# WStarAlgebras have Unique Preduals

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## 0.1 WStarAlgebras and their Topologies

In this section we define the topologies and need to prove that $W^*$–algebras are closed with respect to these. Must also put these dependencies explicitly into the results that use them below.

## 0.2 Normality and Ultraweak Continuity for Positive Functionals

In what follows, let $M$ be a (nonzero) $W^*$–algebra. Let $\mathcal{P}(M)$ denote the projection lattice of $M$. Let $\varphi$ be a positive linear functional $\varphi$ on $M$. We say $\varphi$ is **normal** if whenever $(p_\alpha)$ is an increasing net of projections in $M$ with supremum $p$, we have $\varphi(p_\alpha) \to \varphi(p)$. In this section we show that this property is equivalent to $\sigma(M, M_*)$–continuity. We say that a linear functional $\varphi$ on $M$ is **positive** if $\varphi(x) \geq 0$ whenever $x \geq 0$.

*To do : Net results must be replaced by filter results.*

### 0.2.1 Ultraweakly Continuous Implies Normal

This is 1.7.4 in Sakai. Roughly, taking an increasing net of projections, it converges to the supremum strongly, hence $\sigma$-topology, and the $\sigma$-continuity finishes the proof.

**Theorem 1.** *Every uniforly bounded, increasing directed set converges to its least upper bound in the $\sigma$–topology. Further, if $x = \sup_\alpha x_\alpha$ then $a^*xa = \sup_\alpha a^*xa$.*

*Proof.* □

### 0.2.2 Normal Implies Ultraweakly Continuous

**Lemma 2.** *For all self-adjoint $x \in M$, $\|x\|1 - x \geq 0$.*

*Proof.* □

**Lemma 3.** *For all $p, q \in \mathcal{P}(M)$ such that $q \leq p$, $p - q \in \mathcal{P}(M)$.*

*Proof.* □

**Lemma 4.** *For every increasing bounded net $(p_\alpha)$ of projections in $M$, the supremum $p$ is a projection in $M$*

*Proof.* □

If $P : \mathcal{P}(M) \to \mathrm{Prop}$ is a predicate, the usual ordering "$\leq$" on projections induces an order on the set $\{p \in \mathcal{P}(M) | P(p)\}$. In what follows there will be no confusion if we also denote this induced order by "$\leq$".

The following lemma undoubtedly exists in Mathlib in more generality already.

**Lemma 5.** *For every nonzero element $a \in M$, there is a $\sigma(M, M_*)$–continuous positive linear functional $\psi$ on $M$ such that $\psi(a) \neq 0$.*

*Proof.* □

**Lemma 6.** *For every positive normal linear functional $\varphi$ and nonzero $p \in \mathcal{P}(M)$ there exists a positive $\sigma(M, M_*)$–continuous linear functional $\psi$ such that $\varphi(p) < \psi(p)$.*

*Proof.* □

**Lemma 7.** *For all positive linear functionals $\varphi, \psi$ with $\varphi$ normal and $\psi$ $\sigma(M, M_*)$–continuous and every nonzero $p \in \mathcal{P}(M)$ such that $\varphi(p) < \psi(p)$, there exists a nonzero $p_1 \in \mathcal{P}(M)$ such that $p_1 \leq p$ and for all nonzero $q \in \mathcal{P}(M)$ with $q \leq p_1$, we have $\varphi(q) < \psi(q)$.*

*Proof.* We proceed by contradiction. Suppose the conclusion does not hold. Then, for every nonzero subprojection $p_1$ of $p$ there is a nonzero subprojection $q \leq p_1$ such that $\varphi(q) \geq \psi(q)$. In particular, (letting $p_1 = q$) there is a $q \leq p$ such that $\varphi(q) \geq \psi(q)$. If $(q_\alpha)$ is a chain of such nonzero projections, $q_\alpha \to \sup q_\alpha$ in the $\sigma$–topology by Theorem 1, and by Lemma 4 we know this supremum is a subprojection of $p_1$. Since $\varphi$ is positive and normal, and $\psi$ is positive and $\sigma$–continuous, we have $\varphi(\sup q_\alpha) \geq \psi(\sup q_\alpha)$. Therefore, by Zorn's Lemma, there is a maximal $q_0 \leq p$ such that $\varphi(q_0) \geq \psi(q_0)$. We claim $q_0 = p$. If not, $p - q_0$ is a nonzero subprojection of $p$ by Lemma 3, and there exists a nonzero projection $q_1 \leq p - q_0$ such that $\varphi(q_1) \geq \psi(q_1)$. But then $q_0$ is a proper subprojection of $q_0 + q_1$ and by linearity and positivity $\varphi(q_0 + q_1) \geq \psi(q_0 + q_1)$ contradicting the maximality of $q_0$. Thus $p = q_0$ and so $\varphi(p) \geq \psi(p)$, contradicting the hypothesis $\varphi(p) < \psi(p)$. □

Recall that a compact Hausdorff space is **Stonean** if the closure of every open set is open.

**Lemma 8.** *Let $K$ be a Stonean space. Then every element $a$ in $C(K)$ can be uniformly approximated by finite linear combinations of projections in $C(K)$. If $a \geq 0$ then the coefficients of the approximating linear combinations may be chosen nonnegative.*

*Proof.* □

**Lemma 9.** *If $p \in M$ is a projection, $pMp$ is also a $W^*$ algebra with identity $p$.*

*Proof.* □

**Lemma 10.** *If $C$ is any maximal commutative $C^*$–subalgebra of the $W^*$–algebra $M$, its spectrum space (maximal ideal space) is Stonean.*

*Proof.* □

**Lemma 11.** *Let $\varphi$ be a normal positive linear functional on $M$ and consider the predicate $P : \mathcal{P}(M) \to Prop$ defined, for $p \in \mathcal{P}(M)$, by "$M \ni x \mapsto \varphi(xp)$ is $\sigma(M, M_*)$–continuous". If $(p_\alpha)$ is a chain of projections in $M$ such that $P(p_\alpha)$ is true for each $\alpha$, then $P(\sup(p_\alpha))$ is true. Hence by Zorn's Lemma there is a maximal $p_0 \in \mathcal{P}(M)$ such that $P(p_0)$ is true.*

*Proof.* □

**Lemma 12.** *A linear functional $\rho$ on $M$ is $\sigma$ continuous on the unit sphere (hence $\sigma$–continuous) if and only if it is $s$–continuous on the unit sphere (hence $s$–continuous).*

*Proof.* □

**Lemma 13.** *A maximal abelian $*$–subalgebra of a $W^*$–algebra $M$ is also a $W^*$–algebra.*

*Proof.* □

**Theorem 14.** *Every positive normal linear functional $\varphi$ on $M$ is $\sigma(M, M_*)$–continuous.*

*Proof.* The claim is obvious for the zero functional. Let $\varphi$ be a nonzero positive normal linear functional. By Lemma 11 we have a maximal $p_0 \in \mathcal{P}(M)$ such that $M \ni x \mapsto \varphi(xp_0)$ is $\sigma(M, M_*)$–continuous. Assume for the purposes of finding a contradiction that $p_0 \neq 1$. By Lemma 6 there is a $\sigma(M, M_*)$–continuous positive functional $\psi$ on $M$ such that $\varphi(1-p_0) < \psi(1-p_0)$. By Lemma 7 there is a nonzero subprojection $p \leq 1-p_0$ in $M$ such that $\varphi(q) < \psi(q)$ for every nonzero $q \leq p$ in $M$. Let $x \in pMp$ be on the unit sphere. Then $x^*x$ is positive and hence normal, so the $C^*$–subalgebra of $pMp$ generated by $x^*x$ and $p$ is commutative, and is hence contained in a maximal abelian $*$–subalgebra $A$ of $pMp$. Now $A$ is a $W^*$–subalgebra of $pMp$ by Lemma 13 and hence is a maximal commutative $C^*$–subalgebra of $pMp$. Via the Gelfand Transform, $A$ is star isomorphic to $C(K)$, where $K$ is Stonean by Lemmas 9 and 10. By Lemma 8 it follows that $\varphi(a) \leq \psi(a)$ for every $a \geq 0$ in $A$, which holds a fortiori for $a \geq 0$ in $C^*(x^*x, p)$. In particular, $\varphi(px^*xp) \leq \psi(px^*xp)$. Therefore,

$$\begin{aligned} |\varphi(x(p_0+p))| &\leq |\varphi(xp_0)| + |\varphi(xp)| \\ &\leq |\varphi(xp_0)| + \varphi(1)^{1/2}\varphi(px^*xp)^{1/2} \\ &\leq |\varphi(xp_0)| + \varphi(1)^{1/2}\psi(px^*xp)^{1/2}. \end{aligned} \tag{1}$$

Since $x \mapsto \varphi(xp_0)$ is $\sigma$–continuous, it is $s$-continuous by Lemma 12. The seminorm $x \mapsto \psi(px^*xp)^{1/2}$ is a defining seminorm for the $s$–topology on $M$. It follows that $x \mapsto \varphi(x(p_0+p))$ is $s$-continuous and therefore $\sigma$–continuous. This contradicts the maximality of $p_0$, and therefore $p_0 = 1$ and the result follows. □