THE TOOLS OF ASTEROSEISMOLOGY

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ABSTRACT

One of the twin goals of the *Eddington* mission is to measure stellar oscillations and to use the measured frequencies and line profiles to further our understanding of the structure and evolution of stars. We here cover some of the techniques that can be used to achieve this aim.

Key words: missions: *Eddington* – stars: oscillations, structure, evolution

1. Asteroseismology and stellar evolution

Stars are the sites of chemical evolution in the Universe, they control the chemical and luminosity evolution of galaxies, stellar evolution theory is used to estimate the age of stellar systems and thereby the evolution of our own galaxy. But how do we know that our theories are correct? What empirical checks do we have on the internal structure of stars? This question was posed by Eddington (1926) in the opening paragraph of his monograph "The Internal Constitution of the Stars" as

> *What appliance can pierce through the outer layers of a star and test the conditions within?*

The answer to Eddington's question is stellar oscillations, or asteroseismology. The oscillations are waves that penetrate into the interior and can be seen at the surface through changes in the luminous flux, or in velocity. Different oscillation modes propagate through the interior in different ways, the observed frequencies encoding different combinations of the variation of the stars internal structure with depth. Asteroseismology is the art of using the information encoded in the frequencies (and line shapes) to diagnose the internal structure of the star. By such means we can test and improve our models of stellar structure and evolution with the goal of building a reliable and empirically tested theory of stellar evolution that can be used with confidence in astrophysics, and enhance our understanding of the physical processes in stellar interiors.

The evolution of stars depends crucially on the evolving chemical profile determined by the nuclear reactions in the central core, and by mixing and diffusive processes that redistribute the chemically processed material. A key factor in such mixing is convection and our understanding of convection, particularly the penetration of convective motions from unstable into the surrounding stable layers, is poor, our knowledge of the microphysics such as opacity is poorly tested, the possibility of diffusive mixing by mild turbulence driven by instabilities or rotation is poorly understood. Yet all such processes will affect our estimates of both the chemical and temporal evolution of stars. Asteroseismology provides the tool with which to probe the deep interiors of stars and thereby to test, and develop, our models of stellar evolution.

The aim of this paper is to explain how this can be achieved – what are the tools of asteroseismology that promise such a rich harvest? In the space available to me I cannot hope to give a comprehensive account of the subject, so this paper is restricted to a few examples mostly taken from joint work with Sergei Vorontsov.

2. Stellar oscillations

Most objects oscillate in a set of normal modes when subjected to some disturbance – stars are no exception. The simplest example is that of a sound wave excited by some process (e.g. convection); the wave then propagates through the medium and, if it has the correct phase at the boundary, becomes a standing wave – or a normal mode of oscillation. The phase at the boundary is determined by the frequency of the wave and the properties of the medium through which it propagates, the condition that the wave has the correct phase at the boundary determines which frequencies correspond to normal modes of oscillation. This is exactly the situation with waves on a string, only those waves whose half wavelength is an integer fraction of the length of the string become standing waves, waves excited with different frequencies are dissipated. But a star is three dimensional not one dimensional, so the waves are characterised by both their structure over a sphere, described by a surface harmonic $Y_{\ell m}$, and the number of half wavelengths in radius or the radial order n. The eigenfrequencies $\nu_{n\ell m}$ are in general described by the three "quantum numbers" n, ℓ, m. An example of such an oscillation mode is shown in Fig. 1. For simplicity we shall first only consider the oscillations of a spherical star in which case, as we shall see, the frequencies are independent of the azimuthal "quantum number" m. This degeneracy is lifted by rotation (and magnetic fields) and is important in rapidly, indeed moderately, rotating stars.

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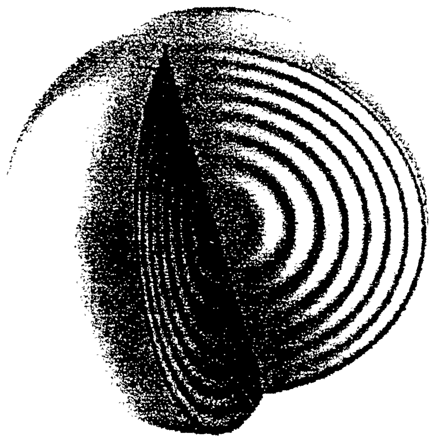

Figure 1. Example of a stellar non-radial oscillation mode with $n = 18, \ell = 2, m = 2$. The surface behaviour of the mode is determined by the relevant spherical harmonic $Y_{2\,2}$ and the cut-out of the interior shows the radial behaviour of the mode.

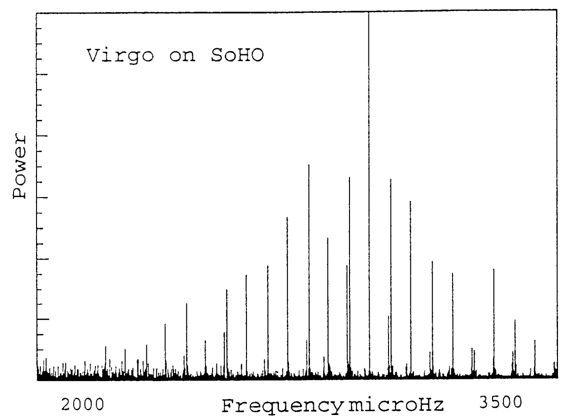

Figure 2. the power spectrum of solar oscilations obtained during the first year of observation of the Sun as a star by the VIRGO experiment on SoHO.

Understanding the effects of rotation remains a challenge to asteroseismology.

The basic observation is a time series of the variation of the flux (or velocity) integrated over the visible hemisphere of the star. Modes with large values of ℓ, m will have many horizontal nodes and are unlikely to be detected due to the cancellation of the signal when averaged over a hemisphere. A Fourier transform of the time series gives a power spectrum that reveals the discrete set of oscillation frequencies. Fig. 2 shows such a power spectrum of the Sun as a star (integrated flux measurements) obtained with the VIRGO instrument on SoHO (Fröhlich et al. 1997).

One can here see a regularity with large peaks, corresponding to modes with $\ell = 0, 1$, and small peaks close to the main peaks corresponding to modes with $\ell = 2, 3$. For future reference the frequency difference between modes with the same ℓ values are called the "large separations", whilst the frequency difference between the large peaks and their neighbouring small peaks ($\ell = 0, 2$ and $\ell = 1, 3$) are called the small separations.

For sound waves the restoring force is primarily pressure and such modes are referred to as p-modes. But in a gravitating fluid the dominant restoring force on low frequency oscillations can be gravity acting on density perturbations, such modes are referred to as g-modes. In a star both types of modes can exist, and also mixed-modes which behave like p-modes in one part of the star and g-modes in other parts. p-modes tend to have larger energies in the outer layers of a star where the sound speed is low, whilst g-modes tend to have larger energies in the inner regions where the density is large.

The Sun is observed to oscillate in p-modes – indeed the search for g-modes is one of the outstanding challenges of helioseismology, other stars e.g. Slowly Pulsating B stars (or SPBs) are observed to be oscillating in g-modes. Yet others (e.g. β Cep stars) are possibly oscillating in both. With the exception of the Sun we can only observe at most a few modes from the ground and only for a small subset of stars, but with *Eddington* we will observe a much larger set of modes which will enable us to probe their internal structure and stage of evolution (Favata et al. 2000).

Each oscillation frequency $\nu_{n\ell}$ of a star is a different functional of the internal structure, characterised by the density $\rho(r)$ and pressure $P(r)$. The art of asteroseismology is to invert these relations to determine the structure (ρ, P) in terms of the measured frequencies.

The frequency of the basic, or fundamental, radial mode of a star (i.e. the mode with no nodes except at the boundaries) is given to order of magnitude by its free fall time under gravity, or more precisely by the round trip travel time of a sound wave from surface to centre and back to the surface which is $\approx 140\,\mu$Hz for the Sun. The frequency of p-modes increases with increasing n whereas the frequency of g-modes decreases with increasing n. Since the radiative diffusion time in stars is very much longer than the free fall time, radiative damping is negligible except in the outermost layers of a star and the oscillations are essentially adiabatic. This simplifies the problem of understanding the relationship between the structure of a star and it eigenmodes, and consequently in developing diagnostic and inversion procedures to determine the internal structure from an observed set of frequencies. As we shall see below it is possible to subtract off, or parameterise the effect of the surface layers in some diagnostic procedures; p-modes corresponding to (n,ℓ) and $(n-1,\ell+2)$ have almost the same frequencies and the same behaviour in the surface layers so the small separations

$$d_{n\ell} = \nu_{n\ell} - \nu_{n-1,\ell+2}$$

give a diagnostic of the central regions of a star.

One should note what can, and what cannot, be obtained from diagnostic and inversion procedures. The density profile $\rho(r)$ inside a star gives the mass distribution $M(r)$ and hence the local acceleration due to gravity $g = GM(r)/r^2$. For a star in hydrostatic equilibrium this gives the pressure profile $P(r)$. The adiabatic oscillations of a star are determined by its hydrostatic structure and the adiabatic exponent $\Gamma_1 = \partial \log P/\partial \log \rho$ at constant entropy, so the eigenfrequencies are determined by a pair of variables e.g. $(\rho(r), \Gamma_1(r))$. Except for massive stars, where radiation pressure is important, the adiabatic exponent in the fully ionised interior is very nearly 5/3 so that the frequencies can in principle give $\rho(r)$, and hence $P(r)$, but not the temperature variation $T(r)$ and the chemical composition profile $X(r)$, only their combination through P/ρ. In massive stars where radiation pressure is important Γ_1 is no longer $\approx 5/3$ and depends on temperature so there is additional information that can be used to determine the temperature and composition profiles. In the outer layers of a star the situation is different. In these regions, where ionisation is important, the composition is essentially constant but Γ_1 depends on ρ and T and the measure frequencies can in principle be used to constrain the entropy and helium abundance. Regions of steep changes in the density and its gradients at the boundary of convective regions produce characteristic signatures in the frequencies which can be used to determine the location of these boundaries.

3. Oscillation equations

The adiabatic oscillations frequencies of a star are determined by the standard equations of adiabatic fluid mechanics which are:

$$\rho\frac{d\mathbf{u}}{dt} + \nabla p + \rho\nabla\phi = 0 \qquad \text{(equation of motion)}$$

$$\frac{d\rho}{dt} + \rho\nabla.\mathbf{u} = 0 \qquad \text{(mass conservation)}$$

$$\frac{1}{p}\frac{dp}{dt} - \Gamma_1\frac{1}{\rho}\frac{d\rho}{dt} = 0 \qquad \text{(adiabatic condition)}$$

$$\nabla^2\Phi = 4\pi G\rho \qquad \text{(Poisson's equation)}$$

where $\mathbf{u}, P, \rho, \Phi, \Gamma_1$ are the velocity, pressure, density, gravitational potential and adiabatic exponent and $d/dt = \partial/\partial t + \mathbf{u}.\nabla$ is the co-moving or Lagrangian time derivative.

If there is no motion $\mathbf{u} = 0$ – the system is in equilibrium and these equations reduce to the standard equations governing the hydrostatic structure of a spherical star

$$\frac{dP_0}{dr} = -g\rho_0, \quad g = \frac{d\Phi_0}{dr} = \frac{GM_r}{r^2}, \quad \frac{dM_r}{dr} = 4\pi G\rho_0 r^2$$

For small amplitude oscillations we linearise about equilibrium and write

$$\mathbf{u} = \frac{d(\delta\mathbf{r})}{dt}, \quad P = P_0 + P', \ \rho = \rho_0 + \rho', \ \Phi = \Phi_0 + \Phi'$$

and express the perturbations in the form

$$\delta\mathbf{r} = \left(\xi(r), \zeta(r)\frac{\partial}{\partial\theta}, \zeta(r)\frac{\partial}{\sin\theta\partial\phi}\right) Y_{\ell m}e^{i\omega t}$$

$$P' = p'(r)Y_{\ell m}e^{i\omega t}, \quad \Phi' = \phi(r)Y_{\ell m}e^{i\omega t}$$

where $Y_{\ell m}$ are spherical harmonics and ω the cyclical frequency ($\omega = 2\pi\nu$) to give the equations governing small amplitude adiabatic oscillations in the form

$$\frac{d\xi}{dr} + \frac{2}{r}\xi - \frac{g}{c^2}\xi + \left(1 - \frac{\ell(\ell+1)c^2}{\omega^2 r^2}\right)\frac{p'}{\rho c^2} - \frac{\ell(\ell+1)}{\omega^2 r^2}\phi' = 0$$

$$\frac{dp'}{dr} + \frac{g}{c^2}p' + (N^2 - \omega^2)\rho\xi + \rho\frac{d\phi'}{dr} = 0$$

$$\frac{d^2\phi'}{dr^2} + \frac{2}{r}\frac{d\phi'}{dr} - \frac{\ell(\ell+1)}{r^2}\phi' - 4\pi G\rho\left(\frac{p'}{\rho c^2} + \frac{N^2}{g}\xi\right) = 0$$

$$c^2 = \Gamma_1\frac{P}{\rho}, \quad N^2 = -\frac{g^2}{c^2}\left(1 - \Gamma_1\frac{d\log\rho}{d\log P}\right)$$

Here c is the sound speed and N the Brunt-Vaisala frequency. Note that for a spherical star these equations are independent of m. This degeneracy is lifted by rotation.

These equations are governed by boundary conditions of regularity at the centre $r = 0$, this requires

$$\phi = Ar^\ell + \ldots, \ p' = Br^\ell + \ldots, \ \xi = (A + B/\rho_c)\omega^2\ell r^{\ell-1} + \ldots,$$

and that at the surface, $r = R$, the Lagrangian pressure perturbation δp vanishes and the potential ϕ' matches onto the corresponding solution of Laplace's equation, which require

$$\frac{d\phi}{dr} + \frac{(\ell+1)}{r}\phi = 0, \quad p' - \rho g\xi = 0 \quad \text{at} \quad r = R$$

More accurate boundary conditions take into account that the wave is reflected high up in the atmosphere, and that the density at the surface is not zero (see e.g. Vorontsov and Zharkov 1989).

This 4th order system of homogeneous equations must satisfy 4 boundary conditions which is only possible for a discrete set of eigenvalues $\omega_{n\ell} = 2\pi\nu_{n\ell}$. In Fig. 3 we show examples of the eigenfunctions, actually the kinetic energy density $r\rho^{1/2}\xi$, for a star of 1.45 $M_\odot$ for a p-mode with $n = 8, \ell = 2$. and a g-mode also with $n = 8, \ell = 2$.

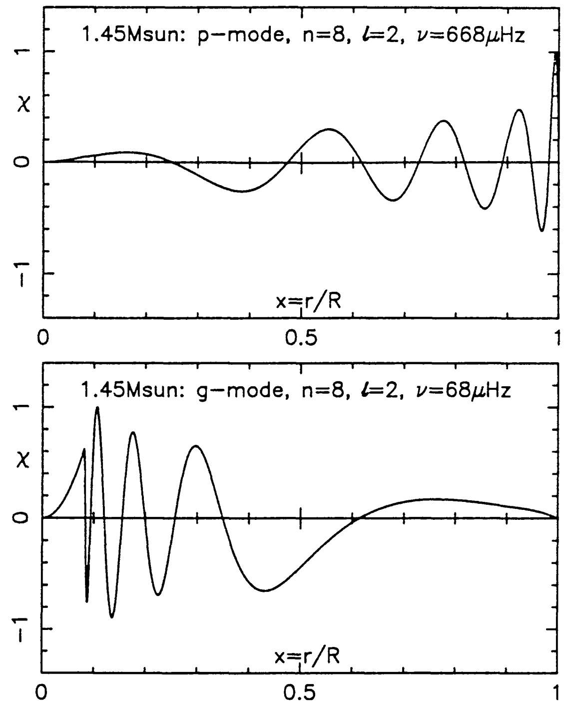

Figure 3. Scaled radial eigenfunction $\chi = r\rho^{1/2}\xi$ for two oscillation modes of an evolved main sequence star of $1.45\,M_\odot$ *with central hydrogen abundance* $X_c = 0.35$. *Top panel p-mode with* $n = 8, \ell = 2$, *bottom panel g-mode with* $n = 8, \ell = 2$.

4. Simple acoustic oscillations

To understand the properties of the solutions of these equations we first consider the simple Rayleigh (1896) problem of oscillations in a non-gravitating sphere. Setting $g = 0$, $N^2 = 0$, $\phi' = 0$, c^2 and ρ constant the oscillaton equations reduce to

$$\frac{d\xi}{dr} + \frac{2}{r}\xi + \left(1 - \frac{\ell(\ell+1)c^2}{\omega^2 r^2}\right)\frac{p'}{\rho c^2} = 0, \quad \frac{dp'}{dr} - \omega^2\rho\xi = 0$$

Setting $\psi = p'r/(\rho c)^{1/2}$ and defining the acoustic radius $t = \int_0^r dr/c$ transforms these equations into the Spherical Bessel equation

$$\frac{d^2\psi}{dt^2} + \left(\omega^2 - \frac{\ell(\ell+1)}{t^2}\right)\psi = 0$$

whose solutions $j_\ell(\omega t)$ are very well documented and which, for large ωt, have the asymptotic behaviour (e.g. Whittaker and Watson, 1927, p. 365)

$$j_\ell(\omega t) \rightarrow \sin(\omega t - \frac{\pi\ell}{2} + \frac{a_\ell}{\omega t})$$

with a_ℓ a constant. At the surface $t = T = R/c$, the pressure perturbation $p' = 0$, hence $\psi(\omega T) = 0$, which gives the "eigenfrequency equation"

$$\omega T - \pi\frac{\ell}{2} + \frac{a_\ell}{\omega T} = n\pi$$

with n an integer. This gives discrete set of eigenfrequencies $\omega_{n\ell}$. With $\nu = \omega/2\pi$ and ωT large this gives $\nu_{n\ell} \approx \nu_{n-1,\ell+2}$ and the large (Δ) and small (d) separations

$$\Delta_{n\ell} = \nu_{n+1,\ell} - \nu_{n\ell} \approx \frac{1}{2T}$$

$$d_{n\ell} = \nu_{n\ell} - \nu_{n-1,\ell+2} \approx \frac{A_\ell}{T}\left(\frac{1}{n+\ell/2}\right)$$

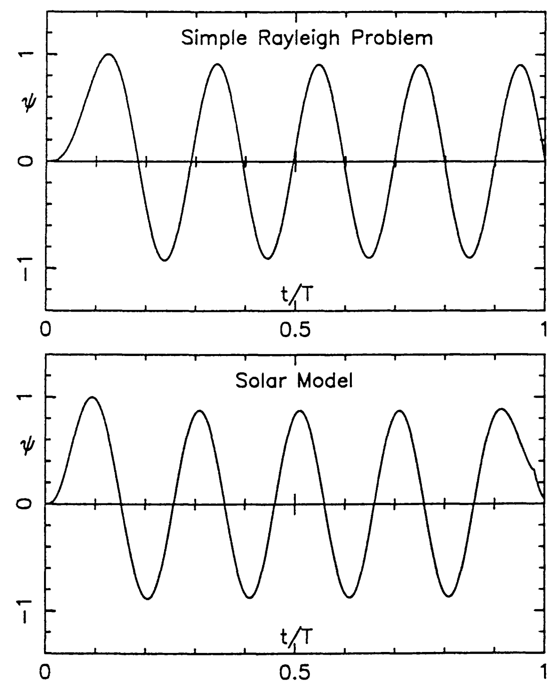

Figure 4. Scaled Eulerian pressure eigenfunctions $\psi(t) = p'r/(\rho c)^{1/2}$ *vs. fractional acoustic radius* t/T *for a p-mode with* $n = 8, \ell = 2$. *Top panel is for the simple Rayleigh problem, bottom panel for a model of the present Sun.*

The solution of this simple problem shows many of the characteristics of the properties of p-mode frequencies in more realistic models of stars, in particular the approximately equal spacings of the large separations Δ seen in the solar power spectrum in Fig. 2. This also extends to the eigenfunctions, and in Fig. 4 we show the scaled pressure eigenfunctions $\psi(t)$ plotted against fractional acoustic radius t/T for a p-mode with $n = 8, \ell = 2$, for the simple Rayleigh case, and for a model of the present Sun (Christensen-Dalsgaard et al. 1996).

5. Acoustic potentials and surface phase shift

The situation for real stars is more complex, and we first consider the properties of the oscillations in the outer layers for p-modes. In these outer layers the sound speed is small so for moderate values of ℓ and ω the combination

$$\frac{\ell(\ell+1)c^2}{\omega^2 r^2} \ll 1$$

and the equation are approximately independent of ℓ. The equations governing the oscillation then reduce to 2nd order and can be rewritten in Schrödinger form with an acoustic depth t_s and an "acoustic potential" $V(t_s)$

$$\frac{d^2\psi}{dt_s^2} + (\omega^2 - V)\psi = 0, \quad \text{where} \quad \psi = r(\rho c)^{1/2}\xi$$

$$V(t_s) = A\frac{d^2}{dt^2}\left(\frac{1}{A}\right) - \frac{4g}{r}, \quad t_s = \int_r^R \frac{dr}{c}, \quad A^2 = \frac{r^2}{\rho c}$$

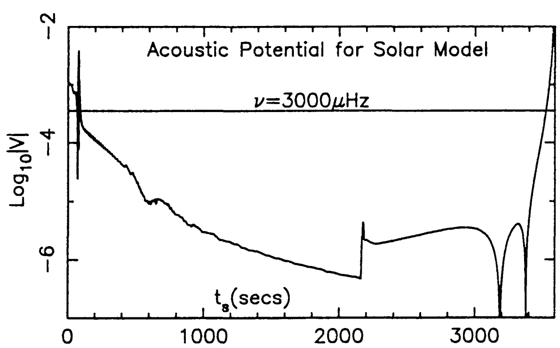

Figure 5. Acoustic potential $V(t_s)$ vs acoustic depth t_s (in secs) for a solar model. The horizontal line corresponds to a frequency of 3000 μHz. Except in the very surface layers and the deep interior $V \ll \omega^2$. The sharp discontinuity at $t_s \approx 2200$ sec is the base of the solar convective zone, the dip at $t_s \approx 600$ sec is caused by the HeII ionization zone.

The acoustic potentials $V(t_s)$ is shown in Fig. 5 for a model of the present Sun (Christensen-Dalsgaard et al. 1996). Below the near surface layers $V/\omega^2 \ll 1$ so the solution for ψ can be represented by a harmonic function

$$\psi \approx A(\omega)\sin[\omega t + \alpha(\omega)]$$

where $\alpha(\omega)$ is the "surface phase shift", and is a function only of frequency and not of ℓ. $\alpha(\omega)$ contains all the uncertainties of the detailed structure of the surface layers and the non-adiabatic properties of the oscillations in these layers. Note also the discontinuity in the potential V at the base of the convective zone and the fluctuation in the helium ionisation zone. Since discontinuities, or variations on a scale length small compared to the wave length of the oscillation, produce a characteristic modulation of the frequencies, if we can determine this modulation we can determine the depth (in acoustic radius t) of the helium ionisation zone and the base of the convective envelope. This wiill be considered in detail below.

6. Internal phase shifts and small separations

In the deeper interior of the star the factor $\ell(\ell+1)c^2/(\omega^2 r^2)$ is no longer small, nor is ϕ' necessarily small, and the full solution for the variable ψ depends on ℓ and the detailed structure of the core. However in the outer regions of the star, the inner solution must be of the same form as solution in outer layers and can be expressed in the form

$$\psi \to B(\omega)\sin[\omega t - \frac{\pi}{2}\ell + \delta_\ell(\omega, t)],$$

where δ is an "internal phase shift" determined by the interior structure and $t = \int_0^r dr/c$ is the acoustic radius. Since the inner and outer solutions must be continuous when ω is an eigenfrequency, they must have the same phase, which gives the eigenfrequency equation (Roxburgh and Vorontsov 2000)

$$\omega T = \pi\left(n + \frac{\ell}{2}\right) + \alpha(\omega) - \delta_\ell(\omega, T) \qquad \text{integer } n$$

With α, δ small and $\omega = 2\pi\nu$ we obtain the results for the large (Δ) and small (d) separations

$$\Delta_{n\ell} = \nu_{n+1,\ell} - \nu_{n\ell} \approx \frac{1}{2T}$$

$$d_{n\ell} = \nu_{n\ell} - \nu_{n-1,\ell+2} \approx \left(\frac{\delta_{\ell+2}(\omega) - \delta_\ell(\omega)}{2\pi T}\right)$$

To relate the internal phase shift δ_ℓ to the internal structure it is necessary to undertake some analysis. This was done by Tassoul (1980) neglecting the gravitational perturbation ϕ' and in more detail by Roxburgh and Vorontsov (1994) using a successive Born approximation about the solution for $\ell = 0$.

We choose the same acoustic variables as in the simple Rayleigh problem above, $\psi = p'r/(\rho c)^{1/2}$, $t = \int_0^r dr/c$ and re-write the oscillation equations in the form

$$\frac{d^2\psi}{dt^2} + \left(\omega^2 - \frac{\ell(\ell+1)}{t^2}\right)\psi = [U_0(\omega, t) + \ell(\ell+1)U_2(\omega, t)]\,\psi$$

where U_0, U_2 are "acoustic potentials" that depend on the the interior structure and solution for ψ, ϕ'. This modified Bessel equation can be solved by perturbation analysis about the solution for $\ell = 0$, taking $U_0 + \ell(\ell+1)U_2$ as a perturbation, and for $\omega t > 1$ this gives

$$\psi \to \sin\left[\omega t - \frac{\pi}{2}\ell + \delta_\ell(\omega, t)\right]$$

where the "internal phase shift"

$$\delta_\ell(\omega, t) = \frac{\ell(\ell+1)}{2\omega t} - \frac{1}{\omega}\int_0^t [U_0 + \ell(\ell+1)U_2]j_\ell^2(\omega t')dt'$$

This process can be repeated with successive approximations (see Roxburgh and Vorontsov 1996) and gives a satisfactory agreement with the numerical solutions of the full oscillation equations. If we take the approximation $\omega^2 >> N^2$ and $\omega^2 >> 4\pi G\rho$ and replace j_ℓ^2 by its average value of 1/2 we obtain $U_2 \approx (c^2/r^2 - 1/t^2)$ and

$$\delta_\ell \approx \frac{A(\omega,t)}{\omega t} + \frac{\ell(\ell+1)B}{2\omega}, \quad B = \left[\frac{c(R)}{R} - \int_0^R \frac{1}{r}\frac{dc}{dr}dr\right]$$

This is turn then gives the Tassoul (1980) result for the small separations $d_{n\ell}$

$$d_{n\ell} = \omega_{n\ell} - \omega_{n-1,\ell+2} \approx \frac{2\ell+3}{\omega T}\left[\frac{c(R)}{R} - \int_0^R \frac{1}{r}\frac{dc}{dr}dr\right]$$

where $T = \int_0^R dr/c$ is the acoustic radius of the star. Whilst this Tassoul approximation is innaccurate except for very high frequencies, it demonstrates that the small separations are sensitive to the central structure of the star where r is small.

7. The Christensen-Dalsgaard diagram

The large separations Δ are determined by the acoustic radius $T = \int_0^R dr/c$ which depends on the run of sound speed inside the star and therefore gives a measure of the gross properties of a star. However the small separations d depend on conditions in the central core and therfore give a measure of the evolution of the star. The C-D diagram (Christensen-Dalsgaard 1988) plots the average values of these separations for stars of different masses and in different stages of evolution. This is shown in Fig. 6.

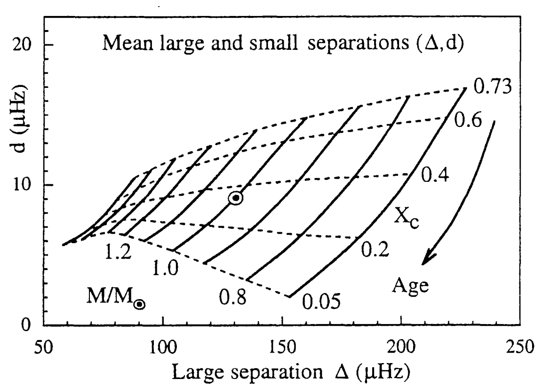

Figure 6. Average values of the small (d) and large (Δ) separations for small mass stars all with initial composition $X = 0.73, Z = 0.02$ The curves are labelled by the central hydrogen abundance so the position on this diagram gives a measure of the age of a star.

If we had a reliable knowledge of all the physical processes governing a star's evolution, then a particular point in this diagram corresponds to a star of a particular mass and a particular age. This gives a flavour of the inferences that can be made from observations of the oscillation frequencies of stars.

8. Differential response inversion

The objective of an inversion procedure is to reconstruct the internal structure of a star using a set of measured frequencies – that is to find the run of density and adiabatic exponent $(\rho(r), \Gamma_1(r))$ that gives a "best fit" to the observed frequencies. This task is hindered by our poor understanding of both the structure of the surface layers and the non-adiabatic nature of the oscillations in these layers so ideally we need to subtract off the effect of the outer regions. This is done in the "Differential Response" inversion technique, originally developed by Vorontsov for solar inversions with a large number of modes (Vorontsov 1998, 2001), but here adapted for small data sets with low ℓ modes such as we can expect to get with *Eddington*.

As we have seen the internal solution for the eigenmode depends on ℓ, ω, r whereas the solution in the outer layers depends only on ω, r. Moreover the surface layers have low density and therefore make a negligible contribution to the perturbation in gravitational potential ϕ'. These observations provide the basis for the present technique.

We define a "partial wave" $\psi_\ell(\omega)$ as solution of the full oscillation equations for a given (ω, ℓ) that satisfies the central conditions of regularity, and the surface gravitational boundary condition that ϕ' matches onto a solution of Laplace's equation at some radius r_f. One can equally think of this as a travelling wave generated by some excitation source. If the wave has the correct phase at the boundary then it becomes a standing wave – or eigenmode – with eigenfrequncy ω. The phase of the partial wave at r_f is

$$\delta_\ell(\omega) = \tan^{-1}\left(\frac{d\psi_\ell/dt}{\omega\psi_\ell}\right)_{r_f}$$

and for arbitrary ω depends on both ω and ℓ. However if ω is an eigenfrequency of the star then $\delta_\ell(\omega)$ collapses to the unknown surface phase shift $\alpha(\omega)$, which is a function of ω only. The requirement that this be the case for the observed frequencies gives a condition that can be used as the basis of an inversion procedure to determine the structure of the star.

The inversion procedure is implemented as follows:

1. Characterise the internal structure of the star by a set parameters C_k, e.g. the value of the density ρ_i and ratio of specific heats Γ_{1i} at a set of mesh of points r_i. In practice we either take Γ_1 from some reference model or set $\Gamma_1 \approx 5/3$. Since with low ℓ modes we are only inverting for the central regions of a star this is not a significant constraint.

2. Using a particular realisation of the observed frequencies $\omega_{n\ell}$ with errors $\sigma_{n\ell}$, calculate the inner phase shifts $\delta_\ell(\omega_{n\ell})$ at some point in the outer layers $r = r_f$ using the model described by the parameters C_k.
3. Characterise the unknown surface phase shift $\alpha(\omega)$ by a set of parameters A_k, eg representing α as a polynomial in ω with coefficients A_k.

Now simultaneously determine corrections to the parameters A_k and C_k by minimising

$$\chi^2(A_k, B_k) = \sum \left(\frac{\delta_\ell(\omega_{n\ell}) - \alpha(\omega_{n\ell})}{\sigma(\omega_{n\ell})} \right)^2,$$

correct the initial guesses of A_k, C_k, and iterate to convergence. The specific details of this procedure using a new adaptive regularisation algorithm of Strakhov and Vorontsov (2001) are given in a paper by Roxburgh and Vorontsov (2002) in these proceedings. This procedure is fully non-linear and the converged solution gives both the internal model and the surface phase function $\alpha(\omega)$

In Fig. 7 we give the results of a such a fully non-linear inversion for a star of $0.8\,M_\odot$ with a central hydrogen abundance $X = 0.1$ and two frequency sets. In the first inversion we took a large frequency set with $\ell = 0$–3 and $\nu = 1000$–$5000\,\mu$Hz, the same as used in an inversion by Roxburgh et al. (1998) using the OLA technique described below. The random errors on the frequencies were assummed to have a standard deviation of $0.3\,\mu$Hz and different realisations of the data were used. In the second inversion the data set was restricted to modes with $\ell = 0$–2, $\nu = 2000$–$4000\,\mu$Hz. In both cases the initial guess to start the iteration was a long way from the converged solution.

In Fig. 8 we give the result of a non-linear inversion using the actual frequency set and error estimates from whole disc solar observations obtained by the ground based BiSON network (Chaplin et al. 1998), but degraded by the addition further random errors with standard deviation of $0.1\,\mu$Hz. The initial guess was taken as a model of the sun at age 2.6×10^9 yr. In this inversion the radius of the star was taken to be unknown and to be determined by the inversion – this gave a radius of 6.95×10^{10} cm, close to recent estimates of the solar radius. In both cases we were able to reproduce the details of the inner structure of the star.

9. Optimised Local Averages (OLA)

The inversion technique most widely used in helioseismology (see e.g. Gough et al. 1996) is the so called "Optimised Local Averages" method (OLA), originally developed by Backus and Gilbert (1968) for terrestrial seismology. This method is based on finding an appropriate combination of the frequencies that gives linearised corrections to the internal structure of a trial model. Gough and Kosovichev (1993) applied this method to model data and were able

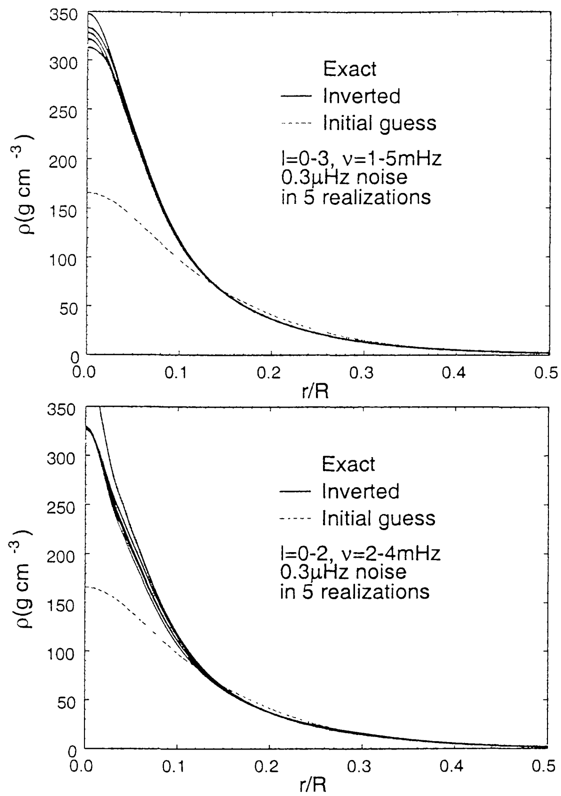

Figure 7. Inversion for a star of $0.8\,M_\odot$ *for different random realisations of the frequencies. Top panel* $\ell = 0$–3, $\nu = 1000$–$5000\,\mu Hz$, *random errors* $\sigma = 0.3\,\mu Hz$; *bottom panel* $\ell = 0$–2, $\nu = 2000$–$4000\,\mu Hz$, *random errors* $\sigma = 0.3\,\mu Hz$.

to see an indication of a small convective core. Since other papers in this volume implement this technique (see e.g. Thompson 2002) I here just sketch the principles of the method.

One starts from a variational principle in which the oscillation equations are formally written as

$$\mathcal{L}\xi_i = \omega_i^2 \xi_i \quad \rightarrow \quad \omega_i^2 = \frac{\int \xi_i^* \mathcal{L}\xi_i \, dm + \mathcal{S}_i}{\int \xi_i^* \xi_i \, dm}$$

where ω_i is an eigenfrequency, ξ_i the corresponding eigenfunction and $\mathcal{S}_i$ the contribution from the surface layers. Given a reference model and corresponding frequency set ω_i, small changes $\delta\rho, \delta c^2$, in the model produce changes in the frequencies $\delta\omega_i$ given by

$$\frac{\delta\omega_i^2}{\omega_i^2} = \int \mathcal{K}_i \frac{\delta\rho}{\rho} dr + \int \mathcal{J}_i \frac{\delta c^2}{c^2} dr + \frac{\mathcal{F}(\omega_i)}{\mathcal{E}_i},$$

where the kernels $\mathcal{K}_i, \mathcal{J}_i$ depend on $\rho(r), c(r)$ in the trial model, and $\mathcal{F}(\omega)$ represents the effect of the surface layers.

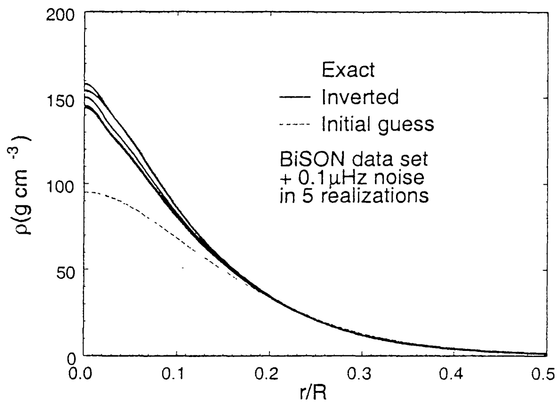

Figure 8. Inversion for a solar model using the observed BiSON frequency set with additional random errors with $\sigma = 0.1\,\mu Hz$. The starting model was a model of the Sun at age 2.6×10^9 yr and the radius was determined by the inversion.

One now seeks a set of constants λ_{ij} such that

$$\sum \lambda_{ij} \mathcal{K}_i = \bar{\mathcal{K}}_j(r_j), \quad \sum \lambda_{ij} \mathcal{J}_i \approx 0$$

where $\bar{\mathcal{K}}_j(r_j)$ is narrowly peaked around r_j, so that

$$\sum \lambda_{ij} \frac{\delta\omega_i^2}{\omega_i^2} = \sum \lambda_{ij} \int \mathcal{K}_i \frac{\delta\rho}{\rho} dr = \int \bar{\mathcal{K}}_j \frac{\delta\rho}{\rho} dr \approx \frac{\delta\rho_j}{\rho_j}$$

Taking the $\delta\omega_i^2$ to be the difference between the model values and an observed data set gives the corrections $\delta\rho(r_j)$, and similarly $\delta c^2(r_j)$, to the trial model.

In principle, though seldom in practice, the corrections $\delta\rho, \delta c^2$ could be used to give an improved model, giving new kernels, and the process repeated until convergence. In practice the inversions are also done with error free data rather than a random realisations of a given (small) data set, but this could be implemented in future research.

10. Inversion using the phase shifts

As shown above matching the internal ℓ dependent phase shifts $\delta_\ell(\omega)$ with the ℓ independent surface phase shift $\alpha(\omega)$ gives the "Eigenfrequency Equation"

$$\omega T + \delta_\ell(\omega, \ell) - \alpha(\omega) = \pi \left(n + \frac{\ell}{2}\right)$$

For $\ell = 0$ the interior and surface phase shift cannot be separated so we write this as

$$F(\omega) + G_\ell(\omega) = \pi(n + \frac{\ell}{2})$$

$$F(\omega) = \omega T + \delta_0(\omega) - \alpha(\omega), \quad G_\ell(\omega) = \delta_\ell(\omega) - \delta_0(\omega)$$

$G_\ell(\omega)$ only contains slowly varying terms from internal phase shifts $\delta_\ell(\omega)$ and can therefore be used to constrain the interior structure, whereas $F(\omega)$ contains the surface phase shift $\alpha(\omega)$ and therefore signatures of surface non adiabatic region, and of regions of sharp change in the acoustic variables and their derivatives, such as the He II ionisation zone and the base of an outer convective envelope.

Since for a given set of identified modes the combination $\pi(n + \ell/2)$ is known, the functions $F(\omega)$ and $G_\ell(\omega)$ can be represented as a finite series in some set of basis functions (here taken as Chebyschev polynomials) by weighted least squares fitting using the input frequencies (or some realisation of the frequencies and errors). This was done for a "data set" from a model of an evolved

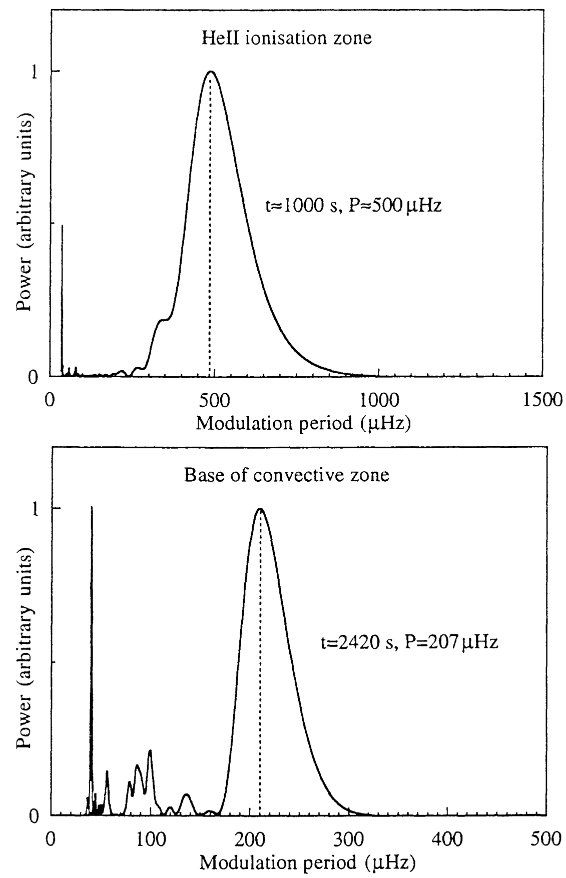

Figure 9. The top panel shows the power spectrum of residuals to low-order polynomial fit to the function $F(\omega)$ of the frequencies showing the signal from the He II ionisation zone; The bottom panel shows the power spectrum of residuals to high-order polynomial fit to $F(\omega)$ showing the signal from the base of the convective zone. Also indicated is the location of these layers in the input model, which is well reproduced by the analysis.

main sequence star of $1.45\,M_\odot$. The frequency set was modes with $\ell = 0, 1, 2, 3$ and frequencies in the range 500–2000 μHz. Taking a low order polynomial fit to $F(\omega)$ and taking a power spectrum of the residuals to the fit gives a broad peak at a "period" of about 500μ Hz coresponding to an acoustic depth of 1000 sec, this is in good agreement with the actual location of the He II ionisation zone which is centred around the vertical line in Fig. 9a.

Taking a higher order polynomial fit to absorb the effect of the HeII ionisation zone, and then taking a power spectrum of the residuals, gives the peak shown in Fig. 9b which agrees well with the actual location of the base of the convective zone in the model. The results in Fig. 9 are for error free data. The ability to resolve these features, especially the location of the base of the convective zone degrades with errors; but for this data set the signal from the convective zone is still detectable with errors with standard deviation $\sigma = 0.25\,\mu$Hz. For the solar case the base of the convective zone is detectable in the actual data set from the BiSON ground based network (see Roxburgh and Vorontsov 1998, 2001; Favata et al. 2000), but rapidly degrades with increasing errors.

To use the information in the $G_\ell(\omega)$ determined from the "observed" frequency set we parameterise a model star by a set of values of some acoustic variable, here taken as

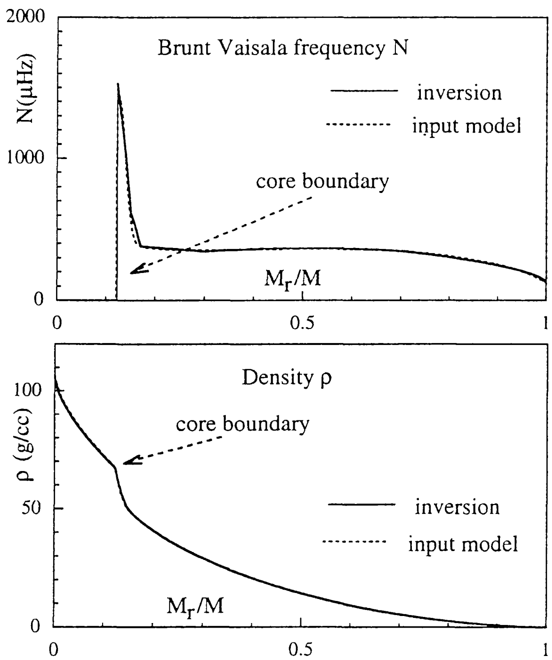

Figure 10. Inversion for the central structure of a star of $1.45\,M_\odot$ using the internal phase shift. The top panel shows the Brunt-Vaisala frequncy in the actual model and the inversion, the bottom panel compares the density.

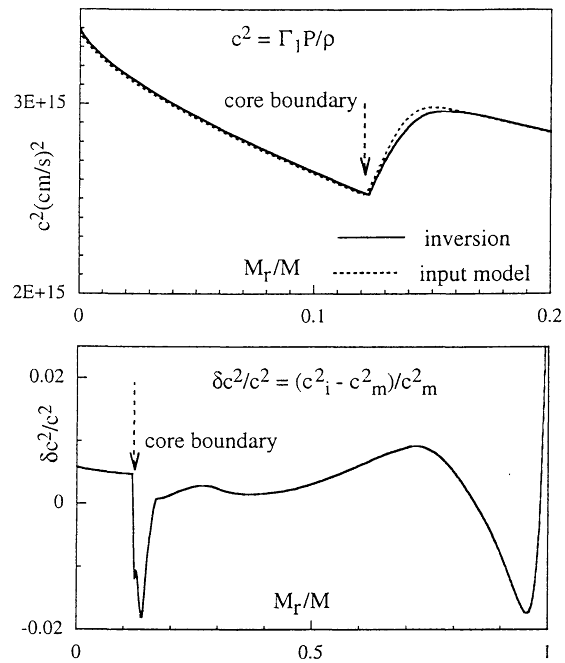

Figure 11. Inversion for the central structure of a star of $1.45\,M_\odot$ using the internal phase shift. The top panel shows the sound speed in the model and the inversion in the neighbourhood of the boundary of the convective core whilst the bottom panel shows the fractional difference in sound speed between the inverision and the mode.

the combination $D_i = \Gamma_1 d\log\rho/d\log P$ at a set of mass points $q_i = M_i/M$; we took $\Gamma_1 = 5/3$. To build a model we also need an interpolation algorithm for the values of D at other values of q; we here took linear interpolation. Using the stellar model constructed with this parameterisation we calculate the model values of $G_\ell^m(\omega)$ and then search for the set of $D_i(q_i)$ that minimises

$$\chi^2 = \sum_\ell \int_\omega \left[G_\ell(\omega) - G_\ell^m(\omega)\right]^2 d\omega$$

For stars with a convective core there is a discontinuity in D at the core boundary so that we also take the location of the core boundary q_c as a parameter to be determined by the minimisation.

This procedure was applied to frequencies from a model of an evolved main sequence star of $1.45\,M_\odot$ with a central hydrogen abundance of $X = 0.35$, and with convective core overshooting of 0.2 of a pressure scale height. The frequency set was modes with $\ell = 0, 1, 2, 3$ and frequencies in the range 500–2000 μHz. The results are shown in Figs. 10 and 11. We reproduce the inner structure and the mass of the convective core with good accuracy.

Whilst this inversion worked, it was not without difficulty. We need a more robust algorithm to search for the solution. One promising line is to modify the differential response technique to include some adjustable mesh points (e.g. the location of the core boundary) within the parameters to be determined.

11. Conclusions

This paper is necessarily of limited scope but I hope it conveys to the reader some of the tools that are available and are being developed to make the maximum use of data from *Eddington*. I have not covered many areas, in particular the g-modes which contain information on the central structure of the star (see Figs. 3a, 3b above). In principle information on g-modes can be incorporated into the inversion techniques described above, but this requires further consideration of the effect of stellar surface layers. Even for very low frequency oscillators such as SPBs the g-modes contain signatures of the convective core.

A major problem area that needs further study is that of rotation. Rotation lifts the m degeneracy in the frequencies, giving many more observable frequencies, and making mode recognition difficult. It also and causes a shift in the frequency of the modes (see Soufi et al. 2000). But the rotationally split modes also provide the information needed to probe the internal rotation of stars (Goupil et al. 1996). There is also information in the line profiles on the excitation mechanism of the oscillations, and on the properties of turbulent convection (see Samadi et al. 2002, these proceedings).

Another approach is that of individual model fitting – which out of a set of stellar models constructed with our present understanding of stellar evolution best fits a set of observed frequencies. Care needs to exercised with such an approach since the actual values of the frequencies are strongly influenced by the structure of surface layers which are poorly understood. As shown above one can approximately subtract off the effect of these layers by using the "small separations" as in the C-D diagram diagnostic.

Model fitting is particularly valuable when applied to a set of stars in a cluster, enhancing the classical cluster fitting technique. If one assumes that the physics of stellar interiors and evolution is known, except for the mixing length, then the addition of measurements of the mean large and small separation to the classical observations gives a substantial improvement in the accuracy of the determination of cluster parameters such as helium abundance and age (Gough and Novotny 1993). This analysis can be extended to include the extent of convective core overshooting, and the entropy in the surface convective zone (or mixing length) of the individual stars and, if we have measurements of individual frequencies for each of the stars, gives improved accuracy on the cluster parameters (Audard and Roxburgh 1998; Favata et al. 2000).

Of course we do not know all the physics of stellar evolution – aquiring that understanding is the goal of the asteroseismology component of the *Eddington* mission!

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