F}(\mathbf{r}')\cdot d\mathbf{r}' = -\int_{r_o}^{r} kr'^n dr' = -kr^{n+1}/(n+1)$ (plus a constant that we can choose to be zero).

7.26 ⋆ From Eq.(7.79), $\Omega'^2 = \omega^2\sin^2\theta_o = \omega^2(1-\cos^2\theta_o)$, and, from (7.76), $\cos\theta_o = g/(\omega^2 R)$. Combining these, we find that $\Omega' = \sqrt{\omega^2 - g^2/(\omega R)^2}$ as claimed.

7.27 ⋆⋆ Let x be the distance from the top pulley down to the lower one and y that from the lower pulley to the mass $3m$. Then

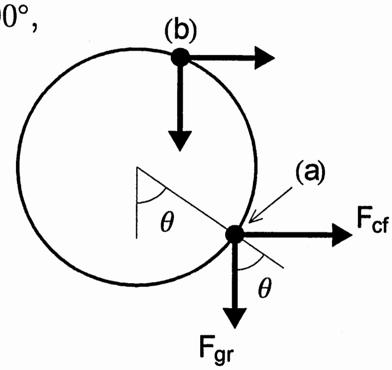
$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}m\left[4\dot{x}^2 + (\dot{x} - \dot{y})^2 + 3(\dot{x} + \dot{y})^2\right] - mg[4x + (y - x) - 3(x + y)]\\ &= 2m(2\dot{x}^2 + \dot{x}\dot{y} + \dot{y}^2) + 2mgy\,.\end{aligned}$$

The x equation is $0 = 4\ddot{x} + \ddot{y}$ and the y equation is $g = \ddot{x} + 2\ddot{y}$, from which we find $\ddot{x} = -g/7$. That is, the mass $4m$ accelerates down at $g/7$.

If the top pulley were stationary, then the same would be true of the mass $4m$. Thus the tension in the top string would be $4mg$. This would mean that the net force on the system consisting of the lower pulley and its two masses would be zero, and hence that its

CM could not be accelerating. But with the upper string stationary, the lower string would behave just like a single Atwood machine and the CM of the two masses m and $3m$ would clearly accelerate down. This is a contradiction, so the top pulley has to move (and, in fact, accelerate).

7.28 ⋆⋆ **(a)** Consider the equilibrium point with $0 < \theta < 90°$, labeled (a) in the picture. As seen in the rotating frame, the bead is subject to three forces, the normal force of the hoop (not shown in the picture), the force of gravity, $\mathbf{F}_{\rm gr} = m\mathbf{g}$, and the centrifugal force $\mathbf{F}_{\rm cf} = m\omega^2\rho$, radially out from the axis of rotation, where $\rho = R\sin\theta$ is the distance of the bead from the axis. The bead will be in equilibrium if and only if the tangential component of the net force is zero. Since the tangential component of the normal force is zero, this condition is,



$$F_{\rm tang} = -(mg)\sin\theta + (m\omega^2 R\sin\theta)\cos\theta = m(\omega^2 R\cos\theta - g)\sin\theta = 0.$$

This condition is satisfied if and only if $\cos\theta = g/\omega^2 R$, which is precisely the condition (7.71).

(b) Suppose the bead has moved a little away from the equilibrium at the top of the hoop, as indicated by (b) in the picture. At this position the tangential components of $\mathbf{F}_{\rm gr}$ and $\mathbf{F}_{\rm cf}$ are both pulling the bead away from equilibrium. Therefore the equilibrium at the top is definitely unstable.

(c) Consider the equilibrium with θ negative [across from (a) in the picture] and suppose the bead moves a little up from the equilibrium (θ more negative). This makes $\cos\theta$ smaller, and the first parenthesis on the right of Eq.(7.73) becomes negative. Since $\sin\theta$ is also negative, $\ddot{\theta}$ is positive, and the bead accelerates back toward equilibrium. Similarly, if the bead moves down from equilibrium, $\ddot{\theta}$ becomes negative and, again, the bead accelerates back toward equilibrium. Therefore, the equilibrium is stable.

7.29 ⋆⋆ Because the angle between the line OP and the horizontal is ωt, the position of P is $(R\cos\omega t, R\sin\omega t)$. Therefore the position of the pendulum's bob is

$$\mathbf{r} = (x, y) = (R\cos\omega t + l\sin\phi, R\sin\omega t - l\cos\phi)$$

and its velocity is

$$\mathbf{v} = (\dot{x}, \dot{y}) = (-\omega R\sin\omega t + \dot{\phi}l\cos\phi, \omega R\cos\omega t + \dot{\phi}l\sin\phi).$$

Therefore the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}mv^2 - mgy = \tfrac{1}{2}m[\omega^2R^2 + \dot{\phi}^2l^2 + 2\omega R\dot{\phi}l\sin(\phi - \omega t)] - mg(R\sin\omega t - l\cos\phi)$$

where I have used a couple of trig identities to combine various terms. The two derivatives of $\mathcal{L}$ are

$$\frac{\partial \mathcal{L}}{\partial \phi} = m\omega R\dot{\phi}l\cos(\phi - \omega t) - mgl\sin\phi \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m[\dot{\phi}l^2 + \omega Rl\sin(\phi - \omega t)]$$

and the Lagrange equation, after a couple of cancellations, is

$$l\ddot{\phi} = -g\sin\phi + \omega^2\cos(\phi - \omega t).$$

As $\omega \to 0$, this becomes $l\ddot{\phi} = -g\sin\phi$, the equation for an ordinary simple pendulum.

7.30 ⋆⋆ **(a)** As in Eq.(7.39) the position of the bob (relative to the ground-based frame) is

$$\mathbf{r} = (l\sin\phi + \tfrac{1}{2}at^2, l\cos\phi) \quad \text{and hence} \quad \mathbf{v} = (l\dot{\phi}\cos\phi + at, -l\dot{\phi}\sin\phi).$$

Therefore, the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}mv^2 + mgy = \tfrac{1}{2}m(l^2\dot{\phi}^2 + 2atl\dot{\phi}\cos\phi + a^2t^2) + mgl\cos\phi$$

and the two derivatives are

$$\frac{\partial \mathcal{L}}{\partial \phi} = -matl\dot{\phi}\sin\phi - mgl\sin\phi \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = ml^2\dot{\phi} + matl\cos\phi.$$

The Lagrange equation is (after a couple of cancellations)

$$l\ddot{\phi} = -g\sin\phi - a\cos\phi.$$

Finally, imagine a right triangle with base g, height a, base angle β, and hypotenuse $\sqrt{g^2 + a^2}$. Multiplying top and bottom of this equation by $\sqrt{g^2 + a^2}$, we find that

$$l\ddot{\phi} = -\sqrt{g^2 + a^2}\,(\cos\beta\sin\phi + \sin\beta\cos\phi) = -\sqrt{g^2 + a^2}\,\sin(\phi + \beta).$$

(b) The condition for equilibrium is that $\ddot{\phi} = 0$, which implies that $\phi = -\beta$. That is, in equilibrium the pendulum hangs at an angle $\beta = \arctan(a/g)$ behind the vertical. If ϕ moves a little to the left of $-\beta$, then $\phi + \beta$ is negative, so $\sin(\phi + \beta)$ is negative, so $\ddot{\phi}$ is positive, and the pendulum accelerates to the right, back toward equilibrium. Similarly if ϕ moves a little to the right. Therefore the equilibrium is stable.

If $\phi = -\beta + \epsilon$, (with ϵ small) then the equation of motion becomes $l\ddot{\epsilon} = -\sqrt{g^2 + a^2}\,\sin\epsilon \approx -\sqrt{g^2 + a^2}\,\epsilon$. Therefore, the frequency of small oscillations is $\omega = \sqrt{\sqrt{g^2 + a^2}/l}$.

7.31 ⋆⋆ **(a)** The position and hence velocity of the mass M are

$$\mathbf{r}_M = (x + L\sin\phi, L\cos\phi) \quad \text{and} \quad \mathbf{v}_M = (\dot{x}_M, \dot{y}_M) = (\dot{x} + L\dot{\phi}\cos\phi, -L\dot{\phi}\sin\phi).$$

Therefore the Lagrangian is

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}(mv_m^2 + Mv_M^2) + Mgy_M - \tfrac{1}{2}kx^2 \\ &= \tfrac{1}{2}(m + M)\dot{x}^2 + \tfrac{1}{2}M(L^2\dot{\phi}^2 + 2\dot{x}L\dot{\phi}\cos\phi) + MgL\cos\phi - \tfrac{1}{2}kx^2\end{aligned}$$

and the two Lagrange equations are (after a little tidying up)