$$\dot{P} = \vec{F}_{ext}, \quad \text{if } \vec{F}_{ext} = 0$$

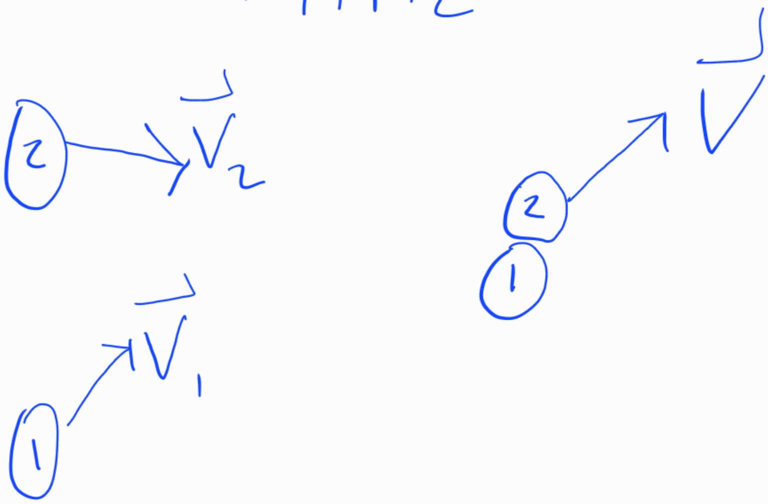
$$\vec{P} = \sum m_\alpha V_\alpha$$

Ex 3.1 (2 Body Perfectly inelastic)

$$P_i = m_1 \vec{V}_1 + m_2 \vec{V}_2$$

$$P_f = m_1 \vec{V} + m_2 \vec{V} = (m_1 + m_2) \vec{V}$$

$$\vec{V} = \frac{m_1 \vec{V}_1 + m_2 \vec{V}_2}{m_1 + m_2}$$



If $\vec{V}_2 = 0$,

$$\vec{V} = \frac{m_1}{m_1 + m_2} \vec{V}_1$$

Rockets

$$P(t) = mV$$

$$P(t+dt) = (m+dm)(V+dV) - dm(V - V_{ex})$$

$$= mV + mdV + dm\, V_{ex}, \quad dm\,dV \approx 0$$

$$dP = P(t+dt) - P(t) = mdV + dm V_{ex}$$

$dP = 0$, $F_{ext} = 0$

$mdV = -dm V_{ex}$, divide by dt

$$m\dot{V} = -\dot{m} V_{ex}$$

Thrust:

$$\text{thrust} = -\dot{m} v_{ex} \quad , (\dot{m} \text{ is negative})$$

$$m\,dv = -dm\, v_{ex}$$

$$dv = -v_{ex} \frac{dm}{m}$$

$$\int dv = -v_{ex} \int \frac{dm}{m}$$

$$v - v_0 = v_{ex} \ln\left(\frac{m_0}{m} \right)$$

Center of Mass

$$\vec{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha = \frac{m\vec{r}_1 + \ldots + m_N\vec{r}_N}{M}$$

$$M = \sum m_\alpha$$

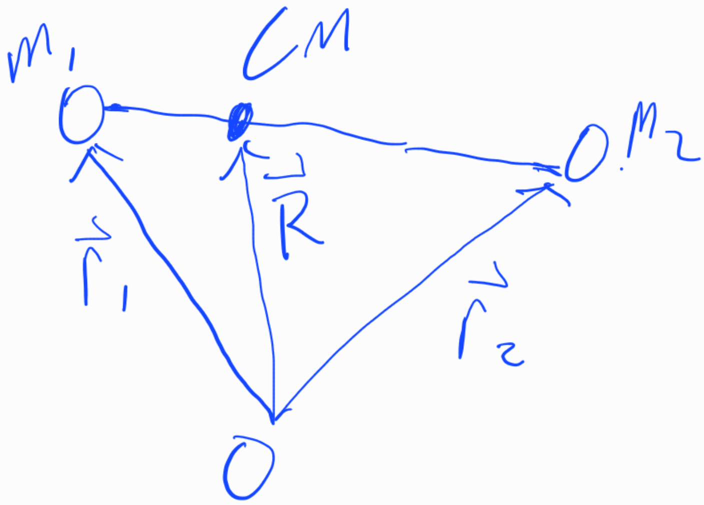
$$X = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha X_\alpha, \quad Y = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha Y_\alpha,$$

$$Z = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha Z_\alpha$$

$N=2$

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

if $m_1 \approx M$, $m_\alpha << M$



Momentum in terms of CM

$$\vec{P} = \sum_\alpha p_\alpha = \sum_\alpha \overbrace{m_\alpha}^{M} \dot{r}_\alpha = M\dot{R}$$

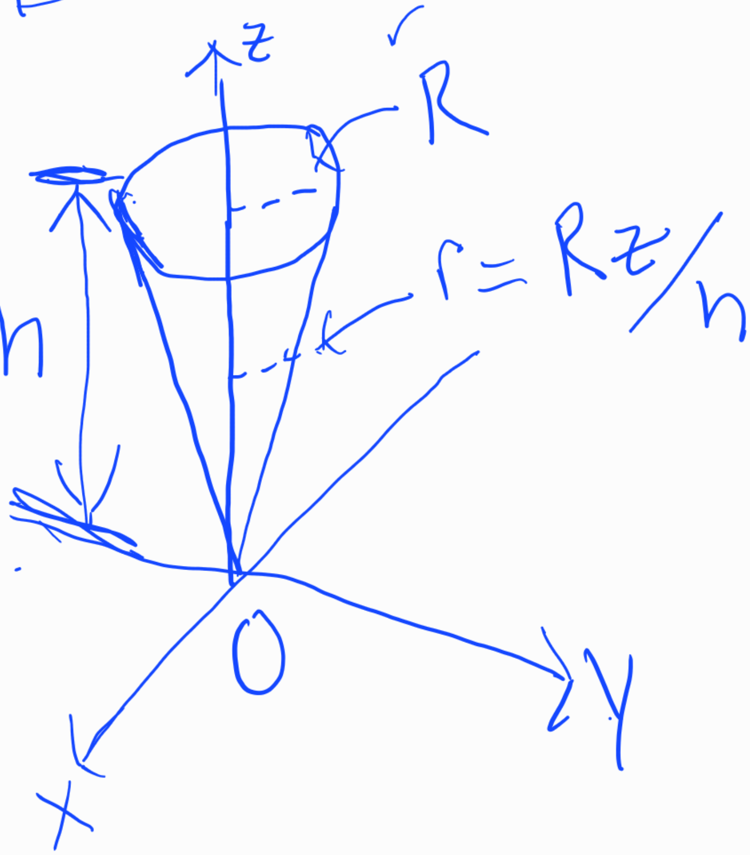
Extending to:

$$F_{ext} = \dot{P} = M\ddot{R}$$

CM for continuos mass distribution:

$$\vec{R} = \frac{1}{M}\int \vec{r}\,dm = \frac{1}{M}\int \underbrace{\underbrace{\rho}_{\text{density}}\,\vec{r}\,dV}_{dm}$$

EX 3.2:



Solid cone symmetric about z

$\therefore \vec{R} = \vec{Z}_{cm}$

$$Z_{cm} = \frac{1}{M}\int \rho z \, dV = \frac{\rho}{M}\int z \, dx \, dy \, dz$$

x, y bound to circle w/ $r = Rz/h$ so

$$\pi r^2 = \pi \frac{R^2 z^2}{h^2} = dx\,dy$$

SO,
$$Z_{cm} = \frac{\rho \pi R^2}{M h^2}\int_0^h h^3 \, dz$$

$$z_{cm} = \frac{\rho \pi R^2}{M h^2} \frac{h^4}{4}$$

$$M = \rho V_{cone} = \frac{1}{3} \rho \pi R^2 h$$

$$\therefore \quad z_{cm} = \frac{3}{4} h$$

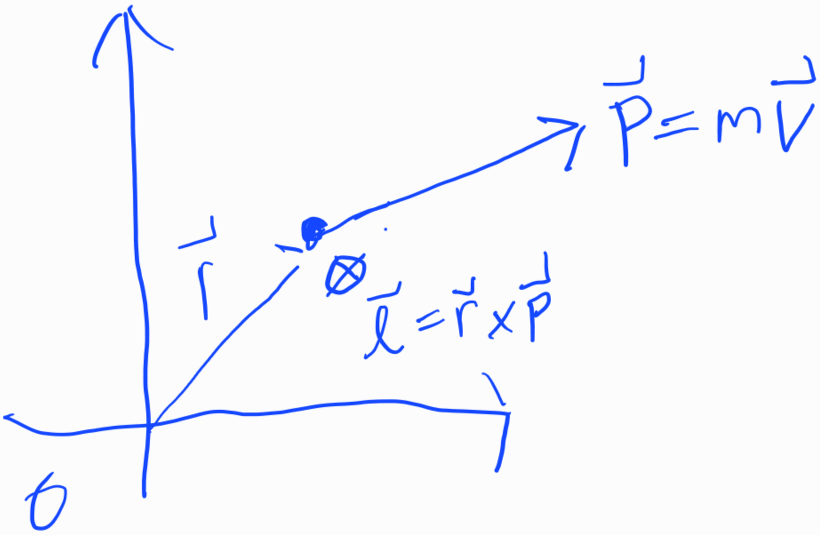
Angular Momentum (single particle)

$$\vec{l} = \vec{r} \times \vec{p}$$

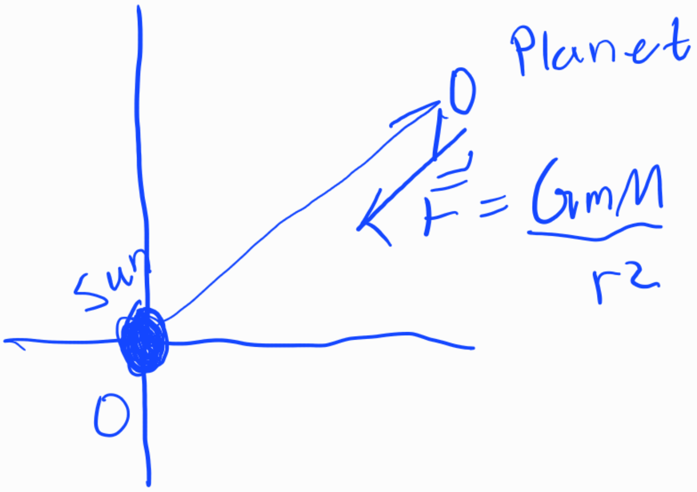
$$\dot{\vec{l}} = \frac{d}{dt}(\vec{r} \times \vec{p}) = (\dot{\vec{r}} \times \vec{p}) + (\vec{r} \times \dot{\vec{p}})$$

$$= \cancel{(\dot{\vec{r}} \times m\dot{\vec{r}})}^{\,0} + (\vec{r} \times \dot{\vec{p}})$$

$$\dot{\vec{l}} = \vec{r} \times \vec{F}_{ext} = \vec{\Gamma}$$



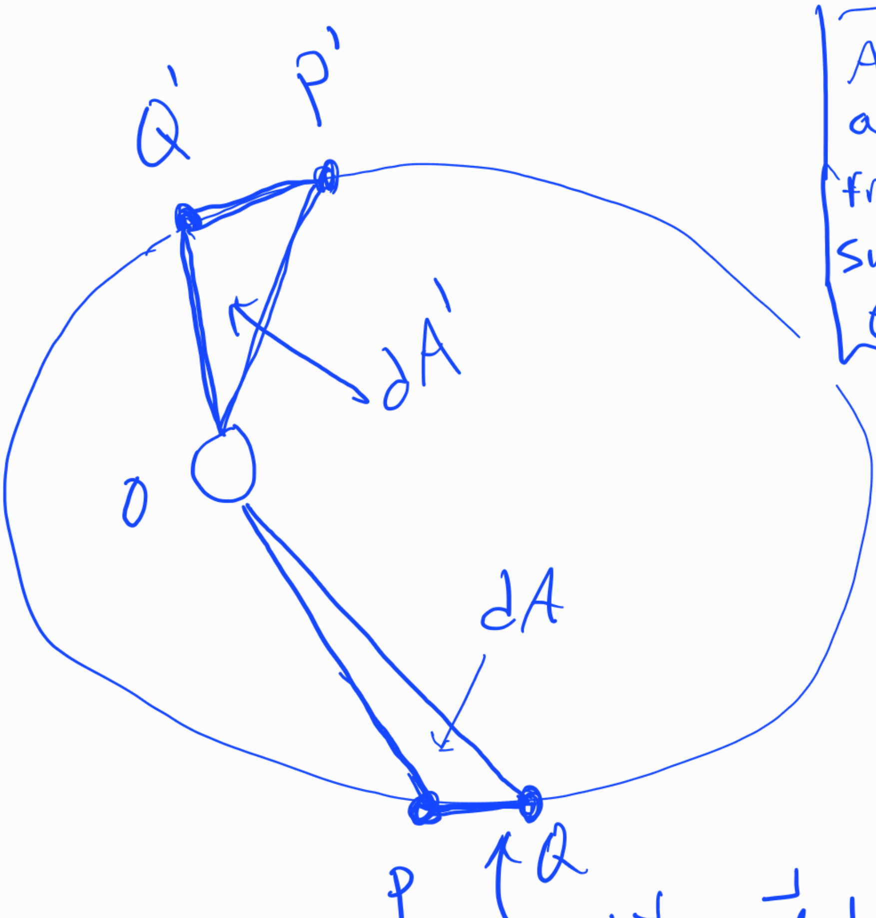
Kepler's Second Law:



$\vec{F} = \frac{GmM}{r^2}$, $\vec{\Gamma} = \vec{r} \times \vec{F} = 0$, $\vec{l} = \text{const}$

Central Force

Since $\vec{r} \times \vec{P}$ is constant ($\vec{l} = \text{const}$)
$\vec{r}$ & $\vec{P}$ must be constrained to a single plane



As each planet moves around the sun, a line drawn from the planet to the sun sweeps out equal areas in equal times *

$dt = dt'$

$d\vec{r} = \vec{v}\,dt$

$dA = \frac{1}{2}\left|\vec{r} \times \vec{V} dt\right|$, Area of triangle OPQ

replace $\vec{V}$ by $\vec{P}/m$ & divide by dt,

$$\frac{dA}{dt} = \frac{1}{2m}\left|\vec{r} \times \vec{P}\right| \frac{\cancel{dt}}{\cancel{dt}} = \frac{1}{2m}\vec{l}$$

$\vec{l}$ is a const.

$$\boxed{\therefore \frac{dA}{dt} \text{ is const}}$$

Angular Momentum for Several Particles:

N particles, $\alpha = 1, 2, 3, \ldots, N$

$$l_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha \quad (\text{all } \vec{r}_\alpha \text{ measured from } O)$$

Total Angular Momentum $\vec{L}$

$$\vec{L} = \sum_{\alpha=1}^{N} l_\alpha = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{p}_\alpha$$

$$\dot{L} = \sum_\alpha \dot{l}_\alpha = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$$

$$F_\alpha = \sum_{\alpha \neq \beta} F_{\alpha\beta} + F_\alpha^{ext}$$

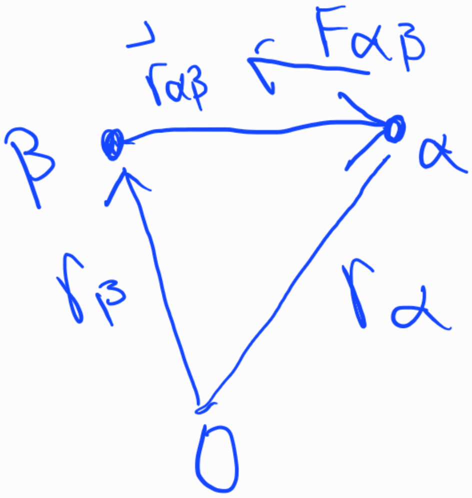
$$\dot{L} = \sum_\alpha \sum_{\beta \neq \alpha} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha^{ext}$$

Remember $F_{21} = -F_{12}$

$$\sum_{\alpha}\sum_{\beta\neq\alpha} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} = \sum_{\alpha}\sum_{\beta>\alpha} \left(\vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \vec{r}_\beta \times \vec{F}_{\beta\alpha}\right)$$

$$= \sum_{\alpha}\sum_{\beta>\alpha} \left(\vec{r}_\alpha - \vec{r}_\beta\right) \times \vec{F}_{\alpha\beta}$$

Look @ $(\vec{r}_\alpha - \vec{r}_\beta) = \vec{r}_{\alpha\beta}$



∴ all $F_{\alpha\beta}$ are central forces

& $\vec{r}_{\alpha\beta} \times \vec{F}_{\alpha\beta} = 0$

$$\dot{L} = \vec{\Gamma}_{ext}$$

If $\Gamma_{ext} = 0$ then

$\vec{L} = \sum \vec{r}_\alpha \times \vec{P}_\alpha$ is constant

Conservation of angular momentum

Moment of Inertia:

from general Phys

$L = I\omega$, I = moment of inertia

Disk (center rotatio)

$$I = \frac{1}{2} M R^2$$

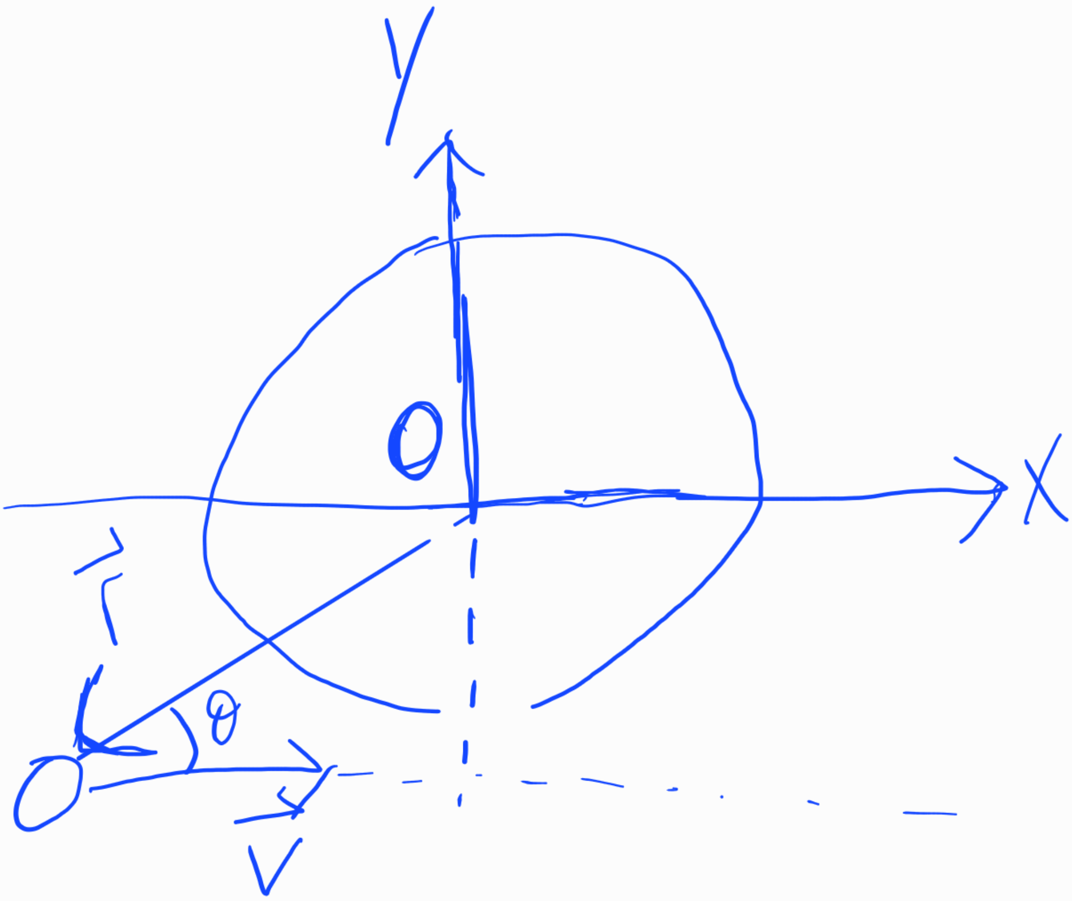
Sphere

$$I = \frac{2}{5} M R^2$$

In general (mult-Particle)

$$I = \sum_\alpha m_\alpha \rho_\alpha^2, \quad \rho_\alpha = \text{distance from axis of rotation}$$

Ex 3.3 Collision of a Lump of Putty w/ Turntable



$\Gamma = 0 \quad L_z = \text{Const.}$

$l_{putty} = \vec{r} \times \vec{P} [\hat{z}]$

$L_{z_0} = l_z = r(mv)\sin\theta$

$= mvb$

$I = (m + M/2) R^2$

$L_{z_f} = I\omega$

$L_{z_0} = L_{z_f}$

$$mvb = \left(m + \frac{M}{2}\right) R^2 \omega$$

$$\omega = \frac{m}{\left(m + \frac{M}{2}\right)} \cdot \frac{vb}{R^2}$$

Angular Momentum about the CM

$$\dot{L} = \Gamma_{ext}$$

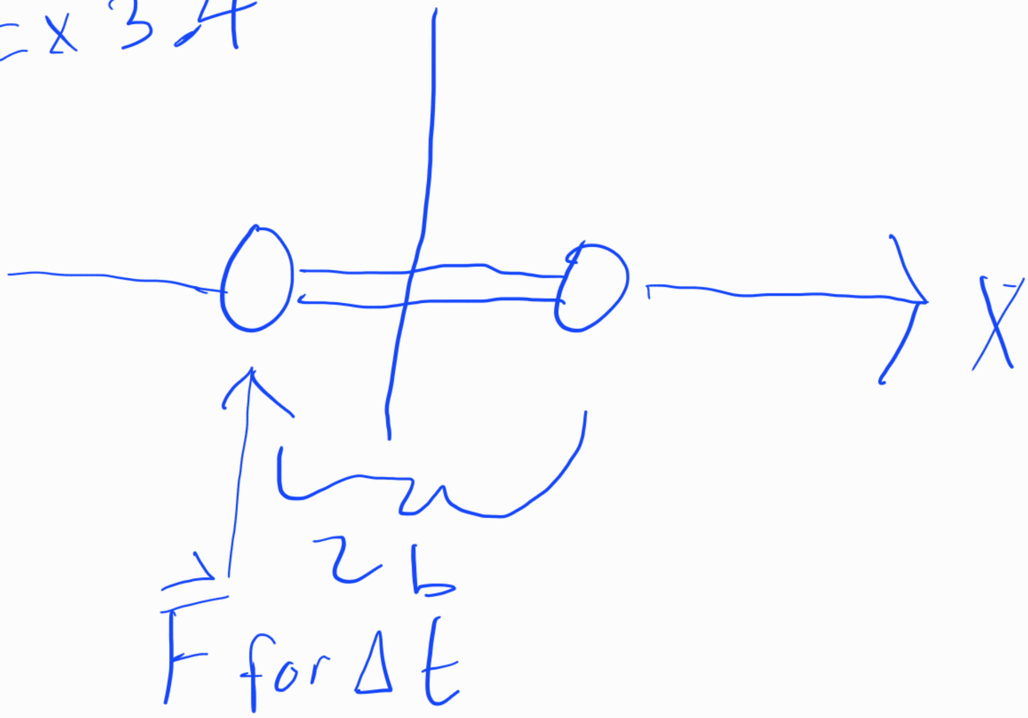
$$\frac{d}{dt} \dot{L}(\text{about CM}) = \Gamma^{ext}(\text{about CM})$$

if $\Gamma^{ext}_{cm} = 0$, $L_{CM} = \text{cons.}$

even if CM is moving

or accelerating

Ex 3.4



$$\dot{\vec{P}} = \vec{F}^{ext}, \quad \vec{P} = \vec{F}\Delta t$$

$$\vec{P} = M\dot{R} = (2m)\dot{R} = F\Delta t$$

$$\vec{V}_{cm} = \dot{R} = F\Delta t / 2m$$

$$\vec{\Gamma}_{ext} = Fb$$

$$\frac{d}{dt} L = \vec{\Gamma}_{ext}$$

$$L = \int \vec{\Gamma}_{ext}\, dt$$

$$L = F b \Delta t$$

$$L = I w \quad , \quad I = 2 m b^2$$

$$L = 2 m b^2 w$$

$$w = F\Delta t / 2mb$$

$$V_{left} = V_{cm} + wb = \frac{F\Delta t}{m}$$

$$V_{right} = V_{cm} - wb = 0$$