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ASYMPTOTIC THEORY OF INTERMEDIATE- AND HIGH-DEGREE SOLAR ACOUSTIC OSCILLATIONS

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ABSTRACT

A second-order asymptotic approximation is developed for adiabatic nonradial p-modes of a spherically symmetric star. The exact solutions of adiabatic oscillations are assumed in the outermost layers, where the asymptotic description becomes invalid, which results in a eigenfrequency equation with model-dependent surface phase shift. For lower degree modes, the phase shift is a function of frequency alone; for high-degree modes, its dependence on the degree is explicitly taken into account.

Subject heading: Sun: oscillations

1. INTRODUCTION

The WKB asymptotic theory of stellar acoustic oscillations is attracting increasing interest in recent years, owing to the rapid progress in solar and stellar seismology. In studying the solar interior with a large number of accurately measured oscillation frequencies, the asymptotic theory serves as a basis for the effective techniques of nonlinear inversion of observational data. (For reviews of helioseismology see Deubner & Gough 1984; Christensen-Dalsgaard, Gough, & Toomre 1985; Libbrecht 1988; Vorontsov & Zharkov 1989). With the simple first-order asymptotic approximation usually used, inversion of the accurate observational frequencies now available is limited by the accuracy of the asymptotic description itself. This situation motivates a development of the second-order asymptotic theory for acoustic modes in a wide degree range.

For low-degree modes, the second-order asymptotic theory was developed by Tassoul (1980, 1990) and Smeyers & Tassoul (1987). The Cowling approximation was used, which neglects gravity perturbations. For a wider range of low- and intermediate-degree modes, the second-order asymptotic theory was developed by Vorontsov (1990, 1991). Gravity perturbations, especially significant for low-degree modes, were taken into account by studying the asymptotic solutions for a complete fourth-order system of governing differential equations.

In the present paper we have two main goals. The first is to find mathematically correct asymptotic expressions for eigenfrequencies of solar acoustical oscillations with accuracy $1/\omega^4$, where ω is the value of the frequency. The standard WKB approximation provides only $1/\omega^2$ accuracy. These expressions can be used for interpretation of high- and intermediate-degree frequencies that extend useful information. Such interpretation is impossible in the framework of the standard asymptotic approach. The second goal is to find a proper approximation for these oscillations in the outermost solar layers. In these layers the reflection of the trapped acoustic waves occurs, and the WKB approximation fails owing to the rapid variations of seismic parameters on a scale that is short compared with radial wavelengths. Such a solution was found by using a parameter which is small near the surface and can be interpreted as the ratio of the horizontal component of the wavenumber to the total wavenumber. The resulting eigenfrequency equation was derived by matching asymptotic solutions in the interior with nonasymptotic solutions near the surface, thus allowing the quantitative description of the outermost solar layers of complicated structure. The solutions in the outer layers are described in the resulting eigenfrequency equation by model-dependent surface phase shift. This phase shift can be considered as a function of frequency alone for low- and intermediate-degree modes (when the curvature of the ray paths in the nonasymptotic region can be neglected). For higher degree modes ($\ell \geq 100$) the dependence of the phase shift on the degree ℓ becomes significant. The present paper extends the theoretical description to high-degree modes. Because the gravity perturbations are negligible for these modes, our present analysis uses the Cowling approximation.

Here is the plan of the paper. In § 2 we find the asymptotic (next after WKB) solution for solar acoustical oscillations in the interior of the Sun. In § 3 we describe how to match an asymptotic solution for the interior region with the exact solution in the exterior region to obtain the equation for the eigenfrequencies. Here we introduce a new definition of the phase shift of an oscillation. In § 4 we obtain an approximation of this phase shift by using the new small parameter in outer layers and get the final equation. In § 5 we show some numerical results supporting the theory we developed and discuss future possibilities for a solution of the helioseismological inverse problem.

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2. ASYMPTOTIC SOLUTIONS

Linear adiabatic oscillations of a spherically symmetric star can be described in the Cowling approximation by the second-order system of ordinary differential equations (e.g., Unno et al. 1989)

$$\frac{d\xi}{dr} = h\left[\frac{\ell(\ell+1)}{\omega^2} - \frac{r^2}{c^2}\right]\eta\,, \qquad \frac{d\eta}{dr} = \frac{1}{r^2 h}(\omega^2 - N^2)\xi\,. \tag{1}$$

The boundary conditions for equations (1) are

$$\eta(r) \text{ and } \xi(r) \text{ are bounded at } r = 0\,, \tag{2}$$

$$A_1\,\xi(R) + A_2\,\eta(R) = 0\,. \tag{3}$$

Here R is the radius of the surface of the Sun, and A_1 and A_2 are constants which will be chosen later. The values of A_1 and A_2 depend on details of the physical model; the form of the asymptotic solution in the solar interior does not depend on A_1 and A_2. In equation (1), ω is the angular frequency, and $\xi(r)$ and $\eta(r)$ determine the distributions of radial displacements and Eulerian pressure perturbations p'. In a spherical coordinate system (r, θ, ϕ) we have

$$\delta r = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{h_1(r)}{r^2}\,\xi_{\ell m}(r) Y_{\ell m}(\theta, \phi)\,,$$

$$p' = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \rho_0(r) h_2(r) \eta_{\ell m}(r) Y_{\ell m}(\theta, \phi)\,;$$

we will omit the indices ℓm in the equations and expressions for $\xi_{\ell m}$ and $\eta_{\ell m}$; $\rho_0(r)$ is the equilibrium density distribution

$$h_1(r) = \exp\int_0^r \frac{g(r)}{c^2(r)}\,dr, \qquad h_2(r) = \exp\int_0^r \frac{N^2(r)}{g(r)}\,dr, \qquad h(r) = \frac{h_2(r)}{h_1(r)}\,, \tag{4}$$

where $g(r)$, $c(r)$, and $N(r)$ denote gravitational acceleration, adiabatic sound speed, and Brunt-Väisälä frequency, respectively, in the equilibrium model.

In the solar interior, the acoustic oscillations satisfy $\omega^2 > N^2$, and we can reduce equations (1) to the single second-order differential equation for $\eta(r)$:

$$\frac{d^2\eta}{dr^2} + C(r, \omega)\frac{d\eta}{dr} - \omega^2 D(r, \omega)\eta = 0\,, \tag{5}$$

where

$$C(r, \omega) = \ln'(hr^2) - \ln'\left(1 - \frac{N^2}{\omega^2}\right) \tag{6}$$

and

$$D(r, \omega) = \left(1 - \frac{N^2}{\omega^2}\right)\left[\frac{\ell(\ell+1)}{\omega^2 r^2} - \frac{1}{c^2}\right]. \tag{7}$$

Here the prime denotes the radial derivative.

Studying the oscillations of different degree ℓ, we define

$$w^2 = \frac{(\ell + 1/2)^2}{\omega^2}\,, \qquad \tilde{w}^2 = \frac{\ell(\ell+1)}{\omega^2}\,. \tag{8}$$

In the asymptotic expansions where $1/\omega$ is a small parameter, we will use w to denote an independent constant parameter. In the first-order asymptotic approximation, the value of w determines the position of the turning point. The use of $(\ell + \frac{1}{2})^2$ instead of $\ell(\ell+1)$ is needed for the asymptotic expansions to be regular as $r \to 0$ (Langer 1934).

We look for uniform asymptotic approximations to the solutions of equation (5) in terms of Airy functions

$$\eta = \left(Y_0 + \frac{1}{\omega}Y_1 + \frac{1}{\omega^2}Y_2 + \cdots\right)\cdot G\,, \tag{9}$$

with two-component vector functions Y_i and

$$G = \begin{pmatrix} \omega^{1/6} & \mathrm{Ai}\,(-\omega^{2/3}\varphi) \\ \omega^{-1/6} & \dot{\mathrm{A}}\mathrm{i}\,(-\omega^{2/3}\varphi) \end{pmatrix}, \tag{10}$$

where $\varphi(r)$ is an unknown function; $\dot{\mathrm{A}}\mathrm{i}$ denotes the derivative of the Airy function with respect to its argument. Here and below we omit the argument r of dependent variables. We expand the coefficients of equation (5) in even powers of $1/\omega$:

$$C = C_0 + \frac{1}{\omega^2} C_2 + \cdots , \qquad D = D_0 + \frac{1}{\omega^2} D_2 + \cdots \tag{11}$$

Here

$$C_0 = \ln'(hr^2) , \tag{12}$$

$$D_0 = \frac{\omega^2}{r^2} - \frac{1}{c^2} , \tag{13}$$

$$D_2 = -N^2\left(\frac{\omega^2}{r^2} - \frac{1}{c^2}\right) - \frac{1}{4r^2} . \tag{14}$$

We substitute the expansions (9) and (11) in equation (5) and collect terms of the same order in $1/\omega$. Using

$$\frac{d}{dr} G = \omega Q G, \qquad Q = \begin{pmatrix} 0 & -\varphi' \\ \varphi\varphi' & 0 \end{pmatrix}, \qquad Q^2 = \begin{pmatrix} -\varphi(\varphi')^2 & 0 \\ 0 & -\varphi(\varphi')^2 \end{pmatrix} , \tag{15}$$

we obtain the system of vector equations

$$[\varphi(\varphi')^2 + D_0] Y_0 = 0 , \tag{16a}$$

$$[\varphi(\varphi')^2 + D_0] Y_1 = C_0 Y_0 Q + Y_0 Q' + 2Y'_0 Q , \tag{16b}$$

$$[\varphi(\varphi')^2 + D_0] Y_2 = C_0(Y'_0 + Y_1 Q) + Y''_0 + 2Y'_1 Q + Y_1 Q' - D_2 Y_0 , \tag{16c}$$

$$[\varphi(\varphi')^2 + D_0] Y_3 = C_0(Y'_1 + Y_2 Q) + C_2 Y_0 Q + Y''_1 + 2Y'_2 Q + Y_2 Q' - D_2 Y_1 . \tag{16d}$$

The solutions will admit arbitrary normalization. The vectors Y_1, Y_2, ... are determined by equations (16) up to a multiplicative constant that we can choose for convenience.

We solve equations (16) successively beginning with equation (16a). For this equation to have a nontrivial solution, we must set

$$\varphi(\varphi')^2 = \frac{1}{c^2} - \frac{w^2}{r^2} , \tag{17}$$

so that the left-hand-sides of all of equations (13) are zero. Define

$$s^2 = \frac{1}{c^2} - \frac{w^2}{r^2} . \tag{18}$$

The root of the equation $s^2(r) = 0$ is the turning point: we denote it by r_1. We will assume further that there is only one turning point in the solar interior, which is equivalent to the assumption that the function of c/r is a monotonic function of the radius. This assumption is satisfied in standard solar models (the appearance of a second turning point corresponds to the appearance of a waveguide and requires special study). The unique solution of equation (17), that is regular at $r = r_1$, is

$$\varphi = \mathrm{sgn}\,(s^2) \left| \frac{3}{2} \int_{r_1}^{r} | s^2 |^{1/2} dr \right|^{2/3} . \tag{19}$$

Since $\varphi \to -\infty$ as $r \to 0$ and the functions Ai (z) and $\dot{\mathrm{A}}\mathrm{i}$ (z) tend to zero if $z \to \infty$, boundary condition (2) is satisfied by solution (9). The next equations of the system (16) are solved separately for their vector components. Using the condition of regularity at $r = r_1$ and fixing the normalization of the resulting solution, from equation (16b) we have

$$y_{01} = (hr^2\varphi')^{-1/2}, \qquad y_{02} = 0 . \tag{20}$$

Here and below we denote $Y_i = (y_{i1}, y_{i2})$.

Equation (16c) gives a homogeneous first-order differential equation for y_{11} with general solution y_{11} = constant times y_{01}. We set the constant to zero, so that

$$y_{11} = 0 . \tag{21}$$

The equation for y_{12} is

$$2\varphi\varphi' y'_{12} + (\varphi\varphi')' y_{12} + \varphi\varphi' \ln'(hr^2) y_{12} + \ln'(hr^2) y'_{01} + y''_{01} - D_2 y_{01} = 0 . \tag{22}$$

The unique solution of this equation that is regular at $r = r_1$ is

$$y_{12} = \frac{1}{2} | hr^2 \varphi\varphi' |^{-1/2} \int_{r_1}^{r} \mathrm{sgn}\,(\varphi\varphi')(hr^2)^{1/2} | \varphi\varphi' |^{-1/2} [-y''_{01} - \ln'(hr^2) y'_{01} + D_2 y_{01}] dr . \tag{23}$$

Equation (16.3) and the condition of regularity at $r = r_1$ give

$$y_{22} = 0 \,. \tag{24}$$

The solution for y_{21} can also be obtained from equation (16d), but we will not use it. We notice only that $y_{21}(r)$ has no singularities on the interval $[r_1, R]$.

The final asymptotic solution for $\eta(r)$ can thus be written

$$\eta = y_{01}\,\omega^{1/6}\mathrm{Ai}\,(-\omega^{2/3}\varphi) + \frac{1}{\omega}\,y_{12}\,\omega^{-1/6}\mathrm{\dot{A}i}\,(-\omega^{2/3}\varphi) + \omega^{-2+1/6}y_{21}\,\mathrm{Ai}\,(-\omega^{2/3}\varphi) + O(\omega^{-3-1/6}) \,. \tag{25}$$

The eigenfrequency equation will be constructed in the next section by matching the asymptotic solutions in the solar interior with the exact solutions in surface layers. Near a matching point r_m, sufficiently far from the turning point, we can replace the Airy function and its derivative with their asymptotic expansions in terms of trigonometric functions (Abramowitz & Stegun 1965). Using solution (19) for the phase function, we obtain

$$\eta_1 = y_{01}\,\varphi^{-1/4}\Bigg[\cos\left(\omega\int_{r_1}^{r} s\,dr - \frac{\pi}{4}\right) + \frac{1}{\omega}\left(\frac{5}{48}\,\varphi^{-3/2} + \frac{y_{12}}{y_{01}}\,\varphi^{1/2}\right)\sin\left(\omega\int_{r_1}^{r} s\,dr - \frac{\pi}{4}\right)$$
$$+ \frac{1}{\omega^2}\cos\left(\omega\int_{r_1}^{r} s\,dr - \frac{\pi}{4}\right)\left(\frac{y_{21}}{y_{01}} + \frac{7}{48}\,\frac{y_{12}}{\varphi y_{01}} - \frac{5\cdot 7\cdot 11}{9\cdot 2^9\varphi^3}\right)\Bigg] + O\!\left(\frac{1}{\omega^3}\right) . \tag{26}$$

(The factor $\pi^{-1/2}$ in the standard normalization of Airy functions was omitted.) Here and below the subscript 1 denotes asymptotic solutions in the interval $(r_1 + r_0, r_m)$ near the matching point r_m, where $s \geq 0$ is $(s^2)^{1/2}$ for $s^2 \geq 0$, $r_0 > 0$.

The exact solutions in the outer layers will be determined using the corresponding second-order differential equation for $\xi(r)$, because equation (5) for $\eta(r)$ has a singular point at $N^2(r) = \omega^2$. Therefore, we transform the asymptotic solution (26) for $\eta(r)$ to $\xi(r)$ using the second equation of the system (1) and obtain for $\omega^2 \gg N^2$

$$\xi = r^2 h\,\frac{1}{\omega^2 - N^2}\,\frac{d\eta}{dr} = \frac{r^2 h}{\omega^2}\,\frac{d\eta}{dr}\left[1 + O\!\left(\frac{1}{\omega^2}\right)\right] .$$

In the interval $(r_1 + r_0, r_m)$ the asymptotic solution (26) has a uniformly bounded remainder term (Fedoruk 1983); that is why we can differentiate $\eta(r)$. After differentiation the order of the remainder term will increase to $O(1/\omega^2)$. Representing the result with a single trigonometric function, we obtain

$$\xi_1 = -\frac{rh^{1/2}s^{1/2}}{\omega}\left[1 + \frac{1}{\omega^2}\left\{N^2 + K + \frac{B^2}{2} - \frac{B'}{s} + \frac{1}{2}\left[\frac{1}{s}\,\ln'(rh^{1/2}s^{1/2})\right]^2\right\}\right]$$
$$\times \sin\left\{\omega\int_{r_1}^{r} s\,dr - \frac{\pi}{4} - \frac{1}{\omega}\left[B - \frac{1}{s}\,\ln'(rh^{1/2}s^{1/2})\right]\right\} + O\!\left(\frac{1}{\omega^4}\right) , \tag{27}$$

$$K(r) = \frac{y_{21}}{y_{01}} + \frac{7}{48}\,\frac{y_{12}}{\varphi y_{01}} - \frac{5\cdot 7\cdot 11}{9\cdot 2^9\varphi^3}\,, \qquad B(r) = \frac{5}{48}\,\varphi^{-3/2} + \frac{y_{12}}{y_{01}}\,\varphi^{1/2}\,.$$

We will now substitute the explicit expressions for y_{01}, y_{12}, and the phase function φ. For the term with y_{12} we have

$$-\frac{y_{12}}{y_{01}}\,\varphi^{1/2} = \frac{1}{2}\int_{r_1}^{r} s^{-1}\left[-\frac{1}{2}\ln'' h - \frac{1}{4}(\ln' h)^2 - \frac{1}{r}\ln' h - \frac{1}{2}\ln''\varphi' + \frac{1}{4}(\ln'\varphi')^2 - s^2N^2 + \frac{1}{4r^2}\right]dr \,. \tag{28}$$

Terms without the phase function φ on the right-hand-side of expression (28) have an integrable singularity at the turning point, because in the vicinity of $r = r_1$ the function $s(r)$ behaves as $(r - r_1)^{1/2}$. The terms with phase function φ have an integrable singularity at the point $r = r_1$ because, in the vicinity of r_1, $\varphi(r) \approx [a_0 + a_2(r - r_1)](r - r_1)$, $a_0 \neq 0$, and therefore

$$\frac{1}{s}\left[\frac{1}{4}(\ln'\varphi')^2 - \frac{1}{2}\ln''\varphi'\right] = b(r - r_1)^{-1/2} + O[(r - r_1)^{1/2}]$$

a_0, a_2, b are constants.

Substitution of expressions (19), (20), and (28), in the asymptotic solution (27) and integration by parts leads to the expression (in a vicinity of the matching point r_m)

$$\xi_1 = -\frac{rh^{1/2}s^{1/2}}{\omega}\sin\left[\omega\int_{r_1}^{r} s\,dr - \frac{\pi}{4} + \frac{1}{\omega}\,\Phi(\omega, r) + \frac{1}{\omega}\,\Psi_1(\omega, r)\right]$$
$$\times \left[1 + \frac{1}{\omega^2}\left\{N^2 + K + \frac{B^2}{2} - \frac{B'}{s} + \frac{1}{2s^2}\,[\ln'(rh^{1/2}s^{1/2})]^2\right\} + O\!\left(\frac{1}{\omega^3}\right)\right] , \tag{29}$$

$$\Phi(w, r) = \frac{1}{2}\int_{r_1}^{r} s^{-1}\left\{-\frac{1}{2}\ln'' h - \frac{1}{4}(\ln' h)^2 - \frac{1}{r}\ln' h - \frac{1}{12r}\left[r\frac{(r^2/c^2)''}{(r^2/c^2)'}\right]' - s^2N^2\right\}dr\,, \tag{30a}$$

$$\Psi_1(w, r) = -\frac{5}{48r^2s^3}\left(\frac{r^2}{c^2}\right)' + \frac{1}{24s}\frac{(r^2/c^2)''}{(r^2/c^2)'} + \frac{1}{24rs} + \frac{1}{s}\ln'(rh^{1/2}s^{1/2})\,, \tag{30b}$$

$$\frac{1}{s}\ln'(rh^{1/2}s^{1/2}) - B(r) = \Phi(w, r) + \Psi_1(w, r)\,. \tag{30c}$$

Let us define now a new independent variable τ,

$$\tau = \int_r^R s\,dr = \int_r^R \left(\frac{1}{c^2} - \frac{w^2}{r^2}\right)^{1/2} dr\,,$$

and a function $V(r)$,

$$V(r) = N^2 + \left[\frac{d\ln(rh^{1/2}s^{1/2})}{d\tau}\right]^2 - \frac{d^2\ln(rh^{1/2}s^{1/2})}{d\tau^2}\,.$$

We will now reduce the asymptotic solutions (26), (29), and (30) to the corresponding expression for $\eta(\tau)$, $\xi(\tau)$ and bring it into a form convenient for matching with the exact solution for outer region. The phase of the asymptotic solution (30) is determined by the integrals from the turning point r_1 to the matching point r_m. We add and subtract the same integrals, but taken from matching point r_m to the surface R. We transform the integrals that are subtracted to the new independent variable τ and obtain

$$\begin{aligned}\eta_1 = \frac{1}{rh^{1/2}s^{1/2}}\cos&\left\{\omega F(w) - \frac{\pi}{4} - \omega\tau + \frac{1}{\omega}[\Phi(w, R) + \Psi(w)]\right.\\ &\left.+ \frac{1}{2\omega}\int_0^\tau\left(V - \frac{1}{4r^2s^2}\right)d\tau + \frac{1}{\omega}\frac{d\ln(rh^{1/2}s^{1/2})}{d\tau}\right\}\left[1 + \frac{1}{\omega^2}\left(K + \frac{B^2}{2}\right) + O\left(\frac{1}{\omega^3}\right)\right],\end{aligned} \tag{31a}$$

$$\begin{aligned}\xi_1 = -\frac{rh^{1/2}s^{1/2}}{\omega}\sin&\left\{\omega F(w) - \frac{\pi}{4} - \omega\tau + \frac{1}{\omega}[\Phi(w, R) + \Psi(w)]\right.\\ &\left.+ \frac{1}{2\omega}\int_0^r\left(V - \frac{1}{4r^2s^2}\right)dr\right\}\left\{1 + \frac{1}{\omega^2}\left(K + \frac{B^2}{2}\right) - \frac{1}{2\omega^2}\left(V - \frac{1}{4r^2s^2}\right)\right.\\ &\left.- \frac{1}{\omega^2}\frac{d^2\ln(rh^{1/2}s^{1/2})}{d\tau^2} + \frac{1}{2\omega^2}\left[\frac{d\ln(rh^{1/2}s^{1/2})}{d\tau}\right]^2 + O\left(\frac{1}{\omega^3}\right)\right\}.\end{aligned} \tag{31b}$$

Here

$$F(w) = \int_{r_1}^R s\,dr\,, \tag{32a}$$

$$\Psi(w) = \left[\frac{1}{2s}\ln' h + \frac{1}{24s}\ln'(r^2s^2)' + \frac{7}{48s}\ln'(r^2s^2) + \frac{13}{24rs}\right]\Bigg|_{r=R}\,. \tag{32b}$$

3. EQUATION FOR THE EIGENFREQUENCIES

In the outer solar layers, we reduce the oscillation equations (1) to the equivalent second-order differential equation for $\xi(r)$:

$$\frac{d}{dr}\left\{\frac{1}{h[\ell(\ell+1)/\omega^2 - r^2/c^2]}\frac{d\xi}{dr}\right\} - \frac{1}{r^2h}(\omega^2 - N^2)\xi = 0\,. \tag{33}$$

In the outer solar layers we have

$$\frac{r^2}{c^2} - \frac{\ell(\ell+1)}{\omega^2} > 0\,,$$

and equation (33) has no singularity. However, we cannot use an asymptotic approximation, that is similar to the one used in the solar interior, in this interval. Gradients of the functions determining the equation are large. That is why we will use exact formulae for solutions.

Reducing equation (33) to the new variables $\tilde{\tau}$ and ζ,

$$\tilde{\tau} = \int_r^R\left(\frac{r^2}{c^2} - \tilde{w}^2\right)^{1/2}\frac{dr}{r} = \int_r^R \tilde{s}\,dr, \qquad \tilde{w}^2 = \frac{\ell(\ell+1)}{\omega^2}\,, \qquad \tilde{s} = \left(\frac{1}{c^2} - \frac{\tilde{w}^2}{r^2}\right)^{1/2}\,,$$

$$\xi = (rh)^{1/2}\left[\frac{r^2}{c^2} - \tilde{w}^2\right]^{1/4}\zeta = rh^{1/2}\tilde{s}^{1/2}\zeta\,. \tag{34}$$

We obtain the Schrödinger-type equation

$$\frac{d^2}{d\tilde{\tau}^2}\zeta + [\omega^2 - \tilde{V}(\tilde{\tau})]\zeta = 0 \tag{35}$$

with "acoustic potential" $\tilde{V}(\tilde{\tau})$,

$$\tilde{V} = N^2 + \frac{1}{4}\left[\frac{d\ln}{d\tilde{\tau}}(r^2h\tilde{s})\right]^2 - \frac{1}{2}\frac{d^2\ln}{d\tilde{\tau}^2}(r^2h\tilde{s})\,. \tag{36}$$

For constant values of $\tilde{w}$ and ω, we are looking for the exact (nonasymptotic) solutions of equation (35) in the form (Babikov 1976)

$$\zeta_2 = A(\tilde{\tau})\cos\left[\omega\tilde{\tau} - \frac{\pi}{4} - \pi\alpha(\tilde{\tau})\right], \tag{37}$$

with the additional requirement that the amplitude function $A(\tilde{\tau})$ satisfies

$$\frac{d\zeta_2}{d\tilde{\tau}} = -\omega A(\tilde{\tau})\sin\left[\omega\tilde{\tau} - \frac{\pi}{4} - \pi\alpha(\tilde{\tau})\right]. \tag{38}$$

The phase function $\alpha(\tilde{\tau})$ is then determined by the first-order nonlinear differential equation

$$\frac{d(\pi\alpha)}{d\tilde{\tau}} = \frac{\tilde{V}}{\omega}\cos^2\left[\omega\tilde{\tau} - \frac{\pi}{4} - \pi\alpha(\tilde{\tau})\right], \tag{39}$$

with corresponding surface boundary condition at $\tilde{\tau} = 0$. The subscript 2 is introduced to denote the solutions in the surface layers.

These nonasymptotic solutions in the outer layers represent the direct generalization of the solutions applicable for low- and intermediate-degree modes (Brodsky & Vorontsov 1988; Vorontsov & Zharkov 1989; Vorontsov 1991). Both the "acoustic potential" $\tilde{V}$ and the "acoustic depth" $\tilde{\tau}$ are now explicitly dependent on the value of $\tilde{w} = [(\ell + 1)\ell]^{1/2}/\omega$, thus accounting for the curvature of the ray path in the reflecting layers. The surface boundary condition for the phase function $\alpha(\tilde{\tau})$ can be taken to be that corresponding to the standard boundary condition in the adiabatic pulsation problem. Our boundary condition is established in the vicinity of the temperature minimum; the acoustic potential is approximately constant there, and the reflection condition given by equation (35) is

$$-\frac{\pi}{4} - \pi\alpha(0) = \arctan\left\{-\left[\frac{\tilde{V}(0)}{\omega^2} - 1\right]^{1/2}\right\} \tag{40}$$

(this boundary condition is a condition of the type in eq. [3]).

To match a solution of the system (1) from the interval $(0, r_m)$ with a solution of equations (1) from the interval (r_m, R), we have to equate the logarithmic derivatives $d\ln\zeta_1/d\tilde{\tau}$ and $d\ln\zeta_2/d\tilde{\tau}$ at the point $\tilde{\tau}_m$ corresponding to r_m and τ_m

$$\tilde{\tau}_m = \int_{r_m}^{R}\tilde{s}\,dr = \tau_m + \frac{1}{\omega^2}\int_0^{\tau_m}\frac{1}{8r^2s^2}\,d\tau + O\left(\frac{1}{\omega^4}\right).$$

Let us note that

$$\frac{1}{\zeta_1}\frac{d\zeta_1}{d\tilde{\tau}} = -\frac{1}{\tilde{s}}\left[\frac{1}{\xi_1}\frac{d\xi_1}{dr} - \frac{d}{dr}\ln(rh^{1/2}\tilde{s}^{1/2})\right] = \left[1 + \frac{1}{8r^2s^2\omega^2} + O\left(\frac{1}{\omega^4}\right)\right]\left[r^2hs\frac{\eta_1}{\xi_1} + \frac{1}{s}\frac{d}{dr}\ln(rh^{1/2}s^{1/2}) + O\left(\frac{1}{\omega^2}\right)\right]. \tag{41}$$

Using equations (31) in equation (41), we obtain

$$\left.\frac{d\ln\zeta_1}{d\tilde{\tau}}\right|_{r=r_m} = -\omega\cot\left\{\omega F(w) - \frac{\pi}{4} - \omega\tau_m + \frac{1}{\omega}[\Phi(w, R) + \Psi(w)] + \frac{1}{2\omega}\int_0^{\tau_m}\left(V - \frac{1}{4r^2s^2}\right)d\tau\right\}\left[1 - \frac{1}{2\omega^2}V + O\left(\frac{1}{\omega^3}\right)\right]. \tag{42}$$

Finally, from equations (37), (38), and (42), we have the equation for the eigenvalues of the system (1) with boundary conditions (2) and (3) or (40):

$$\cot\left\{\omega F(w) - \frac{\pi}{4} - \omega\tau_m + \frac{1}{\omega}[\Phi(w, R) + \Psi(w)] + \frac{1}{2\omega}\int_0^{\tau_m}\left(V - \frac{1}{4r^2s^2}\right)dr\right\}\left(1 - \frac{1}{2\omega^2}V\right) + O\left(\frac{1}{\omega^3}\right)$$
$$= \tan\left[\omega\tau_m - \frac{\pi}{4} + \frac{1}{\omega}\int_0^{\tau_m}\frac{1}{8r^2s^2}\,d\tau - \pi\alpha(\tilde{\tau}_m) + O\left(\frac{1}{\omega^3}\right)\right]. \tag{43}$$

After solving the trigonometrical equation (43), we obtain

$$F(w) + \frac{1}{\omega^2}\Phi(w, R) = \frac{\pi[n + \alpha(\tilde{\tau}_m)]}{\omega} - \frac{1}{\omega^2}\left[\frac{1}{2}\int_0^{\tau_m}V\,d\tau + \Psi(w)\right] - \frac{V(\tau_m)}{4\omega^3}\sin 2\left[\omega\tau_m - \frac{\pi}{4} - \pi\alpha(\tilde{\tau}_m)\right] + O\left(\frac{1}{\omega^4}\right). \tag{44}$$

Here $\Phi(R, w)$ and $\Psi(w)$ are determined by equations (30) and (32); n is the radial order of a p-mode. This number is the number of nodes of the related radial eigenfunction and can be calculated as described in Vorontsov & Zharkov (1989).

The function $\alpha(\tilde{\tau})$ can be calculated for particular values of ω and w using equation (39) with boundary condition (40) if we know the potential $\tilde{V}(\tilde{\tau})$ in the outer region. In the inverse problem $\alpha(\tilde{\tau})$ is the information we use to reconstruct $\tilde{V}(\tilde{\tau})$.

Let us introduce the "phase shift" function $\tilde{\alpha}(w, \omega)$

$$\tilde{\alpha}(w, \omega) = \alpha(\tilde{\tau}_m) - \frac{1}{2\pi\omega} \int_0^{\tau_m} V\,d\tau - \frac{1}{\pi\omega} \Psi(w) - \frac{V(\tau_m)}{4\pi\omega^2} \sin 2\left[\omega\tau_m - \frac{\pi}{4} - \pi\alpha(\tilde{\tau}_m)\right]. \tag{45}$$

This phase shift in equation (44) is a function of both ℓ and ω, while for low- and intermediate-degree modes it is a function of frequency ω alone. The phase shift in equation (44) absorbs the deviation of the exact solution of the wave equation in the outer layers from the asymptotic solution. The function $\tilde{\alpha}$ depends on the matching point, but the particular values of r_m or $\tilde{\tau}_m$ in expression (44) are not very significant. The eigenfrequencies of the solar oscillations almost do not depend on the position of the matching point if this point is chosen sufficiently deep in the region, where the second-order approximation is good enough. Direct computation of $\tilde{\alpha}(w, \omega)$ as a function of depth (for a standard solar model and low-degree modes) shows that this function becomes almost constant just below the hydrogen and helium ionization zone. For the oscillations with cyclic frequency of 3 mHz, below the depth of 4% of solar radius this function is a constant with accuracy better than 10^{-4}.

4. APPROXIMATION IN SURFACE LAYERS

The eigenfrequency equation (44) differs from those appropriate for low- and intermediate-degree modes by the phase shift, which is now dependent not only on the oscillation frequency ω but also on the degree ℓ. For low values of ℓ, the phase shift $\tilde{\alpha}(w, \omega)$ becomes a function of frequency alone. Note that the definition of the phase shift differs from that used in the first-order asymptotic theory (Brodsky & Vorontsov 1988; Vorontsov & Zharkov 1989). In the first-order theory the phase shift described the difference between the exact solution of the wave equations and the asymptotic solution given by the first-order approximation.

When the degree ℓ is sufficiently low, $\ell(\ell + 1)/\omega^2$ can be neglected compared with r^2/c^2 in the surface layers because of low values of the sound speed. To generalize the theoretical description for high-degree modes, the dependence of the solutions in the surface layers on the degree ℓ will now be explicitly taken into account by using new small parameters $\tilde{\delta}$ and δ:

$$\tilde{\delta} = \frac{[\ell(\ell + 1)]^{1/2}}{\omega} \frac{c(R)}{R}, \qquad \delta = \frac{(\ell + 1/2)}{\omega} \frac{c(R)}{R}. \tag{46}$$

We assume that in the surface layers or in the vicinity of $r = R$ the function $c(r)$ is small enough to use δ, $\tilde{\delta}$ as small parameters. It means that the horizontal wavenumber is small compared with the full wavenumber in this area. At a frequency of 3 mHz, and with R taken to be the solar radius at the level of temperature minimum, $c(R)/\omega R \simeq 5 \times 10^{-4}$. At degree $\ell = 1000$, which is approximately an upper limit at which accurate observational data are available, the value of the small parameter δ achieves $\frac{1}{2}$.

To make our results clearer, we would like to point out that

$$\tilde{\delta} = \tilde{w} \frac{c(R)}{R}, \qquad \delta = w \frac{c(R)}{R};$$

furthermore, as we know from equation (45) and the definitions of τ and $\tilde{\tau}$, V and $\tilde{V}$, the parameters $\tilde{\delta}$ and δ can arise in our formulae only in the expressions for s and $\tilde{s}$. It means, for example, for s,

$$\begin{aligned} s &= \left(\frac{1}{c^2} - \frac{w^2}{r^2}\right)^{1/2} = \frac{1}{c}\left\{1 - \delta^2\left[\frac{c^2(r)R^2}{c^2(R)r^2}\right]\right\}^{1/2} \\ &= \frac{1}{c}\left[1 - \frac{1}{2}\delta^2 \frac{c^2(r)R^2}{c^2(R)r^2} + O(\delta^4)\right] \\ &= \frac{1}{c}\left\{1 - \frac{1}{2} w^2 \frac{c^2(r)}{r^2} + O\left[w^4 \frac{c^4(r)}{r^4}\right]\right\}. \end{aligned} \tag{47}$$

We can formally think that w^2 and $\tilde{w}^2$ are new small parameters after appropriate normalization of the function $c(r)/r$ by the factor $C(R)/R$. The result will not depend on this simplification because in all expressions we have only the combination $w\,c(r)/r$. Below we will describe the formula for w, τ, and V, but will remember that the formulae for $\tilde{w}$, $\tilde{\tau}$, and $\tilde{V}$ are the same.

Now both the acoustic potential V and the acoustic depth τ (as a new independent variable) can be expanded in powers of w^2. The resulting expansion for the phase shift is

$$\tilde{\alpha}(w, \omega) = \tilde{\alpha}_0(\omega) + \tilde{\alpha}_2(\omega)w^2 + \cdots, \tag{48}$$

where $\tilde{\alpha}_0(\omega)$ corresponds to low-degree modes. Our aim is to calculate $\tilde{\alpha}_0(\omega)$ and $\tilde{\alpha}_2(\omega)$. So we have

$$s = \frac{1}{c}\left[1 - \frac{1}{2} w^2 \frac{c^2}{r^2} + O(w^4)\right], \qquad \tau = \tau_0 + \tau_2 w^2 + O(w^4), \tag{49}$$

$$\tau_0 = \int_r^R \frac{1}{c}\,dr, \qquad \tau_2 = -\frac{1}{2}\int_r^R \frac{c}{r^2}\,dr = -\frac{1}{2}\int_0^{\tau_0} \frac{c^2}{r^2}\,d\tau_0, \tag{50a}$$

$$\frac{d\tau}{d\tau_0} = 1 - \frac{w^2}{2}\frac{c^2}{r^2}(\tau_0) + O(w^4)\,, \tag{50b}$$

$$\frac{d}{d\tau}[\ln(r^2hs)] = \frac{d}{d\tau_0}\ln\left(\frac{r^2h}{c}\right) + \frac{w^2}{2}\left[\frac{c^2}{r^2}\frac{d\ln(r^2h/c)}{d\tau_0} - \frac{d}{d\tau_0}\left(\frac{c^2}{r^2}\right)\right] + O(w^4)\,, \tag{50c}$$

$$\tilde{V}(\tilde{\tau}) = V_0(\tau_0) + \tilde{w}^2V_2(\tau_0) + O(\tilde{w}^4)\,, \tag{51}$$

$$V_0 = N^2 + \left[\frac{d}{d\tau_0}\ln\left(\frac{rh^{1/2}}{c^{1/2}}\right)\right]^2 - \frac{d^2}{d\tau_0^2}\ln\left(\frac{rh^{1/2}}{c^{1/2}}\right), \tag{52}$$

$$V_2 = \left[\frac{d}{d\tau_0}\ln\left(\frac{rh^{1/2}}{c^{1/2}}\right)\right]^2\frac{c^2}{r^2} - \frac{d}{d\tau_0}\ln\left(\frac{rh^{1/2}}{c^{1/2}}\right)\frac{d}{d\tau_0}\left(\frac{c^2}{r^2}\right) - \frac{d^2}{d\tau_0^2}\ln\left(\frac{rh^{1/2}}{c^{1/2}}\right)\frac{c^2}{r^2} + \frac{1}{4}\frac{d^2}{d\tau_0^2}\left(\frac{c^2}{r^2}\right); \tag{53}$$

$$\alpha(\tilde{\tau}) = \alpha_0(\tau_0) + \tilde{w}^2\alpha_2(\tau_0) + O(\tilde{w}^4) = \alpha_0(\tau_0) + w^2\alpha_2(\tau_0) - \frac{1}{4w^2}\alpha_2(\tau_0) + O(w^4) + O\left(\frac{1}{w^4}\right). \tag{54}$$

Using equation (39) and expressions (51)–(54), we collect terms of the same order in $\tilde{w}$ and obtain equations for α_0 and α_2:

$$\frac{d\alpha_0}{d\tau_0} = \frac{V_0}{2\pi\omega}[1 + \sin(2\omega\tau_0 - 2\pi\alpha_0)] = \frac{V_0}{\pi\omega}\cos^2\left(\omega\tau_0 - \frac{\pi}{4} - \pi\alpha_0\right), \tag{55}$$

$$\frac{d\alpha_2}{d\tau_0} = -\frac{\alpha_2 V_0}{\omega}\cos(2\omega\tau_0 - 2\pi\alpha_0) + \frac{1}{\pi}V_0\tau_2\cos(2\omega\tau_0 - 2\pi\alpha_0) + \frac{1}{\pi}[1 + \sin(2\omega\tau_0 - 2\pi\alpha_0)]\left(\frac{V_2}{2\omega} - \frac{V_0}{4\omega}\frac{c^2}{r^2}\right). \tag{56}$$

The nonlinear equation (55) has the boundary condition

$$\alpha_0(0) = \frac{1}{\pi}\arctan\left[\frac{V_0(0)}{\omega^2} - 1\right]^{1/2} - \frac{1}{4}\,, \tag{57}$$

and equation (56) has the boundary condition

$$\begin{aligned}\alpha_2(0) &= \frac{1}{2\omega\pi}\cos^2\left(\pi\alpha_0 + \frac{\pi}{4}\right)V_2(0)\frac{1}{[V_0(0) - \omega^2]^{1/2}}\\ &= \frac{1}{2}\frac{d\alpha_0}{d\tau_0}(0)\frac{V_2(0)}{V_0(0)[V_0(0) - \omega^2]^{1/2}}\,.\end{aligned} \tag{58}$$

Equation (55) is nonlinear equation for $\alpha_0(\tau_0)$. The solution of this equation and application of the information about $\alpha_0(\tau_{0m})$,

$$\tau_{0m} = \int_{r_m}^{R}\frac{1}{c}\,dr\,,$$

will be the subject of future papers. Here we notice that after equation (55) has been solved, equation (56) is a linear equation for $\alpha_2(\tau_0)$ which can be solved by quadrature. We obtain the result

$$\begin{aligned}\alpha_2(\tau_0) = \alpha_2(0) &+ \frac{1}{\pi}\exp\left[-\int_0^{\tau_0}\frac{V_0}{\omega}\cos(2\omega\tau_0 - 2\pi\alpha_0)d\xi\right]\\ &\times\int_0^{\tau_0}\left\{V_0\tau_2\cos(2\omega\tau_0 - 2\pi\alpha_0) + [1 + \sin(2\omega\tau_0 - 2\pi\alpha_0)]\left(\frac{V_2}{2\omega} - \frac{V_0}{4\omega}\frac{c^2}{r^2}\right)\exp\left[\int_0^{\tau_0}\frac{V_0}{\omega}\cos(2\omega\tau_0 - 2\pi\alpha_0)d\xi\right]\right\}d\xi\,.\end{aligned} \tag{59}$$

Here ξ is a variable of integration.

For the eigenfrequency equations (44) we need information about $\alpha(\tilde{\tau})$ only at the point $\tilde{\tau} = \tilde{\tau}_m$. It means that we need knowledge of $\alpha_0(\tau_{0m})$ and $\alpha_2(\tau_{0m})$. Both of these values are functions only of ω.

To obtain the final result in the surface layers, we have to find the main terms of the decomposition of the phase shift $\tilde{\alpha}(w, \omega)$ in the series on powers of w. We obtain from equation (32)

$$\begin{aligned}\Psi(w) &= \left\{\frac{13}{24}\frac{c}{r} + \frac{c}{2}\ln' h + \frac{c}{24}\ln'\left[\left(\frac{r^2}{c^2}\right)'\right] + \frac{7}{48}c\left[\ln'\left(\frac{r^2}{c^2}\right)\right]\left(1 + w^2\frac{c^2}{r^2}\right)\right\}\left[1 + \frac{w^2}{2}\frac{c^2(r)}{r^2}\right]\Bigg|_{r=R} + O(w^4)\\ &= \Psi(0)\left[1 + \frac{w^2}{2}\frac{c^2(R)}{R^2}\right] + w^2\frac{7}{48}\left[\frac{c^3}{r^2}\ln'\left(\frac{r^2}{c^2}\right)\right]\Bigg|_{r=R} + O(w^4)\,.\end{aligned} \tag{60}$$

From equations (52) and (53) we find

$$\frac{1}{2\omega^2}\int_0^{\tau_m} V\,d\tau = \frac{1}{2\omega^2}\int_0^{\tau_{0m}+\omega^2\tau_{2m}+O(w^4)} [V_0 + w^2 V_2 + O(w^4)]\left[1 - \frac{w^2}{2}\frac{c^2}{r^2}(\tau_0) + O(w^4)\right]d\tau_0$$

$$= \frac{1}{2\omega^2}\left\{\int_0^{\tau_{0m}} V_0\,d\tau_0 + w^2\left[\int_0^{\tau_{0m}} V_2\,d\tau_0 - \frac{1}{2}\int_0^{\tau_{0m}} V_0\frac{c^2}{r^2}\,d\tau_0\right]\right\} + O(w^4)\,, \tag{61a}$$

$$\tau_{2m} = -\frac{1}{2}\int_{r_m}^{R}\frac{c}{r^2}\,dr = -\frac{1}{2}\int_0^{\tau_{0m}}\frac{c^2}{r^2}\,d\tau_0\,. \tag{61b}$$

From equations (52)–(54) we have

$$\frac{-V(\tau_m)}{4\omega^3}\sin 2\left[\omega\tau_m - \frac{\pi}{4} - \pi\alpha(\tilde{\tau}_m)\right] = \frac{1}{4\omega^3}[\![V_0(\tau_{0m})\cos[2\omega\tau_{0m} - 2\pi\alpha_0(\tau_{0m})]$$

$$+ w^2\{V_2(\tau_{0m})\cos[2\omega\tau_{0m} - 2\pi\alpha_0(\tau_{0m})] - 2\omega\tau_{2m}V_0(\tau_{0m})\sin[2\omega\tau_{0m} - 2\pi\alpha_0(\tau_{0m})]$$

$$+ 2\pi\alpha_2(\tau_{0m})V_0(\tau_{0m})\sin[2\omega\tau_{0m} - 2\pi\alpha_0(\tau_{0m})]\}]\!]\,. \tag{62}$$

From equations (44), (54), and (60)–(62) we can finally obtain the equations for the eigenfrequencies with accuracy $O(1/\omega^4)$ and $O(w^4)$.

$$F(w) + \frac{1}{\omega^2}\Phi(R, w) - \left[\!\!\left[\frac{1}{\omega}\pi\alpha_0(\tau_{0m}) - \frac{1}{\omega^2}\left[\frac{1}{2}\int_0^{\tau_{0m}} V_0\,d\tau_0 + \Psi(0)\right] + \frac{1}{4\omega^3}\{V_0(\tau_{0m})\cos[2\omega\tau_{0m} - 2\pi\alpha_0(\tau_{0m})] - \pi\alpha_2(\tau_{0m})\}\right]\!\!\right]$$

$$- w^2\left[\!\!\left[\frac{1}{\omega}\pi\alpha_2(\tau_{0m}) - \frac{1}{2\omega^2}\left\{\int_0^{\tau_{0m}} V_2\,d\tau_0 - \frac{1}{2}\int_0^{\tau_{0m}} V_0\frac{c^2}{r^2}\,d\tau_0\right.\right.\right.$$

$$\left. + \Psi(0)\frac{c^2(R)}{R^2} - \frac{7}{24}\left[\frac{c^3}{r^2}\ln\left(\frac{c^2}{r^2}\right)\right]\Bigg|_{r=R} + \tau_{2m}V_0(\tau_{0m})\sin[2\omega\tau_{0m} - \pi\alpha_0(\tau_{0m})]\right\}$$

$$\left.\left. + \frac{1}{4\omega^3}\{V_2(\tau_{0m})\cos[2\omega\tau_{0m} - \pi\alpha_0(\tau_{0m})] + 2\pi\alpha_2(\tau_{0m})V_0(\tau_{0m})\sin[2\omega\tau_{0m} - \pi\alpha_0(\tau_{0m})]\}\right]\!\!\right] = \frac{\pi n}{\omega}\,. \tag{63}$$

It is very important to notice that to obtain equation (63) for the frequencies we used two small parameters, $1/\omega$ and w, but we used them in different regions and never together. To remain in the first region, we assumed w was not small but finite. However, for low degree ℓ and sufficiently high frequencies in the outer layers, we can assume $w = 0$ (this does not contradict our assumptions) and obtain

$$F(w) + \frac{1}{\omega^2}\Phi(R, w) - \frac{\pi}{\omega}\alpha_0(\tau_{0m}) + \frac{1}{\omega^2}\left[\frac{1}{2}\int_0^{\tau_{0m}} V_0\,d\tau_0 + \Psi(0)\right] - \frac{1}{4\omega^3}\{V_0(\tau_{0m})\cos 2[\omega\tau_{0m} - \pi\alpha_0(\tau_{0m})] - \pi\alpha_2(\tau_{0m})\} = \frac{\pi n}{\omega}\,. \tag{64}$$

5. NUMERICAL EXAMPLE AND CONCLUDING REMARKS

To test the accuracy of the asymptotic description, the eigenfrequencies were computed directly by equations (63) for a standard solar model, and then compared with exact eigenfrequencies. Model 1 of Christensen-Dalsgaard (1982) was used in the computations. The results are shown in Figure 1*a* for computations with α_2 neglected, and in Figure 1*b* for computations with α_2 taken into account. The degree range was limited by $50 \le l \le 200$, and the frequency range by 1 mHz $\le w/2\pi \le$ 5 mHz (the radial order n varies from 1 to 21).

A sharp feature which is seen in both Figures 1*a* and 1*b* at $r_1/R \approx 0.72$ corresponds to the position of the base of the convection zone in the model; the asymptotic description loses accuracy for modes with turning points near this boundary because of the rapid variation of the sound-speed gradient. When the position of the turning point becomes higher, the inclination of the acoustic ray paths in the outermost reflecting layers increases, and Figure 1*b* demonstrates significant improvement of the accuracy of the description when α_2 is taken into account. For modes with $r_1/R < 0.95$, the accuracy is improved by about an order of magnitude. For modes with $r_1/R > 0.95$, however, the improvement is rather poor, because the leading-order correction to the phase shift (α_2) fails to describe modes for which the inclination of the acoustic ray paths in the second helium ionization zone ($r/R \approx 0.98$) can no longer be considered small.

The numerical test shows that equations (63) can be used for the solution of the inverse problem of high-degree solar acoustic oscillations, in particular for the sound-speed inversion in the solar envelope. The logic of the inversion is based on the functional type of these equations, which can be written in the form

$$F(w) + \frac{1}{\omega^2}\Phi(w) - \frac{\pi\tilde{\alpha}_0(\omega)}{\omega} - \pi\frac{\tilde{\alpha}_2(\omega)}{\omega}w^2 \cong \pi\frac{n}{\omega}\,. \tag{65}$$

Four terms in the left-hand side of equation (65) can be determined separately from observational data (represented by the right-hand side) using their different functional dependence on w and ω. This separation for real data is an important problem from

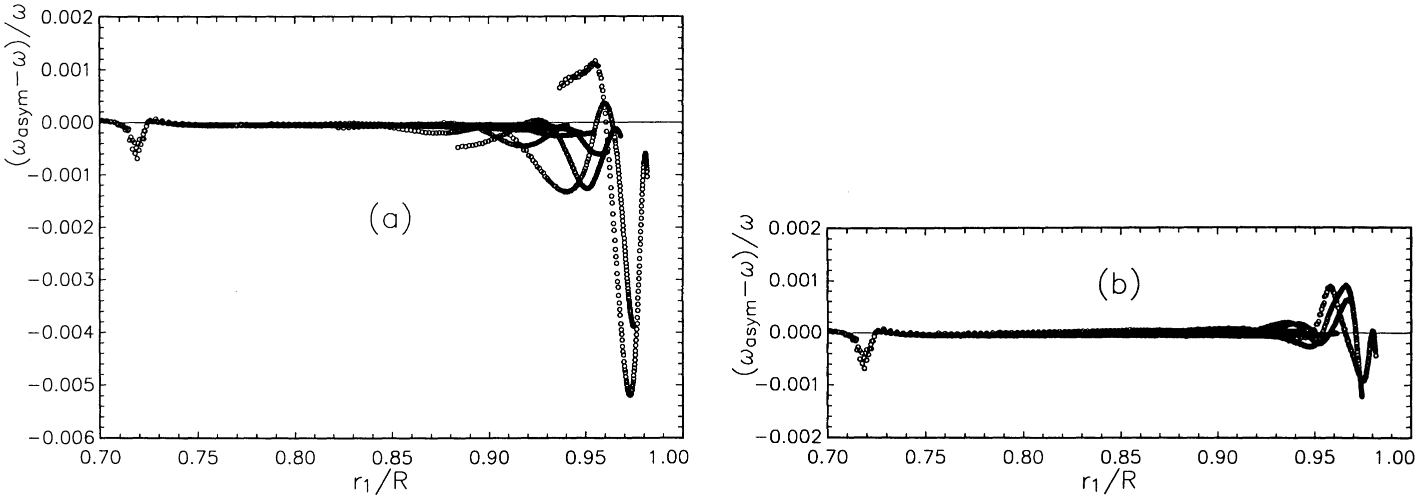

FIG. 1.—Relative differences between solar p-mode frequencies predicted by the asymptotic description and the exact eigenfrequencies of the same solar model. The results are plotted against the position of the inner turning point, for modes with $50 \le l \le 200$ and frequencies between 1 and 5 mHz. (*a*) The phase shift α_2 is neglected in the asymptotic description; (*b*) α_2 is taken into account.

our point of view. Some current steps in this direction are described by Pamyatnykh, Vorontsov, & Däppen (1991) and Vorontsov, Baturin, & Pamyatnykh (1992).

After separation, $F(w)$ will give us the function $c(r)$ and then τ_0, τ_2; the function $\Phi(w)$ is sensitive to $N^2(r)$; the function $\tilde{\alpha}_0(\omega)$ can be used to study $V_0(\tau_0)$ and $\tilde{\alpha}_2(\omega)$ to study $V_2(\tau_0)$. This is the logic of the solution of the problem.

With the formula we can obtain bounds on high-order terms to account for effects of the approximation, in addition to stochastic errors, in the determination of the uncertainty in sound speed in the solar interior; for example, the "strict bounds" technique discussed by Stark (1992) can incorporate such information about possible systematic errors.

A particular interesting application of equation (65) is connected with the study of the second helium ionization zone (at depth of about 2% of the solar radius), which contributes significantly to acoustic scattering (see, e.g., Brodsky & Vorontsov 1988; Pamyatnykh et al. 1991; Vorontsov et al. 1992 and references therein). The dependence of acoustic phase shift on w is already seen in the data of Vorontsov et al. (1992). More accurate frequency measurements will allow us to use this additional source of information about the helium ionization zone to improve the determination of the solar helium abundance and to study the equation of state.

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