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Seismology of the solar envelope: the base of the convective zone as seen in the phase shift of acoustic waves

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ABSTRACT

At the base of the convective zone, discontinuities in the derivatives of the sound speed produce a phase shift in acoustic waves. This phase shift, $\alpha_0(\nu)$, is inferred by matching the measured solar p-modes to a second-order asymptotic description, the contribution from the surface layers and the second helium ionization zone being subracted using modes with turning points well inside the convective zone. The resulting phase shift varies quasi-periodically with frequency with a period ≈ 220 μHz, and has an amplitude $\approx 8(\pm 4)\times 10^{-4}$ at a frequency of 3000 μHz. The phase shift $\alpha_0(\nu)$ is also calculated using the same technique for two solar models, one with no convective penetration and one with penetration extending for 1 per cent of the solar mass.

We estimate the predicted phase shift for models of the convective zone, including possible convective penetration modelled by extending the zone a distance εH_p below the classical boundary; $\alpha(\nu)$ has a quasi-periodic contribution with period $\approx 1/(2\tau_\mathrm{b})$, where $\tau_\mathrm{b}=\int \mathrm{d}r/c$ is the acoustic depth of the base of the zone, and with amplitude

$$A_\nu=\left[\left(\frac{g^2}{16\pi^3 c^2 \nu^2}\right)^2 f^2(\varepsilon)+\left(\frac{g}{4\pi^2 c\nu}\right)^2 h^2(\varepsilon)\right]^{1/2},$$

where $f(\varepsilon)$ and $h(\varepsilon)$ are functions of ε which also depend on the variation of opacity with temperature and density. For $\nu=3000$ μHz, and typical values at the base of the convective zone, $A_\nu=5\times 10^{-4}$ for $\varepsilon=0$, decreases slightly for small ε and then increases to 1×10^{-3} for $\varepsilon=0.25$, and to 1.7×10^{-3} for $\varepsilon=0.5$.

The currently available data are consistent with an overshooting parameter $0\leq\varepsilon\leq 0.25$.

Key words: waves – Sun: interior – Sun: oscillations – stars: oscillations.

1 INTRODUCTION

Convection plays a major role in determining the internal structure and evolution of stars. In stellar interiors, convection is important in maintaining chemical homogeneity, thereby strongly influencing stellar evolutionary time-scales; in the outer regions of stars, convection is thought to be responsible for the generation of magnetic fields, thereby influencing mass and angular momentum loss and stellar activity. Yet our theoretical understanding of convection is rudimentary (at best), and empirical constraints on theoretical speculations are few. One potential source of enhanced knowledge is through the interpretation of the measurement of the frequencies of acoustic oscillations of stars. The propagation of acoustic waves across a region of rapid spatial variation of the sound speed produces a phase shift which can, in principle, be determined from the measurement of oscillation frequencies (cf. Vorontsov & Zharkhov 1989). This phase shift depends on the location and structure of the regions of rapid spatial variation, and can therefore be used to investigate the position and structure of these regions.

An important property of convection, about which we know very little, is the extent of penetration of convective motions from unstable into the surrounding stable regions. Convective penetration below the bases of stellar outer convective zones may well play a crucial role in providing the domain in which stellar dynamos can operate, and penetration from stellar convective cores could lead to a substantial enhancement of the region that is kept chemically homogeneous, and thereby significantly affect the evolution of stars (cf. Maeder 1975; Roxburgh 1976, 1978). In principle, one can probe the structures of the boundaries of both convective cores and convective envelopes through the quasi-periodic signal in the phase shift (Vorontsov & Zharkhov 1989; Roxburgh & Vorontsov 1994a). In practice, at the present time the only data of sufficient accuracy for such an investigation are those of solar oscillations, and we here report on an investigation into the structure of the base of the solar convective zone using modes of intermediate degree obtained by Libbrecht, Woodard & Kaufman (1990).

There are three regions in the solar envelope where a rapid variation in the sound speed may be expected to produce a detectable signal: the surface layers, the second helium ionization zone and the base of the convective zone. We are here concerned with the contribution to the phase shift caused by the discontinuity in temperature derivatives at the base of the convective zone, due to the change from an almost adiabatic stratification to the radiative temperature gradient in the interior. In models of the zone without convective penetration (or overshoot), the base is that depth where all the energy is carried by radiation; the transition from the adiabatic temperature gradient to the radiative gradient leads to a discontinuity in the second derivative of temperature, whose magnitude is determined by the gradient of the opacity. In models with convective penetration, the adiabatic layer extends below the 'classical' base of the zone, the convective energy flux being negative in this penetrative region. At the base of the zone there is a thin boundary layer in which the convective motions are rapidly decelerated and the temperature gradient adjusts to the value in the radiative interior. For very thin boundary layers, this is equivalent to taking the adiabatic gradient down to the base of the overshoot layer and a discontinuity in the temperature gradient at the boundary (cf. Roxburgh 1978); this leads to a corresponding discontinuity in the first derivative of the sound speed. These discontinuities at the base of the zone produce a quasi-periodic phase shift with period $\approx 1/(2\tau_\mathrm{b})$, where $\tau_\mathrm{b}=\int \mathrm{d}r/c$ is the acoustic depth, and with an amplitude that depends on both frequency and the magnitude of the discontinuities. Since the contributions to the amplitude of the phase shift from the discontinuities in the first and second derivatives of the sound speed have different dependences on the frequency of the wave, this raises the possibility of using the phase shift not only to detect the acoustic depth of the base of the zone, τ_b, but also to place some limits on the extent of convective penetration.

The method employed is first to determine the contribution to the phase shift from the surface layers and helium ionization zone using modes with turning points well inside the convective zone. Then, using modes with turning points below the convective zone, and subtracting the contribution from the surface layers and helium ionization zone, we determine the contribution to the phase shift from the discontinuity at the base of the convective envelope; the contribution from the non-radial nature of the ray path is included in this analysis (cf. Brodsky & Vorontsov 1993; Roxburgh & Vorontsov 1993a). We also calculate the predicted phase shift for a standard solar model and for a model with convective penetration, the penetration being for 1 per cent of a solar mass or 0.54 of a pressure scaleheight.

We then present a theoretical calculation of the predicted phase shift for models where convection penetrates a distance εH_p below the unstable region (where H_p is the local pressure scaleheight), and where the local variation of opacity with temperature and density is approximated by a power law. We find that the variation of the amplitude of the phase shift due to convective penetration has a different dependence on frequency than that for models with no penetration, so that in principle the nature of the interface between the convective and radiative regions can be determined through such an investigation. At present, however, the observational data are not accurate enough for this programme to be followed through. The total amplitude of the predicted phase shift (at a given frequency) varies with the penetration parameter ε: for $\varepsilon \ll 1$ the amplitude decreases slightly with increasing ε, the contribution from the decrease in the magnitude of the discontinuity in the second derivative of the sound speed being more important than the effect of the discontinuity in the first derivative; for larger ε the amplitude increases, tending to a limit of 3.5×10^{-3}. The present observational data suggest a limit on ε of the order of 0.25.

2 THE PHASE SHIFT

In an asymptotic analysis of the oscillation equations, the eigenfrequencies satisfy the equation (cf. Vorontsov 1991; Brodsky & Vorontsov 1993; Roxburgh & Vorontsov 1993b, 1994b)

$$F(\tilde{w})+\frac{1}{\omega^2}\Phi(\tilde{w})+\frac{1}{\omega^4}\Psi(\tilde{w})+\ldots=\pi\,\frac{n+\alpha(\nu,\tilde{w})}{\omega}, \qquad (1)$$

where $\omega=2\pi\nu$, $\tilde{w}=(\ell+1/2)/\omega$, $F(\tilde{w})$ is the leading-order asymptotic term usually used for the asymptotic inversion of the sound speed, $\Phi(\tilde{w})/\omega^2$ is the second-order correction and $\Psi(\tilde{w})/\omega^4$ the fourth-order correction to the asymptotic representation. The phase shift $\alpha(\omega,\tilde{w})$ contains the deviation of the exact solution from the asymptotic approximation due to regions of rapid spatial variation. We expand $\alpha(\nu,\tilde{w})=\alpha_0(\nu)+\tilde{w}^2\alpha_2(\nu)$; the leading term corresponds to waves that propagate in the radial direction, while $\tilde{w}^2\alpha_2(\nu)$ is the correction that arises from the (small) inclination of the acoustic ray paths from the normal to the layers that contribute to the acoustic scattering. In the present study we limit our attention to diagnostic properties of $\alpha_0(\nu)$, but we need to include $\alpha_2(\nu)$ in our analysis to obtain a satisfactory description of the oscillation frequencies; the additional diagnostic information contained in $\alpha_2(\nu)$ will be considered in future studies.

To relate $\alpha_0(\nu)$ to the structure of the solar envelope, we note that the adiabatic oscillation equations with $\ell(\ell+1)/r^2 \ll \omega^2/c^2$ reduce to the Schrödinger-type equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\zeta+[\omega^2-V(\tau)]\zeta=0, \qquad (2)$$

where the acoustic depth τ and kinetic energy density of the oscillations ζ^2 are defined by

$$\tau=\int_r^R\frac{\mathrm{d}r}{c(r)}, \qquad \zeta=(\rho c)^{1/2}rU, \qquad (3)$$

and where the radial displacements $\delta r(r, \theta, \phi) = U(r)\, Y_{\ell,m}(\theta, \phi)$. The acoustic potential is

$$V = N^2 - \frac{c}{2}\frac{\mathrm{d}}{\mathrm{d}r}\left[c\left(\frac{2}{r}+\frac{N^2}{g}-\frac{g}{c^2}-\frac{1}{2c^2}\frac{\mathrm{d}c^2}{\mathrm{d}r}\right)\right] + \frac{c^2}{4}\left(\frac{2}{r}+\frac{N^2}{g}-\frac{g}{c^2}-\frac{1}{2c^2}\frac{\mathrm{d}c^2}{\mathrm{d}r}\right)^2 - 4\pi G\rho, \tag{4}$$

where N is the Brunt–Väisälä frequency. The term $-4\pi G\rho$ is due to gravitational perturbations and can be neglected in the solar envelope. The variation of $V(\tau)$ for a solar model is shown in Fig. 1. There are three regions in the envelope where the potential varies on a scale short compared with the radial wavelength and where the asymptotic approximation becomes locally invalid; each of these regions contributes to $\alpha_0(\nu)$. The first ($\tau \lesssim 100$ s) is the region of superadiabatic convection and overlying surface layers; its contribution to $\alpha_0(\nu)$ is difficult to use for diagnostic purposes because it is expected to be influenced by non-adiabatic effects and the interaction of acoustic waves with turbulent convection. The second region ($\tau \approx 600$ s) is the second helium ionization zone; it produces the quasi-periodic component in $\alpha_0(\nu)$ which is used for helium abundance determination (Christensen-Dalsgaard & Pérez Hernández 1991; Vorontsov, Baturin & Pamyatnykh 1991) and for the calibration of the equation of state (Christensen-Dalsgaard & Pérez Hernández 1992; Vorontsov, Baturin & Pamyatnykh 1992). The third region ($\tau \approx 2100$ s) is the base of the convective zone. In a 'standard' model the transition from the adiabatic temperature gradient to the radiative gradient at the base of the convective zone leads to the discontinuity in the second derivative of temperature and hence of $c^2(r)$, which enters directly into the expression for the acoustic potential. The potential for a standard solar model is given in Fig. 1 (solid line) and has a discontinuity at the base of the convective zone; also shown in Fig. 1 is the potential for a model with convective overshooting (dashed line) which has a delta-function-type behaviour at the interface between the convective zone and the radiative interior. (The curves in Fig. 1 are smoothed somewhat by numerical differentiation.)

3 DETERMINATION OF $\alpha(\nu, \tilde{w})$ FROM OSCILLATION FREQUENCIES

The data set used in this analysis is from Libbrecht et al. (1990). To determine $\alpha_0(\nu)$ from these data, we confine our attention to the second-order asymptotic expression and rewrite equation (1) in terms of frequency $\nu = \omega/2\pi$ in dimensionless form as

$$\frac{n}{\nu/\nu_0} = 2\nu_0 F(\tilde{w}) + \left(\frac{\nu_0^2}{\nu^2}\right)\frac{\Phi(\tilde{w})}{2\pi^2\nu_0} - \frac{\alpha_0(\nu)}{\nu/\nu_0} - \left(\frac{\tilde{w}^2}{\tilde{w}_0^2}\right)\frac{\tilde{w}_0^2\alpha_2(\nu)}{\nu/\nu_0}, \tag{5}$$

where $\nu_0 = 3$ mHz and $\tilde{w}_0 = 1000$ s are taken as reference values. The $n/(\nu/\nu_0)$ values are the input data; $F(\tilde{w})$, $\Phi(\tilde{w})$, $\alpha_0(\nu)$ and $\alpha_2(\nu)$ are to be determined. We first confine our attention to modes trapped within the convective zone (733 modes with $3000 \le \tilde{w} \le 8000$ s were used in the inversion), and represent these data by the four terms on the right-hand side of equation (5) using cubic B-spline approximants. [This procedure is described in detail by Gough & Vorontsov (1994).] We thus infer the phase shift $\alpha_0(\nu)/(\nu/\nu_0)$ which is produced by the outer solar layers, including the second helium ionization zone. The main variation of the phase shift is represented by a slowly varying component, produced by the surface layers – it is shown in Fig. 2(a) as a low-order polynomial fit to $\alpha_0/(\nu/\nu_0)$. The remaining low-amplitude

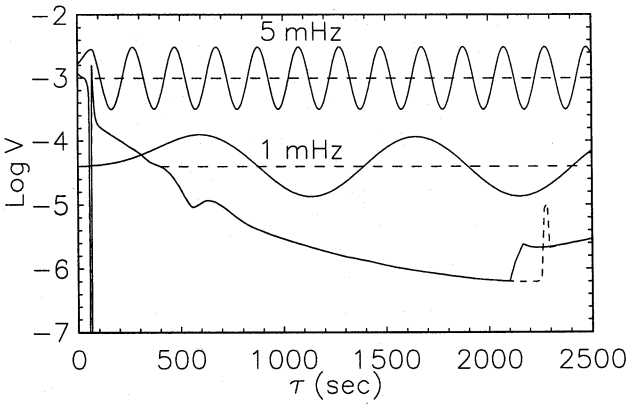


Figure 1. Acoustic potential versus acoustic depth in a standard solar model (solid line), and for a model with convective overshooting (dashed line). Also shown are the wavefunctions $\zeta(\tau)$ for frequencies of 1 and 5 mHz. Axes of zero amplitude (horizontal dashed lines) are placed at $V = \omega^2$. The models were kindly supplied by V. A. Baturin and S. V. Ajukov.

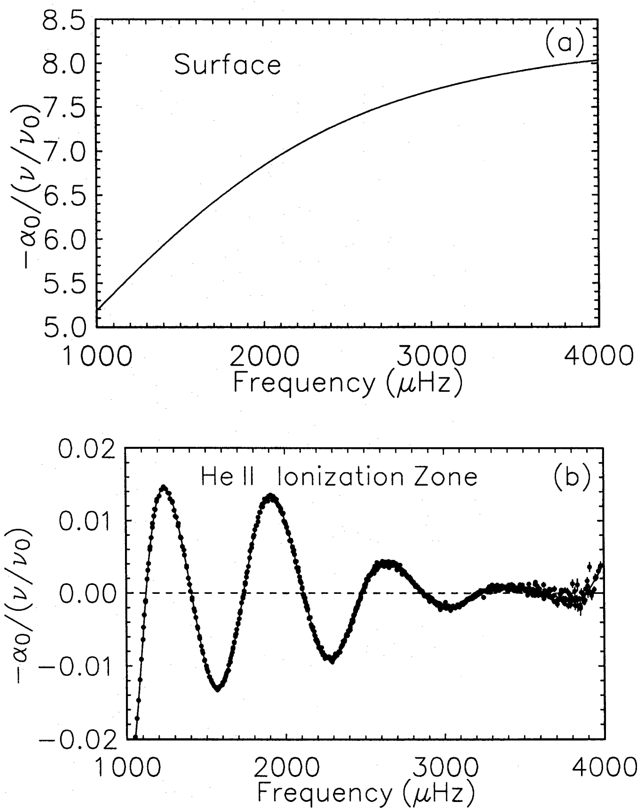


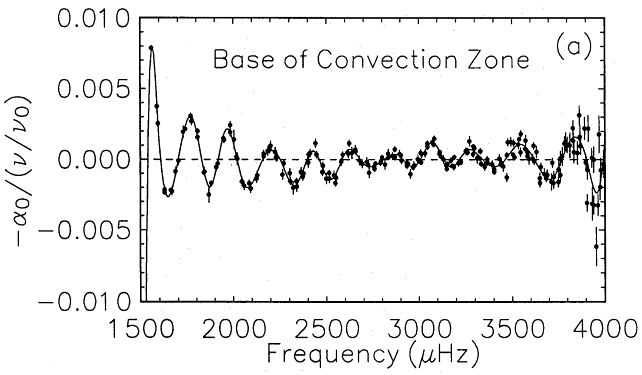
Figure 2. Normalized values of the acoustic phase shift $\alpha_0(\nu)/(\nu/\nu_0)$ [the third term on the right-hand side of equation (5)]: (a) shows the contributions from the surface layers, and (b) the contribution from the second helium ionization zone.

quasi-periodic component is the signal produced by the second helium ionization zone (Fig. 2b).

To determine the contribution to $\alpha_0(\nu)$ and $\alpha_2(\nu)$ from the base of the convective zone, we use modes with inner turning points well below the base of the zone. We first subtract the approximation for the contribution to $\alpha_0/(\nu/\nu_0)$ from the surface layers and helium ionization zone obtained above from the left-hand side of equation (5), and then fit the residual terms on the right-hand side by cubic B-splines, thus obtaining the additional contribution to the phase shift that comes from the base of the zone. The observational data set is limited by $500 \leq \tilde{w} \leq 1200$ s (213 modes); this interval of $\tilde{w}$ corresponds to p-modes with inner turning points $0.25\,R_0 \leq r \leq 0.5\,R_0$, since for modes with $r < 0.25\,R_0$, the second-order asymptotic description begins to lose its accuracy; and for modes with $r > 0.5\,R_0$ the representation of the phase shift by the truncated power expansion $\alpha_0(\nu, \tilde{w}) = \alpha_0(\nu) + \tilde{w}^2\alpha_2(\nu)$ becomes inaccurate. Frequency-dependent terms in equation (6) are approximated by $30+30$ cubic B-splines, and $\tilde{w}$-dependent terms by $7+7$ cubic B-splines; the distribution of breakpoints is uniform in both ν and $\tilde{w}$. Simple weights proportional to the relative accuracy of frequency measurements are used.

The resulting approximation to the phase shift $\alpha_0(\nu)/(\nu/\nu_0)$ produced by the base of the convective zone is shown in Fig. 3(a) as the continuous curve, the points in this figure being the values of $\alpha_0(\nu)/(\nu/\nu_0)$ derived by subtracting the fit to the other terms on the right-hand side of equation (5) from the observed values of $n/(\nu/\nu_0)$. The error bars correspond to the errors in the observed frequencies. Fig. 3(b) shows the result for $\tilde{w}_0^2\alpha_2(\nu)/(\nu/\nu_0)$. The complete phase shift for low-degree modes is the sum of a slowly varying component with variation of order unity on a dimensionless scale, a periodic component with amplitude two orders of magnitude smaller, and a second periodic component with amplitude one order of magnitude smaller still. As can be seen from Figs 3(a) and (b), the accuracy of the approximation of the small signal which comes from the base of the convective zone is limited significantly by the accuracy of the frequency measurements. The amplitude A_ν of $\alpha_0(\nu) \approx 8(\pm 4) \times 10^{-4}$ at $\nu = 3000$ μHz, and the amplitude of $\tilde{w}_0^2\alpha_2(\nu)$ is of a similar order.

Exactly the same procedure is applied to theoretical eigenfrequencies computed for a standard solar model; the results are shown in Fig. 4. In Fig. 5 we show similar results for a theoretical model which includes convective penetration below the base of the unstable layers, the adiabatic layer being extended by 1 per cent of a solar mass or 0.54 of a pressure scaleheight. For the standard solar model we follow exactly the same procedure, using exactly the same modes as are used in the observational data set; for the model with overshooting, 35 splines are used for the frequency-dependent terms. As can be seen from these results, the amplitude of $\alpha_0(\nu)$ for the model with convective penetration (Fig. 5a) is significantly larger than that deduced from the data (Fig. 3a).



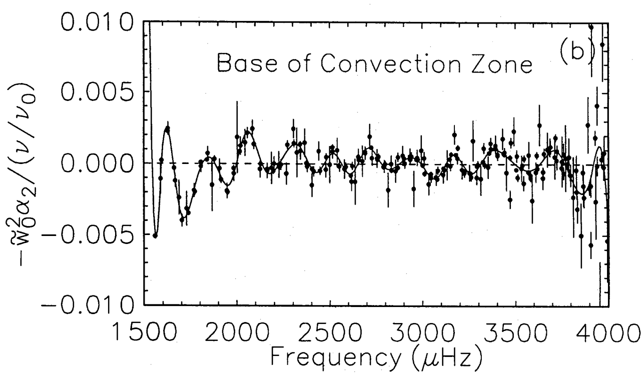


Figure 3. Normalized values of the acoustic phase shift $-\alpha_0(\nu)/(\nu/\nu_0)$ (a), and $-\tilde{w}_0^2\alpha_2(\nu)/(\nu/\nu_0)$ (b), from the base of the convective zone, derived using the 'low-resolution' observational data sets of Libbrecht et al. (1990). Dots with error bars are plotted for individual modes as the difference between the left-hand side of equation (5) and the sum of the approximations for three other terms on the right-hand side.

4 THEORETICAL MODEL OF THE PHASE SHIFT INDUCED BY THE BASE OF THE CONVECTIVE ZONE

The phase shift $\alpha_0(\nu)$ induced by the discontinuities at the base of the convective zone can be estimated as follows. Let the potential $V(\tau)$ vary somewhere within an interval $a < \tau < b$ by $\delta V = V_2 - V_1$, giving rise to the variation of the eigenfunction $\delta\xi = \xi_2 - \xi_1$ (for a given value of the frequency ω). Multiplying equation (2) for ξ_1 by ξ_2, subtracting this from the same equation with ξ_1 and ξ_2 interchanged, and integrating between a and b, we obtain the Wronskian theorem

$$[W(\xi_1, \xi_2)]_a^b = \int_a^b \delta V \xi_1 \xi_2 \, d\tau, \tag{6}$$

where $W(\xi_1, \xi_2) = (\xi_1\, d\xi_2/d\tau - \xi_2\, d\xi_1/d\tau)$. For an arbitrary potential, the eigenfunction $\xi(\tau, \omega)$ is conveniently represented by the method of phase functions (e.g. Babikov 1976; Vorontsov & Zarkhov 1989; Vorontsov 1991) as

$$\xi(\tau, \omega) = A(\tau, \omega) \cos\left[\omega\tau - \frac{\pi}{4} - \pi\tilde{\alpha}(\tau, \omega)\right], \tag{7}$$

subject to the amplitude condition

$$\frac{d}{d\tau}\xi(\tau, \omega) = -\omega A(\tau, \omega) \sin\left[\omega\tau - \frac{\pi}{4} - \pi\tilde{\alpha}(\tau, \omega)\right]. \tag{8}$$

Because the outer boundary conditions at $\tau = 0$ remain unchanged, the eigenfunction remains unchanged for $\tau \leq a$

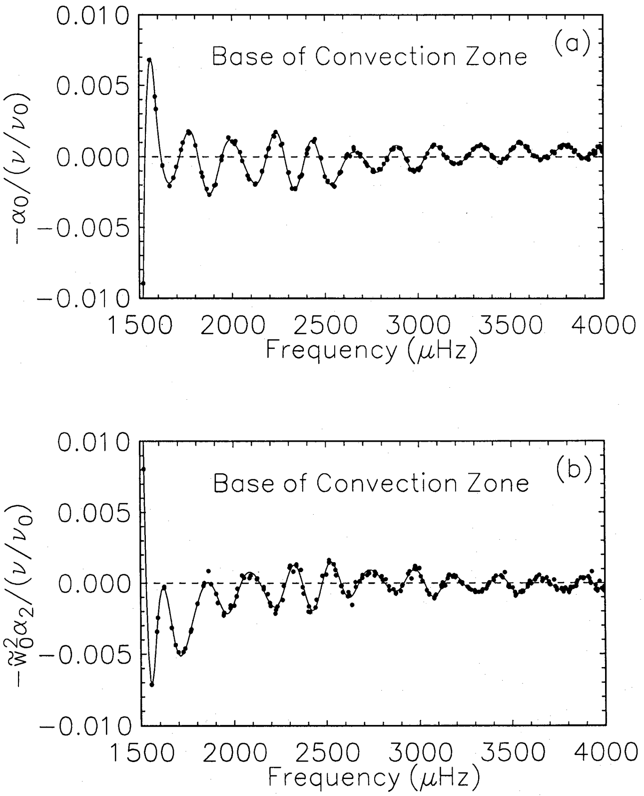


Figure 4. As Fig. 3, but for a standard solar model with no convective penetration below the unstable convective zone.

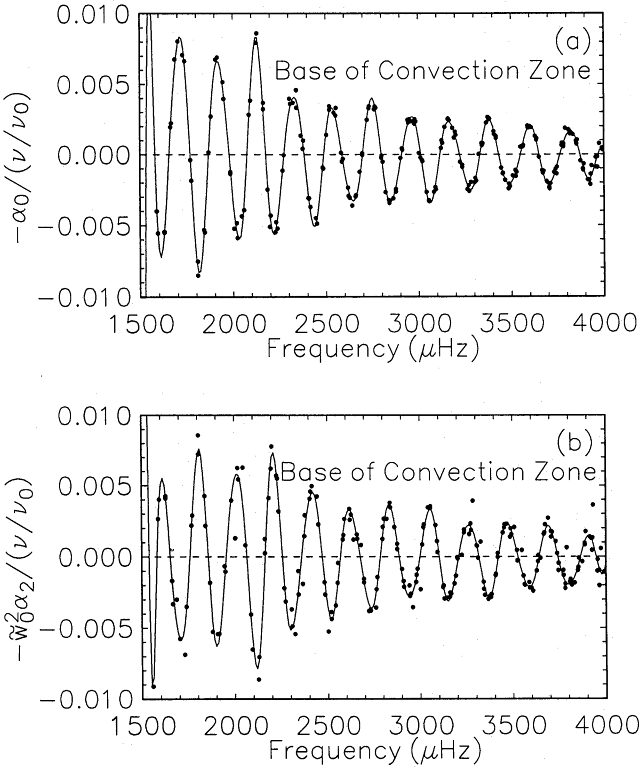


Figure 5. As Fig. 4, but for a solar model with convective penetration. The overshoot region contains 0.01 of a solar mass and extends to a depth of 0.54 H_p below the unstable layer.

and we have

$$[W(\zeta_1, \zeta_2)]_a^b = \omega A_1(b, \omega) A_2(b, \omega) \times \sin[\pi\tilde{\alpha}_2(b, \omega) - \pi\tilde{\alpha}_1(b, \omega)]. \tag{9}$$

Setting $\alpha_0 = \alpha_2 - \alpha_1$ and linearizing, we obtain

$$\pi\omega A^2(\tau, \omega)\alpha_0 \cong \int_a^b \delta V A^2(\tau, \omega) \times \cos^2\left[\omega\tau - \frac{\pi}{4} - \pi\tilde{\alpha}(\tau, \omega)\right] \mathrm{d}\tau. \tag{10}$$

In the propagation zone with $V \ll \omega^2$, the solutions are very close to harmonic functions (see Fig. 1), and neglect of the variation of the amplitude function in the interval (a, b) gives

$$\alpha_0 \cong \frac{1}{\pi\omega} \int_a^b \delta V \cos^2\left[\omega\tau - \frac{\pi}{4} - \pi\tilde{\alpha}(\tau, \omega)\right] \mathrm{d}\tau. \tag{11}$$

The argument of the cosine represents the phase of the unperturbed eigenfunction at a given acoustic depth τ; it increases monotonically with ω. We thus expect that, if δV is localized at an acoustic depth τ_b on a scale short compared to the radial wavelength, $\alpha_0(\nu)$ will vary quasi-periodically with frequency ν, with a period of $1/2\tau_b$. As can be seen from Fig. 2, this period ≈ 220 μHz.

The discontinuity in the acoustic potential at the base of the convective zone has in general two contributions, one from the discontinuity in the second derivative of c^2 that is present in standard models of the convective zone without penetration, and one from the discontinuity in the first derivative that is present in models with convective penetration. The acoustic potential can therefore be represented as (see Section 5)

$$V(\tau) = V_0(\tau) + A_H H(\tau - \tau_b) + A_\delta \delta(\tau - \tau_b), \tag{12}$$

where A_H is the contribution to the discontinuity in V due to the discontinuity in the second derivative of c^2, A_δ is that from the discontinuity in the first derivative, and $\tau = \tau_b$ is the base of the convective zone. [$H(x)$ is the Heaviside function and $\delta(x)$ the Dirac δ-function.]

The contribution from the δ-function is readily evaluated from equation (11), and has a quasi-periodic component

$$\alpha_\delta \cong \frac{A_\delta}{2\pi\omega} \cos 2\left[\omega\tau_b - \frac{\pi}{4} - \pi\tilde{\alpha}(\tau_b, \omega)\right]. \tag{13}$$

The amplitude varies as $1/\omega$. To evaluate the contribution from the Heaviside function, we integrate equation (11) by parts and average the rapidly varying component over the regions of smooth variation of $V(\tau)$ to obtain the quasi-periodic component in the form

$$\alpha_H \cong \frac{A_{\rm H}}{4\pi\omega^2} \sin 2\left[\omega\tau_b - \frac{\pi}{4} - \pi\tilde{\alpha}(\tau_b, \omega)\right]. \tag{14}$$

The amplitude varies as $1/\omega^2$. The combination is therefore a quasi-periodic component with a period of $1/(2\tau_b)$ and with an amplitude

$$A_\nu=\left[\left(\frac{A_H}{16\pi^3\nu^2}\right)^2+\left(\frac{A_\delta}{4\pi^2\nu}\right)^2\right]^{1/2}. \tag{15}$$

If the region near the base of the convective zone is adequately described by standard assumptions (no convective penetration, no magnetic field), $A_\delta=0$, the amplitude A_H is determined by the second derivative of the sound speed and hence by the opacity gradient just below the base of the convective zone. The phase shift can therefore be used to measure the depth of the convective zone and the opacity gradient near its base much more accurately than by the inversion of the sound speed. The periodic signal in the phase shift is sensitive to the phase of the eigenfunction at the base of the convective zone, whereas the resolution of the sound-speed inversions is inherently limited by the finite radial wavelength. The contribution to the phase shift from the surface layers has an altogether different signature, and can therefore be separated out from the periodic signal produced by discontinuities at the base of the convective zone. Convective penetration can produce a discontinuity in the first derivative of the sound speed at the interface between the extended adiabatic layers and the radiative interior, thus producing a perturbation of δ-function type in the acoustic potential. Such a perturbation produces a periodic component in $\alpha(\nu)$ with amplitude variation proportional to $1/\nu$, whereas the standard contribution from the discontinuity in the second derivative is proportional to $1/\nu^2$. In principle, therefore, these two contributions can be separately determined.

A strong magnetic field concentrated in a thin layer near the base of the convective zone can also affect the average frequencies of p-mode multiplets, producing a quasi-periodic component in the phase shift; the dominant influence of such a magnetic field comes from the perturbation of the simple inertial term in the Lagrangian, i.e. from the density perturbation in the magnetized layer (Vorontsov 1988). A density discontinuity leads to a δ-function in the behaviour of $N^2(r)$. If the magnetized layer is thin, it can be considered as a point source, and its influence on the oscillations is determined by the net deficit of inertia and can be described by the appearance of a derivative of a δ-function in the behaviour of $N^2(r)$, or as a second derivative of a δ-function in $V(\tau)$. It is easy to show that in this case the periodic component in $\alpha(\nu)/\nu$ appears with an amplitude that is independent of frequency.

When Fig. 3(a) is compared with Fig. 4(a), no significant difference is seen between the standard model predictions and the solar data, whereas there is a significant difference between Figs 3(a) and 5(a). It would be premature, however, to draw strong conclusions from this. As we show below, for convective penetration distances $\ell=\varepsilon H_p$ (where H_p is the pressure scaleheight), as ε increases from zero the amplitude of α_0 first decreases slightly and then increases, so that the present observational data appear to be consistent with a value of ε in the range $0\le\varepsilon\le0.25$. The amplitude of $\alpha(\nu)/\nu$ also depends on the variation of opacity with temperature and density at the base of the zone but, since the contribution from A_δ has a different dependence on frequency from the contribution from A_H, it is in principle possible to separate the two contributions and hence place constraints on the extent of convective overshooting. The currently available observational data are, however, insufficiently accurate for this separation to be carried out, so the constraint $0\le\varepsilon\le0.25$ should be treated with some caution.

5 DISCONTINUITIES IN THE ACOUSTIC POTENTIAL AT THE BASE OF THE CONVECTIVE ZONES

In a standard model of the convective zone, the base of the zone is the depth at which the energy carried by radiation down the adiabatic temperature gradient equals the total flux, the convective flux going to zero at the boundary. In a simple model incorporating convective penetration, the adiabatic overshoot layer extends a distance $\ell=\varepsilon H_p$ below the 'classical' base of the zone; in the overshoot region the energy carried by radiation exceeds the total, the convective energy flux being negative. At the base of the overshoot region there is a thin boundary layer in which the convective motions are rapidly decelerated and the temperature gradient adjusts to the value in the radiative interior. For very thin boundary layers, this is equivalent to taking the adiabatic gradient down to the base of the overshoot layer and a discontinuity in the temperature gradient at the boundary (cf. Roxburgh 1978). This is illustrated in Fig. 6.

In the overshoot layer and region just below the boundary we take $\Gamma=5/3$ and represent the opacity by a local power law, $\kappa=\kappa_0\rho^\alpha T^{-\beta}$, where $\alpha=\partial\log\kappa/\partial\log\rho$ and $\beta=-\partial\log\kappa/\partial\log T$ (a fit to the Livermore opacity tables gives $\alpha\approx0.5$, $\beta\approx3$). We neglect the mass of the convective zone in comparison with the mass of the Sun, and take the total luminosity L_0 as constant. Inside the adiabatic convective zone with $P/T^{5/2}=$constant, the equation of hydrostatic support gives

$$\frac{\mathrm{d}T}{\mathrm{d}r}=-\frac{2}{5}\frac{\mu GM}{\mathscr{R}r^2},\qquad \frac{\mathrm{d}c^2}{\mathrm{d}r}=-\frac{2}{3}g,$$

$$\frac{\mathrm{d}^2c^2}{\mathrm{d}r^2}=\frac{4}{3r}g,\qquad g=\frac{GM}{r^2}. \tag{16}$$

We denote by r_1 the radius at the classical base of the zone where $L_{\mathrm{rad}}=L_0$, and by r_+ and r_- the radii at either side of the boundary ($r_+>r_-$). At the radiative side of the boundary ($r=r_-$) the temperature gradient is

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_-=-\left(\frac{3\kappa\rho L_0}{16\pi acT^3r^2}\right)_-. \tag{17}$$

The temperature gradient at the 'classical' boundary ($L_{\mathrm{rad}}=L_0$) is given by the same expression evaluated at $r=r_1$. Taking the ratio of these two expressions, we obtain $(\mathrm{d}T/\mathrm{d}r)_-$ in terms of values at r_- and r_1. Use of the result that $\mathrm{d}T/\mathrm{d}r\propto1/r^2$ in the adiabatic region, to eliminate the temperature gradient at r_1 in favour of the value at the base of the zone (r_+), gives

$$\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_-=\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_+\left(\frac{\kappa\rho}{T^3}\right)_+\left(\frac{T^3}{\kappa\rho}\right)_{r_1}. \tag{18}$$

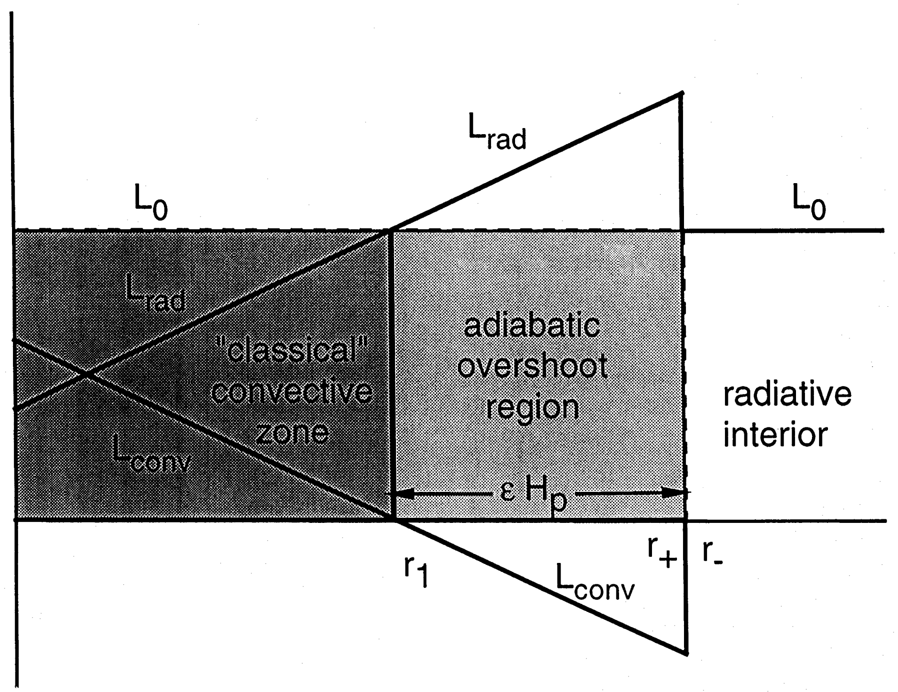


Figure 6. Variation of the radiative, total and convective luminosities near the base of the convective zone in a model with convective penetration distance $\ell = \varepsilon H_p$.

With $\kappa = \kappa_0 \rho^{\alpha} T^{-\beta}$ we find

$$\left(\frac{\kappa\rho}{T^3}\right)_+ \left(\frac{T^3}{\kappa\rho}\right)_{r_1} = \left(\frac{T_1}{T_+}\right)^{3/2+\beta-3\alpha/2}. \tag{19}$$

The temperature in an adiabatic layer is, to a good approximation, proportional to $d = (R - r)$, the distance from the surface, and the pressure scaleheight is $2d/5$. Hence

$$\left(\frac{\kappa\rho}{T^3}\right)_+ \left(\frac{T^3}{\kappa\rho}\right)_{r_1} = \left(\frac{d_1}{d_+}\right)^{3/2+\beta-3\alpha/2} = \left(\frac{d_1}{d_1+\varepsilon H_p}\right)^{3/2+\beta-3\alpha/2}$$

$$= \left(\frac{1}{1+2\varepsilon/5}\right)^{3/2+\beta-3\alpha/2} \tag{20}$$

and so

$$\left(\frac{\mathrm{d}c^2}{\mathrm{d}r}\right)_- = \left(\frac{\mathrm{d}c^2}{\mathrm{d}r}\right)_+ \left(\frac{1}{1+2\varepsilon/5}\right)^{3/2+\beta-3\alpha/2}. \tag{21}$$

The second derivative of the sound speed on the radiative side of the boundary can be calculated by taking the logarithmic derivative of equation (17) to yield

$$\left(\frac{\mathrm{d}^2T}{\mathrm{d}r^2}\right)_- = \left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_- \left[(1+\alpha)\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}r}\right.$$

$$\left. -(4+\alpha+\beta)\frac{1}{T}\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_- - \frac{2}{r}\right]. \tag{22}$$

From hydrostatic support we have $\mathrm{d}P/\mathrm{d}r = -(5/3)Pg/c^2$, and hence

$$\left(\frac{\mathrm{d}^2c^2}{\mathrm{d}r^2}\right)_- = -\left(\frac{\mathrm{d}c^2}{\mathrm{d}r}\right)_-$$

$$\times\left[(1+\alpha)\frac{5g}{3c^2} + (4+\alpha+\beta)\frac{1}{c^2}\left(\frac{\mathrm{d}c^2}{\mathrm{d}r}\right)_- + \frac{2}{r}\right]. \tag{23}$$

The values of $(\mathrm{d}c^2/\mathrm{d}r)_+$ and $(\mathrm{d}^2c^2/\mathrm{d}r^2)_+$ are given by equation (16) with $r = r_+$.

Taking $\alpha = 0.5$ and $\beta = 3$ the values of the derivatives at the base of the zone are

$$\left(\frac{\mathrm{d}c^2}{\mathrm{d}r}\right)_+ = -\frac{2}{3}g, \qquad \left(\frac{\mathrm{d}c^2}{\mathrm{d}r}\right)_- = -\frac{2}{3}g\left(\frac{1}{1+2\varepsilon/5}\right)^{15/4}, \tag{24}$$

$$\left(\frac{\mathrm{d}^2c^2}{\mathrm{d}r^2}\right)_+ = \frac{4}{3}\frac{g}{r}, \qquad \left(\frac{\mathrm{d}^2c^2}{\mathrm{d}r^2}\right)_- = -\frac{5}{3}\frac{g^2}{c^2}\left(\frac{1}{1+2\varepsilon/5}\right)^{15/4}$$

$$\times\left[2\left(\frac{1}{1+2\varepsilon/5}\right)^{15/4} - 1 - \frac{4c^2}{5rg}\right]. \tag{25}$$

Since, at the base of the solar convective zone, $2g/r \ll 5g^2/3c^2$, the discontinuities are

$$\left[\frac{\mathrm{d}c^2}{\mathrm{d}r}\right]_-^+ = -\frac{2}{3}g\left[1-\left(\frac{1}{1+2\varepsilon/5}\right)^{15/4}\right], \tag{26}$$

$$\left[\frac{\mathrm{d}^2c^2}{\mathrm{d}r^2}\right]_-^+ = \frac{5}{3}\frac{g^2}{c^2}\left(\frac{1}{1+2\varepsilon/5}\right)^{15/4}\left[2\left(\frac{1}{1+2\varepsilon/5}\right)^{15/4} - 1\right]. \tag{27}$$

For $\varepsilon=0$ the discontinuity in $\mathrm{d}^2c^2/\mathrm{d}r^2=5g^2/3c^2$; as ε increases, the discontinuity in $\mathrm{d}^2c^2/\mathrm{d}r^2$ first decreases, reaching zero when $\varepsilon=0.5$ and a maximum negative value of $-(5/24)(g/c^2)$ for $\varepsilon\approx 1.1$, and then tends to zero for large ε. The discontinuity in $\mathrm{d}c^2/\mathrm{d}r$ is zero for $\varepsilon=0$ and decreases monotonically, having a value of $-g/3$ at $\varepsilon\approx 0.5$ and tending to a minimum value of $-2g/3$ for large ε.

The acoustic potential is given in equation (4); with $\Gamma=5/3$ and homogeneous composition this can be expressed as

$$V=\frac{2}{3}\frac{g}{c^2}+\frac{g}{c^2}\frac{\mathrm{d}c^2}{\mathrm{d}r}-\frac{c}{2}\frac{\mathrm{d}}{\mathrm{d}r}\left[c\left(\frac{2}{r}-\frac{1}{3}\frac{g}{c^2}+\frac{1}{2c^2}\frac{\mathrm{d}c^2}{\mathrm{d}r}\right)\right]$$
$$+\frac{c^2}{4}\left(\frac{2}{r}-\frac{1}{3}\frac{g}{c^2}+\frac{1}{2c^2}\frac{\mathrm{d}c^2}{\mathrm{d}r}\right)^2-4\pi G\rho\,. \qquad (28)$$

At the base of the zone, g, c^2 and ρ are continuous but $\mathrm{d}c^2/\mathrm{d}r$ and $\mathrm{d}^2c^2/\mathrm{d}r^2$ are discontinuous. V can therefore be represented as

$$V(\tau)=V_0(\tau)+A_H H(\tau-\tau_\mathrm{b})+A_\delta\frac{1}{c}\,\delta(\tau-\tau_\mathrm{b}), \qquad (29)$$

where $V_0(\tau)$ is a smooth function of τ, $H(\tau-\tau_\mathrm{b})$ is a Heaviside function, $\delta(\tau-\tau_\mathrm{b})$ is a Dirac delta-function and $\tau=\tau_\mathrm{b}$ is the base of the zone. The amplitude A_H is the discontinuity in V, which is

$$A_H=[V]_-^+=\frac{5}{6}\frac{g}{c^2}\left[\frac{\mathrm{d}c^2}{\mathrm{d}r}\right]_-^+ +\frac{3}{16c^2}\left[\left(\frac{\mathrm{d}c^2}{\mathrm{d}r}\right)^2\right]_-^+ -\frac{1}{4}\left[\frac{\mathrm{d}^2c^2}{\mathrm{d}r^2}\right]_-^+ . \qquad (30)$$

Use of the results for the values of the discontinuities at the base of the zone gives

$$A_H=[V]_-^+=-\frac{1}{36}\frac{g^2}{c^2}\times\left[33\left(\frac{1}{1+2\varepsilon/5}\right)^{15/2}-35\left(\frac{1}{1+2\varepsilon/5}\right)^{15/4}+17\right]$$
$$=-\frac{g^2}{c^2}f(\varepsilon)\,. \qquad (31)$$

$f(\varepsilon)$ is only weakly dependent on ε. For $\varepsilon=0, f=0.4166\ldots$, as ε increases, f decreases, reaching a minimum of $0.2144\ldots$ at $\varepsilon=0.460$, and then increases, tending to $0.4722\ldots$ as ε becomes large.

The amplitude of the Dirac δ-function is given by the discontinuity in $V(\tau)$ at the base of the zone, and hence by the discontinuity in $\mathrm{d}c^2/\mathrm{d}r$; recalling that A_δ is defined in terms of τ rather than r, we have

$$A_\delta=-\frac{g}{6c}\left[1-\left(\frac{1}{1+2\varepsilon/5}\right)^{15/4}\right]=-\frac{g}{c}\,h(\varepsilon). \qquad (32)$$

$h(\varepsilon)=-\varepsilon/4$ for small ε and increases monotonically to 1/6 for large ε. The amplitude of the normalized phase shift $\alpha(\nu)/(\nu/\nu_0)$ is then given by

$$A_\nu=\left[\left(\frac{\nu_0}{16\pi^3\nu^3}\frac{g^2}{c^2}\right)^2 f^2(\varepsilon)+\left(\frac{\nu_0}{4\pi^2\nu^2}\frac{g}{c}\right)^2 h^2(\varepsilon)\right]^{1/2}. \qquad (33)$$

The contribution from the discontinuity in the second derivative, which is present in the classical model without penetration, has $h(0)=0$ and $f(0)=5/12$. Taking $\nu=\nu_0=3$ mHz, and typical values at the base of the convective zone, we obtain $A_\nu=5.5\times10^{-4}$ for $\varepsilon=0$. This result is in reasonable agreement with the value shown in Fig. 4(a) for a standard solar model, the differences arising from the approximations to the opacity and the smoothing of the discontinuity in the potential over a scale of the order of a radial wavelength. The variation of A_ν with ε (at $\nu=3$ mHz) is shown in Fig. 7; the slight decrease in A_ν for very small ε is due to the fact that the decrease in the magnitude of the discontinuity in the second derivative is initially more important than the effect of the discontinuity in the first derivative, but, as ε increases further, A_ν increases, tending to a limit of 3.5×10^{-3} for large ε. The variation of A_ν with frequency ν is shown in Fig. 8 for $\varepsilon=0$, 0.25 and 0.5. At $\nu=3$ mHz and $\varepsilon=0.25$, $A_\nu=1.07\times10^{-3}$ which is compatible with the accuracy of the available data (Fig. 3a). With

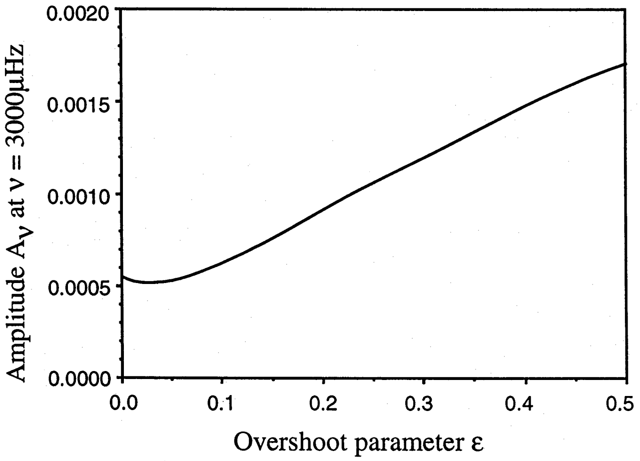


Figure 7. Variation of the amplitude A_ν of the phase shift $\alpha_0(\nu)$ at a frequency of 3 mHz with penetration parameter ε.

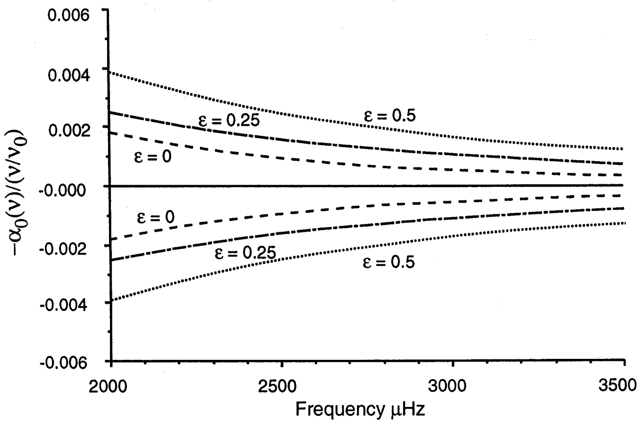


Figure 8. Variation of the amplitude of the acoustic phase shift $-\alpha_0(\nu)/(\nu/\nu_0)$ from the base of the convective zone with frequency and convective overshooting parameter ε. For given ε, the quasi-periodic variation in the phase shift lies between the upper and lower bounds given in this diagram.

$\varepsilon = 0.5$, however, $A_\nu = 1.7 \times 10^{-3}$ at $\nu = 3$ mHz, rising to 4×10^{-3} at 2 mHz, which is already too large to fit the observations but is not inconsistent with the values predicted from the model with an overshoot parameter of 0.52.

The amplitude A_ν depends on both the extent of overshooting, ε, and the variation in the opacity at the base of the convective zone, which we have here approximated by $\kappa = \kappa_0 \rho^{0.5} T^{-3}$; however, since the contributions to A_ν from $f(\varepsilon)$ and $h(\varepsilon)$ have different dependences on frequency, with sufficiently accurate data it should be possible to separate the contribution from convective penetration from the uncertainty in our knowledge of the local dependence of opacity on temperature and density.

6 CONCLUSIONS

The phase shift $\alpha_0(\nu, \tilde{w}) = \alpha_0(\nu) + \tilde{w}^2 \alpha_2(\nu)$ in acoustic waves, caused by the sharp changes in the derivatives of the sound speed at the base of the solar convective zone, has been determined from the observed frequencies of intermediate-degree modes. The contribution to the amplitude of this phase shift from convective penetration has a different frequency dependence from that from a classical convective zone without penetration, and can therefore, in principle, be used to determine the extent of convective penetration below the unstable layers. Current observational data, however, can only place an upper limit on such convective penetration of the order of $0.25H_p$, where H_p is the pressure scaleheight at the base of the zone. This upper limit is somewhat larger than the estimate deduced by Monteiro, Christensen-Dalsgaard & Thompson (1993, 1994) and Basu, Antia & Narasimha (1994), using a somewhat different technique requiring numerical differentiation of a noisy signal on a scale comparable with the scale of variation of the signal, and somewhat less than the estimate obtained by Berthomieu et al. (1993), who compared frequencies from solar models with the observational values.

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