On Nelson-Oppen Techniques

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Abstract. The Nelson-Oppen method [5] allows the modular combination of quantifier free satisfiability procedures of first-order theories into a quantifier-free satisfiability procedure for the union of the theories. However, this method requires the theories to have disjoint signatures and to be stably infinite. Due to the importance of the result, several attempts to extend the method to different and wider classes of theories were made. Recently, two different extensions of the Nelson-Oppen method were proposed, where the stably infinite requirement was replaced by another condition: in [9] it was required that all but one of the theories are shiny, and in [7] it was required that, when combining two theories, one of them is polite. The relationship between shiny and polite theories was analysed in [7]. Later, a stronger notion of polite theory was proposed, see [4], in order to overcome a subtle issue with the proof of the Nelson-Oppen method in [7]. In this paper, we analyse the relationship between shiny and strongly polite theories in the one-sorted case. We show that a shiny theory with a decidable quantifier-free satisfiability problem is strongly polite and we provide two different sufficient conditions for a strongly polite theory to be shiny. Based on these results, we relate the complexity on showing that a theory is shiny with the complexity on showing that a theory is strongly polite.

1 Introduction

The problem of modularly combining satisfiability procedures of two theories into a satisfiability procedure for their union is of great interest in the area of automated reasoning: for instance, verification systems such as CVC4 [1] and SMTInterpol [3] rely on such a combination procedure.

The first and most well-known method for the combination of satisfiability procedures is due to Nelson and Oppen, [5]. In this seminal paper, the authors provide a combination method to decide the satisfiability of quantifier-free formulas in the union of two theories, provided that both theories have their own procedure for deciding the satisfiability problem of quantifier-free formulas. After a correction, see [6], the two main restrictions of the Nelson-Oppen method are:

- the theories are *stably infinite*,
- their signatures are disjoint.

Concerned about the fact that many theories of interest, such as those admitting only finite models, are not stably infinite, Tinelli and Zarba, in [9], showed that the Nelson-Oppen combination procedure still applies when the stable infiniteness condition is replaced by the requirement that all but one of the theories is *shiny*. However, a shiny theory must be equipped with a particular function called mincard, which is inherently hard to compute.

In order to overcome the problem of computing the mincard function and of the shortage of shiny theories, Ranise, Ringeissen and Zarba proposed an alternative requirement, *politeness*, in [7], and analysed its relationship with shininess. A polite theory has to be equipped with a witness function, which was thought to be easier to compute than the mincard function. They show that given a polite theory and an arbitrary one, the Nelson-Oppen combination procedure is still valid when the signatures are disjoint and both theories have their own procedure for deciding the satisfiability problem of quantifier-free formulas. Some time later, in [4], Jovanović and Barrett reported that the politeness notion provided in [7] allowed, after all, witness functions that are not sufficiently strong to prove the combination theorem. In order to solve the problem they provided a seemingly stronger notion of politeness, in the sequel called *strongly politeness*, equipped with a seemingly stronger witness function, s-witness, that allowed to prove the combination theorem. However, the authors left open the relationship between the two notions of politeness and between the strong politeness notion and shininess.

In this paper we investigate the relationship between shiny and strongly polite theories. We show that a shiny theory with a decidable quantifier-free satisfiability problem is strongly polite. For the other direction, we provide two different sets of conditions under which a strongly polite theory is shiny (see Figure 1 for a more detailed global view of the results). Moreover, we show that, under some conditions, a polite theory is also strongly polite and so there is a way to transform a witness function into a strong witness function. Given the constructive nature of the proofs we were able to design such a procedure.

1.1 Organization of the paper

The paper is organized as follows: in Section 2 we recall some relevant definitions. In Section 3 we begin by recalling the definitions of shininess and of (strong) politeness and then we proceed to show the equivalence between these notions. In Section 4 we analyze what was done in the paper and provide directions for further research.

2 Preliminaries

The results in this paper concern first-order logic with equality. We assume given a countably infinite set of variables. We mainly follow the notation in [9].

2.1 Syntax

A *signature* is a tuple $\Sigma = \langle \Sigma^F, \Sigma^P \rangle$ where Σ^F is the set of function symbols and Σ^P is the set of predicate symbols. We use $\cong$ to denote the equality logic symbol and assume the standard definitions of Σ-*atom* and Σ-*term*. A Σ-formula

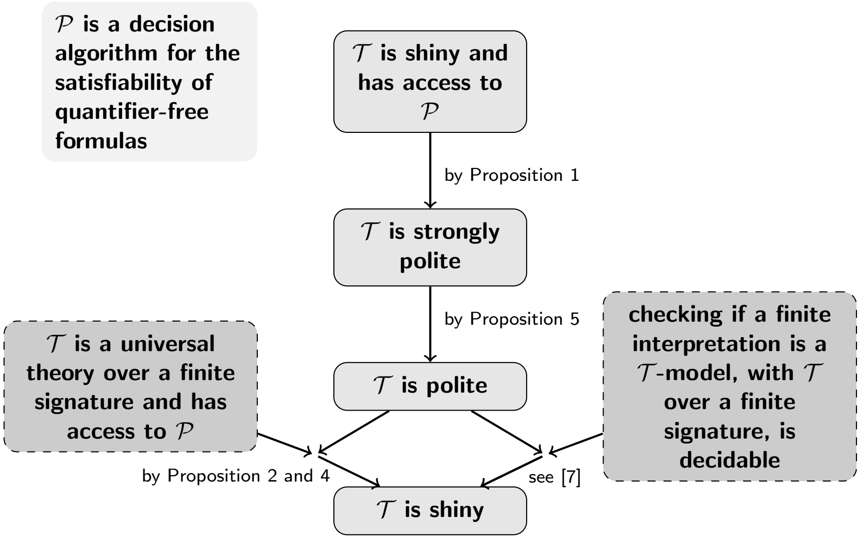

Fig. 1: Schematic representation of the results in the paper

is inductively defined as usual over Σ-atoms and Σ-terms using the connectives $\wedge, \vee, \neg, \rightarrow$ or the quantifiers $\forall$ and $\exists$. We denote by $\mathsf{QF}(\Sigma)$ the set of Σ-formulas with no occurrences of quantifiers and, given a Σ-formula φ, by $\mathsf{vars}(\varphi)$ the set of free variables of φ. We say that a Σ-formula is a *Σ-sentence* if it has no free variables. In the sequel, when there is no ambiguity, we will omit the reference to the signature when referring to atoms, terms, formulas and sentences.

Definition 1 (Arrangement formula). *Given a finite set of variables Y and an equivalence relation $E \subseteq Y^2$, the* arrangement formula *induced by E over Y, denoted by δ_E^Y, is*

$$\bigwedge_{(x,y)\in E} (x \cong y) \wedge \bigwedge_{(x,y)\in Y^2\setminus E} \neg(x \cong y)$$

In the sequel, we may simply denote δ_E^Y by δ_E if there is no confusion to which variable set the formula refers to.

2.2 Semantics

Given a signature Σ, a *Σ-interpretation* $\mathcal{A}$ with domain A over a set of variables X is a map that interprets each variable $x \in X$ as an element $x^{\mathcal{A}} \in A$, each function symbol $f \in \Sigma^F$ of arity n as a map $f^{\mathcal{A}} : A^n \rightarrow A$ and each predicate symbol $p \in \Sigma^P$ of arity n as a a subset $P^{\mathcal{A}}$ of A^n. We denote by $\mathsf{dom}(\mathcal{A})$ the domain of an interpretation $\mathcal{A}$. In the sequel, when there is no ambiguity, we will omit the reference to the signature when referring to interpretations.

Given an interpretation $\mathcal{A}$ and a term t, we denote by $t^{\mathcal{A}}$ the interpretation of t under $\mathcal{A}$. Similarly, we denote by $\varphi^{\mathcal{A}}$ the truth value of the formula φ under the interpretation $\mathcal{A}$. Furthermore, given a set Γ of formulas, we denote by $[\![\Gamma]\!]^{\mathcal{A}}$ the set $\{\varphi^{\mathcal{A}} : \varphi \in \Gamma\}$, and similarly for a set of terms.

A formula φ is *satisfiable* if it is true under some interpretation, and *unsatisfiable* otherwise.

Given a set of variables Y we say that two interpretations $\mathcal{A}$ and $\mathcal{B}$ over a set X of variables are Y*-equivalent* whenever $\mathsf{dom}(\mathcal{A}) = \mathsf{dom}(\mathcal{B})$, $f^{\mathcal{A}} = f^{\mathcal{B}}$ for each function symbol f, $p^{\mathcal{A}} = p^{\mathcal{B}}$ for each predicate symbol p, and $x^{\mathcal{A}} = x^{\mathcal{B}}$ for each variable x in $X \setminus Y$.

We also say that an *interpretation is finite (infinite)* when its domain is finite (infinite).

2.3 Theories

Given a signature Σ, a Σ*-theory* is a set of Σ-sentences and given a Σ-theory $\mathcal{T}$, a $\mathcal{T}$*-model* is a Σ-interpretation that satisfies all sentences of $\mathcal{T}$. We say that a formula φ is $\mathcal{T}$*-satisfiable* when there is a $\mathcal{T}$-model that satisfies it and say that two formulas are $\mathcal{T}$*-equivalent* if they are interpreted to the same truth value in every $\mathcal{T}$-model. In the sequel, when there is no ambiguity, we will omit the reference to the signature when referring to theories.

Given a Σ_1-theory $\mathcal{T}_1$ and a Σ_2-theory $\mathcal{T}_2$, their union, $\mathcal{T}_1 \oplus \mathcal{T}_2$, is a $\Sigma_1 \cup \Sigma_2$-theory defined by the union of the sentences of $\mathcal{T}_1$ with the sentences of $\mathcal{T}_2$.

The following definitions introduce some of the conditions used in the results presented in this paper.

Definition 2 (Smoothness). *We say that a theory $\mathcal{T}$ is* smooth *if for every $\mathcal{T}$-satisfiable quantifier-free formula φ, $\mathcal{T}$-model $\mathcal{A}$ satisfying φ and cardinal $\kappa \geq |\mathcal{A}|$ there exists a $\mathcal{T}$-model $\mathcal{B}$ satisfying φ such that $|\mathcal{B}| = \kappa$.*

Definition 3 (Stable finiteness). *We say that a theory $\mathcal{T}$ is* stably finite *if for every $\mathcal{T}$-satisfiable quantifier-free formula φ there exists a finite $\mathcal{T}$-model of φ.*

Definition 4 (Stable infiniteness). *We say that a theory $\mathcal{T}$ is* stably infinite *if for every $\mathcal{T}$-satisfiable quantifier-free formula φ there exists an infinite $\mathcal{T}$-model of φ.*

Definition 5 (Finite witnessability, [7]). *We say that a theory $\mathcal{T}$ over a signature Σ is* finitely witnessable *if there exists a computable function* $\mathsf{witness} : \mathsf{QF}(\Sigma) \to \mathsf{QF}(\Sigma)$ *such that for every quantifier-free formula φ the following conditions hold:*

- *φ and $\exists \vec{w}\, \mathsf{witness}(\varphi)$ are $\mathcal{T}$-equivalent, where $\vec{w}$ are the variables in $\mathsf{witness}(\varphi)$ which do not occur in φ;*
- *if $\mathsf{witness}(\varphi)$ is satisfiable in $\mathcal{T}$ then there exists a $\mathcal{T}$-model $\mathcal{A}$ such that $\mathcal{A} \Vdash \mathsf{witness}(\varphi)$ and $\mathsf{dom}(\mathcal{A}) = [\![\mathsf{vars}(\mathsf{witness}(\varphi))]\!]^{\mathcal{A}}$.*

A function satisfying the above properties is called a *witness function for $\mathcal{T}$*. In [4], a stronger finite witnessability notion was defined in order to clarify an issue found on [7].

Definition 6 (Strong finite witnessability, [4]). *We say that a theory $\mathcal{T}$ over a signature Σ is* strongly finitely witnessable *if there exists a computable function* $\mathsf{s\text{-}witness} : \mathsf{QF}(\Sigma) \to \mathsf{QF}(\Sigma)$ *such that for every quantifier-free formula φ the following conditions hold:*

- *φ and $\exists \vec{w}\ \mathsf{s\text{-}witness}(\varphi)$ are $\mathcal{T}$-equivalent, where $\vec{w}$ are the variables in $\mathsf{s\text{-}witness}(\varphi)$ which do not occur in φ;*
- *for every finite set of variables Y and relation $E \subseteq Y^2$, if $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y$ is satisfiable in $\mathcal{T}$ then there exists a $\mathcal{T}$-model $\mathcal{A}$ such that $\mathcal{A} \Vdash \mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y$ and $\mathsf{dom}(\mathcal{A}) = [\![\mathsf{vars}(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y)]\!]^{\mathcal{A}}$.*

A function satisfying the above properties is called a *strong witness function for $\mathcal{T}$*. The following notion was introduced by Tinelli and Zarba in [9] and its computability is one of the conditions a theory should satisfy to be shiny.

Definition 7 ($\mathsf{mincard}$ function). *Given a theory $\mathcal{T}$ over a signature Σ, let $\mathsf{mincard}_{\mathcal{T}}$ be the function from $\mathsf{QF}(\Sigma)$ to $\mathsf{QF}(\Sigma)$ such that*

$$\mathsf{mincard}_{\mathcal{T}}(\varphi) = \min\{k : \mathcal{A} \textit{ is a } \mathcal{T}\textit{-model, } \mathcal{A} \Vdash \varphi \textit{ and } |\mathsf{dom}(\mathcal{A})| = k\}$$

if φ is $\mathcal{T}$-satisfiable, otherwise $\mathsf{mincard}_{\mathcal{T}}(\varphi)$ is undefined.

So, when φ is $\mathcal{T}$-satisfiable the function $\mathsf{mincard}_{\mathcal{T}}$ returns the cardinality of the smallest $\mathcal{T}$-model of φ. When there is no ambiguity to which theory the function refers to we will simply write $\mathsf{mincard}$.

3 Shiny and (Strongly) Polite Theories

We now analyze the relationship between shiny and (strongly) polite theories. We start by showing that a shiny theory is strongly polite when assuming that it has a decidable quantifier-free satisfiability problem, but first we recall what is a shiny theory, see [9], and a strongly polite one, see [4].

Definition 8 (Shininess, [9]). *A theory is* shiny *whenever it is smooth, stably finite and its* $\mathsf{mincard}$ *function is computable.*

Several theories were proved to be shiny, such as the theory of equality, the theory of partial orders and the theory of total orders, in [9].

Definition 9 (Strong politeness, [4]). *A theory is* strongly polite *whenever it is smooth and strongly finitely witnessable.*

Proposition 1. *A shiny theory with a decidable quantifier-free satisfiability problem is strongly polite.*

Proof. See [2].

Recall the notion of *politeness* by Ranise, Ringeissen and Zarba [7].

Definition 10 (Politeness). *We say that a theory is* polite *whenever it is smooth and finitely witnessable.*

Proposition 2. *A polite theory is stably finite.*

Proof. See [2]

We now recall a proposition by Ranise, Ringeissen and Zarba in [7] that provides conditions under which a polite theory is shiny.

Proposition 3 ([7]). *Let Σ be a finite signature and $\mathcal{T}$ a Σ-theory. Assume that it is decidable to check if a finite Σ-interpretation is a $\mathcal{T}$-model. Then, if $\mathcal{T}$ is polite then $\mathcal{T}$ is shiny and Algorithm 1 computes its* $\mathsf{mincard}$ *function.*

Algorithm 1 — $\mathsf{mincard}_{\mathsf{witness}}$ algorithm
Input: φ, where φ is a quantifier-free satisfiable formula
Output: k, where k is the cardinality of the smallest $\mathcal{T}$-model of φ
Requires: access to a witness function $\mathsf{witness}$ for $\mathcal{T}$

$n = |\mathsf{vars}(\mathsf{witness}(\varphi))|$;
for $k = 1$ **to** n
  for all non-isomorphic $\mathcal{T}$-models $\mathcal{A}$ s.t. $|\mathsf{dom}(\mathcal{A})| = k$ do
    if $\mathcal{A} \Vdash \varphi$ **then** return k
  end for
end for

Observe that the conditions on the previous proposition are rather weak – for instance, if a theory $\mathcal{T}$ over Σ is finitely axiomatized then it is decidable to check if a finite Σ-interpretation is indeed a $\mathcal{T}$-model.

On the other hand, even if it is not decidable to check whether a finite interpretation is a $\mathcal{T}$-model, it is still possible to construct the $\mathsf{mincard}$ function provided that the theory $\mathcal{T}$ is universal as is stated in the next proposition. This proposition uses a result of Tinelli and Zarba in [9] and requires the notion of simple diagrams which we recall here.

Definition 11 (Simple diagrams). *Given a Σ-interpretation $\mathcal{A}$, define Σ^+ as the signature Σ enriched by a new constant symbol $\bar{d}$ for each $d \in A$. The* simple diagram of $\mathcal{A}$, *denoted by $\Delta(\mathcal{A})$, is the set*

$$\left\{[\varphi]^{x_1,\ldots,x_n}_{\bar{x}_1^{\mathcal{A}},\ldots,\bar{x}_n^{\mathcal{A}}} : \varphi \in \mathsf{QF}(\Sigma), \mathsf{vars}(\varphi) = \{x_1,\ldots,x_n\}, \mathcal{A} \Vdash \varphi\right\}.$$

Proposition 4. *Let Σ be a finite signature and $\mathcal{T}$ a universal Σ-theory with a decidable quantifier-free satisfiability problem. Then, if $\mathcal{T}$ is polite then it is shiny and Algorithm 2 computes its* $\mathsf{mincard}$ *function.*

Algorithm 2 — $\mathsf{mincard}_{\mathcal{P}}$ algorithm, [9]
Input: φ, where φ is a quantifier-free satisfiable formula
Output: k, where k is the cardinality of the smallest $\mathcal{T}$-model of φ
Requires: access to an algorithm $\mathcal{P}$ that decides satisfiability of quantifier-free formulas

while *true* **do**
  $k = 1$
  for all non-isomorphic interpretations $\mathcal{A}$ s.t. $|\mathsf{dom}(\mathcal{A})| = k$ do
    if $\mathcal{P}(\Delta(\mathcal{A}) \wedge \varphi) == 1$ **then** return k
  end for
  $k = k + 1$
end while

Proof. See [2].

Finally, we prove that a strongly finitely witnessable theory is also finitely witnessable.

Proposition 5. *Each strongly finitely witnessable theory is finitely witnessable.*

Proof. See [2].

Combining the previous results, we proved the equivalence between *strong politeness, shininess* and *politeness*, assuming two sets of different conditions on the theory.

Corollary 1. *Let $\mathcal{T}$ be a theory over a finite signature. If either*

- *$\mathcal{T}$ is universal; or*
- *checking whether a finite interpretation is a $\mathcal{T}$-model is decidable,*

then the following statements are equivalent:

1. *$\mathcal{T}$ is shiny;*
2. *$\mathcal{T}$ is strongly polite;*
3. *$\mathcal{T}$ is polite.*

Proof. See [2].

Capitalizing on the previous results on the relationship between strong politeness, shininess, and politeness, we now present a new algorithm, Algorithm 3, that computes a strong witness function for a smooth and finitely witnessable theory.

Theorem 1. *Let Σ be a finite signature and $\mathcal{T}$ a polite Σ-theory with a decidable quantifier-free satisfiability problem. Assume that either $\mathcal{T}$ is universal or it is decidable to check if a finite interpretation is a $\mathcal{T}$-model. Then, Algorithm 3 computes a strong witness function for $\mathcal{T}$.*

Algorithm 3 — Computes a strong witness function for a theory $\mathcal{T}$
Input: φ, where φ is a quantifier-free satisfiable formula
Output: $\mathsf{s\text{-}witness}(\varphi)$
Requires: access to an algorithm $\mathcal{P}$ that decides satisfiability of quantifier-free formulas, and to the function $\mathsf{mincard}$ for $\mathcal{T}$

for $E \subseteq \mathsf{vars}(\varphi)^2$
  $\delta_E^{\mathsf{vars}(\varphi)} = \varepsilon$
  for all pairs $(x, y) \in \mathsf{vars}(\varphi)^2$
    if $(x, y) \in E$
      then $\delta_E^{\mathsf{vars}(\varphi)} = \delta_E^{\mathsf{vars}(\varphi)} \wedge (x \cong y)$
      else $\delta_E^{\mathsf{vars}(\varphi)} = \delta_E^{\mathsf{vars}(\varphi)} \wedge \neg(x \cong y)$
    end if
  end for
  if $\mathcal{P}(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}) == 1$
    then $k_E = \mathsf{mincard}(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)})$
      $\gamma_{k_E} = \varepsilon$
      for $i, j = 1$, $i \neq j$ **to** k_E
        $\gamma_{k_E} = \gamma_{k_E} \wedge \neg(x_i \cong x_j)$
      end for
      $\varphi = \varphi \wedge (\delta_E^{\mathsf{vars}(\varphi)} \to \gamma_{k_E})$
  end if
end for
return φ

Proof. See [2].

Capitalizing on the relationships between the politeness, shininess and strong politeness established in the previous results, we can now establish a combination result very similar to the combination proposition of [4], Proposition 2, but instead of imposing that $\mathcal{T}_2$ is strongly finitely witnessable, imposes that $\mathcal{T}_2$ is

- finitely witnessable;
- either universal or such that checking if a finite Σ_2-interpretation is a model of $\mathcal{T}_2$ is decidable.

Observe that showing these conditions may be more manageable than proving that $\mathcal{T}_2$ is strongly finitely witnessable.

Theorem 2. *Let Σ_2 be a finite signature and $\mathcal{T}_i$ a Σ_i-theory with a decidable quantifier-free satisfiability problem, for $i = 1, 2$, such that $\Sigma_1 \cap \Sigma_2 = \emptyset$. Assume that*

- *$\mathcal{T}_2$ is smooth;*
- *$\mathcal{T}_2$ has a witness function;*
- *either $\mathcal{T}_2$ is universal or checking if a finite Σ_2-interpretation is a model of $\mathcal{T}_2$ is decidable.*

Then, the function $\mathsf{mincard}_{\mathcal{T}_2}$ *is computable and there is a computable strong witness function,* $\mathsf{s\text{-}witness}_{\mathcal{T}_2}$*, for* $\mathcal{T}_2$*, such that the following statements are equivalent:*

1. $\Gamma_1 \wedge \Gamma_2$ *is* $\mathcal{T}_1 \oplus \mathcal{T}_2$ *satisfiable;*
2. *there exists* $E \subseteq Y^2$*, where* Y *is* $\mathsf{vars}(\Gamma_1) \cap \mathsf{vars}(\Gamma_2)$*, such that*
   - $\Gamma_1 \wedge \delta_E^Y \wedge \gamma_\kappa$ *is* $\mathcal{T}_1$*-satisfiable, where* κ *is* $\mathsf{mincard}_{\mathcal{T}_2}(\Gamma_2 \wedge \delta_E^Y)$*;*
   - $\Gamma_2 \wedge \delta_E^Y$ *is* $\mathcal{T}_2$*-satisfiable;*
3. *there exists* $E \subseteq Y^2$*, where* Y *is* $\mathsf{vars}(\mathsf{s\text{-}witness}(\Gamma_2))$*, such that*
   - $\Gamma_1 \wedge \delta_E^Y$ *is* $\mathcal{T}_1$*-satisfiable;*
   - $\mathsf{s\text{-}witness}_{\mathcal{T}_2}(\Gamma_2) \wedge \delta_E^Y$ *is* $\mathcal{T}_2$*-satisfiable;*

for every conjunction Γ_1 *of* Σ_1*-literals and* Γ_2 *of* Σ_2*-literals.*

Proof. See [2].

We now provide an example showing an application of the previous theorem.

Example 1. Consider the theories $\mathcal{T}_1$ and $\mathcal{T}_2$ over the empty signature such that $\mathcal{T}_1$ is axiomatized by $\forall x \forall y\ (x \cong y)$ and $\mathcal{T}_2$ is axiomatized by $\exists x \exists y\ \neg(x \cong y)$. Hence every model of $\mathcal{T}_1$ has cardinality at most one and every model of $\mathcal{T}_2$ has cardinality at least 2. Let φ denote the formula $(x \cong x)$.

Observe that, in [4], it was shown that theory $\mathcal{T}_2$ is smooth and that

$$\mathsf{witness}_{\mathcal{T}_2}(\varphi) := \varphi \wedge (w_1 \cong w_1) \wedge (w_2 \cong w_2)$$

is a witness function for $\mathcal{T}_2$. Hence this condition for the application of Theorem 2 is fullfilled. Taking into account that $\mathsf{mincard}_{\mathcal{T}_2}(\varphi) = 2$, then by Algorithm 3,

$$\begin{aligned}\mathsf{s\text{-}witness}_{\mathcal{T}_2}(\varphi) &= \varphi \wedge (x \cong x) \to \gamma_2\\ &= \varphi \wedge (x \cong x) \to \neg(z_1 \cong z_2)\\ &= (x \cong x) \wedge \neg(z_1 \cong z_2).\end{aligned}$$

Let Γ_1 be the formula tt, Γ_2 the formula φ and Y the set $\mathsf{vars}(\mathsf{s\text{-}witness}(\Gamma_2))$ i.e. $\{x, z_1, z_2\}$. We now would like to check if there is an arrangement of δ_E^Y such that $\Gamma_1 \wedge \delta_E^Y$ is $\mathcal{T}_1$-satisfiable and $\mathsf{s\text{-}witness}(\Gamma_2) \wedge \delta_E^Y$ is $\mathcal{T}_2$-satisfiable. Note that the only arrangement satisfied in $\mathcal{T}_1$ is the one induced by $E = \{(x, z_1), (x, z_2), (z_1, z_2)\}$ since all others would require the interpretation to have cardinality greater than one. However, $\mathsf{s\text{-}witness}(\Gamma_2) \wedge \delta_E^Y$ is clearly not satisfiable. Hence, by Theorem 2, we conclude that φ is not satisfiable in $\mathcal{T}_1 \oplus \mathcal{T}_2$. In this simple case it is no difficult to see that this was the expected conclusion since there are no models that satisfy the theory resulting from the union of $\mathcal{T}_1$ and $\mathcal{T}_2$.

Observe the importance of Algorithm 3 to define in a computable way the strong witnessable function.

4 Conclusion and further research

The first and most well-known method for the combination of satisfiability procedures is due to Nelson and Oppen, [5]. In their paper, the authors provide a combination method to decide the satisfiability of quantifier-free formulas in the union of two theories, provided that both theories have their own procedure for deciding the satisfiability problem of quantifier-free formulas, have disjoint signatures and are stably infinite. At that time, this result had great impact on the area of automated reasoning since it provided a general combination method, which at the time was made on a case by case basis such as in the work of Suzuki and Jefferson [8] on the combination of the theory of arrays and Presburger arithmetic. However, the class of theories to which this method applied seemed too restrictive and several extensions or "migrations" of the method were proposed, namely the extension to shiny theories in [9], to polite theories in [7] and to strongly polite theories in [4]. In this document we answered a question left open by Jovanović and Barrett in [4], and obtained results on the relationship between shiny and strongly polite theories, as well as presenting known results on the relationship between shiny and polite theories. With this set of results, we were able to devise a Nelson-Oppen procedure for the combination of a polite and an arbitrary theory by constructing the $\mathsf{mincard}$ function and a strong witness function from a witness function.

References

1. C. Barrett, C. L. Conway, M. Deters, L. Hadarean, D. Jovanović, T. King, A. Reynolds, and C. Tinelli. CVC4. In *Computer aided verification*, volume 6806 of *Lecture Notes in Computer Science*, pages 171–177. Springer, Heidelberg, 2011.
2. F. Casal. On Nelson-Oppen Techniques. *Master Thesis, Instituto Superior Técnico*, 2013.
3. J. Christ, J. Hoenicke, and A. Nutz. SMTInterpol: An interpolating SMT solver. In *SPIN*, volume 7385 of *Lecture Notes in Computer Science*, pages 248–254. Springer, 2012.
4. D. Jovanović and C. Barrett. Polite theories revisited. In *Proceedings of the Seventeenth International Conference on Logic for Programming, Artificial Intelligence, and Reasoning (LPAR'2010)*, volume 6397 of *LNCS*, pages 402–416, 2010.
5. G. Nelson and D. C. Oppen. Simplification by cooperating decision procedures. *ACM Transactions on Programming Languages and Systems*, 1(2):245–257, 1979.
6. D. C. Oppen. Complexity, convexity and combinations of theories. *Theoretical Computer Science*, 12:291–302, 1980.
7. S. Ranise, C. Ringeissen, and C. G. Zarba. Combining data structures with nonstably infinite theories using many-sorted logic. In *Proceedings of the Fifth International Workshop on Frontiers of Combining Systems (FroCoS'2005)*, volume 3717 of *LNAI*, pages 48–64, 2005.
8. N. Suzuki and D. Jefferson. Verification decidability of Presburger array programs. In *Proceedings of a Conference on Theoretical Computer Science (Univ. Waterloo, Waterloo, Ont., 1977)*, pages 202–212. Univ. Waterloo, Waterloo, Ont., 1978.
9. C. Tinelli and C. G. Zarba. Combining nonstably infinite theories. *Journal of Automated Reasoning*, 34(3):209–238, 2005.