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Inverse Problems of Solar Oscillations

Comments on Helioseismic Inference

Douglas Gough

Institute of Astronomy and
Department of Applied Mathematics and Theoretical Physics,
University of Cambridge, Madingley Road, U.K.
and
Institute for Theoretical Physics, University of California,
Santa Barbara, CA 93106, U.S.A.

Abstract: Helioseismic inference can be made within a wide spectrum of sophistication, from arguments based on the results of very simple and highly idealized model problems which depend on specific limited aspects of the data to a variety of formal numerical inversions of all the data that are available. The idealized problems are relatively simple to analyze, and provide a tool for making immediate qualitative and sometimes even quantitative estimates of certain aspects of the structure of the sun. If well chosen, they are likely to add substantially to our understanding of the situation; indeed, they can be an extremely useful guide to designing the more formal techniques which, though numerically more precise, are frequently also more opaque. Therefore it is often prudent to utilize methods throughout the entire spectrum. In this lecture a selection of the techniques for making immediate inferences will be discussed, and illustrated with examples of topical interest.

1 Introduction

Simple models offer an easy way to obtain a qualitative sense of the behaviour of the oscillations of more realistic and therefore more complicated models of stars. Provided they incorporate the essence of the physics one wishes to describe, they are extremely useful tools for any preliminary investigation. The understanding they provide enables one to appreciate the generality of the qualitative conclusions one may derive from them, particularly when addressing some new physical issue for the first time. Thus they are an invaluable complement to detailed numerical models, which provide but isolated, albeit more accurate examples of the phenomena under investigation.

In this lecture I shall consider two simple physical models, which I shall introduce at the outset. One is spherical, but unstratified, and therefore incorporates the geometry of the star. The other is plane, but stratified. Experience of the two provides an extremely useful introduction to any study of stellar oscillations.

My main motivation is solar diagnostics, and therefore I shall concentrate on adiabatic acoustic modes. I consider only the frequency spectrum, and shall not discuss excitation and damping. I shall start by considering the basic properties of the frequency spectra of my two models, and then relate them to the asymptotic properties of the oscillations of a real star. The ideas are particularly useful for considering the influence of small perturbations, and so can be used to anticipate the results of inversions of frequency data, degeneracy splitting and temporal variations. I shall also use a variational principle, coupled with asymptotic argument, to discuss an oscillatory property of the distribution of oscillation eigenfrequencies that might be used to determine the locations of the edges of convection zones in stars, and which might possibly also provide a seismic calibration of the helium abundance. Finally, I shall apply the ideas that I shall have developed to discuss what can be inferred about the new solar-cycle variations reported at this seminar by Libbrecht and Woodard, Pallé *et al.* and by Jefferies and his colleagues. It is quite evident that there is an acoustically sensitive variation taking place in the very outer layers of the sun, but what that variation is is at present a mystery.

2 Simple Models

I shall discuss two simple models: the isothermal sphere and the complete plane-parallel polytrope. Both are useful for illustrating some of the more basic properties of stellar oscillations. They can each be made more elaborate to address more subtle issues.

2.1 The Isothermal Sphere

Some of the most basic properties of high-order stellar p modes of low and intermediate degree come about because the star is essentially spherical. It is therefore sometimes useful to represent the star by the simplest of spherical models: an isothermal sphere of perfect gas with radius R in the absence of gravity. Its adiabatic acoustic oscillations have been discussed by Rayleigh (1896). A scalar wave variable ψ, such as the pressure perturbation (because the basic state is uniform there is no difference in linear theory between Lagrangian and Eulerian perturbations), satisfies the simple wave equation

$$\nabla^2\psi + \frac{\omega^2}{c^2}\psi = \nabla \ln \rho \cdot \nabla\psi = 0, \tag{2.1}$$

where ω is the frequency of oscillation, ρ is the (constant) density of the basic state and c is the adiabatic sound speed. This equation has separable solutions of the form

$$\psi = r^{-1}\Psi(r)P_l^m(\cos\theta)\cos m\phi\cos\omega t \tag{2.2}$$

with respect to spherical polar coordinates (r, θ, ϕ), where t is time and P_l^m is the associated Legendre function (of the first kind) of degree l and order m. The radial eigenfunction $\Psi(r)$ satisfies

$$\Psi'' + \left(\frac{\omega^2}{c^2} - \frac{L^2-\frac{1}{4}}{r^2}\right)\Psi = 0, \tag{2.3}$$

where $L = l + \frac{1}{2}$ and a prime denotes differentiation with respect to the argument, and can thus be expressed in terms of a Bessel function of the first kind:

$$\Psi = r^{\frac{1}{2}} J_L\left(\frac{\omega r}{c}\right). \tag{2.4}$$

If pressure is presumed to be constant at the outer boundary, which is a good first approximation for a star, then

$$\omega = j_{n,L}\pi^{-1}\omega_0, \tag{2.5}$$

where

$$\omega_0 = \pi c/R \tag{2.6}$$

and $j_{n,L}$ is the nth zero of the Bessel function J_L. In particular, for low-degree modes satisfying $l \ll n$,

$$\omega \sim \left(n + \frac{1}{2}l + \epsilon\right)\omega_0 + [Al(l+1) - B]\omega_0^2/\omega + \ldots, \tag{2.7}$$

where $\epsilon = 0, A = (2\pi^2)^{-1}$ and $B = 0$. By varying the boundary condition at $r = R$ one finds that of the coefficients in the asymptotic expression (2.7) only ϵ and B vary; they depend on the phase jump suffered by a wave on reflection at the surface of the sphere. The quantity A is invariant; it depends on the geometry of the waves near the centre of the star, though this is more easily seen by expressing the solution of Eq. (2.1) as a superposition of plane waves (*cf.* Keller and Rubinow, 1960).

As is well known, the basic structure of the asymptotic expression (2.7) for the eigenfrequencies ω is preserved even when the sphere is stratified under gravity. This was first demonstrated by Tassoul (1980) and was discussed by Shibahashi in the introduction to this seminar. I shall not yet venture further into consideration of more realistic stellar models; I mention Tassoul's result here simply to establish that there is a practical domain of validity of my simple model.

One of the reasons for being interested in so simple a model is that one can quite easily consider what happens to the eigenfrequencies if a perturbation to the simple isothermal sphere is imposed. Suppose, for example, the core of the sun were to have been mixed. What would be the influence on the eigenfrequencies? To get an impression of the general kind of change that would take place one first asks what the effect of the mixing would be. Broadly speaking, it would modify the density ρ and the sound speed c in the core, and produce steep gradients in ρ and c at the interface between the mixed core and its unmixed surroundings. Let us consider its implications.

Because the equilibrium pressure is uniform one can write

$$\rho = \rho_0[1 - \epsilon f(r)], \quad \frac{1}{c^2} = \frac{1}{c_0^2}[1 - \epsilon f(r)] \tag{2.8}$$

and if ϵ [which here is unrelated to the phase factor ϵ appearing in Eq. (2.7)] is small one need consider only perturbations linear in ϵ. Thus one expresses ψ and ω as the sums of zero-order solutions, which I now call ψ_0 and ω_0, obtained from expressions (2.2), (2.4) and (2.5) with c replaced by the unperturbed sound speed c_0, plus small perturbations ψ_1 and ω_1. The perturbation equation, neglecting terms quadratic in ϵ, is then

$$\nabla^2\psi_1 + \frac{\omega_0^2}{c_0^2}\psi_1 = -\frac{\omega_0^2}{c_0^2}\left(2\frac{\omega_1}{\omega_0} - f\right)\psi_0 - \frac{df}{dr}\frac{\partial\psi_0}{\partial r} \tag{2.9}$$

in which the solubility condition [writing ω_0 as ω to avoid confusion with ω_0 of Eq. (2.6)]

$$\omega_1 = \frac{1}{2}\omega \frac{\int \left\{\psi_0^2 + \frac{c_0^2}{2\omega^2}\nabla^2\psi_0^2\right\} f dV}{\int \psi_0^2 dV} \tag{2.10}$$

$$= \frac{\omega \int_0^R \left\{\Psi^2 + \frac{c_0^2}{\omega^2}\frac{d}{dr}\left[r\Psi\frac{d}{dr}\left(r^{-1}\Psi\right)\right]\right\} f(r) dr}{2\int_0^R \Psi^2 dr} \tag{2.11}$$

must be satisfied. In Eq. (2.10) the integrals are over the volume V of the star. Thus we have been able to write down the form of the perturbed eigenfrequency in terms of an arbitrary small spherically symmetrical perturbation to the sound speed. Although this expression cannot be used to make precise quantitative estimates of the frequency perturbation, it can give one a good idea of the general form the perturbation must take: how it depends on n and l. Kosovichev and I used this model in a preliminary investigation of the influence of core mixing in an attempt to explain an anomaly that appeared to have been exhibited in some observations of relatively low-frequency modes with $l = 5$ (Gough and Kosovichev, 1988). Its advantage over the usual asymptotic analyses of more realistic models is that it is valid for low-order modes. What is important is that it does not require the scale of variation of the sound-speed perturbation to be much greater than the characteristic wavelength of the mode, and can therefore be applied to the core. However, it does, of course, require the magnitude of the perturbation to be small. Under some circumstances asymptotic expressions for eigenfunctions of a more realistic solar model can be used in conjunction with perturbation integrals such as those in Eq. (2.10) to obtain more accurate estimates of frequency changes. I shall employ such a hybrid method later to discuss small-scale structure in the outer layers of the sun.

One can also use the isothermal model to study perturbations that are not necessarily small. The price one must pay to retain analytical simplicity, however, is a much tighter restriction on the functional form of the sound-speed variation. If, for example, $c = c_0$ in the envelope: $a < r \leq R$, and $c = c_1$ in the core: $r < a$, then in the core Ψ is given by expression (2.4) with $c = c_1$ but in the envelope there are contributions from Bessel functions of both kinds:

$$\Psi = r^{\frac{1}{2}} J_L\left(\frac{\omega a}{c_1}\right) \frac{J_L\left(\frac{\omega r}{c_0}\right) + K Y_L\left(\frac{\omega r}{c_0}\right)}{J_L\left(\frac{\omega a}{c_0}\right) + K Y_L\left(\frac{\omega a}{c_0}\right)}, \qquad a \leq r \leq R \tag{2.12}$$

for some constant K. I have written the solution in such a way as to ensure continuity of the pressure fluctuation ψ at $r = a$. It is now a straightforward matter to determine K in terms of ω by demanding continuity of the vertical component ξ of the displacement, using the vertical component of the momentum equation to relate ξ to ψ. The eigenvalue ω can then be obtained, as before, by applying the boundary condition at $r = R$. The expression for ω is a little cumbersome, so I refrain from presenting it explicitly. Needless to say, its value reduces to that obtained from Eq. (2.11) when $|c_1/c_0 - 1|$ is small, even though the function Y_L does not appear explicitly in the solubility condition. The perturbed radial eigenfunction ψ_1 which satisfies Eq. (2.9) does, of course, contain the appropriate contribution from Y_L.

In Fig. 1 I illustrate the relative perturbation ω_1/ω to the frequencies ω of several low-degree modes arising from a 10-per-cent augmentation of the sound speed in a core $r < a = \lambda R$ where $\lambda = 0.25$. The results are plotted against $\alpha \equiv \omega R/\pi c_0$ which is approximately $n + \frac{1}{2}l$ when $n \gg l$; it is related to the lower turning point r_t, defined by Eq. (6.3), according to $r_t/R = L/\pi\alpha$. For $r < r_t$ the eigenfunction is evanescent; consequently if $a < r_t$ the perturbation to ω is relatively small. When $a > r_t$, the perturbation is approximately proportional to $(a-r_t)/(R-r_t) \to \lambda = 0.25$ as $\alpha \to \infty$, as one might expect, upon which is an oscillatory contribution associated with the oscillatory structure of the eigenfunction experiencing the discontinuity in the sound speed.

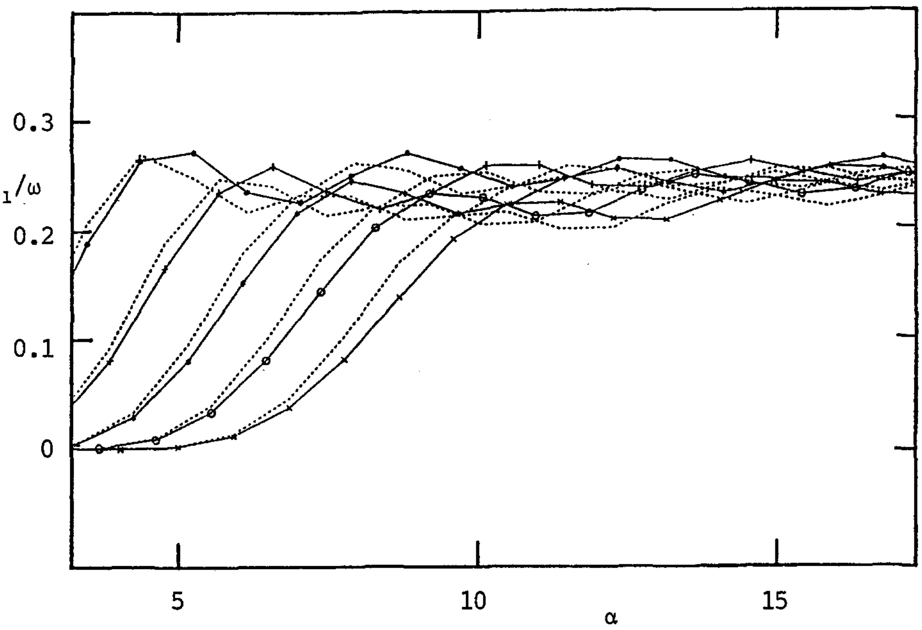

Fig. 1. Scaled relative frequency perturbation ω_1/ω resulting from augmenting the sound speed by ϵc_0 in $r < a = \lambda R$. Here $\epsilon = 0.1$ and $\lambda = 0.25$. The zero-order eigenfrequencies are given by Eq. (2.5) and the symbols joined by continuous lines represent perturbations ω_1 obtained by subtracting them from the exact eigenvalues of Eq. (2.1) and dividing by ϵ. The symbols denote the degree of the mode: $\bullet(l = 1)$, $+(l = 2)$, $*(l = 3)$, $\circ(l = 4)$, $\times(l = 5)$. The dashed lines join linearized perturbations, computed from Eq. (2.11) retaining only the first term in the numerator, which arises from the sound-speed perturbation. If ϵ were 0.01, the dashed and continuous lines would be barely distinguishable.

The simple example of a sound-speed perturbation I have chosen to illustrate is evidently not the best approximation to the outcome of the material mixing in a stellar core that motivated the discussion, but the behaviour of the eigenfrequencies that has been found is generic, and provides a good basis from which to generate more realistic representations. It must also be remembered that I have ignored a sharp spike in the buoyancy frequency N (defined on p. 292) which is produced at the interface between the mixed and unmixed layers. Strictly speaking, it is necessary to introduce gravity to describe it. However, if the interface is very thin, it can be represented as a discontinuity.

Equation (2.1) can still be used above and below the interface, the density discontinuity resulting from the composition discontinuity being accounted for merely in the matching conditions at $r = a$: the vertical component ξ of the displacement remains continuous, but now there is a jump of magnitude $-g\Delta\rho\xi$ in the (Eulerian) pressure perturbation at the unperturbed interface, where $\Delta\rho$ is the difference between the density below and above the interface and g is the acceleration due to gravity. The outcome is the addition of a further oscillatory contribution to ω_1. Such oscillatory components to ω are discussed in more detail in Section 8.

2.2 The Plane-Parallel Polytrope

Near the surface of the star the scale of variation of the background state is comparable with the wavelength of the oscillations. To describe acoustic oscillations in this region it is necessary to take the stratification into account. A plane-parallel polytrope is a convenient model for accomplishing this. In its simplest manifestation, the density ρ and pressure p of the undisturbed state are given by

$$\rho = \rho_0 z^{\mu} \quad , \quad p = (\mu+1)^{-1} g\rho_0 z^{\mu+1}, \tag{2.13}$$

where ρ_0, g and μ are constants, μ being the polytropic index and g the gravitational acceleration. The coordinate z is depth below some fiducial level, which might be regarded as the surface of the star. This is the so-called complete polytrope. The square of the sound speed c increases linearly with depth

$$c^2 = \gamma(\mu+1)^{-1} g z, \tag{2.14}$$

where γ is the adiabatic exponent $(\partial \ln p/\partial \ln \rho)_s$, the thermodynamic derivative being taken at constant specific entropy s.

Adiabatic oscillations of this model have been discussed by Lamb (1932). Since the background state is dependent only on z, separable solutions that are wavelike in the horizontal direction with wave number k can be found. Lamb used the dilatation div$\boldsymbol{\xi}$ for the principal dependent variable, obtaining the confluent hypergeometric equation

$$\zeta\chi'' + (\mu + 2 - \zeta)\chi' + 2\beta\chi = 0 \tag{2.15}$$

for the scaled amplitude χ defined by $\mathrm{div}\boldsymbol{\xi} = \mathrm{Re}[\chi(\zeta)\exp(-\zeta - ikx)]$, x being a horizontal Cartesian coordinate in the direction of variation of the wave and $\zeta = kz$. In this equation

$$2\beta = \frac{\mu+1}{\gamma}\sigma^2 - (\mu+2) + \left(\mu - \frac{\mu+1}{\gamma}\right)\sigma^{-2}, \tag{2.16}$$

and $\sigma^2 = \omega^2/gk$. Lamb's interest was in modelling waves in the terrestrial atmosphere whose horizontal wavelength $2\pi/k$ is much greater than the depth of the atmosphere. Accordingly he needed to approximate the solutions to Eq. (2.15) only for $\zeta \ll 1$. Our interest, however, is in modelling acoustic oscillations of high degree which, as will be seen later [see Eq. (2.20), and Table 1 (p. 302)] are confined to the outer layers of the star where the plane-parallel approximation is valid. Thus we can effectively regard the envelope as being infinitely deep. This is essentially the situation considered by Spiegel and Unno (1962) in their study of convective instability; the analysis of oscillations is essentially the same, save for a few sign changes.

From two independent solutions of Eq. (2.15) a single linear combination can be found for which χ does not diverge as $\zeta \to \infty$. It is normally called $U(-\beta, \mu+2, 2\zeta)$. Moreover, $U(-\beta, \mu+2, 2\zeta) \sim (2\zeta)^{\beta}$ as $\zeta \to \infty$ (*e.g.*, Abramowitz and Stegun, 1964), implying that div$\boldsymbol{\xi}$ decays exponentially to zero at great depths, which justifies regarding the model as being infinitely deep; high-degree p modes do not sense directly conditions at the centre of the star. Indeed, it is evident from the asymptotic expansion that the magnitude of U is substantial only for $\zeta \lesssim C\beta$, where C is of order unity, which implies that the waves are confined to a cavity of depth

$$z_{\mathrm{t}} \simeq C\beta k^{-1}. \tag{2.17}$$

The condition that U does not diverge at the singular point $\zeta = 0$ is that β be a non-negative integer, which implies the eigenvalue equation

$$\omega^4 - 2\gamma(\mu+1)^{-1}(n+\alpha)gk\omega^2 + [\gamma\mu(\mu+1)^{-1} - 1]g^2k^2 = 0, \tag{2.18}$$

where $\alpha = \frac{1}{2}\mu$.

There are two classes of solutions to Eq. (2.18): those corresponding to p modes, with $\omega^2 > \gamma(\mu+1)^{-1}(n+\alpha)gk$, and those corresponding to g modes, with $\omega^2 < \gamma(\mu+1)^{-1}(n+\alpha)gk$. The latter may have $\omega^2 > 0$, when they represent stationary gravity waves, or $\omega^2 < 0$, when they represent convective modes, depending on whether $\mu - (\mu+1)/\gamma$ is positive or negative. I shall discuss only the p modes, which satisfy

$$\frac{\omega^2}{gk} \simeq \frac{2\gamma}{\mu+1}(n+\alpha) - \frac{1}{2}\left(\mu - \frac{\mu+1}{\gamma}\right)(n+\alpha)^{-1}, \tag{2.19}$$

at least when n is large. The second term in the expression on the right-hand side of this equation is the buoyancy correction to acoustics. It is proportional to the square of the buoyancy frequency, defined by Eq. (3.4), through the factor $\mu - (\mu+1)/\gamma$ which is proportional to the subadiabatic temperature gradient. In the case of the sun the outer layers are convective, and except very close to the surface the stratification is very close to being adiabatic, so one would expect to obtain a good approximation to the p-mode eigenfrequencies by retaining just the leading term of the right-hand side of Eq. (2.19), even when n is not large. Notice that in a convectively stable region, where the buoyancy frequency is real, Eqs. (2.18) and (2.19) indicate that the buoyancy acts in opposition to the acoustic restoring force and decreases the frequency of the waves. On the other hand, the acoustic influence on g modes augments ω.

A convenient way of specifying the depth of the acoustical cavity more precisely than expression (2.17) is to adopt the lower turning point of the wave equation, which occurs where the characteristic vertical wave number κ, defined in the case of the isothermal sphere by the square root of the coefficient of the undifferentiated Ψ in the simple wave equation (2.3) and which is given more generally by Eq. (3.9) with $k^2 = l(l+1)/r^2$, vanishes. For p modes in a plane parallel polytrope, this yields

$$z_{\mathrm{t}} \simeq 2(n+\alpha)k^{-1}, \tag{2.20}$$

which shows that high-degree modes are concentrated in the outer layers of the star. It can be verified from the high-degree entries in Table 1 (p. 302) that this result is also approximately true of a more realistic solar model.

It was from the simple eigenvalue equation (2.18) that perhaps the first seismic diagnostic of the structure of the outer layers of the sun was made. From a systematic discrepancy between observed solar oscillation frequencies (Deubner, 1975) and the eigenfrequencies of a model of the solar envelope (Ando and Osaki, 1975) it was realized that the stratification of the upper boundary layer of the convection zone should be more nearly adiabatic than in the theoretical model. Consequently, the mixing length in the model should be increased, implying an increase in the depth of the convection zone to about 2000 Mm (Gough, 1976). Crude as this argument is, the inference is basically correct, and has been confirmed by subsequent more careful analysis. It is important to realize, however, that the rough argument based on this highly simplified model could not have been carried out without the prior existence of numerical computations of a much more realistic model. The simple model served merely as a basis for an approximate perturbation calculation to establish how the theoretical model should be modified to remove the small discrepancy between theory and observation.

A more reliable estimate might have been made by using a hybrid model, composed of a polytropic base supporting an isothermal atmosphere. I shall discuss that only briefly, but before doing so I must consider the isothermal atmosphere.

2.3 Oscillations of an Isothermal Atmosphere

The basic state of a plane-parallel isothermal atmosphere is given by

$$\rho = \rho_a e^{(z-z_0)/H} \quad , \quad p = gH\rho_a e^{(z-z_0)/H} , \tag{2.21}$$

where ρ_a, z_0 and H are constants, H being the scale height of both density and pressure. Once again, the coordinate z measures depth. It has been assumed that the atmosphere is a perfect gas with mean molecular mass μ_0, the scale height being related to the temperature T by

$$H = \frac{\Re T}{\mu_0 g} , \tag{2.22}$$

where $\Re$ is the gas constant.

Adiabatic oscillations of this atmosphere have also been discussed by Lamb (1932). As with the plane-parallel polytrope, one may express oscillation variables in separated form with constant horizontal wave number k. The Lagrangian pressure perturbation δp, for example, can be written

$$\delta p = \mathrm{Re}\left[\Psi(z)\exp(-z/2H + ikx - i\omega t)\right] , \tag{2.23}$$

and the amplitude function Ψ satisfies

$$\Psi'' + \kappa^2 \Psi = 0, \tag{2.24}$$

where

$$\kappa^2 = \frac{\omega^2}{c^2} - \frac{1}{4H^2} - k^2\left[1 - \frac{\gamma - 1}{\omega^2}\left(\frac{c}{\gamma H}\right)^2\right] . \tag{2.25}$$

Evidently, if $\kappa^2 < 0$, the oscillation is evanescent. The atmosphere, if it is considered to extend to $z = -\infty$, cannot transmit waves, and therefore acts as a perfect reflector of waves incident from below.

In the case of purely vertical motion, having $k = 0$, which was first considered by Lamb in 1908, evanescence occurs for frequencies ω below $\omega_c = c/2H$, which is sometimes called Lamb's acoustic cutoff frequency. When ω exceeds this value, vertical oscillations are no longer confined by the atmosphere. Wave energy can propagate outwards until the displacement amplitude, which is proportional to $\rho^{-\frac{1}{2}}$, is so large that the waves become nonlinear and shock.

The critical frequency is modified by horizontal variation of the wave. Its value is given by

$$\omega_c = \left\{ \frac{1}{2}\left(k^2 + \frac{1}{4H^2}\right) + \left[\frac{1}{4}\left(k^2 + \frac{1}{4H^2}\right)^2 - \frac{(\gamma-1)k^2}{\gamma^2 H^2}\right]^{\frac{1}{2}} \right\}^{\frac{1}{2}} c, \tag{2.26}$$

which exceeds $c/2H$ when $k \neq 0$. Thus, as ω increases, vertical oscillations are the first to leak through the atmosphere.

Since κ is constant, the eigensolutions of Eq. (2.24) are either exponential or sinusoidal.

2.4 Simple Hybrid Models

Somewhat more realistic yet simple models of the outer layers of a star can be modelled with an isothermal atmosphere supported by a plane-parallel polytrope. Thus, p and ρ are given by Eq. (2.13) for $z > z_0$ and by Eq. (2.21) for $z < z_0$, with the condition $H\rho_a = (\mu+1)^{-1}\rho_0 z_0^{\mu+1}$ demanded by continuity of pressure at $z = z_0$. Temperature, and hence density, need not be continuous at $z = z_0$, the discontinuity being introduced to represent the thin boundary layer at the top of the convection zone in which the structure of the star is neither polytropic nor isothermal. The atmospheres of stars with coronae, such as the sun, can be modelled with two superposed isothermal regions, the upper high-temperature region extending to $z = -\infty$.

A model of this kind (without the corona) has been used to estimate the importance of the atmosphere on low-degree oscillations at frequencies below the cutoff frequency in the atmosphere (Christensen-Dalsgaard and Gough, 1980) and in assessing the influence on acoustic eigenfrequencies of an atmospheric magnetic field (Campbell and Roberts, 1989). In this case the atmosphere oscillates everywhere in phase with the motion of the fitting surface at the junction with the polytropic interior, and so far as the oscillations of the polytrope are concerned can be regarded simply as a boundary with inertia that resists the pressure fluctuations. Since the displacement eigenfunction increases with height, the atmosphere oscillates with a greater mean amplitude than does the fitting surface, and therefore its apparent inertia is greater than its mass. At frequencies above the cutoff, phase varies with height; waves propagate upwards and energy leakage damps the modes, but the interior of the star continues to exhibit discrete albeit broadened resonant frequencies as a result of reflection at the discontinuities (Balmforth and Gough, 1990).

3 Simple Asymptotics

Asymptotic approximations to the adiabatic acoustic oscillations of a star can be obtained if the characteristic vertical wavelength of the eigenfunction is small compared with the scale of variation of the background state. In that case two approaches present themselves, both of which depend on representing the oscillation locally as a superposition of waves.

The first stage for either method is to transform the equation into a form suitable for easy analysis. This is a very important step, because a blind application of either method to raw equations can sometimes require a considerable amount of unnecessary extra labour to achieve the required accuracy; moreover, it can also shrink the domain of validity of the approximation.

For simplicity, I shall adopt the so-called Cowling approximation, in which the Eulerian perturbation Φ' to the gravitational potential is ignored. The analysis can be generalized to include Φ', and has been carried out for a spherically symmetrical background state by Vorontsov (these proceedings) and by Dziembowski and myself (1990). My independent variables will be spherical polar coordinates (r, θ, ϕ) about the centre of the star. I shall work with a dependent variable which is proportional to the Lagrangian pressure perturbation δp. This is equivalent to the approach taken by Lamb, who based his analysis on div$\boldsymbol{\xi}$, which is also proportional to δp. I find it more convenient to start with a scalar, rather than construct one from a vector, particularly when the full spherical geometry of the problem is taken into account.

Different choices of either dependent or independent variables can lead to somewhat different representations of the solution. For example, Smeyers and Ruymaekers (these proceedings) compare the use of the scalar δp and the radial component of the displacement vector, and Christensen-Dalsgaard *et al.* (1983) obtain a different formula for the acoustic cutoff frequency in a stratified envelope when the more natural acoustical radius $\int c^{-1} dr$ is used for the radial coordinate. A systematic comparison of the different representations has never been published.

To make my presentation yet simpler I ignore local curvature effects. To include them is tedious (*cf.* Gough, 1990), though it probably does not introduce any fundamental difficulties. I then determine the appropriate dependent variable ψ by multiplying δp by that function u which eliminates first derivatives of ψ from the governing differential equation. The outcome, for high-frequency modes, is

$$\psi \sim \rho^{-\frac{1}{2}} \delta p, \tag{3.1}$$

which satisfies

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial t^2} + \omega_\mathrm{c}^2 - c^2\nabla^2\right)\psi - c^2N^2\nabla_\mathrm{h}^2\psi = 0, \tag{3.2}$$

where the acoustic cutoff frequency ω_c and the buoyancy frequency N are given by

$$\omega_\mathrm{c}^2 = \frac{c^2}{4H^2}\left(1 - 2\frac{dH}{dr}\right) = \frac{1}{2}c^2\left[\frac{3}{2}\left(\frac{d\ln\rho}{dr}\right)^2 - \frac{1}{\rho}\frac{d^2\rho}{dr^2}\right], \tag{3.3}$$

$$N^2 = g\left(\frac{1}{H} - \frac{g}{c^2}\right) = -g\left(\frac{d\ln\rho}{dr} + \frac{g}{c^2}\right), \tag{3.4}$$

$H = (-d\ln\rho/dr)^{-1}$ being the density scale height of the equilibrium state, and $\nabla_{\rm h}^2$ is the horizontal Laplacian operator:

$$\nabla_{\rm h}^2 = \nabla^2 - \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{3.5}$$

Since the background state is considered to be independent of time, one can anticipate eigensolutions with sinusoidal time dependence of frequency ω. Equation (3.2) then reduces to

$$\nabla^2\psi + \left(\frac{\omega^2-\omega_{\rm c}^2}{c^2} - \frac{N^2}{\omega^2}\nabla_{\rm h}^2\right)\psi = 0. \tag{3.6}$$

Strictly speaking, Eqs. (3.2) and (3.6) are valid only for spherically symmetrical stars, though if there were a slight asphericity, with horizontal gradients of the background state being everywhere very much smaller than radial gradients, one might expect to be able to use these relatively simple equations as a guide.

If the star can be considered to be spherically symmetrical, one can separate the solution further, into the form

$$\psi \propto \mathrm{Re}\left[r^{-1}\Psi(r)P_l^m(\cos\theta)e^{i(m\phi-\omega t)}\right] \tag{3.7}$$

as was done for the isothermal sphere. Substituting into Eq. (3.6) yields

$$\Psi'' + \kappa^2\Psi = 0, \tag{3.8}$$

where

$$\kappa^2 = \frac{\omega^2-\omega_{\rm c}^2}{c^2} - \frac{l(l+1)}{r^2}\left(1-\frac{N^2}{\omega^2}\right). \tag{3.9}$$

One sees that Eqs. (3.8) and (3.9) reduce immediately to Eq. (2.3) in the absence of stratification ($\omega_{\rm c}=0, N=0$), and that Eq. (3.9) is equivalent to Eq. (2.25) for a plane-parallel isothermal atmosphere, for which one sets $k^2 = l(l+1)/r^2$.

One can easily solve Eq. (3.8) in the JWKB approximation. Except in the vicinity of the radii at which κ vanishes one can well approximate Ψ by the form

$$\Psi = A(r)\exp\left(i\int\kappa dr\right) \tag{3.10}$$

in which it is assumed that the characteristic scale of variation Λ, say, of A and κ greatly exceeds the local inverse wave number κ^{-1}. On substituting this expression into Eq. (3.8) and satisfying the resulting equation separately at each order of $(\kappa\Lambda)^{-1}$, one finds that the leading order is automatically satisfied, as it was designed to do by the choice of the argument of the exponential function in Eq. (3.10), and that the subsequent order implies $A \propto \kappa^{-\frac{1}{2}}$. In regions where $\kappa^2 > 0$ the solution is wavelike; here waves can propagate in the radial direction. Where $\kappa^2 < 0$ the solution is evanescent. Matching across the turning point, where $\kappa = 0$, is accomplished by Olver's method, in which the solution is approximated by an Airy function (*e.g.*, Vandakurov, 1967; Shibahashi, 1979; Tassoul, 1980; Unno *et al.*, 1989). By choosing that solution which, where $\kappa^2 < 0$, decays away from the region of propagation, the integral in the representation (3.10) can be made definite, leading to the expression

$$\delta p \sim \Psi_0 r^{-1} \left(\frac{\rho}{\kappa}\right)^{\frac{1}{2}} \sin\left[\int_{r_t}^{r} \kappa dr + \frac{\pi}{4}\mathrm{sgn}(r - r_t)\right] \tag{3.11}$$

for the pressure fluctuation in the region of propagation, where Ψ_0 is a constant and r_t is a turning point. Identifying the two representations (3.11) obtained by equating r_t separately to the lower and upper turning points r_1 and r_2 of a region of propagation, assuming for the moment that r_2 exists, then yields the eigenvalue equation

$$\int_{r_1}^{r_2} \left[1 - \frac{\omega_c^2}{\omega^2} - \frac{L^2c^2}{\omega^2 r^2}\left(1 - \frac{N^2}{\omega^2}\right)\right]^{\frac{1}{2}} \frac{dr}{c} \sim \frac{\left(n - \frac{1}{2}\right)\pi}{\omega}, \tag{3.12}$$

where now $L^2 = l(l+1)$. This expression indicates that the eigenfunction is such as to fit an integral number of waves within the cavity (r_1, r_2), save for a phase factor $-\frac{\pi}{2}$ which arises from the elongation of the wave resulting from the fact that it penetrates somewhat into the two bounding evanescent regions.

It is instructive to recall the significance of the quantities N and ω_c which appear in the eigenvalue relation (3.12). Let us first note that once again two classes of solution are admitted: the p modes with high frequency for which the first term in the square brackets dominates, and the g modes with low frequency for which the third term dominates. For g modes of moderate order, κ is moderate (I speak loosely, in search of clarity) and therefore as $l \to \infty$ the factor $1 - N^2/\omega^2$ must become small. Thus $\omega \to N$; the buoyancy frequency is the value to which the frequencies of g modes tend as the degree increases at fixed order. The motion becomes almost vertical nearly everywhere, in the form of elongated oscillatory cells. This is evident from the asymptotic forms for the vertical and horizontal amplitude components $(\Xi(r), H(r))$ defining the displacement eigenfunction

$$\boldsymbol{\xi}(r,t) = \mathrm{Re}\left[\left(\Xi P_l^m, L^{-1} H dP_l^m/d\theta, imL^{-1} H \operatorname{cosec}\theta P_l^m\right) e^{im\phi - i\omega t}\right], \tag{3.13}$$

which are given approximately by

$$\Xi \sim \frac{\Psi_0 L}{\omega r^2}(\rho\kappa)^{-\frac{1}{2}} \sin\left(\int_{r_1}^{r} \kappa dr + \frac{\pi}{4}\right), \tag{3.14}$$

$$H \sim \frac{\Psi_0}{\omega r}(\kappa/\rho)^{\frac{1}{2}} \cos\left(\int_{r_1}^{r} \kappa dr + \frac{\pi}{4}\right) \tag{3.15}$$

well away from the turning points in regions where $\kappa^2 > 0$. The pressure fluctuation, which diverts the vertical flow into the horizontal direction near the top and bottom of each cell, is relatively small. Thus, N is the characteristic frequency of a fictitious element of fluid imagined to be moving vertically under the action of buoyancy and otherwise unimpeded by pressure fluctuations. Of course, strictly speaking, buoyancy is produced by a pressure imbalance with gravity, as Archimedes must have known, and which is necessarily dependent on there being horizontal fluctuations, and a consequent horizontal component of the flow; but in fluid dynamics it isn't usually thought of quite in that way. It is more commonly thought of as the force of gravity acting upon the local density deviation from its horizontal average. It follows immediately that there can be no buoyancy in a fluid where the only spatial variation is with height. Therefore the value of N cannot influence the dynamics of radial modes, as is obvious from Eqs. (3.8) and (3.9).

The nature of the acoustic cutoff frequency is quite different. As its name indicates, it is concerned principally with acoustic-wave propagation, and not with the dynamics of gravity waves. It is sometimes associated with the buoyancy frequency, however, and has even been replaced by it (*e.g.*, Shibahashi, 1990), perhaps because in a plane-parallel isothermal atmosphere the numerical values of the two frequencies are similar. [$N^2 = (1-\gamma^{-1})g/H$ and $\omega_c^2 = \frac{1}{4}\gamma g/H$, the two frequencies differing by less than 1% when $\gamma = \frac{5}{3}$. In an adiabatically stratified plane-parallel polytrope, however, $\omega_c^2 = [(2\gamma-1)/4(\gamma-1)]g/z$ whereas $N^2 = 0$.] Equating the two can sometimes be convenient mathematically, but it is important to realize that it is no more than a convenience for calculation. The essential difference should not be forgotten. The influence of the acoustic cutoff becomes important when the scale height H of density becomes comparable with the wavelength of the wave. If density declines (or augments) too abruptly there is too great a mismatch between the inertia of the material in neighboring regions of compression and rarefaction for them to interact continually in opposition. It is evident that this mismatch has nothing to do with the presence of gravity, which is an essential ingredient of buoyancy. Indeed, Eq. (3.2) is still satisfied when $g = 0$; in that case $N = 0$, but if the density of the background state is not uniform ω_c remains nonzero.

The role of the acoustic cutoff can be illustrated by the purely vertical oscillations of a plane-parallel polytrope of index μ. In this case there is no buoyancy, and Eq. (3.8) reduces to

$$\Psi_{zz} + \left[\frac{\omega^2}{c_0^2 z} - \frac{\mu(\mu+2)}{4z^2}\right]\Psi = 0, \tag{3.16}$$

where $c_0^2 = \gamma g/(\mu+1)$. The second term in square brackets is ω_c^2/c^2, and increases in magnitude with μ as the stratification becomes more severe. The solution is

$$\Psi = \Psi_0 \; z^{\frac{1}{2}} J_{\mu+1}\left(\frac{2\omega z^{\frac{1}{2}}}{c_0}\right), \tag{3.17}$$

where Ψ_0 is again a constant amplitude factor. This illustrates that the wave is evanescent near the surface, for $\Psi \propto z^{\mu/2+1}$ where $\omega z/c \ll 1$.

The vertical component ξ of the displacement eigenfunction can be obtained by substituting the solution (3.17) into the momentum equation, yielding

$$\xi = -\xi_0 \; z^{-\mu/2} J_\mu\left(2\omega\sqrt{\frac{(\mu+1)z}{\gamma g}}\right)\cos\omega t, \tag{3.18}$$

where $\xi_0^2 = (\mu+1)\Psi_0^2/(\omega^2\gamma g\rho_0)$. It can be illustrated practically by oscillating horizontally a rope hanging under gravity whose mass ρ per unit length is given by the first of Eqs. (2.13), where now z is height above the free end. In that case the horizontal displacement is given by Eq. (3.18) with $\gamma = 1$, and is illustrated in Fig. 2. It can clearly be seen that near the free end, where the tension is too low to provide adequate propagation, the solution is not wave-like. Above the turning point the wavelength decreases in proportion to the propagation speed $c \propto z^{\frac{1}{2}}$, and the amplitude diminishes in proportion to $(\rho c)^{-\frac{1}{2}}$.

Notice in Fig. 2 that when ξ is expressed with respect to the more natural independent variable $\tilde{\tau} = \int c^{-1}dz$, its spatial variation is very weak at the top of the evanescent region close to the free surface. The behaviour of div$\boldsymbol{\xi}$, which is proportional to $\delta p/p$, can be seen from Eqs. (3.1), (2.13) and (3.17) to be qualitatively similar. I record also that at

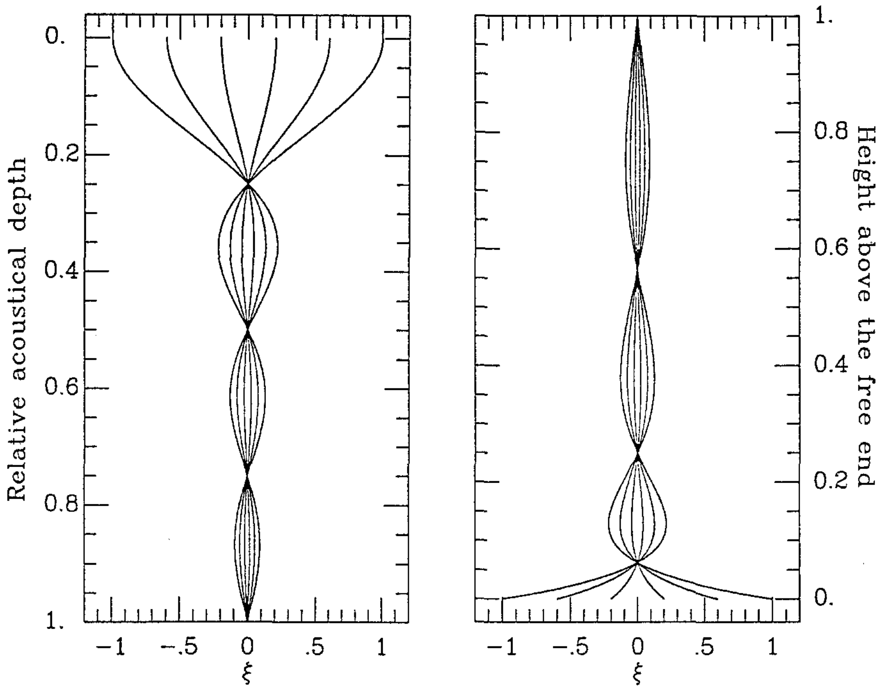

Fig. 2. The left-hand panel illustrates the displacement ξ, given by Eq. (3.18), of a vertically oscillating plane-parallel polytrope plotted as a function of acoustical depth $\tilde{\tau} = \int c^{-1} dz$, normalized such that the acoustical depth of the layer is unity. The displacement is shown at several different instances in time. In this example $\gamma = 5/3$ and $\mu = 0.5$. A small value of μ was chosen because otherwise ξ declines too rapidly with $\tilde{\tau}$ to be clearly visible. The right-hand panel shows the shape of a laterally oscillating rope of unit length, given by Eq. (3.18), hanging under its own weight from a fixed point. The mass per unit length is proportional to z^{μ} where $\mu = 0.5$. Aside from the orientations and the independent variables against which they are plotted, the displacements in the two panels are identical.

the free surface div$\boldsymbol{\xi} \propto \omega^2 \xi$. These properties will be invoked in Section 9 when discussing the solar-cycle frequency variations reported by Libbrecht and Woodard (1990).

The depth (in the case of the polytrope) of the first node in the eigenfunction, which provides some indication of where propagation has begun, increases with μ, as does the extent of the region near the surface where the Lagrangian pressure fluctuation is insignificant. In the almost inert evanescent region the fluid is hardly compressed; the region is simply periodically lifted up bodily and subsequently lowered by the oscillating medium beneath. As can be seen in Fig. 2, however, the displacement is not constant, but decreases linearly with z away from $z = 0$. Thus, as in the case of the isothermal atmosphere, the centre of mass of the evanescent zone oscillates with a greater displacement amplitude than does the material near the turning point: the inertia the zone presents to the wave is again greater than its mass.

It is sometimes useful to think of the oscillations as a resonant superposition of locally plane propagating waves interfering coherently to form a standing wave pattern. This is

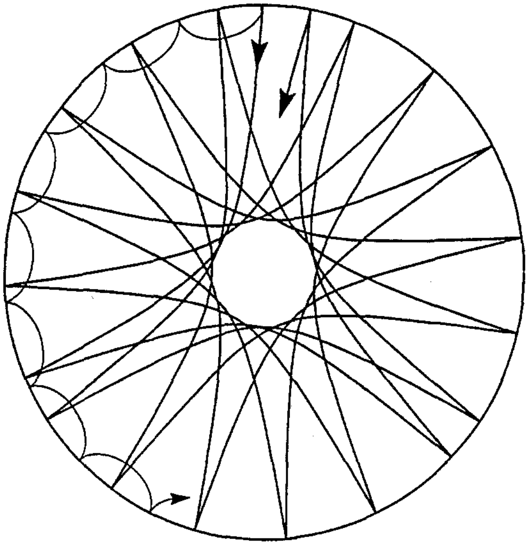

Fig. 3. Ray paths of components of two acoustic modes of an idealized model of the sun. As is almost always the case, the paths are not closed. The number of reflections near the surface per revolution is therefore not integral, and is not, as has sometimes been stated, the value of L (or l); it depends on the radius r_t of the lower turning point, given by Eq. (6.3), at which the waves travel horizontally and beneath which they cannot propagate. Thus, using Table 1 (p. 302) as a guide, one finds that the more deeply penetrating ray path represents a consituent of a mode with $\nu/L \simeq 560\mu$Hz, where $\nu = \omega/2\pi$ is the cyclic frequency; it could be a mode with $l = 2$ and $n = 8$ or perhaps a mode with $l = 6$ and $n = 23$. Similarly one deduces that for the shallower ray $\nu/L \simeq 30\mu$Hz, which corresponds to modes with $L/n \simeq 5$.

the second of the two approaches to which I alluded at the beginning of this section. The paths of all possible rays associated with a particular mode, examples of two of which are shown in Fig. 3, determine the domain within which the mode has substantial amplitude, and satisfying the condition that constructive interference takes place everywhere within that domain, in circumstances in which that is possible, both assures the asymptotic existence of the mode and determines its eigenfrequency. The procedure by which the interference conditions are computed is explained by Keller and Rubinow (1960). They are basically quantization conditions of the form

$$\oint \boldsymbol{K} \cdot d\boldsymbol{r} = 2(n' + m'/4)\pi, \tag{3.19}$$

where $\boldsymbol{K}$ is the total wave number and the integral is taken over an arbitrary closed contour C within the domain of propagation which suffers m' simple grazings with caustic surfaces; n' is an integer quantum number which in the case of a spherically symmetric star is related to the order n, the degree l and the azimuthal order m of the mode in a way that depends on C. For spherical stars the conditions (3.19) imply the existence of the integral quantum numbers l, m and n, and recover the relation (3.12) derived above by JWKB expansion of the radially varying factor $\Psi(r)$ in the separated eigenfunction (3.7) with the sole difference that L is now $l+\frac{1}{2}$. Since the approximation uses the asymptotic wave description in all three dimensions it can be formally valid only for $l \gg 1$, so the difference is immaterial.

This method is potentially more powerful than the JWKB analysis of separable solutions which I described above (at least for high-degree modes) because it does not require separable solutions to exist. In particular, one can formally write down conditions (3.19) that incorporate rotation and other aspherical components of the structure of the star. However, to evaluate them it is usually expedient to make use of the fact that the aspherical influences on the oscillations are small, and to perturb about the spherical state. In that case the results reduce to what one would obtain by applying degenerate perturbation theory about the separable solutions (3.7). Within that approximation, therefore, the two methods are essentially equivalent.

Finally, I record for future use the asymptotic displacement eigenfunctions for high-order p modes. They are given approximately by

$$\Xi \sim \Psi_0 \omega^{-2} r^{-1} (\kappa/\rho)^{\frac{1}{2}} \cos\left(\int_{r_1}^{r} \kappa dr + \frac{\pi}{4}\right), \tag{3.20}$$

$$H \sim \Psi_0 L \omega^{-2} r^{-2} (\rho\kappa)^{-\frac{1}{2}} \sin\left(\int_{r_1}^{r} \kappa dr + \frac{\pi}{4}\right), \tag{3.21}$$

in the propagating region well away from the turning points.

4 Low-degree p Modes

The frequencies of high-order p modes of low degree are given asymptotically by Eq. (2.7), where now

$$\omega_0 = \pi \left(\int_0^R \frac{dr}{c}\right)^{-1}, \tag{4.1}$$

which is an obvious generalization of Eq. (2.6), and ϵ, A and B are now functionals of the equilibrium state. The result was first derived to this order by Tassoul (1980), and the essential features of it are recovered by expanding Eq. (3.12) in inverse powers of ω/ω_0 at fixed l.

An important feature of Eq. (2.7) is that it is influenced separately by conditions in different regions of the star. In particular, ϵ and B depend predominantly on conditions in the surface layers of the star, just as is the case for the isothermal sphere (however, now $\epsilon \simeq \frac{1}{4}+\frac{1}{2}\mu$, where μ is an effective polytropic index in the vicinity of the upper turning point of the mode; B is rather more complicated), whereas A, which according to Tassoul is given by

$$A = \frac{1}{2\pi\omega_0}\left[\frac{c(R)}{R} - \int_0^R \frac{1}{r}\frac{dc}{dr}dr\right], \tag{4.2}$$

is dominated by the second term in the square brackets and hence depends to a large extent on conditions near the centre of the star. Thus by comparing observed low-degree eigenfrequencies with the asymptotic formula one can hope to estimate the parameters ω_0, A, ϵ and B, and so calibrate theoretical models. In particular, because of the different ways in which these parameters depend on the structure of the star, one can infer from discrepancies between theory and observation in what way a theoretical model would need to be adjusted in order to remove those discrepancies.

I mention this simple example first because low-degree modes can in principle be measured in stars other than the sun, so comparison of these simple parameters is likely to be the basis of the earliest asteroseismic inferences.

I should perhaps stress again that such inferences will be made not from asymptotic analysis alone, but from carefully calculated eigenfrequencies of hopefully realistic stellar models which are computed numerically. The role of the asymptotic formulae is to guide the way in which the comparisons are made. In particular, the modelling of conditions near the surfaces of stars is very uncertain, and because all p modes are reflected near the surface they must all suffer from this uncertainty to some degree. However, combining data in such a way as to estimate the parameter A, for example, provides a measure of conditions in the core from which, one hopes, the surface uncertainties have been largely eliminated. Even though such a comparison needs to be carried out entirely numerically, the asymptotic analysis has played an important role in the design of the test and the interpretation of the results. This is one of the justifications for developing the asymptotic analysis yet further.

5 A Variational Principle, and its Application

If a frame of reference exists in which the basic structure of a star, which in general is presumed to be rotating, is independent of time, then one can seek linearized adiabatic oscillations in that frame whose time dependence is sinusoidal with frequency ω. That frequency is related to the displacement eigenfunction $\boldsymbol{\xi}(\boldsymbol{r})e^{-i\omega t}$ through a variational principle of the form

$$\omega^2\mathcal{I} - \omega\mathcal{R} - \mathcal{K} = 0 \tag{5.1}$$

(Lynden-Bell and Ostriker, 1967) provided ω does not exceed the critical cutoff frequency in the atmosphere, where $\mathcal{I}, \mathcal{R}$ and $\mathcal{K}$ are integrals over the volume of the star of functions which are bilinear in the components of $\boldsymbol{\xi}$ or their derivatives and which, of course, depend on the structure of the star but which do not contain ω explicitly. I shall not write down those integrals in their general form, but instead I shall consider some simple special cases. The variational principle is that, considered as a solution of the quadratic Eq. (5.1), ω is stationary with respect to arbitrary variations $\delta\boldsymbol{\xi}$ (satisfying appropriate boundary conditions, which I also refrain from quoting) to $\boldsymbol{\xi}$ when $\boldsymbol{\xi}$ is an eigenfunction and ω an eigenvalue of the adiabatic oscillation problem. This principle can conveniently be used to estimate eigenfrequencies from approximate eigenfunctions, or as the basis of perturbation theory.

In the case of a nonrotating star with no internal motion, $\mathcal{R} = 0$ and

$$\omega^2 = \frac{\mathcal{K}}{\mathcal{I}}. \tag{5.2}$$

Moreover, $\mathcal{I}$ and $\mathcal{K}$ are real. Hence ω^2 is real, and it is possible to choose $\boldsymbol{\xi}$ to be real. Then $\mathcal{I}$ and $\mathcal{K}$ can be written

$$\mathcal{I} = \int \rho\boldsymbol{\xi}\cdot\boldsymbol{\xi}dV, \tag{5.3}$$

where dV is a volume element, and

$$\mathcal{K} \simeq \int \left[\rho c^2(\mathrm{div}\boldsymbol{\xi})^2 + 2(\boldsymbol{\xi}\cdot\nabla p)\mathrm{div}\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla p\;\boldsymbol{\xi}\cdot\nabla\ln\rho\right] dV \tag{5.4}$$

(*cf.* Ledoux and Walraven, 1958; Chandrasekhar, 1964). For simplicity I have not written $\mathcal{K}$ exactly; in particular, I have omitted terms accounting for oscillating perturbations to the gravitational potential which make a relatively small contribution, but which in accurate computations must, of course, be included.

One of the important uses of Eqs. (5.2)–(5.4) in helioseismology is in constructing kernels for computing the small differences between the solar frequencies and the eigenfrequencies of a good theoretical model of the sun. Because the quotient $\mathcal{K}/\mathcal{I}$ is stationary to variations in $\boldsymbol{\xi}$, small perturbations to $\mathcal{K}/\mathcal{I}$ resulting from variations in the model can be computed without requiring explicit knowledge of the resultant perturbation to $\boldsymbol{\xi}$. Thus one can compute, for example, kernels for spherically symmetric structure perturbations for the purposes of inverting frequency discrepancies. Alternatively one can compute kernels for aspherical perturbations, which provide formulae for degeneracy splitting. Similarly, one can introduce rotation as a small perturbation, which yields the formula

$$\delta\omega \simeq \frac{\mathcal{R}}{2\mathcal{I}} \tag{5.5}$$

for rotational splitting $\delta\omega$, where once again $\mathcal{R}$ and $\mathcal{I}$ are evaluated with the unperturbed eigenfunctions. It is important to appreciate that the substitution of unperturbed eigenfunctions into equations such as (5.5) does not imply that perturbations to the eigenfunctions are assumed not to exist; it is merely that the variational principle has permitted the evaluation of the frequency perturbation without requiring the eigenfunction perturbation to be known. Indeed, Eq. (5.5) can be derived without explicit use of the variational principle by expanding the basic oscillation equations and their eigensolutions about the spherically symmetrical nonrotating state, analogously to the expansion that led to Eq. (2.9) resulting from a spherically symmetric perturbation to the isothermal sphere. Equation (5.5) then arises as a solubility condition, analogous to Eq. (2.10), and is thus actually the condition that the perturbation to the eigenfunction exists. That perturbation has not only a different radial dependence but also a different angular dependence from the unperturbed eigenfunction.

6 Asymptotic p-mode Frequency Perturbations

For high-order p modes the term N^2/ω^2 can be ignored in Eq. (3.9) for the vertical component κ of the wave number characterizing the spatial oscillation of the eigenfunction. Moreover, ω_c is comparable with ω only in the vicinity of the upper reflecting layer, so I shall ignore that too. Then I can approximate κ by the very simple acoustic dispersion relation

$$\kappa^2 + \frac{L^2}{r^2} \simeq \frac{\omega^2}{c^2} \tag{6.1}$$

[*cf.* Eq. (2.3)], where $L^2 = l(l+1)$ or $\left(l+\frac{1}{2}\right)^2$. This can be substituted into Eq. (3.12) to yield a simpler eigenvalue equation, which I write as

$$F\left(\frac{\omega}{L}\right) \equiv \int_{r_t}^{R} \left(1 - \frac{L^2c^2}{\omega^2 r^2}\right)^{\frac{1}{2}} \frac{dr}{c} = \frac{(n+\alpha)\pi}{\omega}, \tag{6.2}$$

where r_t is an approximation to r_1 obtained by setting the integrand to zero:

$$\frac{c(r_t)}{r_t} = \frac{\omega}{L}. \tag{6.3}$$

As a result of removing ω_c from the formula, the integrand no longer vanishes near the surface, and I have been forced to replace r_2 by some radius R which represents the radius of the star. The error so introduced can hardly depend on L, because $Lc/\omega r \ll 1$ near the surface [I am ignoring modes of very high degree, with $l \gtrsim 10^3$, for which that is not the case], so I can absorb the transformation into the factor α which could depend on ω but not on L. It is straightforward to show that if the outer layers of the star can be approximated by a polytrope of index μ, then $\alpha = \mu/2$, and in that case is even independent of ω. Indeed, Eq. (6.2) with α constant was first shown by Duvall (1982) to approximate the solar frequencies when $\alpha \simeq 3/2$; a better representation is obtained with a function α that varies weakly with ω. Notice, however, that so far as studying the properties of the oscillations in the vicinity of $r = r_2$ is concerned, if the outer layers of the sun are to be represented by a polytrope one should choose $\mu = 3$; I shall use this result in Section 9 when discussing temporal frequency variations associated with the solar cycle. Values of r_t/R in a standard solar model corresponding to three representative frequencies are listed in Table 1 (p. 302).

One can formally perturb Eq. (6.2) to estimate the frequency difference $\delta\omega$ between two solar models (having the same radius) whose sound speeds differ by δc:

$$\frac{\delta\omega}{\omega} \simeq \left\langle \frac{\delta c}{c} \right\rangle \equiv S^{-1} \int_{\tau_t}^{T} \left(1 - \frac{L^2c^2}{\omega^2 r^2}\right)^{-\frac{1}{2}} \frac{\delta c}{c} d\tau, \tag{6.4}$$

where

$$S = \int_{\tau_t}^{T} \left(1 - \frac{L^2c^2}{\omega^2 r^2}\right)^{-\frac{1}{2}} d\tau, \tag{6.5}$$

$$\tau(r) = \int_0^r c^{-1} dr \tag{6.6}$$

is acoustic radius, $\tau_t = \tau(r_t)$ and $T = \tau(R)$. In obtaining this relation I have ignored the variation of α with both ω and $c(r)$, which appears to be a fairly good approximation (Christensen-Dalsgaard *et al.*, 1988).

Equation (6.4) can also be obtained, more laboriously, by computing a kernel for $c^{-1}\delta c$ from the variational principle (5.2)–(5.4) using the asymptotic eigenfunctions (3.20) and (3.21). When carried out in this way it is necessary to average over the rapidly oscillating part of the eigenfunction, which is valid only when the scales of variation of the models are much greater than κ^{-1}, which is, of course, a necessary condition for the asymptotic equation (6.2) to be valid. It becomes apparent from that analysis that S is proportional to the inertia $\mathcal{I}$ of the mode, defined by Eq. (5.3). One can similarly evaluate the formula

Table 1. Formal lower turning points r_t, in units of the photospheric radius R, of a standard solar model. The turning point is defined by Eq. (6.3) with $L = l + \frac{1}{2}$, and is evaluated at cyclic frequencies ($\nu = \omega/2\pi$) 2, 3 and 4 mHz. Notice that because the sun is roughly isothermal in the core, r_t/R is roughly proportional to $(l + \frac{1}{2})/\nu$ when l is small. It can also be verified that when l is large, the depth $1 - r_t/R$ is approximately proportional to ν^2/L^2 as predicted by the plane-parallel polytropic relations (2.19) and (2.20).

l	$\nu = 2$	$\nu = 3$	$\nu = 4$
1	0.0869	0.0581	0.0436
2	0.1401	0.0963	0.0725
3	0.1856	0.1318	0.1009
4	0.2252	0.1638	0.1275
5	0.2605	0.1926	0.1522
6	0.2929	0.2189	0.1749
7	0.3236	0.2432	0.1960
8	0.3526	0.2660	0.2157
9	0.3799	0.2876	0.2343
10	0.4058	0.3084	0.2519
15	0.5199	0.4016	0.3310
20	0.6124	0.4810	0.3994
25	0.6862	0.5495	0.4604
30	0.7359	0.6096	0.5147
40	0.8036	0.7037	0.6082
50	0.8501	0.7605	0.6830
60	0.8821	0.8027	0.7338
70	0.9052	0.8358	0.7715
80	0.9221	0.8614	0.8022
90	0.9347	0.8816	0.8279
100	0.9444	0.8979	0.8491
150	0.9701	0.9443	0.9138
200	0.9813	0.9642	0.9442
250	0.9867	0.9745	0.9604
300	0.9897	0.9812	0.9700
350	0.9918	0.9854	0.9764
400	0.9935	0.9879	0.9812
450	0.9948	0.9897	0.9845
500	0.9958	0.9912	0.9867
550	0.9967	0.9924	0.9884
600	0.9973	0.9935	0.9897
650	0.9979	0.9944	0.9908
700	0.9984	0.9952	0.9918
750	0.9988	0.9958	0.9927
800	0.9991	0.9964	0.9935
850	0.9994	0.9969	0.9942
900	0.9996	0.9973	0.9948
950	0.9997	0.9977	0.9953
1000	0.9998	0.9981	0.9958

Table 2. Acoustical structure of a solar model: $x = r/R$ where $R = 6.96\times10^{10}$cm; τ is acoustical radius, defined by Eq. (6.6) and $T = \tau(R) = 3518$ s; the sound speed c is quoted in $\mathrm{Mm\,s^{-1}}$, and $c/2\pi r$ in μHz; $\zeta(r) = c^2/r^2$ is the asymptotic depth co-ordinate appearing in Eq. (7.3) and $\zeta_0 = \zeta(R) = 1.29 \times 10^{-10}\,\mathrm{s^{-2}}$; ψ is defined by Eq. (7.4) and $\psi_0 = \psi(R) = 1.31 \times 10^{-12}\,\mathrm{s^{-3}}$; $-x^{-1}dc/dx$ is proportional to the integrand in Eq. (4.2) for the coefficient A in the asymptotic relation (2.7) applied to the sun, and is quoted in $\mathrm{Mm\,s^{-1}}$.

τ/T	r/R	c	$c/2\pi r$	ζ/ζ_0	ψ/ψ_0	$-x^{-1}dc/dx$
0.000	0.0000	0.5067	–	–	–	–3.681
0.025	0.0641	0.5078	1.81×10^3	1.00×10^6	2.25×10^6	0.293
0.050	0.1278	0.4957	8.87×10^2	2.40×10^5	2.93×10^5	3.340
0.075	0.1885	0.4621	5.61×10^2	9.59×10^4	8.43×10^4	3.445
0.100	0.2445	0.4247	3.97×10^2	4.82×10^4	3.28×10^4	2.718
0.125	0.2960	0.3923	3.03×10^2	2.80×10^4	1.51×10^4	1.944
0.150	0.3439	0.3671	2.44×10^2	1.82×10^4	8.04×10^3	1.436
0.175	0.3889	0.3455	2.03×10^2	1.26×10^4	4.80×10^3	1.171
0.200	0.4314	0.3274	1.74×10^2	9.19×10^3	3.03×10^3	0.942
0.225	0.4718	0.3114	1.51×10^2	6.96×10^3	2.06×10^3	0.819
0.250	0.5102	0.2969	1.33×10^2	5.41×10^3	1.46×10^3	0.723
0.275	0.5469	0.2837	1.19×10^2	4.29×10^3	1.06×10^3	0.645
0.300	0.5819	0.2715	1.07×10^2	3.48×10^3	7.94×10^2	0.583
0.325	0.6155	0.2602	9.67×10^1	2.85×10^3	6.14×10^2	0.547
0.350	0.6477	0.2493	8.80×10^1	2.37×10^3	4.89×10^2	0.532
0.375	0.6785	0.2383	8.03×10^1	1.97×10^3	4.08×10^2	0.560
0.400	0.7079	0.2260	7.30×10^1	1.63×10^3	3.71×10^2	0.687
0.425	0.7355	0.2112	6.57×10^1	1.32×10^3	3.11×10^2	0.744
0.450	0.7613	0.1971	5.92×10^1	1.07×10^3	2.46×10^2	0.726
0.475	0.7854	0.1837	5.35×10^1	8.73×10^2	1.96×10^2	0.710
0.500	0.8078	0.1713	4.85×10^1	7.17×10^2	1.56×10^2	0.686
0.525	0.8287	0.1596	4.40×10^1	5.92×10^2	1.27×10^2	0.685
0.550	0.8481	0.1484	4.00×10^1	4.88×10^2	1.06×10^2	0.698
0.575	0.8662	0.1375	3.63×10^1	4.02×10^2	8.82×10^1	0.711
0.600	0.8829	0.1270	3.29×10^1	3.30×10^2	7.32×10^1	0.723
0.625	0.8983	0.1170	2.98×10^1	2.71×10^2	6.14×10^1	0.747
0.650	0.9125	0.1072	2.69×10^1	2.20×10^2	5.19×10^1	0.783
0.675	0.9254	0.0977	2.41×10^1	1.78×10^2	4.38×10^1	0.826
0.700	0.9372	0.0884	2.16×10^1	1.42×10^2	3.70×10^1	0.882
0.725	0.9478	0.0793	1.91×10^1	1.12×10^2	3.12×10^1	0.952
0.750	0.9572	0.0703	1.68×10^1	8.62×10^1	2.60×10^1	1.038
0.775	0.9655	0.0616	1.46×10^1	6.50×10^1	2.17×10^1	1.156
0.800	0.9728	0.0532	1.25×10^1	4.77×10^1	1.70×10^1	1.240
0.825	0.9790	0.0456	1.07×10^1	3.46×10^1	1.25×10^1	1.249
0.850	0.9843	0.0385	8.95×10^0	2.45×10^1	1.11×10^1	1.582
0.875	0.9887	0.0307	7.11×10^0	1.54×10^1	8.59×10^0	1.957
0.900	0.9922	0.0242	5.57×10^0	9.47×10^0	5.02×10^0	1.861
0.925	0.9949	0.0191	4.40×10^0	5.91×10^0	3.16×10^0	1.878
0.950	0.9970	0.0151	3.47×10^0	3.67×10^0	1.96×10^0	1.882
0.975	0.9987	0.0118	2.70×10^0	2.23×10^0	1.38×10^0	2.175
1.000	1.0000	0.0079	1.81×10^0	1.00×10^0	1.00×10^0	3.522

(5.5) for rotational splitting. In the special case when the angular velocity Ω of the star is a function of radius alone, the result is

$$\delta\omega = mS^{-1}\int_{\tau_t}^{T}\left(1-\frac{L^2c^2}{\omega^2r^2}\right)^{-\frac{1}{2}}\Omega d\tau. \tag{6.7}$$

Alternatively, Eqs. (6.4) and (6.7) can be derived from ray theory, by representing the eigenfunctions as resonant superpositions of plane waves as described at the end of Section 3. In that case the relative frequency perturbations are seen to be averages of $c^{-1}\delta c$ and $m\omega^{-1}\Omega$ respectively, the averages being weighted by the time a wave spends in a given interval $d\tau$ of τ. The formulae are then seen to be quite natural.

The variational principle and ray theory can still be carried out when δc is aspherical or when Ω depends on the polar angles θ and ϕ. Associated with such perturbations is likely to be a meridional flow, but I shall ignore its influence on ω. I shall also here restrict attention to axisymmetric perturbations, and take the axis of my spherical polar coordinates to be the axis of symmetry. In that case the functions (3.7) with unique l and m (rather than linear combinations of such functions with different values of m) continue to represent the zero-order eigenfunctions, which simplifies the analysis considerably. One way to proceed with the variational principle is to expand δc or Ω in Legendre functions:

$$c^{-1}\delta c = \sum_k \beta_{2k}(r)P_{2k}(\cos\theta) \tag{6.8}$$

or a similar expression for Ω, and evaluate the angular integrals analytically. Only even terms are included in the sum because, as can easily be seen from the symmetry of the integrals, odd terms provide no contribution. In the case of acoustic asphericity, the frequency splitting between two modes of like n and l is given by

$$\omega^{-1}(\omega_m-\omega_0) \simeq \sum_k Q_{2k,lm}\langle\beta_{2k}\rangle, \tag{6.9}$$

where, provided $k \ll l$, $\langle\beta_{2k}\rangle$ is the average of β_{2k} weighted by $S^{-1}(1-L^2c^2/\omega^2r^2)^{-\frac{1}{2}}$ as in Eq. (6.4) and

$$Q_{2k,lm} = \frac{1}{2}(2l+1)\frac{(l-m)!}{(l+m)!}\int_{-1}^{1}P_{2k}(\mu)[P_l^m(\mu)]^2d\mu. \tag{6.10}$$

The subscripts on ω in Eq. (6.9) denote the value of the azimuthal order, and are included only where they are needed. We note that $Q_{2k,lm}=0$ when $k>l$. When $k \ll l$,

$$Q_{2k,lm} \simeq \frac{(-1)^k(2k)!}{2^{2k}(k!)^2}P_{2k}(m/L); \tag{6.11}$$

this approximate expression can be used as a first guide even when k quite close to l.

If ray theory is employed, the relative frequency perturbation is once again found to be the average of $c^{-1}\delta c$ weighted by the relative time a sound wave spends in a given element of the propagating region. The splitting is given by

$$\omega^{-1}(\omega_m-\omega_0) \simeq \frac{2}{\pi}\int_0^M (M^2-\cos^2\theta)^{-\frac{1}{2}}\left\langle\frac{\delta c}{c}\right\rangle d\cos\theta \tag{6.12}$$

(Gough, 1990), where $M^2 = 1 - m^2/L^2$ and, once again, the angular brackets denote the radial average defined by Eq. (6.4). The formula is valid only for modes of oscillation which vary with θ on a scale much less than the perturbation δc, which is equivalent to the condition that only coefficients with $k \ll l$ contribute substantially to the sum in Eq. (6.9). Equation (6.12) can also be obtained by combining Eq. (6.9) and (6.11) and using an asymptotic expression for $P_{2k}(\mu)$ (Kosovichev and Parchevskii, 1988). Equations similar to (6.9) and (6.12) hold for rotational splitting. Analagous asymptotic formulae for splitting due to magnetic fields with simple geometries are presented by Gough and Thompson (1990).

7 Inversion of Asymptotic Formulae: the Abel Calculus

It is evident that the average $\langle f\rangle$, defined as in Eq. (6.4), of any function $f(r)$, such as $c^{-1}\delta c$, is a function of the parameters defining the mode of oscillation only in the combination ω/L, as also is the integral S. The splitting data must therefore also be functions of only ω/L; they define weighted averages of f over domains beneath the surface which penetrate down to the lower turning points of the modes. By considering combinations of different averages one can deduce the function f.

As is well known, the equation can be transformed into Abel's integral equation [as can Eq. (6.12)] and, provided sufficient data are available, inverted to give f in terms of the data. To carry that out one represents the data, essentially the averages $\langle f\rangle$, by a sufficiently smooth function of ω/L [which one associates with the independent variable $\tau_{\rm t}$ via Eqs. (6.3) and (6.6)] and formally inverts the equation. Although the inversion is apparently an elementary procedure, it is useful to try to acquire some intuition about it, for then one might be able to assess the implications of new data even when one has no computer at hand.

Perhaps the first and crudest approach one might contemplate is to ignore the Abel kernel save for the lower turning point it implies, and approximate Eq. (6.4) by

$$\langle f\rangle \simeq (T-\tau_{\rm t})^{-1}\int_{\tau_{\rm t}}^{T} f\,d\tau. \tag{7.1}$$

As a result of ignoring the singularity at $\tau = \tau_{\rm t}$ the average of f is rather smoother than it should be, though the gross features of the general form are probably still discernable (*cf.* Christensen-Dalsgaard *et al.*, 1988). Evidently some appreciation of the data can therefore be obtained immediately by plotting them as a function of the lower turning point $\tau_{\rm t}$ associated with ω/L. One sees immediately that a slight change in the value of $\langle f\rangle$ with $\tau_{\rm t}$ must be the result of a more severe change in $f(\tau)$. Indeed, f depends on the derivative of $\langle f\rangle$, as can be seen by solving Eq. (7.1) for f:

$$f(\tau_{\rm t}) \simeq \langle f\rangle - (T-\tau_{\rm t})\frac{d\langle f\rangle}{d\tau_{\rm t}}. \tag{7.2}$$

Thus, in the case of rotational splitting, for example, one sees that the value of Ω at some position τ is not simply the value of the rotational splitting divided by m for modes with turning points at τ, as has sometimes been suggested in the literature. Indeed, as was illustrated by Gough (1984), the sun could have rotational splitting always less than the surface angular velocity yet have a rapidly rotating core, provided the splitting were to increase fast enough with ω/L at low L. However, because the average (7.1) provides excessive smoothing, its inverse (7.2) must overestimate the variation of f. Indeed, it can be badly in error in regions where $\langle f\rangle$ varies rapidly. The differentiation in Eq. (7.2) is too severe an operator. As I shall now demonstrate, it is basically the square root of the derivative operator that should be brought into play.

I begin by setting $\xi = \omega^2/L^2$ and introducing the depth coordinate $\zeta(\tau) = r^{-2}c^2$, neither of which should be confused with the depth coordinates used to describe the plane-parallel polytrope and the isothermal atmosphere in Section 2. Then Eq. (6.4) becomes

$$F(\xi) \equiv (\pi\xi)^{-\frac{1}{2}} S(\xi)\langle f\rangle = \pi^{-\frac{1}{2}} \int_{\xi_0}^{\xi} (\xi-\zeta)^{-\frac{1}{2}} \psi^{-1} f d\zeta, \tag{7.3}$$

where $\xi_0 = \zeta(T) = R^{-2}[c(R)]^2$ and

$$\psi = \frac{2c^3}{r^3}\left(1 - \frac{d\ln c}{d\ln r}\right). \tag{7.4}$$

This can be rewritten

$$F(\xi) = \mathcal{A}\psi^{-1} f, \tag{7.5}$$

where $\mathcal{A}$ is the Abel operator:

$$\mathcal{A}u = \pi^{-\frac{1}{2}} \int_{\xi_0}^{\xi} (\xi-\zeta)^{-\frac{1}{2}} u(\zeta) d\zeta. \tag{7.6}$$

Notice now that

$$\begin{aligned} \mathcal{A}^2 u &= \pi^{-1} \int_{\xi_0}^{\xi} (\xi-\zeta)^{-\frac{1}{2}} d\zeta \int_{\xi_0}^{\zeta} (\zeta-\eta)^{-\frac{1}{2}} u(\eta) d\eta \\ &= \pi^{-1} \int_{\xi_0}^{\xi} u(\eta) d\eta \int_{u}^{z} [(\xi-\zeta)(\zeta-\eta)]^{-\frac{1}{2}} d\zeta \\ &= \int_{\xi_0}^{\xi} u(\eta) d\eta. \end{aligned} \tag{7.7}$$

In other words, $\mathcal{A}^2$ is the integration operator. Consequently $\mathcal{A}$ is the square root of integration. Indeed, $\mathcal{A}$ is a special case of the Riemann-Liouville fractional integral. The solution to Eq. (7.5) is now evident. It simply requires inverting the operator $\mathcal{A}$, which yields what one should logically call the square root of differentiation:

$$f = \psi \frac{d^{\frac{1}{2}}}{d\xi^{\frac{1}{2}}} F, \tag{7.8}$$

the right-hand side being evaluated at $\xi = \zeta$. The halfth derivative in Eq. (7.8) can be regarded as the derivative of the halfth integral. Thus

$$\begin{aligned} f(\zeta) &= \psi \frac{d}{d\xi} \mathcal{A} F \\ &= \pi^{-\frac{1}{2}} \psi \frac{d}{d\zeta} \int_{\xi_0}^{\zeta} (\zeta-\xi)^{-\frac{1}{2}} F(\xi) d\xi. \end{aligned} \tag{7.9}$$

Eq. (7.9) can be regarded as a formal explanation of Eq. (7.8). It is what one must use to evaluate f; but perhaps the concept behind Eq. (7.8) is more illuminating. As is well known, differentiation exaggerates the variations in a typical function. The square root of differentiation also exaggerates variation, but less severely so, and can be regarded as performing just half the operation of transforming a function to its derivative. Thus, for example, if D represents differentiation, $D^{\frac{1}{2}}x^{\mu} = [\Gamma(\mu+1)/\Gamma(\mu+\frac{1}{2})]x^{\mu-\frac{1}{2}}$, where Γ is the gamma function. Fractionally differentiating the exponential function is somewhat more complicated, but when the argument is large the result simplifies: $D^{\frac{1}{2}}e^{kx} \sim k^{\frac{1}{2}}e^{kx}$ and $D^{\frac{1}{2}}\cos kx \sim k^{\frac{1}{2}}\cos(kx+\pi/4)$ as $kx \to \infty$. A more complicated function F and its derivatives $D^{\frac{1}{2}}F$ and DF are illustrated in Fig. 4.

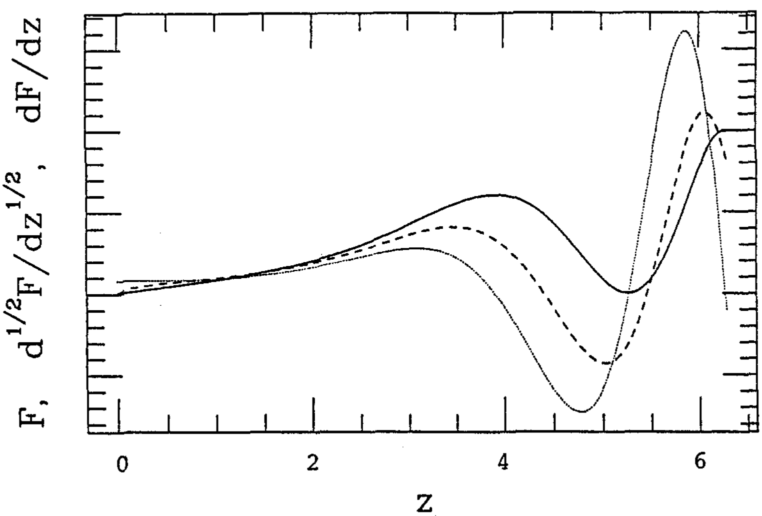

Fig. 4. The first derivative dF/dz (dotted curve) and the halfth derivative $d^{\frac{1}{2}}F/dz^{\frac{1}{2}}$ (dashed curve) of the function $F(z)$ represented by the continuous curve.

Evidently, if one is to assess data in this way one needs as part of one's inverter's tool kit the functions τ, ζ and ψ for the sun. A good solar model provides an adequate substitute, and in Table 2 (p. 303) I supply these, together with the sound speed c and the integrand $-x^{-1}dc/dx$, where $x = r/R$ is normalized radius, in the formula (4.2) for the coefficient A that appears in the asymptotic expression for low-degree p-mode eigenfrequencies.

I conclude this discussion by pointing out that fitting a smooth curve through the data is not a wholly straightforward operation. For example, if one tries to infer from the data the function $F(\omega/L)$ defined by Eq. (6.2) by finding that function $\alpha(\omega)$ which causes $\pi(n+\alpha)/\omega$ to be in some sense most closely a function of ω/L alone, one obtains a result such as that illustrated in Fig. 5. It is evident from the figure that $\pi(n+\alpha)/\omega$ depends also on some other parameter, because the points do not lie on a single curve; this renders it difficult to infer $F(\omega/L)$ precisely because it is not obvious how to draw

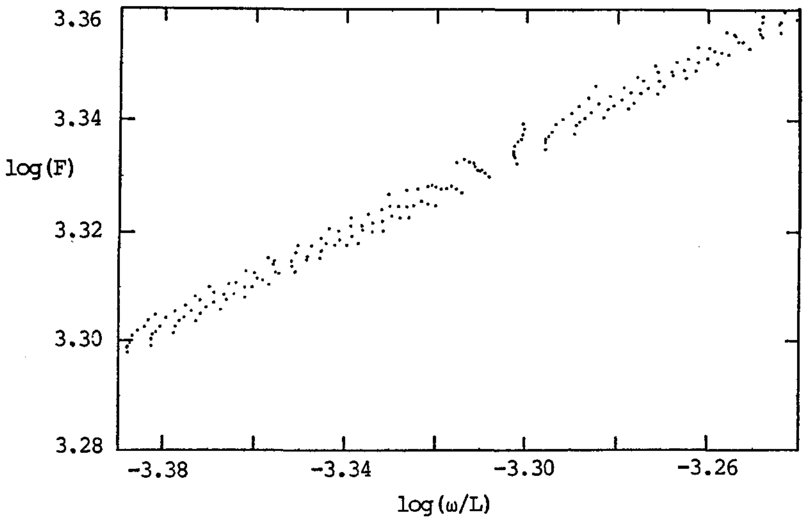

Fig. 5. Log $[\pi(n+\alpha)/\omega]$ computed from adiabatic oscillations of a standard solar model plotted against log $[\omega/(l+\frac{1}{2})]$ over a typical small range, where $\alpha(\omega)$ is chosen such as to minimize the scatter about a single curve for those modes whose lower turning points are in the convection zone.

a curve through the points. However, the deviations of the points from a single curve are systematic, and therefore contain extractable information about the structure of the star. The deviations arise partly from the neglect of N^2 and the inadequate treatment of ω_c^2 in approximating Eq. (3.12) by Eq. (6.2) (and also from the neglect of some relatively small curvature terms), partly from the neglect of the perturbation to the gravitational potential in the differential equations from which the eigenvalue Eq. (3.12) was derived, and partly from the asymptotic truncation adopted in deriving Eq. (3.12) from Eq. (3.6). With a better understanding of these approximations it should be possible to obtain an improved estimate of $F(\omega/L)$ than has hitherto been accomplished merely by adopting some arbitrary procedure to draw a representative curve through the points. This is so also of the formula for the differences in frequency between two similar models used for differential inversions and for the formulae for degeneracy splitting by symmetry-breaking agents. Indeed, the comparison between the asymptotic formula for differential inversions and the corresponding function of exact frequency differences presented by Christensen-Dalsgaard *et al.* (1988) suggests that some envelope of the data rather than a representative mean is more appropriate. More detailed asymptotic analysis will be necessary before such an envelope can be estimated from the frequency data alone.

8 The Signature of Rapid Variation

The JWKB approximation upon which most of the asymptotic analysis is based is valid only where the scale of variation of the background state is very much greater than the wavelength of the oscillations. This is not the case throughout the entire interior of the sun, and therefore in reality there are significant errors in the eigenvalue equation (3.12). One such error, as I have already discussed, comes from the inadequate treatment of the outer layers of the region of propagation where $\omega_c \simeq \omega$. Other regions of rapid variation in the background state are the He II ionization zone and the base of the convection zone. In the He II ionization zone there is a rapid variation of the adiabatic exponent γ which enters directly in the formula for the sound speed. In typical solar models the base of the convection zone is essentially a discontinuity in the second derivative of the sound speed, which produces a discontinuity in ω_c.

One can estimate the influence of localized rapid variation in two ways, and I illustrate both here. The first is via the variational principle discussed in Section 5. The reason I specialize to rapid variation of the background state which is localized is that then I can assume that because the JWKB analysis is valid almost everywhere, the values of the asymptotic p-mode eigenfunctions given by Eqs. (3.20) and (3.21) are adequate for substituting into Eqs. (5.3) and (5.4), as Fig. 1 illustrates. I shall discuss explicitly the localized variation of γ in the He II ionization zone. Noting that away from the turning points the vertical component of $\boldsymbol{\xi}$ dominates the displacement, and that for low-degree modes vertical derivatives of the high-order eigenfunctions are much greater in magnitude than horizontal derivatives, it is a good approximation to represent div$\boldsymbol{\xi}$ by the radial derivative of the radial component ξ of $\boldsymbol{\xi}$. I now consider the contribution from the first term in Eq. (5.4) and integrate it by parts twice, obtaining

$$\int \gamma p(\mathrm{div}\boldsymbol{\xi})^2 dV \simeq 2\pi \int p\frac{d^2\gamma}{dr^2}\xi^2 r^2 dr + \text{integrated parts.} \tag{8.1}$$

In obtaining this equation I have ignored derivatives with respect to r of r, p and ξ compared with the corresponding derivative of γ.

The reason for performing this integration by parts is to express the integral in terms of the undifferentiated eigenfunction ξ, which is much more reliably represented by the JWKB formula than is its derivative. It is evident now that the relatively rapid variation of γ in the He II ionization zone leads to a high value of $d^2\gamma/dr^2$, and hence a contribution to the total integral $\mathcal{K}$ approximately proportional to

$$\mathcal{K}_0 = \cos^2\left(\int_{r_i}^{r_2} \kappa dr + \frac{\pi}{4}\right) \simeq \frac{1}{2}[1 - \sin 2\omega(T - \tau_i)] \tag{8.2}$$

for low-degree modes, where r_i is the radius and $T - \tau_i$ the acoustical depth of the ionization zone. [In deriving the right-hand side of Eq. (8.2) I ignored the small l-dependent terms and removed ω_c from the integrand, incorporating it into the phase by replacing $\pi/4$ by $-\alpha\pi/2$ as in Eq. (6.2) and substituting the observed value $\alpha \simeq 3/2$.] Actually, the contribution $\mathcal{K}_0$ is an appropriate average of the expression (8.2) over the region of variation of γ, but if ω is not too great that average is similar to a point value. Thus there is a contribution to the integral $\mathcal{K}$, and hence to ω given by Eq. (5.2), which is itself an oscillatory function of ω with 'frequency' $2(T - \tau_i)$.

This contribution to $\mathcal{K}$ is quite small in magnitude, but can be recognized by its oscillatory behaviour. Thus, if the major contributions to the frequencies ω_n of order n and their first differences are eliminated by considering the second differences

$$\delta_2\omega \equiv \omega_{n+1} - 2\omega_n + \omega_{n-1}, \tag{8.3}$$

the oscillating term becomes apparent. This is illustrated in Fig. 6, where second differences of low-degree eigenfrequencies of a solar model are plotted against ω.

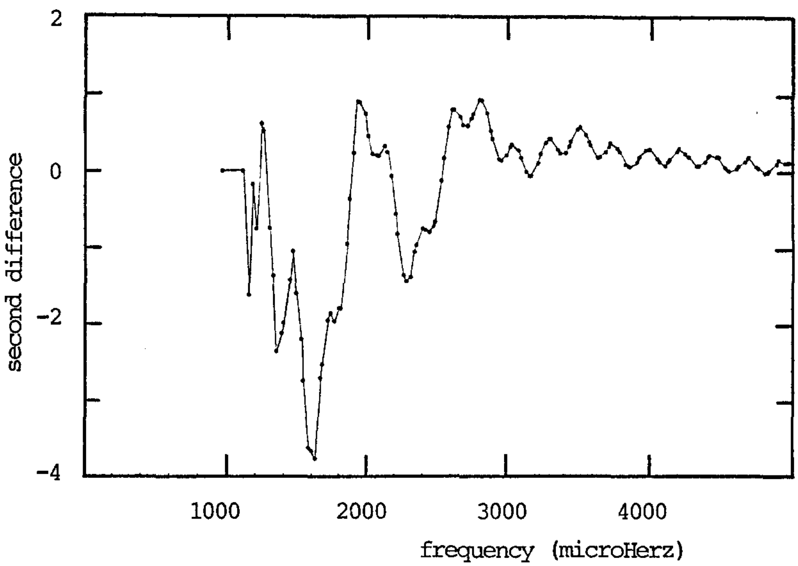

Fig. 6. Second cyclic frequency differences $\delta_2\nu$ defined as in Eq. (8.3) with ω_n replaced by $\nu_n = \omega_n/2\pi$, of modes with $l = 0, 1, 2, 3$ and 4 of a standard solar model, plotted against ν_n. The units are μHz.

The second method of estimating the effect of rapid variation is to regard the localized region as a discontinuity, and at that discontinuity match asymptotic eigenfunctions appropriate to conditions on either side of it. This procedure is perhaps more appropriate for modelling the discontinuity of ω_c^2 at the base of the convection zone, which leads to a discontinuity in the coefficient κ^2 of the undifferentiated term in Eq. (3.8). Rather than present the details of the stellar asymptotics explicitly, I treat a much simpler but mathematically similar problem which exhibits the essence of the influence of the discontinuity that I wish to illustrate. I consider the longitudinal acoustic oscillations of gas in a pipe governed by the simple equation

$$\frac{d^2\Psi}{dz^2} + \frac{\omega^2}{c^2}\Psi = 0, \tag{8.4}$$

in which $c = c_1$ if $0 \le z < \tilde{\lambda}a$ and $c = c_2$ if $\tilde{\lambda}a < z \le a$, where $\tilde{\lambda}$, c_1 and c_2 are constants. Adopting, for simplicity, the boundary conditions $\Psi = 0$ at $z = 0$ and $z = a$, with Ψ and $c^2 d\Psi/dz$ continuous at $z = \tilde{\lambda}a$, then yields the eigenvalue equation

$$c_2 \tan \omega\tau_1 + c_1 \tan \omega\tau_2 = 0, \tag{8.5}$$

in which $\tau_1 = \tilde{\lambda} a/c_1$ and $\tau_2 = (1 - \tilde{\lambda})a/c_2$ are the acoustical lengths of the two uniform regions in the pipe. It is straightforward to show from the eigenvalue equation (8.5) that if $c_2 = (1 + \epsilon)c_1$ with $|\epsilon| \ll 1$, then again there is an oscillatory component to ω, such that

$$\delta_2\omega \propto \epsilon \sin\left(2\omega\tau_1 + \tan^{-1}\sin\frac{2\pi\tau_1}{\tau_1 + \tau_2}\right), \tag{8.6}$$

which also has a 'frequency' that is twice the acoustical length of one of the uniform regions. [Since, in this case, $\omega \simeq n\pi/(\tau_1 + \tau_2)$, τ_1 and τ_2 are interchangeable in the expression on the right-hand side of Eq. (8.6).]

Alternatively, one can consider a mathematical model given by

$$\frac{d^2\Psi}{dz^2} + V\Psi = 0 \tag{8.7}$$

with $V(z;\omega) \equiv (\omega^2 - \omega_c^2)/c^2$ being piecewise constant with respect to z. The model can be considered to be isothermal throughout, with a constant density scale height, the discontinuity in V arising from a discontinuity in the density scale height. This simple model exhibits the character of the base of the convection zone in a standard solar model more closely. Its eigenvalue equation is

$$\tilde{\omega}_2 \;\tan\; \tilde{\omega}_1\tau_1 + \tilde{\omega}_1 \;\tan\; \tilde{\omega}_2\tau_2 = 0, \tag{8.8}$$

where $\tilde{\omega}_i^2 = \omega^2 - \omega_{ci}^2$, ω_{c1} and ω_{c2} being the values of the critical cutoff frequency in regions 1 and 2 respectively. In much the same way as before one can set $\omega_{c2} = (1+\epsilon)\omega_{c1}$ and expand the eigenfrequencies to first order in ϵ. Again an oscillatory component is found, whose 'frequency' tends to $2\tau_1$ for high-order modes for which $\omega^2 \gg \omega_{c1}^2$.

Oscillations due to both the discontinuity at the base of the convection zone and the variation of γ in the He II ionization zone are evident in Fig. 6, where $\delta_2\omega/2\pi$ (evaluated at fixed degree l) is plotted against $\omega/2\pi$ for modes with $0 \leq l \leq 5$. Several low values of l are plotted in order to give a great enough density of points to exhibit the oscillations; high-degree modes are not included, because my approximations demand that the discontinuity be far above the lower turning point, where the actual value of l is immaterial. The most prominent feature of the data is a large-amplitude oscillation, with a 'period' of about 750μHz, which corresponds to a localized variation in the background state at an acoustical depth $T - \tau \simeq (2 \times 750\mu\text{Hz})^{-1} \simeq 670\,$s. Thus, since $T \simeq 3518\,$s, $\tau/T \simeq 0.81$. One can now consult Table 2 (p. 303) to learn that this corresponds to a radius $r \simeq 0.975R$. The oscillation is therefore presumably a consequence of the variation of γ due to He II ionization. (50 per cent of He is doubly ionized at the radius $0.975R$.) Notice that its amplitude decreases with increasing frequency. This must be a result of the variation of γ occurring over an extended region, of thickness d, say, of the star. As ω increases, the wavelength λ of the eigenfunction decreases, as is evident from Eq. (8.2), and once λ becomes comparable with d cancellation of the contribution to the integral (8.1) from the oscillatory component of ξ becomes significant. The value of the amplitude of the large-scale oscillation in the points plotted in Fig. 6 depends on the magnitude of the variation of γ, which in turn depends on the abundance Y of helium. Thus perhaps we are offered a potential diagnostic to determine Y through a measurement of the equation of state which does not depend on the overall structure of the star, and therefore ought to be insensitive to many of the assumptions of the theory of stellar evolution and errors in the uncertain values of the opacity and the nuclear reaction rates. This particular diagnostic is therefore very exciting because, unlike the previous suggestion for making an almost direct seismic measurement of Y in the convection zone (Däppen and Gough, 1986), with sufficiently accurate data it might perhaps be possible to carry out model calibrations using only modes of low degree that can be obtained from integrated whole-disk measurements, which might therefore also be applicable to data from other stars too. A careful analysis of the sensitivity of the oscillation to the value of Y and to uncertainties in the theoretical models will be required before we can judge whether the diagnostic is likely to be of practical use in the foreseeable future.

The other obvious feature in Fig. 6 is the small-scale oscillation with a 'period' of about 230μHz. This has a smaller amplitude than the He II oscillation, at least at low frequency, which does not diminish with increasing ω, indicating that the variation in the background state that causes it is highly localized. It occurs at an acoustical depth of $(2 \times 235\mu\mathrm{Hz})^{-1} \simeq 2130\,\mathrm{s}$, corresponding to $\tau/T \simeq 0.40$ which is situated at a radius $r \simeq 0.71R$, precisely at the base of the convection zone. Notice that this particular property of the oscillation data arises from a discontinuity in ω_c arising from the discontinuity in the second derivative of density in the theoretical solar model, and not from some perturbing agent such as a confined magnetic field which would have had a superficially similar signature. Unlike a magnetic field, however, this property is spherically symmetrical, and therefore does not contribute to degeneracy splitting. Thus there might perhaps be some means of differentiating between the two phenomena.

9 Solar-cycle Frequency Variations

I close with a few remarks about the spectacular new observations reported at this meeting by Libbrecht and Woodard. I shall concentrate on the frequency dependence of the overall frequency change from 1986 to 1988, which was found to be approximately inversely proportional to the inertia $\mathcal{I}$ of the mode, where $\mathcal{I}$ is given by Eq. (5.3) for a displacement eigenfunction $\boldsymbol{\xi}$ normalized such that $\Xi(R) = 1$. What does this imply?

It is useful to discuss the result in terms of the variational principle of Section 5. The effect of a small perturbation to the structure of the sun on the oscillation eigenfrequencies can then be obtained by perturbing Eq. (5.2):

$$\delta\omega = \frac{\delta\mathcal{K} - \omega^2\delta\mathcal{I}}{2\omega\mathcal{I}}, \tag{9.1}$$

where $\delta\mathcal{K}$ and $\delta\mathcal{I}$ are the perturbations to $\mathcal{K}$ and $\mathcal{I}$ at constant $\boldsymbol{\xi}$. It is immediately clear that for $\delta\omega$ to be proportional to $\mathcal{I}^{-1}$, $\delta\mathcal{K} - \omega^2\delta\mathcal{I}$ must be proportional to ω.

Rather than using the variational principle in the form given explicitly in Section 5, it is safer to write it with respect to independent variables that remain constant on the surface of the star, for then it is not necessary to consider variations of the limits of integration when the solar surface is perturbed (*cf.* Gough and Thompson, 1990). It will soon be evident that I shall be discussing only perturbations near the surface of the sun, where all p modes of low and moderate degree have essentially the same spatial structure. Moreover, the perturbation will be predominantly spherically symmetrical. Therefore for the purposes of calculating $\delta\mathcal{K}$ I shall sometimes find it convenient to simplify the discussion to radial modes of a spherical star, replacing $\mathcal{K}$ by (*cf.* Ledoux and Walraven, 1958)

$$\int_0^M \left\{ rc^2 \left(\frac{d\tilde{\xi}}{dr}\right)^2 - \frac{1}{\rho}\frac{d}{dr}\left[(3\gamma-4)p\right]\tilde{\xi}^2 \right\} rdm \tag{9.2}$$

with respect to the Lagrangian variable m, which is the mass enclosed in a sphere of radius r: $dm = 4\pi r^2 \rho dr$. Also $\tilde{\xi} = r^{-1}\xi$, ξ being the radial component of $\boldsymbol{\xi}$, and M is the total mass of the star.

Before continuing I should point out that the frequency dependence of the inertia $\mathcal{I}$ of the mode is quite steep, and depends predominantly on conditions in the outer layers of the sun where the mode is evanescent, and not on the structure of the deep interior. This is because the normalization of the eigenfunction is carried out at the surface of the star, where the motion can be observed, rather than deep in the interior where most of the energy resides. Indeed, $\mathcal{I}$ depends simply on the depth $z_2 = R - r_2$ of the upper turning point. To see this, note that according to the asymptotic relations (3.20) and (3.21), $r^2\rho c\boldsymbol{\xi}\cdot\boldsymbol{\xi}$ is approximately constant in the region of propagation, aside from rapidly oscillating sinusoidal terms, so that the contribution to $\int \rho\boldsymbol{\xi}\cdot\boldsymbol{\xi}r^2dr$ from that region is proportional to $\int c^{-1}dr$, which is the sound travel time between the turning points. For p modes of low and moderate degree, this depends only quite weakly on the radius of those turning points, because the sound speed is relatively high near the lower turning point and conditions vary so rapidly near the upper turning point that the frequency variation of the acoustical radius τ_2 at that point, relative to the total acoustical radius T of the sun, is quite small. The value of $\mathcal{I}$ therefore depends predominantly on the normalization $\Xi(R) = 1$, which requires relating the magnitude of Ξ in the acoustic cavity with the value at the surface. As we learned from the discussion of vertical oscillations of a plane-parallel polytrope, Ξ varies slowly near the very top of the evanescent region where $\omega < \omega_c$, and for a polytrope $\omega_c \propto z^{-\frac{1}{2}}$. Thus, at the depth z_2 at which $\omega_c = \omega, c \propto z^{\frac{1}{2}}$ which is proportional to ω^{-1} and $\rho \propto z^\mu$ which is proportional to $\omega^{-2\mu}$, where μ is a polytropic index characterizing the stratification near $z = z_2$. Consequently, $\mathcal{I} \propto \omega^{-2\mu-1}$, and therefore $\mathcal{I} \propto \omega^{-7}$ if $\mu = 3$, the value suggested by Duvall's analysis (see p. 301). It is straightforward to confirm this result by direct computation using the oscillation eigenfunctions of the plane-parallel polytrope discussed in Section 2. The outer layers of the sun are rather more complicated than a simple polytrope, and consequently $\mathcal{I}$ is not a simple power of ω, but the value of the polytropic exponent $-(2\mu+1) \simeq -7$ is typical of $d\ln\mathcal{I}/d\ln\omega$ for a realistic solar model.

What we have learned from this discussion is that, relative to the surface displacement $\Xi(R)$, the amplitude of the displacement is approximately $\Xi(R)$ well above the upper turning point, and is proportional to $\omega^{-\mu}\Xi(R)$ well below it. Thus, any perturbation to the structure of the star well above the turning points of the modes leads to a weak frequency dependence of $\delta\mathcal{K} - \omega^2\delta\mathcal{I}$, and hence, by Eq. (9.1), to a frequency perturbation $\delta\omega$ which is roughly proportional to $\mathcal{I}^{-1}$. A perturbation well beneath the turning points of the modes, on the other hand, introduces a factor $\omega^{-2\mu}$ into the numerator of Eq. (9.1) which largely cancels the factor $\mathcal{I}$ in the denominator, and the frequency perturbation $\delta\omega$ then varies only weakly with ω. We can conclude with Libbrecht and Woodard, therefore, that the variation in the structure of the sun responsible for the observed frequency changes must lie in the very outer layers where the modes are evanescent. Their effect can be regarded simply as a modification to the reflecting boundary condition that the inert outer layers present to the wave propagating in the cavity beneath.

The precise frequency dependence of $\delta\omega$ depends on the nature of the variation of the background state. Continuing to use the plane-parallel polytrope as a guide, we note from the discussion on pp. 295-296 that in the vicinity of the free surface div$\boldsymbol{\xi} \propto \omega^2\xi$. Therefore, from Eqs. (9.1), (5.3) and (5.4) or (9.2), it appears that a superficial variation in the solar structure leads to a frequency change of the form

$$\delta\omega \propto \frac{Q(\omega^2)}{\omega\mathcal{I}}, \tag{9.3}$$

where $Q(x)$ is a quadratic function of x. Some care must be taken in interpreting this result, however, because the stratification of the sun is essentially hydrostatic, and one cannot arbitrarily modify ρ and c^2 without addressing how that modification might come about. Let us consider a few examples.

If there were a magnetic field introduced into the evanescent layers, which for the moment I shall suppose is concentrated into fibrils by the convective motion, then the characteristic propagation speed of acoustic waves is modified, without necessarily making a serious impact on the mean density stratification. This could be modelled by a change in c^2 appearing in $\mathcal{K}$, and hence $Q \propto \omega^4$ and $\delta\omega \propto \omega^3/\mathcal{I}$, which is the result suggested by Woodard and Libbrecht. If, alternatively, the dominant effect is to change the efficacy of convection and thereby modify principally the scale heights in the rapidly varying superadiabatic boundary layer, then one might expect the modification of the last term in the expression (9.2) for $\mathcal{K}$ to dominate the perturbation, and $\delta\omega \propto (\omega\mathcal{I})^{-1}$. More subtle perturbations, such as modifications to the density that leave the density jump across the superadiabatic boundary layer unmodified, lead to intermediate variation: if the solar radius is unperturbed, they are sensitive mainly to the second term in the integrand in the expression (5.4) for $\mathcal{K}$. Then $\delta\omega \propto \omega/\mathcal{I} \propto \omega^{2(\mu+1)} = \omega^8$ when $\mu = 3$. These results are exhibited by the analyses of the vertical oscillations of the plane-parallel polytrope with a superposed isothermal atmosphere mentioned in Section 2. The variation $\delta\omega \propto \omega^{2\mu}$ arises from variations in a temperature (and density) discontinuity modelling the superadiabatic convective boundary layer, and can be considered to result from the change in the reflection coefficient at the discontinuity. The variation $\delta\omega \propto \omega^{2(\mu+1)}$ arises from variations in the value of the temperature of an atmosphere that matches continuously onto the polytrope (Christensen-Dalsgaard and Gough, 1980), and arises from a variation in the inertia of the oscillating atmosphere.

Of the three examples I have mentioned, the second and third have frequency dependences that are closest to the observations. Indeed, it might be the case that either $\delta\omega \propto (\omega\mathcal{I})^{-1}$ or $\delta\omega \propto \omega/\mathcal{I}$ is not a significantly poorer representation of the data than the $\mathcal{I}^{-1}$ dependence reported by Libbrecht and Woodard. The former might arise from variations in the efficacy of convection, brought about, perhaps, by magnetic-field variations, which modify the structure of only the very outer layers of the sun within and above the superadiabatic boundary layer. Indeed, this was the first idea that came to mind (Gough and Thompson, 1988) to explain solar acoustic asphericity implied by the even component of the degeneracy splitting originally reported by Duvall *et al.* (1986).

The asphericity was modelled by a latitudinal variation in the mixing length used to determine the convective heat and momentum transport, and the structure was computed by integrating the equations governing a spherically symmetrical star inwards from the photosphere, adjusting the mass, radius and effective temperature at the photosphere to ensure that conditions beneath the convection zone were invariant, in order to maintain the horizontal component of hydrostatic balance at greater depths. Of course, this implied a latitudinally varying heat flux, which was presumed to have been accommodated by a small relative horizontal component of the tiny superadiabatic temperature gradient in the essentially adiabatically stratified interior of the convection zone where the thermal capacity of the convecting fluid is very great, thereby permitting the convection zone to be supported by a spherically symmetrical radiative interior.

The response of the radial stratification of the convection zone to the value of the mixing length depends critically on how the zone abuts onto the radiative interior. The computations by Thompson and myself to mimic the latitudinal variation were carried out before any temporal variations had been established, and a model in thermal balance was sought (Gough and Thompson, 1988). The effect of reducing the efficacy of convection in the equatorial regions is to contain the heat more effectively within the star, causing in this case at almost all depths within the convection zone an increase in the sound speed and a consequent expansion, inducing an oblateness of surfaces of constant temperature which is compenstated by a mass transfer from the poles towards the equator. The perturbation to the value of the sound speed c varies weakly with depth beneath the superadiabatic boundary layer, leading to a variation of c^{-1} (upon which the acoustical depth, and hence also the oscillation frequencies depend) which is concentrated near the surface of the model. In the superadiabatic boundary layer the temperature gradient is substantially greater in the equatorial regions, to counteract the reduction in the efficacy of convective transport; this leads to a reduction in the photospheric temperature, despite the fact that the material beneath the boundary layer is hotter.

A temporal change of magnetic activity over all latitudes causing a modulation in convective efficacy is likely also to modify the stratification predominantly in the outer superadiabatic boundary layer, though the overall reaction of the convection zone would be somewhat different. In particular, the mass beneath the photosphere is conserved. Conditions at the base of the convection zone would not be temporally invariant, and the perturbed structure should really be matched onto an essentially adiabatically perturbed radiative interior in hydrostatic equilibrium (*cf.* Gough, 1981). Nevertheless, the amplitude of the perturbation of the interior of the star is very small compared with the perturbation at the top of the convection zone, and the latitudinal variation in the surface layers computed by Gough and Thompson (1988) therefore probably mimics the temporal variation, and can therefore be used for a first estimate of the frequency changes. It is therefore instructive to consider these frequency changes in more detail.

In Fig. 7 is plotted frequency variations for several values of the degree l. Reducing the efficacy of convection increases the frequencies of the modes. The magnitude of the reduction of the mixing length, some 2 per cent, has been chosen to yield frequency variations of about the same magnitude as those reported by Libbrecht and Woodard in these proceedings. The functional form is very similar to the observations too, confirming that a perturbation to the convective boundary layer is likely to be responsible for the frequency changes. At low degree l the frequency changes are independent of l, but as l increases the magnitude of the changes increases. This is largely due to the reduction in the modal inertia $\mathcal{I}$. The frequency changes of 3 mHz modes are plotted as a function of l by Gough and Thompson (1988), and fit the values plotted in Fig. 3 of Libbrecht and Woodard (1990) somewhat better than the inverse modal inertia, though not significantly so.

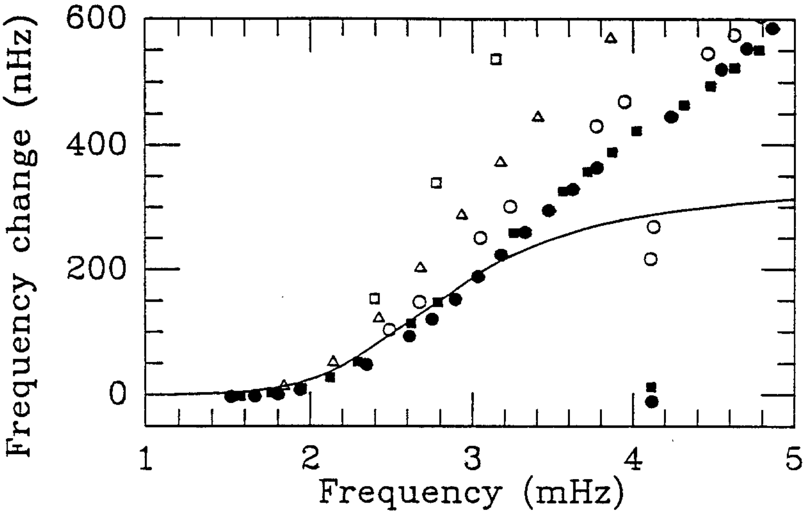

Fig. 7. Frequency dependence of the frequency differences $\Delta\nu$ resulting from a change in mixing length of a solar envelope model (from the computations discussed by Gough and Thompson, 1988) for a selection of modes with $l = 8(\bullet)$, $l = 15(\blacksquare)$, $l = 30(\circ)$, $l = 100(\triangle)$ and $l = 200(\square)$. The units are nHz. The continuous line is proportional to the inverse inertia $\mathcal{I}^{-1}$ corresponding to modes with $l = 0$ of a standard solar model, where $\mathcal{I}$ is defined by Eq. (5.3) normalized such that $\Xi = 1$ at an optical depth of 10^{-2}. This curve does not lie precisely on the low-degree points because $\mathcal{I}$ would be somewhat different, particularly at high frequencies, if it were normalized (more appropriately, for this discussion) at the location where the change in the model is greatest. The modes with $\nu \simeq 4.1$ mHz with small frequency differences resonate with the chromosphere.

Despite the ability of the model to reproduce the observed frequency shifts, it cannot be correct as it stands. The model requires a reduction in effective temperature from 1986, at sunspot minimum, to 1988 of 0.7 per cent. This is contradicted by the observed rise in irradiance reported by Willson and Hudson (1988) from the ACRIM data, implying that an appropriate spatial average of the effective temperature increased during that interval, by only 10^{-2} per cent. Moreover, the latitudinal temperature variation is similarly at variance with the brightness-temperature measurements of Kuhn *et al.* (1988) which is what led to the rejection of the model as an explanation of the acoustic asphericity (Gough and Thompson, 1988). However, the response of the convection zone to a solar-cycle perturbation, varying on a timescale much shorter than the characteristic cooling time of the entire zone (about 10^5 years), might perhaps be rather different from the structure which would result if the zone were given time enough to achieve thermal balance with its surroundings. Therefore, magnetic inhibition of convection is not yet ruled out by this model as a candidate for the frequency variations.

It should perhaps be remarked that Kuhn's (1988a, b) subsequent suggestion of a depth-independent relative temperature variation of about 0.1 per cent extending down to some radius r_0 (where $r_0 \lesssim 0.95R$, and might be as small as $0.2R$ or less) substantially beneath the superadiabatic convective boundary layer also cannot account for Libbrecht and Woodard's observations. As is evident from the discussion above, the frequency dependence of the splitting coefficients predicted by that hypothesis is much weaker than $\mathcal{I}^{-1}$, which is at variance with that reported by Libbrecht and Woodard (1990).

Of course, there are other possibilities for modifying the wave-propagation speed and the location of the upper reflecting boundary of the acoustical cavity, such as the direct influence of the Lorentz force, and these too need to be investigated. Indeed, Gough and Thompson (1988) suggested that the frequency changes may be a result of changes to fibril magnetic fields which might change the propagation speed of large-scale acoustic waves without substantially modifying the mean thermal stratification of the convection zone. Numerical simulations of Boussinesq convection in a magnetic field have shown that at high Rayleigh number the magnetic field is expelled to the interstices between convective cells without modifying greatly the convective flow elsewhere, except possibly to change the preferred horizontal length scale. There is certainly evidence that the scale of solar granulation is different in active regions from that in the quiet sun. Of course, the energy flux is altered, but so too is the magnitude of the inhomogeneities, and since the mean heat flux and the mean acoustic propagation speed depend differently on the inhomogenieties, p-mode oscillation frequency changes could be induced without such large energy-flux changes as the current mixing-length models seem to require. These new observations will certainly provoke considerable further thought.

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