Advanced Probability Theory I

Midterm Exam

Due December 10, 2017 (11:59 P.M.)

1. (3 pt) Andrey Nikolaevich Kolmogorov (1903-1987) is the founder of the modern probability theory. Read the following 40-page review paper. This article can be found either from the link provided below, or from the Blackboard where a pdf file was uploaded.

   - Shafer, Glenn & Vovk, Vladimir (2006). The Sources of Kolmogorovs Grundbegriffe. *Statistical Science* **21** (1), 70–98. doi:10.1214/088342305000000467. https://projecteuclid.org/euclid.ss/1149600847

   Write a two-page essay to discuss Kolmogorov's achievement based on your understanding.

2. (6 pt)

   (a) Read the following 4-page journal article. This paper can be found either from the link provided below, or from the Blackboard where a pdf file was uploaded.

   - Blackwell, David (1946). On an Equation of Wald. *The Annals of Mathematical Statistics* **17** (1), 84–87. doi:10.1214/aoms/1177731028. https://projecteuclid.org/euclid.aoms/1177731028.

   Explain the meaning of Theorem 1, Theorem 2, and Theorem 3 in Blackwell (1946).

   (b) Let

   $$N = \sum_{i=1}^{K} S_i$$

   where S_i be independent and identically distributed (i.i.d.) with distribution S. Assume that S_i and K are independent. Show that

   $$\mathbb{E}[N] = \mathbb{E}[S] \cdot \mathbb{E}[K].$$

   (Hint: Consider the law of total expectation.)

   (c) Choose one and only one subquestion between (c-1) and (c-2).
   Let $\mathbf{X} = (X_1, \cdots, X_m)$ denote a set of real-valued random variables on $\mathcal{Z}^+$ (the set of positive integers), where X_t is the value at t, $t = 1, \ldots, m$.

X_1	X_2	X_3	X_4	X_5	$\cdots$	X_{m-1}	X_m

Suppose that $X_i \sim_{i.i.d} N(0, 1^2)$, say, X_i's are independent and follow standard normal distribution. By given a threshold $u > 0$, we define $U_i = I(X_i \geq u)$, where I is an indicator function, say, $U_i = 1$ if $X_i \geq u$, and $U_i = 0$ if $X_i < u$.

By considering the values of U_i's, the adjacent 1's are considered to form a cluster. The number of 1's in one cluster is the size of this cluster S. The number of clusters is K. The totally number of 1's is N. We can see that $N = \sum_{i=1}^{K} S_i$.

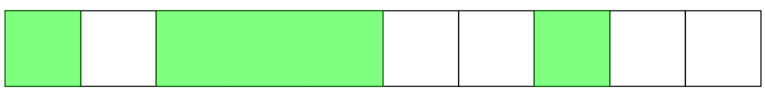
For example, $m = 10$, a realization of X_i's are shown below.

2.6	1.2	2.1	2.7	2.2	-0.5	-3.4	3.8	1.2	0.2

By using threshold $u = 2.0$, the corresponding U_i's are

1	0	1	1	1	0	0	1	0	0

Adjacent 1's form clusters. In this example, $K = 3$ (there are 3 clusters),



$S_1 = 1$, $S_2 = 3$, $S_3 = 1$ (the sizes of clusters are 1,3,1), and $N = 1 + 3 + 1 = 5$.

(c-1) [Theory] Show that when u is large enough and m is large enough, the limiting distribution of K is Poisson and the limiting distribution of S is exponential. Find the corresponding parameters for the Poisson distribution and the exponential distribution.

(c-2) [Simulation] Under two settings

i. $m = 1024$ and $u = 1.036$

ii. $m = 1024$ and $u = 1.645$

Run simulations with number of replica at least 10^4 (the larger the better), and get the empirical distributions of K, S, and N.

(Note: You only need to do either (c-1) or (c-2).)

(d) Based on the results from (c-1) or (c-2), check whether Wald's equation $\mathbb{E}[N] = \mathbb{E}[S] \cdot \mathbb{E}[K]$ holds or not. Explain why (or why not) it holds under question (c)'s setting.

3. (3 pt)

(a) There are three independent uniform distributed random variables X_1, X_2, and X_3, with minimum 0 and maximum 1, say, $X_i \sim_{i.i.d.} \text{Unif}(0,1)$, $i = 1, 2, 3$. Given $0 < \alpha < 1$, show that

$$\Pr\left(\{X_{(1)} \leq \alpha/3\} \cup \{X_{(2)} \leq \alpha/2\} \cup \{X_{(3)} \leq \alpha\}\right) \leq \alpha,$$

where $X_{(1)} = \min\{X_1, X_2, X_3\}$, $X_{(3)} = \max\{X_1, X_2, X_3\}$, and $X_{(2)}$ is the median of $\{X_1, X_2, X_3\}$. Say, $X_{(1)} \leq X_{(2)} \leq X_{(3)}$ is the order statistics of $\{X_1, X_2, X_3\}$.

(b) Check the definition of *stochastic ordering* on page 127, section 4.12 of our textbook *Probability and Random Processes, 3rd edition.* Consider three independent random variables Y_1, Y_2, and Y_3, with minimum 0 and maximum 1. Suppose that $X \sim \text{Unif}(0,1)$, and X dominates Y_i stochastically ($X \geq_{\text{st}} Y_i$) for $i = 1, 2, 3$. Verify whether the inequality

$$\Pr\left(\{Y_{(1)} \leq \alpha/3\} \cup \{Y_{(2)} \leq \alpha/2\} \cup \{Y_{(3)} \leq \alpha\}\right) \leq \alpha$$

holds or not. If the inequality above holds, give a proof; if it does not hold, provide a counterexample. Here $Y_{(1)} \leq Y_{(2)} \leq Y_{(3)}$ is the order statistics of $\{Y_1, Y_2, Y_3\}$, and $0 < \alpha < 1$ is given.

Instructions

- There are 3 questions, each question is between 3-6 points. A perfect score is 10 points.
- Show all work. You will receive partial credit for partially completed problems.
- You may use any references, any texts and any online media.
- Discussion between classmates in Stat 701 is encouraged. But it is not allowed to directly copy solutions from other students. If you have a group discussion (online or face-to-face), please mention it in your solution (including the names of participants in your discussion group). Mentioning the general discussion will not influence your score.

End of the Midterm of Stat 701 (Instructor: Jiangtao Gou)