Mon. Not. R. Astron. Soc. **273**, 573–582 (1995)

Seismology of the solar envelope: measuring the acoustic phase shift generated in the outer layers

D. O. Gough[1,2,4] and S. V. Vorontsov[1,3,4]★

[1] *Institute of Astronomy, Madingley Road, Cambridge CB3 0HA*

[2] *Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge CB3 9EW*

[3] *Institute of Physics of the Earth, B. Gruzinskaya 10, Moscow 123810, Russia*

[4] *Astronomy Unit, Queen Mary and Westfield College, Mile End Road, London E1 4NS*

Accepted 1994 October 19. Received 1994 June 30; in original form 1992 June 25

ABSTRACT

Partial reflection of acoustic waves by regions of rapid spatial variation modulates the solar p-mode frequencies, in a manner that is often described in terms of an acoustic phase α. At low degree, α is a function of frequency alone. At higher degrees, the inclination of the acoustic ray paths from the normal to the reflecting layers becomes significant, and gives rise to a degree dependence. With the current accuracy of frequency measurements, this phenomenon is significant for all the p modes trapped in the convection zone. We describe a technique capable of separating the degree-dependent component from the leading term $\alpha_0(\omega)$. We determine the leading-order contribution to the degree dependence of α from the solar p-mode frequencies reported by Libbrecht, Woodard & Kaufman, and compare the results with direct computations from a solar model based on the formalism described by Brodsky & Vorontsov. The dominant contribution comes from the second helium ionization zone, and can be used as a new source of information relevant to the helioseismic calibration of the equation of state of the solar plasma and to the determination of the solar helium abundance.

Key words: Sun: interior – Sun: oscillations.

1 INTRODUCTION

Modern helioseismic observations have yielded oscillation frequencies with a precision as high as 1 part in 10^5. Given the thousands of diverse modes to be analysed, inverse methods have become the principal instrument for studying the solar interior (for recent reviews see Vorontsov & Zharkov 1989; Gough & Thompson 1991; Gough & Toomre 1991; Libbrecht & Woodard 1991; Vorontsov 1992).

Two basically different approaches are used for helioseismic inversions (Gough & Thompson 1991); both have their own advantages and complement one another. There are methods based on the solutions of the exact (adiabatic) oscillation equations, which are used to construct kernels for relating small structure perturbations to corresponding frequency perturbations, and which draw on a variety of techniques developed in the area of linear inverse theory. In principle, they can be used iteratively to carry out non-linear inversions. Alternatively, there is an asymptotic approach, which is based on the high-frequency asymptotic properties of adiabatic solar p modes and which makes it possible to perform non-linear inversions directly (with sound-speed inversion as a first step). To some extent, both methods permit information in the data pertaining to different regions within the Sun to be separated; this is essential for unambiguously answering questions about the physics of the solar interior.

In this paper we address the separation of information based on asymptotic arguments. With the simplest asymptotics (e.g. Christensen-Dalsgaard et al. 1985), the separation is quite limited, owing to an unresolved near degeneracy in the data. That degeneracy is inherent in stellar p modes with asymptotically high frequencies. It arises from the fact that the simplest asymptotic analyses take little or no account of the spatial oscillations in the eigenfrequencies, and that therefore two asymptotic modes with identical lower turning points, and consequently almost identical ray paths, are almost identically influenced by the stratification

★Permanent address: Institute of Physics of the Earth, B. Gruzinskaya 10, Moscow 123810, Russia.

of the background state. They therefore provide almost the same information. When data errors are substantial, the inclusion of both modes serves merely to improve the signal-to-noise ratio. When measurement errors are small, however, more elaborate analyses can reveal subtle features of the stratification of the Sun that destroy the asymptotic degeneracy. The present paper addresses how some signatures of such features can be isolated.

A highly accurate asymptotic description is needed for the extraction of small signatures from precise observational data. Our analysis is based on the second-order asymptotic description of Brodsky & Vorontsov (1993), which is based on a Liouville–Green expansion of the oscillation equations in the Cowling approximation, and which accounts for the fact that the acoustic waves are influenced somewhat by buoyancy and, separately, by the radial variation of density and of gravity. We ignore degeneracy splitting by rotation and asphericity, and consider only the mean multiplet frequencies $\omega = \omega(n, l)$ of modes of order n and degree l. We therefore restrict attention to the spherically averaged structure of the Sun.

Asymptotic solutions of the adiabatic wave equation are valid throughout most of the convection zone and most of the radiative interior, but, unless special care is taken, they are liable to be invalid where seismically relevant aspects of the structure of the background state vary on a scale short compared with the radial wavelength, such as near the solar surface. For this reason, we match the asymptotic solution of the adiabatic wave equation valid in the interior with the exact solution computed numerically through the outer layers of the solar model, as has been done in the past by, for example, Christensen-Dalsgaard & Frandsen (1983), Brodsky & Vorontsov (1988b), Christensen-Dalsgaard & Pérez Hernández (1988, 1991, 1992, 1994a,b) and Vorontsov (1991). In this sense our procedure is a composite approach. The asymptotic solution is then prescribed in terms of an acoustic phase α, which accounts for distortions and reflections of the sound waves by the complicated outer layers above the matching point.

We must point out that in addition to the breakdown of asymptotics in the surface layers there is also a breakdown of the simple adiabatic wave equation itself. In these layers, non-adiabatic processes resulting from radiative and, more importantly, convective heat transfer influence the oscillations substantially (e.g. Balmforth 1992a,b), as do also the Reynolds stress and lateral inhomogeneity associated with convection. Magnetic fields also play a significant role. Therefore great care must be exercised in interpreting the phase α. Some headway can be made, however, as is discussed in some detail by Christensen-Dalsgaard & Pérez Hernández (1992). Our task here is not even to attempt that intepretation; instead, it is merely to separate α from the data, principally for the purposes of interpreting what remains. As we discuss below, the properties of even the non-adiabatic wave equation (for a spherically symmetrical background state) permit that separation with reasonable confidence; our hope is that the neglected turbulent convection does not significantly degrade the outcome.

In the context of this paper, the 'outer layers' that determine the value of α are considered to extend beneath both helium ionization zones. Indeed, the second helium ionization zone contributes significantly to the acoustic reflection; its influence on the phase of modes that penetrate much more deeply is clearly seen in both theoretical and observational data, and can be used to study the equation of state of the partially ionized solar plasma and to determine the solar helium abundance (see Christensen-Dalsgaard & Pérez Hernández 1991, 1992, 1994b; Vorontsov, Baturin & Pamyatnykh 1991, 1992, and references therein), thereby complementing analyses of modes whose turning points span that zone (Däppen & Gough 1986; Däppen, Gough & Thompson 1988). Throughout the remainder of the convection zone beneath those layers, which is the domain of scrutiny of this paper, the background state varies on a length-scale considerably greater than the acoustic wavelength, and Liouville–Green asymptotic theory is valid. We must point out, however, that the condition does not persist; at and immediately below the base of the convection zone there is essentially a discontinuity in the second derivative of the sound speed, if local mixing-length theory with a non-vanishing mixing length is adopted. (Actually there is a thermal boundary layer which smooths out the discontinuity, formally rendering it of higher order, but which occurs on a length-scale that is very much less than the oscillation wavelength and, in many solar models, is even well beyond the limit of resolution of the numerical difference scheme. If there is adiabatic overshoot, the essential discontinuity is then in the first derivative of the sound speed, and its influence on the acoustic waves is potentially more severe.) This discontinuity introduces an oscillatory component into the eigenfrequencies (Gough 1990; Roxburgh & Vorontsov 1994) which would contribute to the phase α. A magnetic field confined within a thin layer near the base of the convection zone, or elsewhere, can do likewise (e.g. Gough & Thompson 1988; Thompson 1988; Vorontsov 1988). Therefore we restrict attention to only those modes whose lower turning points lie well within the convection zone, and accordingly dub the quantity α determined from them the 'outer phase'.

An important simplifying assumption that is usually adopted (and which will be lifted in the present study) is that the outer phase is a function of frequency alone: $\alpha = \alpha(\omega)$; in particular, it is independent of the degree l of the p modes. It was noted by Gough (1986) that this assumption is valid over a wide range in degree because, provided l is not too high, acoustic ray paths are nearly vertical close to the stellar surface, where the sound speed is low. Generally, the vertical component of the wavenumber near the surface is much greater than the horizontal component. At the upper reflecting level (upper turning point) it formally vanishes, but there reflection is produced locally by the vertical stratification of density, whose inverse scaleheight is also much greater than the horizontal wavenumber of the acoustic modes and therefore has a dominant influence on the dynamics. When l is large, however, the slopes of acoustic ray paths in the outer layers can be significant, and the assumption must be reconsidered.

The ratio of the horizontal component k_h of the wavenumber $\boldsymbol{k}$ to the total magnitude $k = |\boldsymbol{k}|$ at radius r in the second helium ionization zone, predicted by simple ray theory for a mode with lower turning point r_t in the middle of the convection zone, say $r_t \simeq 0.85\,R$, where R is the radius of the Sun (corresponding, for example, to a mode of degree 75 and cyclic frequency 3 mHz), is

$$\frac{k_\mathrm{h}}{k} \simeq \frac{L}{r} \frac{c(r)}{\omega} \simeq \frac{r_\mathrm{t} c(r)}{r c(r_\mathrm{t})} \simeq \frac{1}{4}, \tag{1}$$

where c is the sound speed and $L^2 = l(l+1)$. This value is not small, and can imply observable consequences, because the reflecting properties of the layer are different for waves incident at different angles.

It is the determination from frequency data $\omega = \omega(n, l)$ of the individual terms in the asymptotic eigenfrequency relation, including the phase α, that is the subject of the present paper. In the past, it has been the very clear distinction between the simple functional dependences of α and the remainder $\tilde{F}$ of the asymptotic expression for n/ω, namely $\alpha = \alpha(\omega)$ and $\tilde{F} = \tilde{F}(\omega/L)$, that has permitted a straightforward unambiguous separation to be made from only the frequency data (e.g. Gough 1986; Brodsky & Vorontsov 1988a,b), whose work is a generalization of that of Duvall (1982). From the function $\tilde{F}$ it is then possible to determine the sound speed in the solar interior. Once it is admitted that α does not depend solely on ω, however, such a separation is not immediately possible. The situation is aggravated by recognizing additionally that $\tilde{F}$ is not actually a function of ω/L alone. This presents a potentially serious problem. To separate the individual terms now requires a more careful assessment of their functional forms, from which one can then deduce additional explicit constraints on the outer phase, without which it would not be possible to determined $\tilde{F}$ and thereby infer the acoustic stratification of the interior of the sun.

The next section contains a general description of the acoustic cavity, and a qualitative discussion of reasonable constraints that can be imposed on α. In Section 3, we estimate the degree-dependent effects on α in leading order for a solar envelope model, and verify the validity of the constraints numerically. The asymptotic theory of intermediate- and high-degree modes developed by Brodsky & Vorontsov (1993) is used in the computations. In Section 4, we describe an appropriate separation technique, and apply it to infer the leading-order effect on α from the solar oscillation frequencies reported by Libbrecht, Woodard & Kaufman (1990).

2 THE ACOUSTIC CAVITY

In the Cowling approximation, which is adequate for the intermediate- and high-degree modes (as we explain later, formally relaxing the Cowling approximation does not invalidate our technique), the adiabatic oscillation equations can be reduced to the Schrödinger-type equation (e.g. Brodsky & Vorontsov 1993)

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\zeta + [\omega^2 - V(\tau)]\zeta = 0, \tag{2}$$

with

$$\tau = \mathrm{sgn}(s^2)\int_r^R |s|\,\mathrm{d}r, \qquad s^2 = \frac{1}{c^2} - \frac{1}{w^2 r^2}, \tag{3}$$

$$w = \frac{\omega}{L}, \qquad \zeta = \left(\frac{\rho_0}{s}\right)^{1/2} r\Xi, \tag{4}$$

where Ξ is the radial displacement eigenfunction. The acoustic potential is given by

$$V = N^2 + \frac{1}{4}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\ln(r^2 hs)\right]^2 - \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\ln(r^2 hs), \tag{5}$$

with

$$h = \exp\int_0^r \left(\frac{N^2}{g} - \frac{g}{c^2}\right)\mathrm{d}r. \tag{6}$$

Here, R is the radius of the solar surface, which in this paper is taken to be at the temperature minimum, g is the acceleration due to gravity and N is taken to be the buoyancy frequency. [Note that here and below the notation is different from that of Brodsky & Vorontsov (1993); in particular $\tilde{\tau}$ has been replaced by τ, $\tilde{s}$ by s, etc.]

As we mentioned in the introduction, we need to understand how the phase α characterizing the waves in the outer layers varies with ω and l. To this end we consider the acoustic potential $V(\tau)$. We observe first that it has a singularity at the lower turning point, $s = 0$. For obtaining asymptotic solutions in the interior, one usually casts the oscillation equations into a form with regular coefficients. For the present discussion, however, equation (2) is convenient; it is adopted for the computation of the non-asymptotic solutions near the surface (Brodsky & Vorontsov 1993).

The acoustic potential V for p modes in a standard solar envelope model is shown in Fig. 1, for three different values of the parameter $w^{-1} \simeq (l+1/2)/\omega =: \tilde{w}$ which, in particular, determines the radius of the lower turning point. At $w^{-1} = 0$, the potential pertains to radial modes, and has been discussed in various forms elsewhere (e.g. Christensen-Dalsgaard & Frandsen 1983; Vorontsov 1992; Vorontsov, Baturin & Pamyatnykh 1992). The prominent oscillatory feature at $\tau \simeq 600$ s is a product of the second helium ionization zone, where a significant phase shift occurs. As $\tilde{w}$ increases, the lower turning point, where V is singular, moves upwards to lower τ. Also, closer to the surface, where

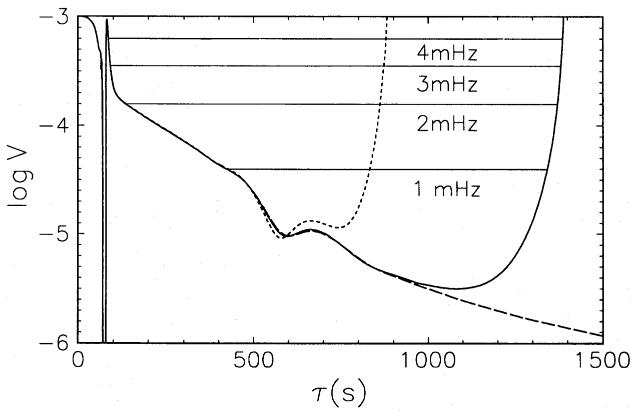

Figure 1. Acoustic potential $V(\tau)$ in a solar model envelope. The continuous line is for $\tilde{w} = 4000$ s (inner asymptotic turning point $r_\mathrm{t} \simeq 0.85R$), the short-dashed line for $\tilde{w} = 8000$ s ($r_\mathrm{t} \simeq 0.94R$), and the long-dashed line is for radial modes ($w^{-1} = 0$). Horizontal bars indicate schematically the trapping regions for p modes with $\tilde{w} = 4000$ s and with the frequencies indicated.

the sound speed is lower, acoustic ray paths are, on the whole, nearer to the vertical: the relative variation of s with $\tilde{w}$ is weaker, and therefore V too is only very weakly dependent on $\tilde{w}$.

We are now in a position to discuss the phase $\alpha(\omega, \tilde{w})$. We observe first that the l-dependent part of α must diminish as ω increases at constant $\tilde{w}$, at least when $\tilde{w}$ is not too high. The reason is that acoustic waves of higher frequencies are reflected at higher layers, where, as we have just explained, the l-dependent part of the acoustic potential varies very slowly; moreover, the variation of $V(\tau)$ in the deeper layers has a smaller influence on the higher frequency waves, because the potential in deeper layers is much smaller than ω^2. We observe also that the symmetry of the system demands that the $\tilde{w}$ dependence of α is an even function of $\tilde{w}$; one therefore expects its deviation from the value at $w^{-1}=0$ to behave as $\tilde{w}^2$ at small $\tilde{w}$.

We summarize these important conclusions thus: (i) at constant ω, the $\tilde{w}$-dependent part of the phase, $\alpha(\omega, \tilde{w})$, is weak at small $\tilde{w}$, and behaves as $\tilde{w}^2$ as $\tilde{w}\rightarrow 0$; and (ii) the $\tilde{w}$-dependent part of $\alpha(\omega, \tilde{w})$ tends to zero when ω increases (at least when $\tilde{w}$ is not too high).

Note that these expectations are essentially independent of any particular solar model; the region with $\tau \gtrsim 80$ s corresponds to the adiabatic part of the solar convection zone, and the generic behaviour of both $c(r)$ and $V(\tau)$ is hardly influenced by detailed model assumptions (cf. Christensen-Dalsgaard, Gough & Thompson 1991).

3 DIRECT COMPUTATIONS FOR A SOLAR MODEL

In designing these computations, we were motivated by the second-order asymptotic description of intermediate- and high-degree modes developed by Brodsky & Vorontsov (1993). Below we present the necessary equations in a form convenient for our computations.

The asymptotic eigenfrequency equation for a mode of order n is

$$F(\tilde{w})+\frac{1}{\omega^2}\Phi(\tilde{w})\simeq\pi\,\frac{n+\alpha(\omega,\tilde{w})}{\omega}, \qquad (7)$$

with

$$F(\tilde{w})=\int_{r_1}^{R}\tilde{s}\,\mathrm{d}r, \qquad \tilde{s}=\left(\frac{1}{c^2}-\frac{\tilde{w}^2}{r^2}\right)^{1/2}, \qquad \tilde{w}=\frac{l+1/2}{\omega}, \qquad (8)$$

in which the lower limit r_1 of the integral is the radius at which $\tilde{s}$ vanishes; it approximates the turning-point radius r_t. The left-hand side of equation (7) is essentially the first two terms of an expansion in powers of $(c/\omega H)^2$ of the product of ω and the phase change of the eigenfunction between $r=r_1$ and the surface $r=R$ of the Sun, where H is a characteristic scaleheight of the equilibrium state. The phase α accounts for both of the phase jumps of $\pi/4$ in the JWKB solution produced at the turning points, together with the phase error, not accounted for by $\omega^{-1}\Phi$, which arises both from the breakdown of the asymptotic description, resulting from rapid variation of the background state, and from a failure of the simple adiabatic wave equation to describe the true motion in the outer layers of the Sun. For modes with r_1 in the convection zone, that breakdown occurs only in the outer layers, in and above the helium ionization zones. The quantity $\Phi(\tilde{w})$ is an integral of structure variables over the region of propagation (r_1, R). It arises as a correction to the leading-order acoustic asymptotics that results essentially from the simple local plane-wave acoustic dispersion relation (Christensen-Dalsgaard et al. 1985). That correction conains contributions from buoyancy and the deviation of the acoustic cut-off frequency from its plane-parallel value (Gough 1984, 1986, 1993). We note further that the perturbation to the gravitational potential, neglected in the analysis of Brodsky & Vorontsov, also contributes to a term of the form $\omega^{-2}\Phi(\tilde{w})$ (Vorontsov 1988, 1991; Gough 1993), so it is not only under the Cowling approximation that our analysis is valid. As mentioned earlier, α is only weakly dependent on $\tilde{w}$. Our intention now is to expand it about a function of ω alone, leaving the rest of equation (7) untouched.

We restrict our analysis to the leading term of the $\tilde{w}$ dependence of $\alpha(\omega, \tilde{w})$. After expanding α in powers of $\tilde{w}$, which we regard as a small parameter (formally, we are expanding in powers of the small dimensionless quantity $\tilde{w}c/r$), and retaining only the first two terms, the eigenfrequency equation becomes

$$F(\tilde{w})+\frac{1}{\omega^2}\Phi(\tilde{w})\simeq\pi\,\frac{n+\alpha_0(\omega)+\tilde{w}^2\alpha_2(\omega)}{\omega}. \qquad (9)$$

The evaluation of α_0 and α_2 can be computed in the manner described by Brodsky & Vorontsov (1993). First, the eigenfunction ζ is expressed as the product of an amplitude and a sinusoidal function of a phase, $\omega\tau-\pi/4-\pi\hat{\alpha}(\tau)$. Specification of the amplitude in such a way that it appears to be constant under the first differentiation of ζ with respect to τ then yields the first-order non-linear differential equation for the phase:

$$\frac{\mathrm{d}\hat{\alpha}}{\mathrm{d}\tau}=(\pi\omega)^{-1}\,V\cos^2(\omega\tau-\pi/4-\pi\hat{\alpha}), \qquad (10)$$

which is then expanded in powers of $\tilde{w}^2$, yielding

$$\frac{\mathrm{d}\hat{\alpha}_0}{\mathrm{d}\tau_0}=\frac{V_0}{2\pi\omega}[1+\sin(2\omega\tau_0-2\pi\hat{\alpha}_0)], \qquad (11)$$

$$\frac{\mathrm{d}\hat{\alpha}_2}{\mathrm{d}\tau_0}=-\frac{\alpha_2 V_0}{\omega}\cos(2\omega\tau_0-2\pi\hat{\alpha}_0)+\frac{1}{\pi}V_0\tau_2\cos(2\omega\tau_0-2\pi\hat{\alpha}_0)$$
$$+\frac{1}{\pi}[1+\sin(2\omega\tau_0-2\pi\hat{\alpha}_0)]\left(\frac{V_2}{2\omega}-\frac{V_0}{4\omega}\frac{c^2}{r^2}\right), \qquad (12)$$

in which $\hat{\alpha}(\tau)=\hat{\alpha}_0(\tau)+\tilde{w}^2\hat{\alpha}_2(\tau)+\ldots$ and $V(\tau)=V_0(\tau)+\tilde{w}^2V_2(\tau)+\ldots$, and where

$$\tau_0=\int_r^R\frac{\mathrm{d}r}{c}, \qquad \tau_2=-\frac{1}{2}\int_r^R\frac{c}{r^2}\mathrm{d}r. \qquad (13)$$

The expansion coefficients of the acoustic potential are given by

$$V_0=N^2-c\frac{\mathrm{d}z_1}{\mathrm{d}r}+z_1^2, \qquad (14)$$

$$V_2=\frac{c^2}{r^2}(V_0-N^2)+z_1z_2-\frac{c}{4}\frac{\mathrm{d}z_2}{\mathrm{d}r}, \tag{15}$$

where

$$z_1=\frac{c}{4}\left[\frac{4}{r}+\frac{N^2}{g}-(3-\gamma_1)\frac{g}{c^2}-\frac{1}{\gamma_1}\frac{\mathrm{d}\gamma_1}{\mathrm{d}r}\right], \tag{16}$$

$$z_2=\frac{c^3}{r^2}\left[\frac{2}{r}-\frac{N^2}{g}-(1-\gamma_1)\frac{g}{c^2}-\frac{1}{\gamma_1}\frac{d\gamma_1}{\mathrm{d}r}\right], \tag{17}$$

and γ_1 is the adiabatic exponent $(\partial \ln p/\partial \ln \rho)_{\mathrm{ad}}$, the partial derivative being taken at constant specific entropy.

Equations (11)–(17) are to be solved subject to an appropriate boundary condition at $r=R$, which, on expansion, becomes

$$\hat{\alpha}_0(0)=\frac{1}{\pi}\tan^{-1}\left[\left(\frac{V_0(0)}{\omega^2}-1\right)^{1/2}\right]-\frac{1}{4}, \tag{18}$$

$$\hat{\alpha}_2(0)=\frac{1}{2\pi}\frac{V_2(0)}{V_0(0)}\left[\frac{V_0(0)}{\omega^2}-1\right]^{-1/2}. \tag{19}$$

The equations are integrated numerically to $\tau_0(r_{\mathrm{m}})=:\tau_{0\mathrm{m}}$, at $r=r_{\mathrm{m}}$, at which point the solutions are matched on to an asymptotic solution that is valid in the deep interior. Provided the matching point is chosen to be beneath the outer layers within which the asymptotic expansion is invalid, the outcome is only very weakly dependent on the precise choice of its location, as it should be. The resulting expressions for α_0 and α_2 are then given by

$$\alpha_0(\omega)=\hat{\alpha}_0(\tau_{0\mathrm{m}})-\frac{1}{2\pi\omega}\int_0^{\tau_{0\mathrm{m}}}V_0\,\mathrm{d}\tau_0-\frac{1}{\pi\omega}\Psi(0)$$
$$+\frac{1}{4\pi\omega^2}V_0(\tau_{0\mathrm{m}})\cos[2\omega\tau_{0\mathrm{m}}-2\pi\hat{\alpha}_0(\tau_{0\mathrm{m}})]-\frac{1}{4\omega^2}\hat{\alpha}_2(\tau_{0\mathrm{m}}), \tag{20}$$

$$\alpha_2(\omega)=\hat{\alpha}_2(\tau_{0\mathrm{m}})-\frac{1}{2\pi\omega}\left[\int_0^{\tau_{0\mathrm{m}}}V_2\,\mathrm{d}\tau_0-\frac{1}{2}\int_0^{\tau_{0\mathrm{m}}}V_0\frac{c^2}{r^2}\mathrm{d}\tau_0\right]$$
$$-\frac{1}{\pi\omega}\left[\frac{1}{2}\Psi(0)\frac{c^2}{r^2}+\frac{7}{48}\frac{c^2}{r^2}\frac{\mathrm{d}\ln}{\mathrm{d}r}\left(\frac{r^2}{c^2}\right)\right]_{r=R}$$
$$+\frac{1}{4\pi\omega^2}\{V_2(\tau_{0\mathrm{m}})\cos[2\omega\tau_{0\mathrm{m}}-2\pi\hat{\alpha}_0(\tau_{0\mathrm{m}})] \tag{21}$$
$$-2\omega\tau_{2\mathrm{m}}V_0(\tau_{0\mathrm{m}})\sin[2\omega\tau_{0\mathrm{m}}-2\pi\hat{\alpha}_0(\tau_{0\mathrm{m}})]$$
$$+2\pi\hat{\alpha}_2(\tau_{0\mathrm{m}})V_0(\tau_{0\mathrm{m}})\sin[2\omega\tau_{0\mathrm{m}}-2\pi\hat{\alpha}_0(\tau_{0\mathrm{m}})]\},$$

$$\Psi(0)=\left[\frac{c}{2}\frac{\mathrm{d}\ln h}{\mathrm{d}r}+\frac{c}{24}\frac{\mathrm{d}\ln}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{c^2}\right)+\frac{7}{48}c\frac{\mathrm{d}\ln}{\mathrm{d}r}\left(\frac{r^2}{c^2}\right)+\frac{13}{24}\frac{c}{r}\right]_{r=R}$$
$$=\left[\frac{c}{2}\left(\frac{N^2}{g}-\frac{g}{c^2}\right)+\frac{z_1}{6}+\frac{1}{6z_2}\left(\frac{c^2}{r^2}(V_0-N^2)-V_2\right)\right. \tag{22}$$
$$\left.+\frac{r^2}{c^2}\frac{z_2}{4}+\frac{1}{2}\frac{c}{r}\right]_{r=R},$$

where $\tau_{2\mathrm{m}}=\tau_2(r_{\mathrm{m}})$.

The leading-order correction, $V_2(\tau_0)$, in the expansion of the acoustic potential in powers of $\tilde{w}$ about $V_0(\tau_0)$ is shown in Fig. 2. It measures the difference between $V(\tau)$, shown in Fig. 1, for $\tilde{w}$ non-zero, and $V_0(\tau_0)$, the latter being essentially the value of V when $w^{-1}=0$, indicated in Fig. 1 by the long-dashed line. There is a prominent oscillatory feature in the second helium ionization zone. The narrow and sharp fluctuation near $\tau_0=80$ s (in the superadiabatic part of the convection zone) is due to the small deformation of the τ-scale. At high values of τ_0, the potential V_2 increases, because τ_0 approaches the lower turning point of equation (2), at which $V(\tau)$ is singular; the phase α is not affected because the asymptotic description is quite adequate there.

The leading-order correction α_2 to the phase is shown in Fig. 3, in normalized units. The prominent oscillatory component is produced by the second helium ionization zone. It is evident also that α_2 tends to zero at high frequencies, in accordance with our expectations from the discussion at the end of Section 2. This property will be used in the inversion of the oscillation frequencies to permit the separate determination of the functions appearing in equation (9).

4 DETERMINATION OF THE PHASE FUNCTIONS FROM FREQUENCY DATA

The principal aim in this paper is to determine the functions α_0, α_2, F and Φ in equation (9) from oscillation frequency

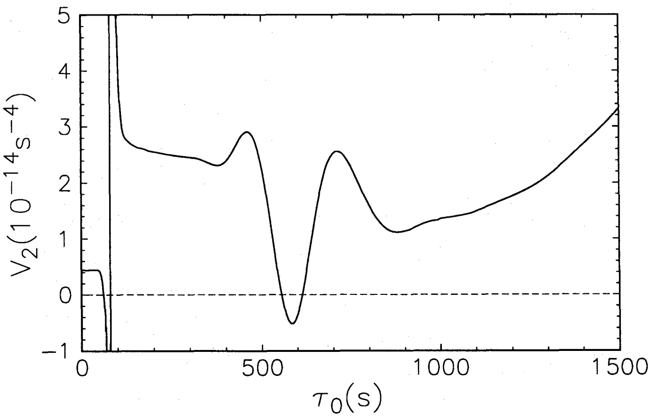

Figure 2. Leading-order correction $V_2(\tau_0)$ to the acoustic potential $V_0(\tau_0)$ for the solar model envelope.

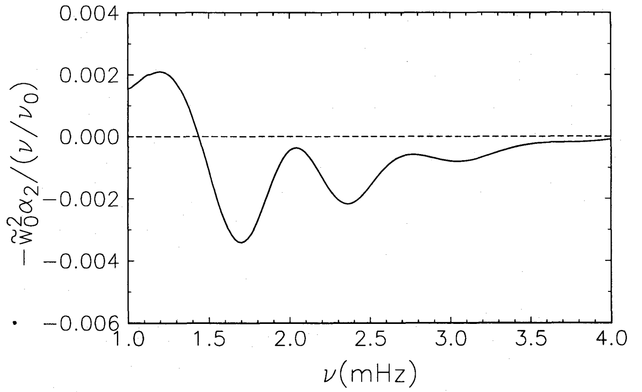

Figure 3. Leading-order correction factor $\psi=(\nu_0/\nu)\,\tilde{w}_0^2\alpha_2(\nu)$ to the surface phase, computed for the solar model envelope.

data. With this accomplished, one can then consider the sound speed $c(r)$ in the interior of the Sun to have been determined, simply as a result of inverting equation (8) (e.g. Gough 1984). Direct information about the density stratification, both in the deep interior and in the surface layers, can be extracted from Φ and α.

4.1 Resolution of an asymptotic ambiguity

As a prelude to describing the separation procedure, we first cast equation (9) into dimensionless form. We work with the cyclic frequency $\nu=\omega/2\pi$, and we adopt fiducial values $\nu_0=3$ mHz and $\tilde{w}_0=4000$ s of ν and $\tilde{w}$ to define new independent variables $x=\nu/\nu_0$ and $y=\tilde{w}/\tilde{w}_0$. We introduce $\mathscr{F}(y)=2\nu_0 F(\tilde{w})$, $\mathscr{P}(y)=(2\pi^2\nu_0)^{-1}\Phi(\tilde{w})$, $\phi(x)=(\nu_0/\nu)\,\alpha_0(\omega)$ and $\psi(x)=(\nu_0/\nu)\,\tilde{w}_0^2\,\alpha_2(\omega)$, and we transfer all the terms that are unknown in the representation to the right-hand side of the equation, which yields

$$\frac{n}{x}=\mathscr{F}(y)+x^{-2}\mathscr{P}(y)-\phi(x)-y^2\psi(x). \tag{23}$$

The left-hand side of equation (23) is to be considered as a function $f(x, y)$ of the two variables x and y, and is determined by the oscillation frequencies on a grid $(\nu_{n,l},\ \tilde{w}_{n,l})$. Were the phase α to have been considered as an arbitrary unknown function of the two independent variables, an unambiguous separation of the right-hand side into its constituent asymptotic components would not have been possible. The separation is made possible by virtue of the different functional dependences on x and y, inferred from our presumption of the form of $\alpha(\nu, \tilde{w})$ at small $\tilde{w}$ and at large ω. Indeed, all helioseismic inversions are based, either explicitly or implicitly, on presumptions of this kind.

It is evident that the explicit functional form of the right-hand side of equation (23) is insufficient to define $\mathscr{F}$, $\mathscr{P}$, ϕ and ψ unambiguously, because the equation is invariant under the transformation $\phi\to\phi+A+Bx^{-2}$, $\psi\to\psi+C+Dx^{-2}$, $\mathscr{F}\to\mathscr{F}+A+Cy^2$, $\mathscr{P}\to\mathscr{P}+B+Dy^2$, for arbitrary constants A, B, C and D. Therefore some additional constraints must be imposed. The ambiguity associated with the constant A has been encountered in the past, for it arises even when only the leading terms $\mathscr{F}$ and ϕ of the asymptotic expansion are retained. However, it has no influence on the primary inversion of the data to obtain $c(r)$, because that depends only on $\mathrm{d}\mathscr{F}/\mathrm{d}y$. When the second-order term $x^{-2}\mathscr{P}(y)$ is also taken into account, the additional constant B arises; the role of that constant has been discussed by Vorontsov (1991), and has no influence on the determination of $\mathscr{F}$. The remaining constants, C and D, arise when the second-order term α_2 is taken into account. The former scales an unknown component of $\mathscr{F}$ that is proportional to y^2, and renders an inversion of the sound speed potentially ambiguous. Even the diagnostic capability of the so-called 'small frequency separation' $\nu_{n,l}-\nu_{n-1,l+2}$ is in danger of being lost, for it is determined, in leading order, by the value of $\mathrm{d}^2\mathscr{F}/\mathrm{d}y^2$ as $y\to 0$ (cf. Vorontsov 1991).

The ambiguity in $\mathscr{F}$ is resolved by invoking the second of the properties of α listed at the end of Section 2, namely that α is essentially independent of $\tilde{w}$ when ω is large, implying that $\psi\to 0$ as x becomes large. That condition determines the

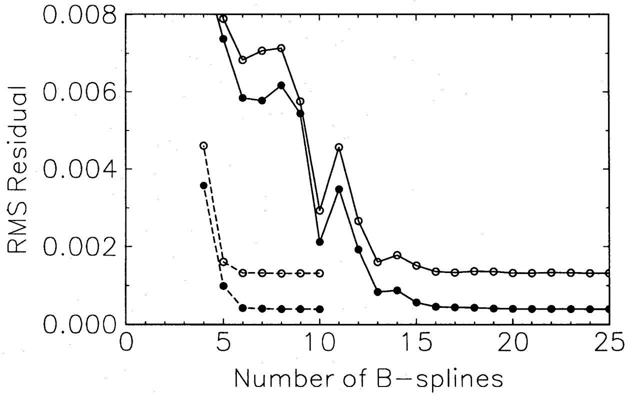

Figure 4. Root-mean-square residual of the approximation of the observational data [left-hand side of equation (23)] versus number of splines. Continuous lines indicate the residual as a function of the number of vector splines in x with 10 splines in y. Dashed lines indicate the dependence on the number of vector splines in y, with 25 splines in x. Open circles indicate the approximation with only the two leading terms, $\mathscr{F}$ and ϕ, in equation (23); filled circles with all four terms taken into account.

constant C. The constant D would be determined by the properties of $\mathscr{P}$, but it is of no serious concern to us here.

4.2 The separation procedure

The decomposition of the function $f(x, y)$, determined for continuous (x, y) by spline interpolation on the grid of n/x, has been accomplished iteratively in the following three stages:

(i) ignore $\mathscr{P}$ and ψ and estimate $\mathscr{F}$ and ϕ by the functions $\tilde{\mathscr{F}}$ and $\tilde{\phi}$ obtained by attempting to separate $f(x, y)$ into the form $\tilde{\mathscr{F}}(y)-\tilde{\phi}(x)$;

(ii) for given estimates of ϕ and ψ, estimate $\mathscr{F}$ and $\mathscr{P}$ by separating $f(x, y)+\phi(x)+y^2\psi(x)$ into the form $\mathscr{F}(y)+x^{-2}\mathscr{P}(y)$;

(iii) with estimates of $\mathscr{F}$ and $\mathscr{P}$, estimate ϕ and ψ by separating $\mathscr{F}(y)+x^{-2}\mathscr{P}(y)-f(x, y)$ into the form $\phi(x)+y^2\psi(x)$.

Stage (i) was carried out first, and was accomplished by iterative back substitution in a manner not unlike the procedures that have been used in the past (e.g. Gough 1986; Vorontsov 1988): with an estimate of $\tilde{\mathscr{F}}(y)$, an iterate of $\tilde{\phi}(x)$ was obtained by fitting to $-f(x, y)+\tilde{\mathscr{F}}(y)$, and then a new iterate of $\tilde{\mathscr{F}}(y)$ was obtained by fitting to $f(x, y)+\tilde{\phi}(x)$, the iteration being repeated until it converged. The fitting here was carried out by least-squares, using representations of $\tilde{\mathscr{F}}$ and $\tilde{\phi}$ in terms of cubic B-splines. The first estimate of $\tilde{\mathscr{F}}(y)$ was a direct evaluation of $\mathscr{F}(y)$ from a theoretical solar model using equations (8).

Stages (ii) and (iii) were then carried out successively and repeatedly, starting from $\mathscr{F}=\tilde{\mathscr{F}}$, $\phi=\tilde{\phi}$ obtained from stage (i), together with $\psi=0$, $\mathscr{P}=0$. Both stages are equivalent to the determination of $u(\xi)$ and $v(\xi)$ from $z(\xi, \eta)$, of the form

$$z(\xi, \eta)=u(\xi)+g(\eta)\,v(\xi) \tag{24}$$

for a known function $g(\eta)$. In stage (ii) $\xi=y$ and $g(\eta)=x^{-2}$, in stage (iii) $\xi=x$ and $g(\eta)=y^2$. The back substitution used in stage (i) diverges here, so some other technique must be applied. We have used instead a vector-spline technique, approximating both $u(\xi)$ and $v(\xi)$ simultaneously by (cubic) B-splines. It is a direct generalization of the usual B-spline approximation technique for a single function of a single variable. The technique is described in more detail by Vorontsov & Shibahashi (1991). It permits an inclusion of a smooth transition to the usual scalar approximation in a region where one of the two functions to be determined is assumed to be small.

A measure of our ability to represent the frequency data in the asymptotic form is provided in Fig. 4. Here, the root-mean-square residual of the approximation of the right-hand side of equation (23) is shown versus the number of splines. In all the computations described below, we used 25 vector splines in x, and 10 in y.

4.3 Application to solar data

Solar oscillation frequencies reported by Libbrecht, Woodard & Kaufman (1990) were used in the inversion. The input data were restricted to the more accurate 'low-resolution' data set. We limited the data set by the constraint $3000 \leq \tilde{w} \leq 8000$ s, which includes only those p modes that are trapped in the convection zone, having inner turning points $r_t \simeq r_1$ lying between about 0.78 and 0.94 R. We did not employ modes that penetrate more deeply, because they are influenced, either in their oscillatory regions or in the evanescent tails, by the base of the convection zone, and are not adequately described by the asymptotic analysis adopted here. Indeed, once the outer phase α has been determined, the procedure could be repeated together with the more deeply penetrating modes in order to study the region spanning the interface between the convection zone and the radiative interior, although in that case it would no doubt be necessary to take more detailed account of the $\tilde{w}$ dependence of α. Our restriction of the range of $\tilde{w}$ excludes also modes with $r_1 > 0.94$ R, whose ray paths in the second helium ionization zone ($r \approx 0.98$ R) are too strongly inclined from the vertical for the leading-order correction to the phase to be adequate. Because the accuracy of frequency measurements differs significantly for different modes, the oscillation frequencies were weighted in inverse proportion to the standard errors in the data. The incorporation of weights into the vector-spline approximation technique is straightforward.

Frequencies in the entire range ($1 \leq \nu \leq 4$ mHz) of the observational data were used. The distribution of the breakpoints of the B-splines was chosen to be uniform in both x and y. In view of the lower accuracy of the data at high frequencies, we performed a smooth transition of the ψ approximant (together with the first two derivatives) to zero by setting the coefficients of all the B-splines for ψ to zero at breakpoints beyond the breakpoint closest to but less than $x = 1.17$ (therefore $\alpha_2 = 0$ for $\nu \gtrsim 3.5$ mHz). A total of 733 modes were used in the inversion.

Fig. 5(a) shows the approximation residuals for individual modes when both $\mathscr{P}$ and ψ [$\Phi(\tilde{w})$ and $\alpha_2(2\pi\nu)$] were neglected. Regular structure whose magnitude is much greater than the observational errors is clearly seen in the residuals. Fig. 5(b) shows the result of the same approximation procedure applied to the adiabatic eigenfrequencies computed from the solar model. The structure is quite similar, indicating that the residuals are due mainly to the inadequacy of the simple leading-order asymptotic approximation, which can therefore benefit from extension.

Similar residuals, this time with all the four terms in the eigenfrequency equation taken into account, are shown in Fig. 6. For the solar data, the residuals are reduced to levels close to the observational errors. When both the second-order asymptotic terms $\mathscr{P}$ and ψ are taken into account, there is a significant improvement in the quality of the approximation. The main contribution to this improvement comes from ψ; the first-order asymptotic description in the deep adiabatically stratified layers of the solar convection zone is quite accurate, and the correction $\mathscr{P}$ is relatively minor. There may perhaps be further information in the data at high frequencies, although the apparently random scatter about zero suggests that that might not actually be the case. We conclude that most of the useful information contained in the currently available observational data has probably been extracted by this approximation. Moreover, we note that the number of parameters that describe the data has been reduced by a factor of 10. The reduction is hardly surprising in view of the near degeneracy of information, which, as we mentioned above, is an asymptotic property of the modes.

The second-order contribution ψ to the phase deduced from the frequencies is shown in Fig. 7(a). Some idea of how well it has been determined can be gauged by comparing Fig. 7(b), in which the values deduced by applying the same procedure to the artificial data are plotted, with the actual non-dimensionalized factor $\alpha_2(2\pi\nu)$ of Fig. 3. On the whole, agreement is good. There are some minor differences, which we believe are predominantly of computational origin. In particular, the amplitude of the oscillatory components in $\alpha_2(2\pi\nu)$ evident in Fig. 3 is slightly underestimated. This is a result of some numerical smoothing of the hump in the helium ionization zone when the acoustic potentials were computed by double differentiation in the model envelope, which was provided on somewhat too coarse a grid. Small differences could also arise from the improper fixing of the unknown constants in the approximation procedure. Those small differences should be largely eliminated when these functions are extracted from the observational and theoretical frequencies by exactly the same procedure.

The non-dimensionalized second-order asymptotic term Φ is shown in Fig. 8. It is small compared with the contribution from the $\tilde{w}$-dependent component of the phase. To fix the arbitrary constant D, the approximation to $\Phi(\tilde{w})$ was set to be zero at $\tilde{w} = \tilde{w}_0 = 4000$ s. We suspect that, aside from near the ends of the region where the spline approximation is unreliable, the small difference between Figs 8(a) and (b) is a result primarily of the inadequacy of the equation of state.

5 CONCLUSIONS

The outer layers of the Sun significantly influence the frequencies of all the observed p modes. Their effect can be represented in terms of an outer phase α whose dependence upon $\tilde{w} = (l + 1/2)/\omega$, at least in leading order, can be determined reliably from currently available oscillation frequency data. The dominant contribution to this dependence comes from the second helium ionization zone. Therefore our procedure permits us to refine solar model calibrations based on asymptotic principles (cf. Gough 1984; Christen-

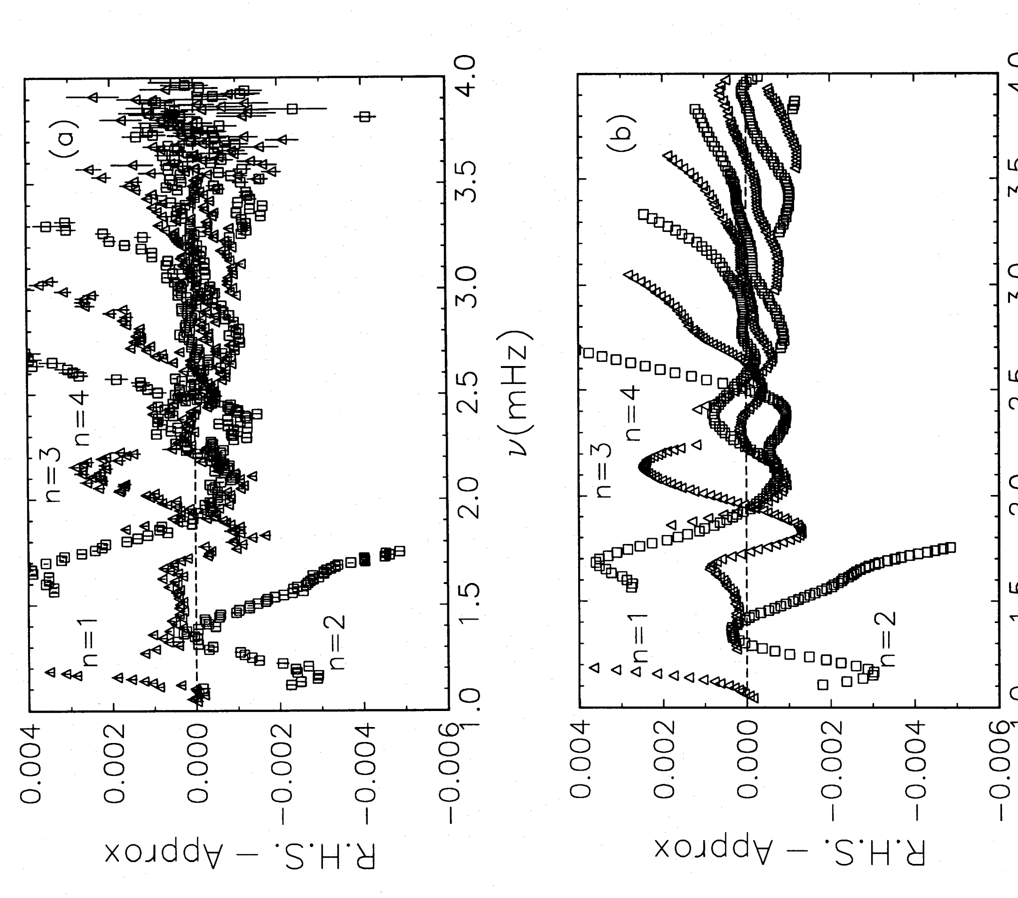

Figure 5. Residuals of the approximation with only the two leading terms $\mathscr{F}$ and ϕ in equation (23) versus frequency. The order n of sequence of modes is shown near some regular features of the residuals (modes of even order are shown by squares, modes of odd order by triangles). The estimated observational errors in the frequency measurements are indicated by vertical bars. Figure (a) is for the observational frequencies, (b) is for the artificial data.

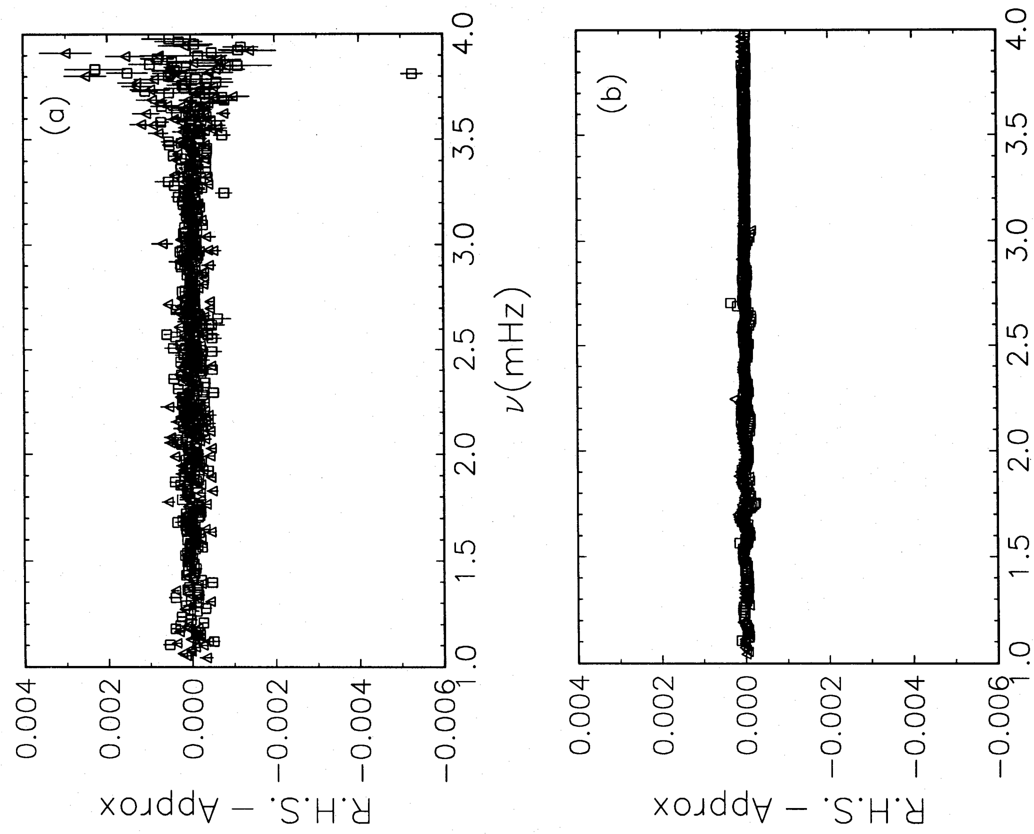

Figure 6. As Fig. 5, but with all four terms in equation (23) included.

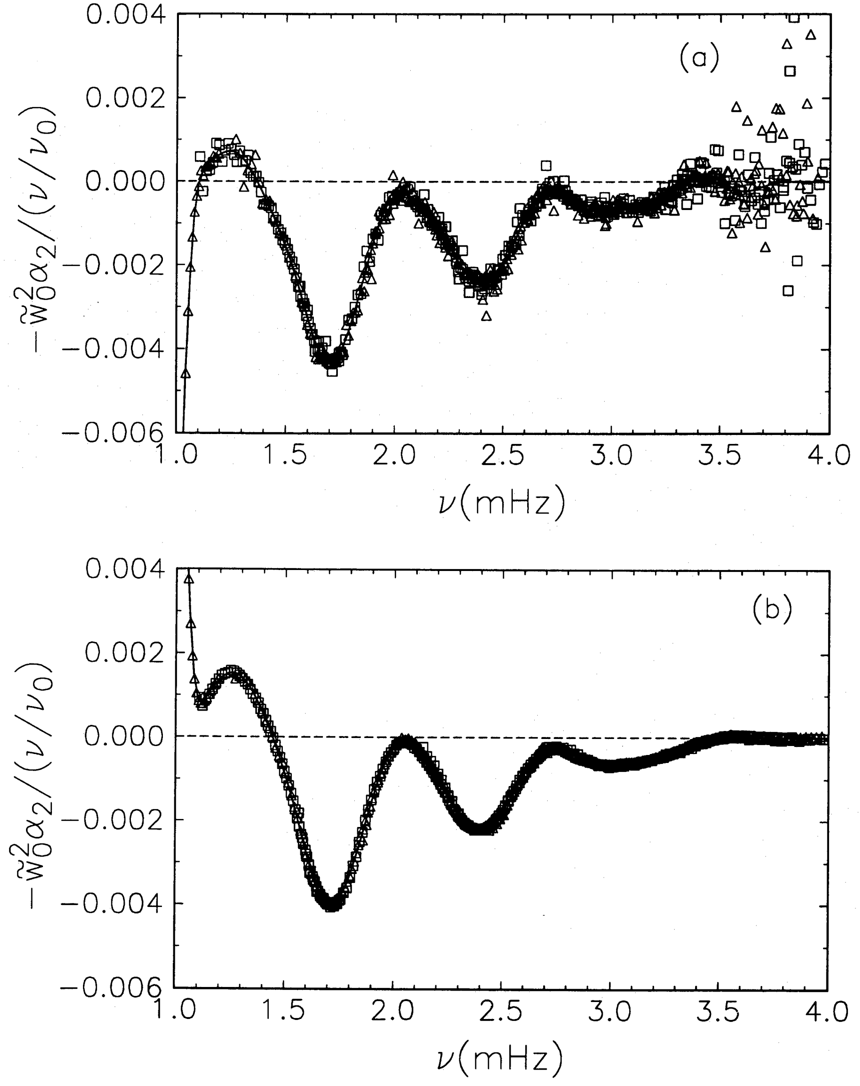

Figure 7. The leading-order term of the $\tilde{w}$-dependence of the acoustic phase α, in normalized units. The resulting spline approximation is shown by the continuous line. The squares and triangles represent values, for modes of even and odd order respectively, of the quantity $y^{-2}(x^{-1}n-\mathscr{F}-x^{-2}\mathscr{P}+\phi)$. Figure (a) is for the observational data, (b) is for the theoretical eigenfrequencies.

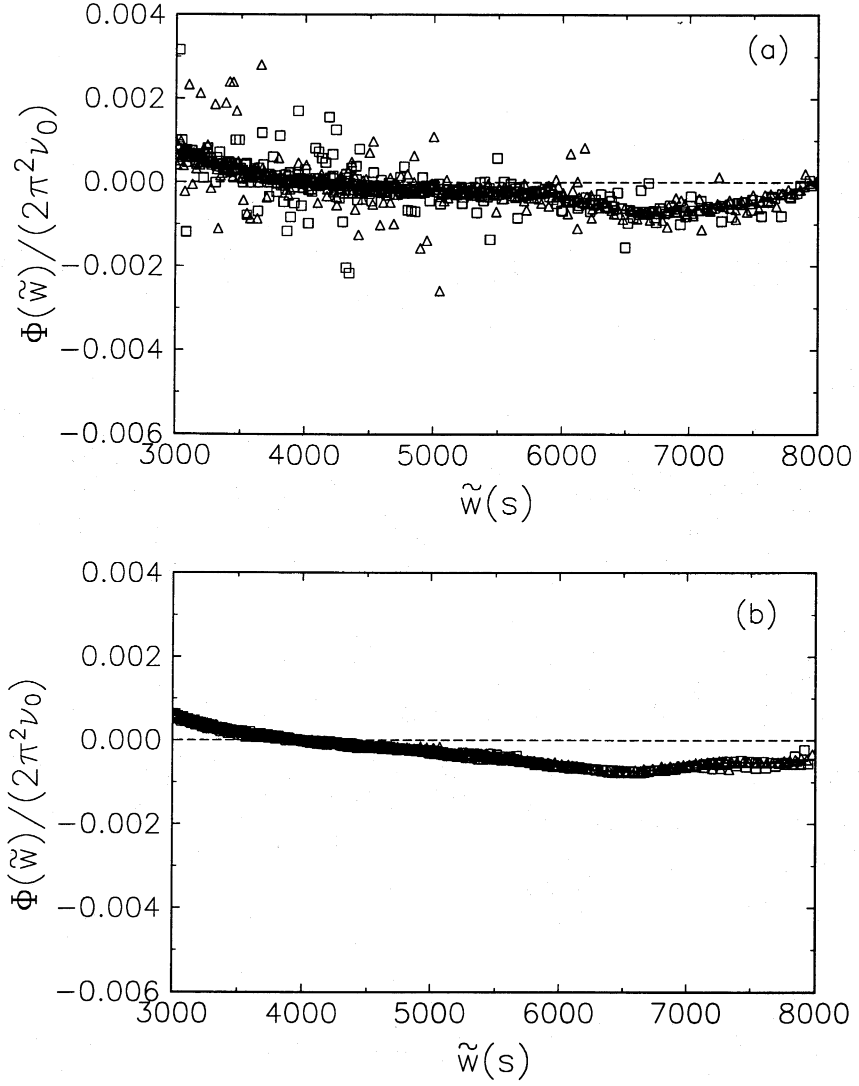

Figure 8. As Fig. 7, but for the second-order asymptotic term Φ.

sen-Dalsgaard & Pérez Hernández 1991, 1992, 1994b; Vorontsov, Baturin & Pamytnykh 1991, 1992), and thereby obtain additional information from acoustic waves that cross this zone at different angles. It is hoped that this will permit more subtle testing of the equation of state, and lead to a more reliable determination of the solar helium abundance. Indeed, we have already demonstrated that the technique can detect the existence of an inadequacy in current equations of state, and can be used to calibrate a proposed improvement (Baturin et al., in preparation). The accurate separation of the phase, which we have demonstrated in this paper, will also permit us to carry out more accurate asymptotic inversions to obtain the sound speed in the solar interior.

ACKNOWLEDGMENTS

We thank V. A. Baturin for the solar model used in the computations. We thank also the referee, N. J. Balmforth, for suggestions leading to a substantial improvement in the presentation. The work was partially supported by a grant of the Science & Engineering Research Council. SVV also thanks the 'Zodiac' Scientific–Methodological Council for their partial support.

REFERENCES

Balmforth N. J., 1992a, MNRAS, 255, 603
Balmforth N. J., 1992b, MNRAS, 255, 632
Baturin V. A., Däppen W., Gough D. O., Vorontsov S. V., 1995, in preparation
Brodsky M. A., Vorontsov S. V., 1988a, in Christensen-Dalsgaard J., ed., Proc. IAU Symp. 123, Advances in helio- and astero-seismology. Reidel, Dordrecht, p. 137
Brodsky M. A., Vorontsov S. V., 1988b, in Rolfe E., ed., Seismology of the sun and sun-like stars. ESA SP-286, Noordwijk, p. 487
Brodsky M. A., Vorontsov S. V., 1993, ApJ, 409, 455
Christensen-Dalsgaard J., Frandsen S., 1983, Sol. Phys., 82, 165
Christensen-Dalsgaard J., Pérez Hernández F., 1988, in Rolfe E., ed., Seismology of the sun and sun-like stars. ESA SP-286, Noordwijk, p. 499
Christensen-Dalsgaard J., Pérez Hernández F., 1991, in Gough D. O., Toomre J., eds, Challenges to Theories of the Structure of Moderate-Mass Stars. Springer-Verlag, Heidelberg, p. 43
Christensen-Dalsgaard J., Pérez Hernández F., 1992, MNRAS, 257, 62
Christensen-Dalsgaard J., Pérez Hernández F., 1994a, MNRAS, 267, 111
Christensen-Dalsgaard J., Pérez Hernández F., 1994b, MNRAS, 269, 475
Christensen-Dalsgaard J., Duvall T. L., Jr, Gough D. O., Harvey J. W., Rhodes E. J., Jr, 1985, Nat, 315, 378
Christensen-Dalsgaard J., Gough D. O., Thompson M. J., 1991, ApJ, 378, 413
Däppen W., Gough D. O., 1986, in Gough D. O., ed., Seismology of the sun and the distant stars. Reidel, Dordrecht, p. 275
Däppen W., Gough D. O., Thompson M. J., 1988, in Rolfe E., ed., Seismology of the sun and sun-like stars. ESA SP-286, Noordwijk, p. 505
Duvall T. L., Jr, 1982, Nat, 300, 242
Gough D. O., 1984, Phil. Trans. R. Soc. Lond, A313, 27
Gough D. O., 1986, in Gough D. O., ed., Seismology of the Sun and the Distant Stars. Reidel, Dordrecht, p. 125
Gough D. O., 1990, in Osaki Y., Shibahashi H., eds, Progress of seismology of the sun and stars. Springer, Berlin, p. 283
Gough D. O., 1993, in Zahn J.-P., Zinn-Justin J., eds, Proc. Les Houches Session XLVII 1987. Astrophysical fluid dynamics. Elsevier, Amsterdam, p. 399
Gough D. O., Thompson M. J., 1988, in Christensen-Dalsgaard J., ed., Proc. IAU Symp. 123, Advances in helio- and astero-seismology. Reidel, Dordrecht, p. 155
Gough D. O., Thompson M. J., 1991, in Cox A. N., Livingston W. C., Matthews M. S., eds, Solar Interior and Atmosphere. Univ. of Arizona Press, Tucson, p. 519
Gough D. O., Toomre J., 1991, ARA&A, 29, 627
Libbrecht K. G., Woodard M. F., 1991, Sci, 253, 152
Libbrecht K. G., Woodard M. F., Kaufman J. M., 1990, ApJS, 74, 1129
Roxburgh I. W., Vorontsov S. V., 1994, MNRAS, 267, 297
Thompson M. J., 1988, in Rolfe E., ed., Seismology of the sun and sun-like stars. ESA SP-286, Noordwijk, p. 321
Vorontsov S. V., 1988, in Rolfe E., ed., Seismology of the sun and sun-like stars. ESA SP-286, Noordwijk, p. 475
Vorontsov S. V., 1991, AZh, 68, 808
Vorontsov S. V., 1992, AZh, 69, 347
Vorontsov S. V., Shibahashi H., 1991, PASJ, 43, 739
Vorontsov S. V., Zharkov V. N., 1989, Ap. Space Phys. Rev., 7, 1
Vorontsov S. V., Baturin V. A., Pamyatnykh A. A., 1991, Nat, 349, 49
Vorontsov S. V., Baturin V. A., Pamyatnykh A. A., 1992, MNRAS, 257, 32