WStarAlgebras have Unique Preduals

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0.1 WStarAlgebras and their Topologies

In this section we define the topologies and need to prove that W^*–algebras are closed with respect to these. Must also put these dependencies explicitly into the results that use them below.

0.2 Normality and Ultraweak Continuity for Positive Functionals

In what follows, let M be a (nonzero) W^*–algebra. Let $\mathcal{P}(M)$ denote the projection lattice of M. Let φ be a positive linear functional φ on M. We say φ is **normal** if whenever (p_α) is an increasing net of projections in M with supremum p, we have $\varphi(p_\alpha) \to \varphi(p)$. In this section we show that this property is equivalent to $\sigma(M, M_*)$–continuity. We say that a linear functional φ on M is **positive** if $\varphi(x) \geq 0$ whenever $x \geq 0$.

To do : Net results must be replaced by filter results.

0.2.1 Ultraweakly Continuous Implies Normal

This is 1.7.4 in Sakai. Roughly, taking an increasing net of projections, it converges to the supremum strongly, hence σ-topology, and the σ-continuity finishes the proof.

Theorem 1. *Every uniformly bounded, increasing directed set converges to its least upper bound in the σ–topology. Further, if $x = \sup_\alpha x_\alpha$ then $a^* x a = \sup_\alpha a^* x a$.*

Proof. □

0.2.2 Normal Implies Ultraweakly Continuous

Lemma 2. *For all self-adjoint $x \in M$, $\|x\| 1 - x \geq 0$.*

Proof. □

Lemma 3. *For all $p, q \in \mathcal{P}(M)$ such that $q \leq p$, $p - q \in \mathcal{P}(M)$.*

Proof. □

Lemma 4. *For every increasing bounded net (p_α) of projections in M, the supremum p is a projection in M*

Proof. □

If $P : \mathcal{P}(M) \to \mathrm{Prop}$ is a predicate, the usual ordering "$\leq$" on projections induces an order on the set $\{p \in \mathcal{P}(M) | P(p)\}$. In what follows there will be no confusion if we also denote this induced order by "$\leq$".

Lemma 5. *For every nonzero element $a \in M$, there is a $\sigma(M, M_*)$–continuous positive linear functional ψ on M such that $\psi(a) \neq 0$.*

Proof. □

Lemma 6. *For every positive normal linear functional φ and nonzero $p \in \mathcal{P}(M)$ there exists a positive $\sigma(M, M_*)$–continuous linear functional ψ such that $\varphi(p) < \psi(p)$.*

Proof. By Lemma 5 there is a $\sigma(M, M_*)$–continuous positive linear functional ψ_0 on M such that $\psi_0(p) \neq 0$. Rescale this functional by a positive constant to obtain ψ such that $\varphi(p) < \psi(p)$. $\square$

Lemma 7. *For all positive linear functionals φ, ψ with φ normal and ψ $\sigma(M, M_*)$–continuous and every nonzero $p \in \mathcal{P}(M)$ such that $\varphi(p) < \psi(p)$, there exists a nonzero $p_1 \in \mathcal{P}(M)$ such that $p_1 \leq p$ and for all nonzero $q \in \mathcal{P}(M)$ with $q \leq p_1$, we have $\varphi(q) < \psi(q)$.*

Proof. We proceed by contradiction. Suppose the conclusion does not hold. Then, for every nonzero subprojection p_1 of p there is a nonzero subprojection $q \leq p_1$ such that $\varphi(q) \geq \psi(q)$. In particular, (letting $p_1 = q$) there is a $q \leq p$ such that $\varphi(q) \geq \psi(q)$. If (q_α) is a chain of such nonzero projections, $q_\alpha \to \sup q_\alpha$ in the σ–topology by Theorem 1, and by Lemma 4 we know this supremum is a subprojection of p_1. Since φ is positive and normal, and ψ is positive and σ–continuous, we have $\varphi(\sup q_\alpha) \geq \psi(\sup q_\alpha)$. Therefore, by Zorn's Lemma, there is a maximal $q_0 \leq p$ such that $\varphi(q_0) \geq \psi(q_0)$. We claim $q_0 = p$. If not, $p - q_0$ is a nonzero subprojection of p by Lemma 3, and there exists a nonzero projection $q_1 \leq p - q_0$ such that $\varphi(q_1) \geq \psi(q_1)$. But then q_0 is a proper subprojection of $q_0 + q_1$ and by linearity and positivity $\varphi(q_0 + q_1) \geq \psi(q_0 + q_1)$ contradicting the maximality of q_0. Thus $p = q_0$ and so $\varphi(p) \geq \psi(p)$, contradicting the hypothesis $\varphi(p) < \psi(p)$. $\square$

Recall that a compact Hausdorff space is **Stonean** if the closure of every open set is open.

Lemma 8. *Let K be a Stonean space. Then every element a in $C(K)$ can be uniformly approximated by finite linear combinations of projections in $C(K)$. If $a \geq 0$ then the coefficients of the approximating linear combinations may be chosen nonnegative.*

Proof. $\square$

Lemma 9. *If $p \in M$ is a projection, pMp is also a W^* algebra with identity p.*

Proof. $\square$

Lemma 10. *If C is any maximal commutative C^*–subalgebra of the W^*–algebra M, its spectrum space (maximal ideal space) is Stonean.*

Proof. $\square$

Note that the following likely contains the most crucial use of the normality of φ for the later proof of Theorem 14.

Lemma 11. *Let φ be a normal positive linear functional on M and consider the predicate $P : \mathcal{P}(M) \to Prop$ defined, for $p \in \mathcal{P}(M)$, by "$M \ni x \mapsto \varphi(xp)$ is $\sigma(M, M_*)$–continuous". If (p_α) is a chain of projections in M such that $P(p_\alpha)$ is true for each α, then $P(\sup(p_\alpha))$ is true. Hence by Zorn's Lemma there is a maximal $p_0 \in \mathcal{P}(M)$ such that $P(p_0)$ is true.*

Proof. Let x be on the unit sphere and let p be the supremum of the p_α. By Theorem 1 and Lemma 4 we know p_α converges in σ–topology to p, and p is a projection. By Lemma 3 we know $p - p_\alpha$ is a projection. By the C^*–property of the norm ($\|x^*x\| = \|x\|^2$), Lemma 2 and the positivity of φ we have $\varphi(x^*x) \leq \varphi(1)$. Now these facts together with Cauchy-Schwartz and the monotonicity of square roots,

$$\begin{aligned} |\varphi(x(p - p_\alpha))| &\leq \varphi(x^*x)^{1/2} \varphi(p - p_\alpha)^{1/2} \\ &\leq \varphi(1)^{1/2} \varphi(p - p_\alpha)^{1/2}. \end{aligned} \tag{1}$$

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Hence, by the definition of operator norm, $\|\varphi(\cdot(p-p_\alpha))\| \leq \varphi(1)^{1/2}\varphi(p-p_\alpha)^{1/2}$. The right hand side converges to 0 with α due to the normality of φ and therefore $\varphi(\cdot p_\alpha)$ converges to $\varphi(\cdot p)$ in norm. Since the set of σ–continuous functionals on M is norm closed, it follows that $\sigma(\cdot p)$ is σ–continuous. We obtain a maximal p_0 by Zorn's Lemma. $\qquad\square$

Lemma 12. *A linear functional ρ on M is σ continuous on the unit sphere (hence σ–continuous) if and only if it is s–continuous on the unit sphere (hence s–continuous).*

Proof. $\qquad\square$

Lemma 13. *A maximal abelian $*$–subalgebra of a W^*–algebra M is also a W^*–algebra.*

Proof. $\qquad\square$

Theorem 14. *Every positive normal linear functional φ on M is $\sigma(M, M_*)$–continuous.*

Proof. The claim is obvious for the zero functional. Let φ be a nonzero positive normal linear functional. By Lemma 11 we have a maximal $p_0 \in \mathcal{P}(M)$ such that $M \ni x \mapsto \varphi(xp_0)$ is $\sigma(M, M_*)$–continuous. Assume for the purposes of finding a contradiction that $p_0 \neq 1$. By Lemma 6 there is a $\sigma(M, M_*)$–continuous positive functional ψ on M such that $\varphi(1-p_0) < \psi(1-p_0)$. By Lemma 7 there is a nonzero subprojection $p \leq 1-p_0$ in M such that $\varphi(q) < \psi(q)$ for every nonzero $q \leq p$ in M. Let $x \in pMp$ be on the unit sphere. Then x^*x is positive and hence normal, so the C^*–subalgebra of pMp generated by x^*x and p is commutative, and is hence contained in a maximal abelian $*$–subalgebra A of pMp. Now A is a W^*–subalgebra of pMp by Lemma 13 and hence is a maximal commutative C^*–subalgebra of pMp. Via the Gelfand Transform, A is star isomorphic to $C(K)$, where K is Stonean by Lemmas 9 and 10. By Lemma 8 it follows that $\varphi(a) \leq \psi(a)$ for every $a \geq 0$ in A, which holds a fortiori for $a \geq 0$ in $C^*(x^*x, p)$. In particular, $\varphi(px^*xp) \leq \psi(px^*xp)$. Therefore,

$$
\begin{aligned}
|\varphi(x(p_0+p))| &\leq |\varphi(xp_0)| + |\varphi(xp)| \qquad\qquad\qquad (2)\\
&\leq |\varphi(xp_0)| + \varphi(1)^{1/2}\varphi(px^*xp)^{1/2}\\
&\leq |\varphi(xp_0)| + \varphi(1)^{1/2}\psi(px^*xp)^{1/2}.
\end{aligned}
$$

Since $x \mapsto \varphi(xp_0)$ is σ–continuous, it is s-continuous by Lemma 12. The seminorm $x \mapsto \psi(px^*xp)^{1/2}$ is a defining seminorm for the s–topology on M. It follows that $x \mapsto \varphi(x(p_0+p))$ is s-continuous and therefore σ–continuous. This contradicts the maximality of p_0, and therefore $p_0 = 1$ and the result follows. $\qquad\square$