THE FREQUENCY DISTRIBUTION OF STELLAR OSCILLATIONS

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The frequency distribution of adiabatic natural oscillations of a nonuniform gravitating gas sphere is investigated. Asymptotic formulas are derived for the periods of oscillations having a large number of nodes along the radius. The frequencies are evaluated numerically for a polytropic configuration. These data are used to analyze the circumstances of resonance excitation of nonradial oscillations in a radially pulsating star, assuming that its structure is polytropic. The conditions are highly favorable for excitation if there is a large central concentration of material but the probability of instability decreases sharply if the polytropic index n falls below a certain value ($\approx$1.8 for a star consisting of monatomic gas, with small radiation). In this range of n a resonance can arise under certain conditions, resulting in the development of two oscillations with similar frequencies. The analysis supports the hypothesis that the density distribution in Cepheids does not differ substantially from a polytrope with $n \approx 1.5$.

Introduction

In a star subject to the influence of a perturbing force varying with time, a resonance excitation of certain natural oscillations is possible. For a radially pulsating star, an instability of this type has been discussed in previous papers [1, 2]. Since the resonance condition takes the form of a relation between oscillation periods, a detailed study of the entire frequency spectrum is required for an analysis of the instability.

The fundamental laws for the frequency distribution of spherically symmetric pulsations are quite well known (a detailed account of the problem has been given, for example, in the survey papers by Ledoux and Walraven [3]). Ledoux [4] has also obtained a formula for determining the periods of the high radial-pulsation modes.

The character of the spectrum of nonradial adiabatic oscillations of a gas sphere has been investigated in several papers [3, 5-12]. A uniform model has been treated by Pekeris [5], and a polytropic configuration was studied by Cowling [6].

Cowling showed [6] that in addition to a high-frequency branch, a stable gas sphere will have a branch whose periods increase without bound as the number of nodes along the radius increases (the radial "quantum number" k). For high-frequency oscillations (p_k modes, in Cowling's terminology) the motion is primarily radial, and is accompanied by large changes in pressure, while in the low-frequency case (g_k modes) the motion is preferentially horizontal. Oscillations of intermediate frequency also exist, with a monotonic dependence of amplitude on radius (the fundamental mode f).

The numerical values of certain frequencies have been determined in [6-12]. Use has been made both of approximate methods of calculation [6, 8, 9] and the exact method of integrating directly in the initial 4th-order differential equation [10-12]. In particular, in a recently published paper [12] the frequencies of radial pulsations and nonradial oscillations in the fundamental mode have been computed for various polytropic configurations.

To carry out an analysis of the conditions for resonance excitation of nonradial oscillations in a radially pulsating star, a further investigation of the frequency spectrum has been found necessary. In this paper we derive asymptotic formulas for the periods of nonradial adiabatic oscillations having a large number of nodes along the radius (Secs. 3-4). The results are presented of a numerical integration of the system describing the oscillations of a gas sphere of polytropic structure (Sec. 5). This material is applied in Sec. 6 to an analysis of the resonance instability.

1. Basic Equations

Let $\rho(r)$ and $p(r)$ represent the density and pressure distributions in an equilibrium state, and let the velocity $\vec{v} = 0$. We have

$$r^2 \frac{dp}{dr} = -4\pi G\rho \int_0^r \rho r^2\, dr. \qquad (1)$$

In the oscillating state, let the perturbation of the gravitational potential in the spherical coordinate system (r, ϑ, φ) be $\Psi(r) Y_{lm}(\vartheta, \varphi) \exp i\omega t$, where $Y_{lm}(\vartheta, \varphi) = P_l^m(\cos\vartheta) \exp im\varphi$, $l = 0, 1, 2, \ldots$, $m = 0, \pm 1, \ldots, \pm l$. Let $X(r)$ and $Q(r)$ denote the radius-dependent amplitudes of the radial component of the displacement and the ratio of the density perturbation to the equilibrium value of the density. The system of oscillation equations may then be written [2] in the form

$$\left.\begin{aligned} \omega^2\rho X - \rho\Psi' + p'Q - \frac{d}{dr}[\gamma p(Q+\chi X)] = 0, \\ \omega^2 r^2 \rho Z - \rho\Psi - \gamma p (Q + \chi X) = 0, \\ \frac{d}{dr}(r^2\Psi') - l(l+1)\Psi - 4\pi G r^2 \rho Q = 0, \end{aligned}\right\} \qquad (2)$$

where

$$Z = \frac{1}{l(l+1)}\left[Q + \frac{1}{r^2\rho}\frac{d}{dr}(r^2\rho X)\right], \quad \chi = \frac{\rho'}{\rho} - \frac{p'}{\gamma p},$$

γ is the adiabatic index, and the prime represents differentiation with respect to r.

The eigenfrequencies $\omega = \omega_{kl}$ correspond to the solutions $X = X_{kl}$, $\Psi = \Psi_{kl}, \ldots$, which satisfy the condition of boundedness for $r = 0$ and approach zero for large r. It is known [3] that the oscillations will be harmonic if $\gamma > 4/3$ and if for all r the function $\chi(r) \leq 0$.

2. Case of a Uniform Sphere

We shall first consider the well known problem of the oscillations of a uniform gas sphere of radius R. In the equilibrium state the density $\rho = \rho_c$ = const, and the pressure decreases from the center by a parabolic law.

By integrating the first and third equations of the system (2), one can establish without difficulty [1] that in the surface layer where the density decreases sharply from ρ_c to zero, the expressions $P[Q + (\rho'/\rho)X]$ and $r^2\Psi' + 4\pi G r^2 \rho X$ remain continuous. Hence in view of the formula $\Psi(r)|_{r \geq R} = \Psi(R)(R/r)^{l+1}$ we obtain boundary conditions for the functions X, Q, and Ψ.

The general solution of Eqs. (2), as was shown by Pekeris [5], may be expressed in terms of the hypergeometric function $F = F(k + l + 3/2, -k-1, l + 3/2, r^2/R^2)$, k = const. For the eigenfunctions the parameter $k = 0, 1, 2, \ldots$, and one finds

$$\left.\begin{aligned} X_{kl} = r^l\left\{\frac{dF}{dr} + \left[1 + \frac{4\pi G\rho_c(l+1)}{3\omega^2_{kl}}\right]\frac{lF}{r}\right\}, \\ \Psi_{kl} = -4\pi G\rho_c r^l F, \quad Q_{kl} = \frac{4(b_{kl}+2)}{\gamma(R^2 - r^2)} r^l F, \end{aligned}\right\} \qquad (3)$$

$$b_{kl} = \gamma(k+1)\left(k + l + \frac{3}{2}\right) - 2,$$

$$\frac{\omega_{kl(p)}}{\sqrt{4\pi G\rho_c}} = \sqrt{\frac{1}{3}[b_{kl} + \sqrt{b^2_{kl} + l(l+1)}]}, \quad k = 0, 1, 2, \ldots, l = 0, 1, 2, \ldots \qquad (4)$$

$$\frac{\omega_{kl(g)}}{\sqrt{4\pi G\rho_c}} = \sqrt{\frac{1}{3}[b_{kl} - \sqrt{b^2_{kl} + l(l+1)}]}, \quad k = 0, 1, 2, \ldots, \quad l = 1, 2, \ldots. \qquad (5)$$

The parameter k is related to the number of nodes along the radius. The indices p and g in Eqs. (4) and (5) are introduced in accordance with Cowling's terminology [6].

There are also solutions of the system (2) for which the function F approaches unity, namely

$$X_l = lrZ_l = \frac{3l\Psi_l}{(3\omega_l^2 - 4\pi G\rho_c l) r} = \text{const}\cdot r^{l-1}, \quad Q_l = 0, \ l \geq 1. \qquad (6)$$

By substituting into the boundary conditions one obtains the known formula [13]

$$\frac{\omega_{l(f)}}{\sqrt{4\pi G\rho_c}} = \sqrt{\frac{2l(l-1)}{3(2l+1)}}, \quad l = 2, 3, \ldots. \qquad (7)$$

Because of the monotonic dependence of the displacement $X_l(r)$ on the radius, the frequencies (7) refer to the fundamental mode (index f), in agreement with Cowling's determination [6].

However, in the case of the lowest-frequency radial oscillations (mode p_0 for $k = 0$, $l = 0$), the function $F = 1 - (r^2/R^2)$ in Eqs. (3), so that the displacement X_{00} will be proportional to r, that is, a monotonic function of the radius. For this reason the mode in question is often called the fundamental mode. To avoid confusion, we shall call the frequency $\omega_{00(p)}$ the basic frequency.

The existence of oscillations $l=1$ in the fundamental mode would result in a motion of the center of inertia of the whole system, so that $\omega_{1(f)}=0$. For the p_k and g_k modes the amplitude of the displacement of points of the medium will be a sign-changing function of the radius, and oscillations $l=1$ will become possible.

3. Asymptotic Behavior of the High-Frequency Branch

We turn now to oscillations with a large number of nodes along the radius ($k \gg 1$). We shall regard the configuration as bounded, with a continuous density gradient at all points except on the boundary surface r = R. For r = R we have $\rho(R)=0$, $p(R)=0$. Let the densitv near the surface r = R be proportional to $(R-r)^{n_R}$, $n_R>0$; then the pressure in the same region will be equal to const $\cdot (R-r)^{n_R+1}$, so that

$$\chi(r)\,|_{R-r\ll R} \approx -\frac{1}{R-r}\left(n_R-\frac{n_R+1}{\gamma}\right), \quad n_R>0. \tag{8}$$

We shall also assume that the derivative ρ' is small near the center r = 0.

For the oscillations under consideration the amplitude of the displacement will change sign many times as a function of radius, so that the perturbation in the potential, as determined by a certain average value of the density perturbation, will be a small quantity. It is known [6, 8, 3] that in the first approximation the perturbation in the potential may be omitted from the equation of motion.

Neglecting Ψ and Ψ' in the first two equations of the system (2), and setting

$$\left.\begin{aligned} p^{1/\gamma}X &= \frac{\sqrt{\mu}}{r^2}T = \frac{1}{\nu}\frac{d}{dr}\left(\frac{\sqrt{\nu}}{r}S\right), \quad p^{-1/\gamma}r^3\rho Z = \sqrt{\nu}\,S \\ &= \frac{r}{\omega^2\mu}\frac{d}{dr}(\sqrt{\mu}\,T), \\ \mu &= p^{2/\gamma}\left[\frac{l(l+1)}{\omega^2\rho}-\frac{r^2}{\gamma p}\right], \quad \nu = p^{-2/\gamma}\left(\rho-\frac{\chi p'}{\omega^2}\right), \end{aligned}\right\} \tag{9}$$

we obtain

$$\frac{d^2S}{dr^2}+\left\{\frac{d}{dr}\left(\frac{\nu'}{2\nu}-\frac{1}{r}\right)-\left(\frac{\nu'}{2\nu}-\frac{1}{r}\right)^2-\frac{\omega^2\mu\nu}{r^2}\right\}S=0, \tag{10}$$

$$\frac{d^2T}{dr^2}+\left\{\frac{d}{dr}\left(\frac{\mu'}{2\mu}\right)-\left(\frac{\mu'}{2\mu}\right)^2-\frac{\omega^2\mu\nu}{r^2}\right\}T=0. \tag{11}$$

We shall first find the solutions describing the high-frequency oscillations (the p_k modes for $k \gg 1$, $k \gg l$). It is convenient to introduce asymptotic expansions separately for two regions, one of which does not include the center r = 0, the other excluding the boundary r = R. The condition that both expansions should coincide in the intermediate region will specify the eigenfrequencies ω_{kl}.

If r is not small and ω is large, then $\gamma\mu \approx -r^2p^{2/\gamma-1}$, and Eq. (11), in view of the relation (8), will take the form

$$\frac{d^2T}{dr^2}+\left\{-\frac{n_R-1}{4(R-r)^2}+\frac{\omega^2\rho}{\gamma p}+f\right\}T=0, \tag{12}$$

where f is a certain function of r whose value near the boundary r = R is of order $1/R(R-r)$. The coefficient of T in Eq. (12) is so written as to emphasize two terms, one of which characterizes the behavior at the singular point r = R, while the other contains the large factor ω^2.

An asymptotic representation of the solution of Eq. (12) may be found by a standard method [14]. One first sets up a "reference" equation which has the same properties but is simpler. Here it will take the form

$$\frac{d^2E}{dx^2}+\left(-\frac{n_R^2-1}{4x^2}+\frac{1}{x}\right)E=0, \tag{13}$$

so that

$$E(x)=\sqrt{x}\,J_n(2\sqrt{x}), \quad E(x)\,|_{x\gg 1} \approx \sqrt[4]{\frac{x}{\pi^2}}\cos\left(2\sqrt{x}-\frac{\pi n_R}{2}-\frac{\pi}{4}\right). \tag{14}$$

We now seek a solution of Eq. (12) in the form of the product of a certain slowly varying function by $E[\omega^2\eta(\psi)]$, with $\psi = R-r$. Requiring that the main terms cancel, we obtain

$$T(r)=\left[\frac{d\eta(\psi)}{d\psi}\right]^{-1/2}E[\omega^2\eta(\psi)], \tag{15}$$

$$\eta(\psi)=\left(\frac{1}{2}\int_0^{\psi}\sqrt{\frac{\rho}{\gamma p}}\,d\psi\right)^2, \quad \psi=R-r. \tag{16}$$

Here we have used the circumstance that because of the factor ω^2 in the argument of the function E, the derivative of E is large. Near the surface r = R, the function $\eta(\psi)$ will approach zero in proportion to ψ.

When finding the solution for a region not including the surface of the sphere, it is convenient to use Eq. (10) for S. Since $\nu \approx \rho p^{-2/\gamma}$, we have

$$\frac{d^2S}{dr^2}+\left\{-\frac{l(l+1)}{r^2}+\frac{\omega^2\rho}{\gamma p}+d\right\}S=0,\quad d\sim\frac{1}{rR}. \tag{17}$$

The "reference" equation and its solution are as follows:

$$\frac{d^2H}{dx^2}+\left\{-\frac{l(l+1)}{x^2}+1\right\}H=0, \tag{18}$$

$$H(x)=\sqrt{x}J_{l+\frac{1}{2}}(x),$$
$$H(x)|_{x\gg1}\approx\sqrt{\frac{2}{\pi}}\cos\left[x-\frac{\pi}{2}(l+1)\right]. \tag{19}$$

The asymptotic representation of the solution of Eq. (17) may be written in the form

$$S(r)=\left[\frac{d\zeta(r)}{dr}\right]^{-1/2}H[\omega\zeta(r)], \tag{20}$$

$$\zeta(r)=\int_0^r\sqrt{\frac{\rho}{\gamma p}}\,dr. \tag{21}$$

We shall now require that the solutions coincide in the region where r and R − r are not small, and hence where $\omega^2\eta$ and $\omega\xi$ are large. The formulas for the function T(r), as obtained from Eq. (15) in view of the asymptotic behavior for $E(\omega^2\eta)$, and from Eqs. (20) and (9), take the form

$$T\approx\sqrt[4]{\frac{\gamma p\omega^2}{\pi^2\rho}}\cos\left\{\omega\int_r^R\sqrt{\frac{\rho}{\gamma p}}\,dr-\frac{\pi}{2}\left(n_R+\frac{1}{2}\right)\right\},$$

$$T\approx i\omega\sqrt[4]{\frac{4\gamma p}{\pi^2\rho}}\sin\left\{\omega\int_0^r\sqrt{\frac{\rho}{\gamma p}}\,dr-\frac{\pi}{2}(l+1)\right\}.$$

These equations will define the same function (to within an unimportant constant factor) if $\omega=\omega_{kl(p)}$, with

$$\omega_{kl(p)}\int_0\sqrt{\frac{\rho}{\gamma p}}\,dr=\frac{\pi}{2}\left(2k+l+n_R+\frac{5}{2}\right). \tag{22}$$

Here k is an integer, with $k\gg l$, $k\gg1$.

For radial oscillations ($l=0$), Eq. (22) was obtained by Ledoux [4]. With very simple examples Ledoux showed that the approximate values of $\omega_{k0(p)}$ for large values of k are quite close to the exact values [for the distribution (23) considered below, with index n = 3, Eq. (22) leads to a 3.5% error for k = 8, 2% for k = 12, and 0.5% for k = 40].

We shall consider the polytropic configuration

$$\rho=\rho_c\theta^n,\quad p=p_c\theta^{n+1},\quad 0\leqslant\theta\leqslant1,\quad 0<n<5. \tag{23}$$

The function $\theta(r)$ satisfies the Emden equation and the conditions $\theta|_{r=0}=1$, $\theta'|_{r=0}=0$. In Eq. (8) the parameter n_R will be equal to n. According to Eq. (22),

$$\omega_{kl(p)}\sqrt{\frac{n+1}{4\pi G\rho_c}}=\frac{\pi\sqrt{\gamma}}{2I_n}\left(2k+l+n+\frac{5}{2}\right), \tag{24}$$
$$k\gg1,\quad k\gg l,$$
$$I_n=\frac{1}{R_E}\int_0^R\frac{dr}{\sqrt{\theta}},\quad R_E^2=\frac{(n+1)p_c}{4\pi G\rho_c^2}.$$

Table 1 presents the values of the integrals I_n, and also the ratios R/R_E and $\rho_c/\bar{\rho}$, where $\bar{\rho}$ is the mean density, for certain n in the range from n = 0 to n = 4.5.

For comparison, Table 2a gives the asymptotic values of the right-hand member of Eq. (24) (row 2) and the exact values (row 1) for n = 3, γ = 5/3, and l = 2. The exact values were obtained by numerical integration of the system (2) (see Sec. 5). Row 3 of the table gives the error in percent. For k > 8, the error becomes less than 3.5%.

In the special case of infinitely small n (that is, a uniform gas sphere), Eq. (24) may be compared with the exact Eq. (4). The first two terms in the series expansion of the frequency with respect to 1/k are found to be correct.

4. Asymptotic Behavior of the Low-Frequency Branch

We turn now to long-period oscillations with a large number of nodes along the radius (g_k modes for $k\gg1$). We shall regard the function $\chi(r)$ appearing in the Schwarzschild convective-stability

TABLE 1

n	$\frac{R}{R_E}$	$\frac{\rho_c}{\bar{\rho}}$	I_n	K_n
0.0	2.44949	1.00000	3.8476	1.2825
0.5	2.75270	1.83516	4.7103	1.2200
1.0	3.14159	3.28987	5.8935	1.1574
1.5	3.65375	5.99071	7.5737	1.0950
2.0	4.35287	11.4025	10.073	1.0332
2.5	5.35528	23.4065	14.033	0.97219
3.0	6.89685	54.1825	20.892	0.91230
3.5	9.53581	152.884	34.517	0.85382
4.0	14.9716	622.408	68.795	0.79714
4.5	31.8365	6189.47	215	0.74

TABLE 2. Values of $\omega_{kl}/(\pi G\rho_c)^{1/2}$ for $n = 3$, $\gamma = 5/3$

a) p_k modes, $l = 2$

k	2	3	4	5	6	7	8
Exact	1.0103	1.2097	1.4082	1.6068	1.8053	2.0036	2.2017
Approx.	1.1163	1.3104	1.5045	1.6986	1.8928	2.0869	2.2810
Error, %	10.5	8.3	6.8	5.7	4.8	4.1	3.6

b) g_k modes, $l = 2$

k	2	3	4	5	6	7
Exact	0.21176	0.17681	0.15180	0.13301	0.11837	0.10666
Approx.	0.19164	0.16325	0.14218	0.12593	0.11302	0.10250
Error,%	9.5	7.7	6.3	5.3	4.5	3.9

c) g_k modes, $l = 3$

k	2	3	4	5	8
Exact	0.26652	0.22693	0.19761	0.17503	0.13043
Approx.	0.24933	0.21494	0.18889	0.16847	0.12722
Error, %	6.4	5.3	4.4	3.7	2.5

criterion as constant in sign and not approaching zero within the interval $0 < r < R$. This means that we will consider only configurations for which all the g_k modes are either stable ($\chi < 0$) or unstable ($\chi > 0$).

The behavior of the function near the boundary is specified by Eq. (8). If $\gamma^{n_R} = n_R + 1$, the equilibrium in the surface layer will be convective. In order to allow for the possibility of this case, we shall assume

$$\chi(r)\,|_{R-r\ll R} \approx \text{const}\cdot(R-r)^{j-1}, \quad j \geqslant 0. \tag{25}$$

The larger j is, the thicker will be the zone within which the equilibrium is nearly convective. For the central region the parameter q will play a similar role if we take

$$\chi(r)\,|_{r\ll R} \approx \text{const}\cdot r^{2q+1}, \quad 2q+1>0. \tag{26}$$

We shall start with the approximate equations (10) and (11). If ω is small, then for a region not including the surface $r = R$ the parameter $\mu \approx [l(l+1)/\omega^2\rho]\,p^{2/\gamma}$, so that we obtain from Eq. (11)

$$\frac{d^2T}{dr^2}+\left\{-\frac{l(l+1)}{r^2}+\frac{l(l+1)\chi p'}{\omega^2r^2\rho}+h\right\}T=0,$$
$$h\sim\frac{\rho''}{\rho}. \tag{27}$$

The "reference" equation will be Eq. (18), in which the unit appearing in the coefficient of H will be replaced by x^{2q}, so that H(x) becomes

$$H_q(x)=\sqrt{x}\,J_{\frac{2l+1}{2(q+1)}}\left(\frac{1}{q+1}x^{q+1}\right). \tag{28}$$

The asymptotic representation of T(r) is

$$T(r)=\left[\frac{d\alpha(r)}{dr}\right]^{-1/2}H_q\left[\omega^{-\frac{1}{q+1}}\alpha(r)\right], \tag{29}$$

$$[\alpha(r)]^{q+1}=(q+1)\int_0^r\sqrt{\frac{l(l+1)\chi p'}{r^2\rho}}\,dr, \tag{30}$$

$$\left.\frac{\alpha(r)}{r}\right|_{r\ll R}\approx\text{const.}$$

For a region not containing the center of the sphere, the parameter $\nu \approx -(\chi p'/\omega^2)p^{-2/\gamma}$, and we obtain from Eq. (10)

$$\frac{d^2S}{dr^2}+\left\{-\frac{n_R^2-1}{4(R-r)^2}+\frac{l(l+1)\chi p'}{\omega^2r^2\rho}+g\right\}S=0,$$
$$g\sim\frac{1}{R(R-r)}. \tag{31}$$

Since the "reference" equation will be Eq. (13), in which E and the second term in parentheses, x^{-1}, are replaced by E_j and x^{j-1}, we obtain

$$E_j(x)=\sqrt{x}\,\frac{J_{n_R}}{j+1}\left(\frac{2}{j+1}x^{\frac{j+1}{2}}\right). \tag{32}$$

The asymptotic representation of S(r) is

$$S(r)=\left[\frac{d\beta(\psi)}{d\psi}\right]^{-1/2}E_j\left[\omega^{-\frac{2}{j+1}}\beta(\psi)\right],\quad \psi=R-r, \tag{33}$$

$$[\beta(\psi)]^{\frac{j+1}{2}}=\frac{j+1}{2}\int_0^{\psi}\sqrt{\frac{l(l+1)\chi p'}{r^2\rho}}\,d\psi, \tag{34}$$

$$\left.\frac{\beta(\psi)}{\psi}\right|_{\psi\ll R}\approx\text{const.}$$

For large values of the argument of the function E_j in Eq. (33), we have

$$S(r)\approx\text{const}\sqrt[4]{\frac{r^2\rho}{\chi p'}}\cos\left\{\frac{1}{\omega}\int_r^R\sqrt{\frac{l(l+1)\chi p'}{r^2\rho}}\,dr-\frac{\pi n_R}{2(j+1)}-\frac{\pi}{4}\right\}.$$

If we find the function S(r) for the same range of intermediate r from Eqs. (29) and (9), we obtain

$$S(r)\approx\text{const}\sqrt[4]{\frac{r^2\rho}{\chi p'}}\sin\left\{\frac{1}{\omega}\int_0^r\sqrt{\frac{l(l+1)\chi p'}{r^2\rho}}\,dr-\frac{\pi}{4}\left(\frac{2l+1}{q+1}+1\right)\right\}.$$

Both of these last equations represent the same solution if $\omega=\omega_{kl(g)}$, where

$$\omega_{kl(g)}=\frac{2\sqrt{l(l+1)}}{\pi\left[2k+2+\dfrac{2l+1}{2(q+1)}+\dfrac{n_R}{j+1}\right]}\times\int_0\sqrt{\frac{\chi p'}{r^2\rho}}\,dr. \tag{35}$$

Here k is an integer, with $k\gg l$, $l=1,2,\ldots$. The parameter n_R characterizes the behavior of the density near the boundary, where ρ is proportional to $(R-r)^{n_R}$; the parameters j and q are given by the relations (25) and (26).

For the polytropic configuration (23), Eq. (35) yields

$$\omega_{kl(g)}\sqrt{\frac{n+1}{4\pi G\rho_c}}=\frac{2\sqrt{l(l+1)}}{\pi\left(2k+l+n+\frac{5}{2}\right)}\times\sqrt{(n+1)\left(n-\frac{n+1}{\gamma}\right)}K_n,$$

$$K_n=R_E\int_0^1\frac{d\theta}{r\sqrt{\theta}},\quad R_E^2=\frac{(n+1)p_c}{4\pi G\rho_c^2}. \tag{36}$$

The values of K_n are given in Table 1 for selected n. Tables 2b and 2c provide a comparison of the asymptotic and exact equations for $n=3$, $\gamma=5/3$, and $l=2$ and 3. The errors are of the same order as in Table 2a.

If we allow the parameter n to approach zero in Eq. (36), we arrive at an equation which is implied by the exact Eq. (5), if in the latter equation we omit quantities of order $1/k^2$ (compared to unity).

5. Frequency Distribution for a Polytropic Configuration

A numerical evaluation of the oscillation frequencies for the bounded polytropic configuration (23) was obtained by integrating Eqs. (2), which for $l\neq 0$ converge to a system of four first-order differential equations. The interval of integration was divided into two parts: $0\le r\le r_1$ and $r_1\le r\le R$. By expanding in powers of r near the center, one can distinguish two solutions bounded at zero (we denote the arbitrary constants of integration by C_1 and C_2). For radii close to the bounding radius, by developing all functions in series in powers of $R-r$ we obtain one unbounded solution and three others for which Ψ, Ψ', X, and Z are finite for $r=R$. The continuation of all the solutions to the fitting point $r=r_1$ was performed by the Runge-Kutta method on the BÉSM-2 electronic computer. From the condition that the functions Ψ, Ψ', X, and Z be continuous at the point $r=r'$, three of the four constants C_i and the eigenfrequency ω_{kl} were determined.

The computed frequencies are given in Table 3 in units of $[4\pi G\rho_c/(n+1)]^{1/2}$, and in Table 4 in units of $\omega_{00(p)}=\sigma$. The computation was performed to an accuracy of several units in the last significant figure.

The order in which the frequencies are distributed over the modes can be checked by computing the corresponding values for nonintegral l. For example, for $n=3$, $\gamma=5/3$, $l=1/4$ we obtain, in the units of Table 3, the number 0.50044, a value intermediate between $\omega_{00}(p)/(\pi G\rho_c)^{1/2}$ and $\omega_{01}(p)/(\pi G\rho_c)^{1/2}$. This calculation confirms that the lowest radial-pulsation frequency σ belongs to modes p_0. The value of σ decreases with decreasing γ.

For large l, the frequencies of the p_k and f modes evidently increase in proportion to $\sqrt{l}$. But

TABLE 3. Values of $\omega_{kl}\,[(n+1)/4\pi G\rho_c]^{1/2}$ for a Polytropic Configuration

a) $n = 3$, $\gamma = 5/3$

Mode \ l	0	1	2	3	4	5	6
g_2	0	0.13816	0.21176	0.26652	0.30790		
g_1	0	0.17785	0.26381	0.32288	0.36389	0.39268	0.41321
g_0	0	0.24883	0.34777	0.40807	0.44378	0.46529	0.47835
f	Not real	0	0.44853	0.48131	0.49736	0.51330	0.53114
p_0	0.47723	0.52976	0.61286	0.67369	0.71462	0.74956	0.78186
p_1	0.64652	0.72826	0.81095	0.87710	0.92673		
p_2	0.8374	0.92704	1.01035	1.07922	1.13400		

b) $l = 0$

n	3	2		1.5
Mode \ γ	1.6	5/3	1.6	5/3
p_0	0.44111	0.59234	0.53317	0.61351
p_1	0.61349	1.08177	1.03963	1.32051
p_2	0.8061	1.52684	1.48140	
p_3	1.003		1.90697	

c) $l = 2$

n	3	2
Mode \ γ	1.6	5.3
g_2	0.19472	0.1269
g_1	0.24343	0.16132
g_0	0.32408	0.22227
f	0.43352	0.52253
p_0	0.59899	1.00667

d)

n	3			2		1.5	
γ	5/3			5/3	1,6	5/3	
Mode \ l	8	11	15	15	15	2	3
g_0	0.49277	0.50360	0.51085	0.36065	0.28438	0	0
f	0.56961		0.69871			0.54397	0.72145

the periods of the g_k modes have a different asymptotic behavior. From an inspection of the sequence of $\omega_{0l(g)}$ given in Tables 3a and 3d ($l = 6$, 8, 11, 15), we find that in the limiting case of large l the frequencies $\omega_{kl(g)}$ approach a finite limit. For the configuration $n = 3$, $\gamma = 5/3$, this limit is greater than the basic radial-pulsation frequency σ.

The values of $\omega_{k\infty(g)}$ will diminish if n decreases. We shall denote by n_0 the value of n for which the greatest frequency $\omega_{0\infty(g)}$ of the g branch is equal to one half σ. An extrapolation from the data of Tables 3 and 4 yields

$$n_0|_{\gamma=5/3} = 1.8, \quad n_0|_{\gamma=1.6} = 1.9. \tag{37}$$

Now let us consider the fundamental mode. The equation $\omega_{1(f)} = 0$ is a consequence of the momentum-conservation law of the system. For oscillations with $l = 2$ the fundamental frequency will be less than the basic frequency if the adiabatic index γ is greater than some number γ_0, and $\omega_{2(f)} > \sigma$ in the opposite case. The value of γ_0, determined by the equation $\omega_{2(f)} = \sigma$, has been computed in [9, 12]:

$$\gamma_0|_{n=3} = 1.5808, \quad \gamma_0|_{n=2} = 1.5863, \quad \gamma_0|_{n=1.5} = 1.5917. \tag{38}$$

For $n < 3$, all the frequencies of the fundamental branch with $l \geq 3$ will lie above the basic frequency σ.

6. Analysis of Resonance Instability

We now shall examine the problem of the resonance self-excitation of nonradial oscillations in a star which initially pulsates radially with the basic frequency $\omega_{00(p)} = \sigma$. If we confine attention to primary resonances in the first approximation, the instability condition will be [2]

$$|\omega_{kl} + \omega_{k'l} - \sigma| < \varepsilon\sigma D_{kk'l}, \quad l \geqslant 1, \tag{39}$$

where ε is the relative amplitude of the radial pulsation, and D is a constant, determined by the integral of the products of the eigenfunctions. Usually D will be of order unity (for example, $D_{kkl} \sim (3\gamma - 1)/4$ for the resonance considered in [1]).

For a polytropic configuration with index $n = 3$, the g_k oscillations will have frequencies distributed over the entire range from zero to σ. Even for a small relative amplitude ε (for example, of the order of one-hundredth) it seems unlikely that resonance excitation would not take place.

TABLE 4. Ratio of Frequency ω_{kl} to Basic Frequency $\omega_{00(p)} = \sigma$

n	3				2			1.5	
γ	5/3			1.6	5/3		1.6	5/3	
Mode \ l	2	3	15	2	2	15	15	2	3
g_0	0.72873	0.85509	1.07045	0.73469	0.37524	0.60886	0.53338	0	0
f	0.93987	1.00856	1.46411	0.98279	0.88215			0.88502	1.17594

As the polytropic index n decreases, the probability of instability does not diminish very appreciably, provided that n does not become less than the value n_0 [see Eq. (37)], for which the greatest frequency of the g branch is equal to one half the basic frequency.

For values of n close to n_0, the resonance condition will be satisfied for the harmonics $k = k' = 0$, $l \gg 1$. The nonradial excited oscillations will have a large amplitude at the surface, and their frequency will be approximately equal to $\frac{1}{2}\sigma$. An analogous instability for other oscillations of the g_k modes ($k \geq 1$) will arise for certain n larger than n_0. As was pointed out in [1, 2], the instability in question could be responsible for the luminosity pattern observed in RV Tauri stars.

In view of the width of the instability band, the resonance $2\omega_{0\infty(g)} \approx \sigma$ will occur for configurations whose index n falls in a certain range: $-\varepsilon C < n - n_0 < \varepsilon C$. Consider now the region $n < n_0 - \varepsilon C$. The only possible resonance in this region will be

$$|\omega_{2(f)} + \omega_{k2(g)} - \sigma| < \varepsilon\sigma D_{fk2}. \tag{40}$$

The other fundamental-mode frequencies lie above the basic frequency σ. For values of γ somewhat smaller than γ_0 [see Eq. (38)], the value of σ will be so small that the condition (40) cannot be satisfied. Such a configuration will be stable relative to excitation of nonradial oscillations.

If $\gamma = 5/3$ and the index n falls in the range $1.5 < n < n_0$, the ratio of the sum $\omega_{2(f)} + \omega_{02(g)}$ to the frequency σ will vary from 0.88502 to 1.1. Thus for n close to 1.5 the left-hand member of the inequality (40) will be negative, and the configuration will be stable. For the value $n \approx 1.62$, the difference in the left-hand member of (40) will be close to zero. In the range of n closer to n_0, resonance self-excitation may be absent only if the relative amplitude ε is sufficiently small and there are frequencies $\omega_{k2(g)}$ such that the condition (40) is satisfied.

The results obtained here support the hypothesis that the density distribution in Cepheids and β Canis Majoris stars does not differ greatly from a polytropic distribution with an index n close to 1.5. For a sounder validation of this claim one would have to investigate the effects that would arise from the resonance instability. The excitation of nonradial surface-type oscillations for which the parameter $l \gg 1$ should probably lead to a strong distortion of the light curve, so that such an instability evidently ought to be absent for regular variables. Adopting a polytropic structure then, we find that n should be somewhat smaller than n_0. This result will be more reliable if ε is large. In the range of n considered, the resonance (40) will be possible, and as a result two oscillations with similar frequencies will be excited (or several such oscillations if the star is rotating or is subject to a field of other perturbing forces).

One hypothesis explaining the phenomenon of beats in some β Canis Majoris stars was advanced by Chandrasekhar and Lebovitz [15]. These authors considered the case $\omega_{2(f)} \approx \sigma$, $\gamma \approx \gamma_0$, for which two oscillations with nearly the same frequencies can arise if rotation is present.

Another possible explanation for the phenomenon of beats in β Canis Majoris stars would be the excitation of nonradial oscillations in the presence of the resonance (40), if the frequency $\omega_{2k(g)}$ is small. The difficulty associated with the need to postulate excessively high rotational velocities for certain stars [16] would then be removed.

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