WStarAlgebras have Unique Preduals

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0.1 WStarAlgebras and their Topologies

In this section we define the topologies and need to prove that W^*–algebras are closed with respect to these. Must also put these dependencies explicitly into the results that use them below.

0.2 Order Lemmas

Let $\mathcal{A}$ be a unital C^*–algebra in this section. We collect lemmas for the ordering of elements in $\mathcal{A}$. Recall that a selfadjoint element $a \in \mathcal{A}$ is said to be **positive** if its spectrum is contained in $\mathbb{R}_{\geq 0}$. This is written $a \geq 0$. If $a, b \in \mathcal{A}$ are selfadjoint then $b \leq a$ if $b - a \geq 0$. (TO DO: adapt this to Jireh's fully general ordering on all elements without mentioning the selfadjoint ones.)

Lemma 1. *Let $h \in \mathcal{A}$. TFAE:*

1. *$h \ \ 0$;*
2. *There exists $x \in \mathcal{A}$ such that $h = x^*x$.*

Proof. □

Corollary 2. *If $h, a \in \mathcal{A}$ with $h \geq 0$ then $a^*ha \geq 0$*

Proof. By Lemma 1, $a^*ha = a^*x^*xa = (xa)^*(xa) \geq 0$. □

Lemma 3. *If $x \in \mathcal{A}$ is selfadjoint then $\|x\|1 - x \geq 0$.*

Proof. □

0.3 Projection Lemmas

Let $\mathcal{A}$ be a unital C^*–algebra in this section. In this section we collect relevant results about (selfadjoint) projections in $\mathcal{A}$. Recall that an element p in a C^*–algebra is a projection if $p^2 = p^* = p$. By the Spectral Mapping Theorem (or using the full Gelfand Duality), a selfadjoint element $a \in A$ is a projection if and only if the spectrum of a is contained in $\{0, 1\}$.

Lemma 4. *For all projections $p \in \mathcal{A}$, $p \ \ 1$.*

Proof. We can get this by Sakai 1.2.3. and Lemma 3 We should include that result here. □

Lemma 5. *Let $\mathcal{A}$ be a C^*–algebra and $p, q \in \mathcal{A}$ be projections. Then $p - q \geq 0$ iff $qp = pq = q$.*

Proof. By Lemma 2, if $p - q \geq 0$ then $q(p - q)q = qpq - q^3 = qpq - q \geq 0$. By Lemma 4, we have $p \leq 1$ and employing Lemma 2 we obtain $qpq \leq q1q = q$. Since $qpq \geq q$ and $qpq \leq q$ we have $qpq = q$, which implies that $q(p - q)q = 0$. By the C^*-property of the norm, we have $\|(p-q)^{1/2}q\|^2 = \|q(p-q)q\| = 0$ and so $(p-q)^{1/2}q = 0$ and therefore $(p-q)q = (p-q)^{1/2}(p-q)^{1/2}q = 0$. It follows that $pq = q$, and taking adjoints, $qp = q$. Conversely, if $qp = pq = q$, one easily checks that $p - q$ is a projection and so its spectrum is contained in $\{0, 1\}$ and it is positive. □

Corollary 6. *For all $p, q \in \mathcal{A}$ projections such that $q \leq p$, $p - q$ is a projection.*

0.4 Positive Linear Functionals

0.5 Normality and Ultraweak Continuity for Positive Functionals

In what follows, let M be a (nonzero) W^*–algebra. Let $\mathcal{P}(M)$ denote the projection lattice of M. Let φ be a positive linear functional φ on M. We say φ is **normal** if whenever (p_α) is an increasing net of projections in M with supremum p, we have $\varphi(p_\alpha) \to \varphi(p)$. In this section we show that this property is equivalent to $\sigma(M, M_*)$–continuity. We say that a linear functional φ on M is **positive** if $\varphi(x) \geq 0$ whenever $x \geq 0$.

0.5.1 Ultraweakly Continuous Implies Normal

This is 1.7.4 in Sakai. Roughly, taking an increasing net of projections, it converges to the supremum strongly, hence σ-topology, and the σ-continuity finishes the proof.

Theorem 7. *Every uniformly bounded, increasing directed set converges to its least upper bound in the σ–topology. Further, if $x = \sup_\alpha x_\alpha$ then $a^*xa = \sup_\alpha a^*xa$.*

Proof. □

0.5.2 Normal Implies Ultraweakly Continuous

Lemma 8. *For every increasing bounded net (p_α) of projections in M, the supremum p is a projection in M*

Proof. □

If $P : \mathcal{P}(M) \to \text{Prop}$ is a predicate, the usual ordering "$\leq$" on projections induces an order on the set $\{p \in \mathcal{P}(M) | P(p)\}$. In what follows there will be no confusion if we also denote this induced order by "$\leq$".

Lemma 9. *For every nonzero element $a \in M$, there is a $\sigma(M, M_*)$–continuous positive linear functional ψ on M such that $\psi(a) \neq 0$.*

Proof. □

Lemma 10. *For every positive normal linear functional φ and nonzero $p \in \mathcal{P}(M)$ there exists a positive $\sigma(M, M_*)$–continuous linear functional ψ such that $\varphi(p) < \psi(p)$.*

Proof. By Lemma 9 there is a $\sigma(M, M_*)$–continuous positive linear functional ψ_0 on M such that $\psi_0(p) \neq 0$. Rescale this functional by a positive constant to obtain ψ such that $\varphi(p) < \psi(p)$. □

Lemma 11. *For all positive linear functionals φ, ψ with φ normal and ψ $\sigma(M, M_*)$–continuous and every nonzero $p \in \mathcal{P}(M)$ such that $\varphi(p) < \psi(p)$, there exists a nonzero $p_1 \in \mathcal{P}(M)$ such that $p_1 \leq p$ and for all nonzero $q \in \mathcal{P}(M)$ with $q \leq p_1$, we have $\varphi(q) < \psi(q)$.*

Proof. We proceed by contradiction. Suppose the conclusion does not hold. Then, for every nonzero subprojection p_1 of p there is a nonzero subprojection $q \leq p_1$ such that $\varphi(q) \geq \psi(q)$. In particular, (letting $p_1 = q$) there is a $q \leq p$ such that $\varphi(q) \geq \psi(q)$. If (q_α) is a chain of such nonzero projections, $q_\alpha \to \sup q_\alpha$ in the σ–topology by Theorem 7, and by Lemma 8 we know this supremum is a subprojection of p_1. Since φ is positive and normal, and ψ is positive and

σ–continuous, we have $\varphi(\sup q_\alpha) \geq \psi(\sup q_\alpha)$. Therefore, by Zorn's Lemma, there is a maximal $q_0 \leq p$ such that $\varphi(q_0) \geq \psi(q_0)$. We claim $q_0 = p$. If not, $p - q_0$ is a nonzero subprojection of p by Corollary 6, and there exists a nonzero projection $q_1 \leq p - q_0$ such that $\varphi(q_1) \geq \psi(q_1)$. But then q_0 is a proper subprojection of $q_0 + q_1$ and by linearity and positivity $\varphi(q_0+q_1) \geq \psi(q_0+q_1)$ contradicting the maximality of q_0. Thus $p = q_0$ and so $\varphi(p) \geq \psi(p)$, contradicting the hypothesis $\varphi(p) < \psi(p)$. □

Recall that a compact Hausdorff space is **Stonean** if the closure of every open set is open.

Lemma 12. *Let K be a Stonean space. Then every element a in $C(K)$ can be uniformly approximated by finite linear combinations of projections in $C(K)$. If $a \geq 0$ then the coefficients of the approximating linear combinations may be chosen nonnegative.*

Proof. □

Lemma 13. *If $p \in M$ is a projection, pMp is also a W^* algebra with identity p.*

Proof. □

Lemma 14. *If C is any maximal commutative C^*–subalgebra of the W^*–algebra M, its spectrum space (maximal ideal space) is Stonean.*

Proof. □

Note that the following likely contains the most crucial use of the normality of φ for the later proof of Theorem 18.

Lemma 15. *Let φ be a normal positive linear functional on M and consider the predicate $P : \mathcal{P}(M) \to Prop$ defined, for $p \in \mathcal{P}(M)$, by "$M \ni x \mapsto \varphi(xp)$ is $\sigma(M, M_*)$–continuous". If (p_α) is a chain of projections in M such that $P(p_\alpha)$ is true for each α, then $P(\sup(p_\alpha))$ is true. Hence by Zorn's Lemma there is a maximal $p_0 \in \mathcal{P}(M)$ such that $P(p_0)$ is true.*

Proof. Let x be on the unit sphere and let p be the supremum of the p_α. By Theorem 7 and Lemma 8 we know p_α converges in σ–topology to p, and p is a projection. By Corollary 6 we know $p - p_\alpha$ is a projection. By the C^*–property of the norm ($\|x^*x\| = \|x\|^2$), Lemma 3 and the positivity of φ we have $\varphi(x^*x) \leq \varphi(1)$. Now these facts together with Cauchy-Schwartz and the monotonicity of square roots,

$$\begin{aligned} |\varphi(x(p - p_\alpha))| &\leq \varphi(x^*x)^{1/2}\varphi(p - p_\alpha)^{1/2} \\ &\leq \varphi(1)^{1/2}\varphi(p - p_\alpha)^{1/2}. \end{aligned} \tag{1}$$

Hence, by the definition of operator norm, $\|\varphi(\cdot(p - p_\alpha))\| \leq \varphi(1)^{1/2}\varphi(p - p_\alpha)^{1/2}$. The right hand side converges to 0 with α due to the normality of φ and therefore $\varphi(\cdot p_\alpha)$ converges to $\varphi(\cdot p)$ in norm. Since the set of σ–continuous functionals on M is norm closed, it follows that $\sigma(\cdot p)$ is σ–continuous. We obtain a maximal p_0 by Zorn's Lemma. □

Lemma 16. *A linear functional ρ on M is σ continuous on the unit sphere (hence σ–continuous) if and only if it is s–continuous on the unit sphere (hence s–continuous).*

Proof. □

Lemma 17. *A maximal abelian $*$–subalgebra of a W^*–algebra M is also a W^*–algebra.*

Proof. □

Theorem 18. *Every positive normal linear functional φ on M is $\sigma(M, M_*)$–continuous.*

Proof. The claim is obvious for the zero functional. Let φ be a nonzero positive normal linear functional. By Lemma 15 we have a maximal $p_0 \in \mathcal{P}(M)$ such that $M \ni x \mapsto \varphi(xp_0)$ is $\sigma(M, M_*)$–continuous. Assume for the purposes of finding a contradiction that $p_0 \neq 1$. By Lemma 10 there is a $\sigma(M, M_*)$–continuous positive functional ψ on M such that $\varphi(1 - p_0) < \psi(1 - p_0)$. By Lemma 11 there is a nonzero subprojection $p \leq 1 - p_0$ in M such that $\varphi(q) < \psi(q)$ for every nonzero $q \leq p$ in M. Let $x \in pMp$ be on the unit sphere. Then x^*x is positive and hence normal, so the C^*–subalgebra of pMp generated by x^*x and p is commutative, and is hence contained in a maximal abelian $*$–subalgebra A of pMp. Now A is a W^*–subalgebra of pMp by Lemma 17 and hence is a maximal commutative C^*–subalgebra of pMp. Via the Gelfand Transform, A is star isomorphic to $C(K)$, where K is Stonean by Lemmas 13 and 14. By Lemma 12 it follows that $\varphi(a) \leq \psi(a)$ for every $a \geq 0$ in A, which holds a fortiori for $a \geq 0$ in $C^*(x^*x, p)$. In particular, $\varphi(px^*xp) \leq \psi(px^*xp)$. Therefore,

$$
\begin{aligned}
|\varphi(x(p_0 + p))| &\leq |\varphi(xp_0)| + |\varphi(xp)| \\
&\leq |\varphi(xp_0)| + \varphi(1)^{1/2}\varphi(px^*xp)^{1/2} \\
&\leq |\varphi(xp_0)| + \varphi(1)^{1/2}\psi(px^*xp)^{1/2}.
\end{aligned} \tag{2}
$$

Since $x \mapsto \varphi(xp_0)$ is σ–continuous, it is s-continuous by Lemma 16. The seminorm $x \mapsto \psi(px^*xp)^{1/2}$ is a defining seminorm for the s–topology on M. It follows that $x \mapsto \varphi(x(p_0 + p))$ is s-continuous and therefore σ–continuous. This contradicts the maximality of p_0, and therefore $p_0 = 1$ and the result follows. ∎