TIF345/FYM345 Project 1: Cosmological models

Andreas Ekström

The *Supernova Cosmology Project* (SCP) and the *High-z Supernova Search Team* demonstrated that Type Ia supernovae (SNIa) are excellent standard candles for measuring the expansion history of the Universe. The leaders of the respective research teams (Saul Perlmutter, Adam Riess, and Brian P. Schmidt) were awarded the Nobel Prize in Physics for 2011 *"for the discovery of the accelerating expansion of the Universe through observations of distant supernovae"*.

In this project you will use the real data from the SCP project to analyze cosmological models of the Universe.

1 Supernova data: SCP 2.1

SNIa explode with a well-defined absolute luminosity L, measured in watts (W). The flux F of an object with luminosity L at distance d_L is given by

$$F_{\text{obs}} = \frac{L}{4\pi d_L^2}. \tag{1}$$

If the flux is strong enough, it can be measured directly. The apparent magnitude

$$m = -2.5 \log_{10}(F(d_L)/F_\circ) \tag{2}$$

is a logarithmic, unit-less, and relative measure of the brightness, i.e. flux, of a star located at a distance d_L from Earth. Here, $F_\circ$ is some reference scale. The absolute magnitude M is as measure of the brightness as if the star was located $d_L = 10$pc from Earth. Thus, we get

$$m - M = -2.5 \log_{10} \frac{F(d_L)}{F(d_L = 10)} = -2.5 \log_{10} \frac{10^2}{d_L^2} = 5 \log_{10} \frac{d_L}{10} = 5 \log_{10}(d_L) - 5. \tag{3}$$

If we instead measure the distance d_L in mega-parsec (Mpc), we get $m - M = 5 \log_{10}(d_L) + 25$. The difference between apparent and absolute magnitudes is called the distance modulus

$$\mu = 5 \log_{10}(d_L) + 25. \tag{4}$$

The SCP measured light curves, i.e. recorded brightness versus time, for a 833 supernovae explosions of type Ia. Every SNIa explodes with the same luminosity. For this reason they are called *standard candles*. The observed luminosity on Earth makes it possible to extract the distance d_L to the exploding star. This light is also cosmologically redshifted due to the expansion of space itself. The measured redshift z of the light, combined with the determined distance modulus provides a firm handle on the large scale expansion of the Universe. The SCP 2.1 dataset contains 580 data for recorded supernova explosions with a well-determined distance modulus and redshift for each explosion. You will use this dataset[1] to infer cosmological knowledge based on different models.

2 General relativity and the Friedman equation

The expansion of space induces a cosmological redshift z of observed light. To be general, we allow the scale factor $a(t)$ of space itself to depend on time t, i.e. the expansion of the universe is not

[1] attached as a separate file in the GitLab directory for project1

necessarily constant. One can show that the cosmological redshift of some emitted light, that is observed today at Earth, is related to the scale factor as

$$1 + z = \frac{\lambda_{\rm obs} - \lambda_{\rm emit}}{\lambda_{\rm emit}} \frac{a(t = t_{\rm obs})}{a(t = t_{\rm emit})} \equiv \frac{a_0}{a}. \tag{5}$$

The scale factor is dimensionless, with time t counted from the birth of the Universe and present time is typically denoted t_0. By convention we fix the scale factor $a(t = t_0) = a_0 = 1$. Einstein's field equations from General Relativity are needed to derive the scale factor as a function of time. Current evidence, e.g. the SCP data, suggests that the expansion rate of space in the Universe is accelerating, i.e. $\ddot{a}_0 > 0$.

On a cosmological scale we can treat the Universe as isotropic and homogeneous. The geometric relationship of space and time is described by the Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 \right]. \tag{6}$$

The above metric allows for a topology with negative, zero, or positive curvature depending on whether k is equal to $-1, 0,$ or $+1$, respectively. These Universes are called, in order, open, flat, or closed. The dynamic equation of the Universe and the scale factor in this metric was first given by Friedman

$$H^2 \equiv \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2}, \tag{7}$$

where we also defined the Hubble parameter H. The current value is denoted H_0 in correspondence with evaluation of the scale factor at $t = t_0$. The dot represents a time derivative as usual. We denote density with ρ and the gravitational constant with G.

The Friedman equation for $k = 0$ defines the *critical density* ρ_c today

$$\rho_c = \frac{3H^2}{8\pi G}. \tag{8}$$

This is the density of matter for which the gravitational effects causes the Universe to become flat. In the following, we do not distinguish between dark and visible matter, we just call it *matter* M, and we neglect radiation. Below this critical density, the Universe has an open geometry and above, a closed geometry.

We now introduce the cosmological constant Λ into the Friedman equation

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho_M + \frac{\Lambda}{3}. \tag{9}$$

The cosmological constant was introduced by Einstein in 1917 to generate a static model of the Universe as it was understood at the time before Hubble discovered cosmic expansion in 1929. A positive value for Λ counteracts the gravitational attraction of matter and renders the Universe static, i.e. spatially and temporally infinite. Formally, the cosmological constant can be associated with an energy density $\rho_\Lambda \equiv \Lambda/(8\pi G)$. We can now use the critical density to define the dimensionless matter density parameter

$$\Omega_M \equiv \frac{\rho_M}{\rho_c}. \tag{10}$$

In this spirit we also define

$$\Omega_k \equiv -\frac{k}{a^2 H^2}, \quad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c}. \tag{11}$$

This enables us to express the Friedman equation as

$$1 = \Omega_M + \Omega_k + \Omega_\Lambda \tag{12}$$

Clearly, in an exactly flat Universe we have $1 = \Omega_M + \Omega_\Lambda$. Matter density is inversely proportional to space cubed. Normalizing at present values, denoted by a subscript $'0'$, the scaling of matter density can therefore be expressed as

$$\rho_M = \rho_{M,0} \left(\frac{a}{a_0} \right)^{-3}. \tag{13}$$

One can now show that

$$H(z) = H_0 \sqrt{\Omega_{M,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}} \equiv H_0 E(z)^{1/2}. \tag{14}$$

This is a reduced version[2] of the so-called ΛCDM cosmology[3]

In a flat Universe, the Hubble parameter is related to the luminosity distance via the relation

$$d_L(z) = c(1+z) \int_0^z \frac{dz'}{H(z')}. \tag{15}$$

This can easily be converted to a distance modulus μ using Eq. 4. We can therefore test cosmological models using measured distance moduli from SNIa explosions at different redshifts. The angular power spectrum of the cosmic microwave background is another, very sensitive, probe for testing cosmological model. Adding several data sources will provide the best constraints. However, in this project we will only use supernova data.

For $z << 1$, which is generally accepted to be $z < 0.5$, we can Taylor expand $E(z)$

$$E(z) \approx \Omega_{M,0}(1+3z) + \Omega_{k,0}(1+2z) + \Omega_{\Lambda,0} = 1 + 2z(q_0+1), \tag{16}$$

where $q_0 \equiv (\Omega_{M,0} - 2\Omega_{\Lambda,0})/2$ is called the deceleration parameter and describes whether the expansion of the universe is decelerating ($q_0 > 0$) or accelerating ($q_0 < 0$). In the small-z domain we can also express Eq. 15 as

$$d_L(z) = c(1+z) \int_0^z \frac{dz'}{H(z')} \approx \frac{c}{H_0} \left(z + \frac{1}{2}(1-q_0)z^2 + \dots \right). \tag{17}$$

Extracting q_0 from SCP data in the small z region gives a first signal of an accelerating Universe.

2.1 Matter density in a flat Universe and other cosmologies

If we now limit ourselves to a flat Universe, i.e. $\Omega_{k,0} = 0$, we obtain

$$E(z) = \Omega_{M,0}(1+z)^3 + \Omega_{\Lambda,0}. \tag{18}$$

This is a rather simple version of the ΛCDM with only one parameter $\Omega_{M,0}$. In this flat Universe, one can also explore other cosmologies, and an extension of the flat ΛCDM is the so-called wCDM which involves a dark-energy equation of state $p_\Lambda = w\rho_\Lambda$ with the additional parameter w. This leads to a Hubble parameter corresponding to

$$E(z) = \Omega_{M,0}(1+z)^3 + \Omega_{\Lambda,0}(1+z)^{3(1+w)}. \tag{19}$$

The ΛCDM model pops out of wCDM for $w = -1$. There exists other models as well. All these models can be tested via Eq. 15, and the redshift dependence of the Hubble parameter is directly sensitive to the parameters of various cosmological models.

[2]we have neglected radiation density.

[3]sometimes referred to as the standard model of Big Bang cosmology

3 Project definition

Although the calibration of the SCP 2.1 dataset relies on a prior determination of the Hubble parameter H_0, we will treat the data as if it didn't.

- Your **first** task is to use Eq. 17 in the small-z regime ($z < 0.5$) to extract a joint probability distribution for H_0 and q_0. You should use data weights proportional to the measurement errors, and an inverse gamma prior for the unknown error scale σ^2. You can use e.g. uniform priors for H_0 and q_0 if you like.
- Do you find convincing evidence that the expansion of the Universe is accelerating?
- How can you check your extraction of H_0? Remember, for even lower values of z one can drop the second order term in the expansion of the luminosity distance.
- Generate a posterior predictive plot of the distance modulus versus redshift to check your model predictions across the entire z-regime in the SCP data. Remember to plot the data as well.
- Discuss how one might improve the inference for q_0 and H_0.

The SCP team actually calibrated their data to a Hubble parameter $H_0 = 70$ km/s/Mpc. Use this value in the next part of the analysis.

- Your **second** task is to compare two cosmological models; ΛCDM and wCDM. Which model, if any, would you select to explain the SCP 2.1 dataset? Now you can use data for all z. The likelihoods become slightly more involved due to the appearance of the z-integral in Eq. 15 and you are not required to do a full Bayesian analysis with computation of marginal likelihoods. Instead, use the AIC and BIC scores and discuss your findings. Here, it is enough to use data weights w_i equal to the measurement variance. That is, we assume that the overall scale is known. Scipy has many good optimization algorithms that can be applied to maximum likelihood estimation.
- Report the central values for the matter and dark energy density parameters today that you extract using the two cosmological models. You do not have to estimate the uncertainty for this number (see next bullet), but make sure to compare with some reference values from other more extensive analyses.
- Extract the posterior probability distribution for $\Omega_{M,0}$ in ΛCDM, i.e. $p(\Omega_{M,0}|\boldsymbol{\mathcal{D}}_{\mathrm{SCP}} M_{\Lambda\mathrm{CDM}} I)$, where I = 'known measurement variance'. Use a uniform prior, i.e. $\Omega_{M,0} \sim \mathcal{U}(0, 1)$.

Reflection: Include a brief paragraph (5 sentences or so) at the end of your report where you describe which part of the project that took the most time and why. Mention briefly any challenges, choices made, or surprises you encountered during code development and analysis.

Python modules: You are free to use any Python modules you want. e.g., Emcee for sampling posteriors. Pymc offers many extra tools for probabilistic programming and built-in pdfs for building statistical models. It requires specific syntax and is not necessary for completing the projects. But feel free to use it if you are curious about exploring its capabilities. Feel free to exploit the functions from the lecture notebooks and choose tools wisely for the tasks at hand.

You should hand in a report, via Canvas, maximum 5 pages (excluding references). Use the report-template.tex for writing your report (see Canvas). Do not attach any appendices. In the report you should present and motivate your data, models (likelihoods,priors), and the final results. Do not forget to visualize your data and results. Tables are also useful for reporting/contrasting results. It is important that you discuss your findings and reflect on the results. In addition you need to hand in your Python code. The code itself will not be graded, but it should run without errors and upon inspection reproduce the results you present in the report. Emphasis will be put on your ability to use appropriate terminology to describe and discuss the employed *statistical* methodologies and results. This project is preferably analyzed using a Jupyter Notebook.

Good Luck and Have Fun!