Mathematical Statistics

Midterm Exam

Due May 3rd, 2016 (5:35 P.M.)

1. (3 pt) Let $X_1, X_2, \cdots$ denote independent and identically distributed real-valued random variables each with a Bernoulli distribution with parameter $\theta = 1/2$. Let $S_n = \sum_{i=1}^n X_i$ and $Y_n = \frac{2}{\sqrt{n}} S_n - \sqrt{n}$. Does Y_n converge in distribution to some random variable Y? If so, please find the distribution of Y.
   Hint: The frequency function of a Bernoulli random variable with parameter θ is $p(z) = \theta$ if $z = 1$ and $p(z) = 1 - \theta$ if $z = 0$.

2. (3 pt) Let $X_1, X_2, \cdots, X_n$ denote independent and identically distributed random variable each absolutely continuous with density function $\theta \exp(-\theta x)$, $x > 0$, $\theta > 0$. Let $a_1, a_2, \cdots, a_n$ denote real-valued nonzero constants and let $Y = \sum_{i=1}^n a_i \log(X_i)$. Specify the conditions on a_i $(i = 1, \cdots, n)$ under which Y and $\sum_{i=1}^n X_n$ are independent.
   Hint: Consider the Gamma distribution.

3. (3 pt) (a). If $\Pr(A|C) = \Pr(A|B, C)$, we say that A and B are conditionally independent given C. The covariance matrix of a trivariate normal random vector is

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 \\ \rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \sigma_3^2 \end{pmatrix}.$$

   Specify the conditions on σ_i's and ρ_{ij}'s under which $X_1|X_3$ and $X_2|X_3$ are conditionally independent.
   (b). $\mathbf{X} = (X_1, X_2, X_3, X_4)^T$ has a multivariate normal distribution with

$$\mu = \begin{pmatrix} \gamma\Delta \\ 0 \\ \Delta \\ 0 \end{pmatrix}, \text{ and } \Sigma = \begin{pmatrix} 1 & \rho & \gamma & \gamma\rho \\ \rho & 1 & \gamma\rho & \gamma \\ \gamma & \gamma\rho & 1 & \rho \\ \gamma\rho & \gamma & \rho & 1 \end{pmatrix},$$

   where $\Delta \in (-\infty, +\infty)$, $\rho \in [-1, 1]$, $\gamma \in [0, 1]$. Let

$$\mathtt{P} = \Pr(X_1 > c_1, X_2 > c_2) + \Pr(X_1 \le c_1, X_3 > c_3, X_4 > c_4).$$

   (b-1). Let $\gamma = \sqrt{1/2}$, $c_1 = 1.876, c_2 = 2.221, c_3 = 1.876, c_4 = 1.570$, find the parameter ρ, Δ which let $\mathtt{P}$ reach its maximum, say

$$(\rho_m, \Delta_m) = \arg\max_{\rho,\Delta} \mathtt{P}.$$

   (b-2). For any fixed $\gamma, c_1, c_2, c_3, c_4$, find $(\rho_m, \Delta_m) = \arg\max_{\rho,\Delta} \mathtt{P}$.

4. (3 pt) Given the pdf

$$f(x;b) = \frac{x}{b^2} e^{-\frac{x^2}{2b^2}}, \quad , x > 0, b > 0.$$

Assume that $X_1, \cdots, X_n$ is an i.i.d. sample from this distribution.
(a) Find the MLE of parameter b.
(b) Find the asymptotic variance of the MLE.
Hint: $\sqrt{n}\left(\widehat{\theta} - \theta_0\right) \rightarrow_d N\left(0, 1/I(\theta_0)\right)$

Instructions

- There are 4 questions, each question is 3 points. A perfect score is 10 points.
- Show all work. You will receive partial credit for partially completed problems.
- You may use any references, any texts and any online media.
- Discussion is allowed. But it is not allowed to directly copy solutions from other students. If you have a group discussion, please mention it in your solution. Mentioning the general discussion will not influence your score.

End of the Midterm of Stat 703 (Instructor: Jiangtao Gou)