

## Solutions to Problems for Chapter 6

**6.1 \*** Imagine first an infinitesimal section of path on the sphere, in which  $\theta$  increases by  $d\theta$  and  $\phi$  by  $d\phi$ . This carries us a distance  $R d\theta$  to the “south,” and  $R \sin \theta d\phi$  to the “east.” The distance  $ds$  along the path is therefore

$$ds = \sqrt{(R d\theta)^2 + (R \sin \theta d\phi)^2} = R \sqrt{1 + \sin^2 \theta \phi'(\theta)^2} d\theta.$$

Therefore, the total path length is  $R \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2 \theta \phi'(\theta)^2} d\theta$ , as claimed.

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**6.2 \*** Imagine first an infinitesimal section of path on the cylinder, in which  $\phi$  increases by  $d\phi$  and  $z$  by  $dz$ . This carries us a distance  $R d\phi$  around the cylinder and  $dz$  up it. The distance  $ds$  along the path is therefore

$$ds = \sqrt{(R d\phi)^2 + (dz)^2} = \sqrt{R^2 \phi'(z)^2 + 1} dz.$$

Therefore, the total path length is  $R \int_{z_1}^{z_2} \sqrt{R^2 \phi'(z)^2 + 1} dz$ .

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**6.3 \*\*** We already know that the actual path is a straight line within one medium. Therefore the segments from  $P_1$  to  $Q$  and from  $Q$  to  $P_2$  are straight and the corresponding distances are  $P_1Q = \sqrt{x^2 + y_1^2 + z^2}$  and  $QP_2 = \sqrt{(x - x_1)^2 + y_2^2 + z^2}$ . Therefore the total time for the journey  $P_1QP_2$  is

$$T = \left( \sqrt{x^2 + y_1^2 + z^2} + \sqrt{(x - x_1)^2 + y_2^2 + z^2} \right) / c.$$

To find the position of  $Q = (x, 0, z)$  for which this is minimum we must differentiate with respect to  $z$  and  $x$  and set the derivatives equal to zero:

$$\frac{\partial T}{\partial z} = \frac{z}{c\sqrt{\dots}} + \frac{z}{c\sqrt{\dots}} = 0 \quad \implies \quad z = 0,$$

which says that  $Q$  must lie in the same vertical plane as  $P_1$  and  $P_2$ , and

$$\frac{\partial T}{\partial x} = \frac{x}{c\sqrt{\dots}} + \frac{x - x_1}{c\sqrt{\dots}} = 0 \quad \implies \quad \sin \theta_1 = \sin \theta_2 \quad \text{or} \quad \theta_1 = \theta_2.$$


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**6.4 \*\*** The lengths of the paths  $P_1Q$  and  $QP_2$  are

$$P_1Q = \sqrt{x^2 + h_1^2 + z^2} \quad \text{and} \quad QP_2 = \sqrt{(x_2 - x)^2 + h_2^2 + z^2}$$

The time for light to traverse each path is the path length divided by the speed of light,  $v = c/n$ . Thus the total time is

$$t = \frac{1}{c} \left( n_1 \sqrt{x^2 + h_1^2 + z^2} + n_2 \sqrt{(x_2 - x)^2 + h_2^2 + z^2} \right).$$

To find where this is minimum we must set  $\partial t/\partial z$  and  $\partial t/\partial x$  equal to zero:

$$\frac{\partial t}{\partial z} = \frac{1}{c} \left( \frac{n_1 z}{\sqrt{x^2 + h_1^2 + z^2}} + \frac{n_2 z}{\sqrt{(x_2 - x)^2 + h_2^2 + z^2}} \right),$$

which is zero if and only if  $z = 0$ . That is, Fermat's principle requires that  $Q$  lie in the plane containing  $P_1$  and  $P_2$  and normal to the interface. Also

$$\frac{\partial t}{\partial x} = \frac{1}{c} \left( \frac{n_1 x}{\sqrt{x^2 + h_1^2 + z^2}} - \frac{n_2 (x_2 - x)}{\sqrt{(x_2 - x)^2 + h_2^2 + z^2}} \right) = \frac{1}{c} (n_1 \sin \theta_1 - n_2 \sin \theta_2)$$

which is zero if and only if  $n_1 \sin \theta_1 = n_2 \sin \theta_2$ , and this is Snell's law.

**6.5 \*\*** The distance from  $A$  to  $P$  is  $AP = 2R \sin[(90^\circ - \theta)/2]$ , where  $R$  is the radius of the mirror. (To see this, drop a perpendicular from the center of the mirror to the line  $AP$ .) Similarly  $PB = 2R \sin[(90^\circ + \theta)/2]$ . Thus the total distance  $APB$  is

$$APB = AP + PB = 2R \left( \sin \frac{90^\circ - \theta}{2} + \sin \frac{90^\circ + \theta}{2} \right) = 4 \sin 45^\circ \cos \frac{\theta}{2}$$

which is *maximum* when  $\theta = 0$ .

**6.6 \*\*** For curves in a plane:

curve	$y = y(x)$	$x = x(y)$	$r = r(\phi)$	$\phi = \phi(r)$
$ds =$	$\sqrt{1 + y'^2} dx$	$\sqrt{1 + x'^2} dy$	$\sqrt{r^2 + r'^2} d\phi$	$\sqrt{1 + r^2 \phi'^2} dr$

For curves on a cylinder (first two) or sphere (last two):

curve	$\phi = \phi(z)$	$z = z(\phi)$	$\theta = \theta(\phi)$	$\phi = \phi(\theta)$
$ds =$	$\sqrt{1 + R^2 \phi'^2} dz$	$\sqrt{R^2 + z'^2} d\phi$	$R \sqrt{\sin^2 \theta + \theta'^2} d\phi$	$R \sqrt{1 + \sin^2 \theta \phi'^2} d\theta$

**6.7 \*** The length of a small arc on the surface of the cylinder is  $ds = \sqrt{R^2 d\phi^2 + dz^2} = \sqrt{R^2 \phi'^2 + 1} dz$ , where the last expression results from thinking of  $\phi = \phi(z)$  as a function of  $z$ . Thus the integrand of the integral  $\int ds = \int f dz$  is  $f = \sqrt{R^2 \phi'^2 + 1}$ , and the Euler-Lagrange equation

$$\frac{\partial f}{\partial \phi} = \frac{d}{dz} \frac{\partial f}{\partial \phi'} \quad \text{becomes} \quad \frac{\partial f}{\partial \phi'} = \frac{\phi'}{\sqrt{R^2 \phi'^2 + 1}} = \text{const.}$$

If you solve this last for  $\phi'$  you will see that  $\phi' = \text{const}$  (a different constant), from which we deduce that  $\phi = az + b$ , with  $a$  and  $b$  chosen so that the path passes through the given points. This equation defines a path which spirals around the cylinder from  $(\phi_1, z_1)$  to  $(\phi_2, z_2)$ , with the angle  $\phi$  changing linearly with  $z$ .

The geodesic connecting points 1 and 2 is certainly not unique. First, we can spiral from 1 to 2 going either way (clockwise or counter-clockwise) around the cylinder, making less than one complete revolution. Further, we could spiral less steeply, making one or more complete revolutions before arriving at point 2. Generally there is a unique *shortest* path, namely the spiral on which  $\phi$  changes by less than  $\pi$ , but if  $\phi_1$  and  $\phi_2$  differ by exactly  $\pi$ , there are two shortest paths with equal lengths. If you unwrap and flatten the cylinder, the spiral paths become straight lines, which are well known to be the shortest paths on a flat surface.

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**6.8 \*** The PE is  $U = -mgy$  and the initial KE and PE are both 0. Therefore, by conservation of energy,  $\frac{1}{2}mv^2 - mgy = 0$  and  $v = \sqrt{2gy}$ .

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**6.9 \*** The integrand is  $f = y'^2 + yy' + y^2$ , so its derivatives are  $\partial f/\partial y = y' + 2y$  and  $\partial f/\partial y' = 2y' + y$  and the Euler-Lagrange equation (6.13) is

$$\frac{\partial f}{\partial y} = \frac{d}{dx} \frac{\partial f}{\partial y'} \implies y' + 2y = 2y'' + y' \implies y'' = y,$$

the general solution of which is  $y(x) = A \sinh(x) + B \cosh(x)$ . Since  $y(0) = 0$  and  $y(1) = 1$ , we see that  $B = 0$  and  $A = 1/\sinh(1)$ , so the solution is  $y(x) = \sinh(x)/\sinh(1)$ .

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**6.10 \*** If  $\partial f/\partial y = 0$ , then the Euler-Lagrange equation (6.13) reduces to  $\frac{d}{dx} \frac{\partial f}{\partial y'} = 0$ , which implies that  $\partial f/\partial y' = \text{const.}$

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**6.11 \*\*** The integrand is  $f(y, y', x) = \sqrt{x}\sqrt{1+y'^2}$ . Since this is independent of  $y$ ,  $\partial f/\partial y = 0$  and the Euler-Lagrange equation (6.13) implies simply that  $\partial f/\partial y'$  is a constant; that is,  $\sqrt{x}y'/\sqrt{1+y'^2} = k$ . This can be solved for  $y'$  to give  $y' = k/\sqrt{x-k^2}$ , which integrates to give  $y = 2k\sqrt{x-k^2} + D$ , where  $D$  is a constant of integration. Rearranging we find that  $x = k^2 + (y-D)^2/4k^2$ , which is a parabola with its axis along the line  $y = D$ .

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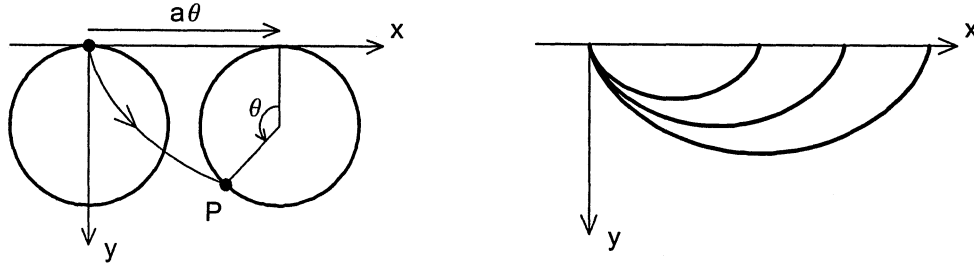
**6.12 \*\*** The integrand is  $f(y, y', x) = x\sqrt{1-y'^2}$ . Since this is independent of  $y$ ,  $\partial f/\partial y = 0$  and the Euler-Lagrange equation (6.13) implies simply that  $\partial f/\partial y'$  is a constant; that is,  $xy'/\sqrt{1-y'^2} = k$ . This can be solved for  $y'$  to give  $y' = k/\sqrt{x^2+k^2}$ , which integrates to give  $y = k \sinh^{-1}(x/k) + c$ , where  $c$  is a constant of integration. (Make the substitution  $x/k = \sinh u$ .) Rearranging we find that  $x = k \sinh[(y-c)/k]$ .

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**6.13 \*\*** If we write the path as  $\phi = \phi(r)$ , the distance from  $O$  to  $P$  is  $\int_O^P ds = \int_0^R f dr$ , where  $f = [2/(1-r^2)]\sqrt{1+r^2\phi'^2}$ . Since  $\partial f/\partial \phi = 0$ , the Euler-Lagrange equation (6.13) implies simply that  $\partial f/\partial \phi'$  is a constant; that is,  $[2/(1-r^2)]r^2\phi'/\sqrt{1+r^2\phi'^2} = k$ . Because the path passes through the origin,  $r = 0$ , the constant  $k$  must in fact be zero, and we find that  $\phi' = 0$ . This defines a straight line through the origin.

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**6.14 \*\*** (a) The left picture shows two positions of a wheel of radius  $a$  rolling on the underside of the  $x$  axis. The point which started in contact with the  $x$  axis has moved to the point  $P$  while the wheel has rotated through an angle  $\theta$  and moved a distance  $a\theta$  to the right. The coordinates of the point  $P$  are easily seen to be  $x = a(\theta - \sin \theta)$  and  $y = a(1 - \cos \theta)$ , in exact agreement with Eq.(6.26). That is, the curve traced by the point  $P$  on the wheel (the cycloid) is the same as the brachistochrone curve of (6.26).



(b) The right picture shows three cycloids with three successively larger values of  $a$ . It is clear from the picture (or from the equations) that as we increase  $a$  from 0 to  $\infty$ , the loop steadily expands, eventually sweeping exactly once across any point in the positive quadrant.

(c) If the point 2 has coordinates  $x_2 = \pi b$  and  $y_2 = 2b$ , it is easy to see that this corresponds to  $a = b$  and  $\theta = \pi$ ; that is, point 2 is at the exact bottom of the cycloid. The time for the journey is

$$t = \int_1^2 \frac{ds}{v} = \frac{1}{\sqrt{2g}} \int_0^\pi \frac{\sqrt{x'^2 + y'^2} d\theta}{\sqrt{y}} = \sqrt{\frac{b}{2g}} \int_0^\pi \frac{\sqrt{(1 - \cos \theta)^2 + \sin^2 \theta} d\theta}{\sqrt{1 - \cos \theta}} = \pi \sqrt{\frac{b}{g}}.$$

If point 2 has coordinates  $x_2 = 2\pi b$  and  $y_2 = 0$ , then again  $a = b$  but now  $\theta = 2\pi$ . That is point 2 is at the top of the cycloid, with point  $P$  back on the  $x$  axis. We can easily calculate the time to reach point 2, but there is no need, because it is obviously twice that to reach the bottom. That is, in this case,  $t = 2\pi \sqrt{b/g}$ .

**6.15 \*\*** The analysis for the case that the car is launched from point 1 with fixed speed  $v_o$  is very similar to the case of Example 6.2 with  $v_o = 0$ , except that, by conservation of energy, the car's speed at height  $y$  is  $v = \sqrt{2gy + v_o^2}$  instead of  $v = \sqrt{2gy}$ . If we make the change of variables from  $y$  to  $\tilde{y} = y + v_o^2/2g$ , the analysis goes through exactly as before and we get the same answer as in Eq.(6.26) except that it is now  $\tilde{y}$  that is equal to  $a(1 - \cos \theta)$ . Therefore  $y = a(1 - \cos \theta) - v_o^2/2g$  (with  $x$  exactly the same as before), and the curve is the same cycloid except that it is shifted up by a height  $h = v_o^2/2g$ .

**6.16 \*\*** By Problem 6.1, the path length is  $L = \int f d\theta$ , with  $f = f(\phi, \phi', \theta) = \sqrt{1 + \sin^2 \theta \phi'^2}$ . Since  $\partial f / \partial \phi = 0$ , the Euler-Lagrange equation reduces to

$$\frac{\partial f}{\partial \phi'} = \frac{\sin^2 \theta \phi'}{\sqrt{1 + \sin^2 \theta \phi'^2}} = c.$$

If we choose our polar axis to go through the point 1, then  $\theta_1 = 0$  and the constant  $c$  has to be zero. Thus the Euler-Lagrange equation implies that  $\phi' = 0$  and hence that  $\phi$  is constant. The curves of constant  $\phi$  are the lines of longitude and are great circles. Therefore, the geodesics are great circles.

**6.17 \*\*** The length of an element of path in cylindrical coordinates is

$$ds = \sqrt{d\rho^2 + \rho^2 d\phi^2 + dz^2} = \sqrt{(1 + \lambda^2)d\rho^2 + \rho^2 d\phi^2} = \sqrt{(1 + \lambda^2) + \rho^2 \phi'^2} d\rho$$

where in the second equality I used the fact that  $dz = \lambda d\rho$  on the cone, and in the last I assumed that the path was written as  $\phi = \phi(\rho)$ . Thus the path length has the standard form  $L = \int f d\rho$ , with  $f = f(\phi, \phi', \rho) = \sqrt{1 + \lambda^2 + \rho^2 \phi'^2}$ . Because  $\partial f / \partial \phi = 0$ , the Euler-Lagrange equation reduces to

$$\frac{\partial f}{\partial \phi'} = \frac{\rho^2 \phi'}{\sqrt{1 + \lambda^2 + \rho^2 \phi'^2}} = K \quad \text{whence} \quad \phi' = \frac{K \sqrt{1 + \lambda^2}}{\rho \sqrt{\rho^2 - K^2}}.$$

This can be integrated (make the substitution  $K/\rho = \cos u$ ) to give

$$\phi - \phi_0 = \sqrt{1 + \lambda^2} \arccos(K/\rho) \quad \text{or} \quad \rho = \frac{K}{\cos[(\phi - \phi_0)/\sqrt{1 + \lambda^2}]}.$$

This is not easily recognized as any simple curve, but in the limit that  $\lambda \rightarrow 0$  the cone approaches the plane  $z = 0$  and the geodesic approaches the curve  $\rho = K/\cos(\phi - \phi_0)$ , which you can show is a straight line perpendicular to the direction  $\phi = \phi_0$ , a distance  $K$  from the origin.

**6.18 \*\*** If we use polar coordinates  $(r, \phi)$  and write the path in the form  $\phi = \phi(r)$ , then the path length takes the form  $L = \int f dr$  with  $f = f(\phi, \phi', r) = \sqrt{1 + r^2 \phi'^2}$ . Because  $\partial f / \partial \phi = 0$ , the Euler-Lagrange equation is

$$\frac{\partial f}{\partial \phi'} = \frac{r^2 \phi'}{\sqrt{1 + r^2 \phi'^2}} = \text{const} \quad \text{whence} \quad \phi' = \frac{K}{r \sqrt{r^2 - K^2}}.$$

This can be integrated (make the substitution  $K/r = \cos u$ ) to give

$$\phi - \phi_0 = \arccos(K/r) \quad \text{or} \quad r = \frac{K}{\cos(\phi - \phi_0)}.$$

This is the equation of a straight line perpendicular to the direction  $\phi = \phi_0$ , a distance  $K$  from the origin. [To see this, note that  $r \cos(\phi - \phi_0)$  is the component of  $\mathbf{r}$  in the direction  $\phi = \phi_0$ ; that this equals the constant  $K$  says that  $\mathbf{r}$  lies on the line indicated.]