# HELIOSEISMOLOGY: SOME CURRENT ISSUES CONCERNING MODEL CALIBRATION

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ABSTRACT

Aspects of helioseismic model calibration pertinent to asteroseismological inference are reviewed, with a view to establishing the uncertainties associated with some of the properties of the structure of distant stars that can be inferred from the asteroseismic data to be obtained by *Eddington*. It is shown that the seismic data to be accrued by *Eddington* will raise our ability to diagnose the structure of stars enormously, even though some previous estimates of the errors in the derived stellar parameters appear likely to have been somewhat optimistic, because the contribution from the imperfect knowledge of the underlying physics was not accounted for.

Key words: helioseismology, asteroseismology, stellar ages

## 1. INTRODUCTION

A variety of approaches will be adopted for using asteroseismic data from *Eddington*, and from the imminent space missions MOST, MONS and COROT that will precede it, to infer internal stellar properties that are inaccessible to other kind of observation. As was the case with the early helioseismological investigations, the most common procedures will be model calibrations in some form or other, because by such procedures one can progress further – although not necessarily in the right direction – than one can by using so-called data inversions. Adequate caution must be exercised when interpreting the results of such calibrations, however, for the results depend, often in a way that is difficult to quantify, on all the assumptions that have been adopted in the construction of the theoretical models. In principle, data inversions have the advantage of depending only weakly on the assumptions underlying the presumed structure of the star under investigation; but I shall not discuss them here.

In this brief presentation I report on some preliminary new work on seismic calibrations of models of the Sun. They are pertinent to the *Eddington* mission, because they point the way towards anticipating what it will be possible to extract from analyses of other stars. However, because the helioseismic data we have in hand today are better and probably more extensive than the asteroseismic data that we can reasonably expect from *Eddington*, and because we have additional information about the Sun, principally its mass and its radius, and also the distribution of magnetic activity over the visible surface (and, less precisely, over the 'backside' too) that is substantially more accurate than corresponding information about any other star, whatever we can measure from calibrating solar models is bound to be more reliable than any corresponding measurement of any other star. What I report here must therefore be viewed as an optimistic, perhaps very optimistic view of what can be achieved by using the same techniques on data from *Eddington*. I emphasize straight away that any asteroseismological conclusion one might draw from the helioseismic calibrations I report here is predicated on the techniques being the same for the two calibrations; substantial improvement may be possible using superior diagnostics. An important message that I wish to get across, therefore, is that it is incumbent upon us not only to do our utmost to ensure that the data are as good ('good' represents a compromise between suitability, quantity and precision) as possible, but also that we must make serious efforts to assess the limitations of the diagnostic procedures we have, and, perhaps more importantly, design better seismic signatures of the quantities we wish to measure.

The most primitive (spherically symmetrical) solar models depend mainly on three parameters: the main-sequence age $t_\odot$ (which is usually presumed to be known), a measure of the initial chemical composition (say the initial helium abundance $Y_0$ or the initial heavy-element abundance $Z_0$) and a parameter $\alpha$ (normally a mixing-length parameter) which controls the relation between heat flux and superadiabatic temperature gradient in the convection zone. Of course the models depend also on the mass $M_\odot$, which is known. The parameter $\alpha$ and a relation between $Y_0$ and $Z_0$ are determined by calibrating the model to the observed luminosity $L_\odot$ and radius $R_\odot$ (or, equivalently, effective temperature), which, we should recall, are known much more accurately than the corresponding values for any other star. One is left with a set of solar models depending on an abundance, say $Z_0$ or $Y_0$, and the main-sequence age $t_\odot$. I report here on trying to calibrate those models with low-degree helioseismic data – data whose stellar analogues we might obtain from *Eddington* – to determine which of them most closely represents the Sun. In so doing, one obtains an estimate of

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$t_\odot$, and a measure of the chemical composition which here I choose to be either $Z_0$ or $Z_0/(1-Y_0-Z_0)$, which I shall label jointly with $\zeta_0$.

As I describe in the next section, age influences the structure of a main-sequence star principally via the conversion of hydrogen into helium. Any robust diagnostic of age is therefore likely to be related directly to helium abundance. The heavy-element abundance $Z_0$ enters principally via the opacity. It thus relates the transport of heat to the temperature gradient via the formula for radiative transfer, assuming that there is no other form of heat transport outside the convection zone. The physics of opacity, and how it relates to chemical composition, can be regarded at best as a little uncertain. I therefore focus my discussion on calibrating the age of the solar model, although it must be realized that formally the calibration determines both $t_\odot$ and $Z_0$ simultaneously.

## 2. Main-sequence solar evolution

According to the so-called standard assumptions of solar evolution, the Sun has been evolving on the main sequence as a result of the change in chemical composition due partly, indeed principally, to the conversion of hydrogen into helium by thermonuclear reactions in the core, and, to a much lesser extent, to the gravitational settling (and potentially to radiatively driven levitation) of heavy elements relative to hydrogen. Convection in the outer envelope maintains uniform (in space) chemical composition in the zone in which it is operative, although the absolute abundances vary with time due to gravitational settling beneath, and it is assumed that no macroscopic motion takes place beneath the convection zone, implying that in the core, where by today's standards gravitational settling is negligible, the products of the nuclear reactions remain essentially in situ. The effect of the augmentation of the helium abundance in the core is to increase the central condensation of the Sun, leading to an increase of pressure, density and temperature. Except at the initial stage of the main-sequence epoch, during which there is a brief yet continuous transition from the Kelvin-Helmholtz gravitational contraction phase – in which the luminous energy flux is supplied gravitationally – to the true main-sequence phase – in which the energy is supplied by nuclear transmutation – (and except also in the final stage of the main-sequence epoch where there will be a corresponding transition in the reverse direction) the rate of diminution of the hydrogen abundance in the core is very nearly constant in time (Gough 1995). Consequently, by backward and forward extrapolation of the central hydrogen abundance (to the initial value in the first case and to zero in the second), one can define quite precisely an effective start and an effective end to the main-sequence evolution. It is then convenient, and reasonably precise, to define the main-sequence age $t_\odot$ of the Sun to be the age relative to the effective main-sequence start.

Although the entire star reacts to the changing profile of chemical composition, it is principally in the core where the structure is related directly to, say, the evolving helium abundance $Y(r,t)$; there $Y$ varies substantially with radius $r$ (cf. Fig. 1). Consequently it is the structure of the core that is the most robust indicator of age.

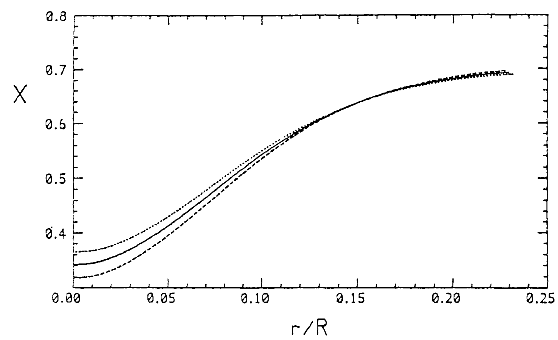


*Figure 1. Fractional hydrogen abundance $X(r,t_\odot)$ in the three solar models of Gough and Novotny (1990) calibrated to the present luminosity and radius of the Sun, all with $Z = 0.02$. The helium abundance is given by $Y = 1 - X - Z$. The ages of the models are 4.15 Gy (dotted curve), 4.60 Gy (solid) and 5.10 Gy (dashed) (from Gough and Novotny, 1990)*

## 3. Seismic diagnosis of age

Helioseismic modes are manifest as a self-interference of resonant acoustic waves: they are essentially standing waves in the acoustic cavity of the Sun. The waves associated with a particular mode of frequency $\omega$ are reflected in the surface layers, at a radius $r = r_t$, where the value of the critical frequency $\omega_c$ (essentially $c/2H$, for elementary diagnostic thinking, where $c$ is the sound speed and $H$ is the pressure scale height) attains the value of $\omega$; and they are refracted at a radius $r = r_b$ near the centre of the Sun (in the case of low-degree modes) where the characteristic acoustic angular frequency $c/r$ is equal to the reduced frequency $w = \omega/L$, where $L = l + \frac{1}{2}$ and $l$ is the degree of the mode, creating a central zone of avoidance. Thus, the waves are confined to an acoustic cavity bounded above by the level at which $\omega_c = \omega$, and bounded below by the level $r = r_b$ at which $\omega_L := Lc(r)/r = \omega$. In the Sun – or at least in standard solar models – $\omega_L$ (which is sometimes called the Lamb frequency) is a monotonic decreasing function of $r$. Consequently its influence on acoustic-wave propagation diminishes as distance from the refracting boundary increases. Similarly, the influence of $\omega_c$ diminishes with increasing depth beneath the upper reflecting boundary, because on the whole $\omega_c$ decreases with depth. Note that because $\omega_c$ is independent of $l$ the upper boundary is determined by frequency $\omega$ alone, whereas the lower boundary is determined by the value of $w$, which depends on both $\omega$ and $l$.

It is evident that the frequency of an acoustic mode depends on the locations $r = r_b$ and $r = r_t$ of the lower and upper boundaries of the resonant cavity and on the sound speed $c(r)$ between those locations. It also depends on the density profile through the Sun, through $\omega_c$ and via buoyancy and the effect of the perturbation to the gravitational potential, all of which are small far from the upper reflecting region $r \simeq r_t$ in which the effect of $\omega_c$ is paramount. Indeed, the eigenvalue equation determining $\omega$ is simply the resonance condition that an integral number of vertical wavelengths (by vertical wavelength I mean $2\pi/k_r$, where $k_r$ is the radial component of the local wavenumber $\mathbf{k}$), less a phase decrement (the physics of which I do not discuss here) fit into the region $(r_b, r_t)$.

Near the upper boundary, acoustic propagation in the radial direction depends essentially only on $\omega$, whereas near the lower boundary it depends principally on $w$. Indeed, the vertical component of the wavenumber is approximately $[(\omega^2 - \omega_c^2)/c^2 - L^2/r^2]^{\frac{1}{2}}$, in which expression it is easy to see the transition between the two extreme dependences. It is evident, therefore, that for diagnosing the structure of the solar core one needs to extract a seismic signature that depends principally on $w$, and hence on $l$ at fixed $\omega$. This is achieved by combinations of mode frequencies $\omega_{n,l}$ (where $n$ is the order of the mode) from which the pure $\omega$ dependence has been (very nearly) eliminated.

## 4. Seismic signatures of age

To proceed with the task of designing a seismic signature of age, it is useful to consult asymptotic theory. The reason – and this applies to calibration for any quantity – is that by having an approximate analytical expression for the frequency of the modes one can relatively easily construct combinations of (observable) frequencies that vary with the property of the models against which one wishes to calibrate (here, age), yet which are relatively insensitive to the other unknown parameters (here $Z_0$ or, equivalently, $Y_0$) that might otherwise degrade the calibration. In the case of observable low-degree modes, the order $n$ is high, so the appropriate asymptotic limit is $n \to \infty$ for $l = \mathrm{O}(1)$. Provided that the background stratification of the Sun is everywhere smooth with respect to the characteristic acoustic scale $k_r^{-1}$ (except at the upper turning point, of course) the cyclic eigenfrequency $\nu_{n,l} = \omega_{n,l}/2\pi$ of such a mode is given asymptotically by

$$\nu_{n,l} \sim (n + \frac{1}{2}l + \varepsilon)\nu_0 - (AL^2 - B)\nu_0^2/\nu_{n,l} \tag{1}$$

as $n \to \infty$ (Tassoul, 1980), where $\varepsilon$, $\nu_0$, $A$ and $B$ are functionals of the solar model and do not depend explicitly on either $n$ or $l$. In practice, the smoothness condition is not satisfied everywhere, such as in the He II ionization zone and at the base of the convection zone, and to the simple formula (1) should be added a component arising from abrupt transitions which oscillates with respect to $\nu$. This component will be discussed in Section 8. In the meanwhile, I simply accept formula (1).

As I have already established, it is the $l$ dependence that needs to be extracted as a signature of the core, and that is measurable through the term $AL^2\nu_0^2/\nu_{n,l}$; there is also an appearance of $l$ in the dominant first term of the approximation, but that arises simply from the spherical geometry of the Sun, and not from the details of the radial variation of sound speed and density. The asymptotic analysis yields

$$\nu_0 = \left(2\int_0^R \frac{\mathrm{d}r}{c}\right)^{-1} \tag{2}$$

(e.g. Tassoul, 1980), and

$$A\nu_0 = \frac{1}{4\pi^2}\left[\frac{c(r_t)}{r_t} - \int_{r_b}^{r_t} \frac{1}{r}\frac{\mathrm{d}c}{\mathrm{d}r}\mathrm{d}r\right] \tag{3}$$

(e.g. Gough, 1993), where $R$ is a fiducial (acoustic) radius of the Sun: it is essentially the radius at which an outward extrapolation of the square of the sound speed from the adiabatically stratified region of the convection zone (which varies almost linearly with depth below $r = R$ in the layers where $(R - r)/r \ll 1$) vanishes (e.g. Lopes and Gough, 2000; cf. Balmforth and Gough, 1990). The quantity $\nu_0$ is the inverse acoustic travel time across a solar diameter. The coefficient $A\nu_0$ is dominated by an integral of the sound-speed gradient weighted preferentially in the solar core (the term $c(r_t)/r_t$ is relatively small), and is evidently a quantity that is sensitive to age. The quantities $\varepsilon$ and $B$ are sensitive principally to the very outer layers of the Sun.

Because decrementing $n$ by 1 and augmenting $l$ by 2 leaves the (dominant) first term in the asymptotic expression unchanged (and consequently the denominator of the second term hardly changed), the frequency combination

$$d_{n,l} := \frac{3}{2l+3}(\nu_{n,l} - \nu_{n-1,l+2}) \simeq 6A\nu_0^2\nu_{n,l}^{-1} \tag{4}$$

(usually called the small separation, either with or without the factor $3/(2l+3)$) is a measure of the core signature $A\nu_0$. It is this quantity (actually some average of it over many low-degree modes) that has been the most widely used simple signature of the structure of the Sun's core. And it is this quantity that forms the basis of the calibration of the solar models that I discuss here to estimate the age of the Sun. In essence, the calibration is carried out simply by finding that theoretical model whose values of $d_{n,l}$ fit those of the Sun most closely. Note that, with the scaling factor $3/(2l+3)$, the frequency differences $d_{n,l}$ depend only weakly on $l$.

Before proceeding with a discussion of the calibration I feel obligated to make two points, the first elementary, the second not quite so evident, in order to clear an obfuscation that has been raised in the literature. I have made these points before, but they are important and can bear repetition. It has been claimed by some that the use of asymptotic analysis is inaccurate and should therefore be avoided. In support of that claim it has been pointed out that the characteristic frequency $\nu_0$, in particular, is poorly defined, because the sound speed $c$ is relatively small in the surface layers of the Sun and that (for those who are not aware of how $R$ is defined) the high sensitivity of the integral in equation (2) to the value of the upper limit of integration renders the value of $\nu_0$ very uncertain. The claim that asymptotic analysis should not be used because it is inaccurate is easily rebutted: the calibration is carried out by fitting numerically computed model eigenfrequencies, which in competent hands can be evaluated as precisely as the theoretical model; asymptotic arguments are used solely to design the observable diagnostic quantities that are used for the calibration, and to appreciate their utility, and although antagonists can rightly argue that in principle those signatures could instead have been discovered numerically, in practice they were not. That $\nu_0$ is not well defined is simply not true, although I accept that the integral defining it would be very uncertain if the value of $R$ were not accurately known. That $r = R$ is the location of a point which, seen from the propagation region beneath $r = r_{\rm t}$, appears to be a singularity of the governing wave equation (where $c^2$ appears to vanish) is a feature of the method by which the asymptotic analysis has been developed. One could equally well develop an analysis in which there appears a similar integral defining $\nu_0$ to $r = r_{\rm t}$, which is also well defined, albeit frequency dependent (Gough, 1986).

## 5. The age calibration

The idea is simply to minimize a suitably weighted sum of the squares of the differences of appropriate seismic signatures of observed solar frequencies from corresponding signatures of theoretical frequencies determined on a grid of solar models. I shall discuss in detail calibrations using only the signatures $d_{n,l}$, which are measures of the core-sensitive quantity $A\nu_0$, whose asymptotic representation is given by equations (2) and (3) but which now is regarded as no more than a parameter in the expression on the right-hand side of equation (4). The calibration is in the two-dimensional parameter space $\{t_\odot, \zeta_0\}$, where $\zeta_0$ is a measure of initial (uniform) chemical composition, which I specify below. Therefore, at least two diagnostic signatures are required, and I have chosen precisely two, namely $d_0$ and $d_1$, where $d_l$ is some average of $d_{n,l}$ over $n$, either unweighted or weighted according to the estimated observational errors. In computing that error estimate I assumed that the errors in $\nu_{n,l}$ and $\nu_{n-1,l+2}$ are independent, which is not obviously correct for at least the higher-frequency modes whose contributions to the power spectrum overlap. The average over $n$ is denoted below by angular brackets: $\langle ... \rangle$. The signatures $d_l$ were evaluated in the manner of Gough and Novotny (1990), according to the formula

$$d_l = 6A\nu_0^2 \left\langle \nu_{n,l}^{-1} \right\rangle, \tag{5}$$

where $A$ and $\nu_0$ were determined by regression of the asymptotic formula (1). Both theoretical and observed frequencies were processed in precisely the same way. The calibration for $t_\odot$ and $\zeta_0$ was then accomplished by equating theoretical signatures $d_0$ and $d_1$ obtained by linear interpolation in a grid of solar models to the corresponding observed (or, for testing purposes, proxy) signatures, using various subsets of the frequencies.

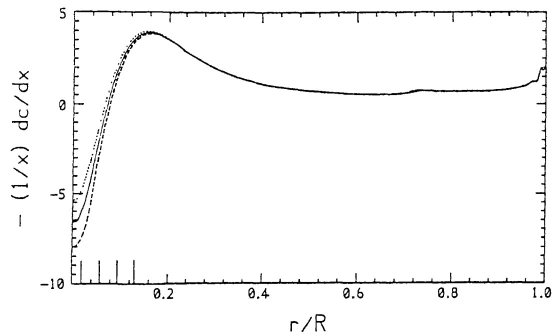


*Figure 2. Functional form of the integrand in the expression (3) for $A\nu_0$, for the three solar models illustrated in Fig. 1, identified with the same linestyles; $x = r/R$ and the units of $c$ are Mm s$^{-1}$. The ticks on the abscissa scale indicate $r_{\rm t}$ for 3-mHz modes with, in increasing order, $l = 0,\ 1,\ 2$ and 3 (from Gough and Novotny, 1990)*

The functional form $x^{-1}{\rm d}c/{\rm d}x$ of the integrand in equation (3) relating $d_l$ to the structure of the Sun, where $x = r/R$, is depicted in Fig. 2 for the same three models as in Fig. 1. It illustrates the sensitivity of the small separation to the stratification of the star. As expected, the magnitude of the integrand is greatest in the energy-generating core. Moreover, its dependence on the age of the model is essentially confined to the core too. The only other place the age sensitivity is (barely) visible in the figure is near the base of the convection zone, the differences between the models arising mainly because the convection zones have somewhat different depths. Corresponding integrands for models with different chemical composition $\zeta_0$ are superficially similar, except that the sensitivity to $\zeta_0$ extends throughout the radiative interior. Elimination of the effects of the structure outside the core is achieved by taking a linear combination of $d_0$ and $d_1$, which, not surprisingly, requires taking a signature that is almost the difference between them. Because the variation with respect to $t_\odot$ is rather similar for the two quantities, there is only little sensitivity to $t_\odot$ remaining in that linear combination, which, unfortunately, implies that the errors in the calibration are magnified substantially relative to the raw errors themselves.

In an attempt to obviate this problem one could consider including the parameter $\nu_0$ explicitly in the calibration, for that weights the central core and the outer envelope quite differently. Such an inclusion was proposed by Christensen-Dalsgaard (1986) for asteroseismic calibrations, and was discussed by Gough and Novotny (1990) in the case of the Sun. Unfortunately, however, the value of $\nu_0$ depends quite strongly on the uncertain outermost layers of the Sun – the turbulent transition layer between the almost adiabatically stratified interior of the convection zone and the atmosphere, and also the horizontally varying nonadiabatic atmospheric layers themselves – which adds more uncertainty to the calibration from not knowing the physics than it removes by increasing the precision that is achieved by reducing the formal random errors.

I have already reported (Gough, 2001) the results of a calibration of solar multiplet frequency data from modes of degrees 0 – 3 obtained from Doppler observations by BiSON (Chaplin et al., 1999). The grid of 9 theoretical models computed by Gough and Novotny (1990) was used to determine the partial derivatives of the seismic signatures with respect to $t_\odot$ and $\zeta_0$, and these were used to calibrate the distance in the parameter space between the Sun and the more modern (relative to the grid) solar model S of Christensen-Dalsgaard et al. (1996). Both $\zeta_0 = Z_0$ and $\zeta_0 = Z_0/X_0$ have been used for the measure of chemical composition, where $X_0 = 1-Y_0-Z_0$ is the initial hydrogen abundance; there was no significant difference in the results obtained using the two different definitions. However, the results were found to be quite sensitive to the mode set that had been adopted for the calibration. Evidently there is either something wrong with the theoretical models or else something systematically wrong with the published frequencies. To select an optimal result a combination of criteria was adopted, requiring that the formal error in the fit be relatively small, that the number of modes used was not too small, that the mode set included modes that penetrate deeply into the core, and that the set did not contain modes of frequency so high that they could be influenced appreciably by magnetic activity, which, through degeneracy splitting, could bias the estimate from the observations of the mean multiplet frequencies of nonradial modes.

To keep the number of calibrations down to a manageable value, only sets of contiguous modes in the $(n, l)$ space were considered. The outcome was

$$t_\odot = 4.57 \pm 0.12 \text{ Gy} \tag{6}$$

and, after converting (theoretically) from initial chemical abundances to present-day surface abundances (indicated by the subscript s),

$$Z_\mathrm{s}/X_\mathrm{s} = 0.032 \pm 0.004. \tag{7}$$

The standard errors are purely formal, obtained by propagating observational error estimates through the calibration process (assuming the errors to be small so that linearization is valid) and taking no account of any bias introduced by the post-hoc selection of only a subset of the modes available. Those errors are substantially smaller than the range of solutions that were obtained as the mode set was varied. Moreover, the discrepancy between the calibrated value (7) of $Z_\mathrm{s}/X_\mathrm{s}$ and the observed value of 0.023, with an uncertainty that 'might be of order 10%', quoted by Grevesse and Sauval (1998) is perhaps uncomfortably large. Therefore there must be substantial uncertainty in the result. It is interesting, however, that the mode set that was preferred by the criteria that had been adopted was found to contain only modes in the vicinity of a local minimum with respect to frequency (near 3 mHz) in estimated observational errors; it did not include the lowest-frequency modes which have the lowest quoted errors, however, perhaps as a consequence of having restricted consideration to sets of contiguous modes only, which requires including modes near a local error maximum when the set containing modes with errors in the basin including the minimum near 3 mHz is augmented with the lowest-frequency modes.

As I pointed out above, there is a considerable degree of cancellation in the linear combination of mean small differences $d_l$ that determines the solar age. For the preferred mode set reported by Gough (2001), the difference $\delta t_\odot$ between the age of the Sun and that of the reference model is given by $\delta \ln t_\odot = -0.75\delta d_0 + 0.66\delta d_1$, where $\delta d_l$ are the corresponding differences in $d_l$. The value of the frequency combination is only about 13% of the mean of the values of $d_0$ and $d_1$, which implies that the standard error in the signature of $\delta \ln t_\odot$ is some eight times greater than the standard errors in the observed values of $d_0$ and $d_1$. Note that all the asteroseismic spacecraft that have been proposed are to make measurements of oscillations in the radiative intensity, rather than Doppler spectrum line shifts, and will therefore be insensitive to modes with $l \geq 3$. Although that does not preclude a measurement of $d_1$, the asteroseismic calibrations must rely on fewer modes than do their helioseismic counterparts, and will therefore average out random data errors less thoroughly.

## 6. Effect of data errors: consistency of the calibration

As a test of the calibration procedure (in which mode frequencies are processed nonlinearly), models of the grid were regarded one-by-one as proxy Suns. Errors were added to the frequencies of each proxy, which was then calibrated against the central model of the grid (rather than Model S which had been used as the standard for the solar calibration). The errors in the frequencies were independent Gaussian-distributed random numbers, each with standard deviations equal to the published error estimates in the BiSON data. Different errors were added to different proxy models. The results of a particular set of realizations are illustrated in Fig. 3, in which the calibrated values $t_{\odot\,\mathrm{cal}}$ and $Z_{\mathrm{cal}}$ of $t_\odot$ and $Z_0$ are plotted against the true values. The mode set adopted in each case was the same as that used in the solar calibration reported by Gough (2001) and discussed in the previous section. The age calibrations are plausibly distributed about the true values; the abundance calibrations, on the other hand, which have much larger relative errors, appear to be systematically low. When more extensive mode sets were used, the errors were naturally greater (and the estimates of $Z_0$ were not always low). All-in-all, I saw no convincing evidence that the results deviated from the correct values by amounts significantly different from statistical expectation.

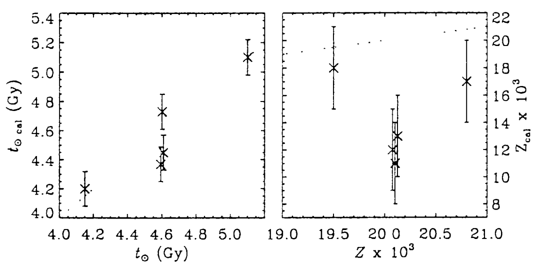


*Figure 3. Calibration, against the central model, of five of the models of the grid. The mode set used was the same as that used for the solar model. Independent Gaussian-distributed errors with standard deviations equal to the standard errors in the BiSON data quoted by Chaplin et al. (1999) were added to the eigenfrequencies of each proxy Sun. The vertical bars represent the standard errors in the calibrations associated with those data errors.*

## 7. A preliminary investigation into the influence of errors in the models

As a preliminary test of the influence of errors in the theoretical solar models, Model S was calibrated against the grid, using accurately computed eigenfrequencies to which no errors had been added. This is equivalent to calibrating the central model of the grid against Model S, and so it is equivalent also to an error-free test of the solar calibration using the central model of the grid as a proxy Sun. The result is an erroneous estimate of the age and chemical composition of the proxy. The magnitudes of the errors are not dissimilar to the variation with mode set of the results of the solar calibrations reported in some detail by Gough (2001). Moreover, the relative errors in $\zeta_0$ are very much greater than those in $t_\odot$. My tentative inference, therefore, is that the spread in the results of the solar calibration could be due principally to errors in the physics of Model S, and, probably to a lesser extent, to errors in the models in the grid (which are used solely for computing derivatives with respect to the calibration parameters $t_\odot$ and $\zeta_0$), and not necessarily to errors in the data. Indeed, I cannot even say with confidence that the solar values of $t_\odot$ and $\zeta_0$ differ significantly from those defining either Model S or the central model of the grid. The central model of the grid, in common with the other models of the grid, was constructed with different opacities and somewhat different nuclear reaction rates from those used in constructing Model S, and, unlike Model S, took no account of gravitational settling of heavy elements. Although one might argue on physical grounds that Model S is a more faithful representation of the Sun than is the central model of the grid, because in the years between the construction of those models there were significant improvements to the physics – and indeed the helioseismological inferences using modes of intermediate and high degree support that point of view – we cannot be sure that Model S is sufficiently accurate to distinguish between its presumed age and the age of the Sun. Results from the calibration of the heavy-element abundance, to which the models are more sensitive, are even less secure. Therefore at present one must conclude that the uncertainty in the age and chemical composition arising from the uncertainty in the theoretical modelling might well exceed the uncertainty resulting from errors in the frequency data, and certainly exceeds the random errors.

## 8. An attempt to measure the helium abundance in the convection zone

Abrupt variation in the stratification of the Sun (relative to the scale of the inverse radial wavenumber $k_r^{-1}$) such as that produced by helium ionization or by the sharp transition from radiative to convective heat transport at the base of the convection zone, induces small oscillatory components (with respect to frequency) in the eigenfrequencies of oscillation. This complicates the evaluation of the smooth component (pertinent to the asymptotic formula (1)) of the derivatives of the seismic signatures obtained by interpolation in the grid of solar models. However, one might wonder whether the variation of the sound speed induced by helium ionization might be used to advantage for determining the solar helium abundance from the low-degree eigenfrequencies, leaving then both $d_0$ and $d_1$ to combine constructively, rather than almost cancel, in determining the age.

A convenient and easily evaluated measure of the oscillatory component is the second difference

$$\Delta_2(\nu_{n,l}) := \nu_{n-1,l} - 2\nu_{n,l} + \nu_{n+1,l} \tag{8}$$

with respect to $n$ (Gough, 1990). Any localized region of rapid variation of either the sound speed or a spatial derivative of it, which here I call an acoustic glitch, induces an oscillatory component in $\Delta_2(\nu)$ with a 'cyclic frequency' approximately equal to twice the acoustic depth

$\tau = \int c^{-1}\mathrm{d}r$ of the glitch, and with an amplitude which depends on the amplitude of the glitch and which decays with $\nu$ once the inverse radial wavenumber of the mode becomes comparable with the radial extent of the glitch. By calibrating a theoretical representation of the effect of glitches against the observations one can learn, in principle, about the characteristics of the glitches. In Fig. 4 I present such a calibrated theoretical second difference, obtained from a pair of oscillatory components to the eigenfunctions arising from glitches associated both with HeII ionization and with the near discontinuity at the base of the convection zone. Because only low-$l$ modes are used, the $l$ dependence can safely be ignored, even down to the base of the convection zone. The acoustic glitch induced by helium ionization was represented by a Gaussian function about the acoustic depth $\tau = \tau_0$ :

$$\frac{\delta c^2}{c^2} \simeq \frac{\delta\gamma_1}{\gamma_1} \simeq -\frac{\Gamma_1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-(\tau-\tau_0)^2/2\sigma^2} \quad , \tag{9}$$

where $\gamma_1$ is the first adiabatic exponent and

$$\tau(r) = \int_r^R \frac{dr}{c} \tag{10}$$

is acoustic depth. The amplitude $\Gamma_1$ and width $\sigma$ are constants, which, together with $\tau_0$, are to be calibrated against the eigenfrequencies. Since ionization reduces $\gamma_1$, the negative sign in the representation (9) ensures that $\Gamma_1$ is positive; one would expect it to be proportional to the helium abundance $Y$ in the convection zone. The acoustic glitch at the base of the convection zone is essentially a discontinuity in the second derivative of density, arising from a discontinuity in the second derivative of temperature caused by the change in the mode of energy transport. Local mixing-length theory with a mixing length that does not vanish at the boundaries of the convection zone – actually proportional to the local pressure scale height – was used in the construction of the theoretical models, leading to a true discontinuity in the second derivatives; in the Sun the transition is smoother but probably on a spatial scale much less than $k_r^{-1}$ for all the frequencies observed, and so acts on the acoustic wave in the manner of a discontinuity. The combined effect of the two glitches is to impart on the eigenfrequencies represented asymptotically by equation (1) an additional, oscillatory component, $\delta\nu$, which can be shown to be given approximately by

$$\delta\nu \sim \pi S^{-1}[\ \Gamma_1 \nu e^{-8\pi^2\delta^2\nu^2}\cos 4\pi(\tau_0\nu + \epsilon) + A_c\nu^{-1}\cos 4\pi(\tau_c\nu + \epsilon)] \quad , \tag{11}$$

where $\epsilon$ is a phase constant which depends on the stratification of the Sun in the vicinity of the upper turning point (and is related to the phase constant $\varepsilon$ appearing in equation (1)), $\tau_c := \tau(r_c)$ is the acoustic depth of the base of the convection zone, and where

$$A_c \simeq -\frac{1}{4}c_c^2\left(\int_{\tau_c}^{\tau_b}(1-\frac{\omega_L^2}{\omega^2})^{3/2}\mathrm{d}\tau\right)\left[\frac{\mathrm{d}^2\ln\rho}{\mathrm{d}r^2}\right]_{r_c-}^{r_c+} \quad , \tag{12}$$

in which $\tau_b = \tau(r_b)$, $c_c = c(r_c)$, $\rho$ is density and

$$S \simeq \int_0^{\tau_b}(1-\frac{\omega_L^2}{\omega^2})^{-1/2}\mathrm{d}\tau \tag{13}$$

is proportional to the mode inertia. The amplitude $A_c$ is a measure of the discontinuity of the second derivative of $\rho$ at the base of the convection zone. Once again a minus sign has been included in the definition (12) to render the value of the amplitude positive.

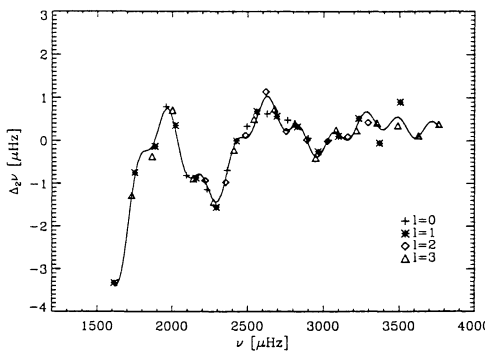


*Figure 4. The symbols are second differences $\Delta_2$, defined by equation (8), of low-degree solar frequencies obtained from MDI. The curve is the second derivative with respect to n of the formula (11) for $\delta\nu$ plus a third-degree polynomial whose parameters $\tau_0$, $\tau_c$, $\Gamma_1$, $\sigma$, $A_c$ and $\epsilon$ have been adjusted to fit the curve to the data by least squares.*

The coefficient $A_c$ and the difference (here zero) between the phase terms $\epsilon$ occurring in the first and the second terms on the right-hand side of equation (11) depend on the nature of the transition between the radiative and the essentially adiabatic stratification. Together with $\tau_c$, the values of these parameters are also calibrated against the data. The formula (11) would be different in the presence of overshoot, for example, and indeed attempts to detect overshooting by seeking what is tantamount to an appropriate deviation from formula (11) have been made (e.g. Berthomieu et al., 1993; Gough and Sekii, 1993; Monteiro, Christensen-Dalsgaard and Thompson, 1993), with negative results. Indeed, if the formula (11), modified by permitting the two phase terms to differ, is fitted to the MDI data plotted in Fig. 4, the resulting two calibrated values of those phase terms are found not to differ significantly. These negative results have been used to set upper limits to the degree of overshooting beneath the unstably stratified region of the solar convection zone. Making that limit tight requires including modes of degree higher than can be detected from whole-disc observations alone, however, so I shall refrain here from discussing the matter further.

The simple theoretical representation shown as a continuous curve in Fig. 4 was obtained by regarding $n$ to be a continuous variable and differentiating the expression (11) twice with respect to $n$, relating $n$ to $\nu$ via the leading term in equation (1), and adding a third-degree polynomial in $\nu$ (to model the smooth trend). The result was calibrated against frequency data obtained from a 360-day MDI time series and kindly supplied by J. Schou (cf. Schou, 1999) from modes with $0 \leq l \leq 3$ (indicated by symbols). Can one from such a fit determine the helium abundance in the convection zone?

In a first attempt to answer that question, the theoretical curve was fitted to sets of frequencies computed from the solar models of the grid of Gough and Novotny (1990). Independent Gaussian distributed random errors with standard deviations equal to the standard errors quoted by Chaplin et al. (1999) were added to the data, which were then given without identification in random order to Günter Houdek to fit the theoretical curve. Unfortunately, the trends that were hoped for could not be established incontrovertibly. The fit cannot be perfect (as one can establish by attempting to fit the curve to precisely computed theoretical eigenfrequencies of a solar model), and some of the difficulty in evaluating the parameters with sufficient accuracy arises from the inability to fit well the low-frequency and the high-frequency ends of the curve simultaneously. We suspect that the reason for this at least in part is that the sound travel time along a ray is not given precisely by the acoustic depth, but is modified by the influence of the acoustical cutoff frequency $\omega_c$; because the degree of every mode considered is low, the ray paths are essentially vertical (the horizontal component $L/r$ of the wavenumber is negligible compared with the vertical component $k_r$), so it should be adequate to replace $\tau$ by $\int(1 - \omega_c^2/\omega^2)^{-1/2}c^{-1}\mathrm{d}r$. The abscissa scale in Fig. 4 would therefore be stretched differentially. Without so doing the parameters specifying the best-fitting curve may depend too sensitively on the frequency range to establish the trends in the properties of the acoustic glitches. Alternatively, it may simply be that the 4% range in $Y$ spanned by the grid is insufficient to be detected unambiguously by analysing frequencies with errors as great as the uncertainties in the BiSON data. Work in collaboration with Houdek to improve the fitting formula is under way. It might be prudent instead to separate the signature of He II ionization differently, such as by a procedure based on the analyses of Pérez Hernández & Christensen-Dalsgaard (1994) or Roxburgh & Vorontsov (1994). But in the meanwhile we must conclude that the simple model calibrations that have been accomplished so far are substantially less accurate than desired.

## 9. Discussion

There are several aspects of the calibrations reported here that would benefit from more sophisticated analysis. An obvious example is in the interpretation of the signatures based on the observed oscillation frequencies that have been used for the calibration. The dependence of $d_l$ on the structure of the solar interior, illustrated in Fig. 2, would be correct only if the Sun were spherically symmetrical, which it is not. Latitudinally dependent magnetic activity splits the degeneracy of the nonradial modes with respect to azimuthal order $m$, which could modify our inferences concerning the mean multiplet frequencies by an amount that near sunspot maximum might be comparable with the differences $\delta d_l$ between the values of the small frequency separations associated with modes of the Sun and those of modes of the theoretical reference solar model used for the calibrations. To be sure, the calibrations reported here have excluded the highest-frequency modes, which are most susceptible to surface activity. But is that sufficient? It is a relatively straightforward matter to make an approximate correction to the frequencies that have been reported from the observed distribution with latitude of long-lived activity, using either the relation between activity and frequency of high-degree modes reported by Woodard et al. (1991), Bachmann and Brown (1993) and Howe et al. (1999), or the phase perturbation on reflection at the upper surface of the acoustic cavity inferred from the time-distance studies of Duvall (1995). That is a task that will certainly be accomplished in the near future. It should make the solar calibration more reliable, although it should be borne in mind that a similar correction is unlikely to be available to asteroseismologists in the immediate future.

On the theoretical side, due account of the oscillatory component of the frequency associated with helium ionization and the abrupt change in stratification at the base of the convection zone must be taken. A study of that is currently under way by Houdek and myself. It is likely that when the oscillatory component is removed from the frequencies, the derivatives of the smooth residual across a grid will be more robust: less sensitive to the precise parameters defining the models and less sensitive to the mode set adopted.

More direct signatures of the amount of hydrogen that has been burned are also likely to lead to a more accurate calibration. Gough and Kosovichev (1993) have presented mode combinations that measure the density distribution in the core, which is closely related, via the assumptions of stellar evolution theory, to the distribution of helium. It must be realized, of course, that interpreting that distribution in terms of solar age is dependent on having correctly taken into account any movement of chemical species, by gravitational settling opposed, to some degree, by radiative levitation, and by macroscopic motion in the form of convection, turbulence generated by fluid instabilities, or waves. It is also dependent on any mass loss or accretion, which is normally assumed to be negligible.

The present status is that by using a six-month-long almost continuous data set from BiSON a calibrated age can be obtained with a formal uncertainty from random data errors of about 2.5%. In the *Eddington* study it was found that the formal error in the age derived from the propagation of uncertainties in the frequency data is some 25 times smaller, even with the shorter proposed observation time in the first (asteroseismological) phase of the mission and notwithstanding the fact that the mass, luminosity and radius of the Sun are known much more accurately than is likely to be the case for most other stars. The investigation reported here suggests, therefore, that the uncertainty in the asteroseismically derived parameters will be dominated by the imperfect knowledge of the physics, and not by statistical error in the data. Solar models have been honed, in the light of helioseismology and in the quest to explain the observed neutrino fluxes. No other star has been studied so assiduously. It is therefore unlikely that asteroseismic calibrations, at least initially, will be as good as their helioseismic counterparts, even when many stars in a cluster are calibrated simultaneously. Nevertheless, even granted the limitations on the accuracy imposed by the incomplete knowledge of the underlying physics, the results from the imminent asteroseismological space missions will indeed raise our knowledge of the structure and evolution of stars to completely new heights.

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## REFERENCES

Bachmann, K. T. & Brown, T. M. 1993, ApJ, 411, L45

Balmforth, N.J. & Gough, D.O. 1990, ApJ, 362, 256

Berthomieu, G., Morel, P., Provost, J. & Zahn, J.-P. 1993, in *Inside the stars; Proc. IAU Colloq. 137*, ed W.W. Weiss & A. Baglin (ASP Conf. Ser. 40), p.60

Chaplin, W. J., Elsworth, Y., Isaak, G.R., Miller, B.A. & New, R. 1999, MNRAS, 308, 424

Christensen-Dalsgaard, J. 1986, Proc. IAU Symp. 123, 295

Christensen-Dalsgaard, J. et al. 1996, Science, 272, 1286

Duvall, Jr, T.L. 1995, in *GONG '94: Helio- and Asteroseismology from the Earth and space*, ed. R.K. Ulrich, E.J. Rhodes Jr & W. Däppen, *ASP Conf. Ser.*, 76, 465

Gough, D.O. 1986, in *Hydrodynamic and magnetohydrodynamic problems in the Sun and stars*, ed. Y. Osaki (Univ. Tokyo Press), p.117

Gough, D. O. 1995, in *GONG '94: Helio- and Asteroseismology from the Earth and Space*, ed R.K. Ulrich, E.J. Rhodes, Jr. & W. Däppen (San Francisco: ASP), 551

Gough, D. O. 1993, in *Astrophysical fluid dynamics*, ed J-P. Zahn & J. Zinn-Justin (Amsterdam: Elsevier), 399

Gough, D.O. 2001, in *Astrophysical Ages and Timescales*, ed T. von Hippel, N. Manset & C. Simpson (ASP Conf. Ser.), in press

Gough, D.O. & Kosovichev, A.G., 1993, in *Inside the Stars, Proc IAU Colloq.*, 137, ed. W. W. Weiss & A. Baglin, *ASP Conf. Ser.*, 40, 541

Gough, D. O. & Novotny, E. 1990, Sol. Phys., 128, 143

Gough, D.O. & Sekii, T. 1993, in *GONG 1992: Seismic investigation of the Sun and stars*, (ASP Conf. Ser.), 42, p.177

Grevesse, N. & Sauval, A. J. 1998, Space Sci. Rev., 85, 161

Howe, R., Komm, R. & Hill, F. 1999, ApJ, 524, 1084

Lopes, I.P. & Gough, D. O. 2001, MNRAS, 322, 473

Monteiro, M.J.P.F.G., Christensen-Dalsgaard, J. & Thompson, M.J. 1993, in *GONG 1992: Seismic investigation of the Sun and stars*, (ASP Conf. Ser.), 42, p. 253

Pérez-Hernández, F. & Christensen-Dalsgaard, J. 1994, MNRAS, 269, 475

Roxburgh, I. W. & Vorontsov, S. V. 1994, MNRAS, 268, 880

Schou, J. 1999 ApJL, 523, 181

Tassoul, M. 1980, ApJS, 43, 469

Woodard, M. F., Kuhn, J. R., Murray, N. & Libbrecht, K. G. ApJ, 373, L81