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ASYMPTOTIC APPROXIMATIONS FOR STELLAR NONRADIAL PULSATIONS

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ABSTRACT

The problem of linear nonradial oscillations in stars is reconsidered using sound asymptotic expansions. High-order *p*- and *g*-modes are investigated. In the case of *g*-modes, the possibility of resonance occurs when the model possesses more than one zone in which spatial oscillations may occur.

Subject heading: stars: pulsation

I. INTRODUCTION

Since the pioneering works of Pekeris (1938) and Cowling (1941), the problem of nonradial oscillations of stars has attracted much attention, from the analytical as well as the numerical side. These studies have revealed many interesting properties of pulsating stars (see, e.g., Wolff 1979, and references therein). Most of these results can be explained merely by examining the defining equations. In particular, it is now known that the classification of modes into *p*-, *g*-, and *f*-modes (introduced by Cowling) is not as well defined as was originally believed: in highly condensed stellar models, the eigenfunctions corresponding to some of the modes oscillate in the acoustic-wave propagation zone as well as in the gravity-wave propagation zone. As noted by Scuflaire (1974), this mixed-mode character is typical of low-order modes in highly condensed stars. It is thus not surprising that this aspect of the problem was discovered numerically. No such ambiguity exists for the higher-order modes, for which the characteristic frequencies are either increasingly large (*p*-modes) or vanishingly small (*g*-modes). As the stability of the gravity modes is intimately linked to the absence of convection, particular attention has been attached to centrally condensed models with radiative and convective zones. Until recently, it was believed that each radiative or convective zone is associated with a different sequence of *g*-modes, and that the corresponding spatial oscillations are almost always confined to that one zone. However, Goossens and Smeyers (1974) have found cases of "resonance" between two noncontiguous wave propagation zones in some numerical models. It is obvious that such a phenomenon may require a precision that is difficult to obtain with the existing numerical methods, especially when it comes to high-order modes.

Asymptotic methods then seem appropriate. These are now well documented (see, e.g., Olver 1974; Perdang 1979). Broadly speaking, the methods used to solve differential equations fall into two main groups. In a first group we find Langer's method (1935) and its extension by Olver (e.g., Olver 1956). In these methods the given differential equation is replaced by a simpler one, which has retained the essential features of the original equation but whose solution is known. This solution is then used to express the unknown function. In the second group, one seeks to represent the solution in terms of an integral, which is then evaluated by a method of stationary phase. The latter approach has been suggested by Perdang in the context of nonradial pulsations. In our opinion, to the same degree of approximation, both approaches should lead to the same results.

The first approach has been applied with great success to the investigation of the *radial* pulsations of gaseous stars (e.g., Tassoul and Tassoul 1968*a*). The same degree of success can easily be achieved for the *nonradial* pulsations of the *homogeneous model* (Iweins and Smeyers 1968): the equations can be reduced to a form that has been considered by Olver (1956). When it comes to more realistic models the situation is quite different. Indeed, in this case, the determination of the eigenfrequencies involves the resolution of a fourth-order differential equation, and no general theory exists for such an equation; when the perturbation of the gravitational potential is neglected, the problem is then reduced to second-order only, but because of possible mobile singularities, the theory expounded by Olver cannot be directly applied. Nevertheless tentative solutions have been constructed using various approximations (Iweins 1964; Vandakurov 1967; Smeyers 1968; Tassoul and Tassoul 1968*b*). Because, in general, the validity of the solutions cannot be inferred from an examination of only the first-order approximation, we shall presently establish second-order approximate solutions by means of the asymptotic technique devised by Olver (1956). In § II, the equations governing the nonradial pulsations are cast into forms to which Olver's method can readily be extended (*p*-modes as well as *g*-modes are considered). It will also be shown that only the approach used by Iweins (1964) and

Vandakurov (1967) is satisfactory. Section III is devoted to a description of the general procedure used to obtain the asymptotic solutions. These formulae are then applied to the investigation of p-modes (§ IV). Because all models are not equivalent from the standpoint of convection, the g-modes are investigated in two steps: we first write down general formulae without reference to any specific models (§ V); particular models are then considered (§ VI), and we analyze the possibility of resonance in some configurations. We assume every physical quantity to be continuous everywhere. The influence of discontinuities on the g-frequencies is deferred to the Appendix.

II. NONRADIAL PULSATIONS

The linearized equations governing the nonradial adiabatic pulsations of a spherical star in hydrostatic equilibrium have been established by Pekeris (1938). If the perturbations are assumed to be proportional to $\exp(i\sigma t)$ and to have an angular dependence given by the spherical harmonics $Y_l^m(\theta, \varphi)$, the frequencies σ are obtained as the eigenvalues associated to a fourth-order differential system of equations. However, for the high-order modes of oscillation, it has been shown (see, e.g., Robe 1968) that the approximation which neglects the Eulerian variation of the gravitational potential is a very good one. In this approximation, the basic equations are reduced to the following second-order system:

$$(L_l^2-\sigma^2)\frac{\delta p}{\rho}=\frac{\sigma^2}{l(l+1)}x^2L_l^2(\xi_r'+B\xi_r), \tag{1}$$

$$(\sigma^2-N^2)\xi_r=\left[\left(\frac{\delta p}{\rho}\right)'+A\left(\frac{\delta p}{\rho}\right)\right]. \tag{2}$$

This system involves the radial component of the Lagrangian displacement ξ_r and the Eulerian variation δp of the pressure. In equations (1) and (2), we have used the following definitions:

$$A=\frac{\rho'}{\rho}-\frac{p'}{\Gamma_1 p}, \tag{3}$$

and

$$B=\frac{p'}{\Gamma_1 p}+\frac{2}{x}, \tag{4}$$

the value N, the Brunt-Väisälä frequency, is given by

$$N^2=\frac{p'}{\rho}A, \tag{5}$$

and the critical acoustical frequency L_l by

$$L_l^2=\frac{l(l+1)}{x^2}\frac{\Gamma_1 p}{\rho}. \tag{6}$$

In these equations, x represents the normalized distance r/R, where R is the radius, and Γ_1 denotes a generalized adiabatic exponent. Displacement, density, pressure, and frequencies are measured in units of R, mean density $\bar{\rho}$, $\pi G\bar{\rho}^2R^2$, and $(\pi G\bar{\rho})^{1/2}$, respectively. Primes denote derivatives with respect to x. As is well known, for the solutions ξ_r and $\delta p/\rho$ to comply with the boundary conditions they must remain finite everywhere, and, in particular, at the center ($x=0$) and at the surface ($x=1$).

The form of the asymptotic solutions of the system given in equations (1) and (2) depends on the number and nature of the transition points. (Following Olver, we extend the definition of "transition points" to include singularities.) These are best analyzed by combining equations (1) and (2) to eliminate one of the unknowns. The resulting second-order equations in ξ_r and δp have already been given by Ledoux (1958, eqs. [16.16] and [16.17]). Both equations exhibit *fixed* transition points: the center, the surface, and (when σ^2 is small, i.e., for high-order g-modes) the possible zeros of N^2. Because of factors such as $(\sigma^2-L_l^2)$ or (σ^2-N^2), they may also exhibit *variable* transition points whose location depend on σ^2.

In order to illustrate the difficulties associated with these second-order equations, let us concentrate on the investigation of (stable or unstable) g-modes by means of the equation in ξ_r. This equation has the form

$$\xi''_r+\xi'_r\left[f_1(x)-\frac{L_l^{2\prime}}{L_l^2-\sigma^2}\right]+\xi_r\left[\frac{l(l+1)}{x^2}\left(\frac{\sigma^2}{L_l^2}+\frac{N^2}{\sigma^2}\right)+f_2(x)-B\frac{L_l^{2\prime}}{L_l^2-\sigma^2}\right]=0, \tag{7}$$

where only the terms containing σ^2 have been written down explicitly. If $\sigma^2>0$, this equation has a mobile singularity at x_m (say), where $\sigma^2=L_l^2$. Tassoul and Tassoul (1968*b*) treated the point x_m like a fixed transition point. However, as they did not take into account that the interval $(x_m, 1)$ shrinks to zero for vanishingly small values of σ^2,

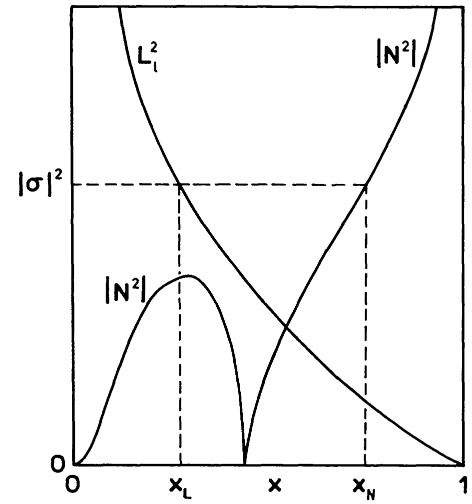

FIG. 1.—Schematic plot of $|N^2|$ and L_l^2 for an idealized model

the solution given by these authors is not uniformly valid. In fact, whether $\sigma^2>0$ or $\sigma^2<0$ (in which case there is no variable singularity) their solutions are not uniformly valid. Indeed, away from the possible variable singularity, the solution of equation (7) was obtained after expanding $1/(\sigma^2-L_l^2)$ in a power series of (σ^2/L_l^2) or (L_l^2/σ^2), depending on the relation of $|\sigma^2|$ with respect to L_l^2, and keeping only the first-order term. From Figure 1, it can be seen that $|\sigma^2/L_l^2|<1$ in the interval $(0, x_L)$, x_L being the point where $|\sigma^2|=L_l^2$; hence, there is no difficulty in extending Olver's method to this modified equation in this interval. By contrast, in the interval $(x_L, 1)$ the series involves increasingly high powers of the ratio $|L_l^2/\sigma^2|$ so that Olver's prescription cannot be applied, i.e., although the ratio $|L_l^2/\sigma^2|$ is smaller than 1, σ^2 is a *small* parameter. Using the series appropriate to the interval $(0, x_L)$ would not resolve the problem either, because then (positive) powers of σ^2/L_l^2 imply an irregular singularity at $x=1$. For these reasons, the calculations of Tassoul and Tassoul (1968*b*) can be considered, at best, as an approximation of the actual solution, certainly not as the first term of an asymptotic expansion.

For sufficiently high-order *g*-modes it is possible to avoid these difficulties. As we just discussed, the construction of the asymptotic expansion of ξ_r by means of equation (7) meets no serious obstacle in the interval $(0, x_L)$, where $|\sigma^2|<L_l^2$; once ξ_r is known the function $\delta p/\rho$ then results from equation (1). On the other hand, the second-order equation in $\delta p/\rho$ has a form analogous to equation (7), with (σ^2-N^2) playing the role of $(\sigma^2-L_l^2)$. The comments we made about equation (7) can be repeated, with minor and obvious changes, about the equation in $\delta p/\rho$: we are not able to solve this equation by current asymptotic methods when $|\sigma^2|>|N^2|$. Thus, let x_N be the point closest to the surface where $|\sigma^2|=|N^2|$. The equation in $\delta p/\rho$ will be considered only in the interval $(x_N, 1)$; equation (2) will then be used to derive ξ_r. From Figure 1 it is clear that for sufficiently high-order modes (i.e., when $|\sigma^2|$ is sufficiently small), we have $x_N<x_L$; hence, both eigenfunctions will be determined over the entire model. This procedure of solving different equations in different regions of the star has actually been used by Iweins (1964) and by Vandakurov (1967) to derive first-order asymptotic *p*- and *g*-frequencies in completely radiative or convective models. However, because of the restriction imposed on x_L and x_N, we should not expect very good agreement for the very first modes of highly condensed models, as these modes have a mixed-mode character (e.g., Scuflaire 1974).

For simplicity of exposition, we now slightly modify the procedure we just described to obtain *g*-modes. Instead of equation (7) we will use the following equation:

$$(\sigma^2\xi_r)''+\left(A+B+\frac{2}{x}\right)(\sigma^2\xi_r)'+\left[\frac{l(l+1)N^2}{\sigma^2x^2}-\frac{l(l+1)}{x^2}+B'+B\left(A+\frac{2}{x}\right)\right](\sigma^2\xi_r)$$
$$=-\frac{l(l+1)\sigma^2}{L_l^2x^2}\left[\left(\frac{\delta p}{\rho}\right)'+\left(A-\frac{L_l^{2\prime}}{L_l^2}\right)\left(\frac{\delta p}{\rho}\right)\right], \quad (8)$$

which results from the partial elimination of the function $\delta p/\rho$ between equations (1) and (2). Because we will use this equation in a region where $|\sigma^2|<L_l^2$, in a very first approximation, its right-hand side can be neglected. For the same reason we will rewrite equation (1) as

$$\frac{\delta p}{\rho}-\frac{x^2}{l(l+1)}\left[(\sigma^2\xi_r)'+B(\sigma^2\xi_r)\right]=\frac{\sigma^2}{L_l^2}\frac{\delta p}{\rho}. \quad (9)$$

In the same way, the second-order differential equation in $\delta p/\rho$ will be replaced by

$$\left(\frac{\delta p}{\rho}\right)''+\left(A+B-\frac{N^{2\prime}}{N^2}\right)\left(\frac{\delta p}{\rho}\right)'+\left[\frac{l(l+1)N^2}{\sigma^2x^2}-\frac{l(l+1)}{x^2}\frac{N^2}{L_l{}^2}+A'+A\left(B-\frac{N^{2\prime}}{N^2}\right)\right]\left(\frac{\delta p}{\rho}\right)=\sigma^2\left[\xi_r'+\left(B-\frac{N^{2\prime}}{N^2}\right)\xi_r\right], \tag{10}$$

and

$$\xi_r+\frac{1}{N^2}\left[\left(\frac{\delta p}{\rho}\right)'+A\left(\frac{\delta p}{\rho}\right)\right]=\frac{\sigma^2}{N^2}\xi_r. \tag{11}$$

Again, the right-hand side of these two equations can be neglected in a first approximation, as they will be solved only where $|\sigma^2|<|N^2|$. Let us note here that Smeyers (1968) used equation (8) (without its right-hand side) to derive an asymptotic solution for ξ_r over the whole interval (0, 1). Because the approximation $|\sigma^2|<L_l{}^2$ breaks down very near the surface, it is not surprising that the solution he derived did not in general satisfy the boundary condition at the surface.

For the p-modes (for which σ^2 is large), the preceding arguments can be repeated with minor and obvious changes. For these modes, equation (7) cannot be solved when $\sigma^2<L_l{}^2$. Hence in the interval $(x_L, 1)$ we replace equations (1) and (2) by

$$\xi_r''+\left[A+B+\frac{2}{x}+\frac{L_l{}^{2\prime}}{L_l{}^2}\right]\xi_r'+\left[\frac{\sigma^2 l(l+1)}{L_l{}^2x^2}-\frac{l(l+1)N^2}{x^2L_l{}^2}+B'+B\left(A+\frac{2}{x}+\frac{L_l{}^{2\prime}}{L_l{}^2}\right)\right]\xi_r$$
$$=\frac{l(l+1)}{\sigma^2x^2}\left[\left(\frac{\delta p}{\rho}\right)'+\left(A+\frac{L_l{}^{2\prime}}{L_l{}^2}\right)\left(\frac{\delta p}{\rho}\right)\right], \tag{12}$$

and

$$\frac{\delta p}{\rho}+\frac{x^2L_l{}^2}{l(l+1)}[\xi_r'+B\xi_r]=\frac{L_l{}^2}{\sigma^2}\frac{\delta p}{\rho}. \tag{13}$$

As before, in a first approximation the right-hand side can be neglected. In the same way the second-order equation in $\delta p/\rho$ cannot be solved when $\sigma^2<|N^2|$. Let us here define x_N as the *first* root of $\sigma^2=|N^2|$. In the interval $(0, x_N)$ we will thus solve the following equations:

$$\left(\frac{\delta p}{\rho}\right)''+(A+B)\left(\frac{\delta p}{\rho}\right)'+\left[\frac{\sigma^2 l(l+1)}{x^2L_l{}^2}-\frac{l(l+1)}{x^2}+A'+AB\right]\frac{\delta p}{\rho}=-\frac{N^2}{\sigma^2}\left[(\sigma^2\xi_r)'+\left(\frac{N^{2\prime}}{N^2}+B\right)(\sigma^2\xi_r)\right], \tag{14}$$

and

$$(\sigma^2\xi_r)-\left[\left(\frac{\delta p}{\rho}\right)'+A\left(\frac{\delta p}{\rho}\right)\right]=\frac{N^2}{\sigma^2}(\sigma^2\xi_r). \tag{15}$$

For sufficiently high-order modes (i.e., if σ^2 is sufficiently large) we have $x_L<x_N$; hence, both eigenfunctions will then be known in the entire model.

We are now ready to solve the basic equations, both for large σ^2 and small σ^2. As usual, the model is divided in several domains in such a way that each domain contains only *one* transition point. In each domain we construct an asymptotic solution, and from the matching of these solutions in their common domain(s) of validity we infer the characteristic frequencies. Consider first the p-modes. In the interval $(0, x_N)$ we solve equations (14) and (15) with the center as a transition point. In the interval $(x_L, 1)$ we use equations (12) and (13) for which the surface is a transition point. For the g-modes, the procedure is similar, although it is quite cumbersome when N^2 has many zeros. (Remember that for g-modes the zeros of N^2 are transition points.) Figure 1 shows that, if $x_N<x_L$, then the interval $(0, x_L)$ contains all the zeros of N^2 and that none is contained in the interval $(x_N, 1)$. Hence, in the interval $(0, x_L)$, the transition points associated with equations (8) and (9) are the center and each zero of N^2, whereas in the interval $(x_N, 1)$ equations (10) and (11) have only the surface as a transition point.

III. ASYMPTOTIC EXPANSIONS

In order to construct asymptotic expansions for the eigenfunctions about each transition point, let us note that in each case the system to be solved has the following symbolic form

$$Z''+M_4Z'+(\varphi\lambda^2+M_2)Z=\frac{1}{\lambda^2}\frac{\phi}{\psi}(X'+M_3X), \tag{16}$$

and

$$X+\psi(Z'+M_1Z)=\frac{1}{\lambda^2}\phi X. \tag{17}$$

The meaning of each symbol can be read by comparison between equations (16) and (17) and each individual equation (8)–(15). The quantities X and Z represent the unknown functions, the other functions depend on x only, and λ^2 is a large positive parameter, proportional to σ^2 in the case of p-modes and proportional to $1/\sigma^2$ for g-modes.

If $h(x)$ satisfies the condition

$$\frac{h'}{h}+\frac{1}{2}M_4=0, \tag{18}$$

then the transformation

$$Z(x,\lambda)=h(x)Z_1(x,\lambda) \tag{19}$$

brings equation (16) to its normal form. For further simplification, we also let

$$X(x,\lambda)=h(x)\psi(x)X_1(x,\lambda). \tag{20}$$

Equations (16) and (17) then become

$$Z_1''+Z_1(\lambda^2\varphi+M_5)=\frac{1}{\lambda^2}\phi(X_1'+M_6X_1) \tag{21}$$

and

$$X_1+Z_1'+Z_1\left(M_1+\frac{h'}{h}\right)=\frac{1}{\lambda^2}\phi X_1, \tag{22}$$

in which we have introduced

$$M_5=M_2+\frac{h''}{h}+\frac{h'}{h}M_4 \tag{23}$$

and

$$M_6=M_3+\frac{h'}{h}+\frac{\psi'}{\psi}. \tag{24}$$

As we mentioned in the preceding section, in a first approximation, the right-hand side of equation (21) may be neglected; this equation is then in a form which has been investigated in detail by Olver (1974; Chapters 10–12). As this last author pointed out, such second-order equations belong to one of three cases, depending on the behavior of the coefficients appearing in the equation. To this end Table 1 gives the behavior, near each transition point, of the (known) functions appearing in equations (21) and (22); n_e is the effective polytropic index of the superficial layers. Basically the same approach is made in all cases. We take a new independent variable u, and new dependent variables X_2 and Z_2 related by

$$Z_1(x,\lambda)=Z_2(u,\lambda)\left[u'^2(x)\right]^{-1/4}, \tag{25}$$

and

$$X_1(x,\lambda)=X_2(u,\lambda)\left[u'^2(x)\right]^{-1/4}u'(x). \tag{26}$$

The new independent variable u satisfies the condition

$$u'^2=|\varphi|u^{-n}, \tag{27}$$

TABLE 1

BEHAVIOR OF THE COEFFICIENTS IN EQUATIONS (16)–(17) NEAR THE TRANSITION POINTS

COEFFICIENT	p-MODES ($x\approx 0$)	g-MODES ($x\approx 0$)	p-MODES AND g-MODES ($x\approx x_j$)	p-MODES AND g-MODES ($x\approx 1$)
φ	$O(1)$	$O(1)$	$O(x-x_j)$	$O[1/(1-x)]$
ψ	$O(1)$	$O(x^2)$	$O(1)$	$O(1-x)$
ϕ	$O(x^2)$	$O(x^2)$	$O(1)$	$O(1-x)$
M_1	$O(x)$	$\frac{2}{x}+O(x)$	$O(1)$	$O[1/(1-x)]$
M_2	$-\frac{l(l+1)}{x^2}+O(1)$	$-\frac{l(l+1)-2}{x^2}+O(1)$	$O(1)$	$O[1/(1-x)]$
M_3	$-\frac{4}{x}+O(x)$	$\frac{2}{x}+O(x)$	$O(1)$	$O[1/(1-x)]$
M_4	$\frac{2}{x}+O(x)$	$\frac{4}{x}+O(x)$	$O(1)$	$-\frac{n_e+1}{1-x}+O(1)$
M_5	$-\frac{l(l+1)}{x^2}+O(1)$	$-\frac{l(l+1)}{x^2}+O(1)$	$O(1)$	$-\frac{(n_e^2-1)}{4(1-x)^2}+O\left(\frac{1}{1-x}\right)$
M_6	$\frac{3}{x}+O(x)$	$\frac{2}{x}+O(x)$	$O(1)$	$O[1/(1-x)]$

where the value of n depends on the transition point: $n=0$ for the center, $n=+1$ for a zero of N^2, and $n=-1$ for the surface. In addition, u is chosen to be positive, except at the transition point where it vanishes. Inserting relations (25)–(27) in equations (21) and (22), we find

$$\frac{d^2Z_2}{du^2}+(\varepsilon\lambda^2u^n+T_2)Z_2=\frac{1}{\lambda^2}\phi\left(\frac{dX_2}{du}+T_3X_2\right), \tag{28}$$

and

$$X_2+\frac{dZ_2}{du}+T_1Z_2=\frac{1}{\lambda^2}\phi X_2, \tag{29}$$

where we define the constant

$$\varepsilon=|\varphi|/\varphi, \tag{30}$$

and the quantities

$$T_1(u)=\frac{1}{u'}\left(M_1+\frac{h'}{h}-\frac{1}{2}\frac{u''}{u'}\right), \tag{31}$$

$$T_2(u)=\frac{1}{u'^2}\left[M_5+\frac{3}{4}\left(\frac{u''}{u'}\right)^2-\frac{1}{2}\frac{u'''}{u'}\right], \tag{32}$$

$$T_3(u)=\frac{1}{u'}\left[M_6+\frac{1}{2}\frac{u''}{u'}\right]. \tag{33}$$

Let us also note that

$$T_3(u)=-T_1(u)+\frac{d\phi}{du}. \tag{34}$$

Let $P_1(u,\lambda)$ be a solution of the comparison equation

$$\frac{\partial^2P_1}{\partial u^2}+\left[\varepsilon\lambda^2u^n-\frac{\tau}{u^2}\right]P_1=0, \tag{35}$$

where τ is a constant. In particular, when $\varepsilon>0$, we have

$$P_1=u^{1/2}J_\nu\left(\frac{2\lambda}{n+2}u^{(n+2)/2}\right), \tag{36}$$

with

$$\nu=\pm\frac{2}{n+2}\left(\tau-\frac{1}{4}\right)^{1/2}, \tag{37}$$

as usual, J_ν denotes standard Bessel function (Magnus, Oberhettinger, and Soni 1966). Then, as a possible solution to equations (28) and (29), we consider the series

$$Z_2(u,\lambda)\sim\sum_{s=0}^{\infty}\frac{E_s(u)}{\lambda^{2s}}P_1(u,\lambda)+\frac{1}{\lambda^2}\sum_{s=0}^{\infty}\frac{F_s(u)}{\lambda^{2s}}\frac{\partial P_1(u,\lambda)}{\partial u}, \tag{38}$$

and

$$X_2(u,\lambda)\sim\sum_{s=0}^{\infty}\frac{G_s(u)}{\lambda^{2s}}\frac{\partial P_1(u,\lambda)}{\partial u}+\sum_{s=0}^{\infty}\frac{H_s(u)}{\lambda^{2s}}P_1(u,\lambda), \tag{39}$$

where E_0 is a constant (which we will choose equal to 1), and the coefficients E_s, F_s, G_s, and H_s are independent of λ. These series formally satisfy equations (28) and (29) if

$$2E_{s+1}=-\frac{dF_s}{du}-\int_0^u T_{20}F_s\,du+\int_0^u\phi\left[H_s+\frac{dG_s}{du}+T_3G_s\right]du, \tag{40}$$

$$G_s=-E_s-\frac{dF_{s-1}}{du}-T_1F_{s-1}+\phi G_{s-1}, \tag{41}$$

$$2\varepsilon F_s=u^{-n/2}\int_0^u u^{-n/2}\mathcal{F}_s\,du, \tag{42}$$

$$H_s=\varepsilon u^nF_s-\frac{\tau}{u^2}F_{s-1}-\frac{dE_s}{du}-E_sT_1+\phi H_{s-1}, \tag{43}$$

where we have introduced

$$T_{20}=T_2+\frac{\tau}{u^2} \tag{44}$$

and

$$\mathcal{F}_s=\frac{2\tau}{u}\frac{d}{du}\left(\frac{F_{s-1}}{u}\right)+\frac{d^2E_s}{du^2}+E_sT_{20}+\phi\left[\varepsilon u^nG_s-\frac{\tau}{u^2}G_{s-1}-\frac{dH_{s-1}}{du}-H_{s-1}T_3\right] \tag{45}$$

($s\geq 0$; we define quantities with negative subscripts to be zero).

The series in expressions (38) and (39) probably diverge. However, in practice, the summations are restricted to the first few terms. If these series represent asymptotic expansions of the solutions, then the error introduced by truncating them is of the order of the first neglected term. For this error to remain uniformly small near $u=0$, it is then necessary for all coefficients E_s, F_s, G_s, and H_s to be of the same order as E_0, F_0, G_0, and H_0, respectively. That this is the case can be readily verified after one chooses τ in such a way that it suppresses the double pole of T_{20} at $u=0$. With this choice of τ the coefficients appearing in equations (40)–(45) exhibit the behavior shown in Table 2. (As in the radial case, we have assumed that every physical equilibrium quantity is an even function of r.) In what

TABLE 2

COEFFICIENTS IN EQUATIONS (40)–(45)

COEFFICIENT	p-MODES ($x\approx 0$)	g-MODES ($x\approx 0$)	p-MODES and g-MODES ($x\approx x_j$)	p-MODES and g-MODES ($x\approx 1$)
τ	$l(l+1)$	$l(l+1)$	0	$(n_e{}^2-1)/4$
n	0	0	1	-1
u	$O(x)$	$O(x)$	$O(x-x_j)$	$O(1-x)$
T_1	$-\frac{1}{u}+O(u)$	$O(u)$	$O(1)$	$O(1/u)$
T_2	$-\frac{l(l+1)}{u^2}+O(1)$	$-\frac{l(l+1)}{u^2}+O(1)$	$O(1)$	$-\frac{1}{4u^2}(n_e{}^2-1)+O\left(\frac{1}{u}\right)$
T_3	$\frac{3}{u}+O(u)$	$\frac{2}{u}+O(u)$	$O(1)$	$O(1/u)$
ϕ	$O(u^2)$	$O(u^2)$	$O(1)$	$O(u)$
E_s	$O(1)$	$O(1)$	$O(1)$	$O(1)$
G_s	$O(1)$	$O(1)$	$O(1)$	$O(1)$
$\mathcal{F}_s$	$O(1)$	$O(1)$	$O(1)$	$O(1/u)$
F_s	$O(u)$	$O(1)$	$O(1)$	$O(u)$
H_s	$O(1/u)$	$O(1/u)$	$O(1)$	$O(1/u)$

follows we will retain only the first term of each series; the relations (40)–(45) then simplify to the following

$$E=-G=1, \tag{46}$$

$$2\varepsilon F=u^{-n/2}\int_0^u u^{-n/2}(T_{20}-\varepsilon\phi u^n)\,du, \tag{47}$$

and

$$H=\varepsilon u^n F-T_1. \tag{48}$$

(For simplicity, we have omitted the subscripts, as there is no more ambiguity.)

Before applying these results to the particular case of p-modes and g-modes, let us make some comments about the auxiliary function P_1 and its derivative, as used in expansions (38) and (39). First, in view of later applications, we let

$$v=\frac{2}{n+2}u^{(n+2)/2}; \tag{49}$$

except for a factor λ, it is precisely the argument of P_1, and it satisfies the simple relation

$$v'^2=\varphi \tag{50}$$

(cf., eq. [27]). Second, because the function J_ν is better known than its derivative, from both the analytical and tabular standpoints, series involving J_ν and $J_{\nu+1}$ are sometimes preferred over series involving J_ν and its derivative. We would then write, neglecting again terms in $1/\lambda^2$ or those smaller,

$$Z_2\sim u^{1/2}J_\nu(\lambda v)-\frac{1}{\lambda}F(u)u^{(n+1)/2}J_{\nu+1}(\lambda v), \tag{51}$$

and

$$X_2\sim\lambda u^{(n+1)/2}J_{\nu+1}(\lambda v)+C(u)u^{1/2}J_\nu(\lambda v), \tag{52}$$

where

$$C(u)=H(u)-\frac{1}{2u}[1+(n+2)\nu] \tag{53}$$

(if $\varepsilon>0$; modified Bessel functions are used when $\varepsilon<0$). When (λv) is large (i.e., in regions where the asymptotic form of the Bessel functions are appropriate), both expansions (truncated eqs. [38] and [39] on the one hand, and eqs. [51] and [52] on the other) are equivalent to the same order of approximation. When (λv) is small, however, the two expansions given for X_2 are not equivalent any more (except when $n=1$, in which case the basic solution is expressible in terms of Airy functions). Indeed, whereas the two terms in the truncated expansion (39) are of the same order of magnitude, in equation (52) the second term is $O[1/(\lambda v)]$ larger than the first. Near the origin $u=0$, the form (39) should thus be preferred.

IV. p-MODES

It is a well known fact that the frequencies associated with p-modes are all real, and that the higher the order of the mode, the higher the frequency. For that reason we let

$$\lambda^2=\sigma^2. \tag{54}$$

We have also shown in § II that for high-order p-modes, equations (1) and (2) have only two fixed transition points: the center and the surface. As we already discussed, near the center we use the system of equations (14) and (15); near the surface we use instead the system of equations (12) and (13). We thus let

$$\varphi=\frac{\rho}{\Gamma_1 p}, \tag{55}$$

and introduce the new variables

$$u_i=v_i=\int_0^x \varphi^{1/2}\,dx, \tag{56}$$

and

$$2u_o^{\ 1/2}=v_o=\int_x^1 \varphi^{1/2}\,dx. \tag{57}$$

Then, in the neighborhood of the center, the eigenfunctions will be approximated by

$$\frac{\delta p}{\rho}\sim k_i\rho^{-1/2}x^{-1}\varphi^{-1/4}v_i^{\ 1/2}\left[J_{l+1/2}(\lambda v_i)-\frac{1}{\lambda}F_i(u_i)J_{l+3/2}(\lambda v_i)\right], \tag{58}$$

and

$$\sigma^2\xi_r\sim -k_i\rho^{-1/2}x^{-1}\varphi^{1/4}v_i^{\ 1/2}\left\{\lambda J_{l+3/2}(\lambda v_i)+\left[-\frac{l+1}{u_i}+H_i(u_i)\right]J_{l+1/2}(\lambda v_i)\right\}. \tag{59}$$

The functions F_i and H_i satisfy equations (47) and (48) with $n=0$ and $\varepsilon=+1$. Similarly, in the neighborhood of the surface, we write

$$\xi_r\sim k_o\rho^{-1/2}x^{-1}\varphi^{1/4}v_o^{\ 1/2}\left[J_{n_e}(\lambda v_o)-\frac{2}{\lambda v_o}F_o(u_o)J_{n_e+1}(\lambda v_o)\right], \tag{60}$$

and

$$\frac{\delta p}{\rho}\sim -k_o\rho^{-1/2}x^{-1}\varphi^{-1/4}v_o^{\ 1/2}\left\{\lambda J_{n_e+1}(\lambda v_o)+\left[H_o(u_o)-\frac{n_e+1}{2u_o}\right]\frac{v_o}{2}J_{n_e}(\lambda v_o)\right\}, \tag{61}$$

and in this case $n=-1$. In these equations, k_i and k_o are arbitrary constants.

One readily verifies that ξ_r and $\delta p/\rho$ remain finite at the surface. In the same way, it can be shown that, near the center

$$\frac{\delta p}{\rho}\propto x^l, \quad \text{and} \quad \xi_r\propto x^{l-1}. \tag{62}$$

For the radial pulsations ($l=0$), the term in square brackets in equation (59) is regular, and hence, in that case $\xi_r \propto x$. Any further rigorous comparison with the radial case (Tassoul and Tassoul 1968*a*) is impaired by the fact that we are presently neglecting the perturbation of the gravitational potential. On the other hand, from the expansions (58) and (59) it is clear that the first node (i.e., the first zero of ξ_r) lies nearer the center than the first loop (i.e., the first zero of $\delta p/\rho$). This fact was already noticed by Cowling (1941) for polytropes. In addition it can also be shown that for the high-order modes the nodes and the loops interlace each other, and this is the basis for the classification of pure p-modes as given by Scuflaire (1974).

In order to obtain the characteristic frequencies we now require the continuity of the solutions (58)–(61) at a point x^* of the interval (0, 1). When (λv) is sufficiently large, the Bessel functions may be replaced by their asymptotic expansions for large arguments. To maintain consistency with the approximation we made in expressing the solutions, we keep only the first two terms in $1/\lambda$ in the developments. We then get, after some manipulations,

$$\cos\left[\lambda(v_i+v_o)-\left(l+n_e+\frac{3}{2}\right)\frac{\pi}{2}-\frac{1}{\lambda}V_{io}+O(\lambda^{-2})\right]=0, \tag{63}$$

where

$$2V_{io}=-F_i{}^*-H_i{}^*-\frac{l(l+1)}{v_i{}^*}-u_0{}^{*-1/2}F_0{}^*-u_o{}^{*1/2}H_0{}^*-\frac{1}{4v_0{}^*}(4n_e{}^2-3). \tag{64}$$

An equivalent expression for the frequencies is given by

$$\sigma\int_0^1\left(\frac{\rho}{\Gamma_1 p}\right)^{1/2}dx=\left(2\kappa+l+n_e+\frac{1}{2}\right)\frac{\pi}{2}+\frac{1}{\sigma}V_{io}+O\left(\frac{1}{\sigma^2}\right), \tag{65}$$

where κ is a positive integer. The constant κ has been chosen in such a way that for the high-order modes ξ_r and δp have κ zeros along the radius.

We note that if we neglect V_{io} in expression (65) we recover the first-order approximation obtained by Iweins (1964) and by Vandakurov (1967). In addition, as in the case of the radial pulsations, the frequencies given by equation (65) are independent of the position of x^*. Indeed, by reverting to the original definitions, one can show that V_{io} is also given by

$$2V_{io}=\lim_{\delta v_i\to 0}\left(\int_{\delta v_i}^{v_i^*}v_i{}'^{-2}\Omega_2\,dv_i-\frac{l(l+1)}{\delta v_i}\right)+\frac{2}{v_i{}^*}\Omega_1(x^*)+\lim_{\delta v_o\to 0}\left(\int_{\delta v_o}^{v_o^*}v_o{}'^{-2}\Omega_3\,dv_o-\frac{1}{4\delta v_o}(4n_e{}^2-3)\right), \tag{66}$$

where

$$\Omega_1=\frac{1}{2}\frac{\rho'}{\rho}-\frac{p'}{\Gamma_1 p}-\frac{1}{x}-\frac{1}{4}\frac{\varphi'}{\varphi}, \tag{67}$$

$$\Omega_2=\frac{l(l+1)}{x^2}-A'-\frac{2}{x}A+\frac{1}{2}\left(\frac{\rho'}{\rho}\right)'+\frac{1}{4}\frac{\rho'^2}{\rho^2}+\frac{\rho'}{x\rho}-\frac{1}{16}\frac{\varphi'^2}{\varphi^2}+\frac{1}{4}\left(\frac{\varphi'}{\varphi}\right)', \tag{68}$$

$$\Omega_3=\Omega_2+2A'-A\frac{\varphi'}{\varphi}-\left(\frac{\rho'}{\rho}\right)'-\frac{1}{2}\left(\frac{\varphi'}{\varphi}\right)'+\frac{\varphi'}{\varphi}\left(\frac{1}{2}\frac{\rho'}{\rho}+\frac{1}{4}\frac{\varphi'}{\varphi}+\frac{1}{x}\right)+\frac{2}{x^2}. \tag{69}$$

Because we also have

$$\Omega_3=\Omega_2+2\varphi^{1/2}(\varphi^{-1/2}\Omega_1)', \tag{70}$$

expression (66) can be written

$$2V_{io}=\lim_{\substack{\delta x_i\to 0\\ \delta x_o\to 0}}\left\{\int_{\delta x_i}^{1-\delta x_o}\varphi^{-1/2}\Omega_2\,dx-\frac{l(l+1)}{[\varphi(\delta x_i)]^{1/2}\,\delta x_i}-\frac{(2n_e+1)(2n_e+3)-16(n_e+1)/\Gamma_1}{8[\varphi(1-\delta x_o)]^{1/2}\,\delta x_o}\right\}. \tag{71}$$

It can obviously be verified that the singularities appearing in this last definition are only apparent.

TABLE 3

THE p-MODES IN THE HOMOGENEOUS MODEL, WITH $\Gamma_1=5/3$

l	κ	σ_{ex}	$\sigma_{(1)}$	$\sigma_{(2)}$	$\sigma_{(1)}'$	$\sigma_{(2)}'$
1	5...	11.7956	12.1221	11.9618	11.9011	11.7952
	10...	22.4897	22.6630	22.5781	22.5452	22.4896
	15...	33.0858	33.2039	33.1461	33.1236	33.0858
	20...	43.6552	43.7448	43.7010	43.6839	43.6552
2	5...	12.7043	13.1762	12.8542	12.9746	12.7015
	10...	23.4572	23.7171	23.5413	23.6048	23.4568
	15...	34.0785	34.2580	34.1368	34.1802	34.0783
	20...	44.6618	44.7989	44.7064	44.7394	44.6617
10	5...	18.4306	21.6089	18.2085	21.4950	18.1231
	10...	30.0997	32.1498	30.0923	32.0697	30.0310
	15...	41.1677	42.6908	41.1874	42.6295	41.1409
	20...	52.0178	53.2317	52.0419	53.1822	52.0045

As an illustration, consider the homogeneous model, in which we assume Γ_1 to be constant and equal to 5/3. In that case equation (65) reduces to

$$(0.9)^{1/2}\sigma=\left(2\kappa+l+\frac{1}{2}\right)-\frac{1}{2(0.9)^{\frac{1}{2}}\sigma}[l(l+1)+1.45]+O(\sigma^{-2}). \tag{72}$$

We applied this formula to obtain approximate frequencies ($\sigma_{(1)}$ and $\sigma_{(2)}$) for a few modes belonging to different spherical harmonics. The results are given in Table 3: the first column gives the exact value σ_{ex}, whereas the second and third columns contain the first- and second-order approximations $\sigma_{(1)}$ and $\sigma_{(2)}$, respectively. A good agreement between the exact and the approximate frequencies is observed, especially for the high-order modes. As is expected, for a fixed value of the spherical harmonics l, the frequencies rapidly converge toward the exact values as the order κ of the mode increases. On the other hand, for fixed values of κ the discrepancy between exact and approximate frequencies increases when we consider higher values of l: this is to be expected since one of the conditions that underlies equation (65) is that $\kappa \gg l$. A different approach must be used to investigate modes for which $l \gg \kappa$ (see, e.g., Denis, Denoyelle, and Smeyers 1975).

Still, it may appear that the errors on the frequencies are sensibly larger than those obtained for the radial pulsations (Tassoul and Tassoul 1968*a*). The reason is that we are presently neglecting the variation of the gravitational potential. As is well known, in the case of the homogeneous model the exact frequencies are obtained as eigenvalues of a second-order differential equation for which Olver's method is immediately applicable: a first-order asymptotic approximation has been derived by Iweins and Smeyers (1968) and to this order their results coincide with equation (72). A closer approximation is obtained if we let

$$\lambda^2=\frac{2}{\Gamma_1}\left[\frac{3\sigma^2}{4}-\frac{4l(l+1)}{3\sigma^2}+4\right], \tag{73}$$

rather than $\lambda^2=\sigma^2$. Parenthetically this enables us to obtain simultaneously approximate p-modes and g-modes. With this definition, which is practical only for the homogeneous model, a straightforward application of Olver's method then gives

$$\lambda=\left(2\kappa+l+\frac{1}{2}\right)-\frac{1}{2\lambda}\left[\frac{1}{4}+l(l+1)\right]+O\left(\frac{1}{\lambda^2}\right). \tag{74}$$

The frequencies obtained by means of these two formulae are denoted $\sigma'_{(1)}$ and $\sigma'_{(2)}$ (first- and second-order approximations, respectively; see Table 3). As is obvious, the convergence toward the exact values is now greatly improved.

In order to investigate the effect of central condensation on the approximate frequencies, we now consider polytropes of increasing index n_e; Γ_1 is again assumed constant and equal to 5/3. We computed the first-order approximation for p_3-, p_5-, and p_{10}-modes belonging to $l=2$. The results are given in Table 4. (The required integral has been evaluated by using an analytical approximation to the Emden functions; see Service 1977.) We have compared our approximations with the exact numerical results obtained by Robe (1968); the relative, percent error $(\sigma_{(1)}{}^2/\sigma_{ex}{}^2-1)$ also appears in Table 4. In order to interpret these results, we must keep in mind that the differences observed between the two sets of frequencies stem from two distinct approximations: the first when we neglected the

TABLE 4

THE p-MODES IN POLYTROPES, WITH $\Gamma_1=5/3$

n_e	p_3		p_5		p_{10}	
	$\sigma_{(1)}{}^2$	Relative Error	$\sigma_{(1)}{}^2$	Relative Error	$\sigma_{(1)}{}^2$	Relative Error
1.0...........	70.3094	+23.1	141.9822	+10.8	430.2313	+3.4
2.0...........	67.9458	25.4	129.5747	13.7	369.9272	4.8
3.0...........	67.5208	22.1	122.6606	14.2	331.9877	6.4
3.25...........	67.5264	20.0	121.3274	13.5	324.3040	7.0
3.5...........	67.5638	16.7	120.1134	12.5	317.1744	6.8
4.0...........	67.6293	−20.3	117.8372	1.1	303.9529	5.5

perturbation of the gravitational potential, and the second when we constructed the asymptotic expansion. As pointed out by Robe, except for the lowest order modes or for models with nearly constant density, the approximation by which one neglects the variation of the gravitational potential only slightly overestimates the characteristic frequencies. On the other hand we expect the asymptotic approximation to give better and better results as the order κ increases. That this is not always the case (see, e.g., the first-order modes of the more condensed polytropes) can be understood by the fact that, then, some of the assumptions underlying the present expansion cease to be verified: we must remember that we must match the inner and the outer solutions in a domain where $\sigma^2 > L_l{}^2$ and $\sigma^2 > |N^2|$, and that this domain does not always exist for the lowest order modes (see § II).

V. GENERAL RESULTS FOR g-MODES

As is well known, the stability of gravity modes is intimately linked to Schwarzschild's criterion. Convectively stable stars possess stable g-modes of oscillation, and convectively unstable stars possess unstable g-modes. If the star contains both radiative and convective zones, then the g-spectra are split into real eigenvalues (stable modes) and imaginary eigenvalues (unstable modes). Thus in order to investigate stable modes, we will let $\lambda^2 = 1/\sigma^2$, whereas in the case of unstable modes we let $\lambda^2 = -1/\sigma^2$. We then define

$$\varphi = \frac{l(l+1)}{\lambda^2\sigma^2}\frac{N^2}{x^2}. \tag{75}$$

Thus φ will be positive in radiative zones for (stable) g^+-modes, or in convective zones for (unstable) g^--modes; φ will be negative otherwise.

As we noted in § II, the behavior of the eigenfunctions depends strongly on the nature of the transition points. For g-modes, these transition points are the center ($x=0$), the surface ($x=1$), and, possibly, each point ($x=x_j$, say; $j=1,2,\ldots$) where N^2 vanishes; and near each transition point φ can be positive or negative. Because the procedure we adopted in § III requires considering domains which contain *only one* transition point and because all stellar models are not equivalent as far as the sign of φ is concerned, we establish, in this section, some general formulae without any reference to a specific model; the implications of these formulae for a few models will be considered in § VI.

In principle, in a stellar model, there are at most eight types of domains to be considered; each of these domains is characterized by the position in the model of the transition point it contains, by the sign of φ, and by the sign of u'. The theory set forth in § III is applied in the neighborhood of the surface where one uses equations (10) and (11), and also in the neighborhood of all other transition points where equations (8) and (9) are used. In order to condense the expressions for the solutions we introduce the vector $\boldsymbol{S}$ whose components are defined by

$$\sigma^2\xi_r = \rho^{-1/2}x^{-2}|\varphi|^{-1/4}S_1, \tag{76}$$

and

$$l(l+1)\frac{\delta p}{\rho} = \lambda\rho^{-1/2}|\varphi|^{1/4}S_2. \tag{77}$$

We then obtain the following asymptotic expansions.

a) Near $x=0$, we let

$$v_i = u_i = \int_0^x |\varphi|^{1/2}\,dx. \tag{78}$$

Then, if $\varphi>0$, we have

$$\begin{Bmatrix} S_1 \\ S_2 \end{Bmatrix}=\begin{Bmatrix} 1 & -F_i^+/\lambda \\ -\left(H_i^+-\dfrac{l+1}{u_i}\right)\Big/\lambda & -1 \end{Bmatrix}\begin{Bmatrix} k_i v_i^{1/2} J_{l+1/2}(\lambda v_i) \\ k_i v_i^{1/2} J_{l+3/2}(\lambda v_i) \end{Bmatrix}. \tag{79}$$

On the other hand, if $\varphi<0$, we have

$$\begin{Bmatrix} S_1 \\ S_2 \end{Bmatrix}=\begin{Bmatrix} 1 & +F_i^-/\lambda \\ -\left(H_i^--\dfrac{l+1}{u_i}\right)\Big/\lambda & 1 \end{Bmatrix}\begin{Bmatrix} k_i v_i^{1/2} I_{l+1/2}(\lambda v_i) \\ k_i v_i^{1/2} I_{l+3/2}(\lambda v_i) \end{Bmatrix}. \tag{80}$$

b) Near $x=1$, we let

$$v_o=2u_o^{1/2}=\int_x^1|\varphi|^{1/2}\,dx. \tag{81}$$

Then, if $\varphi>0$, we get

$$\begin{Bmatrix} S_1 \\ S_2 \end{Bmatrix}=\begin{Bmatrix} u_o^{1/2}\left[H_0^+-\dfrac{1}{2u_o}(1+n_e)\right] & 1 \\ -1 & +u_o^{-1/2}F_o^+/\lambda \end{Bmatrix}\begin{Bmatrix} k_o v_o^{1/2} J_{n_e}(\lambda v_0) \\ k_o v_o^{1/2} J_{n_e+1}(\lambda v_0) \end{Bmatrix}. \tag{82}$$

On the other hand, if $\varphi<0$, we obtain

$$\begin{Bmatrix} S_1 \\ S_2 \end{Bmatrix}=\begin{Bmatrix} -u_0^{1/2}\left[H_o^--\dfrac{1}{2u_o}(1+n_e)\right] & 1 \\ -1 & -u_o^{-1/2}F_o^-/\lambda \end{Bmatrix}\begin{Bmatrix} k_o v_o^{1/2} I_{n_e}(\lambda v_o) \\ k_o v_o^{1/2} I_{n_e+1}(\lambda v_o) \end{Bmatrix}. \tag{83}$$

c) Near $x=x_j$ (jth zero of N^2) we let

$$v_{ja}=\int_x^{x_j}|\varphi|^{1/2}\,dx, \quad v_{jb}=\int_{x_j}^{x}|\varphi|^{1/2}\,dx, \tag{84}$$

and

$$u_{j\alpha}=\left(\frac{3}{2}v_{j\alpha}\right)^{2/3}, \quad \alpha=a,b. \tag{85}$$

Suppose first $\varphi>0$ when $x<x_j$ and $\varphi<0$ when $x>x_j$. We then have

$$\begin{Bmatrix} S_1 \\ S_2 \end{Bmatrix}=\begin{Bmatrix} 1 & u_{ja}^{1/2}F_{ja}^+/\lambda \\ u_{ja}^{-1/2}H_{ja}^+/\lambda & -1 \end{Bmatrix}\begin{Bmatrix} Q_1(v_{ja}) & Q_3(v_{ja}) \\ Q_2(v_{ja}) & Q_4(v_{ja}) \end{Bmatrix}\begin{Bmatrix} k_{j1} \\ k_{j2} \end{Bmatrix} \tag{86}$$

when $x\le x_j$; and

$$\begin{Bmatrix} S_1 \\ S_2 \end{Bmatrix}=\begin{Bmatrix} 1 & u_{jb}^{1/2}F_{jb}^-/\lambda \\ -u_{jb}^{-1/2}H_{jb}^-/\lambda & 1 \end{Bmatrix}\begin{Bmatrix} \mathcal{Q}_1(v_{jb}) & \mathcal{Q}_3(v_{jb}) \\ \mathcal{Q}_2(v_{jb}) & \mathcal{Q}_4(v_{jb}) \end{Bmatrix}\begin{Bmatrix} k_{j1} \\ k_{j2} \end{Bmatrix} \tag{87}$$

when $x \geq x_j$. In these equations, we have introduced the functions

$$Q_1(v) = (v/3)^{1/2}\left[J_{-1/3}(\lambda v) + J_{1/3}(\lambda v)\right], \tag{88}$$

$$Q_2(v) = (v/3)^{1/2}\left[J_{-2/3}(\lambda v) - J_{2/3}(\lambda v)\right], \tag{89}$$

$$Q_3(v) = v^{1/2}\left[J_{-1/3}(\lambda v) - J_{1/3}(\lambda v)\right], \tag{90}$$

$$Q_4(v) = -v^{1/2}\left[J_{-2/3}(\lambda v) + J_{2/3}(\lambda v)\right], \tag{91}$$

$$\mathcal{Q}_1(v) = (v/3)^{1/2}\left[I_{-1/3}(\lambda v) - I_{1/3}(\lambda v)\right], \tag{92}$$

$$\mathcal{Q}_2(v) = (v/3)^{1/2}\left[I_{2/3}(\lambda v) - I_{-2/3}(\lambda v)\right], \tag{93}$$

$$\mathcal{Q}_3(v) = v^{1/2}\left[I_{1/3}(\lambda v) + I_{-1/3}(\lambda v)\right], \tag{94}$$

$$\mathcal{Q}_4(v) = v^{1/2}\left[I_{-2/3}(\lambda v) + I_{2/3}(\lambda v)\right]. \tag{95}$$

The choice of the constants $k_{j\alpha}$ ($\alpha = 1,2$) insures the continuity of S_1 and S_2 across $x = x_j$. This continuity is manifest if the functions Q_i and $\mathcal{Q}_i$ are expressed in terms of Airy functions Ai and Bi, and their derivatives. If, on the other hand, $\varphi < 0$ when $x < x_j$ and $\varphi > 0$ when $x > x_j$, the solutions are then

$$\begin{Bmatrix} S_1 \\ S_2 \end{Bmatrix} = \begin{Bmatrix} 1 & u_{ja}{}^{1/2}F_{ja}{}^{-}/\lambda \\ u_{ja}{}^{-1/2}H_{ja}{}^{-}/\lambda & -1 \end{Bmatrix} \begin{Bmatrix} \mathcal{Q}_1(v_{ja}) & \mathcal{Q}_3(v_{ja}) \\ \mathcal{Q}_2(v_{ja}) & \mathcal{Q}_4(v_{ja}) \end{Bmatrix} \begin{Bmatrix} k_{j1} \\ k_{j2} \end{Bmatrix}, \tag{96}$$

when $x \leq x_j$; and

$$\begin{Bmatrix} S_1 \\ S_2 \end{Bmatrix} = \begin{Bmatrix} 1 & u_{jb}{}^{1/2}F_{jb}{}^{+}/\lambda \\ -u_{jb}{}^{-1/2}H_{jb}{}^{+}/\lambda & 1 \end{Bmatrix} \begin{Bmatrix} Q_1(v_{jb}) & Q_3(v_{jb}) \\ Q_2(v_{jb}) & Q_4(v_{jb}) \end{Bmatrix} \begin{Bmatrix} k_{j1} \\ k_{j2} \end{Bmatrix}, \tag{97}$$

when $x \geq x_j$.

Let us recall that in these expressions for $\boldsymbol{S}$, the functions F and H satisfy equations (47) and (48); the superscript indicates the sign of ε and the subscripts characterize the transition point. As in the case of p-modes, it can be easily verified that ξ_r and $\delta p/\rho$ are finite everywhere, and that near the center

$$\xi_r \propto x^{l-1}, \quad \text{and} \quad \delta p/\rho \propto x^l, \tag{98}$$

as we expected.

Here again, the eigenfrequencies are obtained by matching the individual solutions in their common domain of validity. At the matching points, we use, of course, the asymptotic expression of the comparison functions valid for large arguments. These continuity relations all involve a quantity $W_{\alpha\beta}$ which, in principle, must be evaluated at some point of the interval (x_α, x_β) (cf. eq. [64] in the case of p-modes; x_α and x_β are the positions of two successive transition points; $\alpha, \beta = i, o$, or $1,2,\ldots$). However, it can be shown, along the same lines as before, that the values of the $W_{\alpha\beta}$ do not depend on the actual position of the matching points, and that, in particular

$$2W_{\alpha\beta} = \lim_{\substack{\delta x_\alpha \to 0 \\ \delta x_\beta \to 0}} \left\{ \int_{x_\alpha + \delta x_\alpha}^{x_\beta - \delta x_\beta} |\varphi|^{-1/2}\Omega_4\, dx + \frac{c_\alpha}{|\varphi(x_\alpha + \delta x_\alpha)|^{1/2}\delta x_\alpha} + \frac{c_\beta}{|\varphi(x_\beta - \delta x_\beta)|^{1/2}\delta x_\beta} \right\}, \tag{99}$$

where

$$c_i = -l(l+1), \tag{100}$$

$$c_j = 5/24, \quad j = 1, 2, \ldots, \tag{101}$$

$$c_o = -\frac{1}{8}(2n_e - 1)(2n_e - 3) - \frac{2(n_e+1)}{\Gamma_1}, \tag{102}$$

and

$$\Omega_4 = \frac{l(l+1)}{x^2} - \frac{1}{2}\left(\frac{\rho'}{\rho}\right)' + \frac{1}{4}\frac{\rho'^2}{\rho^2} + A' + \frac{1}{4}\left(\frac{\varphi'}{\varphi}\right)' - \frac{1}{16}\frac{\varphi'^2}{\varphi^2}. \tag{103}$$

Because all stellar models are not equivalent from the standpoint of convection, we now write down all the possible continuity relations, without any reference to a particular model. Let us first consider the matching of *oscillating* solutions ($\varphi > 0$). We distinguish the following cases: (a) Matching of the solutions (79) and (86) in the interval $(0, x_1)$

$$k_{11} \sin \Phi_{i1} + k_{12} \cos \Phi_{i1} = 0, \tag{104}$$

$$k_i = k_{11} \cos \Phi_{i1} - k_{12} \sin \Phi_{i1}, \tag{105}$$

where

$$\Phi_{i1} \sim \lambda \int_0^{x_1} \varphi^{1/2}\, dx - \left(l + \frac{3}{2}\right)\frac{\pi}{2} - \frac{1}{\lambda} W_{i1}. \tag{106}$$

(b) Matching of the solutions (97) and (86) in the interval (x_j, x_{j+1}):

$$k_{j,1} = k_{j+1,1} \cos \Phi_{j,j+1} - k_{j+1,2} \sin \Phi_{j,j+1}, \tag{107}$$

$$k_{j+1,1} = k_{j,1} \cos \Phi_{j,j+1} - k_{j,2} \sin \Phi_{j,j+1}, \tag{108}$$

where

$$\Phi_{j,j+1} \sim \lambda \int_{x_j}^{x_{j+1}} \varphi^{1/2}\, dx - \frac{\pi}{2} - \frac{1}{\lambda} W_{j,j+1}. \tag{109}$$

(c) Matching of solutions (97) and (82) in the interval $(x_J, 1)$, x_J being the last zero of N^2:

$$k_{J1} \cos \Phi_{Jo} - k_{J2} \sin \Phi_{Jo} = 0, \tag{110}$$

$$k_o = k_{J1} \sin \Phi_{Jo} + k_{J2} \cos \Phi_{Jo}, \tag{111}$$

where

$$\Phi_{Jo} \sim \lambda \int_{x_J}^{1} \varphi^{1/2}\, dx - (n_e + 1)\frac{\pi}{2} - \frac{1}{\lambda} W_{Jo}. \tag{112}$$

(d) Matching of the solutions (79) and (82) in the interval $(0, 1)$:

$$\cos \Phi_{io} = 0, \tag{113}$$

$$k_i = k_o \sin \Phi_{io}, \tag{114}$$

where

$$\Phi_{io} \sim \lambda \int_0^1 \varphi^{1/2}\, dx - \left(l + n_e + \frac{3}{2}\right)\frac{\pi}{2} - \frac{1}{\lambda} W_{io}. \tag{115}$$

This case presupposes that φ does not change sign in the interval (0, 1). Compare these last equations with equation (63) derived for *p*-modes.

We next turn to the *exponential-like* solutions ($\varphi<0$). Let us introduce the quantity

$$\Psi_{\alpha\beta}\sim\lambda\int_{x_\alpha}^{x_\beta}|\varphi|^{1/2}\,dx+\frac{1}{\lambda}W_{\alpha\beta}. \tag{116}$$

(Compare with the definition of $\Phi_{\alpha\beta}$). We then have the following continuity relations: (*a*) Matching of the solutions (80) and (96) in the interval $(0, x_1)$:

$$k_{12}=0, \tag{117}$$

$$k_{11}=k_i\exp\Psi_{i1}. \tag{118}$$

(*b*) Matching of the solutions (87) and (96) in the interval (x_j, x_{j+1}):

$$k_{j,1}=2k_{j+1,2}\exp\Psi_{j,j+1}, \tag{119}$$

$$k_{j+1,1}=2k_{j,2}\exp\Psi_{j,j+1}. \tag{120}$$

(*c*) Matching of the solutions (87) and (83) in the interval $(x_J, 1)$:

$$k_{J,2}=0, \tag{121}$$

$$k_{J,1}=k_o\exp\Psi_{Jo}. \tag{122}$$

x_J is obviously the last zero of N^2.

VI. APPLICATIONS FOR *g*-MODES

As we already mentioned, all stellar models are not equivalent from the standpoint of convection. In addition, we have seen that, for the *g*-modes, the properties of the eigenfunctions depend essentially on the sign of the function φ as defined by equation (75), and not on the separate signs of σ^2 and of N^2. We have also shown that the changes of sign of φ (if any) subdivide every model into spatially oscillating regions ($\varphi>0$) and nonoscillating regions ($\varphi<0$). The sign of φ in these zones thus defines the signature of the particular *g*-spectrum (g^+ or g^-) in the model, and it is this signature that determines the behavior of the eigenfunctions. The only patterns we will consider here are: a one-zone model (pattern 1), two-zone models (patterns 2 and 3), and three-zone models (patterns 4 and 5). These simple models will illustrate (*a*) the cases of isolated oscillating regions and (*b*) one case where two spatially oscillating regions interact with one another. The generalization to more complex models is trivial and is not expected to produce any new features.

Consider first models which are either entirely radiative ($N^2>0$) or entirely convective ($N^2<0$). It is well established that in the former case the models possess only a g^+-spectrum ($\sigma^2>0$), whereas in the latter case all *g*-modes are unstable ($\sigma^2<0$). In each case φ, as defined by equation (75), is positive everywhere. Hence the eigenfunctions ξ_r and δp are to be approximated by equation (79) near the center and by equation (82) near the surface; the relevant continuity relations are given by equations (113)–(115). These relations imply that the constants k_i and k_o are equal in magnitude and that, in a *first approximation*, the frequencies are given by

$$\frac{[l(l+1)]^{1/2}}{|\sigma_{(1)}|}\int_0^1\frac{|N|}{x}\,dx=\left(2\kappa+l+n_e+\frac{1}{2}\right)\frac{\pi}{2}, \tag{123}$$

where the positive integer κ is the order of the mode. This expression has been applied to polytropic models (in which $\Gamma_1=5/3$), to estimate the frequencies associated to the g_3-, g_5-, and g_{10}-modes belonging to $l=2$. Because the approximate expressions for the Lane-Emden functions given by Service (1977) are not even functions of x, we used Padé approximants (Pascual 1977) in the immediate neighborhood of the center, in order to calculate the integrand in equation (123). The squared estimated frequencies are tabulated in Table 5, along with the relative errors $(\sigma_{(1)}{}^2/\sigma^2-1)$ in percent; the exact frequencies σ^2 have been taken from Robe (1968). As can be seen, the approximation is quite good. Except for the fact that now the frequencies are generally underestimated, the same remarks apply here as for the *p*-modes.

TABLE 5

THE g-MODES IN POLYTROPES, WITH $\Gamma_1=5/3$

	g_3		g_5		g_{10}	
n_e	$\sigma_{(1)}{}^2$	Relative Error	$\sigma_{(1)}{}^2$	Relative Error	$\sigma_{(1)}{}^2$	Relative Error
1.0...........	−0.0950	−11.2	−0.0470	−5.7	−0.0155	−1.9
2.0...........	+0.2148	−12.4	+0.1126	−7.3	+0.0395	−2.9
3.0...........	1.9900	−18.1	1.0955	−12.2	0.4047	−5.7
3.25...........	3.3886	−19.9	1.8860	−14.0	0.7056	−7.0
3.5...........	6.0229	−21.3	3.3879	−15.8	1.2830	−8.4
4.0...........	24.6230	+2.6	14.1317	−17.0	5.4786	−11.6

Each of the next three models also includes a single oscillating region, but now φ is no longer positive everywhere. Models of type 2 consist of two regions, φ being positive in the inner region and negative in the outer region. The pertinent matching conditions are thus provided by equations (104)–(106) and (121)–(122). We immediately deduce that, in a first approximation,

$$\frac{[l(l+1)]^{1/2}}{|\sigma_{(1)}|}\int_0^{x_1}\frac{|N|}{x}\,dx=\left(2\kappa_i+l-\frac{1}{2}\right)\frac{\pi}{2}, \tag{124}$$

and that

$$|k_i|=|k_{11}|,\quad k_{12}=0, \tag{125}$$

and

$$k_o=k_{11}\exp(-\Psi_{1o}). \tag{126}$$

One may interpret these results by saying that the oscillations present in the inner part of this model are damped in the outer part.

Similarly, in models of type 3, which consist of an inner core where $\varphi<0$ and an outer envelope where $\varphi>0$, we find that, in a first approximation,

$$\frac{[l(l+1)]^{1/2}}{|\sigma_{(1)}|}\int_{x_1}^{1}\frac{|N|}{x}\,dx=(2\kappa_o+n_e)\frac{\pi}{2}, \tag{127}$$

and that

$$|k_o|=|k_{11}|,\quad k_{12}=0, \tag{128}$$

and

$$k_i=k_{11}\exp(-\Psi_{i1}). \tag{129}$$

These relations are a direct consequence of the matching conditions (117)–(118) and (110)–(112). Let us mention at this point that equations (124) and (127) were also obtained by Tassoul and Tassoul (1968*b*), although, as we mentioned in § II, their approach was not satisfactory as far as the asymptotic expansion is concerned.

Pattern 4 consists of three zones: a region where $\varphi>0$ between two regions where $\varphi<0$. It has thus only one oscillating zone, but has two damping zones. As is expected, the results are quite similar to those of patterns 2 and 3. In a first approximation, the frequencies are given by

$$\frac{[l(l+1)]^{1/2}}{|\sigma_{(1)}|}\int_{x_1}^{x_2}\frac{|N|}{x}\,dx=(2\kappa+1)\frac{\pi}{2}. \tag{130}$$

This last result is in perfect agreement with the work of Denis and Smeyers (1977) who introduced Weber functions valid over both zeros of N^2 instead of Airy functions near each zero, as we did here.

The last pattern we will now consider consists likewise of three zones, but in contrast to pattern 4, it has two oscillating zones separated by a region where $\varphi<0$. By combining the relevant continuity relations between the individual solutions we can show that

$$k_i^2=k_{11}^2+k_{12}^2, \quad k_o^2=k_{21}^2+k_{22}^2, \tag{131}$$

$$k_{22}=\frac{1}{2}\exp(-\Psi_{12})k_{11}, \quad k_{12}=\frac{1}{2}\exp(-\Psi_{12})k_{21}, \tag{132}$$

$$\frac{k_i^2}{k_o^2}=-\frac{\sin 2\Phi_{2o}}{\sin 2\Phi_{i1}}, \tag{133}$$

and

$$\tan\Phi_{i1}\cot\Phi_{2o}+\frac{1}{4}\exp(-2\Psi_{12})=0. \tag{134}$$

As a first approach to this system, let us neglect the exponentially small terms, as is reasonable because, from its definition (116), Ψ_{12} is a large positive quantity. Hence, the constants k_{12} and k_{22} vanish and the inner and the outer zones oscillate independently of each other; we thus recover the results of patterns 2 and 3, i.e., the case of isolated oscillating regions. It was this fact that prompted Tassoul and Tassoul (1968*b*) to conclude that, in a stellar model, each zone where φ is positive possesses its own *g*-spectrum and that the corresponding spatial oscillations are almost confined to that one zone. Let, thus, σ_i and σ_o be the natural frequencies derived by means of equations (124) and (127), respectively. (In the present context, the lower limit of integration in eq. [127] should obviously be replaced by x_2.)

Of course, the asymptotic frequencies must satisfy the full equation (134). As the second term in this equation is small, the corrected frequencies differ but little from the natural frequencies. Let us concentrate for the moment on the sequence of frequencies associated with the inner region: for each of these values we have $\tan\Phi_{i1}=0$. In most cases, for the values of σ close to σ_i, of the two factors in equation (134) only $\tan\Phi_{i1}$ is small, $\cot\Phi_{2o}$ being of order unity. As a consequence, the natural frequency is corrected by a quantity proportional to $\exp(-2\Psi_{12})$. Furthermore, if one assumes the oscillations in the core to be of order unity ($k_i=1$), equations (131)–(133) show that the amplitudes of the oscillations in the envelope do not vanish, but are of order $\exp(-\Psi_{12})$.

On the other hand, it happens sometimes that a natural frequency of the core is close to one associated with the envelope, i.e., $\tan\Phi_{i1}$ and $\cot\Phi_{2o}$ vanish for approximately the same value of σ. It is not surprising that in this case the corrections to both natural frequencies are of order $\exp(-\Psi_{12})$. What is perhaps more surprising is that these corrections tend to separate the nearly equal frequencies. A similar phenomenon was observed by Aizenman, Smeyers, and Weigert (1977) and by Shibahashi (1979), although in a different context: the bumping of the lowest order nonradial modes of highly evolved stellar models. It should be noted that, while the corrections to the frequencies are small, they are nevertheless larger than in the previous case. The effect on the eigenfunctions is even more dramatic. Whereas in the approximation of uncoupled modes the oscillations are confined to only one zone, when the coupling between the different regions is taken into account, one finds that the amplitudes of the oscillations in the core and in the envelope are comparable in magnitude. It is then very difficult to associate a frequency to a particular zone. This is a case of resonance.

For that reason it is convenient to merge the two sets of corrected frequencies in such a way that they now constitute a single sequence of ordered values. The spatial oscillations in both the inner and the outer regions then correspond to each of these values. In most cases the oscillations are important in only one of these zones, but in case of resonance there are two different modes of oscillation with nearly equal frequencies and for which the amplitudes in both zones are comparable. Another argument in favor of our combining the individual sequences of frequencies will be given later.

As an illustration we now consider the g^+-modes oscillations in one of the composite polytropic models constructed by Goossens and Smeyers (1974). As they pointed out, in their models N^2 is not continuous across the boundaries of the convective zones; in particular, it does not vanish at x_1 and x_2 as we have assumed here. For that reason the general solutions must be slightly modified: the Airy functions are replaced by simple trigonometric functions and the continuity of ξ_r and δp across the boundaries is a more delicate matter. The detailed calculations will not be given here; we only quote the final results:

$$\frac{k_i^2}{k_o^2}\propto-\frac{\sin\left(\Phi_{2o}-\frac{\pi}{12}\right)\cos\left(\Phi_{2o}+\frac{\pi}{12}\right)}{\cos\left(\Phi_{i1}-\frac{\pi}{12}\right)\sin\left(\Phi_{i1}+\frac{\pi}{12}\right)}\equiv\frac{\mathfrak{R}}{1-\mathfrak{R}}, \tag{135}$$

TABLE 6

THE g^+-MODES IN A COMPOSITE POLYTROPE

κ	σ_{ex}	κ_i	κ_o	σ_i, σ_o	$\sigma_{(1)}$	$\mathfrak{R}$
1...........	1.20383	0	.	1.34757	1.32238	0.2351
2...........	0.87309	1	.	0.92202	0.95632	0.5253
3...........	0.67842	2	.	0.70073	0.69544	0.1137
4...........	0.62656	3	.	0.56511	0.56715	0.1661
5...........	0.55258	.	1	0.52275	0.51996	0.9453
6...........	0.46569	4	.	0.47347	0.47189	0.0766
7...........	0.40226	5	.	0.40740	0.40744	0.0357
8...........	0.40152	.	2	0.36975	0.36993	0.9690
9...........	0.35375	6	.	0.35752	0.35682	0.1062
10...........	0.31569	7	.	0.31852	0.31847	0.0104
11...........	0.29965	8	.	0.28719	0.28780	0.4486
12...........	0.28499	.	3	0.28603	0.28534	0.8676
13...........	0.25971	9	.	0.26147	0.26144	0.0043
14...........	0.24013	10	.	0.23998	0.24000	0.0115

and

$$\cos\left(\Phi_{2o}+\frac{\pi}{12}\right)\sin\left(\Phi_{i1}+\frac{\pi}{12}\right)+\cos\left(\Phi_{i1}-\frac{\pi}{12}\right)\sin\left(\Phi_{2o}-\frac{\pi}{12}\right)\exp(-2\Psi_{12})=0. \tag{136}$$

As these relations are similar to equations (133) and (134), the preceding arguments still apply. In particular, Table 6 shows the effect of the coupling between the inner and the outer regions. The exact frequencies σ_{ex} refer to the g^+-modes belonging to $l=3$ in model 2 by Goossens and Smeyers (1974). Except for an additional term $(-\pi/12)$, the uncoupled frequencies σ_i and σ_o are obtained by means of equations (124) and (127), with the constants κ_i and κ_o, respectively. The (first-order) asymptotic frequencies $\sigma_{(1)}$ result, of course, from equation (136).

From an inspection of this table it is clear that the frequencies are not greatly affected by the coupling. This is especially true for the smaller frequencies (i.e, the higher-order modes). Of course, had the intermediate zone been thinner, the effect on the frequencies would have been larger. In any case, the second-order asymptotic approximation is expected to bring corrections to the frequencies that are greater than those due to the coupling between the zones. However, the conclusions about resonance will not be modified by the next asymptotic approximation because only the Φ and the Ψ are affected by this approximation, not the general form of the matching relations.

A last point of interest is the number of zeros of the eigenfunctions. Let us exclude for the moment the cases of resonance. Then, as we have seen, each zone in which $\varphi>0$ oscillates more or less independently of the others, and the number of nodes in that region increases with the order of the mode. By comparison with numerical results we conclude that for the modes associated with the inner zone the number of zeros of ξ_r in that zone is given by κ_i; similarly for the modes associated with the outer zone the number of nodes in that zone is given by κ_o. Of course, these eigenfunctions also oscillate, though with much smaller amplitudes, in the other region where $\varphi>0$, and it is the total number of these nodes that we are interested in.

To fix our ideas, let us consider a mode which is associated mainly with the inner region. Its frequency (σ_i) is bracketed in the merged sequence (σ) by two frequencies (σ_o) of the outer region. From the asymptotic form of the solutions one can convince oneself that the number of zeros of this mode in the outer region is likewise bracketed by the values of κ_o corresponding to the two frequencies σ_o. The same reasoning applies for modes that are associated mainly with the outer region and which also oscillate in the inner region. In addition, even though there are no oscillations in the intermediate region, one cannot exclude the presence of one zero in that zone. In fact, a careful analysis of the number of nodes in each zone shows that the number of nodes over the whole model increases by one unit with the order of the mode. (In the case of the model implied in Table 6, it is just the constant κ.) This regularity is a convincing argument in favor of the existence of a single g-spectrum, even when the model possesses more than one oscillating region. This regularity is contradicted by some of the results of Goossens and Smeyers (1974), who found that beyond a certain mode, the total number of nodes fluctuates in an irregular manner. In our opinion this conclusion is a consequence of inevitable numerical imprecision: indeed, in the region where, according to these authors, the nodes disappear, the amplitude of the eigenfunctions is many orders of magnitude smaller than in the other region.

VII. CONCLUSION

By dividing stellar models in domains that contain only one transition point, we have been able to study their high-order p- and g-modes. In particular, for g-modes we have confirmed the possibility of resonance between two noncontiguous regions. The restriction that the subregions contain only one transition point is not essential. We could have subdivided the model in only three domains: an inner domain, an outer domain, and an intermediate domain which contains all the zeros of N^2, with a concomitant change of the comparison equation. Such an approach, although theoretically workable (see, e.g., Lynn and Keller 1970) becomes rapidly impracticable. Indeed, except for the particular case of only one turning point (or zero of N^2) in the intermediate interval, the asymptotic solutions are no longer expressible in terms of the simple variable v (defined by eqs. [84]). In addition, if the asymptotic properties of Airy functions (*one* turning point) are well known, those of Weber functions (*two* turning points) are more complex, and the properties of the other unnamed comparison functions (more than two turning points) remain to be established. In any case, as long as the turning points are well separated, both approaches seem to be equivalent, as was indicated by Denis and Smeyers (1977) when they studied the g-modes in a three-zone model (our pattern 4) by means of Weber functions.

APPENDIX

g-MODES AND DISCONTINUITIES

For the sake of simplicity in exposition, we have assumed in the main text that every physical quantity is continuous, as well as its derivatives. This is not always the case in real stars where, even though N^2 may vanish at the boundaries of a convective zone, its first derivative is not necessarily continuous. Near each turning point (or zero of N^2) let us thus introduce q_{ja} and q_{jb}, the left-hand and right-hand limits of the slope of N^2; e.g., we define

$$q_{ja}{}^2 = \lim_{x\to x_j^-} \frac{|N^2|}{x_j - x}. \tag{A1}$$

These two limits are not always equal. We cannot assume either that ρ is continuous. Let us likewise define ρ_{ja} and ρ_{jb} as the left-hand and right-hand limits of ρ near the boundary. The discontinuity at each turning point may then be characterized by the quantity Δ_j,

$$\Delta_j = \left(\frac{\rho_{ja}}{\rho_{jb}}\right)^{1/2}\left(\frac{q_{ja}}{q_{jb}}\right)^{1/3}. \tag{A2}$$

In what follows it will appear convenient to introduce the angles θ_j and θ_{jr} which satisfy the relations

$$\Delta_j{}^2 = \frac{\sin(\pi/6 - \theta_j)}{\sin(\pi/6 + \theta_j)}, \tag{A3}$$

and

$$\tan\theta_{jr} = 3\tan\theta_j. \tag{A4}$$

Note that in the ideal case, when there is no discontinuity, we have $\Delta_j = 1$ and $\theta_j = \theta_{jr} = 0$.

Except for the fact that we can no longer use the same two constants (k_{j1} and k_{j2}) to express the solutions on both sides of a turning point (see, e.g., eqs. [86] and [87]) the expressions for the eigenfunctions are not modified by the presence of a discontinuity (as long as N^2 vanishes at the turning point). To *first-order* in λ (or in $1/|\sigma|$) the requirement that ξ_r and δp be continuous across a turning point now implies

$$k_{j1}{}^a + (3)^{1/2} k_{j2}{}^a = \Delta_j\left(k_{j1}{}^b + (3)^{1/2} k_{j2}{}^b\right), \tag{A5}$$

$$\Delta_j\left(k_{j1}{}^a - (3)^{1/2} k_{j2}{}^a\right) = k_{j1}{}^b - (3)^{1/2} k_{j2}{}^b. \tag{A6}$$

(The superscripts a and b refer, respectively, to solutions valid inside and outside a turning point.) These relations, along with the matching conditions given in § V, enable us to determine the g-spectrum of more realistic stars. We now examine in turn the expressions for these frequencies in the models we discussed in § VI.

(*a*) *Models for which* φ (*see eq.* [75]) *has a* $(+,-)$ *sign*. In this case equation (124) is replaced by

$$\frac{[l(l+1)]^{1/2}}{|\sigma_{(1)}|}\int_0^{x_1}\frac{|N|}{x}\,dx=\left(2\kappa_i+l-\frac{1}{2}\right)\frac{\pi}{2}+\theta_1, \tag{A7}$$

and the amplitudes k_i and k_o satisfy the relation

$$|k_o|=2\Delta_1\sin(\pi/6+\theta_1)\exp(-\Psi_{1o})|k_i|. \tag{A8}$$

In models with a deep-seated convective core (with N^2 close to zero), θ_1 is very near $\pi/6$, and in this limit the g^--frequencies become

$$\frac{[l(l+1)]^{1/2}}{|\sigma_{(1)}|}\int_0^{x_1}\frac{|N|}{x}\,dx=\left(\kappa_i+\frac{l}{2}-\frac{1}{12}\right)\pi. \tag{A9}$$

This is exactly the relation derived by Smeyers (1968), although this author used an approximation not valid near the surface.

(*b*) *Models for which* φ *has the* $(-,+)$ *sign*. Instead of equations (127) and (129), we now have

$$\frac{[l(l+1)]^{1/2}}{|\sigma_{(1)}|}\int_{x_1}^{1}\frac{|N|}{x}\,dx=(2\kappa_o+n_e)\frac{\pi}{2}-\theta_1 \tag{A10}$$

and

$$|k_i|=2\Delta_1\sin(\pi/6+\theta_1)\exp(-\Psi_{i1})|k_o|. \tag{A11}$$

For the g^+-modes in a model with a convective core (in which θ_1 is close to $\pi/6$) relation (A10) simplifies to

$$\frac{[l(l+1)]^{1/2}}{|\sigma_{(1)}|}\int_{x_1}^{1}\frac{|N|}{x}\,dx=\left(\kappa_o+\frac{n_e}{2}-\frac{1}{6}\right)\pi. \tag{A12}$$

Zahn (1970) obtained the same result by assuming that N^2 *exactly vanishes* in the convective core.

(*c*) *Models for which* φ *has the* $(-,+,-)$ *sign*. Equation (130) is replaced by

$$\frac{[l(l+1)]^{1/2}}{|\sigma_{(1)}|}\int_{x_1}^{x_2}\frac{|N|}{x}\,dx=\left(\kappa+\frac{1}{2}\right)\pi-\theta_1+\theta_2. \tag{A13}$$

(Compare with eqs. [A7] and [A10].) We also have

$$\left|\frac{k_{21}{}^b}{k_{11}{}^a}\right|=\frac{\Delta_2}{\Delta_1}\frac{\sin(\pi/6+\theta_2)}{\sin(\pi/6+\theta_1)}. \tag{A14}$$

(*d*) *Models for which* φ *has the* $(+,-,+)$ *sign*. The frequencies are now obtained as the solutions of

$$\sin(\Phi_{i1}-\theta_1)\cos(\Phi_{2o}+\theta_2)+\frac{1}{4}\frac{\cos\theta_1\cos\theta_2}{\cos\theta_{1r}\cos\theta_{2r}}\cos(\Phi_{i1}+\theta_{1r})\sin(\Phi_{2o}-\theta_{2r})\exp(-2\Psi_{12})=0, \tag{A15}$$

while the ratio of the amplitudes in the inner and outer regions is given by

$$\frac{k_i{}^2}{k_o{}^2}=-\frac{\sin(\pi/6-\theta_1)\sin(\pi/3+\theta_{1r})}{\sin(\pi/6-\theta_2)\sin(\pi/3+\theta_{2r})}\times\frac{\cos(\Phi_{2o}+\theta_2)\sin(\theta_{2o}-\theta_{2r})}{\sin(\Phi_{i1}-\theta_1)\cos(\Phi_{i1}+\theta_{1r})}. \tag{A16}$$

(Compare with eqs. [133] and [134].)

REFERENCES

Aizenman, M., Smeyers, P., and Weigert, A. 1977, *Astr. Ap.*, **58**, 41.
Cowling, T. G. 1941, *M.N.R.A.S.*, **101**, 367.
Denis, J., Denoyelle, J., and Smeyers, P. 1975, *Astr. Ap.*, **37**, 221.
Denis, J., and Smeyers, P. 1977, *Ap. Space Sci.*, **52**, 435.
Goossens, M., and Smeyers, P. 1974, *Ap. Space Sci.*, **26**, 137.
Iweins, P. 1964, Mémoire de Licence, Université de Liège.
Iweins, P., and Smeyers, P. 1968, *Bull. Classe Sci. Acad. Roy. Belgium* 5th ser., **54**, 164.
Langer, R.E. 1935, *Trans. Amer. Math. Soc.*, **37**, 397.
Ledoux, P. 1958, in *Handbuch der Physik*, **LI**, 605.
Lynn, R. Y. S., and Keller, J. B. 1970, *Comm. Pure Applied Math.*, **23**, 379.
Magnus, W., Oberhettinger, F., and Soni, R. P. 1966, *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3d edition (New York: Springer Verlag Inc.).
Olver, F. W. J. 1956, *Phil. Trans. Roy. Soc. London A*, **249**, 65.
______. 1974, *Asymptotics and Special Functions* (New York: Academic Press).
Pascual, P. 1977, *Astr. Ap.*, **60**, 161.
Pekeris, C. L. 1938, *Ap. J.*, **88**, 189.
Perdang, J. 1979, *Stellar Oscillations: The asymptotic approach*, Troisième Cycle Interuniversitaire en Astronomie et Astrophysique sous les auspices du F.N.R.S. (Belgium).
Robe, H. 1968, *Ann. d'Ap.*, **31**, 475.
Scuflaire, R. 1974, *Astr. Ap.*, **36**, 107.
Service, A. T. 1977, *Ap. J.*, **211**, 908.
Shibahashi, H. 1979, *Pub. Astr. Soc. Japan*, **31**, 87.
Smeyers, P. 1968, *Ann. d'Ap.*, **31**, 159.
Tassoul, M., and Tassoul, J. L. 1968*a*, *Ap. J.*, **153**, 127.
______. 1968*b*, *Ann. d'Ap.*, **31**, 251.
Vandakurov, Yu. V. 1967, *Astr. Zh.*, **44**, 786.
Wolff, C. L. 1979, *Ap. J.*, **227**, 943.
Zahn, J. P. 1970, *Astr. Ap.*, **4**, 452.

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