PS5

October 30, 2021

[2]:
```
import numpy as np
import matplotlib.pyplot as plt
```

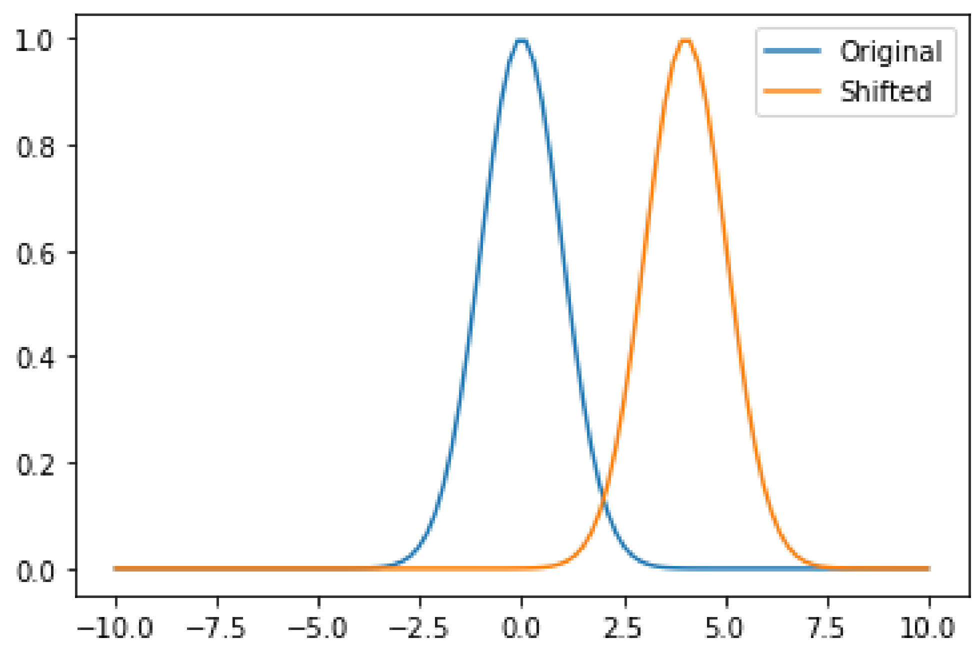
0.1 Question 1

[3]:
```
def Shift(array, dx):
    N = len(array)
    k = np.arange(N)
    ft = np.fft.fft(array)
    ft_new = ft*np.exp(-2j*np.pi*k*dx/N)
    new_array = np.fft.ifft(ft_new)
    return new_array
```

[4]:
```
x = np.linspace(-10, 10, 100)
y = np.exp(-x**2/(2*1**2))
dx = len(y)/5
y_new = Shift(y, dx)
```

[5]:
```
plt.plot(x, y, label = 'Original')
plt.plot(x, y_new, label = 'Shifted')
plt.legend()
plt.show()
```

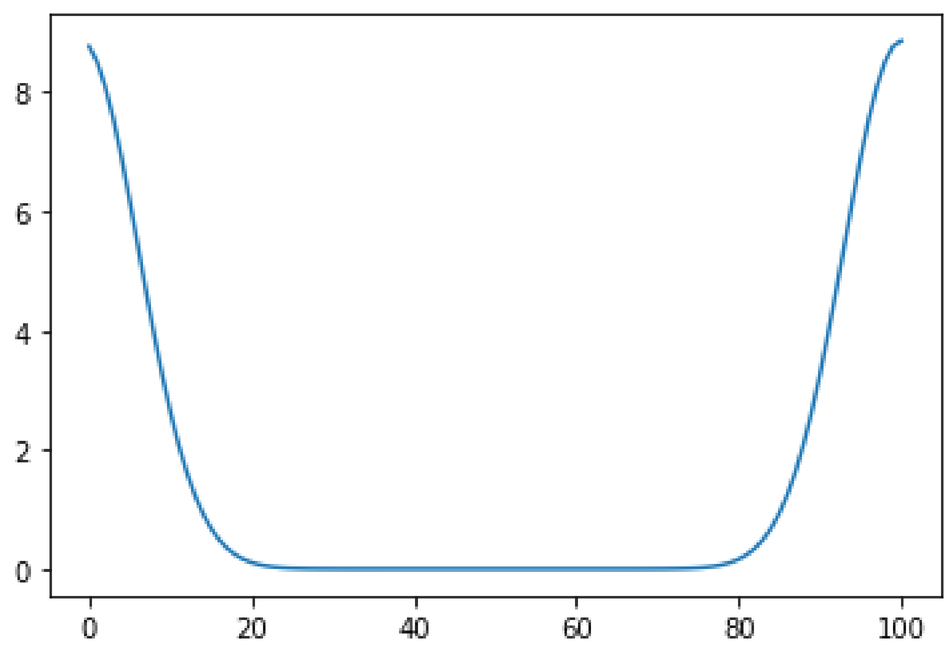
```
//anaconda3/lib/python3.7/site-packages/numpy/core/_asarray.py:102:
ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order)
```



0.2 Question 2

[6]:
```
def Corr(fun1, fun2):
    ft1 = np.fft.fft(fun1)
    ft2 = np.fft.fft(np.flip(fun2))
    return np.real(np.fft.ifft(ft1*ft2))
```

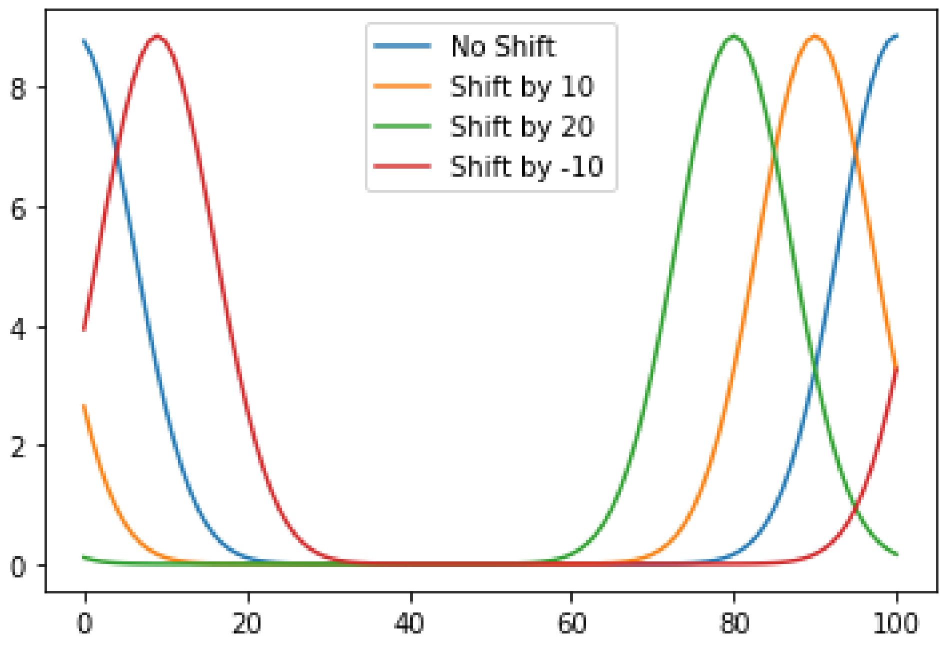
[7]:
```
x = np.linspace(-10,10,101)
fun1 = np.exp(-x**2/(2))
corr = Corr(fun1, fun1)
plt.plot(corr)
plt.show()
```



0.3 Question 3

[8]:
```python
def ShiftCorr(fun, dx):
    shift = Shift(fun, dx)
    corr = Corr(fun, shift)
    return corr
```

[9]:
```python
x = np.linspace(-10, 10, 101)
gauss = np.exp(-x**2/2)
shift0 = ShiftCorr(gauss, 0)
plt.plot(shift0, label = 'No Shift')
shift1 = ShiftCorr(gauss, 10)
plt.plot(shift1, label = 'Shift by 10')
shift2 = ShiftCorr(gauss, 20)
plt.plot(shift2, label = 'Shift by 20')
shift2 = ShiftCorr(gauss, -10)
plt.plot(shift2, label = 'Shift by -10')
plt.legend()
plt.show()
```



0.4 Note:

0.4.1 Here we took the correlation of the original function with its shifted function and we observe shift in the OPPOSITE direction; however, if we took the correlation of the shifted with the original function (SWAP ORDER) then we will observe shift in the SAME direction.

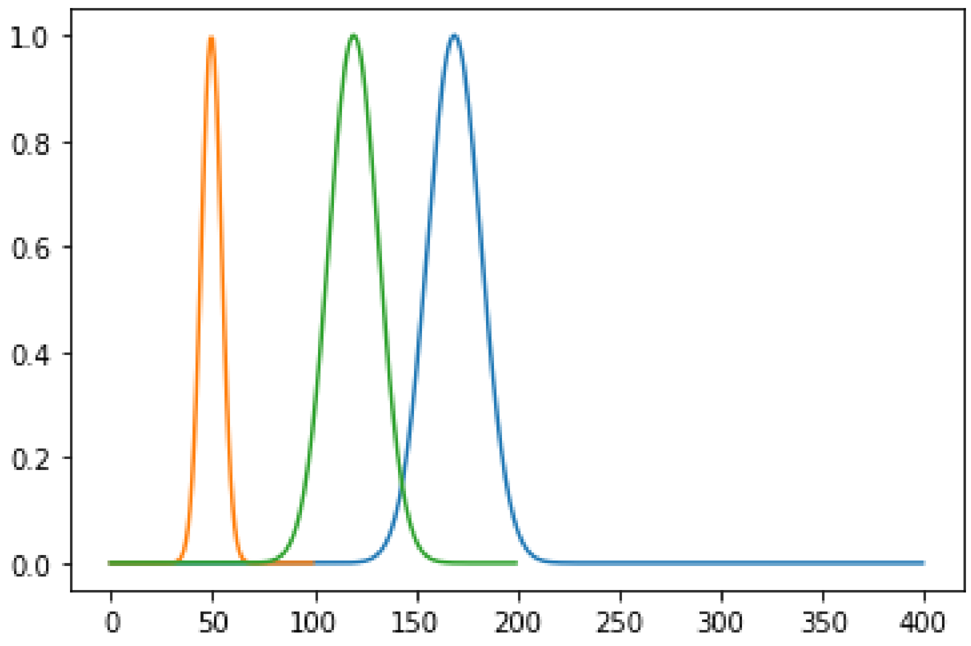
0.5 Question 4

[10]:
```python
def conv_safe(f,g):
    diff = abs(len(f)-len(g))
    n = len(f)
    if len(f) > len(g):
        g = np.concatenate((g,np.zeros(diff)))
        n = round(len(f))
    elif len(f) < len(g):
        f = np.concatenate((f,np.zeros(diff)))
        n = round(len(g))
    f = np.concatenate((f,np.zeros(n)))
    g = np.concatenate((g,np.zeros(n)))
    fft = np.fft.fft(f)
    gft = np.fft.fft(g)
    conv = np.real(np.fft.ifft(fft*gft))
    return conv
```

[11]:
```python
x = np.linspace(-10, 10, 100)
x2 = np.linspace(-10, 10, 200)
gauss = np.exp(-x**2/2)
gauss2 = np.exp(-(x2-2)**2/3)
#sin1 = np.sin(x)
#sin2 = np.sin(x2)
conv = conv_safe(gauss, gauss2)
print(len(conv))
plt.plot(conv/max(conv), label = 'Conv')
plt.plot(gauss, label = 'Function 1')
plt.plot(gauss2, label = 'Function 2')
```

```
400
```

[11]:
```
[<matplotlib.lines.Line2D at 0x11d5dffd0>]
```



Here we add first pad the smaller array with enough zeros so that the 2 arrays are the same size as we need for future operations. Then since we need both arrays to be zero padded, we add zeros equal to the size of the array to the arrays so that our output array is twice the size of the longest array passed in.

0.6 Question 5

0.6.1 a.) $\sum_{x=0}^{N-1} exp(-2\pi ikx/N) = \sum_{x=0}^{N-1} exp(-2\pi ik/N)^x$

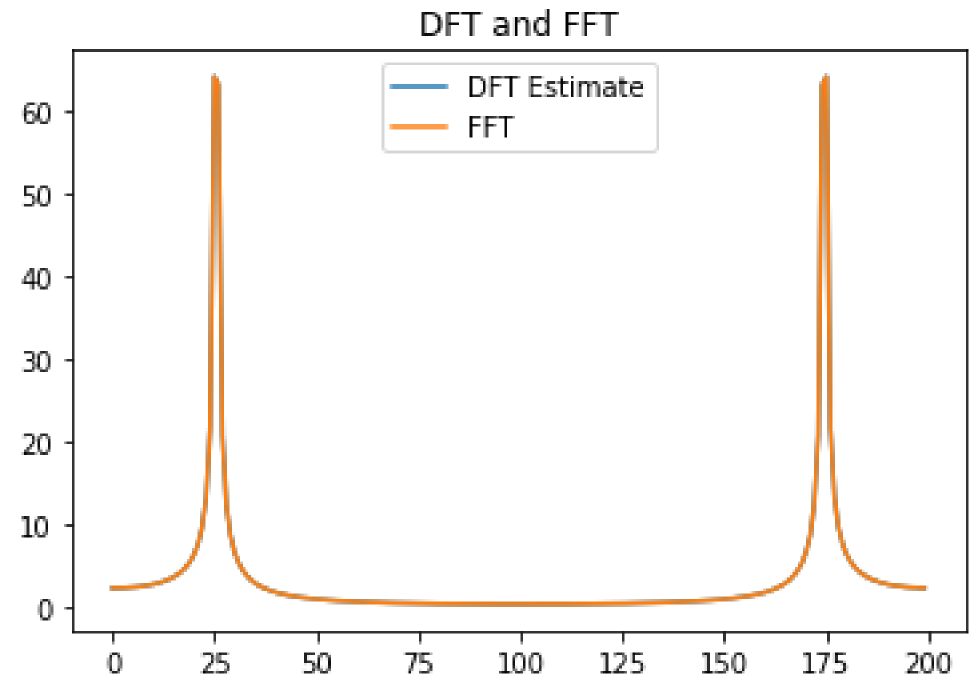
Recall Geometric Series: $\sum_{k=0}^{n} ar^k = a(\frac{1-r^{n+1}}{1-r})$ $\sum_{x=0}^{N-1} exp(-2\pi ik/N)^x = \frac{1-exp(-2\pi ik/N)^{N-1+1}}{1-exp(-2\pi ik/N)} = \frac{1-exp(-2\pi ik)}{1-exp(-2\pi ik/N)}$

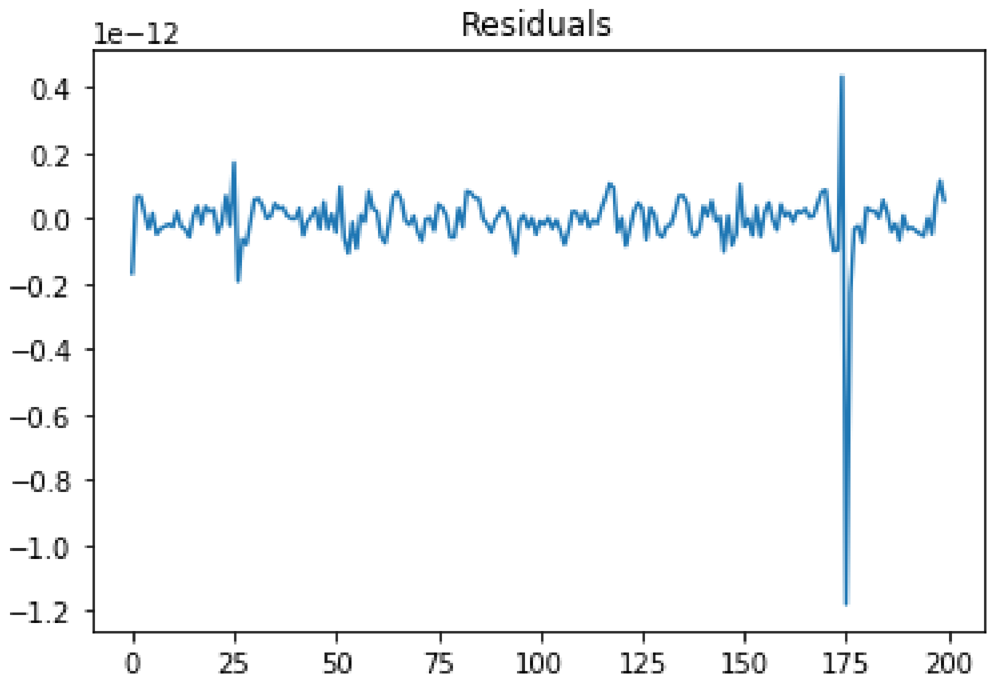
0.6.2 b.)

Using l'hopital's rule: $\lim_{k\to 0} \frac{1-exp(-2\pi ik)}{1-exp(-2\pi ik/N)} = \lim_{k\to 0} \frac{2\pi iexp(-2\pi ik)}{2\pi i/Nexp(-2\pi ik/N)} = N \; 1-exp(2\pi ik) = 0, k \in \mathbb{Z}$ Therefore, if k is not a multiple of N, $1-exp(-2\pi ik/N) \neq 0$ Hence, $\frac{1-exp(-2\pi ik)}{1-exp(-2\pi ik/N)} = 0$ if k is an integer, but not a multiple of N.

0.6.3 c.)

[12]:
```python
k2 = np.arange(200)
k = 51/2
N = len(k2)
def dft_sin(k, k2):
    N = len(k2)
    num_1 = 1-np.exp(2j*np.pi*(k-k2))
    bot_1 = 1-np.exp(2j*np.pi*(k-k2)/N)
    num_2 = 1-np.exp(-2j*np.pi*(k+k2))
    bot_2 = 1-np.exp(-2j*np.pi*(k+k2)/N)
    return np.real(1/(2j)*((num_1/bot_1)-(num_2/bot_2)))
dft = dft_sin(k=k, k2=k2)
plt.plot(abs(dft), label = 'DFT Estimate')
fft = np.fft.fft(np.sin(2*np.pi*k*k2/N))
plt.plot(abs(fft), label = 'FFT')
plt.title('DFT and FFT')
plt.legend()
plt.show()
plt.plot(abs(fft)-abs(dft))
plt.title('Residuals')
plt.show()
```



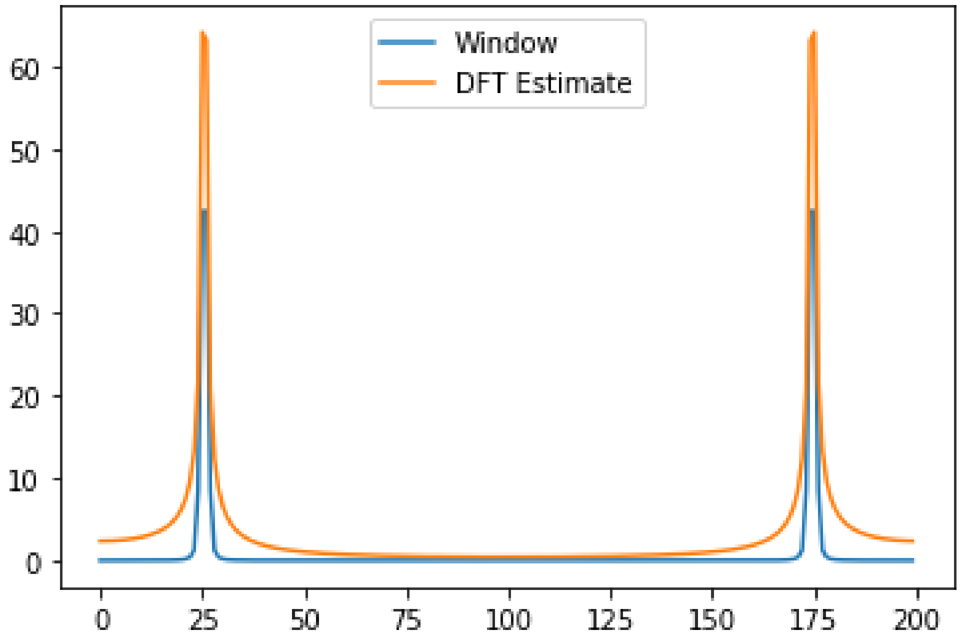


By plotting the residuals between the our analytic estimate of the DFT and FFT, we see that we have errors on the order of magnitude of -12. We plot our DFT and observe that the curve looks like a wide delta function.

0.6.4 d.)

[13]:
```
k2 = np.arange(200)
k = 51/2
N = len(k2)
window = 0.5 - 0.5*np.cos(2*np.pi*k2/N)
fft = np.fft.fft(window*np.sin(2*np.pi*k*k2/N))
plt.plot(abs(fft), label = 'Window')
plt.plot(abs(dft), label = 'DFT Estimate')
plt.legend()
```

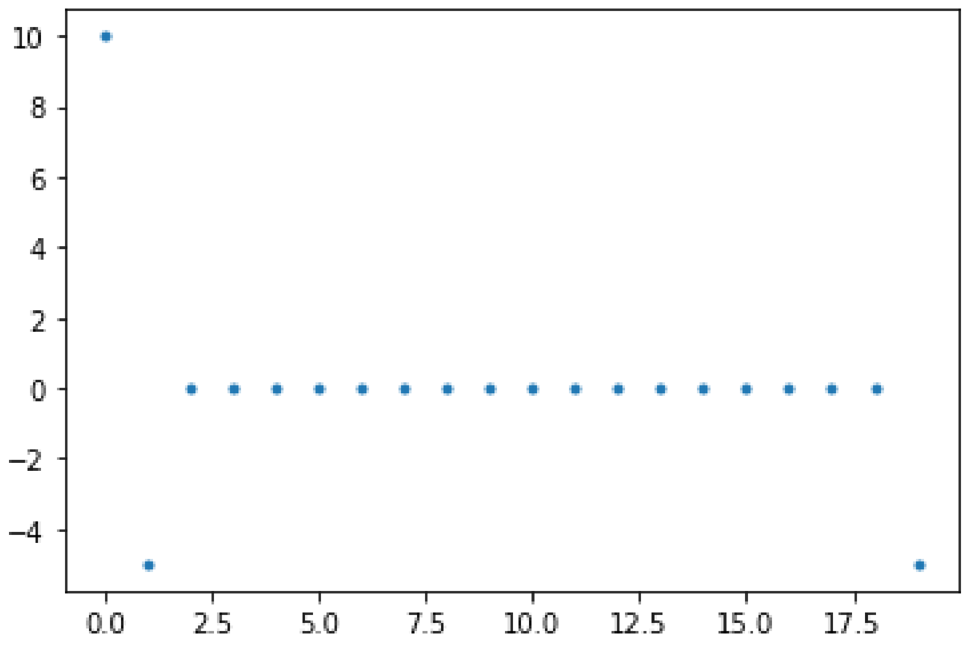
[13]:
```
<matplotlib.legend.Legend at 0x11d77b6d8>
```



0.6.5 e.)

[14]:
```
k2 = np.arange(20)
N = len(k2)
window = 0.5 - 0.5*np.cos(2*np.pi*k2/N)
winfft = np.fft.fft(window)
plt.plot(np.real(winfft), '.')
print(np.round(np.real(winfft)))
```

```
[10. -5.  0.  0.  0.  0.  0. -0.  0.  0.  0.  0.  0.  0.  0.  0.  0. -0.
  0. -5.]
```

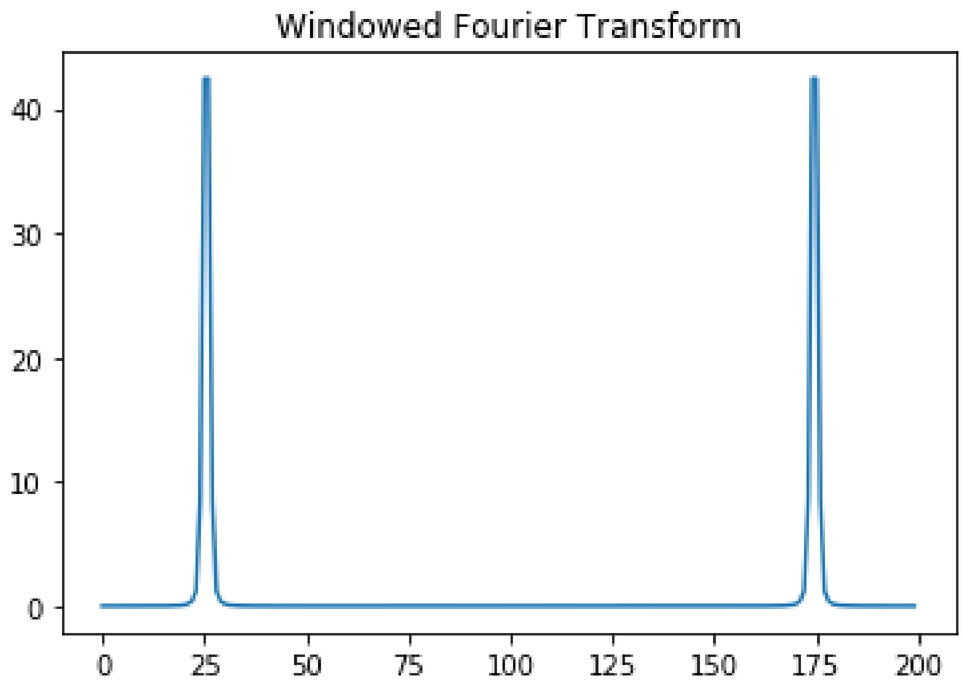


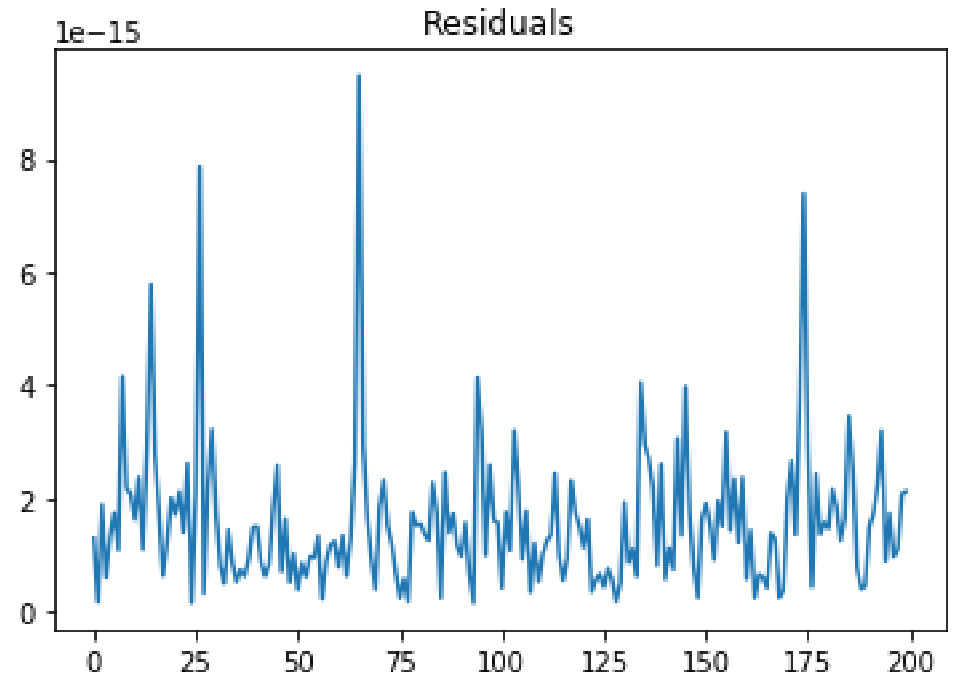
Since we have N = 20, we expect the first, second and last elements of the array to have values of 20/2, -20/4, -20/4 respectively, which we observe in the plot/print statement. For this next part, we know that the Fourier Transform (FT) of the window tells us how to sample points in order to apply the window in Fourier Space. So our windowed sin function is simply: $F(N) = -\frac{F(N-1)}{4} + \frac{F(N)}{2} - \frac{F(N+1)}{4}$ where $F(N)$ is the FT of our function.

[16]:
```
x = np.arange(200)
k = 51/2
N = len(x)


rfft = np.fft.fft(np.sin(2*np.pi*k*x/N))
windowed = -np.roll(rfft, -1)/4 + rfft/2 - np.roll(rfft, 1)/4
plt.plot(np.abs(windowed))
plt.title('Windowed Fourier Transform')
plt.show()


window = 0.5 - 0.5*np.cos(2*np.pi*x/N)
fft = np.fft.fft(window*np.sin(2*np.pi*k*x/N))
plt.plot(np.abs(windowed - fft))
plt.title('Residuals')
plt.show()
```


Windowed Fourier Transform
40
30
20
10
0
0
25
50
75
100
125
150
175
200

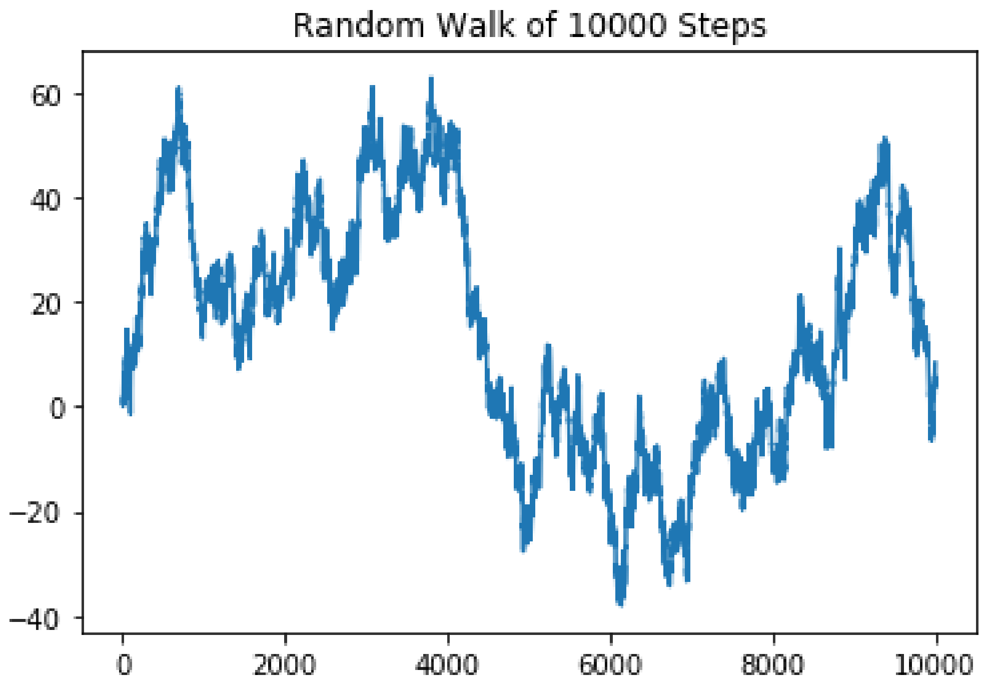

Residuals
1e−15
8
6
4
2
0
0
25
50
75
100
125
150
175
200

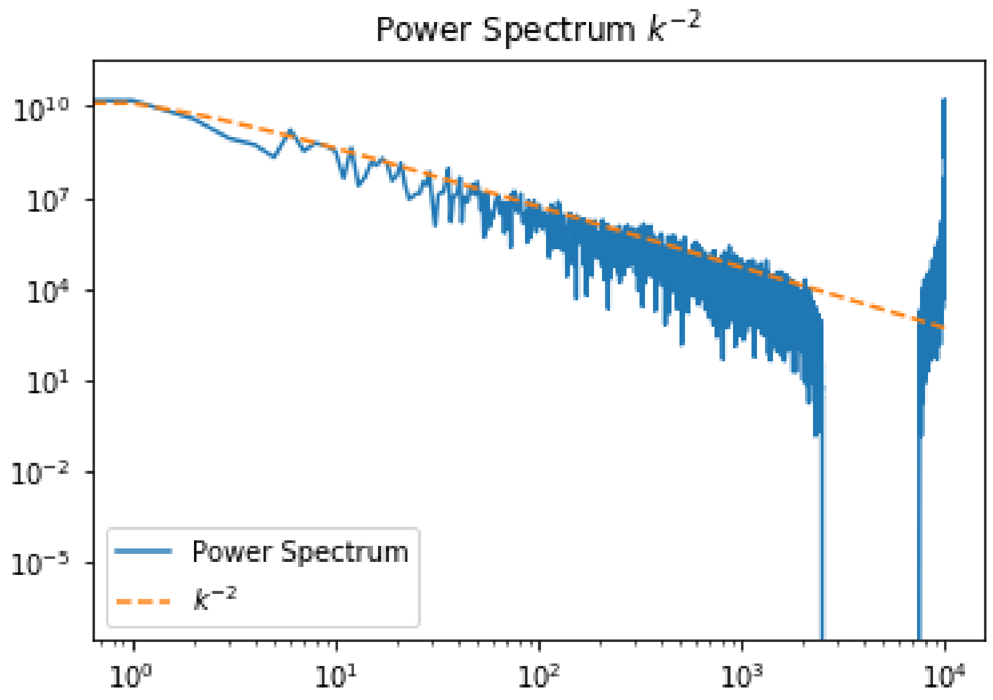
0.7 Question 6.)

0.7.1 a.)

0.7.2 b.)

[17]:
```
n = 10000
walk = np.cumsum(np.random.randn(n))
plt.plot(walk)
plt.title('Random Walk of 10000 Steps')
plt.show()
N = np.arange(n, dtype = float)[1:]**-2 # Take [1:] to bypass divide by zero
corr = Corr(walk, walk)
plt.loglog(np.fft.fft(corr), label = 'Power Spectrum')
plt.loglog(N*10e10/2, '--', label = r'$k^{-2}$') # 10e10/2 is arbitrary␣
 ↪coefficient to show scaling
plt.title('Power Spectrum $k^{-2}$')
plt.legend()
plt.show()
```




Power Spectrum k^{-2}
10^{10}
10^{7}
10^{4}
10^{1}
10^{-2}
10^{-5}
Power Spectrum
k^{-2}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}