SPRING 2016

HUNTER COLLEGE

STAT 703

Mathematical Statistics

Final Exam

Last Name: ______________________________

First Name: ______________________________

May 24, 2016

Instructions

- There are 4 questions, each question is between 5-7 points. The maximal score is 24 points. A perfect score is **20** points.
- There are two versions of Question 2, 3, 4 with different points. Please choice one version and only one to answer.
- Show all work. You will receive partial credit for partially completed problems.
- You may use any references.

Q1. (5 pts) Let $Z_{(1)} \leq Z_{(2)} \leq Z_{(3)} \leq Z_{(4)}$ denote the order statistics in a sample of 4 from the exponential distribution with the density function

$$p(z) = e^{-z} \cdot \mathbf{1}_{[0,+\infty)}(z).$$

(1) Show that $Z_{(1)}$, $Z_{(2)} - Z_{(1)}$, $Z_{(3)} - Z_{(2)}$, $Z_{(4)} - Z_{(3)}$ are independent.
(2) Find $\mathbb{E}[Z_1 Z_2 - Z_1^2]$.
(Hint: The expected value of an exponentially distributed random variable with density function $\frac{1}{\theta} e^{-x/\theta} \cdot \mathbf{1}_{[0,+\infty)}(x)$ is θ).

Q2. Please **choose only one** question from 2-A and 2-B.
2-A. (5 pts) Consider a distribution with density function

$$f(x) = \frac{e^{-\frac{x-1}{2}}}{2(1+e^{-\frac{x-1}{2}})^2}$$

(1) Find the distribution function $F(x) = \int_{-\infty}^{x} f(u)du$.
(2) The moment generating function is $M(t) = \frac{2\pi t e^t}{\sin(2\pi t)}$. Find the mean and the variance.
2-B. (6 pts) Consider a distribution with density function

$$f(x) = \frac{1}{2b} e^{-\frac{|x-a|}{b}}$$

(1) Verify the density function is well-defined. Say, show that $\int_{-\infty}^{+\infty} f(x)dx = 1$.
(2) The moment generating function is $M(t) = \frac{e^{at}}{1-b^2t^2}$. Find the mean and the variance.
(Hint: $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$, $\sin x = x - x^3/3! + x^5/5! - \cdots$)

Q3. Please **choose only one** question from 3-A and 3-B.
3-A. (5 pts) Let $X = (X_1, \cdots, X_n)$ is a sample from Gamma distribution with density function $\frac{1}{\Gamma(5)\beta^5} x^4 e^{-\frac{x}{\beta}} \cdot \mathbf{1}_{[0,+\infty)}(x)$, where β is the parameter we are interested in.
(1) Find the maximum likelihood estimator of β.
(2) Find the Fisher information $I(\beta)$.
(3) Verify that $f(x_1, \cdots, x_n \mid \beta)$ belongs to the exponential family of distributions.
(4) Show that $\sum_{i=1}^{n} x_i$ is a complete sufficient statistic. Briefly explain the sufficiency and completeness.
(5) Fine the minimum variance unbiased estimator (MVUE) of β. Is it unique?
3-B. (6 pts) Let $X = (X_1, \cdots, X_n)$ is a sample from Inverse Gaussian (Wald) distribution with density function

$$\sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right) \cdot \mathbf{1}_{[0,+\infty)}(x),$$

where λ is known, and μ is unknown.
(1) Find the maximum likelihood estimator of μ.
(2) Find the Fisher information $I(\mu)$.
(3) Verify that $f(x_1, \cdots, x_n \mid \mu)$ belongs to the exponential family of distributions.
(4) Find a complete sufficient statistic for estimating μ. Briefly explain the sufficiency and completeness.
(5) Fine the minimum variance unbiased estimator (MVUE) of μ. Is it unique?

Q4. Please **choose only one** question from 4-A and 4-B.

4-A. (5 pts) Let $X \sim$ Bernoulli(p), (X_1, X_2) is a sample of 2. Consider a hypothesis test $H_0 : p = 1/4$ v.s. $H_1 : p = 1/2$.

(1) Give the expression of the most powerful (MP) test.

(2) Given the loss function $L(p, a)$ as

$L(p,a)$	$a = 1/4$	$a = 1/2$
$p = 1/4$	1	2
$p = 1/2$	3	0

where p is the true parameter, and a is the decision

(2-i) Find the minimax solution of this test.

(2-ii) Given the prior distribution $\Pr(p = 1/4) = 1/3$ and $\Pr(p = 1/2) = 2/3$, find the Bayes solution of this test.

4-B. (7 pts) Let $X \sim$ Bernoulli(p), (X_1, X_2, X_3) is a sample of 3. Consider a hypothesis test $H_0 : p = 1/4$ v.s. $H_1 : p = 1/2$.

(1) Give the expression of the most powerful (MP) test.

(2) Given the loss function $L(p, a)$ as

$L(p,a)$	$a = 1/4$	$a = 1/2$
$p = 1/4$	0	1
$p = 1/2$	2	0

where p is the true parameter, and a is the decision

(2-i) Find the minimax solution of this test.

(2-ii) Given the prior distribution $\Pr(p = 1/4) = 1/3$ and $\Pr(p = 1/2) = 2/3$, find the Bayes solution of this test.

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End of the final exam of Stat 703 (Instructor: Jiangtao Gou)