Closed Form Solution

April 1, 2014

For simplicity, let's assume that we are working with a net with only 1 attacker "A" and 1 node that can possibly be infected "B". This makes it simple because there are only 2 possible "infection orderings". One is when the node gets infected before the observation window ends at T and one where it doesn't get infected. The latter is trivial to compute. First some notation, Let λ_1 be the rate that "B" sends messages given it is not infected. Let λ_2 be the rate that "B" sends messages given it is infected. Let γ be the rate at which "A" sends infected messages to "B". Let k_x be the number of messages that "B" sends before x and let N be the total number of messages that "B" sends. Since only one node can be infected drop the subscript and let $\bar{z}$ be the infection time of "B". Here we go...

$$\int_0^T P(\text{ data } |\bar{z})P(\bar{z}|s)d\bar{z}$$

$$= \int_0^T P(\text{ messages before } \bar{z}|\bar{z}) \times P(\text{ messages after } \bar{z}|\bar{z}) \times P(\bar{z}|s)d\bar{z} \tag{1}$$

$$= \int_0^T \frac{(\lambda_1\bar{z})^{k_{\bar{z}}}e^{-\lambda_1\bar{z}}}{k_{\bar{z}}!} \times \frac{(\lambda_2(T-\bar{z}))^{N-k_{\bar{z}}}e^{-\lambda_2(T-\bar{z})}}{(N-k_{\bar{z}})!} \times \gamma e^{-\gamma\bar{z}}d\bar{z} \tag{2}$$

Denote m_i as the time of the i th message sent by "B" (3)

and $m_0 = 0$ and $m_{N+1} = T$ (4)

$$= \sum_{i=0}^{N} \int_{m_i}^{m_{i+1}-\Delta t} \frac{(\lambda_1\bar{z})^{k_{\bar{z}}}e^{-\lambda_1\bar{z}}}{k_{\bar{z}}!} \times \frac{(\lambda_2(T-\bar{z}))^{N-k_{\bar{z}}}e^{-\lambda_2(T-\bar{z})}}{(N-k_{\bar{z}})!} \times \gamma e^{-\gamma\bar{z}}d\bar{z} \tag{5}$$

Note that for each integral, k is a constant. This is because we subtract an arbitrarily small positive number $\Delta t,$ from the upper limit of the integral

Therefore, let k_i be the number of messages sent by node "B" at or before m_i

$$= \lim_{\Delta t\to 0^+} \sum_{i=0}^{N} \frac{1}{k_i!(N-k_i)!} \int_{m_i}^{m_{i+1}-\Delta t} (\lambda_1\bar{z})^{k_I}e^{-\lambda_1\bar{z}} \times (\lambda_2(T-\bar{z}))^{N-k_i}e^{-\lambda_2(T-\bar{z})} \times \gamma e^{-\gamma\bar{z}}d\bar{z}$$

$$= \lim_{\Delta t\to 0^+} \sum_{i=0}^{N} \frac{\lambda_1^{k_i}\lambda_2^{N-k_i}}{k_i!(N-k_i)!} \int_{m_i}^{m_{i+1}-\Delta t} \bar{z}^{k_I}e^{-\lambda_1\bar{z}} \times (T-\bar{z})^{N-k_i}e^{-\lambda_2(T-\bar{z})} \times \gamma e^{-\gamma\bar{z}}d\bar{z}$$

$$= \lim_{\Delta t\to 0^+} \sum_{i=0}^{N} \frac{e^{-\lambda_2 T}\lambda_1^{k_i}\lambda_2^{N-k_i}}{k_i!(N-k_i)!} \int_{m_i}^{m_{i+1}-\Delta t} \bar{z}^{k_I}e^{-\lambda_1\bar{z}} \times (T-\bar{z})^{N-k_i}e^{\lambda_2\bar{z}} \times \gamma e^{-\gamma\bar{z}}d\bar{z}$$

$$= \lim_{\Delta t\to 0^+} \sum_{i=0}^{N} \frac{e^{-\lambda_2 T}\lambda_1^{k_i}\lambda_2^{N-k_i}\gamma}{k_i!(N-k_i)!} \int_{m_i}^{m_{i+1}-\Delta t} \bar{z}^{k_I}(T-\bar{z})^{N-k_i}e^{\bar{z}(\lambda_2-\lambda_1-\gamma)}d\bar{z}$$

Let's simplify notation a bit here. Let $N - k_i = \eta_i$. Therefore, we can rewrite the integral as

$$= \lim_{\Delta t\to 0^+} \sum_{i=0}^{N} \frac{e^{-\lambda_2 T}\lambda_1^{k_i}\lambda_2^{N-k_i}\gamma}{k_i!(N-k_i)!} \int_{m_i}^{m_{i+1}-\Delta t} \bar{z}^{k_I}(T-\bar{z})^{\eta_i}e^{\bar{z}(\lambda_2-\lambda_1-\gamma)}d\bar{z}$$

Note that since $\frac{\partial^y}{\partial_c}e^{x(a+b-c)} = (-1)^y e^{-x(a+b-c)}x^y,$ we can rewrite

(ignoring the terms before the integral) the integral as

$$= (-1)^k \lim_{\Delta t\to 0^+} \int_{m_i}^{m_{i+1}-\Delta t} (T-\bar{z})^{\eta_i}\left[\frac{\partial^k}{\partial_\gamma}e^{\bar{z}(\lambda_2-\lambda_1-\gamma)}\right]d\bar{z}$$

By Leibniz rule, we can rewrite this as

$$\frac{\partial^k}{\partial_\gamma}(-1)^k \lim_{\Delta t\to 0^+} \int_{m_i}^{m_{i+1}-\Delta t} (T-\bar{z})^{\eta_i}\left[e^{\bar{z}(\lambda_2-\lambda_1-\gamma)}\right]d\bar{z}$$

Multiplying and dividing by $e^{T(\lambda_2-\lambda_1-\gamma)}$ allows us to rewrite the integral as

$$\frac{\partial^k}{\partial_\gamma}(-1)^k e^{T(\lambda_2-\lambda_1-\gamma)} \lim_{\Delta t\to 0^+} \int_{m_i}^{m_{i+1}-\Delta t} (T-\bar{z})^{\eta_i}\left[e^{-(T-\bar{z})(\lambda_2-\lambda_1-\gamma)}\right]d\bar{z}$$

Substituting $w = (T-\bar{z})$ and $-(\lambda_2-\lambda_1-\gamma) = \Lambda$ The integral left to evaluate is

$$\lim_{\Delta t\to 0^+} \int_{m_i}^{m_{i+1}-\Delta t} w^{\eta_i}e^{\Lambda w}dw \tag{6}$$

Now integrate by parts and let :

$$U = w^{\eta_i} \; du = \eta_i w^{\eta_i - 1} \; V = \frac{e^{\Lambda w} w}{\Lambda} \; dV = e^{\Lambda w}$$

$$\implies \lim_{\Delta t \to 0^+} \int_{m_i}^{m_{i+1}-\Delta t} w^{\eta_i} e^{\Lambda w} dw$$

$$= \frac{w_i^{\eta} e^{\Lambda w}}{a} \Big|_{m_i}^{m_{i+1}-\Delta t} - \frac{n}{a} \int_{m_i}^{m_{i+1}-\Delta t} w^{\eta_i - 1} e^{\Lambda w} dw$$

Taking advantage of the fact that η_i is an integer, we can repeat the integration by parts η_i times until we just need to take the integral of $e^{\Lambda w}$. This yields a solution of (7)

$$e^{\Lambda w} \sum_{j=0}^{j=\eta_i} (-1)^k \frac{\eta_i!}{(\eta_i - j)!} \cdot \frac{w^{\eta_i - j}}{\Lambda^{j+1}}$$

Remembering that $dw = -d\bar{z}$ and plugging all of the terms back into the integral we have a closed form solution of

$$\sum_{i=0}^{N} \frac{e^{-\lambda_2 T} \lambda_1^{k_i} \lambda_2^{N-k_i} \gamma}{k_i!(N-k_i)!} \Big[\frac{\partial^k}{\partial_\gamma} (-1)^{k+1} e^{\Lambda w} \sum_{j=0}^{j=\eta_i} (-1)^k \frac{\eta_i!}{(\eta_i - j)!} \cdot \frac{w^{\eta_i - j}}{\Lambda^{j+1}} \Big] \Big|_{m_i}^{m_{i+1}-\Delta t}$$

plugging back in for w, Λ and η_i, we get

$$\sum_{i=0}^{N} \frac{e^{-\lambda_2 T} \lambda_1^{k_i} \lambda_2^{N-k_i} \gamma}{k_i!(N-k_i)!} \Big[\frac{\partial^{k_i}}{\partial_\gamma} (-1)^{k+1} e^{(\gamma+\lambda_1-\lambda_2)(T-\bar{z})} \sum_{j=0}^{j=(N-k_i)} (-1)^k \frac{(N-k_i)!}{((N-k_i-j)!} \cdot \frac{(T-\bar{z})^{(N-k_i-j)}}{(\gamma+\lambda_1-\lambda_2)^{j+1}} \Big] \Big|_{m_i}^{m_{i+1}-\Delta t}$$

There are two ways to proceed from here since we still need to take the kth derivative of everything in brackets One approach is to rewrite the entire term in the brackets as a partial gamma function and then take the derivative of that with resepct to γ. However, if we consider all terms that don't depend on γ to be constants denoted by , c_{x_y}, then we can rewrite the entire solution as

$$\sum_{i=0}^{N} c_1 \times \Big[\frac{\partial^{k_i}}{\partial_\gamma} c_{2_i} e^{\gamma(T-\bar{z})} \sum_{j=0}^{j=(N-k_i)} c_{3_{i,j}} \cdot \frac{c_{4_{i,j}}}{(\gamma+\lambda_1-\lambda_2)^{j+1}} \Big] \Big|_{m_i}^{m_{i+1}-\Delta t}$$

$$= \sum_{i=0}^{N} c_1 \times \Big[\frac{\partial^{k_i}}{\partial_\gamma} c_{2_i} \sum_{j=0}^{j=(N-k_i)} c_{3_{i,j}} \cdot \frac{c_{4_{i,j}} e^{\gamma(T-\bar{z})}}{(\gamma+\lambda_1-\lambda_2)^{j+1}} \Big] \Big|_{m_i}^{m_{i+1}-\Delta t}$$

$$= \sum_{i=0}^{N} c_1 \times \Big[c_{2_i} \sum_{j=0}^{j=(N-k_i)} c_{3_{i,j}} \cdot (-1)^k \frac{c_{4_{i,j}} e^{\gamma(T-\bar{z})} (j+k_i)!}{(\gamma+\lambda_1-\lambda_2)^{j+1+k_i} j!} \Big] \Big|_{m_i}^{m_{i+1}-\Delta t}$$

Cancelling some of the $(-1)^k$terms and plugging back in we get

$$= \sum_{i=0}^{N} \frac{e^{-\lambda_2 T} \lambda_1^{k_i} \lambda_2^{N-k_i} \gamma}{k_i!(N-k_i)!} (-1)^{k+1} \Big[e^{(\gamma+\lambda_1-\lambda_2)(T-\bar{z})} \sum_{j=0}^{j=(N-k_i)} \frac{(N-k_i)!(T-\bar{z})^{N-k_i-j)}}{(N-k_i-j)!} \frac{(j+k_i)!}{(\gamma+\lambda_1-\lambda_2)^{j+1+k_i} j!} \Big] \Big|_{m_i}^{m_{i+1}-\Delta t} \tag{8}$$