A GENERAL VARIATIONAL PRINCIPLE GOVERNING THE RADIAL AND THE NON-RADIAL OSCILLATIONS OF GASEOUS MASSES

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ABSTRACT

In this paper a general variational principle, applicable to radial as well as non-radial oscillations of gaseous masses, is formulated and proved. And it is, further, shown that when the normal modes are analyzed in vector spherical harmonics, the variational principle requires that the square of the characteristic frequency of oscillation, σ^2, belonging to a particular spherical harmonic, is stationary with respect to simultaneous variations of two independent radial functions. A consequence of this result is that σ^2 (belonging to a particular harmonic) emerges as a characteristic root of a 2×2 matrix.

Two simple illustrations of the variational principle are given.

I. INTRODUCTION

A general variational principle applicable to radial as well as non-radial oscillations of gaseous masses has been formulated recently (Chandrasekhar 1963). In this paper the principle will be cast in a form in which it can be applied, directly, to determining the characteristic frequencies of the normal modes of oscillation belonging to the different spherical harmonics. For the sake of completeness, the principle will be rederived with, however, some amplifications.

II. THE VARIATIONAL PRINCIPLE

Consider a spherically symmetric configuration in equilibrium under its own gravitation and governed by the equations

$$\frac{dp}{dr} = \rho \frac{d\mathfrak{B}}{dr} = -\frac{GM(r)}{r^2}\rho\,, \tag{1}$$

and

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\mathfrak{B}}{dr}\right) = -4\pi G\rho\,, \tag{2}$$

where ρ is the density, p is the pressure, $\mathfrak{B}$ is the gravitational potential, $M(r)$ is the mass interior to r, and G is the constant of gravitation.

The boundary condition on the equilibrium configuration requires that

$$p = 0 \quad \text{on} \quad r = R\,, \tag{3}$$

where R denotes the radius of the configuration. We shall suppose, in addition, that

$$\rho = 0 \quad \text{on} \quad r = R\,; \tag{4}$$

from equation (1) it then follows that also

$$\frac{dp}{dr} = 0 \quad \text{on} \quad r = R\,. \tag{5}$$

Let the equilibrium configuration considered be subject to a small perturbation; let $\delta\rho$, δp, and $\delta\mathfrak{B}$ be the resulting Eulerian changes in the respective variables; and, finally, let u_i denote the velocity of the ensuing motions. The linearized form of the equation of motion which governs the perturbation is

$$\rho \frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_i} \delta p + \delta \rho \frac{\partial \mathfrak{B}}{\partial x_i} + \rho \frac{\partial}{\partial x_i} \delta \mathfrak{B} ; \tag{6}$$

or using the equilibrium value $\partial p/\rho \partial x_i$ for $\partial \mathfrak{B}/\partial x_i$ we have

$$\rho \frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_i} \delta p + \frac{\delta \rho}{\rho} \frac{\partial p}{\partial x_i} + \rho \frac{\partial}{\partial x_i} \delta \mathfrak{B} . \tag{7}$$

Let the perturbed state be described by a Lagrangian displacement of the form

$$\xi(x) e^{i\sigma t} , \tag{8}$$

where σ is a characteristic frequency to be determined. Equation (7) then becomes

$$\sigma^2 \rho \xi_i = \frac{\partial}{\partial x_i} \delta p - \frac{\delta \rho}{\rho} \frac{\partial p}{\partial x_i} - \rho \frac{\partial}{\partial x_i} \delta \mathfrak{B} , \tag{9}$$

where a common factor $e^{i\sigma t}$ in δp, $\delta \rho$, and $\delta \mathfrak{B}$ has been suppressed.

The Eulerian change in the density ensuing a Lagrangian displacement is given (quite generally) by

$$\delta \rho = -\xi_k \frac{\partial \rho}{\partial x_k} - \rho \operatorname{div} \xi = -\operatorname{div} \rho \xi . \tag{10}$$

Equation (10) is no more than the expression of the conservation of mass. In terms of $\delta \rho$, the change in the gravitational potential is given by Poisson's integral formula,

$$\delta \mathfrak{B}(x) = G \int_V \frac{\delta \rho(x')}{|x - x'|} dx' = -G \int_V \frac{[\operatorname{div}(\rho \xi)]_{x'}}{|x - x'|} dx' . \tag{11}$$

To express the Eulerian change in the pressure, similarly, in terms of the Lagrangian displacement, some additional assumption concerning the nature of the oscillations is necessary. We shall assume that the oscillations take place adiabatically in accordance with the laws appropriate to a gas with a ratio of the specific heats, γ. Then,

$$\delta p = -\xi_k \frac{\partial p}{\partial x_k} - \gamma p \operatorname{div} \xi . \tag{12}$$

The boundary conditions with respect to which equations (9)–(12) must be solved are that none of the physical variables has a singularity at the origin (or anywhere else); and, also, that

$$\delta p = 0 \quad \text{on} \quad r = R . \tag{13}$$

The particular form of the boundary condition (13) depends, explicitly, on the assumption (that we have made) that ρ vanishes on the boundary: for what is required, strictly, is that the Lagrangian change in the pressure vanish; that the Eulerian change vanishes follows from the fact that, when the density vanishes on the boundary, the gradient of the pressure (in addition to the pressure) vanishes on the boundary.

In the usual treatments of the problem (cf. Pekeris 1938; also Ledoux and Walraven 1958, p. 513) the boundary conditions on $\delta \mathfrak{B}$ (determined in those treatments with the aid of Poisson's and Laplace's equations) are also specified. We do not need to specify these conditions since, by expressing $\delta \mathfrak{B}$ in terms of $\delta \rho$ by Poisson's integral formula (11), the requisite conditions have been satisfied already.

Returning to equation (9), let $\sigma^{(\lambda)}$ denote a particular characteristic frequency; and let the proper solutions belonging to it be distinguished by the same superscript. Consider equation (9) belonging to $\sigma^{(\lambda)}$ and after multiplication by $\xi_i^{(\mu)}$, belonging to a different characteristic value $\sigma^{(\mu)}$, integrate over the volume V occupied by the fluid to obtain

$$[\sigma^{(\lambda)}]^2 \int_V \rho \boldsymbol{\xi}^{(\lambda)} \cdot \boldsymbol{\xi}^{(\mu)} d\boldsymbol{x} = \int_V \xi_i^{(\mu)} \frac{\partial}{\partial x_i} \delta p^{(\lambda)} d\boldsymbol{x} - \int_V \frac{\delta \rho^{(\lambda)}}{\rho} \xi_i^{(\mu)} \frac{\partial p}{\partial x_i} d\boldsymbol{x} - \int_V \rho \xi_i^{(\mu)} \frac{\partial}{\partial x_i} \delta \mathfrak{B}^{(\lambda)} d\boldsymbol{x} . \tag{14}$$

Integrating by parts the first and the last of the three integrals on the right-hand side of equation (14), and remembering that (under the assumed conditions) both $\delta p^{(\lambda)}$ and ρ vanish on the boundary of V, we obtain

$$[\sigma^{(\lambda)}]^2 \int_V \rho \boldsymbol{\xi}^{(\lambda)} \cdot \boldsymbol{\xi}^{(\mu)} d\boldsymbol{x} = -\int_V \delta p^{(\lambda)} \operatorname{div} \boldsymbol{\xi}^{(\mu)} d\boldsymbol{x} - \int_V \frac{\delta \rho^{(\lambda)}}{\rho} \xi_i^{(\mu)} \frac{\partial p}{\partial x_i} d\boldsymbol{x} + \int_V \delta \mathfrak{B}^{(\lambda)} \operatorname{div} \rho \boldsymbol{\xi}^{(\mu)} d\boldsymbol{x} . \tag{15}$$

Now substituting for $\delta\rho^{(\lambda)}$ and $\delta p^{(\lambda)}$ in accordance with equations (10) and (12), we have

$$[\sigma^{(\lambda)}]^2 \int_V \rho \boldsymbol{\xi}^{(\lambda)} \cdot \boldsymbol{\xi}^{(\mu)} d\boldsymbol{x} = \int_V \left[\xi_k^{(\lambda)} \frac{\partial p}{\partial x_k} + \gamma p \operatorname{div} \boldsymbol{\xi}^{(\lambda)} \right] \operatorname{div} \boldsymbol{\xi}^{(\mu)} d\boldsymbol{x} + \int_V \left[\frac{\xi_k^{(\lambda)}}{\rho} \frac{\partial \rho}{\partial x_k} + \operatorname{div} \boldsymbol{\xi}^{(\lambda)} \right] \xi_i^{(\mu)} \frac{\partial p}{\partial x_i} d\boldsymbol{x} - \int_V \delta \rho^{(\mu)} \delta \mathfrak{B}^{(\lambda)} d\boldsymbol{x} . \tag{16}$$

Regrouping the terms on the right-hand side of equation (16) and expressing $\delta\mathfrak{B}^{(\lambda)}$ in terms of $\delta\rho^{(\lambda)}$ by Poisson's integral formula, we obtain

$$[\sigma^{(\lambda)}]^2 \int_V \rho \boldsymbol{\xi}^{(\lambda)} \cdot \boldsymbol{\xi}^{(\mu)} d\boldsymbol{x} = \gamma \int_V p \operatorname{div} \boldsymbol{\xi}^{(\lambda)} \operatorname{div} \boldsymbol{\xi}^{(\mu)} d\boldsymbol{x} + \int_V \frac{1}{r} \frac{d p}{d r} \{ [\boldsymbol{x} \cdot \boldsymbol{\xi}^{(\lambda)}] \operatorname{div} \boldsymbol{\xi}^{(\mu)} + [\boldsymbol{x} \cdot \boldsymbol{\xi}^{(\mu)}] \operatorname{div} \boldsymbol{\xi}^{(\lambda)} \} d\boldsymbol{x} + \int_V \frac{[\boldsymbol{x} \cdot \boldsymbol{\xi}^{(\lambda)}][\boldsymbol{x} \cdot \boldsymbol{\xi}^{(\mu)}]}{r^2 \rho} \frac{d\rho}{dr} \frac{dp}{dr} d\boldsymbol{x} - G \int_V \int_V \frac{\delta \rho^{(\lambda)}(\boldsymbol{x}') d\rho^{(\mu)}(\boldsymbol{x})}{|\boldsymbol{x} - \boldsymbol{x}'|} d\boldsymbol{x} d\boldsymbol{x}' , \tag{17}$$

where the fact that p and ρ are functions of r only has been incorporated.

The right-hand side of equation (17) is manifestly symmetric in λ and μ; accordingly,

$$\int_V \rho \boldsymbol{\xi}^{(\lambda)} \cdot \boldsymbol{\xi}^{(\mu)} d\boldsymbol{x} = 0 \qquad (\lambda \neq \mu) . \tag{18}$$

The underlying characteristic value problem is, therefore, self-adjoint; and the equation obtained by setting $\lambda = \mu$ in equation (17) and suppressing the distinguishing superscripts, namely,

$$\sigma^2\int_V \rho\,|\,\xi\,|^2 dx=\int_V\left[\gamma p(\operatorname{div}\xi)^2+\frac{2}{r}\frac{dp}{dr}(x\cdot\xi)\operatorname{div}\xi\right]dx$$
$$+\int_V\frac{(x\cdot\xi)^2}{r^2\rho}\frac{d\rho}{dr}\frac{dp}{dr}dx-G\int_V\int_V\frac{(\operatorname{div}\rho\xi)_x(\operatorname{div}\rho\xi)_{x'}}{|x-x'|}dx\,dx' \tag{19}$$

provides a variational base for determining the characteristic frequencies.

While the fact that equation (19) provides a variational base for determining σ^2 is clear on general grounds, a direct proof that this is the case is of some interest; and such a proof is given below.

Proof of the Variational Principle

Rewriting equation (19) in the form

$$\sigma^2=\frac{I_2}{I_1}, \tag{20}$$

where

$$I_1=\int_V \rho\,|\,\xi\,|^2 dx \tag{21}$$

and

$$I_2=\int_V\left[\gamma p(\operatorname{div}\xi)^2+\frac{2}{r}\frac{dp}{dr}(x\cdot\xi)\operatorname{div}\xi\right]dx$$
$$+\int_V\frac{(x\cdot\xi)^2}{r^2\rho}\frac{d\rho}{dr}\frac{dp}{dr}dx-G\int_V\int_V\frac{(\operatorname{div}\rho\xi)_x(\operatorname{div}\rho\xi)_{x'}}{|x-x'|}dx\,dx', \tag{22}$$

we shall show that σ^2 given by these equations has a stationary property when I_1 and I_2 are evaluated in terms of the true proper solutions.

To prove the stationary property, let σ^2 be evaluated in accordance with equations (20)–(22) in terms of a displacement ξ which is arbitrary except for the requirement (apart from boundedness and continuity) that it satisfy the boundary condition (13) (with δp given by eq. [12]). And let $\delta\sigma^2$ be the change in σ^2 (evaluated in accordance with the same equations) when ξ is subject to a small variation $\delta\xi$ which is, again, restricted to be compatible only with the boundary conditions. According to equation (20)

$$\delta\sigma^2=\frac{1}{I_1}(\delta I_2-\sigma^2\delta I_1), \tag{23}$$

where δI_1 and δI_2 are the changes in I_1 and I_2 consequent to the variation $\delta\xi$ in ξ. We have

$$\delta I_1=2\int_V \rho\xi_i\delta\xi_i dx \tag{24}$$

and (cf. eq. [10])

$$\delta I_2=\int_V\Big(2\gamma p\operatorname{div}\xi\operatorname{div}\delta\xi+2\delta\xi_i\frac{\partial p}{\partial x_i}\operatorname{div}\xi+2\xi_k\frac{\partial p}{\partial x_k}\operatorname{div}\delta\xi$$
$$+\frac{1}{\rho}\delta\xi_i\frac{\partial\rho}{\partial x_i}\xi_k\frac{\partial p}{\partial x_k}+\frac{\xi_k}{\rho}\frac{\partial\rho}{\partial x_k}\delta\xi_i\frac{\partial p}{\partial x_i}\Big)dx \tag{25}$$
$$+2G\int_V dx\operatorname{div}(\rho\delta\xi)\int_V dx'\frac{\delta\rho(x')}{|x-x'|}.$$

Regrouping the terms, we can write

$$\delta I_2=\int_V\Big[2\left(\gamma p \operatorname{div} \xi+\xi_k \frac{\partial p}{\partial x_k}\right)\operatorname{div}\delta\xi+2\delta\xi_i\frac{\partial p}{\rho\partial x_i}\left(\xi_k\frac{\partial\rho}{\partial x_k}+\rho\operatorname{div}\xi\right)$$
$$+\frac{\xi_k\delta\xi_i}{\rho}\frac{\partial(\rho, p)}{\partial(x_i, x_k)}\Big]dx+2G\int_V dx\operatorname{div}(\rho\delta\xi)\int_V dx'\frac{\delta\rho(x')}{|x-x'|}. \tag{26}$$

Since p and ρ are functions only of r, the Jacobian $\partial(\rho, p)/\partial(x_i, x_k)$ clearly vanishes; and making further use of equations (10)–(12) (all of which are valid as *definitions*), we can rewrite the expression for δI_2 in the form

$$\delta I_2=-2\int_V\delta p\operatorname{div}\delta\xi dx-2\int_V\frac{\delta\rho}{\rho}\delta\xi_i\frac{\partial p}{\partial x_i}dx+2\int_V\delta\mathfrak{B}\operatorname{div}\rho\delta\xi dx. \tag{27}$$

Now integrating by parts the first and the last integrals on the right-hand side of equation (27), we observe that the integrated parts vanish in both cases by virtue of the boundary conditions on δp and ρ; and we are left with

$$\delta I_2=2\int_V\delta\xi_i\left(\frac{\partial}{\partial x_i}\delta p-\frac{\delta\rho}{\rho}\frac{\partial p}{\partial x_i}-\rho\frac{\partial}{\partial x_i}\delta\mathfrak{B}\right)dx. \tag{28}$$

Equations (23), (24), and (28) now give

$$\delta\sigma^2=\frac{2}{I_1}\int_V\delta\xi_i\left(\frac{\partial}{\partial x_i}\delta p-\frac{\delta\rho}{\rho}\frac{\partial p}{\partial x_i}-\rho\frac{\partial}{\partial x_i}\delta\mathfrak{B}-\sigma^2\rho\xi_i\right)dx. \tag{29}$$

From equation (29) it follows that $\delta\sigma^2=0$ if

$$\sigma^2\rho\xi_i=\frac{\partial}{\partial x_i}\delta p-\frac{\delta\rho}{\rho}\frac{\partial p}{\partial x_i}-\rho\frac{\partial}{\partial x_i}\delta\mathfrak{B}, \tag{30}$$

that is, if ξ is a proper solution belonging to σ^2; and, conversely, if $\delta\sigma^2=0$ for an arbitrary $\delta\xi$, restricted only by the boundary conditions of the problem, then equation (30) must hold and the displacement ξ in terms of which σ^2 was initially evaluated must have been a proper solution belonging to it. This completes the proof of the variational principle.

III. THE FORM OF THE VARIATIONAL PRINCIPLE FOR THE NORMAL MODES BELONGING TO THE DIFFERENT SPHERICAL HARMONICS

It is known that, when the normal modes of the Lagrangian displacement are analyzed in vector spherical harmonics (for a brief account of the basic properties of these harmonics see Chandrasekhar 1961), the solution involves two functions which determine the radial dependence of the components, ξ_r, ξ_ϑ, and ξ_φ, of the displacement in spherical coordinates. In conformity with what is required, we shall find it convenient to express ξ_r, ξ_ϑ, and ξ_φ in the forms

$$\xi_r=\frac{\psi(r)}{r^2}Y_l^m(\vartheta, \varphi),$$

$$\xi_\vartheta=\frac{1}{l(l+1)r}\frac{d\chi(r)}{dr}\frac{\partial Y_l^m(\vartheta, \varphi)}{\partial\vartheta}, \tag{31}$$

and

$$\xi_\varphi=\frac{1}{l(l+1)r\sin\vartheta}\frac{d\chi(r)}{dr}\frac{\partial Y_l^m(\vartheta, \varphi)}{\partial\varphi},$$

where $\psi(r)$ and $\chi(r)$ are the two radial functions and $Y_l^m(\vartheta, \varphi)$ is a spherical harmonic.

For ξ given by equations (31)

$$x \cdot \xi = r \xi_r = \frac{\psi}{r} Y_l^m , \tag{32}$$

$$\operatorname{div} \xi = \frac{1}{r^2} \frac{d}{dr} (\psi - \chi) Y_l^m , \tag{33}$$ [1]

and

$$\delta \rho(x) = - \operatorname{div} \rho \xi = - \delta \rho(r) Y_l^m , \tag{34}$$

where

$$\delta \rho(r) = \frac{1}{r^2}\left[\frac{d}{dr}(\rho\psi) - \rho \frac{d\chi}{dr}\right] = \frac{1}{r^2}\left[\rho \frac{d}{dr}(\psi - \chi) + \psi \frac{d\rho}{dr}\right]. \tag{35}$$

Also, when averaged over a unit sphere,

$$\int_0^{2\pi}\int_0^{\pi} |\xi|^2 \sin \vartheta d\vartheta d\varphi = \frac{N_l^m}{r^2}\left[\frac{\psi^2}{r^2} + \frac{1}{l(l+1)}\left(\frac{d\chi}{dr}\right)^2\right], \tag{36}$$

where

$$N_l^m = \frac{4\pi}{2l+1} \frac{(l+|m|)!}{(l-|m|)!}. \tag{37}$$

Making use of the foregoing results, we find that, for ξ of the chosen form, equation (19) becomes

$$\sigma^2 \int_0^R \rho\left[\frac{\psi^2}{r^2} + \frac{1}{l(l+1)}\left(\frac{d\chi}{dr}\right)^2\right] dr = \gamma \int_0^R p\left[\frac{d}{dr}(\psi - \chi)\right]^2 \frac{dr}{r^2}$$
$$+ \int_0^R \left[2\frac{dp}{dr}\psi \frac{d}{dr}(\psi - \chi) + \frac{\psi^2}{\rho}\frac{dp}{dr}\frac{d\rho}{dr}\right]\frac{dr}{r^2} - \frac{G}{N_l^m}\int_V\int_V \frac{\delta\rho(x)\delta\rho(x')}{|x - x'|} dx dx'. \tag{38}$$

Since $\delta\rho(x)$ is now expressed as a product of a radial function and a spherical harmonic (cf. eq. [34]), the last term on the right-hand side of equation (38) can be reduced, in the usual manner, by expanding $|x - x'|^{-1}$ in spherical harmonics and carrying out the integrations over the angles. We thus find

$$\delta^2 W = -G\int_V\int_V \frac{\delta\rho(x)\delta\rho(x')}{|x - x'|} dx dx'$$
$$= -\frac{4\pi G N_l^m}{2l+1}\int_0^R dr\, r^2 \delta\rho(r)\left[\frac{1}{r^{l+1}}\int_0^r ds\, s^{l+2}\delta\rho(s) + r^l \int_r^R \frac{ds}{s^{l-1}}\delta\rho(s)\right]. \tag{39}$$

The two double integrals over r and s which occur in this expression for $\delta^2 W$ are equal: this becomes apparent when the order of the integrations in one of them is inverted. We can, therefore, write

$$\delta^2 W = -\frac{8\pi G N_l^m}{2l+1}\int_0^R dr\, r^{l+2}\delta\rho(r)\int_r^R \frac{ds}{s^{l-1}}\delta\rho(s)$$
$$= +\frac{4\pi G N_l^m}{2l+1}\int_0^R dr\, r^{2l+1}\frac{d}{dr}\left[\int_r^R \frac{ds}{s^{l-1}}\delta\rho(s)\right]^2 \tag{40}$$
$$= -4\pi G N_l^m \int_0^R dr\, r^{2l}\left[\int_r^R \frac{ds}{s^{l-1}}\delta\rho(s)\right]^2.$$

[1] Note that the condition for the velocity field to be solenoidal is $\psi = \chi$.

With this reduction of the last term of equation (38), we can write

$$\sigma^2\int_0^R \rho\left[\frac{\psi^2}{r^2}+\frac{1}{l(l+1)}\left(\frac{d\chi}{dr}\right)^2\right]dr=\gamma\int_0^R p\left[\frac{d}{dr}(\psi-\chi)\right]^2\frac{dr}{r^2}$$
$$+\int_0^R\left[\frac{1}{\rho}\frac{dp}{dr}\frac{d}{dr}(\rho\psi^2)-2\frac{dp}{dr}\psi\frac{d\chi}{dr}\right]\frac{dr}{r^2}-4\pi G\int_0^R dr\,r^{2l}\left[\int_r^R\frac{ds}{s^{l-1}}\delta\rho(s)\right]^2, \tag{41}$$

where it may be recalled that

$$\delta\rho(r)=\frac{1}{r^2}\left[\frac{d}{dr}(\rho\psi)-\rho\frac{d\chi}{dr}\right]. \tag{42}$$

The variational principle requires that σ^2 determined by equation (41) be stationary with respect to *simultaneous* variations in ψ and χ. In other words, the variations to be effected are not only with respect to the chosen *forms* of ψ and χ but also with respect to their *relative amplitudes*. The latter variation, with respect to the amplitudes, can be effected independently of the forms. Thus, writing $A\psi$ and $B\chi$ in place of ψ and χ in equation (41), we find that the requirement that σ^2 be stationary with respect to variations in A and B (for fixed ψ and χ) leads to the characteristic equation

$$\left\|\begin{matrix}
\begin{array}{l}-\sigma^2\int_0^R\rho\psi^2\frac{dr}{r^2}+\int_0^R\left[\gamma p\left(\frac{d\psi}{dr}\right)^2+\frac{1}{\rho}\frac{dp}{dr}\frac{d}{dr}(\rho\psi^2)\right]\frac{dr}{r^2}\\ \qquad -4\pi G\int_0^R dr\,r^{2l}\left[\int_r\frac{ds}{s^{l+1}}\frac{d}{ds}(\rho\psi)\right]^2\end{array} &
\begin{array}{l}-\int_0^R\left(\gamma p\frac{d\psi}{dr}\frac{d\chi}{dr}+\frac{dp}{dr}\psi\frac{d\chi}{dr}\right)\frac{dr}{r^2}\\ \qquad +4\pi G\int_0^R dr\,r^{2l}\left[\left(\int_r^R\frac{ds}{s^{l+1}}\frac{d}{ds}\rho\psi\right)\left(\int_r^R\frac{ds}{s^{l+1}}\rho\frac{d\chi}{ds}\right)\right]\end{array}\\
\begin{array}{l}-\int_0^R\left(\gamma p\frac{d\psi}{dr}\frac{d\chi}{dr}+\frac{dp}{dr}\psi\frac{d\chi}{dr}\right)\frac{dr}{r^2}\\ \qquad +4\pi G\int_0^R dr\,r^{2l}\left[\left(\int_r^R\frac{ds}{s^{l+1}}\frac{d}{ds}\rho\psi\right)\left(\int_r^R\frac{ds}{s^{l+1}}\rho\frac{d\chi}{ds}\right)\right]\end{array} &
\begin{array}{l}-\frac{\sigma^2}{l(l+1)}\int_0^R\rho\left(\frac{d\chi}{dr}\right)^2dr+\gamma\int_0^R p\left(\frac{d\chi}{dr}\right)^2\frac{dr}{r^2}\\ \qquad -4\pi G\int_0^R dr\,r^{2l}\left[\int_r^R\frac{ds}{s^{l+1}}\rho\frac{d\chi}{ds}\right]^2\end{array}
\end{matrix}\right\|=0\,. \tag{43}$$

The fact that the variational principle leads to a characteristic equation which is quadratic in σ^2 corresponds to the circumstance (first noted by Cowling 1942) that the characteristic frequencies belonging to a particular spherical harmonic, apparently fall into two distinct spectra with normal modes of widely different attributes. Thus, it appears that one may distinguish between a set of modes which are principally radial—the p-modes of Cowling—and a set of modes which are principally transversal— the g-modes of Cowling. From the present point of view, the extreme aspects of these two sets of modes will be described by the following equations which are obtained from equation (43) by setting $\chi = 0$ and $\psi = 0$, respectively:

$$\begin{aligned}\sigma^2 \int_0^R \rho\psi^2 \frac{dr}{r^2} = \int_0^R \left[\gamma p \left(\frac{d\psi}{dr}\right)^2 + \frac{1}{\rho}\frac{dp}{dr}\frac{d}{dr}(\rho\psi^2)\right]\frac{dr}{r^2} \\ - 4\pi G \int_0^R dr\, r^{2l}\left[\int_r^R \frac{ds}{s^{l+1}}\frac{d}{ds}(\rho\psi)\right]^2,\end{aligned} \tag{44}$$

and

$$\begin{aligned}\frac{\sigma^2}{l(l+1)}\int_0^R \rho \left(\frac{d\chi}{dr}\right)^2 dr = \gamma \int_0^R p \left(\frac{d\chi}{dr}\right)^2 \frac{dr}{r^2} \\ - 4\pi G \int_0^R dr\, r^{2l}\left[\int_r^R \frac{ds}{s^{l+1}}\rho \frac{d\chi}{ds}\right]^2.\end{aligned} \tag{45}$$

The approximation considered by Cowling and others, of neglecting the variation of the gravitational potential during the oscillations, is equivalent to suppressing the last term ($\delta^2 W$) in equation (38); and when this term is suppressed, equation (43) becomes

$$\begin{Vmatrix} \displaystyle\int_0^R \left[\gamma p \left(\frac{d\psi}{dr}\right)^2 + \frac{1}{\rho}\frac{dp}{dr}\frac{d}{dr}(\rho\psi^2)\right]\frac{dr}{r^2} - \sigma^2 \int_0^R \rho\psi^2 \frac{dr}{r^2} & \displaystyle -\int_0^R \left(\gamma p \frac{d\psi}{dr}\frac{d\chi}{dr} + \frac{dp}{dr}\psi\frac{d\chi}{dr}\right)\frac{dr}{r^2} \\ \displaystyle -\int_0^R \left(\gamma p \frac{d\psi}{dr}\frac{d\chi}{dr} + \frac{dp}{dr}\psi\frac{d\chi}{dr}\right)\frac{dr}{r^2} & \displaystyle \gamma \int_0^R p\left(\frac{d\chi}{dr}\right)^2 \frac{dr}{r^2} - \frac{\sigma^2}{l(l+1)}\int_0^R \rho\left(\frac{d\chi}{dr}\right)^2 dr \end{Vmatrix} = 0\,. \tag{46}$$

IV. TWO ILLUSTRATIONS

Detailed applications of equation (43) to the problem of determining the characteristic frequencies of non-radial oscillations of polytropes will be considered in a later paper. In this paper we shall limit ourselves to illustrating its applications by deriving two known results.

a) The Variational Principle for Purely Radial Oscillations

The case of purely radial oscillations is obtained by setting $l = 0$ and $\chi = 0$. Equation (43) then becomes

$$\begin{aligned}\sigma^2 \int_0^R \rho\psi^2 \frac{dr}{r^2} = \int_0^R \left[\gamma p \left(\frac{d\psi}{dr}\right)^2 + \frac{1}{\rho}\frac{dp}{dr}\frac{d}{dr}(\rho\psi^2)\right]\frac{dr}{r^2} \\ - 4\pi G \int_0^R dr \left[\int_r^R \frac{ds}{s}\frac{d}{ds}(\rho\psi)\right]^2.\end{aligned} \tag{47}$$

The last term on the right-hand side of equation (47) can be reduced in the following manner:

$$
\begin{aligned}
4\pi G\int_0^R dr\left[\int_r^R \frac{ds}{s}\frac{d}{ds}(\rho\psi)\right]^2 &= 8\pi G\int_0^R dr\,\frac{d}{dr}(\rho\psi)\int_r^R \frac{ds}{s}\frac{d}{ds}(\rho\psi) \\
&= 8\pi G\int_0^R \frac{ds}{s}\frac{d}{ds}(\rho\psi)\int_0^s dr\,\frac{d}{dr}(\rho\psi) \\
&= 4\pi G\int_0^R \frac{dr}{r}\frac{d}{dr}(\rho\psi)^2 = 4\pi G\int_0^R \rho^2\psi^2\frac{dr}{r^2} \\
&= -\int_0^R \rho\frac{\psi^2}{r^4}\frac{d}{dr}\left(r^2\frac{d\mathfrak{B}}{dr}\right)dr = \int_0^R r^2\frac{d\mathfrak{B}}{dr}\frac{d}{dr}\left(\rho\frac{\psi^2}{r^4}\right)dr \\
&= \int_0^R \frac{r^2}{\rho}\frac{dp}{dr}\frac{d}{dr}\left(\rho\frac{\psi^2}{r^4}\right)dr\,.
\end{aligned}
\tag{48}
$$

Inserting the result of this reduction in equation (47) and simplifying, we are left with

$$
\sigma^2\int_0^R \rho\psi^2\frac{dr}{r^2} = \int_0^R\left[\gamma p\left(\frac{d\psi}{dr}\right)^2 + \frac{4}{r}\frac{dp}{dr}\psi^2\right]\frac{dr}{r^2}\,. \tag{49}
$$

It can be readily verified that this last equation is equivalent to the one which was first derived by Ledoux and Pekeris (cf. Ledoux and Walraven 1958, p. 465) for treating radial oscillations.

b) *The "Kelvin Modes"*

As a second illustration, consider the following particular case which leads to a simple class of solenoidal velocity fields (cf. n. 1 above):

$$
\psi = \chi = r^{l+1}\,. \tag{50}
$$

For this choice of the radial functions

$$
\delta\rho = r^{l-1}\frac{d\rho}{dr}\,. \tag{51}
$$

With the substitutions (50) and (51), equation (41) becomes

$$
\begin{aligned}
\frac{2l+1}{l}\sigma^2\int_0^R \rho r^{2l}dr &= \int_0^R \frac{r^{2l}}{\rho}\frac{dp}{dr}\frac{d\rho}{dr}dr - 4\pi G\int_0^R \rho^2 r^{2l}dr \\
&= -G\int_0^R \frac{d\rho}{dr}M(r)r^{2l-2}dr - 4\pi G\int_0^R \rho^2 r^{2l}dr\,.
\end{aligned}
\tag{52}
$$

On further simplification, we obtain

$$
\sigma^2 = \frac{2l(l-1)}{2l+1}G\frac{\int_0^R \rho r^{2l-3}M(r)dr}{\int_0^R \rho r^{2l}dr} = \frac{2l(l-1)(2l-1)}{2l+1}\frac{\int_0^R p r^{2l-2}dr}{\int_0^R \rho r^{2l}dr}\,. \tag{53}
$$

The analogy with the Kelvin modes of non-radial oscillations of an incompressible sphere is manifest.

Equation (53) is a generalization of the results known for the cases $l = 2$ and $l = 3$ (Chandrasekhar and Lebovitz 1962, eq. [3], and 1963, eq. [64]).

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APPENDIX

THE FORM OF THE VARIATIONAL PRINCIPLE FOR CYLINDRICAL SYSTEMS

The variational principle formulated in the paper, in the context of spherical systems, can be adapted readily to cylindrical systems in which, in the equilibrium state, the physical variables are functions only of the distance ϖ from the axis. The appropriately modified form of equation (19) is clearly

$$\sigma^2\int_V \rho\,|\,\xi\,|^2 dx = \int_V\left[\gamma p(\operatorname{div}\xi)^2 + 2\frac{dp}{d\varpi}\xi_\varpi \operatorname{div}\xi\right]dx + \int_V \frac{\xi_\varpi^2}{\rho}\frac{d\rho}{d\varpi}\frac{dp}{d\varpi}dx - \int_V \delta\rho(x)\,\delta\mathfrak{B}(x)\,dx\,, \tag{A.1}$$

where the integrations over V are now to be interpreted as averages per unit length of the cylinder; also, in equation (A.1), ξ_ϖ is the ϖ-component of the Lagrangian displacement $\xi = (\xi_\varpi, \xi_\varphi, \xi_z)$ in a cylindrical system of coordinates (ϖ, φ, z).

Without loss of generality, we may assume that, under the circumstances considered, a normal mode of oscillation has a (φ, z)-dependence given by $\cos kz \cos m\varphi$ where $0 \leq |k| < \infty$ and m is an integer (positive or zero). The Lagrangian displacement belonging to such a normal mode can be expressed in the form

$$\xi_\varpi = \frac{\psi(\varpi)}{\varpi}\cos kz\cos m\varphi\,,\qquad \xi_\varphi = \frac{m}{\varpi}\chi(\varpi)\cos kz\sin m\varphi\,,\qquad \text{and } \xi_z = k\chi(\varpi)\sin kz\cos m\varphi\,, \tag{A.2}$$

where ψ and χ are two radial functions. For ξ of this chosen form

$$\operatorname{div}\xi = \left[\frac{1}{\varpi}\frac{d\psi}{d\varpi} + \left(\frac{m^2}{\varpi^2} + k^2\right)\chi\right]\cos kz\cos m\varphi \tag{A.3}$$

and

$$\delta\rho(x) = -\operatorname{div}\rho\,\xi = -\delta\rho(\varpi)\cos kz\cos m\varphi\,, \tag{A.4}$$

where

$$\delta\rho(\varpi) = \frac{1}{\varpi}\frac{d}{d\varpi}(\rho\psi) + \left(\frac{m^2}{\varpi^2} + k^2\right)\rho\chi\,. \tag{A.5}$$

For $\delta\rho(x)$ given by equation (A.4), $\delta\mathfrak{B}$, determined as the solution of Poisson's equation, is given by

$$\delta\mathfrak{B} = -4\pi G\left[I_m(k\varpi)\int_\varpi^R \varpi'\delta\rho(\varpi')K_m(k\varpi')d\varpi' + K_m(k\varpi)\int_0^\varpi \varpi'\delta\rho(\varpi')I_m(k\varpi')d\varpi'\right]\cos kz\cos m\varphi\,, \tag{A.6}$$

where I_m and K_m are the Bessel functions of order m for a purely imaginary argument and R is the radius of the cylinder.

With the foregoing expressions for the various quantities which occur in equation (A.1), we find that the present analogue of equation (41) is

$$
\begin{aligned}
\sigma^2 \int_0^R \rho \left[\frac{\psi^2}{\varpi^2} + \left(\frac{m^2}{\varpi^2} + k^2 \right) \chi^2 \right] \varpi d\varpi \\
= \gamma \int_0^R p \left[\frac{1}{\varpi} \frac{d\psi}{d\varpi} + \left(\frac{m^2}{\varpi^2} + k^2 \right) \chi \right]^2 \varpi d\varpi \\
+ 2 \int_0^R \frac{dp}{d\varpi} \psi \left[\frac{1}{\varpi} \frac{d\psi}{d\varpi} + \left(\frac{m^2}{\varpi^2} + k^2 \right) \chi \right] d\varpi + \int_0^R \frac{\psi^2}{\rho} \frac{d\rho}{d\varpi} \frac{dp}{d\varpi} \frac{d\varpi}{\varpi} \\
- 8\pi G \int_0^R d\varpi \varpi \delta\rho(\varpi) K_m(k\varpi) \int_0^{\varpi} d\varpi' \varpi' \delta\rho(\varpi') I_m(k\varpi') .
\end{aligned}
\qquad (\mathrm{A.7})
$$

As in Section III of the paper, we may, by writing $A\psi$ and $B\chi$ in place of ψ and χ in equation (A.7), derive a characteristic equation of order 2 for σ^2 from the requirement that σ^2 be stationary with respect to small variations in A and B; but we shall not write out the resulting equation explicitly.

REFERENCES

Chandrasekhar, S. 1961, *Hydrodynamic and Hydromagnetic Stability* (Oxford: Clarendon Press), Appendix iii, pp. 622–25.

———. 1963, *Ap. J.*, **138**, 896.

Chandrasekhar, S., and Lebovitz, N. R. 1962, *Ap. J.*, **136**, 1105.

———. 1963, *ibid.*, **138**, 185.

Cowling, T. G. 1942, *M.N.*, **101**, 367.

Ledoux, P., and Walraven, Th. 1958, *Hdb. d. Phys*, ed. S. Flügge (Berlin: Springer-Verlag), pp. 509–26.

Pekeris, C. L. 1938, *Ap. J.*, **88**, 189.