THE NON-RADIAL OSCILLATIONS OF POLYTROPIC STARS

T. G. Cowling, M.A., D.Phil.

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1. The problem of the non-radial oscillations of a fluid globe is of interest in the theory of the tidal distortion of binary stars. If the periods of rotation of the components differ from the orbital period, the tidal force produces a periodic disturbance at any point of either component. Provided that the period of this disturbance is long compared with the free periods of the stars, the tidal distortion differs little from that given by equilibrium theory. When, on the other hand, the period of the disturbance is comparable with the free periods, resonance is possible; this leads to an enhanced tidal distortion, resulting, in extreme cases, in the fission of a component.

The oscillations of a homogeneous incompressible liquid globe were considered by Kelvin.* The problem for a similar globe in rotation was considered by Bryan † and Love.‡ This problem is relatively tractable; but when a heterogeneous liquid globe is considered, the mathematical difficulties are much greater. For this reason, we consider here only the oscillations of non-rotating stars.

The oscillations of a heterogeneous compressible fluid sphere present a problem which is, in any case, of great mathematical complexity. As Pekeris has recently shown §, the problem is equivalent to the solution of two simultaneous second-order differential equations. Pekeris considered in detail the case of an initially homogeneous sphere of compressible fluid. This is clearly dynamically unstable, since, by transferring material from the outer layers to the centre, a mass of greater central density and smaller gravitational potential energy is obtained; and Pekeris' results indicated this instability.

Emden || considered approximately the oscillations of a polytropic gas sphere. His work was, however, vitiated by the approximations made in considering the equation of continuity, which he reduced to the form appropriate to a homogeneous incompressible fluid; the terms neglected and those retained are of the same order of magnitude. He obtained only one free period for oscillations in which the radial displacement is proportional to a surface harmonic of given order; there are, in fact, an infinite number of such periods.

In this paper we shall consider a polytropic sphere, whose specific heat may differ from that corresponding to convective equilibrium with the corresponding polytropic index. We shall show that oscillations of indefinitely large period are possible, and consider the bearing of this result on the tidal problem.

2. Consider a star of polytropic index n. In equilibrium the density ρ and pressure p at a distance r from the centre are connected with the central density ρ_c and pressure p_c by equations of the form

$$\rho=\rho_c y^n, \qquad p=p_c y^{1+n}, \tag{1}$$

where y satisfies Emden's equation

$$\frac{d}{dx}\left(x^2\frac{dy}{dx}\right)=-x^2y^n \tag{2}$$

* *Collected Papers*, **3**, 384.
† *Phil. Mag.*, (5), **27**, 254, 1889.
‡ *Phil. Trans.*, (A), **153**, 187, 1889.
§ *Ap. J.*, **88**, 189, 1939.
|| *Gaskugeln*, p. 448.

and x is given by

$$r = r_0 x, \qquad r_0^2 = (1+n)p_c/4\pi G \rho_c^2 \tag{3}$$

(G is the gravitational constant).

In small oscillations, let the density and pressure, and the gravitational potential V at any point, be altered to $\rho + \delta\rho$, $p + \delta p$, $V + \delta V$, and let the (vector) displacement of material at any point from its equilibrium position be $\mathbf{h}$. Then, with the approximations usual in the theory of small oscillations, the equation of motion is

$$\rho\sigma^2 \mathbf{h} = \text{grad}\, \delta p + \delta\rho\, \text{grad}\, V + \rho\, \text{grad}\, \delta V, \tag{4}$$

where $2\pi/\sigma$ is the period; and the equation of continuity is

$$\delta\rho = -\text{div}\,(\rho \mathbf{h}). \tag{5}$$

If the ratio of specific heats of the material is $1 + 1/N$, the relation between the changes in pressure and density of a given element of fluid is

$$\left(\delta p + R\frac{\partial p}{\partial r}\right)\Big/ p = \left(1 + \frac{1}{N}\right)\left(\delta\rho + R\frac{\partial \rho}{\partial r}\right)\Big/\rho, \tag{6}$$

where R is the radial component of $\mathbf{h}$.

3. If the mass is strongly concentrated near the centre of the star, as is generally the case, the variations in the density do not produce great variations in the gravitational potential in the outer regions of the star. These regions have a considerable influence on the values of σ. We therefore ignore δV in deriving a first approximation to the periods.* The effect of this is to reduce the mathematical problem to the solution of a single differential equation of second order.

Eliminating $\mathbf{h}$ between (4) and (5), and remembering that grad $V = -(1/\rho)$ grad p, we now get

$$\sigma^2 \delta\rho = -\nabla^2 \delta p + \text{div}\left(\frac{\delta\rho}{\rho}\,\text{grad}\, p\right). \tag{7}$$

Also the radial component of (4) becomes

$$\rho\sigma^2 R = \frac{\partial \delta p}{\partial r} - \frac{\delta\rho}{\rho}\frac{\partial p}{\partial r}. \tag{8}$$

We shall consider oscillations in which each of $\delta\rho$, δp and R is the product of a surface harmonic of order s and a function of r. Then (7) becomes

$$\begin{aligned} \sigma^2 \delta\rho &= -\frac{\partial}{r^2 \partial r}\left(r^2 \frac{\partial \delta p}{\partial r} - r^2 \frac{\delta\rho}{\rho}\frac{\partial p}{\partial r}\right) + \frac{s(s+1)}{r^2}\delta p, \\ &= -\frac{\partial}{r^2 \partial r}(\rho\sigma^2 r^2 R) + \frac{s(s+1)}{r^2}\delta p, \end{aligned} \tag{9}$$

by (8).

We introduce new non-dimensional quantities X, θ, ϕ, α, defined by

$$R = r_0 X, \qquad \delta p = p_c \theta, \qquad \delta\rho = \rho_c \phi, \qquad \sigma^2 = 4\pi G \rho_c \alpha/(1+n). \tag{10}$$

Using these definitions and (1) and (3), we can transform (6), (8) and (9) to the forms

$$y\phi(1+N) = N\theta - (n-N)Xy^n y', \tag{11}$$

$$y^n \alpha X = \frac{\partial \theta}{\partial x} - (1+n)\phi y', \tag{12}$$

$$\alpha\phi = -\frac{\partial}{x^2 \partial x}(\alpha y^n x^2 X) + \frac{s(s+1)}{x^2}\theta. \tag{13}$$

* This was one of the approximations made by Emden.

Substituting for ϕ from (11) into (12) and (13), we get

$$\frac{\partial\theta}{\partial x}-\frac{(1+n)Ny'}{(1+N)y}\theta=X\left(y^n\alpha-\frac{(1+n)(n-N)}{1+N}y^{n-1}y'^2\right),$$

$$\frac{\partial}{x^2\partial x}(y^nx^2X)-\frac{n-N}{1+N}y^{n-1}y'X=\theta\left(\frac{s(s+1)}{\alpha x^2}-\frac{N}{(1+N)y}\right),$$

and if

$$z=Xy^{N(1+n)/(1+N)}, \qquad w=\theta y^{-N(1+n)/(1+N)}, \qquad Q=-n+\frac{2N(1+n)}{1+N}, \tag{14}$$

these can be reduced to

$$\frac{\partial w}{\partial x}=zy^{-Q}\left(\alpha-\frac{(1+n)(n-N)}{1+N}\frac{y'^2}{y}\right), \tag{15}$$

$$\frac{\partial}{\partial x}(x^2z)=wy^Q\left(\frac{s(s+1)}{\alpha}-\frac{Nx^2}{(1+N)y}\right), \tag{16}$$

We require solutions of these equations which make $R=0$ at the centre and $\delta p=0$ at the boundary of the stars. These conditions are equivalent to $z=0$ at $x=0$, and $w=0$ at the boundary. It is possible to satisfy these conditions only for a certain set of values $\alpha_1, \alpha_2, \ldots$ of α; the corresponding values of $\mathbf{h}$, etc., are denoted by $\mathbf{h}_1, \mathbf{h}_2, \ldots$, etc.

4. Equations (15) and (16) are equivalent to a single second-order equation in the variable w or z. If α is very large,

$$\left(1+\frac{1}{N}\right)\frac{\partial}{\partial x}\left(y^{1-Q}\frac{\partial}{x^2\partial x}(x^2z)\right)\doteqdot-\alpha zy^{-Q},$$

and if α is very small,

$$\frac{1+N}{(1+n)(n-N)}\frac{\partial}{\partial x}\left(\frac{x^2y^{1+Q}}{y'^2}\frac{\partial w}{\partial x}\right)\doteqdot-\frac{s(s+1)}{\alpha}wy^Q.$$

Each of these is of the Sturm-Liouville type; the first has an infinite number of solutions with α indefinitely large, the second an infinite number with α indefinitely small. Hence an infinite number of oscillations with very long periods are possible, as well as an infinite number with indefinitely small periods.

In the oscillations of very short period δp is large, and the motion, which is chiefly radial, is largely actuated by the pressure variations. In those of very long period δp and $\delta\rho$ are small, $\delta\rho$ being the larger of the two unless $n-N$ is small; the motion is chiefly horizontal and is due to the action of gravity in attempting to smooth out the density differences on any sphere concentric with the star. In the cases which have been studied in detail there is a single oscillation of intermediate period such that R and $-\delta\rho$ have the same sign at all points of any assigned radius; this may be called the fundamental (f-) oscillation. The oscillations of greater and less period may be called respectively gravitational (g-) and pressure (p-) oscillations. The g- and p-oscillations of high order (those whose periods are very different from that of the f-oscillation) possess a large number of nodes and loops (spheres concentric with the star on which $R=0$ and $\delta p=0$ respectively). The node of least radius is nearer the centre than the loop of least radius for the p-oscillations, and *vice versa* for the g-oscillations.

The free periods all decrease if either n increases (greater central condensation) or N decreases (smaller compressibility). If $n=N$ the periods of all the g-oscillations are infinite; the equilibrium of the star is then clearly neutral for indefinitely slow internal displacements of the material. When $n<N$, the star is unstable. If $N=0$ (incompressible fluid), the periods of all the p-oscillations vanish. The variations in density,

$\delta\rho$, are very small for the g-oscillations when $n-N$ is small, and for the p-oscillations when N is small.

5. Equations (15) and (16) are of a form suitable for numerical integration, and after a number of trial solutions a fairly accurate value of α can be obtained. We can use the corresponding values of w and z to correct the error involved in the neglect of δV, using the usual methods of perturbation theory. Suppose that the retention of δV in (4) results in an increase in the value $\mathbf{h}_m$ of $\mathbf{h}$ corresponding to $\alpha=\alpha_m$ of amount $\Sigma a_l\mathbf{h}_l$, where the quantities a_l are so small that their squares and products can be neglected.* Then, by (5) and (6), $\delta\rho_m$ and δp_m are increased by $\Sigma a_l\delta\rho_l$ and $\Sigma a_l\delta p_l$, and (4) can be written

$$\rho\sigma^2(\mathbf{h}_m+\Sigma a_l\mathbf{h}_l)=\text{grad}\,(\delta p_m+\Sigma a_l\delta p_l)+(\delta\rho_m+\Sigma a_l\delta\rho_l)\,\text{grad}\,V+\rho\,\text{grad}\,\delta V,$$

or, using the unperturbed equation and neglecting second-order quantities,

$$\rho(\sigma^2-\sigma_m{}^2)\mathbf{h}_m=\rho\Sigma a_l(\sigma_l{}^2-\sigma_m{}^2)\mathbf{h}_l+\rho\,\text{grad}\,\delta V.$$

Take the scalar product of this into $\mathbf{h}_m$ and integrate throughout the volume τ of the star. The orthogonality condition satisfied by the eigen-functions $\mathbf{h}_m$ is of the form †

$$\int\rho\mathbf{h}_l\,.\,\mathbf{h}_m d\tau=0 \qquad (l \neq m), \tag{17}$$

where the notation $\mathbf{a}\,.\,\mathbf{b}$ is used for the scalar product of the vectors $\mathbf{a}$, $\mathbf{b}$. Hence

$$(\sigma^2-\sigma_m{}^2)\int\rho\mathbf{h}_m\,.\,\mathbf{h}_m d\tau=\int\rho\mathbf{h}_m\,.\,\text{grad}\,\delta V d\tau$$
$$=\int\delta\rho_m\delta V d\tau \tag{18}$$

using an integration by parts and substituting from (5). This equation gives the correction to the period $2\pi/\sigma_m$.

The value of δV is found from the Poisson equation

$$4\pi G\delta\rho_m=\nabla^2\delta V=\frac{\partial}{r^2\partial r}\left(r^2\frac{\partial\delta V}{\partial r}\right)-\frac{s(s+1)}{r^2}\delta V.$$

The solution of this is

$$\delta V=-\frac{4\pi G}{2s+1}\left\{r^s\int_r^\infty\delta\rho_m r^{1-s}dr+r^{-1-s}\int_0^r\delta\rho_m r^{s+2}dr\right\}, \tag{19}$$

the integration being along the radius.

In terms of the non-dimensional variables y, z, w and x, equations (18) and (19) can be written

$$(\alpha-\alpha_m)\int_0^\infty\left(y^{-Q}x^2z^2+\frac{s(s+1)}{\alpha^2}y^Q w^2\right)dx=\int_0^\infty\phi x^2 v dx, \tag{20}$$

where

$$v=\frac{\rho_c}{p_c}\delta V=-\frac{1+n}{2s+1}\left\{x^s\int_x^\infty\phi x^{1-s}dx+x^{-1-s}\int_0^x\phi x^{s+2}dx\right\}. \tag{21}$$

It is clear from (18) and (19) that the errors in period due to the neglect of δV are small for higher oscillations of both p- and g-types, because of the large number of changes of sign of $\delta\rho_m$ on any radius. Especially is this true for the higher g-oscillations, since for these $\delta\rho_m$ is small in absolute value. Also, as (19) shows, the error is small when s is large.

* The time-factor in $\mathbf{h}_l$ is supposed omitted.

† For a system with a finite number of degrees of freedom whose kinetic energy is $\frac{1}{2}\Sigma\Sigma a_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta$, the orthogonality condition is of the form $\Sigma\Sigma a_{\alpha\beta}q_\alpha{}^{(l)}q_\beta{}^{(m)}=0$ $(l\neq m)$ (see Lamb, *Higher Mechanics*, 2nd ed., p. 228). Equation (17) is the generalisation of this to a continuous system.

6. The case $s=2$ is the important one for the tidal problem, and this alone has been considered numerically. The periods of the f-oscillation and the g-oscillation of largest period ($g1$) have been found for a polytrope of index 3 in the cases (i) $N=\frac{3}{2}$, corresponding to a monatomic gas, (ii) $n-N$ very small and positive, corresponding to a star nearly in neutral equilibrium. The first approximations obtained by solving (15) and (16), and the corrections given by (18), are listed below: M denotes the total mass, r_1 the radius of the star. For comparison, the results for the f-oscillation of a homogeneous incompressible star ($n=0=N$) are also given.

Oscillation	α_m	$\alpha-\alpha_m$	α	σ^2
$N=\frac{3}{2}, f$	0·2325	−0·0186	0·2139	$8{\cdot}69GM/r_1^3$
$N=\frac{3}{2}, g1$	0·1308	−0·0142	0·1166	$4{\cdot}72GM/r_1^3$
$n-N$ small, f	0·1583	−0·0099	0·1484	$6{\cdot}03GM/r_1^3$
$n-N$ small, $g1$	$0{\cdot}06(n-N)$	$O(n-N)^2$	$0{\cdot}06(n-N)$	$2{\cdot}44(n-N)GM/r_1^3$
$n=0=N, f$	$\frac{2}{3}$	$\frac{2}{5}$	$\frac{-4}{15}$*	$\frac{4}{5}GM/r_1^3$*

It is clear from this table that the first approximations to α found by solving (15) and (16) are not greatly in error for a polytrope of index 3. The values of $\mathbf{h}$, $\delta\rho$, etc., obtained from this approximation are not very accurate near the centre of the star, but are not greatly in error at distances from the centre greater than half the radius.

The value of σ^2 given by Emden was $2GM/r_1^3$. This is much smaller than the true values for the f-oscillation.

7. It remains to consider the bearing of these results on the tidal problem in double stars. Since the components of a double star rotate, their free periods will differ somewhat from those given above, the longest periods being most affected: but the general conclusion that indefinitely long periods are possible is still valid.† We cannot, therefore, rule out the possibility of resonance, however long may be the period of the tidal disturbance at any point of the star. But reasons may be urged why this should not necessarily produce important effects in actual stars.

(A) The oscillation with which there is resonance contributes relatively little to the equilibrium distortion of the star, and so can be responsible for a greatly enhanced total tide only if there is very close resonance.

(B) In the oscillation in question, the fluid undergoes considerable horizontal displacements even when the density-variations are small. Thus, when the oscillation is highly stimulated by resonance, second-order effects will appear, which may alter the free period so that there is no longer exact resonance. In this case the increase in amplitude of the displacements will be checked before the density-changes and the consequent distortion of the surface become large.

We consider (B) first. The period of the tidal disturbance at any point of the star is $\pi/(\omega-\omega_1)$, where ω is the orbital angular velocity and ω_1 the angular velocity of the star. This is not likely to be much less than $2\pi/\omega$, and ω^2 can hardly exceed $GM/4r_1^3$, its value for two equal stars in contact. Comparing this with the values of σ^2 given above, we see that the tidal disturbance can only be in resonance with a g-oscillation, which will be one of high order unless $n-N$ is small. In a high-order g-oscillation, $\delta\rho$ is small unless the horizontal displacements are considerable; the same is true of all the g-oscillations if $n-N$ is small. Argument (B) follows.

To discuss (A), suppose the star is subjected to a (small) periodic tidal force of potential $\Omega\cos\lambda t$. Using an argument similar to that of § 5, we can show that the

* These are exact values for α and σ^2.

† The general effect of the rotation is that, corresponding to each free period of a non-rotating star, there are several free periods of the rotating star, some greater, some smaller than the original period. The character of the corresponding motions is also altered: for large periods, the horizontal motions are nearly orthogonal to the horizontal pressure gredients.

resulting displacement at any point can be expressed in terms of the eigen-functions $\mathbf{h}_m$ by a series $\Sigma b_m \mathbf{h}_m \cos \lambda t$, where

$$b_m(\sigma_m{}^2 - \lambda^2)\int \rho \mathbf{h}_m \cdot \mathbf{h}_m d\tau = \int \delta\rho_m \Omega d\tau.$$

The resulting change in potential energy of the star is

$$\cos^2 \lambda t \int \Sigma b_m \delta\rho_m \Omega d\tau = \cos^2 \lambda t \Sigma \frac{c_m}{1 - \lambda^2/\sigma_m{}^2},$$

where

$$c_m = \left(\int \delta\rho_m \Omega d\tau\right)^2 \Big/ \sigma_m{}^2 \int \rho \mathbf{h}_m \cdot \mathbf{h}_m d\tau. \tag{22}$$

Thus in the equilibrium tide ($\lambda = 0$) the part of the change in potential energy due to coupling with the oscillation of period $2\pi/\sigma_m$ is $c_m \cos^2 \lambda t$; this is increased in the ratio $1/(1 - \lambda^2/\sigma_m{}^2)$ in a periodic tide.

For the case $n = 3$, $N = \frac{3}{2}$, the coefficient c_m for the $g1$ oscillation is only one-fifth as large as for the f-oscillation. For the higher g-oscillations c_m is much smaller, since $\delta\rho_m$ changes sign repeatedly on any radius. Smaller values of c_m are also obtained for the lower g-oscillations when $n - N$ is small, since then $\delta\rho_m$ is small for these. Argument (A) follows.

Thus, in spite of the possibility of resonance, the tidal distortion of double stars may be expected not to differ seriously from the equilibrium distortion, save, perhaps, in rare cases. There seems to be no serious likelihood that tidal disruption of a star will take place through resonance.

8. Recently the asymmetry of light-curves of eclipsing variables has been studied by P. H. Taylor.* He found that for stars exhibiting a marked asymmetry, approximately

$$P/\bar{\rho}^{\frac{1}{2}} = \text{const.}, \tag{23}$$

where P is the orbital period and $\bar{\rho}$ the mean density of the component of lower density. Now, the free periods of stars built on a similar model satisfy a homology relation of the form (23). He therefore suggested that the asymmetry was due to resonance between a free oscillation of this component and the periodic variation of the tidal force on it, due to the orbital variation of the distance from its companion. If this interpretation is correct, resonance is more important than the above arguments would suggest.

The interpretation seems, however, to be questionable. It rests on the assumption that the stars have a single free period with which there is resonance. In fact, however, in the range in question (which is that of the higher g-oscillations) the free periods lie fairly close together, and, for resonance, $P/\bar{\rho}^{\frac{1}{2}}$ can take a whole series of values not very widely separated. Thus, allowing for observational errors, no actual relationship between P and $\bar{\rho}$ would be observed.

An explanation of the existence of asymmetry can be given on the following lines. It is known that the variations in the light emitted by a star pulsating radially lag behind the variations in radius.† When the orbital and rotational angular velocities ω, ω_1 are different, the tidal distortion of a component is a forced oscillation; we may expect that the light variations at any point in such a forced oscillation will also lag behind the vertical displacement. If so, the observed light variations will show asymmetry whose sign will depend on the sign of $\omega - \omega_1$. A relation of the form (23) will be found if the lag of the light variations is enhanced for very close pairs.

9. Our results can also be related to the resonance theory of the origin of the Moon, studied by Darwin and Jeffreys, among others. Jeffreys‡ considered (in a two-

* *Ap. J.*, **94**, 46, 1941.
† See, especially, Eddington, *M.N.*, **101**, 182, 1941.
‡ *M.N.*, **78**, 116, 1917.

dimensional model) the free periods of oscillation of the primitive Earth, supposed to consist of a core of heavy liquid and an outer layer of lighter liquid. He found free periods considerably larger than those of a homogeneous mass of the same radius and mean density, so that resonance between the solar tide and the corresponding oscillations was possible. By a generalisation of the theory given above, the same result would be obtained if the Earth were supposed to consist of a compressible liquid whose density varied continuously with depth, provided that the mass was stable for internal displacements (corresponding to the condition $n > N$ above).

In a later paper* Jeffreys concluded that frictional forces due to turbulance at the surface of the core were so large that the tidal distortion could not be amplified by resonance up to the point where fission occurs. This argument need not apply, however, if the Earth's density varies continuously, since the stability of the mass for internal displacements prevents turbulence from appearing until large velocities of streaming are present.

The arguments (A) and (B) given above (§ 7) can be urged in this connection, but they have less weight for the primitive Earth than for gaseous stars. The greater uniformity of density in the Earth results in an increase in all the free periods. In consequence, resonance can occur with a g-oscillation not of high order; the oscillation with which Jeffreys found resonance corresponds roughly to our $g1$ oscillation. Such an oscillation contributes more to the total tide than a high-order g-oscillation, and the horizontal motions corresponding to a given surface distortion are smaller. Thus, while second-order effects must be taken into account, they do not wholly rule out the possibility of fission. It would be misleading to assert that these considerations clear up the difficulties of the resonance theory of the Moon's origin; but they suggest that further work on the problem is desirable.

Summary

The non-radial oscillations of a polytropic gas sphere are discussed. Periods are obtained numerically in certain special cases by an approximate method, and the error of the approximation is estimated. An infinite number of indefinitely long periods is shown in general to exist. The possibility of resonance between long-period oscillations and the tidal force in double stars is considered, and it is concluded that such resonance will not in general greatly affect the tidal distortion. The bearing of the results on the resonance theory of the Moon is discussed.

Appendix.—Tables giving the variation of $X(=R/r_0)$, $\theta(=\delta p/p_c)$, and $\phi(=\delta\rho/\rho_c)$ along any given radius. The constant of proportionality in the solution is adjusted to make X tend to equal $x(=r/r_0)$ as x becomes small. In the numerical work, the British Association tables of Emden functions were used.

* *M.N.*, 91, 169, 1930.

[TABLES.

$n=3, \quad N=\frac{3}{2}$

	f-oscillation			*g*1-oscillation		
x	X	$10^3\theta$	$10^2\phi$	X	$10^3\theta$	$10^3\phi$
0·2	0·1982	4·442	1·040	0·1951	2·441	6·137
0·4	0·3851	15·45	3·737	0·3633	7·889	21·44
0·6	0·5546	28·07	7·115	0·4843	12·34	38·68
0·8	0·6981	36·97	10·040	0·5504	12·95	51·12
1·0	0·8168	39·97	11·845	0·5661	9·647	56·37
1·2	0·9124	37·29	12·335	0·5412	4·359	53·46
1·4	1·001	31·23	11·882	0·4911	−0·850	46·59
1·6	1·068	24·09	10·642	0·4253	−4·708	38·05
1·8	1·143	17·28	9·196	0·3505	−6·880	29·69
2·0	1·223	11·64	7·713	0·2736	−7·506	22·31
2·2	1·311	7·341	6·343	0·2017	−7·120	16·49
2·4	1·420	4·763	5·220	0·1330	−6·246	12·09
2·6	1·551	3·031	4·285	0·0691	−5·168	8·790
2·8	1·713	2·072	3·561	0·0114	−4·132	6·452
3·0	1·914	1·611	2·995	−0·0442	−3·223	4·769
3·2	2·162	1·444	2·571	−0·0940	−2·469	3·572
3·4	2·473	1·440	2·255	−0·1430	−1·873	2·722
3·6	2·856	1·509	2·017	−0·1913	−1·407	2·106
3·8	3·327	1·581	1·840	−0·2401	−1·051	1·655
4·0	3·908	1·629	1·700	−0·2888	−0·7807	1·327
4·2	4·618	1·640	1·590	−0·3383	−0·5772	1·082
4·4	5·487	1·597	1·492	−0·3938	−0·4239	0·8878
4·6	6·549	1·508	1·402	−0·4525	−0·3091	0·7362
4·8	7·845	1·375	1·309	−0·5168	−0·2224	0·6105
5·0	9·426	1·208	1·211	−0·5870	−0·1575	0·5059
5·2	11·34	1·021	1·102	−0·6664	−0·1089	0·4153
5·4	13·68	0·8198	0·979	−0·7523	−0·0730	0·3355
5·6	16·59	0·6210	0·844	−0·8487	−0·0468	0·2641
5·8	19·99	0·4363	0·695	−0·9579	−0·0281	0·1994
6·0	23·96	0·2757	0·535	−1·078	−0·0154	0·1418
6·2	28·82	0·1488	0·362	−1·222	−0·0073	0·0912
6·4	34·42	0·0621	0·221	−1·397	−0·0027	0·0494
6·6	41·52	0·0152	0·091	−1·552	−0·0006	0·0189
6·8	51·41	0·0006	0·010	−1·670	−0·0000	0·0022

$n=3, \quad n-N$ small

	f-oscillation			*g*1-oscillation		
x	X	$10^2\theta$	$10^2\phi$	$10X$	$10^3\theta/(n-N)$	$10^3\phi/(n-N)$
0·2	0·2022	0·3119	0·2355	1·964	6·723	8·268
0·4	0·4178	1·196	0·9204	3·686	21·92	27·74
0·6	0·6601	2·504	1·991	5·009	35·03	47·92
0·8	0·9461	4·060	3·373	5·861	38·06	58·13
1·0	1·289	5·646	4·952	6·275	30·16	55·38
1·2	1·713	7·114	6·646	6·311	15·82	42·29
1·4	2·244	8·341	8·370	6·080	0·96	24·50
1·6	2·913	9·292	10·08	5·671	−10·61	7·41
1·8	3·745	9·938	11·71	5·178	−17·48	− 6·31
2·0	4·808	10·312	13·26	4·650	−20·11	−15·54
2·2	6·143	10·432	14·71	4·142	−19·66	−20·48
2·4	7·805	10·348	16·03	3·653	−17·44	−22·52
2·6	9·890	10·085	17·22	3·203	−14·97	−22·22
2·8	12·49	9·703	18·30	2·802	−12·13	−20·68
3·0	15·71	9·215	19·24	2·444	− 9·473	−18·32
3·2	19·69	8·652	20·03	2·144	− 7·246	−15·82
3·4	24·61	8·035	20·67	1·871	− 5·436	−13·37
3·6	30·63	7·375	21·13	1·633	− 4·011	−11·10
3·8	38·01	6·691	21·41	1·420	− 2·917	− 9·078
4·0	47·01	5·995	21·48	1·241	− 2·091	− 7·333
4·2	57·95	5·293	21·31	1·078	− 1·481	− 5·860
4·4	71·20	4·598	20·89	0·9393	− 1·033	− 4·628
4·6	87·20	3·922	20·21	0·8123	− 0·7087	− 3·613
4·8	106·5	3·271	19·24	0·6964	− 0·4761	− 2·776
5·0	129·7	2·657	17·98	0·6027	− 0·3125	− 2·100
5·2	157·6	2·087	16·41	0·5113	− 0·1992	− 1·557
5·4	191·1	1·575	14·56	0·4217	− 0·1222	− 1·126
5·6	230·9	1·127	12·47	0·3467	− 0·0713	− 0·787
5·8	278·3	0·7498	10·16	0·2806	− 0·0389	− 0·525
6·0	334·1	0·4511	7·736	0·2192	− 0·0192	− 0·329
6·2	401·0	0·2331	5·316	0·1629	− 0·0082	− 0·186
6·4	479·5	0·1133	3·074	0·1116	− 0·0027	− 0·089
6·6	567	0·0219	1·248	0·0644	− 0·0005	− 0·030
6·8	675	0·0008	0·152	0·0209	− 0·0000	− 0·003

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