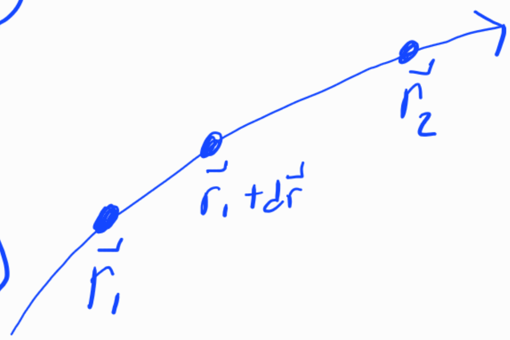
Kinetic Energy: (KE)



$$T = \frac{1}{2} m v^2$$

$$\frac{dT}{dt} = \frac{1}{2} m \frac{d}{dt}(\vec{v} \cdot \vec{v}) = \frac{1}{2} m (\dot{\vec{v}} \cdot \vec{v} + \vec{v} \cdot \dot{\vec{v}})$$

$$= \underbrace{m \dot{\vec{v}}}_{F!} \cdot \vec{v}$$

$$\therefore \frac{dT}{dt} = \vec{F} \cdot \vec{v}, \quad dT = \vec{F} \cdot d\vec{r}$$

Work-KE theorem:

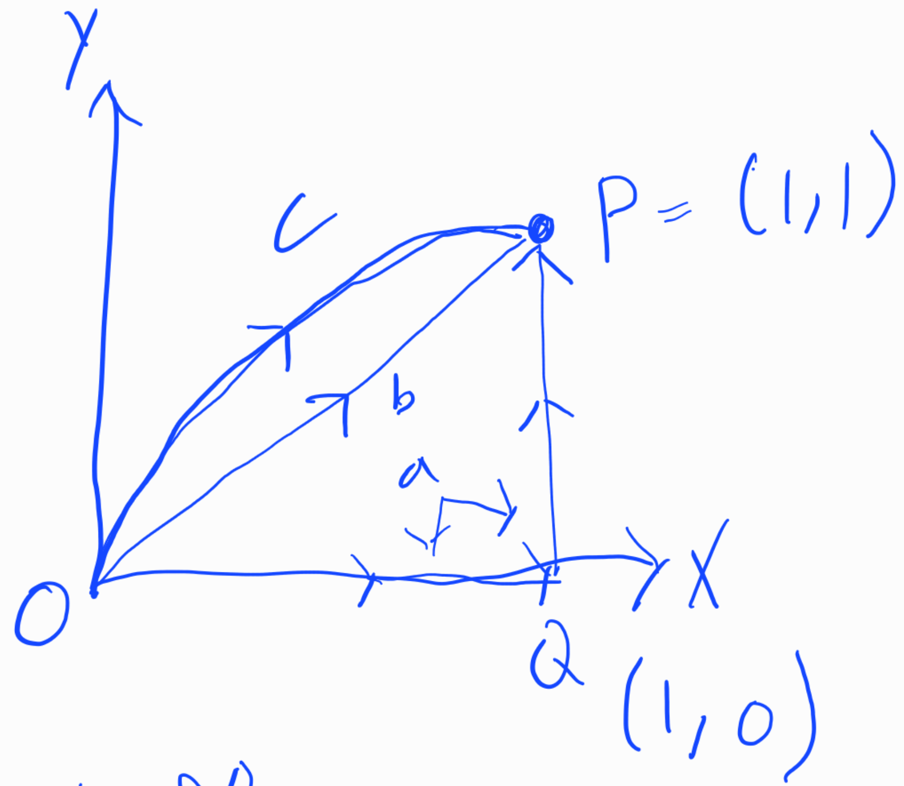
$$dT = \vec{F} \cdot d\vec{r}, \quad \Delta T = F |\Delta \vec{r}|$$

$$\Delta T \equiv T_2 - T_1 = \sum \vec{F} \cdot d\vec{r}$$

$$\sum \vec{F} \cdot d\vec{r} \rightarrow \int_1^2 \vec{F} \cdot d\vec{r}$$

Ex4.1: line integral

$\vec{F} = (y, 2x)$



$$W_a = \int_a \vec{F} \cdot d\vec{r} = \int_O^Q \vec{F} \cdot \underbrace{d\vec{r}}_{(dx,0)} + \int_Q^P \vec{F} \cdot \underbrace{d\vec{r}}_{(0,dy)}$$

$$= \int_0^1 F_x(x,0)\,dx + \int_0^1 F_y(1,y)\,dy$$

$$= \int_0^1 0\,dx + \int_0^1 2(1)\,dy = 2\int_0^1 dy$$

$$= 2$$

$$W_b = \int_b \vec{F} \cdot d\vec{r} = \int_b (F_x dx + F_y dy), \quad x = y \;\&\; dx = dy$$

$$= \int_0^1 (x + 2x)\, dx = 1.5$$

$$\vec{r} = (x, y) = (1 - \cos\theta, \sin\theta), \quad 0 \le \theta \le \pi/2$$

$$d\vec{r} = (dx, dy) = (\sin\theta, \cos\theta)\, d\theta$$

$$W_c = \int_c \vec{F}_c \cdot d\vec{r} = \int_c (F_x dx + F_y dy)$$

$$= \int_0^{\pi/2} \left[\sin^2\theta + 2(1 - \cos\theta)\cos\theta \right] d\theta$$

$$= 2 - \pi/4 = 1.21$$

$$\Delta T \equiv T_2 - T_1 = \int_1^2 \vec{F} \cdot d\vec{r} \equiv W(1 \to 2)$$

$$W(1\to 2) = \int_1^2 \vec{F} \cdot d\vec{r} = \int_1^2 \sum_i \vec{F}_i \cdot d\vec{r}$$

$$= \sum_i \int_1^2 \vec{F}_i \cdot d\vec{r} = \sum_i W_i(1\to 2)$$

$$\therefore \quad T_2 - T_1 = \sum_i^n W_i(1-2)$$

Conservative Forces & Potential Energy:

Two rules for conservative forces

1st $\vec{F}$ must depend only on position $\vec{r}$

$$\vec{F} = \vec{F}(\vec{r})$$

Ex: Gravity, Electric Force

$$\vec{F}_G = G\frac{Mm}{r^2}\hat{r}, \quad \vec{F}_E = q\vec{E}(\vec{r})$$

not conservative

$$F_{f_{drag}} \propto v, \quad \vec{F}_E(\vec{r},t), \quad \vec{F}_{mag} \propto \vec{v}$$

2nd Rule

For any two Points 1 to 2, the $W(1 \to 2)$ done by $\vec{F}$ is the Same for all Paths

If all forces are conservative we can define a quantity called Potential Energy $U(\vec{r})$, only depending on $\vec{r}$

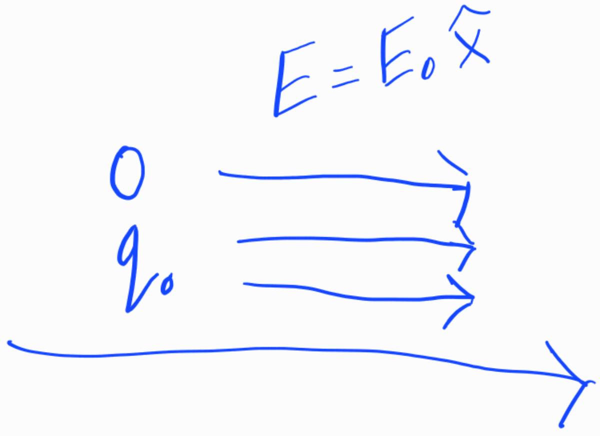
$$E_{Mech} = KE + PE = T + U(\vec{r})$$

if $E = $ const. Energy is conserved

$$U(\vec{r}) = -W(r_0 \to \vec{r}) \equiv -\int \vec{F}(\vec{r}\,') \cdot d\vec{r}$$

$U(\vec{r})$ is only unique if $\vec{F}$ is conservative

Ex 4.2: Finding $U(\vec{r})$



$$\vec{F} = q\vec{E} = qE_0\hat{x}$$

$$W(1\to 2) = \int_1^2 \vec{F}\cdot d\vec{r} = qE_0\int_1^2 \hat{x}\cdot d\vec{r}$$

$$= qE_0\int_1^2 dx = \underline{qE_0(x_2 - x_1)}$$

$$U(\vec{r}) = -W(0\to\vec{r}), \quad U(\vec{r}) = -qE_0 x$$

$$W(r_0\to r_2) = W(r_0\to r_1) + W(r_1\to r_2)$$

$$W(r_1\to r_2) = W(r_0\to r_2) - W(r_0\to r_1)$$

$$\Delta T = W(r_1 \to r_2), \quad \Delta T = -\Delta U$$

$$\Delta(T+U) = 0, \quad E = T + U$$

$$\therefore \Delta E = 0$$

If a force is conservative mechanical Energy is conserved

For multiple conservative forces:

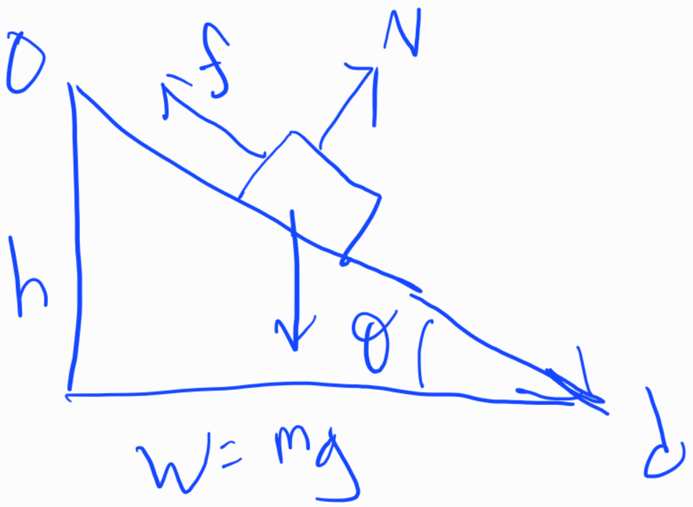
$$E \equiv T + U \equiv T + U_1(\vec{r}) + \dots + U_n(\vec{r})$$

For nonconservative forces:

$$\Delta T = W = \underbrace{W_{cons}}_{-\Delta U} + W_{nc}$$

$$\Delta T + \Delta U = W_{nc}, \quad \therefore \Delta E \equiv \Delta(T+U) = W_{nc}$$

E4.3 Block sliding down an Incline



what is V as the block reaches the end of the slope?

$$U = mgy, \quad W_{fric} = -fd = -\mu mgd\cos\theta$$

$$\Delta T = T_f - T_i = \frac{1}{2}mv^2$$

$$\Delta U = U_f - U_i = -mgh = mgd\sin\theta$$

$$\Delta T + \Delta U = W_{fric}$$

$$\frac{1}{2}mv^2 - mgd\sin\theta = -\mu mgd\cos\theta$$

$$V = \sqrt{2gd(\sin\theta - \mu\cos\theta)}$$

Force as the Gradient of Potential Energy:

$$W(\vec{r} \to \vec{r}+d\vec{r}) = \vec{F}(\vec{r}) \cdot d\vec{r}$$

$$= F_x dx + F_y dy + F_z dz$$

$$W(\vec{r} \to \vec{r}+d\vec{r}) = -dU = -\left[U(\vec{r}+d\vec{r}) - U(\vec{r})\right]$$

$$= -\left[U(x+dx, y+dy, z+dz) - U(x,y,z)\right]$$

$$df = f(x+dx) - f(x) = \frac{df}{dx} dx$$

$$dU = U(x+dx, y+dy, z+dz) - U(x,y,z)$$

$$= \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz$$

$$W(\vec{r} \to \vec{r} + d\vec{r}) = -\left[\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz\right]$$

$$\therefore F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}$$

$$\vec{F} = -\hat{x}\frac{\partial U}{\partial x} - \hat{y}\frac{\partial U}{\partial y} - \hat{z}\frac{\partial U}{\partial z}$$

$$\nabla f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$

$$\therefore \vec{F} = -\nabla U$$

any conservative force is derivable from a Potential Energy

Ex 4.4

$$U = Axy^2 + B\sin cz$$

$$\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z}\right)$$

$$\vec{F} = -\left(Ay^2\hat{x} + 2Axy\,\hat{y} + cB\cos cz\,\hat{z}\right)$$

Stoke's Theorem

$\nabla \times \vec{F} = 0$ (Work Done is indepecting of Path if & only if)

$\vec{A} \times \vec{B}$:

	$\hat{x}$,	$\hat{y}$,	$\hat{z}$
$\vec{A}$	A_x	A_y	A_z
$\vec{B}$	B_x	B_y	B_z
$\vec{A} \times \vec{B}$	$A_yB_z - A_zB_y$	$A_zB_x - A_xB_z$	$A_xB_y - A_yB_x$

$\nabla \times \vec{F}$:

∇	$\partial/\partial x$	$\partial/\partial y$	$\partial/\partial z$
$\vec{F}$	F_x	F_y	F_z
$\nabla \times \vec{F}$	$\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y$	$\frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z$	$\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x$

Time Dependant Potential Energy:

$$U(\vec{r},t) \to F(\vec{r},t)$$

$$\nabla \times \vec{F}(\vec{r},t) = 0$$

$$\vec{F}(\vec{r},t) = -\nabla U(\vec{r},t) \checkmark$$

but looking at $E = T + U$

$$dT = \frac{dT}{dt}dt = (m\dot{\vec{v}} \cdot \vec{v})dt = \vec{F} \cdot d\vec{r}$$

$$dU = \underbrace{\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}\partial z}_{\nabla U \cdot d\vec{r} = -\vec{F} \cdot d\vec{r}} + \frac{\partial U}{\partial t}dt$$

$$dU = -\vec{F} \cdot dr + \frac{\partial U}{\partial t}dt$$

$$d(T+U) = \vec{F}\cdot d\vec{r} - \vec{F}\cdot d\vec{r} + \frac{\partial U}{\partial t}\,dt$$

$$d(T+U(r,t)) = \frac{\partial U}{\partial t}\,dt$$

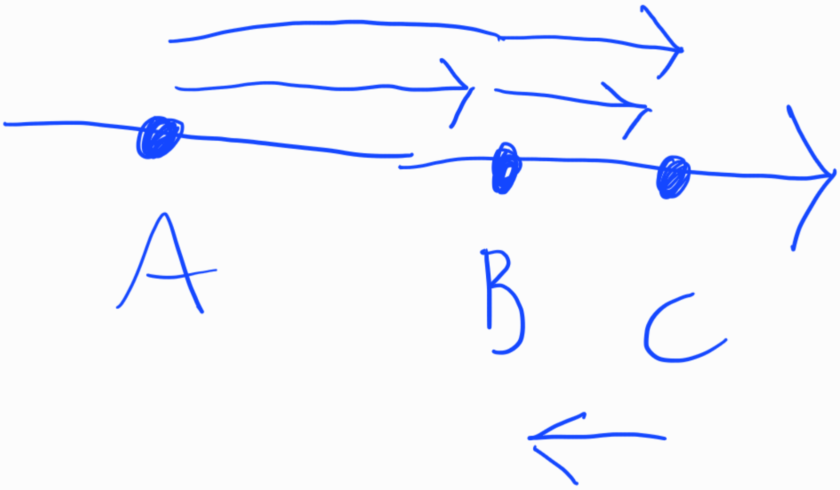
$$E = T+U$$

$$dE = d(T+U) \neq 0 = \frac{\partial U}{\partial t}\,dt$$

$\therefore$ U must be indeperdant of t to conserve mechanical Energy

One-Dimensional Systems

$$W(x_1 \to x_2) \int_{x_1}^{x_2} F_x(x)\,dx$$



$$W(BC) = -W(CB)$$

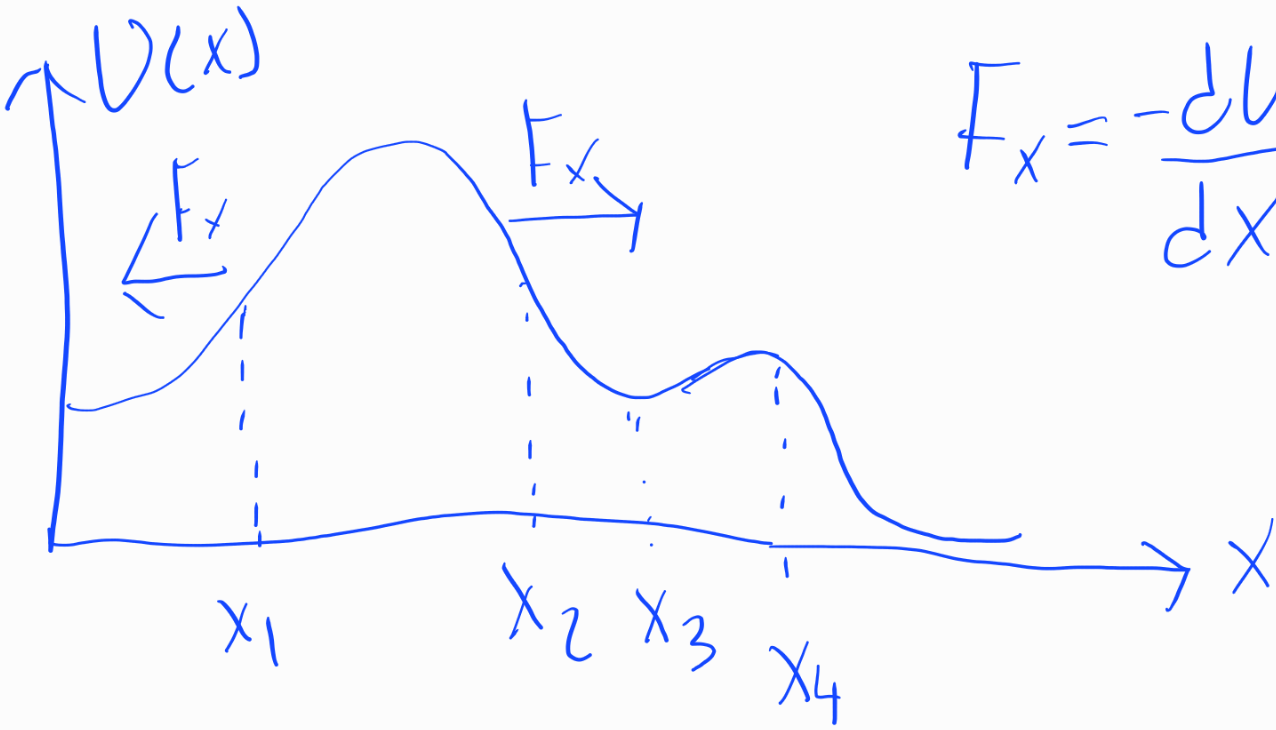
$$W(ABCB) = W(AB) + W(BC) + W(CB)$$

$$W(ABCB) = W(AB)$$

$$U(x) = \int_{x_0}^{x} F_x(x')\,dx'$$

$F_x = -\frac{dU}{dx}$, set $F_x = -kx$

$U(x) = \frac{1}{2}kx^2$, $x_0 = 0$



$F_x = -\frac{dU}{dx}$

F_x pushes object downhill

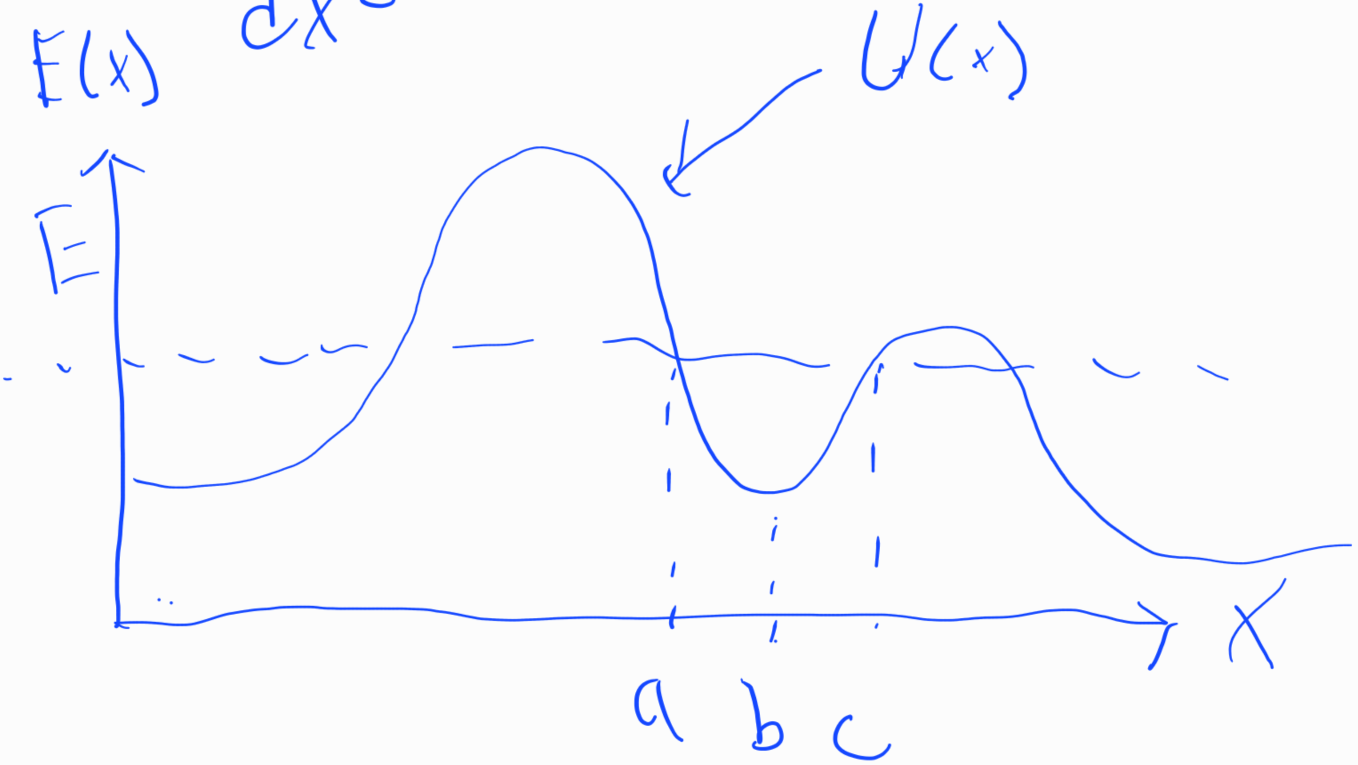
x_3 & x_4 are equilibrium points

X_4 is unstable

$$\frac{d^2U}{dx^2} < 0$$

X_3 is stable

$$\frac{d^2U}{dx^2} > 0$$



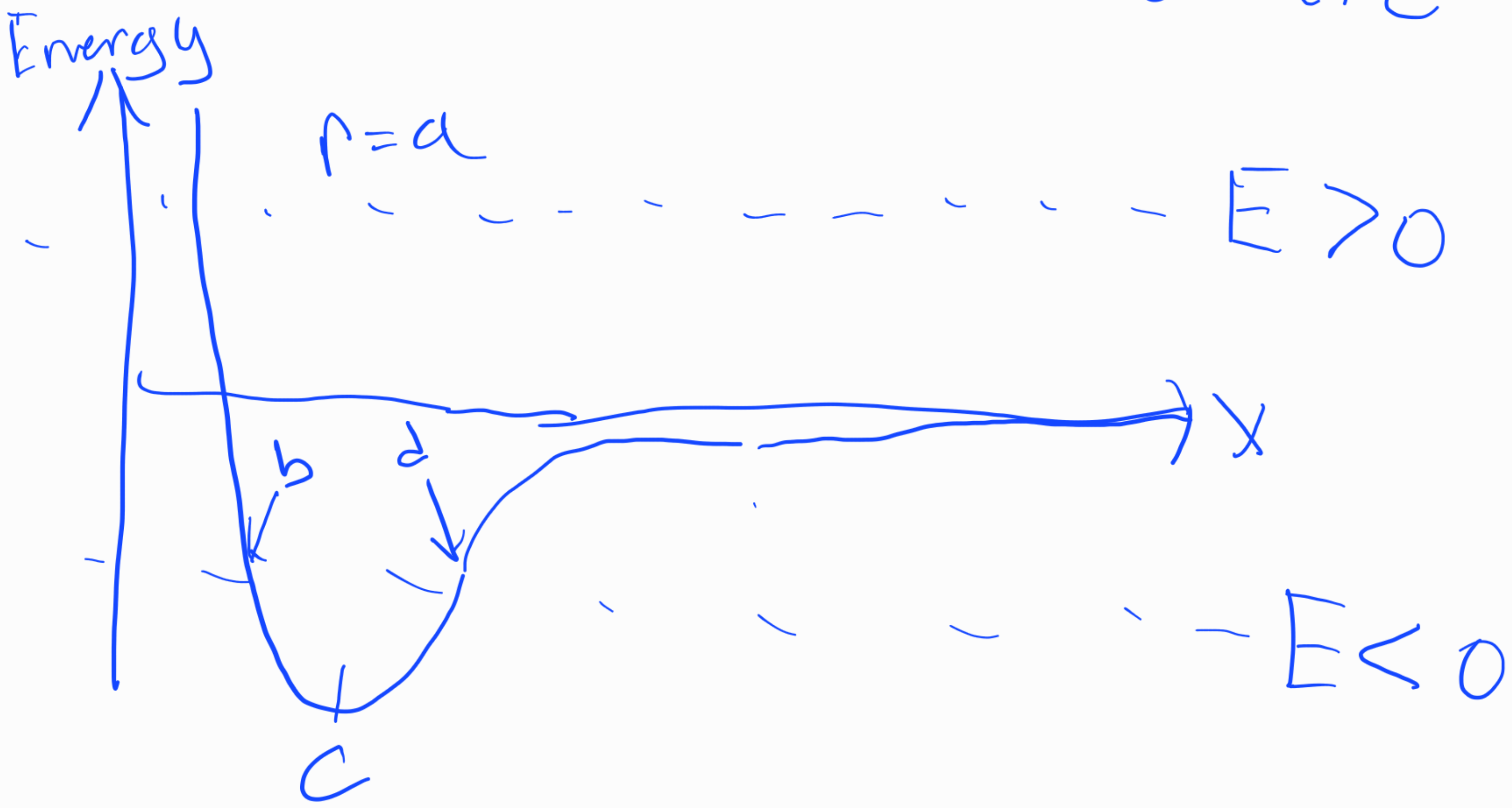
If object is moving $E > U(x)$

$T > 0$

a & c are turning points

If Object is near or d
b it will be trapped
Oscillating between a & c

Diatomic HCl molecule



Complete Solution of Motion

$$E = T + U(x)$$

$$T = \frac{1}{2} m \dot{x}^2 = E - U(x)$$

$$\dot{x}(x) = \pm \sqrt{\frac{2}{m}} \sqrt{E - U(x)}$$

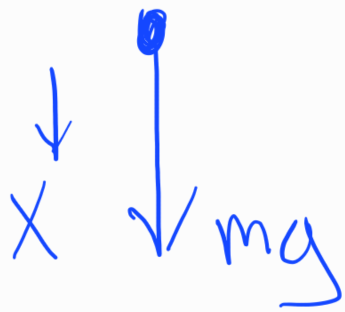
determine by inspection

$$\dot{x} = \frac{dx}{dt} \rightarrow dt = \frac{dx}{\dot{x}}$$

$$t_f - t_i = \int_{x_i}^{x_f} \frac{dx}{\dot{x}}$$

$$\therefore t = \int_{x_0}^{x} \frac{dx'}{\dot{x}(x')} = \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx'}{\sqrt{E - U(x')}}$$

Ex 4.6 Free Fall



$$F = mg = -\frac{dU}{dx}$$

$$U(x) = -mgx$$

$$E = 0 \text{ @ } x = 0$$

$$\dot{x}(x) = \sqrt{\frac{2}{m}} \sqrt{E - U(x)}$$

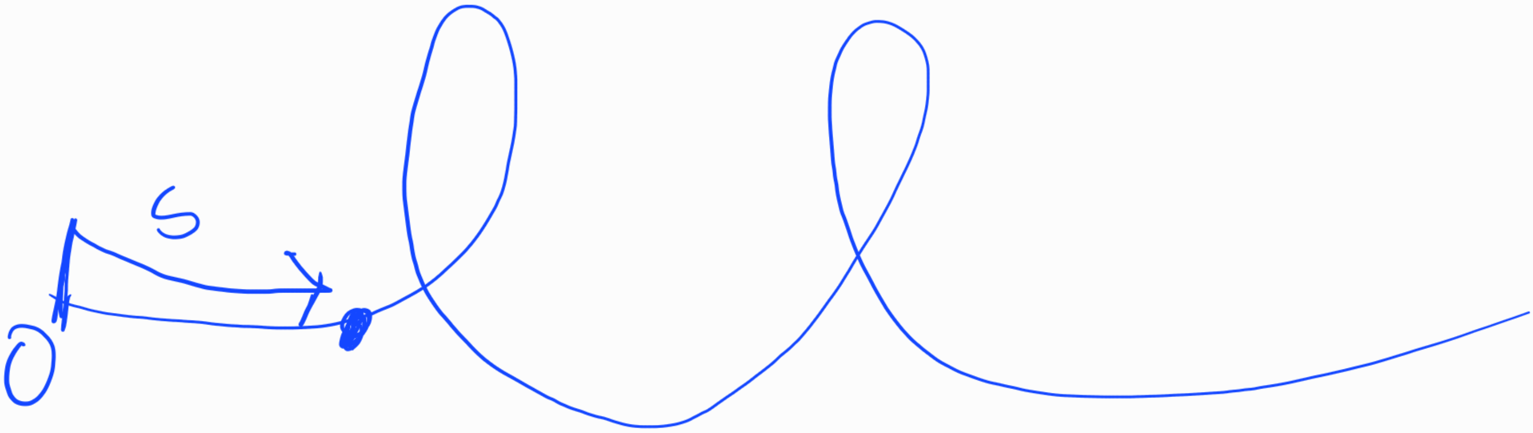
$$= \sqrt{2gx}$$

want this

$$t = \int_0^x \frac{dx'}{\dot{x}(x')} = \int_0^x \frac{dx'}{\sqrt{2gx'}} = \sqrt{\frac{2x}{g}}$$

$$\therefore x = \frac{1}{2} g t^2$$

Curvilinear One-Dimensional systems



$$T = \frac{1}{2} m \dot{s}^2$$

(any perpendicular component of force does no work)

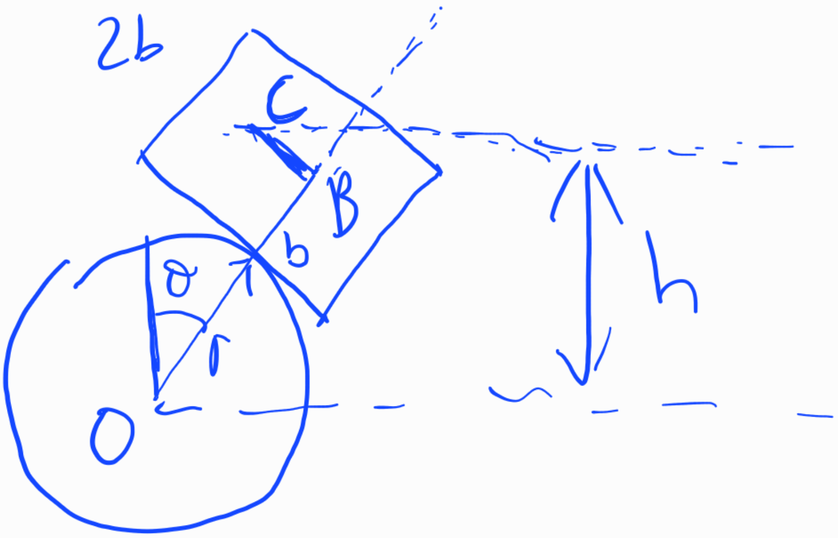
$$F_{tang} = m \ddot{s}$$

$$F_{tang} = -\frac{dU}{ds}$$

Ex 4.7

Cube of side $2b$

Center : C



What are the equilibrium positions?

$U = mgh$

Length OB
$= r + b$

Length $CB = r\theta$

$\therefore h = (r+b)\cos\theta + r\theta\sin\theta$

$U = mg\left[(r+b)\cos\theta + r\theta\sin\theta\right]$

$$\frac{dU}{d\theta} = mg\left[r\theta\cos\theta - b\sin\theta\right]$$

$$\left[\theta = 0 \quad \frac{dU}{d\theta} = 0\right]$$

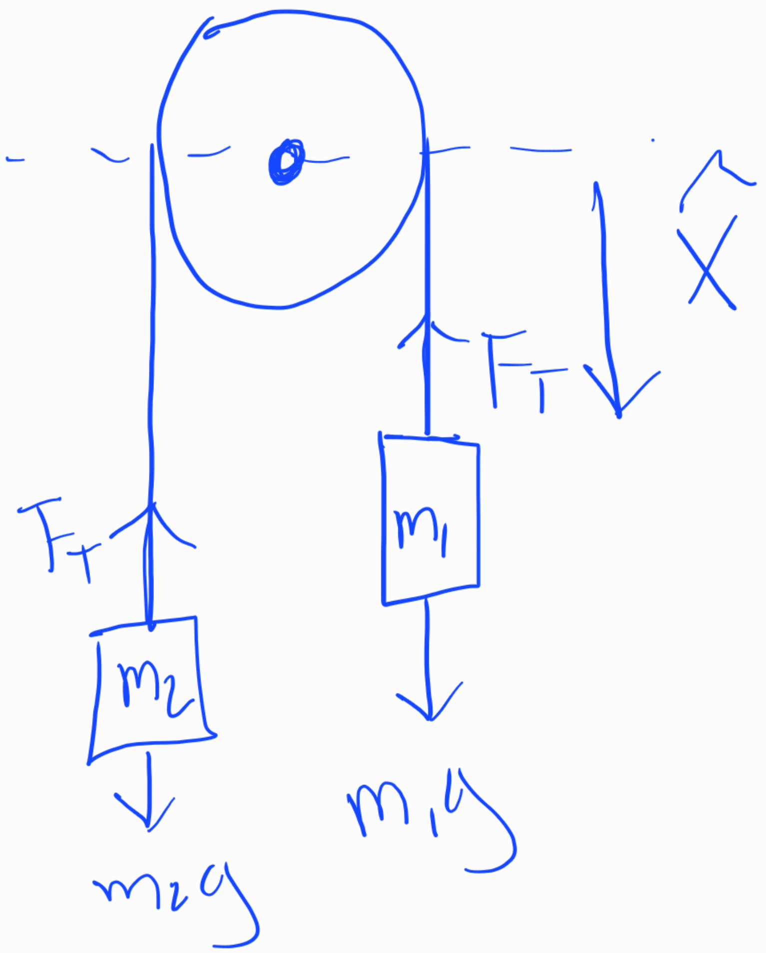
Stable or unstable?

$$\frac{d^2U}{d\theta^2} = mg(r-b)$$

$+$ if $r>b$, $-$ if $r<b$

Stable Unstable

Atwood machine:



$$\Delta T_1 + \Delta U_1 = \Delta E_1 = W_1^{ten}$$

$$\Delta T_2 + \Delta U_2 = \Delta E_2 = W_2^{ten}$$

$$W_1^{ten} = -W_2^{ten}$$

$$\Delta E_1 + \Delta E_2 = \Delta(T_1 + U_1 + T_2 + U_2) = 0$$

$$T_1 + U_1 + T_2 + U_2 = E_{tot}$$

$$\therefore \quad E = \sum_{\alpha=1}^{N} (T_\alpha + U_\alpha), \text{(one dimension)}$$

4.8 Central Forces

$$F(\vec{r}) = f(\vec{r})\hat{r}$$

+ if Outward
− if Inward

Ex: Coulomb Force

$$\vec{F}(\vec{r}) = \underbrace{k\frac{qQ}{r^2}}_{f(\vec{r})}\hat{r}$$

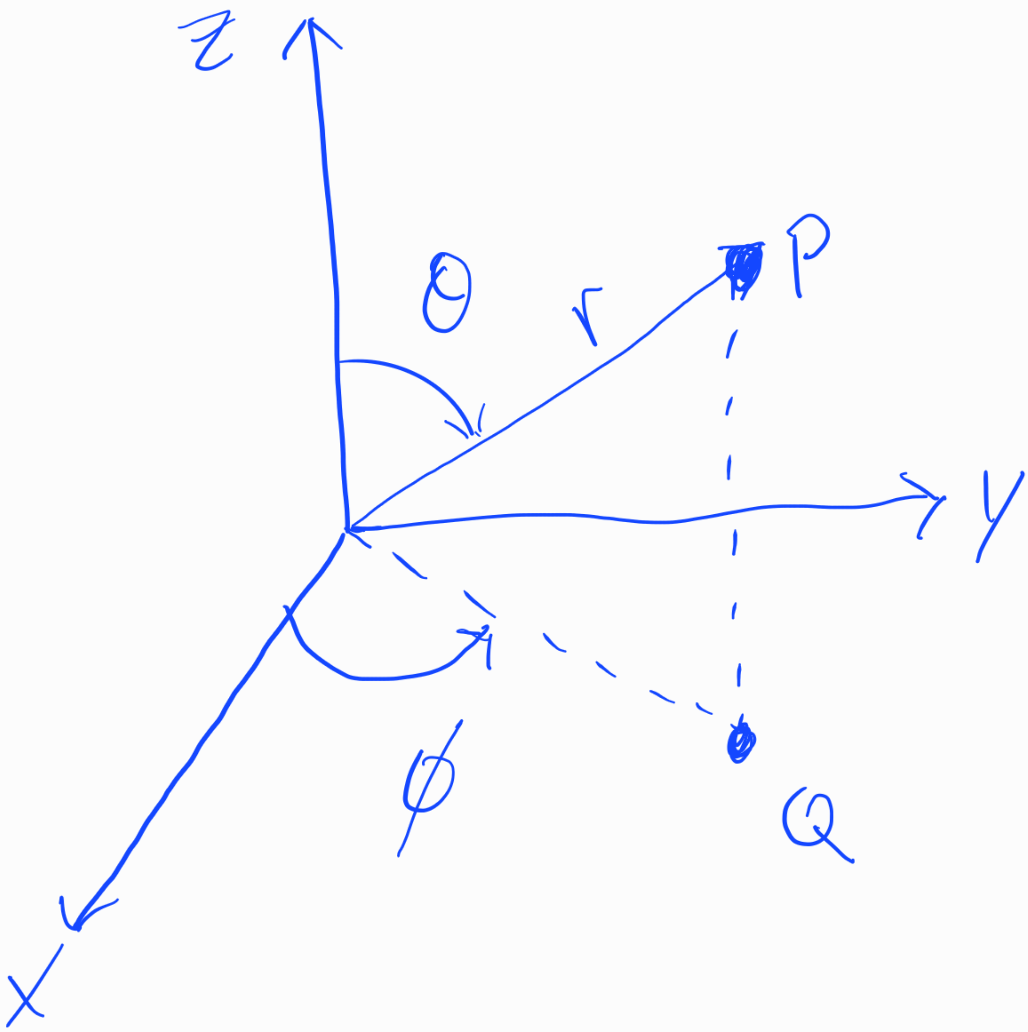
spherically symeteric

$f(\vec{r})$ is independant of direction

So $f(\vec{r}) = f(r)$

If a central force is conservative it is also spherically symetric and visa versa.

Spherical Coordinates



ϕ : azimuth

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

For Earth r = radius of earth

so any point P defined

by (θ, ϕ), z aligned north

θ: latitude (colatitude)

ϕ: longitude

For $f(\vec{r}) = f(r)$

$f(r)$ is independant of

(θ, ϕ)

$$\therefore \frac{\partial f(\vec{r})}{\partial \theta} = 0 \quad \& \quad \frac{\partial f(\vec{r})}{\partial \phi} = 0$$

for spherical symmetry

$\hat{\theta}$ Points towards increasing θ

$\hat{r}$ Points radially outward

$\hat{\phi}$ Points towards increasin ϕ

what is $\vec{a} \cdot \vec{b}$ in spherical coordinates?

$$\vec{a} = a_r \hat{r} + a_\theta \hat{\theta} + a_\phi \hat{\phi}$$

$$\vec{b} = b_r \hat{r} + b_\theta \hat{\theta} + b_\phi \hat{\phi}$$

$$\vec{a} \cdot \vec{b} = a_r b_r + a_\theta b_\theta + a_\phi b_\phi$$

Gradient in Spherical Coordinates

$$\nabla f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$

recall $\quad df = \nabla f \cdot d\vec{r}$

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$df = (\nabla f)_r\,dr + (\nabla f)_\theta\,r\,d\theta + (\nabla f)_\phi\,r\sin\theta\,d\phi$$

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}\partial\phi$$

$$(\nabla f)_r = \frac{\partial f}{\partial r}, \quad (\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}$$

$$(\nabla f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

$$\therefore \quad \nabla f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

Conservative and Spherically Symetric Central Forces

A central Force is conservative if & only if it is spherically symetric

Assume $F(\vec{r})$ is Conservative

$$F(\vec{r}) = -\nabla U = -\hat{r}\frac{\partial U}{\partial r} - \hat{\theta}\frac{1}{r}\frac{\partial U}{\partial \theta} - \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}$$

$$\frac{\partial U}{\partial \theta} = \frac{\partial U}{\partial \phi} = 0$$

$$F(\vec{r}) = -\hat{r}\frac{\partial U}{\partial r}$$

only dependant on r!

4.9 Energy of Interaction of Two Particles

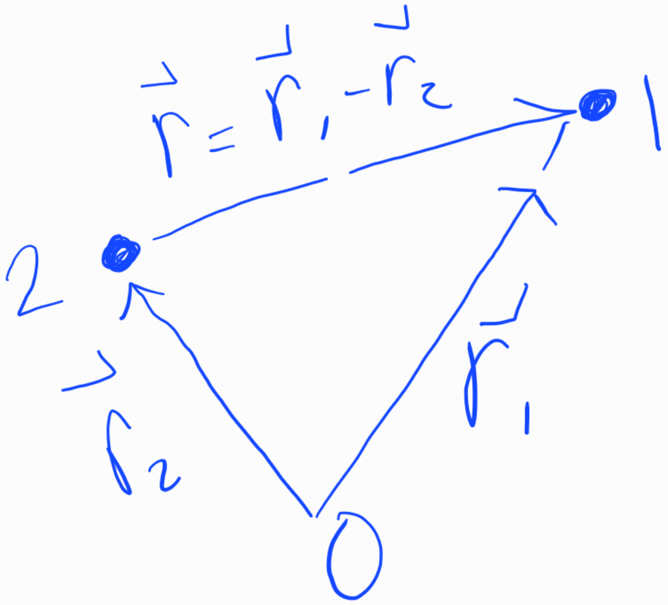
2 Particles interact through Forces: $\vec{F}_{12} = \vec{F}_{12}(\vec{r}_1, \vec{r}_2)$

dependent on the two particles position

Remember Newton's 3rd Law:

$$\vec{F}_{12} = -\vec{F}_{21}$$

Example of two stars



$$\vec{F}_{12} = \frac{-Gm_1m_2}{r^2}\hat{r}$$

$$= \frac{-Gm_1m_2}{r^3}\vec{r}$$

$(\vec{r}_1 - \vec{r}_2)$

$$\vec{F}_{12} = -\frac{Gm_1m_2}{|\vec{r}_1 - \vec{r}_2|^3}\cdot(\vec{r}_1 - \vec{r}_2)$$

$$\vec{F}_{12} = \vec{F}_{12}(\vec{r}_1 - \vec{r}_2)$$

If $\vec{r}_2$ is fixed @ the origin

$$\vec{F}_{12} = \vec{F}_{12}(\vec{r}_1)$$

so,

$$\nabla_1 \times \vec{F}_{12} = 0$$

$$\nabla_1 = \hat{x}\frac{\partial}{\partial x_1} + \hat{y}\frac{\partial}{\partial y_1} + \hat{z}\frac{\partial}{\partial z_1}$$

$$\vec{F}_{12} = -\nabla_1 U(\vec{r}_1)$$

moving r_2 to an arbitrary position replace r_1 w/ $\vec{r}_1 - \vec{r}_2$

$$\vec{F}_{12} = -\nabla_1 U(\vec{r}_1 - \vec{r}_2)$$

∇_1 is uneffected by the addition of a constant

using $F_{12} = -F_{21}$

$$\nabla_1 U(\vec{r}_1 - \vec{r}_2) = -\nabla_2 U(\vec{r}_1 - \vec{r}_2)$$

$$\nabla_2 = \hat{x}\frac{\partial}{\partial x_2} + \hat{y}\frac{\partial}{\partial y_2} + \hat{z}\frac{\partial}{\partial z_2}$$

So instead of changing sign we can swap ∇_1 & ∇_2

$$\vec{F}_{21} = -\nabla_2 U(\vec{r}_1 - \vec{r}_2)$$

$$\vec{F}_{12} = -\nabla_1 U(\vec{r}_1 - r_2)$$

$$(\text{Force on } 1) = -\nabla_1 U$$

$$(\text{Force on } 2) = -\nabla_2 U$$

One Potential energy function can derive both forces!

Conservation of Energy through the work-KE theorem shows:

$$dT_1 = (\text{work on } 1) = d\vec{r}_1 \cdot \vec{F}_{12}$$

$$dT_2 = (\text{work on } 2) = d\vec{r}_2 \cdot \vec{F}_{21}$$

$$dT = dT_1 + dT_2 = W_{tot}$$

$$W_{tot} = d\vec{r}_1 \cdot \vec{F}_{12} + d\vec{r}_2 \cdot \vec{F}_{21}$$

* Replace $\vec{F}_{21}$ by $-\vec{F}_{12}$

$$W_{tot} = (d\vec{r}_1 - d\vec{r}_2) \cdot \vec{F}_{12}$$

* Replace $\vec{F}_{12}$ by $[-\nabla_1 U(\vec{r}_1 - \vec{r}_2)]$

$$W_{tot} = (d\vec{r}_1 - d\vec{r}_2) \cdot [-\nabla_1 U(\vec{r}_1 - \vec{r}_2)]$$

* Replace $\vec{r}_1 - \vec{r}_2$ with $\vec{r}$

$$W_{tot} = d\vec{r} \cdot \nabla U(\vec{r}) = -dU$$

$$dT = W_{tot} = -dU$$

$$d(T+U) = 0$$

$$E = T + U = T_1 + T_2 + U$$

Elastic Collisions

a collision between two particles that interact by a conservative force, that goes to zero as $\vec{r}_1 - \vec{r}_2$ increases.

$$\lim_{\vec{r}_1 - \vec{r}_2 \to \infty} F = 0$$

if $F = 0$, $\nabla U = 0$, $U = \text{const.}$

set const. to 0

Total Energy is conserved

$T + U = \text{const.}$ $(T = T_1 + T_2)$

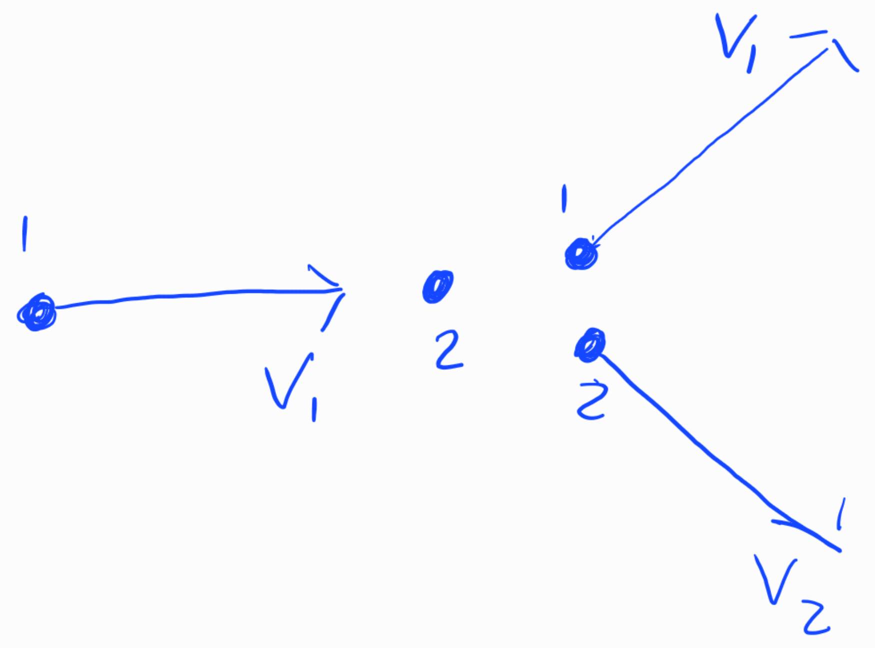
@ $\vec{r} = \vec{r}_1 - \vec{r}_2$ is large $U = 0$

$T_{in} + 0 = T_{fin} + 0$, $T_{in} = T_{fin}$

That is when the Object are seperates by far enough PE is negligable and KE is conserved

EX 4.8

Elastic Collision of two particles of mass $m = m_1 = m_2$, where Particle 2 is initially at rest. Prove the angle between the two outgoing velocities is $\theta = 90^\circ$



Conservation of momentum ($\vec{p}$)

$$\cancel{m}\vec{V}_1 = \cancel{m}\vec{V}_1' + \cancel{m}\vec{V}_2'$$

$$\vec{V}_1 = \vec{V}_1' + \vec{V}_2' \quad (\mathrm{I})$$

Elastic collision KE is conserved

$$\cancel{\tfrac{1}{2}}\cancel{m}\vec{V}_1^2 = \cancel{\tfrac{1}{2}}\cancel{m}\vec{V}_1'^2 + \cancel{\tfrac{1}{2}}\cancel{m}\vec{V}_2'^2$$

$$V_1^2 = \vec{V}_1'^2 + \vec{V}_2'^2 \quad (\mathrm{II})$$

Squaring (I)

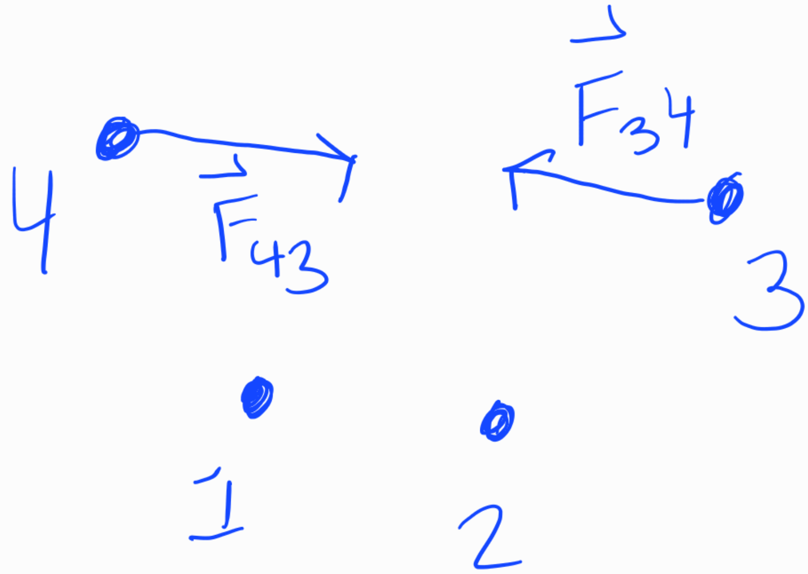
$$V_1^2 = (\vec{V}_1' + \vec{V}_2')^2 = \vec{V}_1'^2 + 2\vec{V}_1' \cdot \vec{V}_2' + \vec{V}_2'^2$$

$$\therefore \vec{V}_1' \cdot \vec{V}_2' = 0 \;, \; \theta = 90^\circ$$

if $V_1' \neq 0$ & $V_2' \neq 0$

The Energy of a Multi-Particle System

Four Particles:



$$T = T_1 + T_2 + T_3 + T_4$$

$$T_\alpha = \frac{1}{2} m_\alpha v_\alpha^2$$

$F_{\alpha\beta}$ unaffected by particles other than α, β

$$U_{34} = U_{34}(\vec{r}_3 - \vec{r}_4)$$

$$F_{34} = -\nabla_3 U_{34} \quad \& \quad F_{43} = -\nabla_4 U_{34}$$

6 Pairs 12, 13, 14, 23, 24, 34

U_{12}, U_{34} (internal)

$$F_\alpha^{ext} = -\nabla_\alpha U_\alpha^{ext}(\vec{r}_\alpha)$$

$$U = U^{int} + U^{ext} = (U_{12} + \dots + U_{34})$$

$$+ (U_1^{ext} + \dots + U_4^{ext})$$