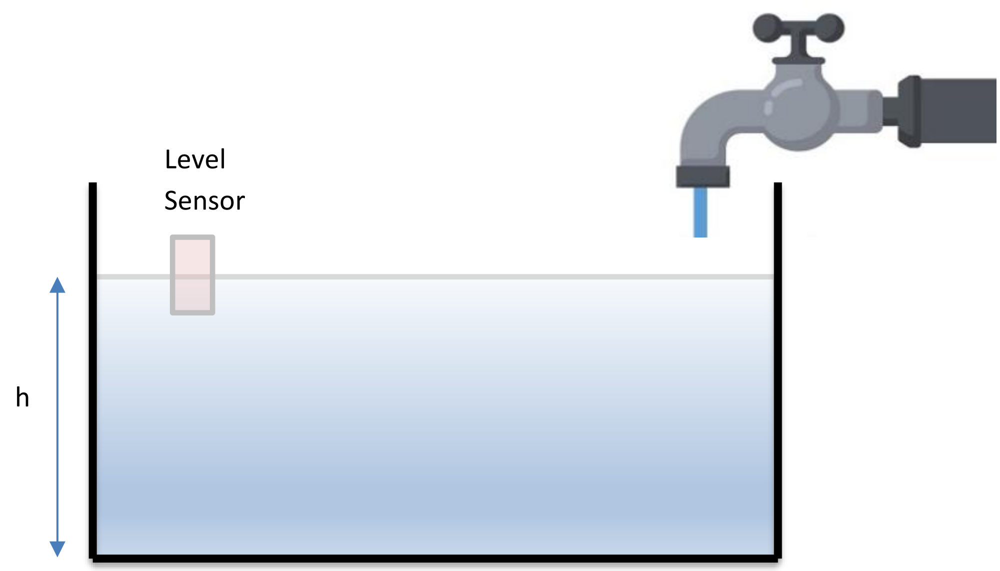
Mini Project # 6 **Due: Nov. 19/2025**

Consider the water tank system shown in the following figure. The tank has an input, a water tap, which can add water to the tank and increase the water level. Water level is measured by a level sensor.



The goal is to use Kalman filter to estimate the level of the water in the tank using the noisy measurements coming from the level sensor. Assume that the dynamics and the measurements are modeled by the following equations:

$$x(n+1) = A\,x(n) + B\,u(n) + \varepsilon_p$$
$$y(n) = C\,x(n) + \varepsilon_m$$

Here, $u(n)$ is the input to the system. In Kalman estimation we have:

$$K(n) = P^{n|n-1} C^T \left[\, C P^{n|n-1} C^T + R \,\right]^{-1}$$

$$x^{n|n} = x^{n|n-1} + K(n)\left[\, y(n) - C x^{n|n-1} \,\right]$$

$$P^{n|n} = \left[\, I - K(n)\,C \,\right] P^{n|n-1}$$

$$x^{n+1|n} = Ax^{n|n} + Bu(n+1)$$

$$P^{n+1|n} = AP^{n|n}A^T + Q$$

Here Q and R are the covariance matrices of the process and measurement noises.

1. Assume that the tap is not adding any water to the tank and the level of the water is 1.0. Also assume that $x(n)$ is the level of the water in the tank at time n.
   a. The system is static. What are A and $u(n)$?
   b. We have only one sensor directly measuring the water level. What is C ?
   c. At the beginning, before making any measurement, we have no idea about the water level in the tank. So, we assume that $x^{0|0} = 0.0$. We also have no confidence in this initial point. So, let's assume that $P^{0|0} = 1{,}000.0$, a big number. Since our state variable, water level, and the measurement, water level, are both scalars, then noise covariance matrices are scalars or: $Q = q$ and $R = r$. We are confident that we are not adding water to the tank. So, let's assume that $q = 0.0001$ but sensor is very noisy $r = 0.1$. We make the first measurement and the number we read is: $y(1) = 0.9$. Use the Kalman filter to find the best first estimation of the water level using the result of this measurement.
   d. Since we have very minimum confidence in our initial estimation of the water level use the Maximum likelihood theory to find your estimation of the water level and the related covariance matrix? Are these numbers close to what you calculated in the previous question using the Kalman filter? Do they get closer if you increase $P^{0|0} = 10{,}000$?
   e. Now assume that we have made one measurement after another for a total of 10 measurements. The outcomes of these measurements are: $y(1) = 0.9$, $y(2) = 0.8, y(3) = 1.1, y(4) = 1.0, y(5) = 0.95, y(6) = 1.05, y(7) = 1.2, y(8) = 0.9, y(9) = 0.85, y(10) = 1.15$. Use Kalman filter iterations to update your estimation of the water level and the corresponding variance after each measurement. Plot the true value of the water level, all measurements, and the Kalman estimations as a function of the iteration number in one graph. Measurements are made once every second. How many seconds does it take for the estimations to be within the 5% range of the accurate level? Notice that the measurements are about 20% above or below the actual level.

2. Now assume that someone opens the tap to add water to the tank without telling us. At the beginning the tank is empty, $x(0) = 0.0$, but the level goes up by 0.1 unit per second. However, we still assume that the system is static.

a. As before we initialize the iteration with $x^{0|0} = 0$ and $r = 0.1, q = 0.0001$. We use Kalman estimation filter to update our estimations based on the recoded measurements. This time measurements are:

$$y(1) = 0.11, y(2) = 0.29, y(3) = 0.32, y(4) = 0.50, y(5) = 0.58, y(6) = 0.54, y(7) = 0.63, y(8) = 0.64, y(9) = 0.78, y(10) = 1.1, y(11) = 0.95, y(12) = 1.4, y(13) = 1.4, y(14) = 1.6, y(15) = 1.42.$$

Actual water level changes as: $x(0) = 0.0, x(1) = 0.1, x(2) = 0.2, \ldots, x(15) = 1.5.$ Use the Kalman filter to estimate the level of water using these measurements and the model you have in mind for the dynamics of the system. Plot the actual water level, measurements, and Kalman estimations as a function of time in one graph? What is the problem here? Seems estimations are not close to the actual values and the system has a biased error! Why? Can we do better? What would you change to make it better (without changing your dynamics)?

b. A problem in the previous section is that you have a wrong impression about what is happening in the system (you assume that the system is static, but the water tap is actually adding water to the tank). But the main challenge is that we have good confidence in that wrong assumption reflected in out very low $q = 0.0001$ value. Let's say that we are not very sure about the dynamics (because we cannot monitor the system closely and it is possible that someone without our knowledge goes and open that tap). To add this uncertainty, we change $q = 0.01$. Now, re-run the Kalman estimations and compute the water level estimations after each measurement? Is it better now? Increase the value to $q = 0.1$. Is it even better than last trial? How about $q = 1.0$? Seems now we can get better results that measurements getting closer to the actual values. Plot the actual values, measurements, and Kalman estimations for these different cases in one plot. What did you learn?

3. Let consider the case that we know that we are adding water to the system with the rate of 0.1 increase in the water level per second. We can consider this as an input to the system or use this trick. Let's assume that our state variable is a vector with two elements: $x = [\,x1\;\; x2]^T$. $x1$ is the water level in the tank and $x2 = dx1/dt$ or the rate of change in the water level. Then we can assume that:

$$A = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$$

Now, we don't need to add the input to the system, and we assume that $u(n) = 0.0$. Here, our time steps are Δt = 1.0 second. Measurements are scalar as before and we are directly measuring the water level.

a. Assume that once we initialize the iterations with: $x^{0|0} = \left[x1^{0|0}\ x2^{0|0}\right]^T = [0\ \ 0]^T, r = 0.1$ and:
$P^{0|0} = \begin{bmatrix} 1{,}000 & 0 \\ 0 & 1{,}000 \end{bmatrix}$, $Q = \begin{bmatrix} q/3 & q/2 \\ q/2 & q \end{bmatrix}$ where $q = 0.0001$. Use the previous measurements and run the Kalman filter estimation filter. Plot the actual values, measurements, and the estimation values in one plot. How do you evaluate the results?

b. Now assume that someone closed the tap when the level was at 1.0 without telling you. So, you still think that you are adding water to the tank. Do you think with this vector arrangement you can tolerate this problem and come up with good estimations? Can you test it?

Have fun with Kalman Filter

Wish you a wonderful Thanksgiving!