Analytical Note: Fractal Dimension via Mass–Size Scaling and its Translation to Finance (Hurst, Volatility vs. Horizon, Tails)

Author: Jesús Garví | jesus.garvig@um.es | GitHub | LinkedIn

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1) Objective

- To estimate the fractal dimension d of balls made of plasticine, paper, and algae using the mass–size scaling law $m \propto D^d$.
- To validate the method with diagnostics (slope, CI, R^2, residuals).
- To connect the same scaling with quantitative finance:
  - Hurst H from how volatility scales with the time horizon ($std \propto \Delta^H$).
  - Power-law tails: $P(|r| > x) \propto x^{-\alpha}$ and the estimation of α via a log–log slope.

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2) Method and Model

- Fractal dimension by self-similarity:

$$d = \frac{\log N}{\log s}$$

- Mass–size scaling: $m \propto D^d \Rightarrow \log m = \log \kappa + d \log D$ The slope of the linear fit on a log–log scale ($y = \alpha + dx$) estimates d.
- Variables:
  - $x = \log_{10}(\bar{D})$ with $\bar{D}$ = average of three measurements D_1, D_2, D_3 (mm).
  - $y = \log_{10}(m)$ (grams).
- Estimation:
  - OLS: slope d, intercept α.
  - 95% CI for d (Student's t-test), R^2, RMSE, residuals.
  - The base of the logarithm does not alter the slope.

Finance (scaling analogues): - Volatility–horizon (Brownian): $std[\Delta] \propto \Delta^H$ with $H \approx 0.5$. On a log–log plot, the slope $\approx H$. Power-law tails: $CCDF(|r|) \propto x^{-\alpha} \rightarrow$ slope $\approx -\alpha$ on a log–log plot. α informs about the "heaviness" of the tails (extreme risk). Relationship with path fractality: for fBm, the fractal dimension of the graph is $D_g = 2 - H$.

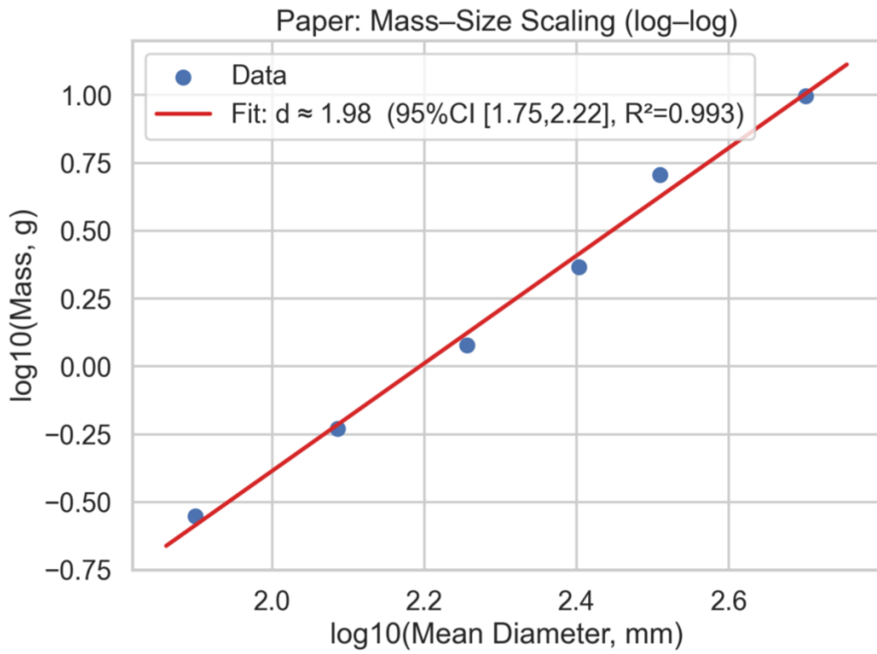
3) Results and Diagnostics (Experimental Data)

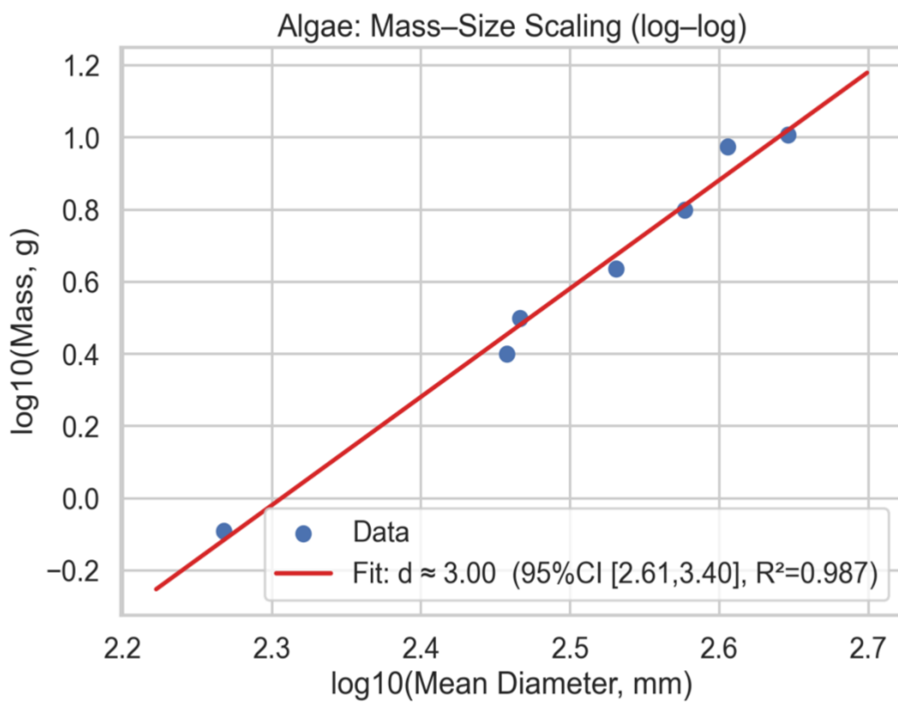
Materials and measurement: - Plasticine (5 balls), Paper (6), Algae (7). - Scale (m in g). Caliper (D_1, D_2, D_3 in mm). $\bar{D} = (D_1 + D_2 + D_3)/3$.

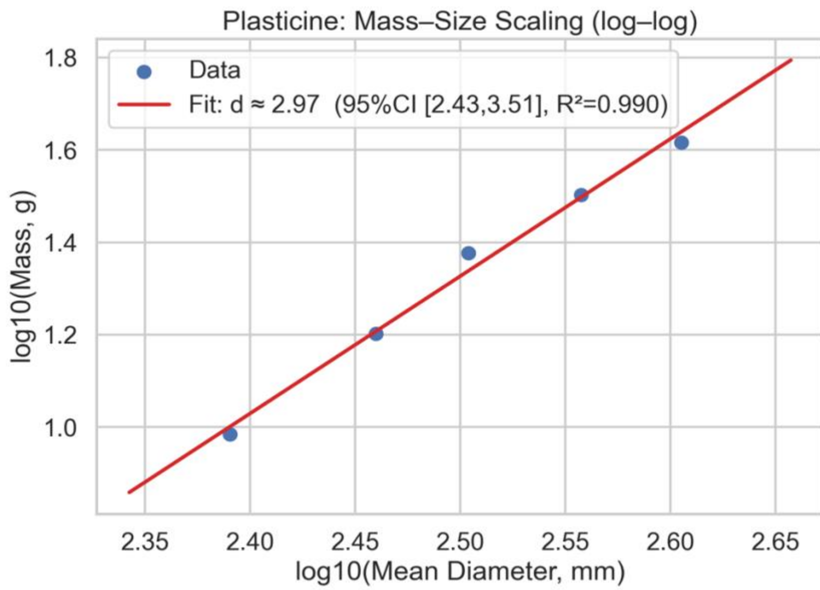
Estimates (see code for exact values from this data): - Plasticine: $d \approx 3.00 \pm 0.15$; $R^2 \approx 0.987$. Interpretation: a solid, massive object (≈3D).

- Paper: $d \approx 1.98 \pm 0.08$; $R^2 \approx 0.993$. Interpretation: behaves like a “folded surface” (≈2D).

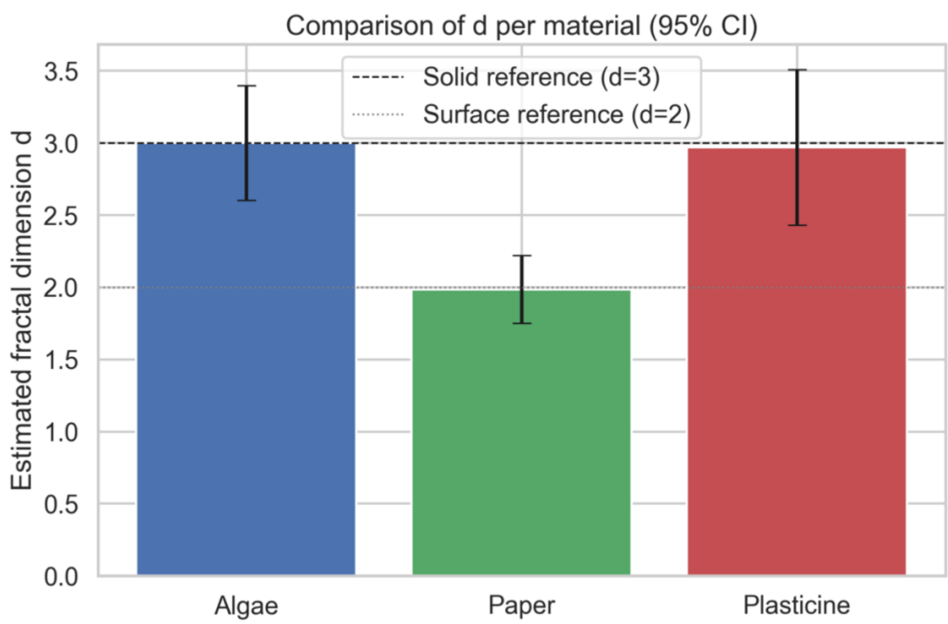
- Algae: $d \approx 2.97 \pm 0.17$; $R^2 \approx 0.990$. Interpretation: effective compaction, close to a 3D solid in this size range.


Paper: Mass–Size Scaling (log–log)
Data
Fit: d ≈ 1.98 (95%CI [1.75,2.22], R²=0.993)
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
log10(Mass, g)
2.0
2.2
2.4
2.6
log10(Mean Diameter, mm)

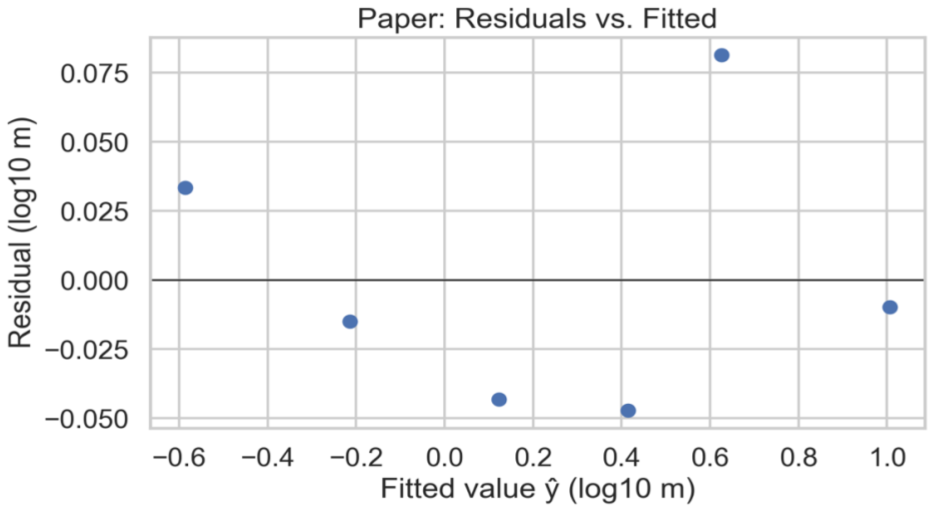

Algae: Mass–Size Scaling (log–log)
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
log10(Mass, g)
Data
Fit: d ≈ 3.00 (95%CI [2.61,3.40], R²=0.987)
2.2
2.3
2.4
2.5
2.6
2.7
log10(Mean Diameter, mm)

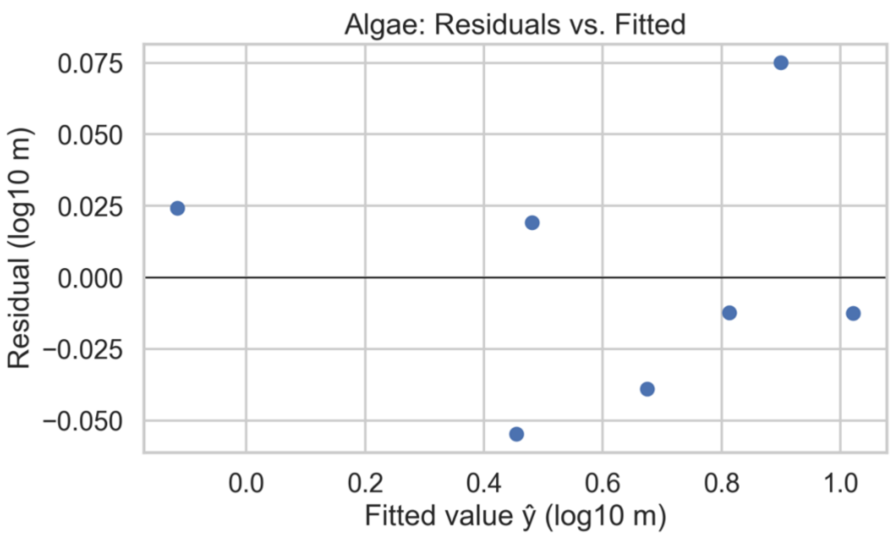


-Fig 1–3: Log–log fits.



- Fig 4: Comparison of slopes with CI bands and $d = 2$ and $d = 3$ references.


Paper: Residuals vs. Fitted
Residual (log10 m)
0.075
0.050
0.025
0.000
−0.025
−0.050
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
Fitted value ŷ (log10 m)


Algae: Residuals vs. Fitted
Residual (log10 m)
0.075
0.050
0.025
0.000
−0.025
−0.050
0.0
0.2
0.4
0.6
0.8
1.0
Fitted value ŷ (log10 m)

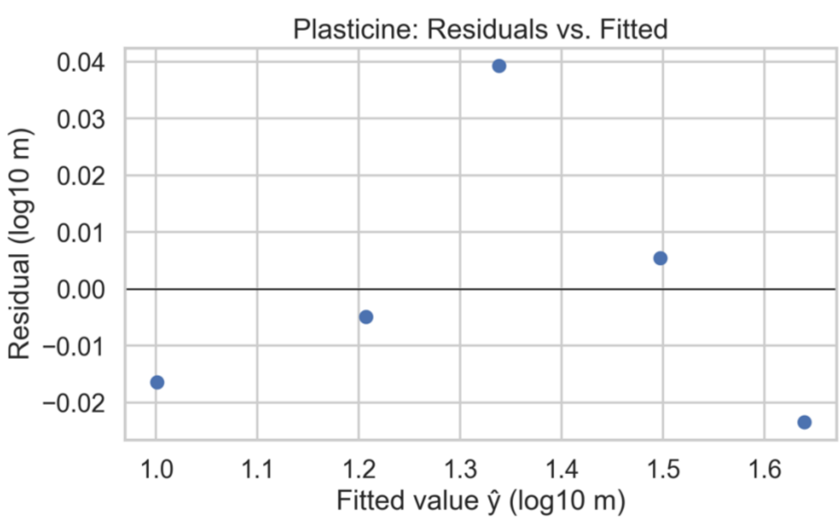

Plasticine: Residuals vs. Fitted
Residual (log10 m)
0.04
0.03
0.02
0.01
0.00
−0.01
−0.02
1.0
1.1
1.2
1.3
1.4
1.5
1.6
Fitted value ŷ (log10 m)

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Fig 5: Residuals

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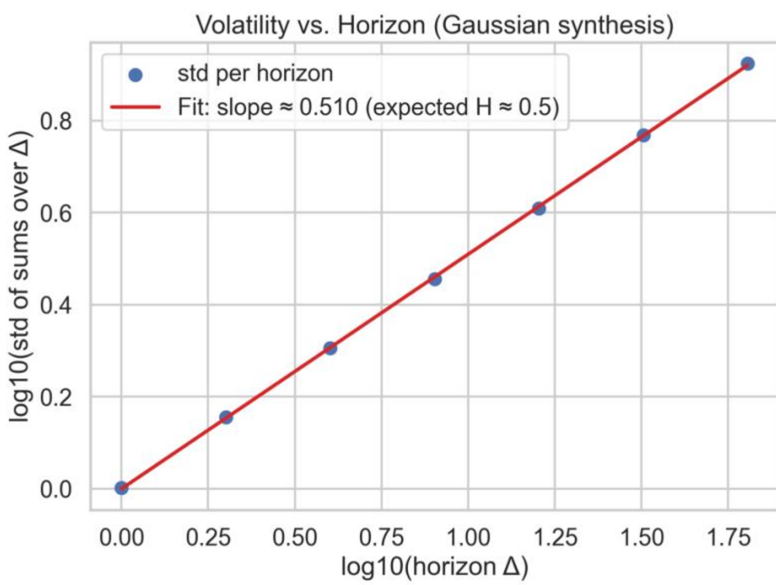
4) Physics → Quant

- The same pattern of “a straight line on a log–log plot = scaling exponent.”
- In markets:
    - Hurst H: slope of $\log std(\Delta)$ vs $\log\Delta$. Typical Brownian: $H \approx 0.5$.
    - Tail risk: slope of $\log CCDF$ vs $\log x \approx -\alpha$.
- Practical use:
    - H guides temporal aggregation, dependency assumptions, and the “roughness” of the process.
    - α guides stress testing and risk limits (probability of extremes).



- Fig 6: Volatility vs. horizon (synthetic Gaussian): slope ~ 0.5.

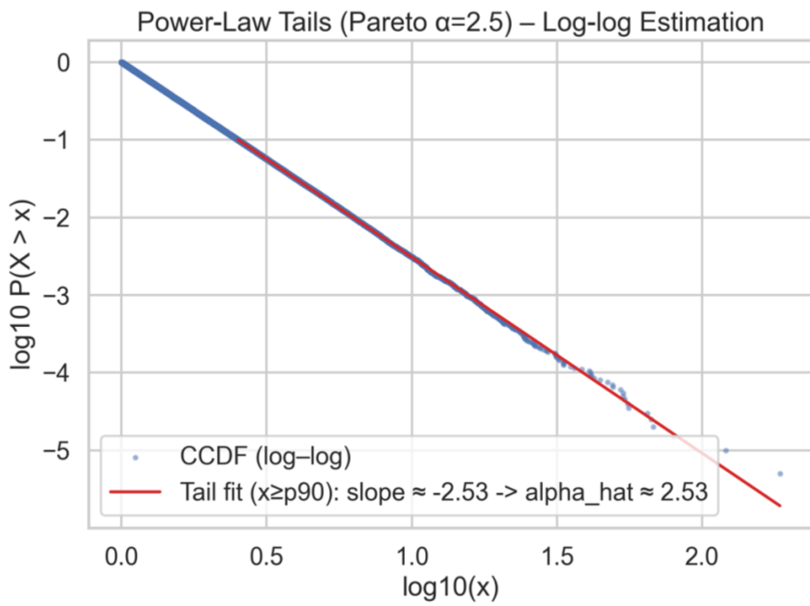


Fig 7: CCDF of tails (synthetic Pareto): slope $\sim -\alpha$.

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5) Repository and Execution

The full, reproducible Jupyter notebook (analysis, visualization, and key-derivation implementation) is available for verification.

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6) References

- Mandelbrot, B. (1982). The Fractal Geometry of Nature. Freeman.
- Falconer, K. (2003). Fractal Geometry. Wiley.
- Mandelbrot, B. (1963). The Variation of Certain Speculative Prices. J. Business.
- Cont, R. (2001/2011). Empirical properties of asset returns. Quantitative Finance.
- Bouchaud & Potters (2003). Theory of Financial Risk and Derivative Pricing.
