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The asymptotic theory of stellar acoustic oscillations: a fourth-order approximation for low-degree modes

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ABSTRACT
The asymptotic description of low-degree stellar acoustic modes is extended to fourth order in inverse powers of the cyclical frequency. The accuracy of the asymptotic approximation is tested by comparing the predicted values of the eigenfrequencies and the 'small frequency separations' with those calculated by solving the full equations governing the adiabatic oscillations of a standard solar model.

Key words: stars: interiors – stars: oscillations.

1 INTRODUCTION

Recent progress in solar seismology and the possibility of applying similar techniques to other stars has stimulated interest in detailed studies of the asymptotic properties of high-frequency stellar p-modes (for reviews, see e.g. Gough & Toomre 1991; Libbrecht & Woodard 1991; Vorontsov 1992). Asymptotic analysis gives a convenient description of the observed oscillation frequencies and hence provides insight into the diagnostic capability of the observed frequencies. In solar seismology, such an asymptotic description serves as the basis for developing efficient non-linear inversion techniques.

The simplest leading-order asymptotic approximation is of very limited applicability, and a more accurate second-order asymptotic description is widely used. Such second-order expansions incorporate modifications to the acoustic oscillations due to the effects of buoyancy forces and gravitational perturbations; the additional mathematical complexity is not severe and the properties of this asymptotic description can still be used in an inverse analysis. Second-order asymptotic sound-speed inversions in the solar interior were described by, for example, Vorontsov & Shibahashi (1991) and Vorontsov et al. (in preparation); their application to the seismology of the helium ionization zone was discussed by Gough & Vorontsov (1994), and to the study of penetrative convection at the base of the solar convection zone by Roxburgh & Vorontsov (1993).

A first-order asymptotic description of low-degree stellar p-modes was developed by Vandakurov (1967). For modes of low degree a second-order description was developed by Tassoul (1980) (see also Smeyers & Tassoul 1987) using the Cowling approximation, in which the effects of gravity perturbations are neglected and the governing system of differential equations is reduced from fourth to second order. This analysis was subsequently extended to include gravitational perturbations (Tassoul 1990), the asymptotic expansions being developed in terms of Bessel functions. In the outer layers of a star, where the reflection of trapped acoustic waves takes place, an asymptotic description becomes locally invalid due to the rapid variation of seismic parameters on a scale short compared with the radial wavelength. These outer layers were approximated by a polytrope in Tassoul's analysis, an approximation that enables one to solve the adiabatic wave equation analytically.

A second-order asymptotic description for acoustic oscillations over a wide range of degrees was developed by Vorontsov (1991), using a uniform asymptotic approximation in terms of Airy functions for the complete fourth-order problem (including gravitational perturbations). In this analysis asymptotic solutions for the interior were matched to exact (non-asymptotic) solutions in the outer layers, the structure of the outer layers being quite general and not restricted to a polytropic approximation. The resulting description is thus not purely asymptotic, but composite; it remains useful, however, for developing an inverse analysis, due to the property that the solutions in the surface layers depend only on frequency and not on degree (at least when the degree is not too high). The complicated structure of the surface layers contributes a frequency-dependent 'surface phase shift' to the resulting eigenfrequency equation. This phase shift can be separated from input data in an inverse analysis, the essential feature of this separation being that non-adiabatic effects, which are widely believed to be localized in the outer layers, are separated together with the surface phase shift. The frequency dependence of this phase shift represents a valuable source of information in itself. Specific features in the phase shift produced by the helium ionization zone can be used to give a seismic calibration of solar envelope models (Pamyatnykh, Vorontsov & Däppen 1991; Christensen-Dalsgaard & Pérez

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Hernández 1992) including the helium abundance (Christensen-Dalsgaard & Pérez Hernández 1991; Vorontsov, Baturin & Pamyatnykh 1991) and the equation of state (Vorontsov, Baturin & Pamyatnykh 1992). Low-degree modes are covered as a limiting case of this more general description (Vorontsov 1991). For higher degree modes, a second-order asymptotic description that allows for a variation of the phase shift with degree has been developed by Brodsky & Vorontsov (1993).

Whereas a second-order asymptotic description is satisfactory for intermediate-degree modes, the accuracy for low-degree modes ($\ell=0$–3) is poor, and far from being satisfactory for diagnostic purposes (Vorontsov 1991). For low-degree modes the effects of gravitational perturbations, induced by the perturbations in the high-density stellar core, become large and difficult to treat as a small perturbation. Further, in the central regions of evolved stars, the seismic parameters vary on a scale short compared to the radial wavelength. The diagnostic capabilities of low-degree modes are, however, of particular interest since these are the modes that penetrate into the energy-generating core. For stars other than the Sun these are the only modes that can be detected (due to lack of spatial resolution). An improvement in the asymptotic description of low-degree modes is thus highly desirable.

The present paper presents a straightforward attempt to improve the accuracy, by extending the asymptotic description to include higher order terms. The first-order approximation retains terms of order $1/\omega$, where ω is the angular frequency, but higher order terms in the eigenfrequency equation depend only on even powers of $1/\omega$ (see Vorontsov 1991). Thus the second-order approximation retains terms up to order $1/\omega^2$, but to proceed further we need to retain terms of fourth order. Although the use of such a high-order asymptotic expansion may seem unusual there are a number of reasons, specific to this particular problem, for estimating higher terms.

(1) The contribution from gravitational perturbations is large for low-degree modes, and only enters the asymptotic expansion starting with terms of order $1/\omega^2$; a second-order description thus only incorporates gravitational perturbations in the leading order. The description of the effects of gravitational perturbations by two terms can lead to a significant improvement.

(2) One of the most interesting applications of asymptotic analysis is the simple theoretical description of the 'small frequency separations'

$$\delta\omega_{n,\ell}=\omega_{n,\ell}-\omega_{n-1,\ell+2}, \tag{1}$$

which are sensitive predominantly to the structure of the stellar core. The small frequency separations are known to be influenced strongly by the effects of gravitational perturbations and buoyancy forces (e.g. Gabriel 1989). When a second-order asymptotic description is applied, however, the effects of gravity are cancelled in the frequency differences (Tassoul 1990) (the effects of buoyancy also cancel at this order; Vorontsov 1991). An adequate asymptotic description of the small frequency separations thus explicitly requires higher order terms to be taken into account.

(3) The possibility of evaluating the next-order terms of the asymptotic expansions is of general interest in itself since it gives an indication of the accuracy of the second-order asymptotic descriptions that are currently in use in practical applications.

In Section 2, we derive the fourth-order asymptotic solutions of the adiabatic oscillation equations in the stellar interior. The asymptotic expansions are developed in terms of Bessel functions. In this sense, our analysis is an extension of that of Tassoul (1980, 1990); but the technique that we use is quite different, being more transparent and straightforward. The matching of the asymptotic solutions in the interior with the non-asymptotic solutions in the surface layers and the resulting eigenfrequency equation are described in Section 3. In Section 4, we develop the corresponding higher order expression for the small frequency separations. In Section 5, our theoretical description is compared with earlier works. In Section 6, we compare the exact eigenfrequencies of a standard solar model with those predicted by the asymptotic analysis, and in Section 7 we summarize our conclusions.

2 ASYMPTOTIC SOLUTIONS IN THE STELLAR INTERIOR

By an appropriate choice of dependent variables, the fourth-order system of ordinary differential equations governing the linear adiabatic oscillations of a spherically symmetric star is reduced to a form convenient for asymptotic analysis:

$$\frac{\mathrm{d}}{\mathrm{d}r}\begin{pmatrix}\omega\xi\\ \eta\\ P\\ \omega S\end{pmatrix}=\omega\begin{pmatrix}0 & h\left[\dfrac{\ell(\ell+1)}{\omega^2}-\dfrac{r^2}{c^2}\right] & -\dfrac{1}{h_1}\dfrac{\ell(\ell+1)}{\omega^2} & 0\\ \dfrac{1}{r^2h}\left(1-\dfrac{N^2-4\pi G\rho_0}{\omega^2}\right) & 0 & 0 & \dfrac{\ell(\ell+1)}{h_2r^2\omega^2}\\ \dfrac{4\pi G\rho_0 h_1}{r^2\omega^2} & 0 & 0 & \dfrac{\ell(\ell+1)}{r^2\omega^2}\\ 0 & -\dfrac{4\pi G\rho_0 h_2}{\omega^2} & 1+\dfrac{4\pi G\rho_0}{\omega^2} & 0\end{pmatrix}\begin{pmatrix}\omega\xi\\ \eta\\ P\\ \omega S\end{pmatrix} \tag{2}$$

(Vorontsov 1991). Here $\xi(r)$, $\eta(r)$ and $P(r)$ are related to the radial displacement δr and the Eulerian perturbations of pressure p' and gravitational potential ψ' (in spherical coordinates r, θ, φ) by

$$\delta r = \frac{h_1(r)}{r^2}\,\xi(r)\,Y_{\ell m}(\theta,\varphi)\exp(\mathrm{i}\omega t), \tag{3}$$

$$p' = \rho_0(r)\,h_2(r)\,\eta(r)\,Y_{\ell m}(\theta,\varphi)\exp(\mathrm{i}\omega t), \tag{4}$$

$$\psi' = -P(r)\,Y_{\ell m}(\theta,\varphi)\exp(\mathrm{i}\omega t), \tag{5}$$

$$\ell(\ell+1)\,S(r) = r^2\frac{\mathrm{d}P}{\mathrm{d}r} - 4\pi G\rho_0(r)\,h_1(r)\,\xi(r), \tag{6}$$

where $\rho_0(r)$ is the equilibrium density and

$$h_1(r) = \exp\int_0^r \frac{g(r)}{c^2(r)}\,\mathrm{d}r, \qquad h_2(r) = \exp\int_0^r \frac{N^2(r)}{g(r)}\,\mathrm{d}r, \qquad h(r) = \frac{h_2(r)}{h_1(r)}, \tag{7}$$

where $g(r)$, $c(r)$ and $N(r)$ are the equilibrium distributions of gravitational acceleration, adiabatic sound speed, and Brunt–Väisälä frequency.

The special structure of the matrix of coefficients on the right-hand side of equation (2) allows us to rewrite this equation as a system of two vector equations of the first order:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}r}\begin{pmatrix}\eta\\ P\end{pmatrix} = \omega\mathbf{A}(\omega;r)\begin{pmatrix}\omega\xi\\ \omega S\end{pmatrix}, \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}r}\begin{pmatrix}\omega\xi\\ \omega S\end{pmatrix} = \omega\mathbf{B}(\omega;r)\begin{pmatrix}\eta\\ P\end{pmatrix}, \end{cases} \tag{8}$$

where the elements of the matrices $\mathbf{A}$ and $\mathbf{B}$ correspond to elements of the original matrix in equation (2). By multiplying the first of these equations from the left by $\mathbf{A}^{-1}$ and differentiating with respect to r, this system is transformed to the single second-order vector differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\begin{pmatrix}\eta\\ P\end{pmatrix} + \mathbf{C}(\omega;r)\frac{\mathrm{d}}{\mathrm{d}r}\begin{pmatrix}\eta\\ P\end{pmatrix} - \omega^2\mathbf{D}(\omega;r)\begin{pmatrix}\eta\\ P\end{pmatrix} = 0, \tag{9}$$

which is convenient for asymptotic analysis. The matrices $\mathbf{C} = \mathbf{A}\,\mathrm{d}\mathbf{A}^{-1}/\mathrm{d}r = -(\mathrm{d}\mathbf{A}/\mathrm{d}r)\,\mathbf{A}^{-1}$ and $\mathbf{D} = \mathbf{AB}$ are

$$\mathbf{C}(\omega;r) = \begin{pmatrix} \ln'(r^2h_2) - \dfrac{\{h_1[1-N^2/\omega^2+(4\pi G\rho_0)/\omega^2]\}'}{h_1(1-N^2/\omega^2)} & \dfrac{\{h_1[1-N^2/\omega^2+(4\pi G\rho_0)/\omega^2]\}'}{h_1h_2(1-N^2/\omega^2)} \\[2ex] -\dfrac{h_2\{h_1[(4\pi G\rho_0)/\omega^2]\}'}{h_1(1-N^2/\omega^2)} & \ln'(r^2) + \dfrac{\{h_1[(4\pi G\rho_0)/\omega^2]\}'}{h_1(1-N^2/\omega^2)} \end{pmatrix}, \tag{10}$$

$$\mathbf{D}(\omega;r) = \begin{pmatrix} \left(1-\dfrac{N^2}{\omega^2}\right)\left[\dfrac{\ell(\ell+1)}{\omega^2r^2} - \dfrac{1}{c^2}\right] - \dfrac{4\pi G\rho_0}{\omega^2c^2} & \dfrac{N^2}{\omega^2h_2}\dfrac{\ell(\ell+1)}{\omega^2r^2} \\[2ex] -\dfrac{4\pi G\rho_0h_2}{\omega^2c^2} & \dfrac{\ell(\ell+1)}{\omega^2r^2} \end{pmatrix}, \tag{11}$$

where the prime denotes a radial derivative.

Equation (9) is used to derive the asymptotic solutions in the stellar interior, where we assume $N^2 < \omega^2$ for the high-frequency acoustic modes. In the outer stellar layers, which can include the singular points of equation (9) where $N^2 = \omega^2$, a different transformation of equation (2) will be used.

We look for regular solutions of the vector equation (9) in terms of Bessel functions:

$$\begin{pmatrix}\eta\\ P\end{pmatrix} = \left[\mathbf{Y}_0(r) + \frac{1}{\omega}\mathbf{Y}_1(r) + \frac{1}{\omega^2}\mathbf{Y}_2(r) + \frac{1}{\omega^3}\mathbf{Y}_3(r) + \dots\right]\mathbf{G}(\omega;r), \tag{12}$$

with matrices $\mathbf{Y}_i$ representing the unknown amplitude functions and with the vector

$$\mathbf{G}(\omega;\, r)=\begin{pmatrix} J_{\ell+1/2}(\omega t) \\ \dot{J}_{\ell+1/2}(\omega t)\end{pmatrix}, \tag{13}$$

where $t(r)$ is the unknown phase function, and $\dot{J}_{\ell+1/2}$ denotes the derivative of the Bessel function with respect to its argument. $\mathbf{G}(\omega;\, r)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathbf{G}(\omega;\, r)=\left[\omega\mathbf{Q}_1(r)+\mathbf{Q}_2(r)+\frac{1}{\omega}\mathbf{Q}_3(r)\right]\mathbf{G}(\omega;\, r), \tag{14}$$

where

$$\mathbf{Q}_1(r)=t'\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \qquad \mathbf{Q}_2(r)=\frac{t'}{t}\begin{pmatrix} 0 & 0 \\ 0 & -1\end{pmatrix}, \qquad \mathbf{Q}_3(r)=\left(\ell+\frac{1}{2}\right)^2\frac{t'}{t^2}\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}. \tag{15}$$

The matrices $\mathbf{C}(\omega;\, r)$ and $\mathbf{D}(\omega;\, r)$ are expanded in even powers of $1/\omega$:

$$\mathbf{C}(\omega;\, r)=\mathbf{C}_0(r)+\frac{1}{\omega^2}\mathbf{C}_2(r)+\ldots, \tag{16}$$

$$\mathbf{C}_0(r)=\begin{pmatrix} \ln'(hr^2) & (1/h_2)\ln' h_1 \\ 0 & \ln'(r^2)\end{pmatrix}, \tag{17}$$

$$\mathbf{C}_2(r)=\begin{pmatrix} (N^2)'-4\pi G\rho_0'-4\pi G\rho_0\ln' h_1 & (1/h_2)[-(N^2)'+4\pi G\rho_0'+4\pi G\rho_0\ln' h_1] \\ -h_2(4\pi G\rho_0'+4\pi G\rho_0\ln' h_1) & 4\pi G\rho_0'+4\pi G\rho_0\ln' h_1\end{pmatrix}, \tag{18}$$

$$\mathbf{D}(\omega;\, r)=\mathbf{D}_0(r)+\frac{1}{\omega^2}\mathbf{D}_2(r)+\frac{1}{\omega^4}\mathbf{D}_4(r), \tag{19}$$

$$\mathbf{D}_0(r)=\begin{pmatrix} -\dfrac{1}{c^2} & 0 \\ 0 & 0\end{pmatrix}, \tag{20}$$

$$\mathbf{D}_2(r)=\begin{pmatrix} \dfrac{N^2}{c^2}+\dfrac{\ell(\ell+1)}{r^2}-\dfrac{4\pi G\rho_0}{c^2} & 0 \\ -\dfrac{4\pi G\rho_0 h_2}{c^2} & \dfrac{\ell(\ell+1)}{r^2}\end{pmatrix}, \tag{21}$$

$$\mathbf{D}_4(r)=\begin{pmatrix} -N^2\dfrac{\ell(\ell+1)}{r^2} & \dfrac{N^2}{h_2}\dfrac{\ell(\ell+1)}{r^2} \\ 0 & 0\end{pmatrix}. \tag{22}$$

Note that the matrix $\mathbf{C}(\omega;\, r)$ is independent of degree ℓ, and the expression (19) for $\mathbf{D}(\omega;\, r)$ with three terms is exact.

We substitute the expansions (12), (16) and (19) into equation (9), collect terms of the same order in $1/\omega$, and obtain the system of matrix equations

$$(\mathbf{Q}_1^2-\mathbf{D}_0)\,\mathbf{Y}_0=0, \tag{23a}$$

$$(\mathbf{Q}_1^2-\mathbf{D}_0)\,\mathbf{Y}_1+2\mathbf{Y}_0'\mathbf{Q}_1+\mathbf{Y}_0\mathbf{Q}_1'+\mathbf{Y}_0(\mathbf{Q}_1\mathbf{Q}_2+\mathbf{Q}_2\mathbf{Q}_1)+\mathbf{C}_0\mathbf{Y}_0\mathbf{Q}_1=0, \tag{23b}$$

$$\begin{aligned}(\mathbf{Q}_1^2-\mathbf{D}_0)\,\mathbf{Y}_2&+\mathbf{Y}_0''+2\mathbf{Y}_0'\mathbf{Q}_2+\mathbf{Y}_0\mathbf{Q}_2'+\mathbf{Y}_0(\mathbf{Q}_1\mathbf{Q}_3+\mathbf{Q}_3\mathbf{Q}_1)+\mathbf{Y}_0\mathbf{Q}_2^2+2\mathbf{Y}_1'\mathbf{Q}_1+\mathbf{Y}_1\mathbf{Q}_1'+\mathbf{Y}_1(\mathbf{Q}_1\mathbf{Q}_2+\mathbf{Q}_2\mathbf{Q}_1)+\mathbf{C}_0\mathbf{Y}_0'+\mathbf{C}_0\mathbf{Y}_0\mathbf{Q}_2\\&+\mathbf{C}_0\mathbf{Y}_1\mathbf{Q}_1-\mathbf{D}_2\mathbf{Y}_0=0,\end{aligned} \tag{23c}$$

$$\begin{aligned}(\mathbf{Q}_1^2-\mathbf{D}_0)\,\mathbf{Y}_3&+\mathbf{Y}_1''+2\mathbf{Y}_1'\mathbf{Q}_2+\mathbf{Y}_1\mathbf{Q}_2'+\mathbf{Y}_1(\mathbf{Q}_1\mathbf{Q}_3+\mathbf{Q}_3\mathbf{Q}_1)+\mathbf{Y}_1\mathbf{Q}_2^2+2\mathbf{Y}_0'\mathbf{Q}_3+\mathbf{Y}_0\mathbf{Q}_3'+\mathbf{Y}_0(\mathbf{Q}_2\mathbf{Q}_3+\mathbf{Q}_3\mathbf{Q}_2)+2\mathbf{Y}_2'\mathbf{Q}_1+\mathbf{Y}_2\mathbf{Q}_1'\\&+\mathbf{Y}_2(\mathbf{Q}_1\mathbf{Q}_2+\mathbf{Q}_2\mathbf{Q}_1)+\mathbf{C}_0\mathbf{Y}_1'+\mathbf{C}_0\mathbf{Y}_0\mathbf{Q}_3+\mathbf{C}_0\mathbf{Y}_1\mathbf{Q}_2+\mathbf{C}_0\mathbf{Y}_2\mathbf{Q}_1+\mathbf{C}_2\mathbf{Y}_0\mathbf{Q}_1-\mathbf{D}_2\mathbf{Y}_1=0,\end{aligned} \tag{23d}$$

$$\begin{aligned}(\mathbf{Q}_1^2-\mathbf{D}_0)\,\mathbf{Y}_4&+\mathbf{Y}_2''+2\mathbf{Y}_2'\mathbf{Q}_2+\mathbf{Y}_2\mathbf{Q}_2'+\mathbf{Y}_2(\mathbf{Q}_1\mathbf{Q}_3+\mathbf{Q}_3\mathbf{Q}_1)+\mathbf{Y}_2\mathbf{Q}_2^2+2\mathbf{Y}_1'\mathbf{Q}_3+\mathbf{Y}_1\mathbf{Q}_3'+\mathbf{Y}_1(\mathbf{Q}_2\mathbf{Q}_3+\mathbf{Q}_3\mathbf{Q}_2)+2\mathbf{Y}_3'\mathbf{Q}_1+\mathbf{Y}_3\mathbf{Q}_1'\\&+\mathbf{Y}_3(\mathbf{Q}_1\mathbf{Q}_2+\mathbf{Q}_2\mathbf{Q}_1)+\mathbf{Y}_0\mathbf{Q}_3^2+\mathbf{C}_0\mathbf{Y}_2'+\mathbf{C}_0\mathbf{Y}_1\mathbf{Q}_3+\mathbf{C}_0\mathbf{Y}_2\mathbf{Q}_2+\mathbf{C}_0\mathbf{Y}_3\mathbf{Q}_1+\mathbf{C}_2\mathbf{Y}_0'+\mathbf{C}_2\mathbf{Y}_0\mathbf{Q}_2+\mathbf{C}_2\mathbf{Y}_1\mathbf{Q}_1-\mathbf{D}_2\mathbf{Y}_2-\mathbf{D}_4\mathbf{Y}_0=0.\end{aligned} \tag{23e}$$

Here and below, the radial dependence of all the functions is omitted for conciseness. The original equations are homogeneous, and the solutions admit an arbitrary normalization. The matrices $\mathbf{Y}_1$, $\mathbf{Y}_2, \ldots$ are determined by equations (23) to within an arbitrary multiple of $\mathbf{Y}_0$, so that some orthogonality constraints need to be imposed. We will solve these equations consecutively for $\mathbf{Y}_i$, $i=0, 1, 2, 3$. Equation (23e) contains $\mathbf{Y}_4$, but is needed to determine $\mathbf{Y}_3$. The matrix $\mathbf{Q}_1^2 = -t'^2\mathbf{I}$ is diagonal, and commutes with all $\mathbf{Y}_i$; this property was used in deriving equations (23).

Equation (23a) is homogeneous and has a non-trivial solution if (and only if)

$$\det(\mathbf{Q}_1^2 - \mathbf{D}_0) = -t'^2\left(\frac{1}{c^2} - t'^2\right) = 0. \tag{24}$$

We define

$$t = \int_0^r \frac{\mathrm{d}r}{c}, \tag{25}$$

the constant of integration being fixed by the regularity of the solutions at $r=0$ (see equation 29 below). A second solution with $t'=0$ is irrelevant to our normal-mode analysis as long as there are no gravitational perturbations from the outside of the star; this second solution will be discussed in more detail at the end of this section.

We thus have

$$\mathbf{Y}_0 = \begin{pmatrix} y_{011} & y_{012} \\ 0 & 0 \end{pmatrix} \tag{26}$$

with arbitrary y_{011} and y_{012} (here and below we denote $\mathbf{Y}_i = \{y_{ijk}\}$).

Equation (23b) is equivalent to a system of four scalar differential equations:

$$2t'y'_{012} + \left[t'' - \frac{t'^2}{t} + \ln'(hr^2)\, t'\right] y_{012} = 0, \tag{27a}$$

$$2t'y'_{011} + \left[t'' - \frac{t'^2}{t} + \ln'(hr^2)\, t'\right] y_{011} = 0, \tag{27b}$$

$$\frac{1}{c^2}\, y_{121} = 0, \tag{27c}$$

$$\frac{1}{c^2}\, y_{122} = 0. \tag{27d}$$

Equation (27a) for y_{012} has the general solution $y_{012} = \text{constant} \times (hr^2 t'/t)^{-1/2}$. Near the origin, regular solutions of the governing equations (2) behave as $\eta(r) \sim r^\ell$, $P(r) \sim r^\ell$. With $J_{\ell+1/2}(\omega t) \sim t^{\ell+1/2}$, $\dot{J}_{\ell+1/2}(\omega t) \sim t^{\ell-1/2}$ and $t \sim r$, we have $y_{i11} \sim t^{-1/2}$ and $y_{i12} \sim t^{1/2}$. Regularity of the solutions thus requires

$$y_{012} = 0. \tag{28}$$

Equation (27b) for y_{011} is similar to (27a). Fixing the norm of the solution, we obtain

$$y_{011} = \left(hr^2\,\frac{t'}{t}\right)^{-1/2}. \tag{29}$$

Equations (27c) and (27d) impose two constraints on the matrix $\mathbf{Y}_1$:

$$\mathbf{Y}_1 = \begin{pmatrix} y_{111} & y_{112} \\ 0 & 0 \end{pmatrix}, \tag{30}$$

the elements y_{111} and y_{112} being undetermined.

Equation (23c) gives the four scalar equations

$$2t'y'_{112} + \left[t'' - \frac{t'^2}{t} + \ln'(hr^2)\, t'\right] y_{112} = y''_{011} + \ln'(hr^2)\, y'_{011} + \left[\left(\ell + \frac{1}{2}\right)^2 \frac{t'^2}{t^2} - d_{211}\right] y_{011}, \tag{31a}$$

$$2t'y_{111}'+\left[t''-\frac{t'^2}{t}+\ln'(hr^2)\,t'\right]y_{111}=0, \tag{31b}$$

$$\frac{1}{c^2}\,y_{221}=-d_{221}y_{011}, \tag{31c}$$

$$\frac{1}{c^2}\,y_{222}=0, \tag{31d}$$

where d_{211} and d_{221} are elements of the matrix $\mathbf{D}_2=\{d_{2ij}\}$.

The inhomogeneous equation (31a) gives y_{112}. Dividing this equation by $t'y_{011}$, and using the explicit expressions for $t(r)$ and $y_{011}(r)$, we obtain

$$2\left(\frac{y_{112}}{y_{011}}\right)'=\frac{1}{2}\,c''-\frac{1}{4}\frac{c'^2}{c}+\ell(\ell+1)\left(\frac{1}{ct^2}-\frac{c}{r^2}\right)+c\left(-\frac{1}{2}\ln''h-\frac{1}{4}\ln'^2h-\frac{1}{r}\ln'h-\frac{N^2}{c^2}+\frac{4\pi G\rho_0}{c^2}\right). \tag{32}$$

Near the centre, $c(r)$ and $\rho_0(r)$ can be expanded in even powers of r. We thus have $t(r)=c^{-1}(0)\,r+O(r^3)$; the right-hand side of equation (32) is therefore regular, and can also be expanded in even powers of r. The central boundary conditions require $y_{112}/y_{011}\sim r$ for small r; thus when equation (32) is solved by quadratures the constant of integration is zero. Using the identity

$$\int_0^r\left(\frac{1}{ct^2}-\frac{c}{r^2}\right)\mathrm{d}r=-\int_0^r\frac{\mathrm{d}c}{\mathrm{d}r}\frac{\mathrm{d}r}{r}+\frac{c}{r}-\frac{1}{t}, \tag{33}$$

which can be verified by differentiation, we obtain

$$\frac{y_{112}}{y_{011}}=\frac{\ell(\ell+1)}{2}\left(-\int_0^r\frac{c'\mathrm{d}r}{r}+\frac{c}{r}-\frac{1}{t}\right)+\frac{c'}{4}-\frac{1}{8}\int_0^r\frac{c'^2\mathrm{d}r}{c}+\frac{1}{2}\int_0^r c\left(-\frac{1}{2}\ln''h-\frac{1}{4}\ln'^2h-\frac{1}{r}\ln'h-\frac{N^2}{c^2}+\frac{4\pi G\rho_0}{c^2}\right)\mathrm{d}r. \tag{34}$$

Equation (31b) gives

$$2\left(\frac{y_{111}}{y_{011}}\right)'=0, \tag{35}$$

or $y_{111}=\text{constant}\times y_{011}$. We set constant $=0$ from considerations of orthogonality, so that

$$y_{111}=0. \tag{36}$$

Equations (31c) and (31d) give the elements of the second row of the matrix $\mathbf{Y}_2$:

$$y_{221}=4\pi G\rho_0h_2y_{011}, \tag{37}$$

$$y_{222}=0, \tag{38}$$

while y_{211} and y_{212} remain arbitrary.

Equation (23d) is equivalent to the next quartet of scalar equations:

$$2t'y_{212}'+\left[t''-\frac{t'^2}{t}+\ln'(hr^2)\,t'\right]y_{212}=0, \tag{39a}$$

$$\begin{aligned}2t'y_{211}'+\left[t''-\frac{t'^2}{t}+\ln'(hr^2)\,t'\right]y_{211}=&-y_{112}''+\left[2\frac{t'}{t}-\ln'(hr^2)\right]y_{112}'+\left[\left(\frac{t'}{t}\right)'-\left(\ell+\frac{1}{2}\right)^2\frac{t'^2}{t^2}-\frac{t'^2}{t^2}+\ln'(hr^2)\frac{t'}{t}+d_{211}\right]y_{112}\\&-(4\pi G\rho_0\ln'h_1+c_{211})\,t'y_{011},\end{aligned} \tag{39b}$$

$$\frac{1}{c^2}\,y_{321}=0, \tag{39c}$$

$$\frac{1}{c^2}\,y_{322}=-d_{221}y_{112}+2(4\pi G\rho_0h_2y_{011})'t'+\left(4\pi G\rho_0h_2t''-4\pi G\rho_0h_2\frac{t'^2}{t}+4\pi G\rho_0h_2\ln'r^2t'+c_{221}t'\right)y_{011}, \tag{39d}$$

where c_{221} is the corresponding element of the matrix $\mathbf{C}_2=\{c_{2ij}\}$.

Equation (39a) for y_{212} is the same as equation (27a) for y_{012}; regularity requires

$$y_{212}=0. \tag{40}$$

Equation (39b) for y_{211} can be integrated to yield

$$\frac{y_{211}}{y_{011}}=-\frac{c}{2}\left(\frac{y_{112}}{y_{011}}\right)'+\frac{1}{2t}\frac{y_{112}}{y_{011}}-\frac{1}{2}\left(\frac{y_{112}}{y_{011}}\right)^2-\frac{1}{2}N^2+2\pi G\rho_0. \tag{41}$$

Equations (39c) and (39d) give two elements of matrix $\mathbf{Y}_3$:

$$\mathbf{Y}_3=\begin{pmatrix} y_{311} & y_{312} \\ 0 & y_{322}\end{pmatrix}, \tag{42}$$

with

$$y_{322}=c(4\pi G\rho_0 h_2)'y_{011}+4\pi G\rho_0 h_2 y_{112}, \tag{43}$$

y_{311} and y_{312} remaining undetermined.

Equation (23e) also gives four scalar equations, but only two of them will be used below:

$$\begin{aligned}2t'y_{312}'+\left[t''-\frac{t'^2}{t}+\ln'(hr^2)\,t'\right]y_{312}=&-\frac{1}{h_2}\ln'h_1t'y_{322}+y_{211}''+\ln'(hr^2)\,y_{211}'+\left[\left(\ell+\frac{1}{2}\right)^2\frac{t'^2}{t^2}-d_{211}\right]y_{211}+2\left(\ell+\frac{1}{2}\right)^2\frac{t'}{t^2}y_{112}'\\&+\left[\left(\ell+\frac{1}{2}\right)^2\left(\frac{t'}{t^2}\right)'-\left(\ell+\frac{1}{2}\right)^2\frac{t'^2}{t^3}+\left(\ell+\frac{1}{2}\right)^2\ln'(hr^2)\frac{t'}{t^2}-t'c_{211}\right]y_{112}+\frac{1}{h_2}\ln'h_1(4\pi G\rho_0h_2y_{011})'+c_{211}y_{011}'-d_{411}y_{011},\end{aligned} \tag{44}$$

$$2t'y_{311}'+\left[t''-\frac{t'^2}{t}+\ln'(hr^2)\,t'\right]y_{311}=0. \tag{45}$$

We divide equation (44) by $t'y_{011}$, giving $2(y_{312}/y_{011})'$ on the left-hand side. After somewhat lengthy manipulations of the right-hand side, we obtain

$$\begin{aligned}\frac{y_{312}}{y_{011}}=&-\frac{c}{4}\left[c\left(\frac{y_{112}}{y_{011}}\right)'\right]'-c\frac{\ell(\ell+1)}{2}\left(\frac{1}{t}\frac{y_{112}}{y_{011}}\right)'-\frac{c}{2}\left(\frac{y_{112}}{y_{011}}\right)'\frac{y_{112}}{y_{011}}-\frac{N^2-4\pi G\rho_0}{2}\frac{y_{112}}{y_{011}}-\frac{1}{6}\left(\frac{y_{112}}{y_{011}}\right)^3\\&+\frac{1}{2}\int_0^r c\left\{-\left(\frac{y_{112}}{y_{011}}\right)'^2+\frac{\ell(\ell+1)}{ct}\left[c\left(\frac{y_{112}}{y_{011}}\right)'\right]'-\frac{1}{2}\ln'(hr^2)(N^2-4\pi G\rho_0)'-\frac{1}{2}(N^2-4\pi G\rho_0)''+\ell(\ell+1)\frac{N^2}{r^2}\right\}\mathrm{d}r.\end{aligned} \tag{46}$$

Near the centre, $(y_{112}/y_{011})'$ can be expanded in even powers of r, which makes the second term under the integral regular. The constant of integration is fixed by the central boundary condition $y_{312}/y_{011}\sim r$ at small r. Equation (45), together with considerations of orthogonality, gives

$$y_{311}=0. \tag{47}$$

Collecting the results obtained so far, the matrices $\mathbf{Y}_i$ in the asymptotic expansion (12) for $i=0, 1, 2, 3$ are

$$\mathbf{Y}_0=\begin{pmatrix} y_{011} & 0 \\ 0 & 0\end{pmatrix},\quad \mathbf{Y}_1=\begin{pmatrix} 0 & y_{112} \\ 0 & 0\end{pmatrix},\quad \mathbf{Y}_2=\begin{pmatrix} y_{211} & 0 \\ y_{221} & 0\end{pmatrix},\quad \mathbf{Y}_3=\begin{pmatrix} 0 & y_{312} \\ 0 & y_{322}\end{pmatrix}, \tag{48}$$

with the non-zero elements determined by the expressions (29), (34), (37), (41), (43) and (46). The solutions for $\eta(r)$ and $P(r)$ in the stellar interior are

$$\eta=\left(\frac{ct}{hr^2}\right)^{1/2}\left[1+\frac{1}{\omega^2}\frac{y_{211}}{y_{011}}+O\left(\frac{1}{\omega^4}\right)\right]J_{\ell+1/2}(\omega t)+\left(\frac{ct}{hr^2}\right)^{1/2}\left[\frac{1}{\omega}\frac{y_{112}}{y_{011}}+\frac{1}{\omega^3}\frac{y_{312}}{y_{011}}+O\left(\frac{1}{\omega^5}\right)\right]\dot{J}_{\ell+1/2}(\omega t), \tag{49}$$

$$P=\left(\frac{ct}{hr^2}\right)^{1/2}\left[\frac{1}{\omega^2}4\pi G\rho_0h_2+O\left(\frac{1}{\omega^4}\right)\right]J_{\ell+1/2}(\omega t)+\left(\frac{\cdot ct}{hr^2}\right)^{1/2}\left\{\frac{1}{\omega^3}\left[4\pi G\rho_0h_2\frac{y_{112}}{y_{011}}+c(4\pi G\rho_0h_2)'\right]+O\left(\frac{1}{\omega^5}\right)\right\}\dot{J}_{\ell+1/2}(\omega t). \tag{50}$$

In the outer stellar layers, the density is small, and the gravitational perturbations ψ' induced by the oscillations behave as solutions of Laplace's equation, which are regular at infinity: $P=\text{constant}\times r^{-\ell-1}$. This condition is satisfied identically by the solution (49), (50) with constant $=0$, since $P(r)$ tends to zero when $\rho_0(r)$ becomes small.

The fourth-order system of differential equations (2) has a second solution, which is regular at the centre of the star; this corresponds to the second solution for the phase function $t'(r)=0$ (equation 24). Using the same technique, this solution is determined as

$$\eta=-\frac{1}{\omega^2}\frac{g\ell}{h_2}r^{\ell-1}+O\left(\frac{1}{\omega^4}\right), \tag{51}$$

$$P=r^\ell+O\left(\frac{1}{\omega^2}\right). \tag{52}$$

The leading term is now represented by gravitational perturbations. This solution represents the response of a star to externally induced gravitational perturbations and does not satisfy the condition on the behaviour of gravitational perturbations in the outer layers of the star; it is set equal to zero in the subsequent analysis. This additional solution is not present in the Cowling approximation, since the governing system of differential equations is then only of second order and therefore has just the one solution that is regular at the origin.

3 EIGENFREQUENCY EQUATION

The eigenfrequency equation is obtained by matching the asymptotic solutions of the wave equations in the stellar interior with non-asymptotic solutions near the surface. The matching of the solutions is performed in the low-density envelope, where the Cowling approximation is locally valid. In the neighbourhood of the matching point and above, we neglect $4\pi G\rho_0$ compared with ω^2:

$$\frac{4\pi G\rho_0}{\omega^2}\ll 1, \qquad r\geq r_{\text{match}}. \tag{53}$$

In the outer layers, the sound speed becomes relatively small due to the low temperature, and, since the present analysis is restricted to low-degree modes, we also have

$$\frac{\ell(\ell+1)}{\omega^2}\frac{c^2}{r^2}\ll 1, \qquad r\geq r_{\text{match}}. \tag{54}$$

This condition expresses the approximation that, in the reflecting layers near the surface, trapped acoustic waves propagate almost vertically. For higher degree modes, this approximation can be relaxed (Brodsky & Vorontsov 1993).

In the Cowling approximation, the complete fourth-order system of differential equations (2) degenerates to second order, and will be solved as one scalar second-order equation. It is not convenient, however, to use η as an independent variable because of the apparent singularities in the coefficients of the differential equation at $N^2=\omega^2$. We therefore introduce the new variable

$$\zeta=\left(\frac{c}{hr^2}\right)^{1/2}\xi \tag{55}$$

to describe the solutions in the neighbourhood of the matching point and above. We use subscripts 1 to denote the internal solutions, $r\leq r_{\text{match}}$, and 2 to designate the external solutions, $r\geq r_{\text{match}}$. The eigenfrequency equation follows from the condition that the Wronskian be zero,

$$\zeta_1\frac{d}{dt}\zeta_2-\zeta_2\frac{d}{dt}\zeta_1=0, \qquad r=r_{\text{match}}, \tag{56}$$

where the 'acoustic radius' $t=t(r)$ defined by (25) is taken as the independent variable instead of r.

3.1 Internal solution near the matching point

We transform the internal asymptotic solution (49) to the new variable ζ defined by (55), using

$$\frac{d}{dr}\eta=\frac{\omega^2}{hr^2}\left(1-\frac{N^2}{\omega^2}\right)\xi, \tag{57}$$

one of two equations to which the complete system (2) degenerates in the Cowling approximation. Bessel functions are replaced by Hankel's asymptotic expansions for large arguments in terms of harmonic functions. The resulting expression for ζ_1, which is a linear combination of $\sin(\omega t-\pi\ell/2)$ and $\cos(\omega t-\pi\ell/2)$ with slowly varying coefficients, is then transformed to a single harmonic function – such a representation will be more convenient for matching with the outer, non-asymptotic solution. The analysis is straightforward, although rather lengthy, and leads to

$$\zeta_1=\frac{1}{\omega}\left\{1+\frac{1}{\omega^2}\left[\frac{1}{4}V_0(t)+3\pi G\rho_0(t)-\frac{\ell(\ell+1)}{4}\frac{c^2(t)}{r^2(t)}\right]+O\left(\frac{1}{\omega^4}\right)\right\}\times\cos\left\{\omega t-\frac{\ell}{2}\pi+\frac{1}{\omega}[\tilde{A}_0(t)+\ell(\ell+1)\tilde{A}_\ell(t)]\right.$$
$$\left.+\frac{1}{\omega^3}[\tilde{B}_0(t)+\ell(\ell+1)\tilde{B}_\ell(t)+\ell^2(\ell+1)^2\tilde{B}_{\ell\ell}(t)]+O\left(\frac{1}{\omega^5}\right)\right\},\qquad(58)$$

where

$$V_0(t)=N^2-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}\ln}{\mathrm{d}t}\left(\frac{rh^{1/2}}{c^{1/2}}\right)+\left[\frac{\mathrm{d}\ln}{\mathrm{d}t}\left(\frac{rh^{1/2}}{c^{1/2}}\right)\right]^2,\qquad(59)$$

$$\tilde{A}_0(t)=-\frac{1}{2}\int_0^t\left(V_0-4\pi G\rho_0-\frac{2}{t^2}\right)\mathrm{d}t+\frac{1}{t},\qquad(60)$$

$$\tilde{A}_\ell(t)=-\frac{1}{2}\int_0^t\left(\frac{c^2}{r^2}-\frac{1}{t^2}\right)\mathrm{d}t+\frac{1}{2t},\qquad(61)$$

$$\tilde{B}_0(t)=\frac{1}{8}\frac{\mathrm{d}}{\mathrm{d}t}(V_0-4\pi G\rho_0)-\frac{1}{8}\int_0^t\left[(V_0-4\pi G\rho_0)^2-\frac{4}{t^4}-\frac{b_0}{t^2}\right]\mathrm{d}t+\frac{1}{6t^3}+\frac{b_0}{8t},\qquad(62)$$

$$b_0=4\lim_{t\to0}\left(V_0-4\pi G\rho_0-\frac{2}{t^2}\right),\qquad(63)$$

$$\tilde{B}_\ell(t)=-\frac{1}{8}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{c^2}{r^2}\right)+\frac{1}{2}\frac{c^2}{r^2}\frac{\mathrm{d}\ln}{\mathrm{d}t}\left(\frac{rh^{1/2}}{c^{1/2}}\right)+\frac{1}{4}\int_0^t\left[\frac{c^2}{r^2}\left\{V_0+4\pi G\rho_0-2\left[\frac{\mathrm{d}\ln}{\mathrm{d}t}\left(\frac{rh^{1/2}}{c^{1/2}}\right)\right]^2\right\}-\frac{b_\ell}{t^2}\right]\mathrm{d}t-\frac{b_\ell}{4t},\qquad(64)$$

$$b_\ell=\lim_{t\to0}\left\{V_0+4\pi G\rho_0-2\left[\frac{\mathrm{d}\ln}{\mathrm{d}t}\left(\frac{rh^{1/2}}{c^{1/2}}\right)\right]^2\right\},\qquad(65)$$

$$\tilde{B}_{\ell\ell}(t)=-\frac{1}{8}\int_0^t\left(\frac{c^4}{r^4}-\frac{1}{t^4}-\frac{b_{\ell\ell}}{t^2}\right)\mathrm{d}t+\frac{1}{24t^3}+\frac{b_{\ell\ell}}{8t},\qquad(66)$$

$$b_{\ell\ell}=2\lim_{t\to0}\left(\frac{c^2}{r^2}-\frac{1}{t^2}\right).\qquad(67)$$

Note that V_0, $\tilde{A}_0$, $\tilde{A}_\ell$, $\tilde{B}_0$, $\tilde{B}_\ell$ and $\tilde{B}_{\ell\ell}$ are singular at $t=0$. Near the origin, $V_0(t)$ can be expanded in even powers of t starting with $2/t^2$, c^2/r^2 can be expanded in even powers of t starting with $1/t^2$, and $\mathrm{d}\ln(rh^{1/2}/c^{1/2})/\mathrm{d}t$ can be expanded in odd powers starting with $1/t$. The constants b_0, b_ℓ and $b_{\ell\ell}$ are defined such that the integrands in (62), (64) and (66) are regular.

In deriving expression (58), functions $\tilde{A}_0(t)$ and $\tilde{A}_\ell(t)$ were defined such that

$$\tilde{A}_0+\ell(\ell+1)\tilde{A}_\ell=\frac{y_{112}}{y_{011}}+\frac{\ell(\ell+1)}{2t}-\frac{c}{2}\ln'\left(\frac{c}{hr^2}\right),\qquad(68)$$

and functions $\tilde{B}_0(t)$, $\tilde{B}_\ell(t)$ and $\tilde{B}_{\ell\ell}(t)$ such that

$$\tilde{B}_0+L\tilde{B}_\ell+L^2\tilde{B}_{\ell\ell}=\frac{y_{312}}{y_{011}}-c\left(\frac{y_{211}}{y_{011}}\right)'+\frac{c}{2t}\left(\frac{y_{112}}{y_{011}}\right)'-\frac{L+1}{2t^2}\frac{y_{112}}{y_{011}}+\frac{L(L+6)}{24t^3}-\frac{y_{112}}{y_{011}}\frac{y_{211}}{y_{011}}+\frac{1}{2t}\left(\frac{y_{112}}{y_{011}}\right)^2-c\frac{y_{112}}{y_{011}}\left(\frac{y_{112}}{y_{011}}\right)'-\frac{1}{3}\left(\frac{y_{112}}{y_{011}}\right)^3$$
$$-\frac{c}{2}\ln'\left(\frac{c}{hr^2}\right)\frac{L}{2t^2}+\frac{c}{2}\ln'\left(\frac{c}{hr^2}\right)c\left(\frac{y_{112}}{y_{011}}\right)'+\frac{1}{3}\left[\frac{c}{2}\ln'\left(\frac{c}{hr^2}\right)\right]^3,\qquad(69)$$

where L denotes $\ell(\ell+1)$. The representation of the right-hand sides of equations (68) and (69) by two and three terms, respectively, on the left-hand sides is such as to separate the dependence on the degree ℓ. The equivalence of expressions (60–62), (64) and (66) for $\tilde{A}_0$, $\tilde{A}_\ell(t)$, $\tilde{B}_0(t)$, $\tilde{B}_\ell(t)$, $\tilde{B}_{\ell\ell}(t)$ and the corresponding expressions in terms of y_{ijk} is established by comparing their derivatives with respect to t, and their behaviours at $t\to 0$, to ensure that the expressions do not differ by additive constants.

3.2 External solution

In the outer layers, where the approximations (53) and (54) are valid, we are left with two first-order differential equations. One is (57), and the other is

$$\frac{\mathrm{d}}{\mathrm{d}r}\xi=-\frac{r^2h}{c^2}\eta, \qquad r\geq r_{\mathrm{match}}. \tag{70}$$

Using t and $\zeta(t)$ (defined by equation 55) as independent and dependent variables, respectively, these two equations are transformed to the Schrödinger-type equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\zeta+[\omega^2-V_0(t)]\,\zeta=0, \qquad r\geq r_{\mathrm{match}}, \tag{71}$$

where the 'acoustic potential' $V_0(t)$ is defined by (59).

We solve equation (71) by the method of phase functions (e.g. Babikov 1976), representing the exact (non-asymptotic) solutions as

$$\zeta_2=A(\omega;\,t)\cos\left[\omega(T-t)-\frac{\pi}{4}-\pi\tilde{\alpha}(\omega;\,t)\right], \tag{72}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta_2=\omega A(\omega;\,t)\sin\left[\omega(T-t)-\frac{\pi}{4}-\pi\tilde{\alpha}(\omega;\,t)\right], \tag{73}$$

where $T=t(R)$ denotes the total acoustic radius of the star. For the phase function $\tilde{\alpha}$, we obtain the first-order non-linear equation

$$\frac{\mathrm{d}(\pi\tilde{\alpha})}{\mathrm{d}t}=-\frac{V_0(t)}{\omega}\cos^2\left[\omega(T-t)-\frac{\pi}{4}-\pi\tilde{\alpha}(\omega;\,t)\right]. \tag{74}$$

The external solution is determined by the initial-value problem for equation (74), with the appropriate surface boundary condition. This boundary condition is usually taken to be given by the approximate analytical solution of the wave equation in an isothermal atmosphere, applied at the temperature minimum. This gives

$$-\frac{\pi}{4}-\pi\tilde{\alpha}(\omega;\,T)=\arctan\left.\left\{\frac{c}{\omega}\left[-\frac{1}{2}\frac{g}{c^2}+\left(\frac{\omega^2}{\Gamma_1}-\frac{N^2}{2}\right)\frac{1}{g}+\frac{4-\Gamma_1}{\Gamma_1}\frac{1}{r}\right]\right\}\right|_{t=T} \tag{75}$$

(Vorontsov & Zharkov 1989), where Γ_1 is the adiabatic exponent.

3.3 The eigenfrequency equation

The matching of the internal (58) and the external (72, 73) solutions using equation (56) leads to the eigenfrequency equation

$$\begin{aligned}1&+\frac{1}{\omega^2T}[\tilde{A}_0(t)+\ell(\ell+1)\,\tilde{A}_\ell(t)]+\frac{1}{\omega^4T}[\tilde{B}_0(t)+\ell(\ell+1)\,\tilde{B}_\ell(t)+\ell^2(\ell+1)^2\tilde{B}_{\ell\ell}(t)]\\
&+\frac{1}{4\omega^3T}\left[\ell(\ell+1)\frac{c^2}{r^2}+V_0-4\pi G\rho_0\right]\sin 2\left[\omega(T-t)-\frac{\pi}{4}-\pi\tilde{\alpha}(\omega;\,t)\right]\\
&+\frac{1}{8\omega^4T}\frac{\mathrm{d}}{\mathrm{d}t}\left[\ell(\ell+1)\frac{c^2}{r^2}-V_0-12\pi G\rho_0\right]\left\{1+\cos 2\left[\omega(T-t)-\frac{\pi}{4}-\pi\tilde{\alpha}(\omega;\,t)\right]\right\}+O\left(\frac{1}{\omega^5}\right)=\frac{\pi}{\omega T}\left[n+\frac{\ell}{2}+\frac{1}{4}+\tilde{\alpha}(\omega;\,t)\right],\end{aligned} \tag{76}$$

where n is the radial order (the number of nodes in radial eigenfunctions can be easily counted as described in, for example, Vorontsov & Zharkov 1989).

We now simplify the eigenfrequency equation (76) by taking into account the property that, in the vicinity of the matching point t and above, conditions (53) and (54) are satisfied. Two terms with $\pi G\rho_0$, and $1/(4\omega^3 T)\,\ell(\ell+1)\,c^2/r^2 \sin 2[\omega(T-t)-\pi/4-\pi\tilde{\alpha}(\omega;\,t)]$ can thus be neglected in equation (76), and $\tilde{A}_\ell(t)$, $\tilde{B}_{\ell\ell}(t)$ can be replaced with $\tilde{A}_\ell(T)$, $\tilde{B}_{\ell\ell}(T)$:

$$\frac{\ell(\ell+1)}{\omega^2 T}\left|\tilde{A}_\ell(T)-\tilde{A}_\ell(t)\right| = \frac{\ell(\ell+1)}{2\omega^2 T}\left|\int_t^T \frac{c^2}{r^2}\,\mathrm{d}t\right| < \frac{\ell(\ell+1)}{2\omega^2}\frac{c^2(t)}{r^2(t)} \ll 1, \tag{77}$$

$$\frac{\ell^2(\ell+1)^2}{\omega^4 T}\left|\tilde{B}_{\ell\ell}(T)-\tilde{B}_{\ell\ell}(t)\right| = \frac{\ell^2(\ell+1)^2}{8\omega^4 T}\left|\int_t^T \frac{c^4}{r^4}\,\mathrm{d}t\right| < \frac{\ell^2(\ell+1)^2}{8\omega^4}\frac{c^4(t)}{r^4(t)} \ll 1. \tag{78}$$

We assume further that the periods of the high-order p-modes are significantly smaller than the dynamical time-scale of the star (which corresponds roughly to the periods of the fundamental modes, being approximately 1 h for the Sun):

$$\omega^2 \gg \frac{g(R)}{R}. \tag{79}$$

Using this assumption, in the thin outer domain above the matching point we have $c/r\,|\mathrm{d}\ln c^2/\mathrm{d}t|$, $c/r\,|\mathrm{d}\ln \rho_0/\mathrm{d}t|$ and $c/r\,|\mathrm{d}\ln p_0/\mathrm{d}t|$ much smaller than ω^2, and it is then straightforward to show that $\tilde{B}_\ell(t)$ can be replaced in the eigenfrequency equation (76) by $\tilde{B}_\ell(T)$. Since the dynamical time-scale of the star is of the order of T, this simplification of the eigenfrequency equation means that we neglect terms of order $1/(\omega^4 T^4) \approx 1/n^4$ compared with unity. A term with $\mathrm{d}/\mathrm{d}t[\ell(\ell+1)\,c^2/r^2]$ in equation (76) can also be neglected.

We add $1/(\omega^2 T)[\tilde{A}_0(T)-\tilde{A}_0(t)]+1/(\omega^4 T)[\tilde{B}_0(T)-\tilde{B}_0(t)]$ to both sides of equation (76), thus replacing $\tilde{A}_0(t)$ and $\tilde{B}_0(t)$ with $\tilde{A}_0(T)$ and $\tilde{B}_0(T)$ on the left-hand side, and introduce a new definition for the phase shift on the right-hand side. The final eigenfrequency equation is

$$T+\frac{1}{\omega^2}[A_0+\ell(\ell+1)\,A_\ell]+\frac{1}{\omega^4}[B_0+\ell(\ell+1)\,B_\ell+\ell^2(\ell+1)^2 B_{\ell\ell}] \simeq \frac{\pi}{\omega}\left[n+\frac{\ell}{2}+\frac{1}{4}+\alpha(\omega)\right], \tag{80}$$

with

$$A_0 = -\frac{1}{2}\int_0^T \left(V_0-4\pi G\rho_0-\frac{2}{t^2}\right)\mathrm{d}t+\frac{1}{T}, \tag{81}$$

$$A_\ell = -\frac{1}{2}\int_0^T \left(\frac{c^2}{r^2}-\frac{1}{t^2}\right)\mathrm{d}t+\frac{1}{2T}, \tag{82}$$

$$B_0 = -\frac{1}{8}\int_0^T \left[(V_0-4\pi G\rho_0)^2-\frac{4}{t^4}-\frac{b_0}{t^2}\right]\mathrm{d}t+\frac{1}{6T^3}+\frac{b_0}{8T}, \tag{83}$$

$$B_\ell = \frac{1}{4}\int_0^T \left[\frac{c^2}{r^2}\left\{V_0+4\pi G\rho_0-2\left[\frac{\mathrm{d}\ln}{\mathrm{d}t}\left(\frac{rh^{1/2}}{c^{1/2}}\right)\right]^2\right\}-\frac{b_\ell}{t^2}\right]\mathrm{d}t-\frac{b_\ell}{4T}, \tag{84}$$

$$B_{\ell\ell} = -\frac{1}{8}\int_0^T \left(\frac{c^4}{r^4}-\frac{1}{t^4}-\frac{b_{\ell\ell}}{t^2}\right)\mathrm{d}t+\frac{1}{24T^3}+\frac{b_{\ell\ell}}{8T}, \tag{85}$$

where constants b_0, b_ℓ and $b_{\ell\ell}$ defined by (63), (65) and (67) respectively ensure the regularity of the integrands, and

$$\pi\alpha(\omega) = \pi\tilde{\alpha}(\omega;\,t)-\frac{1}{2\omega}\int_t^T V_0(t)\,\mathrm{d}t-\frac{1}{8\omega^3}\int_t^T V_0^2(t)\,\mathrm{d}t-\frac{V_0(t)}{4\omega^2}\sin 2\left[\omega(T-t)-\frac{\pi}{4}-\pi\tilde{\alpha}(\omega;\,t)\right]$$
$$+\frac{1}{8\omega^3}\frac{\mathrm{d}V_0(t)}{\mathrm{d}t}\cos 2\left[\omega(T-t)-\frac{\pi}{4}-\pi\tilde{\alpha}(\omega;\,t)\right]. \tag{86}$$

The phase shift $\alpha(\omega)$ absorbs the deviation of the exact solutions of equation (71) from the fourth-order asymptotic approximation. This phase shift does not depend on the position of the matching point (i.e. on the value of t on the right-hand side of equation 86), provided that it is taken to be sufficiently deep where the asymptotic approximation becomes locally valid.

4 SMALL FREQUENCY SEPARATIONS

In the first-order asymptotic approximation (when terms of order $1/\omega^2$ and $1/\omega^4$ in the eigenfrequency equation are neglected), modes with the same values of $n+\ell/2$ appear to have the same frequencies. This degeneracy is lifted by the higher order terms. For the 'small frequency separations' defined by equation (1), the fourth-order description gives

$$\delta\omega_{n,\ell}T \simeq \frac{2(2\ell+3)}{\omega}A_\ell\left\{1+\frac{1}{\omega^2 T}[A_0+\ell(\ell+1)A_\ell]\right\}+\frac{2\ell+3}{\omega^3}\{2B_\ell+[(2\ell+3)^2+3]\,B_{\ell\ell}\}. \tag{87}$$

The term of order $1/\omega^2$ in the first set of curly brackets on the right-hand side of the expression (87) contributes formally a term of order $1/\omega^3$ to the result. We can simplify the result by neglecting this contribution, which is small whenever the second-order term in the eigenfrequency equation (80) is significantly smaller than the leading-order term T. In deriving expression (87), we have also neglected a small variation of $\alpha(\omega)$ when ω varies by $\delta\omega$. These two simplifications are valid for the Sun, but can be easily relaxed if inappropriate.

The 'small frequency separations' are more conveniently described as

$$D_{n\ell}=\frac{\delta\nu_{n,\ell}}{2\ell+3}=\frac{\delta\omega_{n,\ell}}{2\pi(2\ell+3)}. \tag{88}$$

From expression (87), neglecting the term of order $1/\omega^2$ in the first curly brackets, we obtain

$$2\pi T D_{n,\ell}\simeq\frac{A_\ell}{\pi\nu}+\frac{1}{(2\pi\nu)^3}\{2B_\ell+[(2\ell+3)^2+3]\,B_{\ell\ell}\}. \tag{89}$$

5 COMPARISON WITH EARLIER WORKS

We first compare our description with the second-order asymptotic analysis of Tassoul (1990, hereafter T90). In both works, the asymptotic solutions in the inner region were constructed using Bessel functions [$J_{\ell+1/2}(\omega t)$ and $\dot{J}_{\ell+1/2}(\omega t)$ in the present work, $J_{\ell+1/2}(\omega t)$ and $J_{\ell+3/2}(\omega t)$ in T90]. In T90, the radial displacement function was used as one of the dependent variables. From the governing equations (2), the radial displacements $h_1\xi/r^2$ can be expressed through our variables η, P as

$$\frac{h_1}{r^2}\xi=\frac{h_2}{\omega^2}\left(1-\frac{N^2}{\omega^2}\right)^{-1}\left(\frac{\mathrm{d}\eta}{\mathrm{d}r}-\frac{1}{h_2}\frac{\mathrm{d}P}{\mathrm{d}r}\right). \tag{90}$$

By substituting the asymptotic solutions (49) and (50), and truncating the result to the second order, we obtain

$$\frac{h_1}{r^2}\xi=\frac{t^{1/2}}{(\rho_0 c)^{1/2}r}\left\{\left[-\frac{1}{\omega}+O\left(\frac{1}{\omega^3}\right)\right]\dot{J}_{\ell+1/2}(\omega t)+\left[\frac{1}{\omega^2}\frac{y_{112}}{y_{011}}-\frac{1}{2\omega^2}\frac{\mathrm{d}\ln}{\mathrm{d}t}\left(\frac{ct}{hr^2}\right)+O\left(\frac{1}{\omega^4}\right)\right]J_{\ell+1/2}(\omega t)\right\} \tag{91}$$

[the norm of the solution has been changed by a factor of $-\rho_0^{-1/2}(0)$]. Tassoul's asymptotic solution for the radial displacements is governed by her variable $C_1^{(\mathrm{T90})}$, defined by expression (49) of T90. It is straightforward to show that

$$C_1^{(\mathrm{T90})}=-\frac{\ell+1}{t}+\frac{y_{112}}{y_{011}}+\frac{\mathrm{d}\ln}{\mathrm{d}t}\left(\frac{rh^{1/2}}{c^{1/2}}\right). \tag{92}$$

Using the recurrence relations for Bessel functions to transform our solution (91) to $J_{\ell+1/2}(\omega t)$, $J_{\ell+3/2}(\omega t)$, we obtain

$$\frac{h_1}{r^2}\xi=\frac{t^{1/2}}{(\rho_0 c)^{1/2}r}\left\{\left[\frac{1}{\omega}+O\left(\frac{1}{\omega^3}\right)\right]J_{\ell+3/2}(\omega t)+\left[\frac{1}{\omega^2}C_1^{(\mathrm{T90})}+O\left(\frac{1}{\omega^4}\right)\right]J_{\ell+1/2}(\omega t)\right\}, \tag{93}$$

which is Tassoul's solution for radial displacements [equation (68) of T90 with $k_f=0$, $k_{\mathrm{in}}=1$]. To second order, our asymptotic solutions in the inner region are thus equivalent to those of T90.

Comparison with T90 is limited to this level and cannot be extended to the eigenfrequency equation, because of the quite different approach of T90 to the description of the effects of the surface layers. The analysis of T90 assumes a polytropic stratification of the outer layers, and thus admits analytic solutions of the adiabatic wave equations in the outer region; our analysis is free from this assumption and requires numerical integration of equation (74). For radial modes, the surface phase shift for a polytropic model has a constant value, which is determined by the polytropic index; in our analysis, it is a function of frequency (which can be a rather complicated function for realisic models of stellar envelopes: in solar seismology it is currently in use for a number of diagnostic purposes).

Our non-asymptotic solutions of the adiabatic equations in the outer layers assume the conditions (53) and (54) to be satisfied. When working with non-radial modes, the second assumption can be lifted, as described by Brodsky & Vorontsov (1993), leading to the surface phase shift being described explicitly as a function of both ω and ℓ. Until some additional constraints are found, however, the phase shift as an arbitrary function of two variables will prevent us from developing any constructive description of the oscillation frequencies – constructive in the sense of diagnostic capability, because the contribution of the inner structure to the oscillation frequencies cannot be separated from the surface effects (the observational data set is also, at most, a function of two variables). For realistic stellar models (temperature, and hence c^2 in the outermost layers much smaller than in the interior), however, when ℓ is not too large, the surface phase shift admits the functional expansion in powers of $\ell(\ell+1)/\omega^2$, with coefficients being functions of frequency alone (Brodsky & Vorontsov 1993). The physical meaning of the parameter $\ell(\ell+1)/\omega^2$ is that $\ell(\ell+1)/\omega^2 \times c^2/r^2$ (which is just the left-hand side of inequality 54) describes the inclination of the acoustic ray paths to the radial direction: when ℓ is small, the acoustic ray paths in the outermost layers are nearly vertical due to small values of c^2. The phase shift $\alpha(\omega)$ used in the present study of the lowest degree modes is just the leading term of this functional expansion. For solar p-modes, the asymptotic description loses its accuracy in the outer 2 per cent of the solar radius, i.e. in the second He ionization zone and above. Just these layers contribute to the 'surface' phase shift; at a frequency of 3 mHz and with $\ell \leq 3$, we have $\ell(\ell+1)/\omega^2 \times c^2/r^2 < 2\times10^{-4}$. In solar seismology, within the current accuracy of frequency measurements, the ℓ-dependence of the surface phase shift becomes significant at $\ell \approx 100$. It thus seems unreasonable to complicate the present study by lifting assumption (54) and to include the next term in the functional expansion for the surface phase shift. This can be done, however, in a rather straightforward way as described by Brodsky & Vorontsov (1993).

Assumption (53) is just the local Cowling approximation, which is valid in the low-density surface layers of the realistic stellar models (for solar p-modes at a frequency of 3 mHz, we have $4\pi G\rho_0/\omega^2 < 3\times10^{-6}$ in the outer layers which contribute to the phase shift). This assumption reduces the fourth-order system of the governing differential equations to second order, simplifying the analysis significantly when the ℓ-dependence of the surface phase shift is to be taken into account. At low degrees, the governing equations degenerate in the outer layers to second order (to the equations of radial adiabatic oscillations), even without the Cowling approximation. Assumption (53) can thus be easily lifted when working with low-degree modes; the additional negative ('attractive') term $-4\pi G\rho_0$ will then appear in the acoustic potential (e.g. Vorontsov & Zharkov 1989). We retain assumption (53) in the present analysis just for simplicity, and in order that the description of the surface effects remain coherent with other works in which the complicated non-polytropic structure of the outer stellar layers was included (Vorontsov & Zharkov 1989 and references therein; Vorontsov 1991; Brodsky & Vorontsov 1993).

6 NUMERICAL RESULTS FOR A STANDARD SOLAR MODEL

The solar p-mode frequencies were computed directly from the asymptotic eigenfrequency equation (80) for the standard solar model 1 of Christensen-Dalsgaard (1982), and then compared with the exact eigenfrequencies determined by solving numerically the full system of the oscillation equations.

Fig. 1 shows the dependence of the phase shift $\alpha(\omega)$ on the position of the matching point. The phase shift achieves its constant value just below the second helium ionization zone, which produces a wave-like perturbation in the acoustic potential at $t \approx 0.82T$, and contributes significantly to the phase shift. Below this layer, numerical computations reveal no significant deviation of the phase shift from a constant value, which indicates that the fourth-order asymptotic approximation becomes

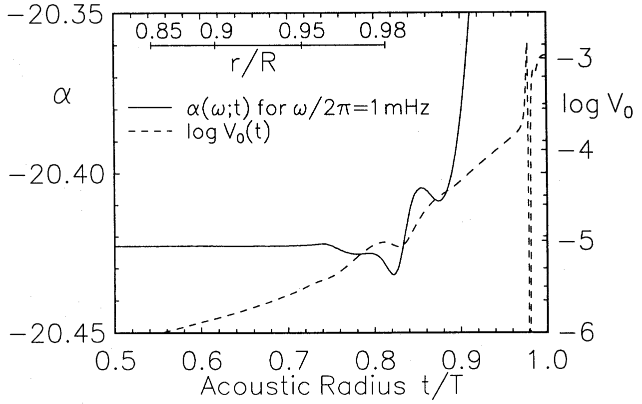

Figure 1. The acoustic phase shift $\alpha(\omega)$ computed for the solar model at a frequency of 1 mHz, versus the position of the matching point. The dashed line shows the acoustic potential.

locally valid to very high accuracy. The computations were carried out at a frequency of 1 mHz, i.e. at the lower frequency end of the observable solar *p*-modes; at higher frequencies, the convergence of the phase shift to a constant value is more rapid.

The kernels of the integrals that determine all the terms on the left-hand side of the eigenfrequency equation are shown in Fig. 2. For ease of comparison, they have been normalized to a frequency of 3 mHz, which is roughly the central frequency of the observable solar *p*-modes. All the kernels have large amplitudes in the solar core, which indicates rather poor convergence of the asymptotic expansions. The fourth-order kernels have almost the same amplitudes as the second-order kernels, and a significant improvement of the description of the eigenfrequencies by including the fourth-order terms can hardly be expected. The convergence will be much better at higher frequencies.

Fig. 3 shows the absolute errors of the asymptotic eigenfrequencies, when compared with the exact eigenfrequencies computed for the same solar model. For radial modes ($\ell=0$) at about 3 mHz, the accuracy is improved by a factor of 4 when using the fourth-order approximation, and the improvement is much better at higher frequencies. There is, however, no improvement below 2 mHz. For non-radial modes, the overall convergence is somewhat weaker.

The asymptotic predictions for the 'small frequency separations' are compared with the exact separations in Fig. 4. At low and intermediate frequencies, the results are very poor, and are far from being useful for diagnostic purposes. At high frequencies, however, there is a systematic difference between the exact values of D_{n0} and D_{n1}, which tends to be described by the fourth-order approximation. This difference (which might be called the 'fine frequency separations') is described by the value of $B_{\ell\ell}$, whose kernel is sharply localized in the energy-generating core. Observational measurement of this quantity would be very informative for diagnostics of the solar core, but requires an accuracy of frequency measurements significantly higher than those that are currently available.

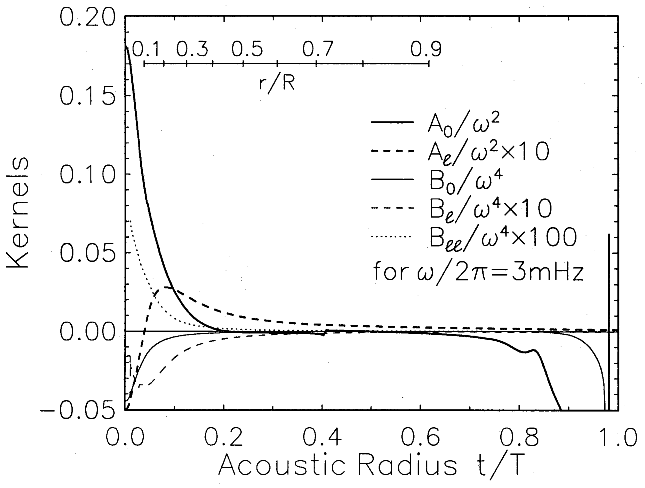

Figure 2. Kernels of the integrals of the second-order terms (thick lines) and of the fourth-order terms (thin lines). The kernels are normalized in such a way that their contributions to the eigenfrequency equation (80) can easily be compared amongst themselves and with the leading-order term. The numerical values of constants in the eigenfrequency equation are $A_0=-3.17\times10^{-2}$ s^{-1}, $A_\ell=7.37\times10^{-4}$ s^{-1}, $B_0=-1.60\times10^{-5}$ s^{-3}, $B_\ell=-2.11\times10^{-7}$ s^{-3}, $B_{\ell\ell}=1.80\times10^{-8}$ s^{-3} and $T=3.599\times10^3$ s.

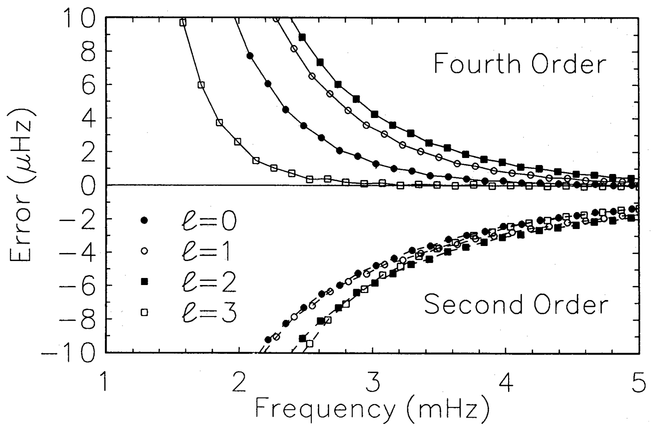

Figure 3. Differences between the asymptotic eigenfrequencies and the exact eigenfrequencies of the standard solar model.

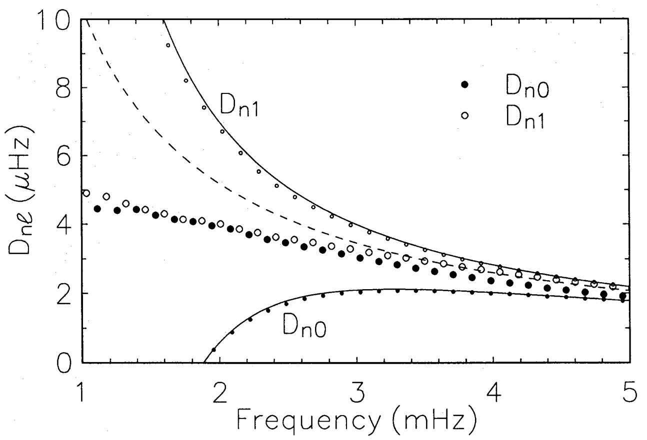

Figure 4. 'Small frequency separations' of solar p-modes. Large circles are separations between the exact eigenfrequencies, and small circles those between the fourth-order asymptotic eigenfrequencies. Solid lines are separations calculated with expression (89). The dashed line corresponds to the second-order approximation.

7 CONCLUSIONS

The fourth-order asymptotic analysis, although rather lengthy, leads to an eigenfrequency equation that is quite compact and readily interpreted. It is especially simple for radial modes, for which all the terms are expressed through the acoustic potential (note that $V_0 - 4\pi G\rho_0$ is the acoustic potential for radial waves when gravitational perturbations are taken into account).

For solar p-modes, in the frequency band of the observable oscillations between 1 and 5 mHz, the convergence of the asymptotic expansions varies significantly, being very poor at low frequencies. The accuracy of the asymptotic eigenfrequencies is substantially improved by using the fourth-order approximation rather than the second-order approximation for frequencies higher than 3 mHz only. The convergence of the asymptotic description will probably be better for less evolved stars, with a lower gradient of chemical composition in the core.

The fourth-order approximation is capable of describing the asymptotic behaviour of the difference $D_{n1} - D_{n0}$ at high frequencies, relating it to the value of $B_{\ell\ell}$ (equation 85). An alternative expression for this quantity is

$$B_{\ell\ell} = -\frac{1}{48}\int_0^R \frac{(c^3)'''}{r}\,dr + \frac{1}{48}\left[\frac{(c^3)''}{r} + \frac{(c^3)'}{r^2} + 2\frac{c^3}{r^3}\right]_{r=R}. \tag{94}$$

The sensitivity of this quantity to the stellar stratification is localized, being much more sharply peaked in the energy-generating core than

$$A_\ell = -\frac{1}{2}\int_0^R \frac{c'}{r}\,dr + \frac{c(R)}{2R}, \tag{95}$$

which determines (in the leading order) the 'small frequency separations' themselves. Precise measurements of high-frequency modes is needed, however, to employ the diagnostic capability of these 'fine' frequency separations.

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REFERENCES

Babikov V. V., 1976, Method of Phase Functions in Quantum Mechanics. Nauka, Moscow
Brodsky M. A., Vorontsov S. V., 1993, ApJ, 409, 455
Christensen-Dalsgaard J., 1982, MNRAS, 199, 735
Christensen-Dalsgaard J., Pérez Hernández F., 1991, in Gough D. O., Toomre J., Challenges to Theories of the Structure of Moderate-Mass Stars. Springer-Verlag, Heidelberg, p. 43
Christensen-Dalsgaard J., Pérez Hernández F., 1992, MNRAS, 257, 62
Gabriel M., 1989, A&A, 226, 278

Gough D., Toomre J., 1991, ARA&A, 29, 627
Gough D. O., Vorontsov S. V., 1994, MNRAS, in press
Libbrecht K. G., Woodard M. F., 1991, Science, 253, 152
Pamyatnykh A. A., Vorontsov S. V., Däppen W., 1991, A&A, 248, 263
Roxburgh I. W., Vorontsov S. V., 1993, in Brown T. M., ed., ASP Conf. Ser. 42, Seismic Investigations of the Sun and Stars. Astron. Soc. Pac., San Francisco, p. 169
Smeyers P., Tassoul M., 1987, ApJS, 65, 429
Tassoul M., 1980, ApJS, 43, 469
Tassoul M., 1990, ApJ, 358, 313
Vandakurov Yu. V., 1967, AZh, 44, 786
Vorontsov S. V., 1991, AZh, 68, 808 (English translation in SvA, 35, 400)
Vorontsov S. V., 1992, AZh, 69, 347 (English translation in SvZ, 36, 175)
Vorontsov S. V., Shibahashi H., 1991, PASJ, 43, 739
Vorontsov S. V., Zharkov V. N., 1989, Sov. Sci. Rev. E., Astrophys. Space Phys., 7, 1
Vorontsov S. V., Baturin V. A., Pamyatnykh A. A., 1991, Nat, 349, 49
Vorontsov S. V., Baturin V. A., Pamyatnykh A. A., 1992, MNRAS, 257, 32