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SECOND-ORDER ASYMPTOTIC APPROXIMATIONS FOR STELLAR NONRADIAL ACOUSTIC MODES

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ABSTRACT

The problem of linear, nonradial pulsations in spherical stellar models is reconsidered using second-order asymptotic expansions. Acoustic modes of high radial order and low degree are investigated without making use of the so-called Cowling approximation. Detailed solutions are obtained for these modes. Comparison is made between these new results and those obtained when Cowling's approximation has been used. The connection between Olver's asymptotic theory and the so-called plane-wave method is also discussed.

Subject headings: stars: pulsation — Sun: oscillations — wave motions

I. INTRODUCTION

Asymptotic theory has proved useful in the analysis and interpretation of the many acoustic modes of oscillation that are observed at the surface of the Sun (e.g., Jiménez *et al.* 1988; Duvall *et al.* 1988). To calculate these modes, one must solve the fourth-order system of differential equations that governs the linear, adiabatic oscillations of a self-gravitating sphere in hydrostatic equilibrium. As was originally pointed out by Cowling (1941), however, for modes of high radial order or high degree the Eulerian perturbation in the gravitational potential could be neglected without significantly modifying the frequencies. This is the so-called Cowling approximation, which reduces the order of the system of differential equations from four to two. Thence, by making use of this approximation, one can easily apply Olver's (1974) theory to obtain second-order asymptotic solutions for the acoustic modes of high radial order and low degree (Tassoul 1980; Smeyers and Tassoul 1987; hereafter T80 and ST87, respectively).

Yet, recent work (e.g., Gabriel 1989) seems to indicate that Cowling's approximation may not be adequate to discuss the solar acoustic modes. This is the reason why I shall present in this paper second-order asymptotic solutions that are based on the full fourth-order system of differential equations, thus without making use of Cowling's approximation. Not unexpectedly, these new solutions for the nonradial acoustic modes generalize the second-order asymptotic solutions that were obtained for the radial pulsations (Tassoul and Tassoul 1968, hereafter TT68). However, since §§ III–V present highly technical matters, I recommend that one start the first reading of this paper with § VI, which can be read without going through all the mathematics. This final section also presents a detailed comparison between Olver's asymptotic theory and the so-called plane-wave approach.

II. BASIC EQUATIONS

We restrict ourselves to a star in hydrostatic equilibrium, without rotation or magnetic field. The linearized equations governing the nonradial, adiabatic pulsations of such an object have been established by Pekeris (1938). Because these equations are linear, one can search for normal-mode solutions. In other words, one may assume that the time dependence of the perturbations is given by a factor exp $(i\sigma t)$. Moreover, the dependence on the angular variables (θ, φ) can be separated from that on the radial variable r by means of spherical harmonics Y_l^m. For example, the r and θ components of the Lagrangian displacement $\boldsymbol{\xi}$ are written as follows:

$$\xi_r(r, \theta, \varphi, t) = \xi(r) Y_l^m(\theta, \varphi) \exp(i\sigma t) ,$$

$$\xi_\theta(r, \theta, \varphi, t) = \eta(r) \frac{\partial}{\partial \theta} Y_l^m(\theta, \varphi) \exp(i\sigma t) . \tag{1}$$

If one inserts this separation of variables into the equations of motion, of continuity, of adiabaticity, and in Poisson's equation, one obtains a fourth-order differential system involving, besides the Lagrangian displacement, the Eulerian perturbations in the pressure δP, in the density $\delta\rho$, and in the gravitational potential $\delta\Phi$. If one defines

$$X = \operatorname{div} \boldsymbol{\xi} = \frac{1}{r^2} \frac{d}{dr} (r^2 \xi) - \frac{l(l+1)}{r} \eta , \tag{2}$$

these equations can be combined to give

$$K_1 \frac{d\xi}{dr} + \frac{d^2 X}{dr^2} + K_2 \frac{dX}{dr} + \left(\sigma^2 \varphi + K_3 + \frac{K_4}{\sigma^2}\right) X = 0 , \tag{3}$$

$$\frac{d^2\xi}{dr^2} + \frac{4}{r}\frac{d\xi}{dr} + \frac{2 - \Lambda}{r^2} \xi - \frac{dX}{dr} - \left(\frac{2}{r} - \frac{\Lambda N^2}{\sigma^2 r^2} \frac{c^2}{g}\right) X = 0 , \tag{4}$$

where we have introduced the following quantities

$$\varphi = \frac{\rho}{\Gamma_1 P} = \frac{1}{c^2}, \tag{5}$$

$$K_1 = -\frac{2g}{c^2}\left(\frac{1}{g}\frac{dg}{dr} - \frac{1}{r}\right), \tag{6}$$

$$K_2 = \frac{2}{c^2}\frac{dc^2}{dr} + \frac{1}{\rho}\frac{d\rho}{dr} + \frac{2}{r}, \tag{7}$$

$$K_3 = -\frac{\Lambda}{r^2} + \frac{2}{c^2}\left(\frac{g}{r} + \frac{dg}{dr}\right) + \frac{1}{c^2}\frac{d^2c^2}{dr^2} + \frac{1}{\rho}\frac{d^2\rho}{dr^2} + \frac{1}{\rho}\frac{d\rho}{dr}\left(\frac{1}{c^2}\frac{dc^2}{dr} - \frac{1}{\rho}\frac{d\rho}{dr}\right) + \frac{2}{r}\left(\frac{1}{c^2}\frac{dc^2}{dr} + \frac{1}{\rho}\frac{d\rho}{dr}\right), \tag{8}$$

$$K_4 = \Lambda\frac{N^2}{r^2}, \tag{9}$$

and where $\Lambda = l(l+1)$, g is gravity, c is the sound speed, N is the Brunt-Väisälä frequency, and Γ_1 is one of the generalized adiabatic coefficients.

Let us first recall a result that will be useful later: the "horizontal component" η of the Lagrangian displacement is related to δP, $\delta\Phi$, ξ, and X by

$$r\eta = \frac{1}{\sigma^2}\left(\delta\Phi + \frac{\delta P}{\rho}\right) = \frac{1}{\sigma^2}(\delta\Phi + g\xi - c^2X). \tag{10}$$

Note also that for the homogeneous model, gravity is proportional to r, implying the vanishing of K_1. Equation (3) is then decoupled from equation (4) (since it involves the function X only) and, as a result, σ is thus the eigenvalue of a second-order problem. In the case of more complex configurations, the problem is a genuine fourth-order problem.

To be physically acceptable, the solution of equations (3) and (4) has to satisfy the usual regularity boundary conditions: ξ and X have to be regular everywhere, and in particular, at the center ($r = 0$) and at the surface ($r = R$) of the star. (Needless to say, we assume here the so-called zero-boundary conditions, i.e., we assume that the pressure and the density vanish at the surface.) In addition, because Poisson's equation has been used to eliminate the perturbation in the gravitational potential, one must also express the continuity of gravity across the surface of the star, i.e.,

$$\frac{d}{dr}\delta\Phi + (l+1)\frac{\delta\Phi}{R} = -4\pi G\rho\xi, \quad \text{at} \quad r = R. \tag{11}$$

In general, since we are considering configurations for which ρ vanishes at the surface, the right-hand side of this equation may be neglected. Furthermore, it is also worth noticing that, with the help of equations (2) and (10), condition (11) can be easily translated into a condition on ξ and X. There is no additional constant of integration involved.

The elimination of ξ between equations (3) and (4), although possible, leads to an extremely cumbersome equation that has the following structure

$$\mathscr{P}X + \sigma^2\mathscr{Q}X + \frac{1}{\sigma^2}\mathscr{R}X = 0, \tag{12}$$

where $\mathscr{P}$, $\mathscr{Q}$, and $\mathscr{R}$ are differential operators, $\mathscr{P}$ being of the fourth order, and $\mathscr{Q}$ and $\mathscr{R}$ being of the second order. Note, however, that these three operators are independent of σ^2. As pointed out by Ledoux and Walraven (1958), this is the only fourth-order differential equation that can be derived from the basic equations and which is free of variable singularities.

III. TENTATIVE SOLUTION OF EQUATION (12)

The singular points of equation (12) are $r = 0$ and $r = R$. Near $r = 0$, this equation reduces to

$$\frac{d^4X}{dr^4} + \frac{4}{r}\frac{d^3X}{dr^3} + \left(\omega^2 - \frac{2\Lambda}{r^2}\right)\frac{d^2X}{dr^2} + \frac{2\omega^2}{r}\frac{dX}{dr} + \left[-\frac{\Lambda\omega^2}{r^2} + \frac{\Lambda(\Lambda-2)}{r^4}\right]X = 0, \tag{13}$$

where

$$\omega^2 = \left(\frac{\sigma^2\rho}{\Gamma_1 P} + \frac{\Lambda N^2}{r^2\sigma^2}\right)_{r=0}. \tag{14}$$

(For convenience, we have assumed $N^2 > 0$, so that ω is always real, whether σ^2 is large or small. For high enough p-modes, the sign of N^2 is irrelevant, since the second term in equation (14) can then be assumed to be small with respect to the first one.) Linearly independent solutions of equation (13) are the following

$$X = r^l, \quad r^{-l-1}, \quad r^{-1/2}J_{l+1/2}(\omega r), \quad r^{-1/2}Y_{l+1/2}(\omega r), \tag{15}$$

where J and Y are the standard Bessel functions. Of course, the singular solutions (i.e., r^{-l-1} and $r^{-1/2}Y_{l+1/2}$) have to be discarded.

Near $r = R$, we let

$$x = R - r \,, \tag{16}$$

and we define, instead of equation (14),

$$\omega^2 = \lim_{r \to R} \left[(R - r)\left(\frac{\sigma^2 \rho}{\Gamma_1 P} + \frac{\Lambda N^2}{\sigma^2 r^2} \right) \right] . \tag{17}$$

Obviously, we assume very simple conditions near the surface of the star. As is usual in this kind of analysis, we assume that the surface of the star can be assimilated to a nearly polytropic atmosphere of index n. Close to the surface thus, equation (12) reduces to

$$\frac{d^4 X}{dx^4} + \frac{n+4}{x}\frac{d^3 X}{dx^3} + \frac{\omega^2}{x}\frac{d^2 X}{dx^2} = 0 \,, \tag{18}$$

whose general solution is a combination of the following four

$$X = 1 \,, \qquad x \,, \qquad x^{-(n+1)/2} J_{n+1}(2\omega x^{1/2}) \,, \qquad x^{-(n+1)/2} Y_{n+1}(2\omega x^{1/2}) \,. \tag{19}$$

Here again only the regular solutions have to be retained.

These results suggest that, for large (positive) values of σ^2, the asymptotic expansion for X can be written as the sum of two parts: a first one in which one expresses X as a straightforward series in $1/\sigma^2$ (generalizing the power-like solutions in equations [15] and [19]) and a second one for which one adapts the method that Olver (1974) devised for second-order differential equations with a large parameter.

a) *The Nonoscillating Part*

To approximate this part of the eigenfunction, we look for solutions of the form

$$X(r, \sigma) = \sum_{i=0}^{N} \frac{1}{\sigma^{2i}} X_i(r) \,. \tag{20}$$

After inserting this series into equation (12) and identifying the terms in the same powers of σ^2, we find the conditions

$$\mathscr{Q} X_0 = 0 \,, \tag{21}$$

$$\mathscr{Q} X_i = -\mathscr{P} X_{i-1} - \mathscr{R} X_{i-2} \tag{22}$$

($i > 0$; by convention, a quantity with a negative index is zero). Two particular solutions of equation (21) are the following

$$X_0 = (r^{l-2} \quad \text{and} \quad r^{-l-3}) \left(\frac{dg}{dr} - \frac{g}{r} \right) . \tag{23}$$

Obviously, only the first one is acceptable. Thence, in principle, the next X_i's can be derived: they are the solution of successive inhomogeneous second-order differential equations (22). Note also that $(dg/dr - g/r)$ is $O(r^2)$; so that X_0 is always regular, whatever the value of l.

To complete the solution, we must evaluate the corresponding solution for ξ. Expanding ξ in the same manner as in equation (20) and inserting this expansion into equation (3) or (4), we find

$$\xi = \frac{\sigma^2}{2(l-1)} r^{l-1} \left[1 + \frac{1}{\sigma^2} O(1) \right] . \tag{24}$$

Let us note here that, for $l = 0$, ξ is not regular; therefore this solution must be discarded (together with the corresponding X_0). On the other hand, since the amplitude in the partial solutions (23) and (24) are not specified, in the case when $l = 1$ the only acceptable solution is $X_0 \equiv 0$ (and $X \equiv 0$, as well) and $\xi =$ constant. (In fact, this is probably the reason why it has been prematurely concluded that the $l = 1$ p-modes had to be disregarded because they consisted only in a lateral displacement of the star as a whole.)

Expressions (23) and (24) are reminiscent of the f-modes in the homogeneous model, for which we have indeed $X \equiv 0$ and $\xi \propto r^{l-1}$. In what follows, we will therefore denote this part of the solution as the f-like part. (Obviously, there is no such part for the radial pulsations!)

b) *The Oscillating Part*

The third solutions in equations (15) and (19) suggest, on the other hand, that in order to derive another type of expansion of the solution of equation (12), one proceeds along the same lines as in T80 and ST87. We will thus refer to this part of the solution as the p-like part. In this approach, thus, one divides the radius into two domains, each one of which contains only one "turning" point (here a singularity). We thus have an inner domain extending outward from the center, and an outer domain extending inward from the surface. In each domain, one introduces new dependent and independent variables. The independent variable u is chosen in such a way that it is proportional either to r (near $r = 0$) or to x (near $r = R$). We then introduce a new dependent variable by writing

$$X(r, \sigma) = h(r) X_1(u, \sigma) \,. \tag{25}$$

For the moment, the function $h(r)$ is arbitrary. We also assume that X_1 can be written as

$$X_1(u, \sigma) = \sum_i \frac{A_i(u)}{\sigma^{2i}} P_1(u, \sigma) + \frac{1}{\sigma^2} \sum_i \frac{B_i(u)}{\sigma^{2i}} \frac{\partial P_1(u, \sigma)}{\partial u},$$

with

$$P_1 = u^{1/2} J_\nu\left(\frac{2}{j+2} \sigma u^{1+j/2}\right), \tag{26}$$

where ν is again arbitrary; $j = 0$ in the inner domain, and $j = -1$ in the outer domain. (In what follows we will use the short-hand notation A when we actually mean the whole series of the A_i/σ^{2i}'s; this applies also to B and similar series that will be introduced later.) These changes of variables are inserted into equation (12), together with formal solution (26). By using Bessel differential equation, we can express the resulting expression in terms of P_1 and its first derivative. Because these two functions are linearly independent, we thus obtain two conditions for the two unknown series A and B. Now the function $h(r)$ is determined by requiring that the first term in series A (i.e., A_0) be a constant. Thus we find

$$h(r) = r^{-1} \rho^{-1/2} \varphi^{3/4}. \tag{27}$$

Next we determine the index ν of the Bessel function in such a way that each term in series A and B are of the same order (with respect to u) near $u = 0$. (Obviously, this value of ν has to be the same as in eqs. [15] or [19].) In practice, the two recurrence relations between the individual A_i's and B_i's are so cumbersome that practically nothing can be derived. For example, one cannot be assured that all A_i's are $O(1)$. Indeed, even though $h(r)$ is chosen so that A_0 is $O(1)$, the expression for A_1 contains terms that are $O(u^{-4})$; if ν is suitably chosen one can suppress this singularity, but we are still left with terms that are $O(u^{-2})$, and these cannot be shown to be regular!

It thus seems more promising to use equations (3) and (4), rather than equation (12). The scanty information we gained in equations (26) and (27) will nevertheless prove very helpful in the next sections.

IV. PARTIAL SOLUTION OF SYSTEM (3)–(4)

As we have seen in the previous section, the f-part of the solution does not present any practical problem. We thus turn our attention to the p-part of the solution.

As we have recalled, the method involves first the need to subdivide the radius in an inner and an outer domain. In each domain one then performs changes of dependent and independent variables. We thus write

$$X = h(r) X_1(z, \sigma), \quad \text{and} \quad \xi = f_\xi(r) \xi_1(z, \sigma). \tag{28}$$

(To simplify the notation we will often omit the index "in" or "out": either when it is obvious from the context, or when it is immaterial.) We will depart from the conventional approach in two minor respects. A moment of reflection will convince the reader that these modifications are of no consequence. In particular, we will choose the independent variable z as follows

$$z_{\rm in} = \int_0^r \varphi^{1/2}\, dr; \qquad z_{\rm out} = \int_r^R \varphi^{1/2}\, dr. \tag{29}$$

Obviously, $z_{\rm in}$ satisfies the requirement that $z \propto r$ as $r \to 0$. This is not true for $z_{\rm out}$! However, the present choice of independent variable is supported by the following considerations. First, past experience (T80; ST87) has shown that it is indeed (σz) that is the argument of the Bessel functions, in spite of the use of the more conventional variable. Second, because the nonstandard variable $z_{\rm out}$ is related to the standard variable u_o (see T80, eq. [57]) by $u_o = (z_{\rm out}/2)^2$, any regular function $A(u_o)$ will also be regular if considered a function of $z_{\rm out}$! (In this connection let us recall that the asymptotic approximation to the radial pulsations has indeed been studied in terms of $z_{\rm out}$, instead of u_o.) As far as the functions $h(r)$ and $f_\xi(r)$ are concerned, we will define $h(r)$ as in equation (27), and we will also choose $f_\xi(r)$ in such a way that

$$h(r) = f_\xi(r) \frac{dz}{dr}. \tag{30}$$

This last definition will later prove convenient, but is obviously not essential. Indeed, since z is a function of r only, if $f_1(r)$ and $f_2(z)$ are two arbitrary functions, their product may also be considered a function of r.

We next assume that X_1 and ξ_1 can be formally written as follows

$$X_1 = \sum_{i=0}^{\infty} \frac{A_i(z)}{\sigma^{2i}} P_1(z, \sigma) + \frac{1}{\sigma} \sum_{i=0}^{\infty} \frac{B_i(z)}{\sigma^{2i}} P_2(z, \sigma),$$

$$\xi_1 = \sum_{i=0}^{\infty} \frac{C_i(z)}{\sigma^{2i}} P_1(z, \sigma) + \frac{1}{\sigma} \sum_{i=0}^{\infty} \frac{D_i(z)}{\sigma^{2i}} P_2(z, \sigma). \tag{31}$$

The functions P_1 and P_2 are defined as

$$P_1 = z^{1/2} J_{a-1/2}(\sigma z), \qquad P_2 = z^{1/2} J_{a+1/2}(\sigma z), \tag{32}$$

and satisfy the following recurrence relations

$$\frac{dP_1}{dz} = \frac{a}{z} P_1 - \sigma P_2 , \qquad \frac{dP_2}{dz} = \sigma P_1 - \frac{a}{z} P_2 . \tag{33}$$

(Compare with eq. [26]. The present form is preferred because, as we will see later, in the case when P_1 is the first approximation to X_1, P_2 is the first approximation to ξ_1.) In practice, the series in the formal solution (31) are limited to the first few terms.

We insert these transformations into equations (3) and (4) and use the recurrence relations (33) between P_1 and P_2 and their derivatives. Again, because these two functions are linearly independent, equation (3) leads to the two sets of equations

$$2A_{i+1,z} = B_{i,zz} - \frac{2a}{z} B_{i,z} + \left[\frac{a(a+1)}{z^2} + L_3\right]B_i + L_4 B_{i-1} + L_1\left[-C_{i+1} + D_{i,z} + \left(-\frac{a}{z} + L_0\right)D_i\right] = 0 ; \tag{34}$$

$$-2B_{i,z} = A_{i,zz} + \frac{2a}{z} A_{i,z} + \left[\frac{a(a-1)}{z^2} + L_3\right]A_i + L_4 A_{i-1} + L_1\left[D_i + C_{i,z} + \left(\frac{a}{z} + L_0\right)C_i\right] = 0 , \tag{35}$$

where we have introduced the following quantities:

$$L_0 = \frac{1}{z'}\left(-\frac{1}{2}\frac{\rho'}{\rho} + \frac{1}{4}\frac{\varphi'}{\varphi} - \frac{1}{r}\right), \tag{36}$$

$$L_1 = -2g' + \frac{2g}{r} , \tag{37}$$

$$L_3 = \frac{1}{z'^2}\left[\frac{1}{2}\frac{\rho''}{\rho} - \frac{3}{4}\frac{\rho'^2}{\rho^2} + \frac{1}{r}\frac{\rho'}{\rho} - \frac{1}{4}\frac{\varphi''}{\varphi} + \frac{5}{16}\frac{\varphi'^2}{\varphi^2} - \frac{\Lambda}{r^2} + 2g\varphi\left(\frac{1}{r} + \frac{g'}{g}\right)\right], \tag{38}$$

$$L_4 = \frac{1}{z'^2}\frac{\Lambda N^2}{r^2} . \tag{39}$$

(A prime denotes a derivative with respect to r, while an index z denotes a derivative with respect to z.) These relations are valid for any (positive or negative) value of i: we define quantities with negative subscripts to be zero. (This also applies to the other relations involving individual A_i, B_i, ...). In the same way, equation (4) is equivalent to the following two sets of conditions:

$$D_{i+1} = A_{i+1} - B_{i,z} + \left(\frac{a}{z} - L_5\right)B_i - L_6 B_{i-1} - 2C_{i+1,z} - L_7 C_{i+1} + D_{i,zz} + \left(-\frac{2a}{z} + L_7\right)D_{i,z} + \left[\frac{a(a+1)}{z^2} - \frac{a}{z} L_7 + L_8\right]D_i = 0 ; \tag{40}$$

$$C_{i+1} = -B_i - A_{i,z} + \left(-\frac{a}{z} - L_5\right)A_i - L_6 A_{i-1} + 2D_{i,z} + L_7 D_i + C_{i,zz} + \left(\frac{2a}{z} + L_7\right)C_{i,z} + \left[\frac{a(a-1)}{z^2} + \frac{a}{z} L_7 + L_8\right]C_i = 0 , \tag{41}$$

where

$$L_5 = \frac{1}{z'}\left(-\frac{1}{2}\frac{\rho'}{\rho} + \frac{3}{4}\frac{\varphi'}{\varphi} + \frac{1}{r}\right), \tag{42}$$

$$L_6 = -\frac{1}{z'}\frac{\Lambda N^2}{r^2 \varphi g} , \tag{43}$$

$$L_7 = \frac{1}{z'}\left(-\frac{\rho'}{\rho} + \frac{\varphi'}{\varphi} + \frac{2}{r}\right), \tag{44}$$

$$L_8 = \frac{1}{z'^2}\left[-\frac{1}{2}\frac{\rho''}{\rho} + \frac{3}{4}\frac{\rho'^2}{\rho^2} - \frac{1}{r}\frac{\rho'}{\rho} + \frac{1}{4}\frac{\varphi''}{\varphi} - \frac{3}{16}\frac{\varphi'^2}{\varphi^2} - \frac{1}{4}\frac{\varphi'\rho'}{\varphi\rho} + \frac{1}{2r}\frac{\varphi'}{\varphi} - \frac{\Lambda}{r^2}\right]. \tag{45}$$

The expressions (34), (35), (40), and (41) can be considered as recurrence relations between A_i, B_i, C_i, D_i. A glance at these equations implies the following order: relations (41), (34), (40), and (35) will determine C_i, A_i, D_i, and B_i, respectively. Obviously, C_0, in the notation used in equation (31), always vanishes. Next, because equation (34) is a differential one, one is at liberty to assume A_0 to vanish or not. Of course, if one chooses $A_0 \equiv 0$, then one cannot assume later that B_0 also vanishes: we are considering linear perturbations, so that the amplitudes are arbitrary.

To proceed any further, we now particularize these relations in each of the two domains defined before. In order to facilitate future discussion, we give in Table 1, the behavior, near the center and the surface, of the many known functions that appear in equations (34), (35), (40), and (41).

a) Solution in the Inner Region

From § III, we know that the dominant solution for X_1 involves $J_{l+1/2}$. On the other hand, since we want to recover the results of the radial pulsations (TT68), we expect the dominant solution for ξ_1 to be $J_{l+3/2}$. We thus let $a = l + 1$ in relations (31)–(35), and

TABLE 1

BEHAVIOR OF VARIOUS FUNCTIONS NEAR THE SINGULAR POINTS[a]

Function	Defined in Equation:	$r \approx 0$	$x = (R - r) \approx 0$
z	(27)	$O(r)$	$O(x^{1/2})$
z/z'		$r[1 + O(z^2)]$	$-2x[1 + O(z^2)]$
L_0	(36)	$-1/z + O(z)$	$-(n + 1/2)/z + O(z)$
L_1	(37)	$O(z^2)$	$O(1)$
L_3	(38)	$-\Lambda/z^2 + O(1)$	$-(n + 1/2)(n + 3/2)/z^2 + O(1)$
L_4	(39)	$O(1)$	$O(1)$
L_5	(40)	$1/z + O(z)$	$-(n + 3/2)/z + O(z)$
L_6	(41)	$O(1/z)$	$O(z)$
L_7	(42)	$2/z + O(z)$	$-2(n + 1)/z + O(z)$
L_8	(43)	$-\Lambda/z^2 + O(1)$	$(n + 1/2)(n + 5/2)/z^2 + O(1)$
L_9	(59)	$-1/z + O(z)$	$-(n - 1/2)/z + O(z)$
W_1	(46)	$-\Lambda/r^2 + O(1)$	$-(n + 1/2)(n + 3/2)/4x^2 + O(1)$
W_2	(49)	$-(\Lambda + 2)/r^2 + O(1)$	$-(n + 1/2)(n - 1/2)/4x^2 + O(1)$

[a] $\Lambda \equiv l(l + 1)$.

(40)–(42). The first nonvanishing coefficient is A_0. From equation (34), we conclude that A_0 is indeed a constant. Obviously, there is no loss of generality in letting $A_0 = 1$. From equation (40), one then derives $D_0 = A_0 = 1$. (Of course, such a simple solution rests upon the particular choice for $f_\xi(r)$; see equation [30].) Equation (35) can then be integrated to give

$$-2B_0 = \int_0^z \left[\frac{l(l + 1)}{z^2} + \frac{W_1}{z'^2}\right] dz \,, \tag{46}$$

where we have defined

$$W_1 = z'^2(L_1 + L_3) = \frac{1}{2}\frac{\rho''}{\rho} - \frac{3}{4}\frac{\rho'^2}{\rho^2} + \frac{1}{r}\frac{\rho'}{\rho} - \frac{1}{4}\frac{\varphi''}{\varphi} + \frac{5}{16}\frac{\varphi'^2}{\varphi^2} - \frac{\Lambda}{r^2} + \frac{4}{r} g\varphi \,. \tag{47}$$

From the behavior of z with respect to r near the center (see Table 1), it is obvious that the integrand in equation (46) is regular, and thus that B_0 is $O(z)$. Of course, this last statement is true only because of the lower bound in the integral defining B_0. (In an analogous problem, ST87 showed explicitly that it is essential that B_0 vanishes at $z = 0$. The same proof applies here.) Finally, C_1 may be derived from equation (41). We find

$$C_1 = -\frac{l}{z} - B_0 + \left(L_7 - L_5 - \frac{1}{z}\right) . \tag{48}$$

Here again, from the definitions of L_5 and L_7, it is obvious that the term in parenthesis is regular. We thus recover the result that C_1 is regular in the radial case. Since we want to compare the present results with those of TT68 (the radial case) which involve the function ξ only, we eliminate B_0 from equation (47). By using the definition of B_0 and integrating by parts, one can show that C_1 can also be written as

$$2C_1 = -\frac{2l}{z} + \int_0^z \left(\frac{\Lambda + 2}{z^2} + \frac{W_2}{z'^2}\right) dz \,, \tag{49}$$

where the quantity W_2 is defined as

$$W_2 = z'^2[L_1 + L_3 + 2(L_7 - L_5)_{,z}] = -\frac{1}{2}\frac{\rho''}{\rho} + \frac{1}{4}\frac{\rho'^2}{\rho^2} + \frac{1}{2}\frac{\varphi'}{\varphi}\frac{\rho'}{\rho} + \frac{1}{4}\frac{\varphi''}{\varphi} - \frac{7}{16}\frac{\varphi'^2}{\varphi^2} - \frac{1}{r}\frac{\varphi'}{\varphi} + \frac{1}{r}\frac{\rho'}{\rho} + \frac{4}{r} g\varphi - \frac{\Lambda + 2}{r^2} \,. \tag{50}$$

This quantity is the same as the quantity $h(x)$ introduced in TT68 (eq. [40]), except of course for the term in Λ.

In principle, the next coefficients in the series can be evaluated. Obviously, the expressions for those coefficients become rapidly cumbersome. With the help of Table 1, it can be shown however that each term in their respective series have the same behavior near $z = 0$. We proceed by recurrent induction. We assume that up to a given value of i, all A_i's and all D_i's are even and $O(1)$, that all B_i's are odd and $O(z)$, and that all C_i's are odd and $O(1/z)$. We have to show that this is also true for the next value of i. For example, since $a = l + 1$ and with the help of Table 1, it can be shown that in equation (41), which determines C_{i+1}, the first few terms are either $O(z)$ and or $O(1/z)$. However, the terms involving C_i and its derivatives, i.e.,

$$C_{i,zz} + \left[\frac{2(l + 1)}{z} + L_7\right] C_{i,z} + \left[\frac{l(l + 1)}{z^2} + \frac{l + 1}{z} L_7 + L_8\right] C_i \,, \tag{51}$$

are each $O(z^{-3})$. On the other hand, they may also be written as

$$\frac{1}{z}(zC_i)_{,zz} + \frac{1}{z}\left(\frac{2l}{z} + L_7\right)(zC_i)_{,z} + \frac{1}{z}\left[\frac{l(l - 1)}{z^2} + \frac{l}{z} L_7 + L_8\right](zC_i) \,. \tag{52}$$

Because (zC_i) is regular and even, and from Table 1, it is obvious that each of the three terms is now $O(1/z)$ near $z = 0$. Hence the conclusion that C_{i+1} is indeed $O(1/z)$, as were the preceding ones. An analogous manipulation (noting that B_i/z is even and regular) can be used to prove that $A_{i+1} = O(1)$, $D_{i+1} = O(1)$, and $B_{i+1} = O(z)$.

A final comment concerns the behavior of C near $z = 0$. The fact that C is $O(1/z)$ does not imply that the radial displacement ξ is not finite at the origin. Indeed, if $l \geq 1$ the term CP_1 in equation (31) is $O(1/z)O(z^{l+1})$. We thus recover the well known result that when $l \geq 1$, $\xi \propto r^{l-1}$ near the center. What is more worthy of note is that the first nontrivial term in the expansion for ξ_1, i.e., $D_0 P_2/\sigma$, is not the dominant term near $z = 0$. In fact, it can be shown that in this neighborhood the ratio of $C_1 P_1/\sigma^2$ (the next term) to $D_0 P_2/\sigma$ is equal to $-l(2l + 3)/(\sigma^2 z^2)$. This behavior was already apparent in the results obtained in the context of Cowling's approximation (see T80 and ST87), and it is certainly related to the presence of a variable singularity in the differential equation which determines ξ. We are thus led to the conclusion that in order to achieve the same degree of accuracy in the eigenfunctions as in the eigenfrequencies, one must retain the same number of terms in each series. In other words, only even-order asymptotic approximations are satisfactory in every respect.

b) *Solution in the Outer Region*

From § III, we know that the dominant solution for X_1 involves J_{n+1}. On the other hand, from the results of the radial pulsations (TT68), we expect the dominant solution for ξ_1 to be J_n. We thus let $a = n + 1/2$ in relations (32), (34), (35), (40), and (41). Since we want the first approximation for X_1 to contain J_{n+1}, we must let $A_0 = 0$. As a result, C_0 and D_0 also vanish. Thus, the first nonvanishing coefficient is B_0. By using equation (34), we derive that B_0 is indeed a constant. Again, there is no loss of generality in letting $B_0 = 1$. Equation (41) immediately implies that $C_1 = -B_0 = -1$. Equation (34) can then be integrated to give

$$2A_1 = \int_0^z \left[\frac{(2n+1)(2n+3)}{4z^2} + \frac{W_1}{z'^2}\right] dz , \tag{53}$$

where W_1 has been defined in equation (47). From Table 1, it is obvious that the integrand in equation (53) is regular, and thus that A_1 is $O(z)$. Of course, this last conclusion is true because of the lower bound of integration. Finally, D_1 may be derived from equation (40). We find

$$D_1 = A_1 + \left(L_7 - L_5 + \frac{2n+1}{2z}\right) . \tag{54}$$

Here again, because of the behavior of L_5 and L_7 near the surface (see Table 1), it is obvious that the term in parenthesis is regular. Hence D_1 is regular and $O(z)$. By using definition (53) and integrating by parts, we can also write

$$2D_1 = \int_0^z \left[\frac{(2n+1)(2n-1)}{4z^2} + \frac{W_2}{z'^2}\right] dz . \tag{55}$$

The quantity W_2 has already been defined in equation (50).

To be consistent, it remains to show that all B_i's and C_i's are $O(1)$, and that all A_i's and D_i's are $O(z)$. Here again, when solving equation (35) for B_i, the lower bound of integration is arbitrary. (A different lower bound corresponds to a different amplitude.) On the other hand, when solving equation (34) for A_{i+1}, it is imperative that the lower bound in the integral vanishes. As a result, it can be shown that B_i and C_i are even functions and that A_i and D_i are odd in z. By using the same recurrent induction reasoning as before, we can show that C_{i+1} is indeed $O(1)$, and that A_{i+1} is $O(z)$. However, when it comes to D_{i+1}, we conclude that D_{i+1} cannot be $O(z)$ unless

$$C_{i+1} + B_i + \frac{2n}{z} D_i = O(z^2) \tag{56}$$

($i \geq 1$). It is not obvious whether this is true or not. However, we know that for the radial pulsations, all C_i's and D_i's are regular (see TT68; in that paper our D_1 was denoted C_0.) Furthermore, we do not expect terms that vanish when $l = 0$ to play a significant role in the outer solution. Equation (2) thus suggests that we use η as an intermediate variable.

We therefore express η in a manner analogous to equations (28) and (31) and introduce two new functions E and F, in such a way that

$$\eta = f_\eta\left(EP_1 + \frac{1}{\sigma} FP_2\right) \quad \text{with} \quad f_\eta = h(r)/(rz'^2) . \tag{57}$$

Equation (2) then gives

$$-D_i = -A_i + C_{i,z} + \left(L_7 - L_5 + \frac{a}{z}\right)C_i - \frac{\Lambda}{z'^2 r^2} E_i ; \tag{58}$$

$$C_{i+1} = -B_i + D_{i,z} + \left(L_7 - L_5 - \frac{a}{z}\right)D_i - \frac{\Lambda}{z'^2 r^2} F_i . \tag{59}$$

We have not replaced a by $n + 1/2$ since these relations can also be used to compute η in the inner domain. We insert these relations into equations (38) and (39) and we observe that

$$L_8 - \frac{d}{dz}(L_7 - L_5) - L_5(L_7 - L_5) = -\frac{1}{z'^2}\frac{\Lambda}{r^2}. \tag{60}$$

After dividing by a common factor $\Lambda \equiv l(l+1)$ we obtain two new sets of recurrence relations

$$-E_{i+1} = -F_{i,z} + \left(\frac{a}{z} + L_9\right)F_i - \frac{N^2}{z'g}B_{i-1} + D_i\ ; \tag{61}$$

$$F_i = -E_{i,z} + \left(-\frac{a}{z} + L_9\right)E_i - \frac{N^2}{z'g}A_{i-1} + C_i\ ; \tag{62}$$

where we have defined

$$L_9 = \frac{1}{z'}\left(\frac{1}{4}\frac{\varphi'}{\varphi} + \frac{1}{2}\frac{\rho'}{\rho} + \frac{1}{r}\right). \tag{63}$$

In a more transparent way, this last set of relations may be seen as a consequence of the following equation

$$\eta' + \frac{\eta}{r} - \frac{\xi}{r} + \frac{1}{\sigma^2}\frac{N^2c^2}{rg}X = 0. \tag{64}$$

Incidentally it should be noted that in the case of radial pulsations, although the horizontal component ξ_θ of the Lagrangian displacement vanishes, from equation (1) there is no such constraint on η.

In the outer domain, we thus have the six sets of recurrence relations (34), (35), (58), (59), (61), and (62) to determine the six functions A, B, C, D, E, and F. It is easy to show that if we assume $B_0 = 1$, we will also have $C_1 = -B_0 = -1$, $F_1 = C_1 = -1$, A_1 given by equation (50), D_1 given by equation (52), $E_1 = 0$, and

$$E_2 = \frac{2n+1}{2z} + L_9 - \frac{N^2}{z'g} - D_1 = -A_1 - \frac{1}{z'}g\varphi. \tag{65}$$

The second equality follows from equation (54). Obviously, E_2 is $O(1/z)$.

It is now a simple matter to show that B, C, and F are $O(1)$, that A is $O(z)$, and that E is $O(1/z)$. The proof follows the same lines as described before. Here we have to use the fact that the already known A_i's and E_i's are such that (A_i/z) and $(E_i z)$ are even and regular functions of z. For example, expression (62) can also be written as

$$F_i = -\frac{1}{z}\frac{d}{dz}(zE_i) + \left(-\frac{a-1}{z} + L_9\right)E_i - \frac{N^2}{z'g}A_{i-1} + C_i. \tag{66}$$

We have thus constructed genuine asymptotic expansions for X, ξ, and η.

V. FULL SOLUTION OF EQUATIONS (3) AND (4)

The results we have obtained so far can be summarized as follows.

In the inner domain the general solution of equations (3) and (4) is

$$X = k_f 2l(l-1)r^{l-2}\left(g' - \frac{g}{r}\right)\left[1 + O\left(\frac{1}{\sigma^2}\right)\right] + k_{\text{in}} r^{-1}\rho^{-1/2}\varphi^{3/4}z_{\text{in}}^{1/2}\left[J_{l+1/2} + \frac{1}{\sigma}B_0 J_{l+3/2}\right]\left[1 + O\left(\frac{1}{\sigma^2}\right)\right]; \tag{67}$$

$$\xi = k_f\sigma^2 l r^{l-1}\left[1 + O\left(\frac{1}{\sigma^2}\right)\right] + k_{\text{in}} r^{-1}\rho^{-1/2}\varphi^{1/4}z_{\text{in}}^{1/2}\left[\frac{1}{\sigma}J_{l+3/2} + \frac{1}{\sigma^2}C_1 J_{l+1/2}\right]\left[1 + O\left(\frac{1}{\sigma^2}\right)\right]. \tag{68}$$

The argument of the Bessel functions is (σz_{in}). B_0 and C_1 are functions of z_{in} and are defined in equations (46) and (48). The arbitrary constants k_f and k_{in} may each depend on σ^2. Of course, since we are considering linear pulsations, the global amplitudes are arbitrary, so that k_f/k_{in} is the relevant quantity; this last quantity may also depend on σ^2. For completeness, let us also quote the solution for η. We have

$$\eta = k_f\sigma^2 r^l\left[1 + O\left(\frac{1}{\sigma^2}\right)\right] + k_{\text{in}} r^{-2}\rho^{-1/2}\varphi^{-1/4}z_{\text{in}}^{1/2}\left[-\frac{1}{\sigma^2}J_{l+1/2} + \frac{1}{\sigma^3}\left(-B_0 + \frac{1}{z'}g\varphi\right)J_{l+3/2}\right]\left[1 + O\left(\frac{1}{\sigma^2}\right)\right]. \tag{69}$$

On the other hand, the general solution in the outer domain is given by

$$X = k_f 2l(l-1)r^{l-2}\left(g' - \frac{g}{r}\right)\left[1 + O\left(\frac{1}{\sigma^2}\right)\right] + k_{\text{out}} r^{-1}\rho^{-1/2}\varphi^{3/4}z_{\text{out}}^{1/2}\left[\sigma J_{n+1} + A_1 J_n\right]\left[1 + O\left(\frac{1}{\sigma^2}\right)\right]; \tag{70}$$

$$\xi = k_f\sigma^2 l r^{l-1}\left[1 + O\left(\frac{1}{\sigma^2}\right)\right] - k_{\text{out}} r^{-1}\rho^{-1/2}\varphi^{1/4}z_{\text{out}}^{1/2}\left[-J_n + \frac{1}{\sigma}D_1 J_{n+1}\right]\left[1 + O\left(\frac{1}{\sigma^2}\right)\right]; \tag{71}$$

$$\eta = k_f\sigma^2 r^l\left[1 + O\left(\frac{1}{\sigma^2}\right)\right] + k_{\text{out}} r^{-2}\rho^{-1/2}\varphi^{-1/4}z_{\text{out}}^{1/2}\left[-\frac{1}{\sigma}J_{n+1} + \frac{1}{\sigma^2}E_2 J_n\right]\left[1 + O\left(\frac{1}{\sigma^2}\right)\right]. \tag{72}$$

Here the argument of the Bessel functions is $(\sigma z_{\rm out})$, and the functions A_1, D_1, and E_2, which can be considered as functions of $z_{\rm out}$, are defined by equations (53), (54), and (65), respectively.

Before proceeding any further, let us compare solutions (67) and (70) with the limiting solutions (15) and (19). The inner solution is compatible with equation (15): one part of the solution is proportional to r^l and the other one is oscillating with the correct "period" (inasmuch as one can define a period in a Bessel function). As far as the outer solution is concerned, equation (19) has revealed, besides the oscillating solution, *two* polynomial solutions. Only one is present in solution (70). The other one, which is proportional to r^{-l-3} (see eq. [23]), has been discarded since it is not regular at the center. In this connection, let us note that the same constant k_f has been used in the inner and the outer solutions: this part of the solution does not require a different treatment near the center and near the surface.

Because some of the solutions obtained so far are not valid all the way from the center to the surface, we have to match them in their common domain of validity. If we were to solve equation (12) numerically, we would follow up the solutions in equations (15) and (19). From this point of view, there would be five arbitrary constants involved. We therefore need five conditions to close the problem. Since equation (12) is a fourth-order differential equation we must thus express the continuity of X and its first three derivatives at an intermediary point. The fifth condition is condition (11) on the perturbation of the gravitational potential across the surface. Parenthetically note that since we solved equations (3) and (4) instead of equation (12), the four conditions on X and its first three derivatives may be replaced by the requirement that X and ξ together with their first derivative be continuous at the junction point.

Manifestly, in solutions (67)–(72) only three independent constants are involved. We still have five conditions to satisfy. Therefore, we will have to show that two of these conditions are redundant, and, hence, that the assumptions made in solutions (67)–(72) were legitimate. Incidentally note that the f-part of the solution requires no matching since it is valid throughout the star.

Thus, we see the problem as follows: the matching between the inner and outer solutions will determine the possible values of σ (the solution of an eigenvalue problem) whereas the requirement that gravity be continuous across the perturbed surface of the star will determine the importance of the f-part of the solution with respect to the p-part.

a) Matching of X

As we mentioned before, we only have to match the oscillating part of X. In the notation of equation (25), we thus have to express that

$$(hX_1)_{\rm in} = (hX_1)_{\rm out}\ ; \qquad \frac{d}{dr}(hX_1)_{\rm in} = \frac{d}{dr}(hX_1)_{\rm out} \tag{73}$$

at an arbitrary chosen point in the common domain of validity of the inner and outer solutions. Because h(in) $= h$(out), and because $dz_{\rm in}/dr = -dz_{\rm out}/dr = \varphi^{1/2}$, these two conditions can be translated immediately into the following

$$X_1({\rm in}) = X_1({\rm out})\ ; \qquad \left(\frac{dX_1}{dz}\right)_{\rm in} = -\left(\frac{dX_1}{dz}\right)_{\rm out} . \tag{74}$$

If the junction point is not "too" close to the center and to the surface, $\sigma z_{\rm in}$ and $\sigma z_{\rm out}$ are large compared to 1, and we may use Hankel's asymptotic expansions for Bessel functions (fixed index and $\sigma z \to \infty$; see Abramowitz and Stegun 1970). In agreement with the number of terms kept in solutions (67) and (70), we will only keep the first two terms in this expansion: terms proportional to $1/\sigma^2$ are neglected before terms independent of σ. We thus find the condition

$$\sigma \tan\left[\sigma Z - \frac{\pi}{2}\left(n + l + \frac{1}{2}\right)\right] = T_l + O\left(\frac{1}{\sigma}\right) , \tag{75}$$

where

$$Z = z_{\rm in} + z_{\rm out} = \int_0^R \frac{dr}{c} , \tag{76}$$

$$2T_l = -\frac{l(l+1)}{z_{\rm in}} - \frac{(2n+1)(2n+3)}{4z_{\rm out}} + 2B_0({\rm in}) - 2A_1({\rm out}) . \tag{77}$$

Reverting to the original definitions (46) and (53), one can also write

$$2T_l = -\frac{l(l+1)}{z_{\rm in}} - \frac{(2n+1)(2n+3)}{4z_{\rm out}} - \int_0^{z_{\rm in}} \left[\frac{l(l+1)}{z^2} + \frac{W_1}{z'^2}\right] dz - \int_0^{z_{\rm out}} \left[\frac{(2n+1)(2n+3)}{4z^2} + \frac{W_1}{z'^2}\right] dz . \tag{78}$$

As in similar problems (see TT68 and T80), it can be shown that this expression does not depend on the chosen junction point. Note also that in each integrand the first term cancels out the singularity from the second term (see Table 1).

b) Matching of ξ

One proceeds along the same lines as for the matching of X. Here, however, we have f_ξ(in) $= -f_\xi$(out), because f_ξ contains the factor z' (see eq. [30]). The conditions resulting from the matching of ξ are thus

$$\xi_1({\rm in}) = -\xi_1({\rm out})\ ; \qquad \left(\frac{d\xi_1}{dz}\right)_{\rm in} = \left(\frac{d\xi_1}{dz}\right)_{\rm out} . \tag{79}$$

By using the asymptotic expansions of the Bessel functions and retaining only terms that differ by less than two orders of magnitude in σ, we obtain condition (75), where T_l is now defined as

$$2T_l = -\frac{(l+1)(l+2)}{z_{\text{in}}} - \frac{(2n+1)(2n-1)}{4z_{\text{out}}} - 2C_1(\text{in}) - 2D_1(\text{out}) . \tag{80}$$

By using definitions (48) and (54) for C_1(in) and D_1(out) and by noting that $L_5(\text{in}) = -L_5(\text{out})$, it can be shown that this expression is identical to expression (77). On the other hand, we could have used the equivalent definitions (49) and (55) to eliminate C_1(in) and D_1(out). The result is

$$2T_l = -\frac{l(l+1)+2}{z_{\text{in}}} - \frac{(2n+1)(2n-1)}{4z_{\text{out}}} - \int_0^{z_{\text{in}}} \left[\frac{l(l+1)+2}{z^2} + \frac{W_2}{z'^2}\right] dz - \int_0^{z_{\text{out}}} \left[\frac{(2n+1)(2n-1)}{4z^2} + \frac{W_2}{z'^2}\right) dz . \tag{81}$$

As in definition (78), both integrands are regular (see Table 1). We have thus two apparently different expressions for T_l although they are strictly equivalent.

We have thus shown that four of the conditions actually reduce to two.

c) *Boundary Condition on* $\delta\Phi$

It only remains to express the continuity of gravity across the perturbed surface of the star. From equation (10) we have

$$\delta\Phi = \sigma^2 r\eta - g\xi + c^2 X . \tag{82}$$

If we insert solutions (70), (71), and (72) into this relation, we find

$$\delta\Phi = k_f \sigma^4 r^l \left[1 + O\left(\frac{1}{\sigma^2}\right)\right] + k_{\text{out}} r^{-1} p^{-1/2} \varphi^{-1/4} z^{1/2} \left[\left(E_2 + A_1 + \frac{g\varphi}{z'}\right) J_n + O\left(\frac{1}{\sigma}\right)\right] . \tag{83}$$

From definition (65), it is obvious that the first term (i.e., the only one that has been explicited) in the oscillating part vanishes. This does not mean that $\delta\Phi$ contains no p-part! Indeed it can be shown that the next term in the expansion, which was implicitly included in $O(1/\sigma)$, is given by

$$\frac{1}{\sigma}\left(F_2 + B_1 - \frac{g\varphi}{z'} D_1\right) J_{n+1} . \tag{84}$$

By using various recurrence relations and by noting that

$$W_1 + \frac{\Lambda}{r^2} + z'^2(L_7 - L_5)^2 + (L_7 - L_5)_{,z} = \frac{4}{r} g\varphi , \tag{85}$$

one can show that

$$F_2 + B_1 - \frac{g\varphi}{z'} D_1 = \frac{1}{z'^2}\left(g' + \frac{2g}{r}\right)\varphi = 4\pi G\rho \ ! \tag{86}$$

Note that this result does not depend on A_1 (or on any other second-order quantity), and in that respect, it is a first-order term. Indeed, had we solved directly Poisson's equation, instead of solving first for X, ξ, and η, this fact would have been evident. How come, then, that we had to wait till the third-order approximation in order to derive a first-order approximation? It only means that $(\delta P/\rho + \delta\Phi)$ and $(\delta P/\rho)$ differ by terms proportional to $1/\sigma^2$.

Note also that if we assume ρ to vanish at the surface, the p-part of $\delta\Phi$ also vanishes there, as well as its first derivative. Boundary condition (11) then implies that k_f vanishes. We therefore come to the conclusion that, to the order of approximation considered here, there is no f-part present in the eigenfunctions associated to high-order p-modes. The same conclusion is reached for the homogeneous model, although it is less obvious!

VI. DISCUSSION

Thus far we have shown how to construct a uniformly valid asymptotic expansion for the nonradial pulsations of stellar models, without having to invoke Cowling's approximation. We have restricted ourselves to high-order p-modes, for which the frequency σ may be assumed to be large. It now remains to compare the present results with other previously known results: high-order radial oscillations (i.e., TT68), high-order nonradial oscillations in the case when Cowling's approximation has been used (i.e., T80 and ST87), results based on plane-wave theory (e.g., Gough 1986). We will also comment on the relevance of these results in the context of the observed solar acoustic modes.

Let us first discuss the connection between the plane-wave approach and Olver's method (1974). For the sake of simplicity, consider the case when the pulsations satisfy a second-order differential equation (i.e., radial pulsations and nonradial pulsations for which one neglects the perturbation in the gravitational potential). By a change of dependent variable it is always possible to bring this equation to its normal form. In the particular case of nonradial pulsations, in a region not too close to the surface, this equation reduces to

$$X_*'' + \left[\frac{\sigma^2}{c^2} - \frac{l(l+1)}{r^2} + O(1)\right] X_* = 0 . \tag{87}$$

The term $O(1)$ contains quantities that are much smaller than those made explicit.

The plane-wave approach consists in comparing equation (87) with the simple wave equation

$$X''_{**} + K^2 X_{**} = 0 \, . \tag{88}$$

One thus assumes that K^2 is a slowly varying function of r. (It depends also on σ and l.) The solution of equation (87) is then expressed in terms of sinusoidal functions (if $K^2 > 0$), or in terms of exponentials (if $K^2 < 0$). Obviously, such solutions break down in the vicinity of a zero of K^2, or when K^2 becomes singular. These points are the turning points of the equation. For a fixed value of l, the zeros of K^2 define what is called the propagation zone. We thus have an inner turning point at r_t for which σ is equal to the local acoustic (or Lamb) frequency. (The outer turning point depends on terms that were not written down explicitly in equation [87].) According to the plane-wave approach, thus, for $r \gg r_t$ the solution of equation (87) is oscillatory, and for $r \ll r_t$ the solution is exponential. There is no such simple solution when $r \approx r_t$. The purpose of the so-called WKB method is to provide a connection between these two asymptotic behaviors, even though an adequate description near the turning point is lacking. In the particular case of equation (87), however, if σ is sufficiently large (but l not too large), r_t is so close to the center that nowhere in the interval (0, r_t) can the solution of equation (87) be approximated by an exponential. Indeed, if c^2 was a constant, the solution of equation (87) would then be

$$X_* = r^{1/2} J_{l+1/2}\left(\frac{\sigma r}{c}\right) . \tag{89}$$

In this case, therefore, r_t has lost its meaning of a transition point between an exponential behavior and an oscillatory one. A similar conclusion is reached for the outer turning point: it lies so close to the surface that the plane-wave approximation cannot be made anywhere near $r = R$. It is not surprising, therefore, that the plane-wave approach leads to results that are slightly different from those derived from a more appropriate comparison equation (e.g., Vandakurov 1967). To sum up, the plane-wave approach has some weaknesses: among others, it does not provide a complete description of the solution (in particular near the transition points), the error bounds are difficult to ascertain, and an asymptotic series based on this first approximation is not easily derived.

Olver's method has been devised to overcome all these difficulties. In this approach (if σ is the only large quantity in eq. [87]) the only genuine turning points are the center and the surface, since these are singular points. However, there is no need for introducing an r_t because σ^2/c^2 does not vanish anywhere. Except for a few essential details, the procedure is similar to the plane-wave approach. One defines a new independent variable which, at large distance from the turning point, takes into account that c^2 is not constant but which, at short distance, is proportional to this distance. Equation (87) is thus replaced by

$$\frac{d^2 X_{**}}{du^2} + \left[\sigma^2 - \frac{l(l+1)}{u^2} + O(1)\right] X_{**} = 0 \, . \tag{90}$$

If one neglects altogether the term $O(1)$ in this equation, its solution is given by expression (89), where r is replaced by u (and c by 1). In this connection, compare the ideal solution (89) with the corresponding leading term in solution (26). It should be pointed out that, although the emphasis has been put here on the behavior of the solution near the turning point, it naturally merges with the expected oscillatory behavior far away from these turning points.

Obviously, far enough from the transition points there is no difference between the plane-wave approach and Olver's asymptotic theory. What is more satisfactory in the latter approach is that the comparison equation is now allowed to be more complex than equation (88), so that one can describe the solution in the whole domain without any restriction. As already mentioned, the point r_t, which is essential in the plane-wave approach, does not play any role here, nor does the point where $u^2 = l(l+1)/\sigma^2$. Hence, there is no reason for the integrals defining the correcting term T_l (see eq. [77]) to be limited to the "propagation zone." On the contrary, we have shown that one *must* extend the integration over the whole star, i.e., from $r = 0$ to $r = R$.

Basically, in this paper we have followed Olver's procedure. We have introduced new dependent and independent variables in such a way that the comparison equation be as simple as possible, while keeping the major features of the original equation. Of course, in the case of a fourth-order differential equation, it is not obvious how to define a normal form: it is not possible to derive from equation (12) an equation that is free from the two odd-order derivatives. However, guided by the limiting form of the solution close to the transition points, we knew the comparison solutions, without having to write down the comparison equation! We thus came to the conclusion that, in principle, the nonradial stellar pulsations were the superposition of two parts, a nonoscillating one (the f-part) and an oscillating one (the p-part), and that each of these two parts required a different asymptotic treatment. Only in a homogeneous model are these parts independent of each other. We have shown that, at least to second-order approximation, there is no f-part in the solution describing high-order p-modes (i.e., modes for which σ^2 is large, but l is not too large). To this order, the p-modes pulsations are thus purely oscillatory. Of cource, we could not have obtained the f-modes, since their frequencies cannot be large, unless the degree l is also large!

We are now in a position to compare the present results with the asymptotic approximations derived for the radial pulsations (TT68) as well as for the high-order p-modes (T80 and ST87). When $l = 0$, it is obviously not necessary to solve for X, as we did here. Indeed, these modes are described by a simple second-order equation in the radial displacement ξ, and this equation has been studied from an asymptotic point of view in TT68. We have already mentioned that the quantity W_2, defined in equation (50), reduces exactly to the quantity $h(x)$ defined in TT68 (eq. [40]). As a result, if allowance is made for the fact that the inner solution is expressed here in terms of $J_{3/2}$ and $J_{1/2}$ (it was expressed in terms of $J_{3/2}$ and $J_{5/2}$ in TT68), we obtain exactly the same expansions. And since the expansions are the same, the matching condition between the inner and outer solutions gives exactly the same expression for the frequencies.

As far as the nonradial oscillations are concerned, we do not expect a perfect agreement between the present results and those that were obtained after having neglected the perturbation in the gravitational potential. In T80, the p-modes were described in terms of

ξ and $\delta P/\rho$ (or is it $\delta P/\rho + \delta\Phi$, since $\delta\Phi$ was assumed to be negligible?). As we have shown in § V*c*, up to second-order in $1/\sigma$, $\delta P/\rho$ and $(\delta P/\rho + \delta\Phi)$ are identical. On the other hand, from equation (10) it follows that $(\delta P/\rho + \delta\Phi)$ is precisely $\sigma^2 r\eta$. The comparison between the present results and those of T80 is therefore easy to make. Thus, the present solutions (68) [ξ(in)], (69) [η(in)], (71) [ξ(out)], and (72) [η(out)] must be compared to solutions (59), (58), (60), and (61) of T80, respectively. Not surprisingly, in a first approximation, the agreement is excellent. In other words, to this order of approximation, Cowling's approximation is not determinant. However, a closer examination of the eigenfunctions reveals that at second-order in $1/\sigma$ these functions already depend on whether Cowling's approximation has been made or not. Indeed, it can be shown that the present inner solutions differ from the corresponding solutions in T80 by terms that are proportional to

$$\int_0^r \frac{1}{c}\left(g' + \frac{2}{r}\,g\right)dr = \int_0^r \frac{1}{c}\,(4\pi G\rho)dr\,. \tag{91}$$

Similarly, the respective outer solutions differ from each other by terms that are proportional to

$$\int_r^R \frac{1}{c}\left(g' + \frac{2}{r}\,g\right)dr = \int_r^R \frac{1}{c}\,(4\pi G\rho)dr\,. \tag{92}$$

Since in a realistic star ρ decreases outward, one thus confirms the conclusion that Cowling's approximation has more impact in the inner region than in the outer layers. It is perhaps also worth noting that the error introduced by neglecting the perturbation in the gravitational potential involves the use of Laplace's equation instead of Poisson's equation in the equilibrium state.

Of greater interest is the impact of Cowling's approximation on the frequencies themselves. It is immediately apparent that the dispersion relation (75) and the one derived in T80 (or in ST87; see their eq. [120]) have the same structural form. In both cases, the second-order approximation involves a quantity that is an integrated characteristic of the model (it is denoted T_l here, eqs. [78] or [81]; it was V_{io} in T80, eq. [71]; and γ in ST87, eq. [123]). Here too the dependence of T_l on l may be expressed as

$$T_l = -\alpha l(l+1) + \beta\,, \tag{93}$$

where α and β are two constants. The constant β depends only on the model and is the same as the quantity T that was defined for the radial pulsations (see TT68, eq. [39]; we have already noted that the present results naturally generalize the case $l = 0$). As to the constant α, a comparison between expression (93) and definition (78) yields, after an integration by parts,

$$2\alpha = -\int_0^R \frac{1}{r}\frac{dc}{dr}\,dr = -\int_0^R \frac{1}{c}\frac{d(c^2)}{d(r^2)}\,dr\,. \tag{94}$$

This result is not different from what was already known. Parenthetically, it has often been argued that the constant Z (see eq. [76]; essentially the travel time of a sound wave between the center and the surface) is sensitive to the superficial layers because c is small there, and that the constant α is sensitive to the central regions, because c is large there (e.g., Gilliland and Däppen 1988). This is partly true. The second equality in equation (94) makes the point clear: for a homogeneous model $d(c^2)/d(r^2)$ is a constant, and except for normalization constants, 2α and Z are exactly the same integrals! What is important in equation (94) is not the *amplitude* of c but the *variation of its amplitude.*

In view of the differences noted between the eigenfunctions describing the nonradial oscillations (computed with and without $\delta\Phi$) it is not surprising to find that T_l differs from the corresponding quantity (V_{io} in T80, γ in ST87) by a term

$$\Delta T_l = \int_0^R \frac{1}{c}\,(4\pi G\rho)dr\,. \tag{95}$$

One concludes therefore that ΔT_l is independent of the degree l of the mode. Now, it is well known that Cowling's approximation is a good approximation for sufficiently high order k, or for sufficiently high degree l. Furthermore, it is expected to give better results in more centrally condensed objects. These properties are apparent in the asymptotic expressions.

Let us now concentrate on the dispersion relation (75) itself. Since we are considering high-order p-modes, T_l/σ in this equation is small compared to 1, and we may use the expansion of tan( ) around $k\pi$ (k is an integer). In agreement with the order of approximation used for the eigenfunctions, we thus write

$$\sigma Z = \frac{\pi}{2}\left(2k + n_e + l + \frac{1}{2}\right) + \frac{T_l}{\sigma} + O\left(\frac{1}{\sigma^2}\right). \tag{96}$$

(We now use the more explicit notation n_e for the superficial effective polytropic index, in order to avoid any confusion with the order of the mode, which is sometimes also denoted by n in the literature). To second order, this last expression may also be written as

$$\sigma Z = \frac{\pi}{2}\left(2k + n_e + l + \frac{1}{2}\right) + \frac{2}{\pi}\frac{T_l Z}{(2k + n_e + l + 1/2)} + O\left(\frac{1}{\sigma^2}\right). \tag{97}$$

In a first approximation, $(k-1)$ is the number of nodes in the eigenfunction X. This conclusion is immediate if the junction between the inner and the outer solutions is made at node number s counting from the center, and at node number t counting from the surface. We have therefore

$$\sigma Z = \pi\left[s + \frac{1}{2}\left(l + \frac{1}{2}\right) - \frac{1}{4}\right] + \pi\left[t + \frac{1}{2}\,(n_e + 1) - \frac{1}{4}\right]. \tag{98}$$

By comparing this expression with expression (96), we conclude that the number of nodes along the radius (i.e., $s + t - 1$) is indeed equal to $(k - 1)$. Counting the number of nodes of ξ is a much more delicate matter since, as we have seen in § IVa, close to the center the second term in its expansion dominates over the first one.

From a practical point of view, one of the main concerns is the following: when may we use expression (96) or (97), and what accuracy may we expect to reach? Since the method we used to construct the asymptotic expansion is nothing but a generalization of the WKB method, the more nodes there are along the radius, the more accurate the results. Therefore, we must necessarily have $k \gg 1$. In addition, the terms $l(l + 1)/r^2$ were considered not more than a correction when we solved for X in the outer domain; this means that we have implicitly assumed that $\sigma^2 Z^2 \gg l(l + 1)$, (or $k \gg l + 1/2$). Furthermore, an inspection of Hankel's expansion of Bessel function $J_\nu(\sigma z)$ (which has been used to derive eq. [75]) suggests that it is valid whenever $\sigma z \gg \nu^2/2$. Since n_e is not expected to exceed 3 or 3.5 in stars, this condition is easily satisfied by the outer solution. On the contrary the situation in the inner solution can be more stringent. Indeed, it would seem that when $2\sigma z < (l + 1/2)^2$ we cannot hope to have reliable results. Again for the lowest degree modes, this condition is easily avoided. We now have a more precise limitation. By comparing expressions (96) and (97), which, from an asymptotic point of view, are equivalent, we find that they indeed lead to nearly the same results provided

$$|T_l Z| \ll \left[\frac{\pi}{2}\left(2k + n_e + l + \frac{1}{2}\right)\right]^2, \tag{99}$$

or, if we limit T_l to its part that is proportional to $l(l + 1)$,

$$2k \gg (l + 1/2)\left(1 + \frac{4}{\pi^2}\alpha Z\right). \tag{100}$$

For polytropic models, the product αZ (which is independent of any normalization) is an increasing function of the concentration, ranging from $\pi^2/4$ ($n_e = 0$) to ~ 9.5 ($n_e = 3$). In these two particular cases, condition (100) therefore translates into $k \gg (l + 1/2)$ in the homogeneous model and into $k \gg 3.9(l + 1/2)$ for the more centrally condensed polytrope. Hence if we want to achieve the same accuracy in both models for modes of degree $l = 10$ (say) we have to consider modes of order $k \gg 10$ in the homogeneous model and modes of order $k \gg 40$ in the polytrope $n_e = 3$. It thus seems that the more centrally condensed the object, the less accurate the asymptotic approximation.

The fact that the limitation (100) depends on l is not surprising. It has nothing to do with a supposedly inappropriate comparison equation near the "turning points" (as suggested by Gough 1986, p. 138). Indeed, as we have said before, the method devised by Olver (1974) guarantees that the solution is valid *everywhere*, including near the turning points. How then is it possible to lift the restriction imposed by condition (100)? First, we have assumed that the outer solution satisfies an equation in which terms proportional to $l(l + 1)$ are not essential, and second, at the junction point we approximated the Bessel functions appearing in the inner solution by their Hankel's expansion (valid for large values of the argument, but for fixed value of the index). These two essential assumptions cannot be made for high-degree modes. But then, we are faced with two-parameter expansions.

Let us now turn to more quantitative considerations. Since only in a homogeneous model are the frequencies computable with an infinite accuracy, we give in Table 2 the exact values and the approximations derived by means of equations (96) and (97) for this

TABLE 2

THE p-MODES IN THE HOMOGENEOUS MODEL[a]

		WITHOUT COWLING'S APPROXIMATION				$\delta\phi$ NEGLECTED	
l	k	σ_1	$\sigma_2(96)$	$\sigma_2(97)$	Exact	"Exact"	σ_2(T80)
0.....	5	11.0680	10.8084	10.8145	10.8115	10.9949	10.9947
0.....	10	21.6089	21.4783	21.4791	21.4787	21.5716	21.5716
0.....	15	32.1498	32.0623	32.0626	32.0624	32.1248	32.1247
0.....	20	42.6907	42.6249	42.6250	42.6250	42.6719	42.6719
1.....	5	12.1221	11.7899	11.7990	11.7956	11.9639	11.9618
1.....	10	22.6630	22.4888	22.4902	22.4897	22.5784	22.5781
1.....	15	33.2039	33.0855	33.0860	33.0858	33.1462	33.1461
1.....	20	43.7448	43.6551	43.6553	43.6552	43.7010	43.7010
2.....	5	13.1762	12.6925	12.7102	12.7043	12.8607	12.8542
2.....	10	23.7171	23.4554	23.4582	23.4572	23.5423	23.5413
2.....	15	34.2580	34.0779	34.0788	34.0785	34.1371	34.1368
2.....	20	44.7989	44.6615	44.6619	44.6618	44.7065	44.7064
3.....	5	14.2302	13.5302	13.5646	13.5526	13.6992	13.6842
3.....	10	24.7712	24.3827	24.3888	24.3865	24.4684	24.4658
3.....	15	35.3121	35.0418	35.0439	35.0431	35.1001	35.0992
3.....	20	45.8530	45.6455	45.6464	45.6461	45.6899	45.6895
10.....	5	21.6089	18.0721	18.6510	18.4306	18.5386	18.2085
10.....	10	32.1498	30.0207	30.1617	30.0997	30.1661	30.0923
10.....	15	42.6907	41.1370	41.1935	41.1677	41.2163	41.1875
10.....	20	53.2317	52.0026	52.0309	52.0178	52.0562	52.0419

[a] With $\Gamma_1 = 5/3$. The frequencies σ are in units of $\sqrt{(\pi G \rho)}$.

model. All frequencies are normalized by $(\pi G\rho)^{1/2}$. Thus, σ_1 is the first approximation, obtained by neglecting the correcting term T_l; $\sigma_2(96)$ and $\sigma_2(97)$ are second-order approximations, obtained by solving equations (96) and (97), respectively. (In the second-degree eq. [96] only the largest solution is relevant.) For comparison, we recall the corresponding values obtained in the context of Cowling's approximation. (Of course, in the case of a homogeneous model, it is not necessary to use Cowling's approximation; we include it only as a reference.) The "exact" values were computed by Sauvenier-Goffin (1951) and the approximations σ_2(T80) are those derived in T80.

From Table 2, the asymptotic character of the approximations is immediately apparent: the errors decrease as the order of the mode increases. We also observe that for a given order k, the error increases as the degree l increases. Note also how the second-order correction greatly improves the accuracy of the approximations. We now have very good agreement, even for the unfavorable case $l = 10$ and $k = 10$! As we have mentioned before, by comparing the frequencies derived by means of equations (96) and (97) [i.e., $\sigma_2(96)$ and $\sigma_2(97)$] one can estimate the error made in the approximations. In the present case these two values have approximately the same relative error, and the mean value would be more accurate; however, I would not generalize to other models and say that the exact value always lies between $\sigma_2(96)$ and $\sigma_2(97)$! Of particular interest is the fact that, for the lowest-degree modes illustrated in Table 2, the present second-order approximations are much more accurate than the "exact" frequencies computed with Cowling's approximation. This is not true for the high-degree modes since, as we have already mentioned, the asymptotic approximation looses accuracy when l increases, whereas the converse is true for Cowling's approximation.

We now come to the implication of the results discussed so far on the 5 minute solar oscillations. It is sometimes claimed that these observations are not well represented by an asymptotic relation of the type (97) above. First, numerical fits, through the solar data, of the form suggested by equation (97) are not convincing, and the identification of the various constants with physical quantities is hazardous. (Some fits seem to imply a negative n_e.) However, it should be emphasized that there is a conceptual difference between an *asymptotic* approximation and a *numerical* approximation. In the former case, the error is of the order of the first neglected term in the expansion. (In the case of the p-modes, this error decreases as the frequency increases.) On the contrary, in a numerical approximation (such as a least-squares fit), one tries to minimize the mean relative error over a definite range. In this connection it is well known that a good polynomial approximation to $J_0(x)$ in a given range is not identical to its Taylor expansion (see, e.g., Abramowitz and Stegun 1970, eqs. [9.4.1] and [9.1.12]). In the case of the solar 5 minute oscillations, comparison between the observed modes and an asymptotic expression is difficult to make, since at too high a frequency (i.e., when the asymptotic expression should be more accurate) the observational errors tend to increase.

Another example illustrating the distinction between a numerical approximation and an asymptotic one is the following. If we adopt expression (97) to estimate the frequencies, we conclude that

$$S_{k,l} = (\sigma_{k,l} - \sigma_{k-1,l+2}) \frac{2k + l + n_e + 1/2}{2(2l + 3)} \tag{101}$$

should be a constant, equal to $2\alpha/\pi$. This is not observed in the Sun (see Table 3, last column; the numbers in parentheses are the probable errors on $S_{k,l}$). Contrary to the expectation expressed by Gough (1986) and Gabriel (1989), our taking $\delta\Phi$ into account has not modified an earlier (and similar) conclusion. The solar data we have used to compute $S_{k,l}$ are those obtained by Jiménez *et al.* (1988): on the one hand, these appear to be the most extensive ones for the low-degree solar oscillations and, on the other hand, these are the only data (in this range of l) that include error bars (see, e.g., Duvall *et al.* 1988). In close agreement with Duvall (1982) we have let $n_e = 1.43$. Obviously, $S_{k,l}$ is not a constant for the observed solar modes. However, as we have said before, one should not forget about the term $O(1/\sigma^2)$. Indeed, from equation (96), which is asymptotically equivalent to equation (97), we do not conclude at an exact proportionality.

TABLE 3

THE QUANTITY $S_{k,l}$[a]

		$n_e = 0$				
l	k	Exact	Approximation	$n_e = 1.5$ ("exact")	$n_e = 3$ ("exact")	SUN (observed)
0.......	5	46.6108	49.3076	...	...	...
0.......	10	45.8227	46.4418	...	...	...
0.......	15	45.6688	45.9418	59.43	101.91	56.7(2.2)
0.......	20	45.6140	45.7675	59.74	107.37	66.5(2.2)
0.......	25	45.5884	45.6867	59.90	110.74	80.9(9.8)
0.......	30	45.5744	45.6427	60.00	112.88	85.5(9.5)
0.......	35	45.5659	45.6162	...	114.35	...
1.......	5	47.1577	50.4155	...	...	...
1.......	10	45.9973	46.7810	...	...	...
1.......	15	45.7534	46.1062	59.21	97.48	59.1(1.7)
1.......	20	45.6637	45.8644	59.60	103.79	68.3(2.7)
1.......	25	45.6211	45.7506	59.80	107.86	76.6(3.8)
1.......	30	45.5975	45.6880	59.92	110.56	109.4(8.4)
1.......	35	45.5831	45.6500	...	112.44	...
$2\alpha/\pi$ ...	...	45.5422	45.5422	60.15	119	???

[a] Frequencies are expressed in μHz.

For comparison purposes we have evaluated some theoretical $S_{k,l}$'s: in a homogeneous model and in two centrally condensed polytropes, $n_e = 1.5$ and $n_e = 3$. Following Mullan and Ulrich (1988), the scaling of the frequencies has been chosen so that these idealized models have the solar mass and radius ($G = 6.67 \times 10^{-8}$, $M = 1.989 \times 10^{33}$, $R = 6.9627 \times 10^{10}$, in cgs units). The frequencies are now expressed in μHz. For the homogeneous model we have used both the exact frequencies (Pekeris 1938) and the approximate values σ_2(96) above. For the polytropes $n_e = 1.5$ and $n_e = 3$ the "exact" frequencies are those computed by Mullan and Ulrich (1988); the fact that these frequencies were obtained with the help of Cowling's approximation is of no consequence, since, as we have shown, the quantity α does not depend on this approximation, at least to first order. The bottom line in Table 3 gives the value to which $S_{k,l}$ must tend, according to asymptotic theory. For the three polytropes, the convergence is obvious. As expected, it is slower in the most centrally condensed model. What is perhaps significant is that in the homogeneous model (which is convectively unstable), the convergence of $S_{k,l}$ toward $2\alpha/\pi$ proceeds from above, whereas in the polytrope $n_e = 3$ (convectively stable) it proceeds from below. As far as the solar results are concerned, although the $S_{k,l}$'s increase with k, their convergence to a particular value is far from obvious. This could be explained by different factors. Among others, observational errors (either random or systematic) could play a role. On the theoretical side, nonadiabatic effects could have an important effect on the $S_{k,l}$'s. Another possibility is that the assumptions we have made about the structure of the Sun are too crude: ρ and c^2 (and their first two derivatives) are continuous everywhere, the atmosphere is nearly polytropic, c^2 vanishes at the surface as $(R-r)$, to name a few.

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