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Physics honours thesis

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Abstract

This is my thesis, yada yada yada.

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Chapter 1

Introduction

The aim of the thesis is to study the gravitational field within orbiting compact binary objects near the point of merger. The primary objective is to explore a poorly studied phenomenon, gravitational reconnection. This concept is analogous to its magnetic counterpart, which leads to phenomena such as solar flares and coronal mass ejections. Similarly, gravitational reconnection could result in significant mass ejection or acceleration in the context of merging compact binary objects. By investigating the conditions under which gravitational reconnection can occur, the project aims to gain a deeper understanding of the astrophysical processes involved in the final stages of compact binary mergers. If gravitational reconnection is found to occur in this context, this could have important implications for our understanding of the formation and evolution of astrophysical jets in the universe.

The potential discovery of gravitational reconnection leading to jet formation would imply an electromagnetic counterpart to black hole mergers, much like the post-merger electromagnetic counterparts of neutron stars. This would open up exciting opportunities for further research into the behavior of compact binary mergers and the physics of extreme astrophysical environments. Additionally, this would inform future efforts to perform electromagnetic follow-ups after gravitational wave detections between binary black holes. By combining observations of gravitational waves with electromagnetic radiation at different wavelengths, we can obtain a more complete picture of the physical processes involved in the most extreme environments in the universe, and gain insights into the fundamental laws that govern the behavior of matter and energy on a cosmic scale. Furthermore, these investigations could reveal exotic shapes of the event horizon during these extreme events, providing a new perspective on the dynamical nature of spacetime in the vicinity of merging compact objects.

1.1 Status (literature review)

Since Einstein published his general theory of relativity, it has become evident that gravitation and electromagnetism are intricately intertwined. In the weak field limit of general relativity, the Einstein field equations, governing gravity, simplify to the gravitomagnetic equations, which bear a striking resemblance to Maxwell's equations. Consequently, this remarkable similarity between gravity and electromagnetism has given rise to various analogies between gravitational and electromagnetic phenomena.

In recent years, the field of astronomy has witnessed a significant transformation, empowering researchers to explore general relativistic phenomena with unprecedented precision. Particularly, multimessenger astronomy has emerged as a powerful approach, combining data from diverse detectors such as gravitational wave detectors, telescopes operating across different wavelengths, and neutrino detectors

Nonetheless, for the situation of interest in our study - the merger of binary black holes - it is currently unknown if there exists an electromagnetic counterpart **?**. However, an intriguing possibility arises if gravitational field lines were to reconnect at a stable Lagrange point of orbiting Kerr black holes. In such a scenario, any matter trapped at that Lagrange point - which would be subjected to high pressure - would briefly become liberated from the field, potentially resulting in the ejection of material in the form of a jet. This mechanism could give rise to an electromagnetic counterpart and offer plausible explanations for jet phenomena with currently unknown origins.

The possibility of electromagnetic counterparts to binary black hole mergers has been proposed, with potential signals already detected **?**. Various theories have been put forth to explain the mechanisms behind such signals **?**. To the best of my knowledge, investigation into gravitational reconnection has not been undertaken. Nevertheless, recent studies on magnetic reconnection in the vicinity of black holes have emerged **??**, led by researchers such as Felipe A. Asenjo and Luca Comisso.

Chapter 2

Field Theory

In this chapter, we provide a brief review of Field Theory, which forms the foundation of this thesis.

2.1 The Basis of Modern Physics

The concept of a field, as we understand it today, took root with the groundbreaking work of Michael Faraday in the 19th century. Faraday's vision was revolutionary: it proposed that physical phenomena occur within fields that permeate all of space, a radical departure from the Newtonian perspective of action-at-a-distance.

This field concept has been refined and expanded upon over the centuries, eventually leading to the development of quantum field theory and becoming the cornerstone of modern physics. Today, it underpins our understanding of both particle physics and gravitation. Every particle, every force we encounter, is a manifestation of various fields interacting with each other.

The construction of a field theory often begins with an educated guess of the Lagrangian of a system. The dynamics of the system are then derived by minimizing the action, a principle that stems from the calculus of variations. This approach has been successfully applied to derive numerous modern field theories, including but not limited to, Yang-Mills theory, quantum electrodynamics, and general relativity. In this thesis, we will focus on general relativity.

2.2 Special relativity

We can build our field theory based on some simple axioms:

1. The laws of physics are the same in all inertial frames of reference (Principle of relativity).

2. The speed of light in a vacuum is the same for all observers, regardless of their state of motion or the state of motion of the source of light.

These postulates were first introduced by Einstein when he noticed some inconsistencies between Galileo's principle of relativity, and Maxwell's equations. Maxwell's which state that the speed of light is the same in all reference frames, but Gelileo's principle of relativity would imply that different observers will measure different speeds of light.

Figure 2.1: An inscription from the pre-Socratic philosopher, Democritus, welcomes visitors at the entrance of the UWA Physics building. This quote from over two millennia ago stands as one of the earliest written recognition's of atoms as nature's foundational building blocks. It reads, "By convention sweet and by convention bitter, by convention hot, by convention cold, by convention color; but in reality atoms and void." Not bad for a 2400-year-old perspective. However, hopefully by the end of this chapter you will agree that Democritus's 'atom' might find a more accurate representation in the concept of a 'field'.

Following these postulates to their logical conclusion leads to famous phenomena, such as time dialtion and length contraction. We can encode all the information we know from special relativity by considering consequence of these postulates is the metric, which is an invariant in all reference frames.

$$s^2 = -(t_1 - t_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \tag{2.1}$$

Notice that there is a negative in front of the time contribution, this means distance in spacetime is not the same as distance in a general Euclidean space, so our intuition about distance in space does not exactly carry over to spacetime. This leads to many of the non-intuitive concepts of special and general relativity. The reason the metric is chosen in this way, is because all observers will agree on these distances, no matter where observers are or how they are moving. In general relativity, the metric changes in the presence of masses, i.e. distances between points is no longer the flat space distance, and therefore, the straightest path between two points is no longer a straight line.

2.3 Special Relativity and the Metric

Special relativity, developed by Albert Einstein in the early 20th century, revolutionized our understanding of space and time. Its theoretical foundation is built upon two simple, yet profound postulates:

1. The laws of physics are the same in all inertial frames of reference (Principle of relativity).
2. The speed of light in a vacuum is the same for all observers, regardless of their state of motion or the state of motion of the source of light.

These seemingly straightforward axioms have far-reaching implications for our understanding of the universe. They necessitate the unification of space and time into a four-dimensional spacetime, a framework within which events and interactions are described.

One of the cornerstones of special relativity is the concept of Lorentz invariance, which states that the laws of physics are unchanged under Lorentz transformations. These transformations describe how measurements of space and time by two observers moving at a constant speed relative to each other are related.

Within this framework, the concept of the metric arises naturally. The metric, defined in the context of special relativity as the Minkowski metric, is a mathematical function that provides a measure of the interval - an invariant distance - between two events in spacetime. The metric is a fundamental aspect of the geometry of spacetime, and the invariance of this interval under Lorentz transformations is a reflection of the symmetries of spacetime under special relativity.

The notion of the metric and its properties are crucial when we extend our understanding from special to general relativity. In general relativity, the metric becomes a dynamic quantity that encodes the curvature of spacetime due to mass and energy. This curvature is what we perceive as gravity. Thus, the simple axioms of special relativity lay the foundation for the modern understanding of the universe at a fundamental level.

2.4 Electromagnetism as a Field Theory

Electromagnetism, the theory describing the interactions between charged particles, is a prime example of a field theory. This theory, encapsulated in Maxwell's equations, is formulated around the electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields. These fields are assigned a value at every point in space and time, and their dynamics are dictated by Maxwell's equations, which in Gaussian-cgs units [1], are expressed as:

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{2.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.4}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{2.5}$$

Where ρ is the charge density, $\mathbf{J}$ is the current density, and c is the speed of light.

The field theory approach to electromagnetism further shines when we consider the principle of least action. We can derive the dynamics of the electric and magnetic fields from a simple Lagrangian density $\mathcal{L}$, which is a function of the fields and their derivatives. The choice of the Lagrangian is guided by the need to obtain Maxwell's equations as the equations of motion and the requirement of Lorentz invariance:

$$\mathcal{L} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu \tag{2.6}$$

[1] The choice to use Gaussian-cgs units eliminates the nonphysical factor $k = \frac{1}{4\pi\epsilon_0}$ from the electrostatic force law, leading to the more elegant expression $\vec{F} = \frac{q_1 \cdot q_2}{r^2}$. This system uses the unit of charge called the electrostatic unit (esu), force in dyne, energy in ergs, among others - More details to be filled in later.

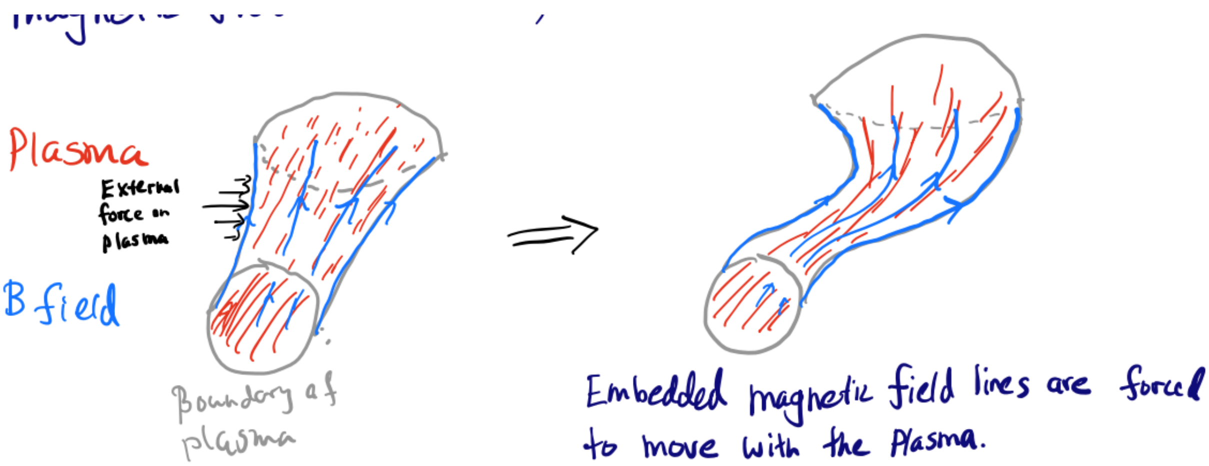


Figure 2.2: Illustration of Alfven's theorem - make better diagram

Where $F_{\mu\nu}$ is the electromagnetic field tensor and A_μ is the four-potential. The first term represents the kinetic energy of the electromagnetic field, and the second term represents the interaction of the field with matter. The Lagrangian is constructed to be Lorentz invariant, reflecting the fact that the laws of electromagnetism are the same in all inertial frames of reference.

Applying the Euler-Lagrange equations to this Lagrangian, we obtain Maxwell's equations. This demonstrates how field theory and the principle of least action can lead us to the laws governing electric and magnetic fields. These principles provide a unified description of electric and magnetic phenomena and form the basis for quantum electrodynamics, the quantum field theory of electromagnetic interactions.

2.5 Magnetic Reconnection and Physicality of Fields

Magnetic reconnection is a fundamental process in plasma physics that vividly illustrates the physicality of fields. In essence, magnetic reconnection is a phenomenon in which the magnetic topology is rearranged and magnetic energy is converted into kinetic and thermal energy of the plasma.

The process begins when two oppositely directed magnetic field lines come into contact. The field lines break and then reconnect with each other, forming a new set of field lines. This process changes the topology of the magnetic field and accelerates plasma particles, leading to phenomena such as solar flares and auroras.

2.5.1 Alfven's theorem

This reconnection process is a direct consequence of the properties of the magnetic field in a plasma. According to Alfvén's theorem, the magnetic field lines are "frozen" into the plasma under ideal conditions, meaning the magnetic field embedded within the plasma must move with the plasma - you can not move either the plasma or the magnetic field individually. This is demonstrated in figure **??**.

Alfvén's theorem can be intuitively understood by considering Lenz's law, which states that an induced electromotive force (or EMF) will always generate a current that creates a magnetic field opposing the change in the magnetic field. Consider a blob of Plasma, like that shown in figure

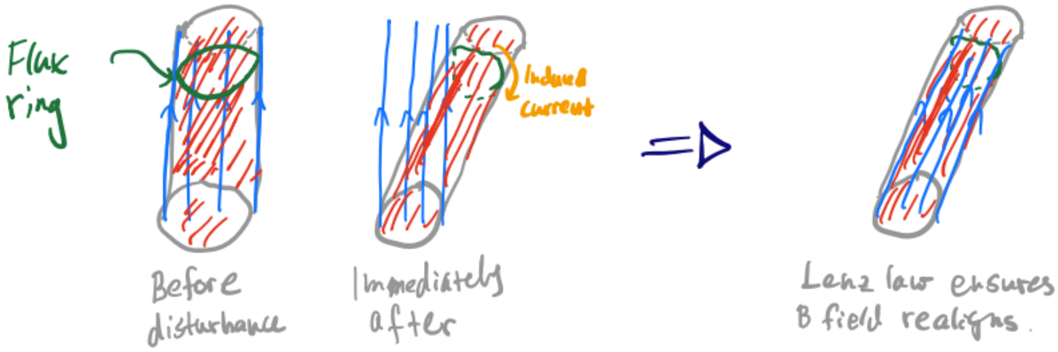


Figure 2.3: Explanation of Alfven's theorem through Lenz law. Obviously not an actual full proof.

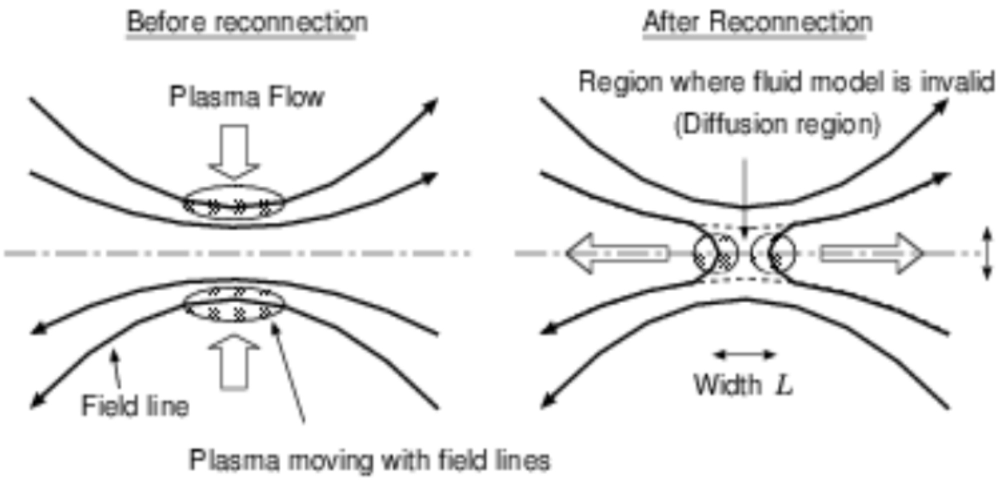


Figure 2.4: Illustration of magnetic reconnection process. From https://rnumata.org/research/recon

??. When we move a blob of plasma, the magnetic field lines embedded in it would initially remain stationary, becoming disconnected with the original configuration in the Plasma. However, this will result in a changing magnetic field through the Plasma, this change in the magnetic field will induce an electric field (Faraday's law of electromagnetic induction), which in turn drives a current. The induced current will then create a magnetic field that opposes the original change (assuming the idealized case of infinite conductivity of the Plasma - which is a good approximation in most cases).

As a result, any attempt to separate the plasma from the magnetic field lines is opposed by the induced magnetic field. Therefore, the magnetic field lines are effectively "frozen" into the plasma and move along with it, which is the essence of Alfvén's theorem.

2.5.2 Magnetic reconnection

To see how magnetic reconnection occurs, imagine two regions of Plasma next to one another, with each region having magnetic fields oriented in the opposite direction. Setting up a pressure gradient from each region towards the centre will lead each Plasma to move towards the centre, which by Alfven's theorem means the two magnetic fields will approach the centre - as illustrated on the left hand side of figure **??**. Due to the continuity of the magnetic field, the field lines must both vanish at this central point - which indeed does occur due to a process called magnetic diffusivity, a consequence of the fact that there are regions in a Plasma (called "current sheets") where conductivity is not infinite. This means there is a magnetic null point at the central region, which allows the magnetic field to reconnect into a new topology, by connecting with field lines on the opposite end of the boundary - as shown in the right hand side of figure **??**. By flux conservation, the inflowing Plasma is pushed outwards in the opposite direction, dragging the newly configured magnetic fields out with it. This results in a large release of energy, due to the energy stored within the magnetic field that vanished at the reconnection point.

This magnetic reconnection happens frequently in Astrophysical situations. For example, magnetic reconnection in the sun results in ejection of huge amounts of energy in the form of relativistic particles and Plasma, this is what is responsible for solar flares and coronal mass ejections.

2.5.3 Physical nature of fields

Alfvén's theorem and the concept of magnetic reconnection make it clear that fields are not just abstract mathematical constructs for calculations, but they have tangible physical effects. They store/release energy,

The magnetic field lines in a plasma are physical entities that affect and are affected by the motions of the plasma. As we will see in later sections, understanding the behavior of these field lines is crucial for understanding many phenomena in plasma physics and astrophysics. We will also come back to magnetic reconnection later on in this thesis, so stay tuned.

2.6 General relativity

2.6.1 Additional postulates

After Einstein discovered special relativity, he noticed it was limited. To go beyond special relativity, we need to add two more postulates:

1. The principle of covariance.
2. The equivalence principle.

The theory of general relativity (GR) was developed by Albert Einstein to reconcile the laws of mechanics with the laws of the electromagnetic field. Special relativity, Einstein's previous formulation, was constrained to inertial frames and flat spacetime. To extend the theory to all coordinate systems—both inertial and non-inertial—Einstein added two more postulates: the principle of covariance and the equivalence principle.

The principle of covariance, also known as the principle of general relativity, asserts that the laws of physics must take the same mathematical form in all coordinate systems (i.e. we must write equations in tensor notation). This led to the concept of a four-dimensional spacetime, where the geometry can be curved by the presence of mass and energy.

The equivalence principle, on the other hand, states that locally the laws of physics reduce to that of special relativity. Einstein famously came up with this principle while imagining a man falling off a building, and observing that in his perspective he is free of any forces. The principle also states that this man would would not be able to tell the difference between the acceleration he was experiencing due to gravity from what he would experience being uniformly accelerated through space by a jet pack (assuming of course that the man had no other form of sensation other then his acceleration, as if he was inside a dark chamber with no view of the world outside). This principle leads to the notion of a *locally inertial reference frame*, a frame that follows the motion of a freely falling particle in a small region of spacetime. This is required because there is no such thing as a globally inertial reference frame in general relativity, since we can imagine a particle that begins at rest with respect to this global frame, but begins accelerating and hence begins to move according to this frame.

Since there is no global inertial reference frame in GR, we must define a new geometry for spacetime then the one described by $\eta_{\mu\nu}$ in special relativity - i.e. we need a spacetime that changes from point to point. From Einstein's equivalence principle, this geometry must be locally flat, i.e. for a given point in spacetime we can choose coordinates such that in a small enough region the spacetime can be described by the metric $\eta_{\mu\nu}$. A geometry that is locally flat is known as a *manifold* - for example, the surface of the Earth is a manifold since from out in space it looks like a sphere, but to an ant - who can only observe a small chunk of the Earth at a time - it is flat. Einstein then postulated that gravity emerges as a result of the curvature of this spacetime - like how a ball rolled over the surface of a trampoline distorted by a large bowling ball in the centre will be deflected towards the balling ball - any particle in spacetime will be deflected in the direction of other massive objects. We will make this intuitive notion more precise in the following sections.

2.6.2 Why general relativity is hard

Recall that the electromagnetic field is sourced by the current four vector $J^\mu = \left(\rho, \vec{j}\right)$, produced by charged particles moving through space, leading to a field that can be described by another four vector $A^\mu = (\phi, \vec{A})$. In general relativity, the source is now the energy-momentum/stress tensor $T_{\mu\nu}$, which contains information about particle momentum (described by a single four vector) as well as particle energy (described by another four vector) and hence is a rank two tensor. This results in a field that is also rank two, called the metric $g_{\mu\nu}$, which turns out to be a generalization of the space-time interval we found in special relativity. In fact, the spacetime metric $\eta_{\mu\nu}$ is the metric for flat spacetime, while in general $g_{\mu\nu}$ describes the distance between events in curved spacetime. $ds^2 = g_{\mu\nu} dx_\mu dx_\nu$ gives the infinitesimal displacement of two events in the spacetime described by the metric $g_{\mu\nu}$.

When we say space-time is curved, what we mean is that the coordinate system we use to describe events changes as we move between points in spacetime. When a light ray passes by the gravitational field of the sun, the mass-energy of the sun curves spacetime more strongly closer to its surface, meaning . This is very similar to how light rays travelling through a medium with a smoothly varying index of refraction will curve towards the side with the lower index of refraction, since light travels faster in a higher index of refraction - but now light (or anything with mass/energy) is bending towards in the direction of the decreasing metric.

So general relativity requires studying how fields change along space-time, with the coordinates describing the space-time also changing from point to point. This means our standard derivative from calculus is not going to cut it, we need to account for both the change in the field and the change in the coordinates. So to describe GR, we need tools from the mathematical subject called *differential geometry.*

2.6.3 Differential geometry

Consider a function f defined on a curve $x^\mu(\lambda)$, then the ordinary derivative of f can be found by the chain rule

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda}. \tag{2.7}$$

We call $\frac{\partial f}{\partial x^\mu} := \partial_\mu f$ the gradient of f, this has a lower index and is called a contravariant derivative, because with a change of coordinates it transforms in the opposite way that a vector does. Indeed, using the chain rule we see that a change of coordinates $x^\alpha \to x^{\alpha'}(x^\alpha)$, then

$$\frac{\partial f}{\partial x^{\alpha'}} = \frac{\partial f}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial x^{\alpha'}} \implies \partial_{\alpha'} f = \partial_\alpha f \cdot \frac{\partial x^\alpha}{\partial x^{\alpha'}} \tag{2.8}$$

While a vector A^μ transforms covariantly as

$$\frac{\partial A^\mu}{\partial x^{\nu'}} = \frac{\partial A^\mu}{\partial x^\alpha}\frac{\partial x^\rho}{\partial x^{\nu'}}. \tag{2.9}$$

In general, a tensor $T^{\alpha\ldots\beta}{}_{\gamma\ldots\delta}$ of type (n, m) transforms with a change of coordinates according to the rule

$$T^{\alpha'\ldots\beta'}{}_{\gamma'\ldots\delta'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\cdots\frac{\partial x^{\beta'}}{\partial x^\beta}\cdot\frac{\partial x^\gamma}{\partial x^{\gamma'}}\cdots\frac{\partial x^\delta}{\partial x^{\delta'}}\cdot T^{\alpha\ldots\beta}{}_{\gamma\ldots\delta}. \tag{2.10}$$

However, the gradient of an arbitrary tensor is not itself a tensor, this is a problem as it violates the principle of covariance. This we need a new type of derivative, one that is covariant. In GR one of these derivatives is known as the *covariant derivative*, which is built from the standard partial derivative, but contains a correction term such that the covariant derivative of a tensor is also a tensor. The covariant derivative is given by

$$\nabla_\beta A^\alpha = \partial_\beta A^\alpha + \Gamma^\alpha_{\mu\beta} A^\mu \tag{2.11}$$

. The coefficients $\Gamma^\mu_{\nu\sigma}$ are called the connection. Einstein's principle of equivalence demands the connection be symmetric $\Gamma^\alpha_{\mu\beta} = \Gamma^\alpha_{\beta\mu}$ and metric compatible $\nabla_\gamma g_{\alpha\beta} = 0$, this allows us to uniquely define the metric according to the formula

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\alpha\mu}(\partial_\nu g_{\alpha\sigma} + \partial_\sigma g_{\nu\alpha} - \partial_\alpha g_{\nu\sigma}). \tag{2.12}$$

These gamma coefficients are now called the *Christoffel symbols*, they give the change in the tensor due to the change in the coordinates between points. A tensor field is said to be *parallel transported* along a curve γ if its covariant derivative $\nabla_\gamma T^{\alpha\ldots}_{\beta\ldots} = 0$ vanishes. This intuitively means that the tensor remains parallel with the surface it is moving under - this allows us to compare two tensors located at different points in spacetime - which is not trivial in curved spacetime.

The paths followed by particles in general relativity correspond to shortest paths through (curved) space-time. These shortest paths are known as geodesics, and the formula for determining these paths is know as the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \tag{2.13}$$

This equation is derived by minimizing the action S defined by

$$S = \int \mathcal{L}(x_\mu, \dot{x}_\mu)d\lambda = \int \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu d\lambda \tag{2.14}$$

Using the Euler Lagrange equations

$$\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}_i} - \frac{\partial\mathcal{L}}{\partial x_i} = 0 \tag{2.15}$$

We can physically motivate this definition by considering that this is equivalent to extremizing the *proper time* τ between two events[2]. The proper time is the time a clock carried by a particle will experience, while a distant observer (at a relative velocity and different gravitational field) will measure a different time between two events. So just find the extremal value of

$$\tau_{AB} = \int \sqrt{-ds^2} = \int_0^1 \sqrt{-g_{\mu\nu}(x(\lambda))\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda \tag{2.16}$$

Where λ is the parametrization of the curve in space. Hence finding the maxima of equation **??** is equivalent to finding the minima of the action given in equation (**??**), as required.

Another way to view equation (**??**) is as a path that is locally straight in curved spacetime. The particles four velocity

$$\boldsymbol{u} := \frac{dx^\mu}{d\tau} \tag{2.17}$$

Is always tangent to its path in space, the four velocity will therefore be straight if the four vector $\boldsymbol{u}$ does not change after an infinitesimal displacement.

2.6.4 Einstein's field equation

Doing the same thing we did with electromagnetism, we can derive the central equation of GR. We do this by noting that ...

Varying this action gives (derivation in appendix).

General relativity describes gravity as the result of mass curving spacetime, and the resulting curved spacetime causing particles to be deflected along there trajectory toward the mass - manifesting as the attractive Newtonian force we see in the classical world. So anything with mass, in fact anything with energy[3] causes spacetime to curve, as shown in the illustration of in figure **??**. Then, any other object with energy within that spacetime will travel in straight lines across this spacetime, resulting in for example orbital motion.

The way we precisely calculate how energy curves this spacetime, and how particles respond to this curvature, is with the *Einstein field equations* (EFEs).

[2]In special relativity, i.e. flat spacetime, time dilation means that the worldine of free particles moving between two events is the worldline of largest proper time between those events. General relativity says that this principle is extended to curved spacetime.

[3]Since by Einstein's mass energy equivalence, anything with energy has mass, and vice-versa.

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{2.18}$$

Where

- $G_{\mu\nu}$ is the Einstein tensor, representing the curvature of space-time.
- $T_{\mu\nu}$ is the stress energy tensor, representing the density and flux of energy/momentum in space-time.
- Λ is the cosmological constant, which gives rise to the accelerating expansion of the universe.
- $\kappa = \frac{8\pi G}{c^4}$ is a constant.
- $g_{\alpha\beta} = \boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta$ is the metric tensor, given by the dot product of the basis vectors, and $g^{\alpha\beta}$ is the inverse metric tensor. The metric tensor allows for measurements of distance in space-time, and can be thought of as representing the gravitational field in general relativity.

The Einstein tensor can be written as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{2.19}$$

Where the R tensor are contractions of the Ricci tensor defined by equation

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\rho}_{\mu\lambda} - \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\rho}_{\nu\lambda} \tag{2.20}$$

- $R_{\mu\nu} = R^{\alpha}_{\nu\alpha\beta}$ is the Ricci curvature tensor, which roughly represents how much the curved space-time deviates from flat space-time.
- R $= R^{\alpha}_{\alpha}$ is the Ricci scalar, the contraction of the Ricci tensor, a measure of curvature.

And gamma coefficients $\Gamma^{\mu}_{\nu\sigma}$ describes the Christoffel symbols (or affine connection), which itself is given by the formula

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2} g^{\alpha\mu}(\partial_\nu g_{\alpha\sigma} + \partial_\sigma g_{\nu\alpha} - \partial_\alpha g_{\nu\sigma}) \tag{2.21}$$

These gamma coefficients represent how basis vectors change between points in space.
TODO: Go more into the intuition behind the terms in the EFEs.

2.6.5 The weak field limit

To elucidate the connection between the gravitational and electromagnetic field, we will approximate Einstein's field equations in the weak field limit. This introduces two new concepts, gravitoelectromagnetism, and gravitational waves.

TODO - Derivation of gravitomagnetic equations from weak field limit of general relativity.

$$\nabla \cdot \mathbf{E_g} = -4\pi G \rho_g \tag{2.22}$$

$$\nabla \times \mathbf{E_g} = -\frac{\partial \mathbf{B_g}}{\partial t} \tag{2.23}$$

$$\nabla \cdot \mathbf{B_g} = 0 \tag{2.24}$$

$$\nabla \times \mathbf{B_g} = -\frac{4\pi G}{c^2}\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E_g}}{\partial t} \tag{2.25}$$

Where $\mathbf{E_g}$ is the gravitoelectric field, $\mathbf{B_g}$ is the gravitomagnetic field, and ρ_g is the mass density. Almost identical to Maxwell's equations in equations (**??**).

Imagine a charged sphere rotating about its axis. The rotating charges will form current loops, and hence an electric dipole will be generated. This electric dipole will exert effectively a magnetic force on an incoming charged particle, with the direction given by the right hand rule. Consider now, the analogous gravitational situation, i.e. a sphere of matter rotating about its axis. The rotation of the mass will result in a gravitomagnetic force acting on an incoming moving mass, this is known as the Lens-Thirring effect, and has been experimentally verified. This situation resembles very well the concept of frame dragging, a black hole rotating drags spacetime with it, and indeed the direction of the force from frame dragging can be found by using the right hand rule.

Chapter 3

Black holes

3.1 Schwarzschild's discovery

Sketch of Schwarzschild's derivation and consequence of singularity. Show that matter from an observing falling into BH never passes event horizon according to outside observer.

The distance between two points in space-time is given by the invariant line element ds^2. In Schwarzschild space-time, the line is given by equation (**??**):

$$ds^2 = -d\tau^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.1}$$

3.2 Kerr black holes

Long after Kerr's discovery, other black hole solutions where found. It was shown (no hair theorem) that all black holes can be characterised by a charge and spin (like an electron).

The metric for a Kerr black hole (spherically symmetric, stationary, black hole of mass M rotating at constant angular momentum J) is given by:

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}d\phi dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin\theta^2 d\phi^2 \tag{3.2}$$

Where $a := \frac{J}{M}$, $\rho^2 := r^2 + a^2\cos\theta$, and $\Delta := r^2 - 2Mr + a^2$ **?**.

It is easy to see that setting $a = 0$ leaves us with the Schwarzschild solution (the case of zero spin is the same as Schwarzschild solution in equation (**??**)), and in the limit that $r \gg a$ we get the weak field metric (in terms of gravitational potential $\Phi(r) = -\frac{GM}{r}$)

$$ds^2 = -(1 + 2\Phi(r))dt^2 + (1 - 2\Phi(r))dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.3}$$

Which as $r \to \infty$ approaches flat (Minkowski) spacetime

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{3.4}$$

Explain Kerr black holes, frame dragging (relate back to weak field limit).

Show plots from Kerr simulation.

3.3 Binary black holes

3.3.1 Mergers

Mention how when two black holes orbit, GW emission results in energy losses, so by conservation of energy gravitational binding energy must decrease leading to inspiral and eventual merger.

3.3.2 Analogy with magnetic reconnection

Looking at magnetic reconnection, it requires a dipole

3.3.3 Accreting matter

. Look at L1 Lagrange point (for matter infalling outside event horizon) or L4 (stable orbits - show simulation)

3.3.4 Consequences

This could cause an ejection, explain how.

Chapter 4

Modelling

4.1 How to study the consequences

By studying effect of

4.2 First model

Gravitomagnetism model. Give quick derivation and show some numerical results.

4.3 Adding higher order terms

Use higher PN terms to improve accuracy. Does this change much?

4.4 Numerical relativity

4.4.1 What is numerical relativity

Explain numerical relativity and why we need it

4.4.2 Theoretical basis

Briefly explain 3+1 decomposition of spacetime.

4.4.3 Our model

Explain what packages we used, and how we setup the model, link code.

4.4.4 Results

Chapter 5

Results and discussion

5.1 Results

5.2 Limitation

5.3 Future work

5.4 Discussion

5.5 Conclusion

Chapter 6

Thesis preliminaries

6.1 Introduction

Black holes are astronomical bodies that have an escape velocity at the surface greater than the speed of light. A consequence of this, is that anything crosses a certain region called the **event horizon** of the black hole can not escape.

Black holes do not make sense in classical Physics perspective, since light in classical mechanics is not affected by classical gravity. Thus we need to understand a subject called **General Relativity**, which explains the motions of objects in space and time, as a result of the geometry of **spacetime**, with this geometry being the result of massive objects existing in this spacetime (i.e. planets, stars, humans etc.).

Throughout this document, we will first give a brief introduction to general relativity (GR) in section **??**, followed by a a discussion of how black holes emerge from GR in section **??**. After this introduction, we will then introduce some concepts in plasma physics, also relevant for the thesis. Equations will be given throughout, but they will just be stated and described intuitively. For a more rigorous introduction to GR, their are many great texts introductory texts **??**, as well as more rigorous treatments **?**.

6.2 Introduction to general relativity

General relativity can be summed up with the quote of John Wheeler, "Spacetime tells matter how to move; matter tells spacetime how to curve". This introduction to GR section will explain this statement in slightly more detail.

GR is built on the notion of spacetime. Spacetime is what you get when you combine space and time into a single unifying object. In order to describe an event, such as the birth of a child, you need to describe both its location in space (which hospital), and when in time the event occurred (the birthday). Thus, events can be parameterized by four numbers (t, x, y, z), called a *world point*. This space then needs a definition of distance between two events in spacetime. This notion of distance should match our intuition of distance in (Euclidean) space for two events located at the same point in time, i.e. the Pythagorean theorem $s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$. But we need to involve time in some way, in flat spacetime, the arena of *special relativity*, the distance between two events (t_1, x_1, y_1, z_1) and

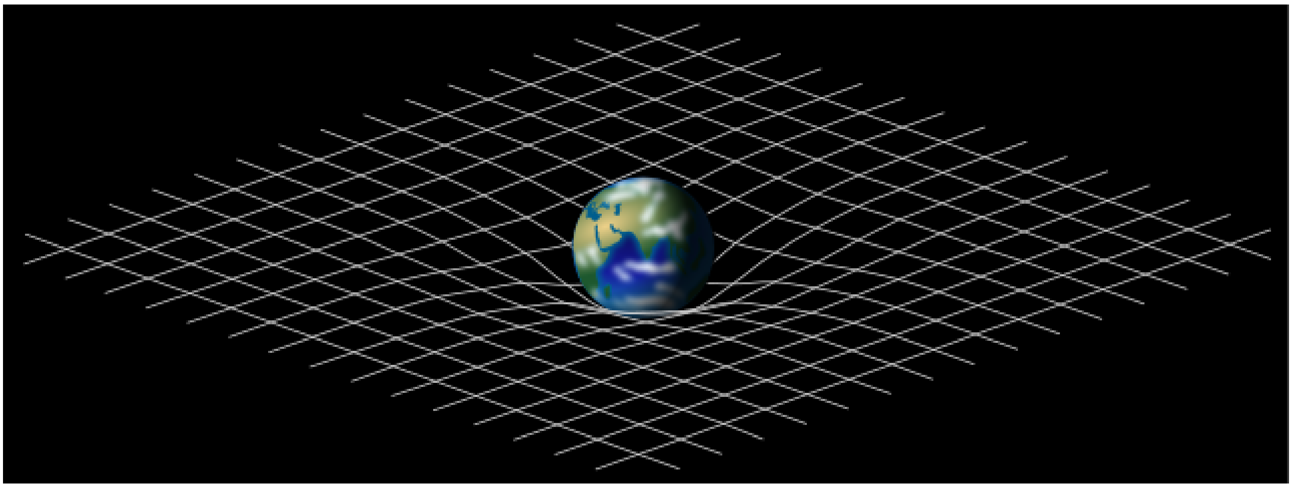


Figure 6.1: General relativity illustration - credit NASA

(t_2, x_2, y_2, z_2), is given by the *metric* s, given by

$$s^2 = -(t_1 - t_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \tag{6.1}$$

Notice that there is a negative in front of the time contribution, this means distance in spacetime is not the same as distance in a general Euclidean space, so our intuition about distance in space does not exactly carry over to spacetime. This leads to many of the non-intuitive concepts of special and general relativity. The reason the metric is chosen in this way, is because all observers will agree on these distances, no matter where observers are or how they are moving. In general relativity, the metric changes in the presence of masses, i.e. distances between points is no longer the flat space distance, and therefore, the straightest path between two points is no longer a straight line.

General relativity describes gravity as the result of mass curving spacetime, and the resulting curved spacetime causing particles to be deflected along there trajectory toward the mass - manifesting as the attractive Newtonian force we see in the classical world. So anything with mass, in fact anything with energy[1] causes spacetime to curve, as shown in the illustration of in figure **??**. Then, any other object with energy within that spacetime will travel in straight lines across this spacetime, resulting in for example orbital motion.

The way we precisely calculate how energy curves this spacetime, and how particles respond to this curvature, is with the *Einstein field equations* (EFEs).

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{6.2}$$

Where

- $G_{\mu\nu}$ is the Einstein tensor, representing the curvature of space-time.
- $T_{\mu\nu}$ is the stress energy tensor, representing the density and flux of energy/momentum in space-time.

[1]Since by Einstein's mass energy equivalence, anything with energy has mass, and vice-versa.

- Λ is the cosmological constant, which gives rise to the accelerating expansion of the universe.
- $\kappa = \frac{8\pi G}{c^4}$ is a constant.
- $g_{\alpha\beta} = \boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta$ is the metric tensor, given by the dot product of the basis vectors, and $g^{\alpha\beta}$ is the inverse metric tensor. The metric tensor allows for measurements of distance in space-time, and can be thought of as representing the gravitational field in general relativity.

The Einstein tensor can be written as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{6.3}$$

Where the R tensor are contractions of the Ricci tensor defined by equation

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\lambda_{\nu\sigma}\Gamma^\rho_{\mu\lambda} - \Gamma^\lambda_{\mu\sigma}\Gamma^\rho_{\nu\lambda} \tag{6.4}$$

- $R_{\mu\nu} = R^\alpha_{\nu\alpha\beta}$ is the Ricci curvature tensor, which roughly represents how much the curved space-time deviates from flat space-time.
- R $= R^\alpha_\alpha$ is the Ricci scalar, the contraction of the Ricci tensor, a measure of curvature.

And gamma coefficients $\Gamma^\mu_{\nu\sigma}$ describes the Christoffel symbols (or affine connection), which itself is given by the formula

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\alpha\mu}(\partial_\nu g_{\alpha\sigma} + \partial_\sigma g_{\nu\alpha} - \partial_\alpha g_{\nu\sigma}) \tag{6.5}$$

These gamma coefficients represent how basis vectors change between points in space.

TODO: Go more into the intuition behind the terms in the EFEs.

6.2.1 Derivation of EFEs from action

TODO: Derive the Einstein field equations from the Einstein-Hilbert action. Also motivate the action (along with the field equations) by considering the basic principles of general relativity (i.e. the strong equivalence principle, principle of general covariance). Also describe some basics of differential geometry.

6.3 Black hole solutions

Black holes are simultaneously extremely complicated, and yet incredibly simple. The complexity comes from the difficulty of solving Einstein's field equations, the simplicity comes from a result known as the "No hair theorem". The no hair theorem states that black holes are completely characterised by three parameters, the mass M, the spin J, and the charge Q. These free parameters give rise to different types of black holes.

1. Schwarzschild Black Holes: The Schwarzschild black hole is the simplest and most fundamental black hole solution. It is spherically symmetric and has no spin or charge (J = 0, Q = 0).

The mass parameter M is the only defining characteristic of a Schwarzschild black hole. It is surrounded by an event horizon, beyond which gravity is so intense that nothing, not even light, can escape.

2. Kerr Black Holes: The Kerr black hole is characterized by both mass and angular momentum or spin (J 0). It possesses a rotating singularity and an event horizon known as the ergosphere, which is larger than the event horizon of a Schwarzschild black hole. The rotation of a Kerr black hole introduces frame-dragging effects, where spacetime itself is dragged along with the rotating black hole. These effects have profound implications for phenomena such as accretion disks, jets, and the extraction of energy from black holes through processes like the Penrose process. Kerr black holes are the most astrophysically relevant of the black hole solutions, as all black holes in nature are expected to be Kerr black holes.

3. Reissner-Nordström Black Holes: The Reissner-Nordström black hole incorporates electric charge (Q 0) along with mass (M) and has no spin (J = 0). Electrically charged black holes are relatively theoretical, as astrophysical objects are typically electrically neutral. However, they play a crucial role in understanding the interplay between gravity and electromagnetism. The presence of charge alters the structure of the event horizon and affects the behavior of charged particles near the black hole. Reissner-Nordström black holes provide insights into the behavior of charged matter falling into a black hole and are crucial in studies involving the cosmic censorship hypothesis.

4. Kerr-Newman black holes: The Kerr-Newman black hole extends the Kerr black hole solution by incorporating both mass, angular momentum (spin), and electric charge (J 0, Q 0). Like Reissner-Nordström black holes, they are not expected to be Astrophysically relevant.

6.3.1 Schwarzschild metric

The distance between two points in space-time is given by the invariant line element ds^2. In Schwarzschild space-time, the line is given by equation (**??**) (given in units c=1).

$$ds^2 = -d\tau^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{6.6}$$

I derived the Schwarzschild metric in schwarzschild_solution.nb, a Mathematica notebook based on the OGRe Mathematica package **?** for GR calculations.

6.3.2 Kerr metric

The metric for a Kerr black hole (spherically symmetric, stationary, black hole of mass M rotating at constant angular momentum J) is given by (in units G=c=1)

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}d\phi dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin\theta^2 d\phi^2 \tag{6.7}$$

Where $a := \frac{J}{M}$, $\rho^2 := r^2 + a^2\cos\theta$, and $\Delta := r^2 - 2Mr + a^2$ **?**.

It is easy to see that setting $a = 0$ leaves us with the Schwarzschild solution (the case of zero spin is the same as Schwarzschild solution in equation (**??**)), and in the limit that $r \gg a$ we get the weak field metric (in terms of gravitational potential $\Phi(r) = -\frac{GM}{r}$)

$$ds^2 = -(1 + 2\Phi(r))dt^2 + (1 - 2\Phi(r))dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{6.8}$$

Which as $r \to \infty$ approaches flat (Minkowski) spacetime

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{6.9}$$

6.3.3 Reissner–Nordström metric

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)dt^2 + \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{6.10}$$

where G is the gravitational constant, M is the mass of the black hole, Q is its electric charge, and (t,r,,) are the coordinates in the metric.

6.3.4 Kerr-Newman

6.4 Orbits of Black holes

See black hole orbits document.

6.5 Gravitomagnetism

6.5.1 Gravitomagnetic equations

TODO - Derivation of gravitmagnetic equations from weak field limit of general relativity.

$$\nabla \cdot \mathbf{E_g} = -4\pi G\rho_g \tag{6.11}$$

$$\nabla \times \mathbf{E_g} = -\frac{\partial \mathbf{B_g}}{\partial t} \tag{6.12}$$

$$\nabla \cdot \mathbf{B_g} = 0 \tag{6.13}$$

$$\nabla \times \mathbf{B_g} = -\frac{4\pi G}{c^2}\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E_g}}{\partial t} \tag{6.14}$$

Where $\mathbf{E_g}$ is the gravito-electric field, $\mathbf{B_g}$ is the gravitomagnetic field, and ρ_g is the mass density. Almost identical to Maxwell's equations in equations (**??**).

6.5.2 Frame dragging

Imagine a charged sphere rotating about its axis. The rotating charges will form current loops, and hence an electric dipole will be generated. This electric dipole will exert effectively a magnetic force on an incoming charged particle, with the direction given by the right hand rule. Consider now, the analogous gravitational situation, i.e. a sphere of matter rotating about its axis. The rotation of the mass will result in a gravitomagnetic force acting on an incoming moving mass, this is known as the Lens-Thirring effect, and has been experimentally verified. This situation resembles very well the concept of frame dragging, a black hole rotating drags spacetime with it, and indeed the direction of the force from frame dragging can be found by using the right hand rule.

6.6 Magnetohydrodynamics

Magnetohydrodynamics (MHD) is the study of magnetic fields in conductive fluids, hence it is a combination of fluid mechanics with electromagnetism. In this section, we will cover the minimum amount of MHD theory necessary to understand magnetic reconnection. The reference used for all of this is the text "The Physics of Fluids and Plasmas, An Introduction for Astrophysicists" by ARNAB RAI CHOUDHURI **?**.

6.6.1 Fluid mechanics and Navier-Stokes equation

Fluid mechanics is governed by the Navier-stokes equation

$$\underbrace{\frac{D\vec{v}}{Dt}}_{\text{Acceleration}} = \underbrace{\Sigma_i \vec{F}_i}_{\text{External forces}} - \underbrace{\frac{1}{\rho}\nabla p}_{\text{Pressure gradient}} + \underbrace{\nu\nabla^2\vec{v}}_{\text{Viscosity}} \tag{6.15}$$

Where the left hand side is the Lagrangian derivative, which is just the full derivative of the velocity field of the fluid $\vec{v}(t, \vec{x})$ obtained by using the chain rule

$$\frac{D\vec{v}}{Dt} := \frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} \tag{6.16}$$

Which we can see comes with a contribution from the time dependence through the partial with respect to t, as well as the directional derivative of the velocity field in the direction of the velocity. Hence, the total derivative represents the change in the velocity of the fluid after an infinitesimal step in time accompanied by an infinitesimal step in the direction of the fluid velocity. This means intuitively, the total derivative, which is also called the material derivative in fluid mechanics, is the rate of change of fluid velocity according to an observer moving with the fluid flow at a point. So, the left hand side describes an acceleration. Hence, by Newton's second law, the right hand side represents a force (divided by mass, and divided by volume since we are only considering an infinitesimal fluid element with a given density). The first term on the right hand side represents external forces applied to the fluid, such as gravity, electric fields etc. The next term is the acceleration due to a pressure gradient, since pressure is force divided by area acting in all directions at a point, if there is a difference in pressure between two points (a pressure gradient) there will be a net force accelerating any object in the direction opposite the pressure gradient. The final term represents viscocity, which is essentially how hard it is to move through the fluid. Note that viscosity is represented by a difussion term, similar to heat in the heat wquation. This term represents the diffusion of velocity within the fluid. To see why this makes sense, consider that a difference in velocity of a fluid element compared to its negibours (given by the laplacian of the velocity field) will result in the element giving some of its velocity to its neighbours due to frictional effects caused by its viscosity. This has the result of spreading the velocity out through the fluid, exactly like a classic diffusion process.

The Navier-Stokes equation applies

6.6.2 Electromagnetism

Electromagnetism is governed by the Maxwell equations, given below in Gaussian-cgs units

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{6.17}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{6.18}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{6.19}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{6.20}$$

where ρ is the charge density, $\mathbf{J}$ is the current density, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, and c is the speed of light. Note that Gaussian-cgs (centimeters, grams, seconds) units are used in the above equations in order to get rid of the nonphysical factor $k = \frac{1}{4\pi\epsilon_0}$ in the electrostatic force law. Setting $k = 1$ gives the simpler $\vec{F} = \frac{q_1 \cdot q_2}{r^2}$, which is done in Gaussian-cgs units. In this new coordinate system we now use the unit of charge called the electrostatic unit (esu), force has the unit dyne, energy ergs, etc[2].

6.6.3 Magnetohydrodynamics basic equation

We get the basics of Magnetohydrodynamics by combining the equations of fluid dynamics with Maxwell's equations, which leads to various new phenomena. For example, by adding a Coulomb force term, from equation **??**

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = \Sigma_i \vec{F}_i - \frac{1}{\rho}\nabla p + \frac{1}{\rho c}\vec{J} \times \vec{B} + \nu \nabla^2 \vec{v} \tag{6.21}$$

But from equation **??**, when setting the displacement current $\frac{1}{c}\frac{\partial \vec{E}}{\partial t}\ \frac{v^2}{c^2} = 0$, and using the vector product identity

$$(\nabla \times \vec{B}) \times \vec{B} = (\vec{B} \cdot \nabla)\vec{B} - \nabla(\frac{B^2}{2}) \tag{6.22}$$

We arrive at the analogous Navier-Stokes equation for Magnetohydrodynamics

$$\underbrace{\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}}_{\text{Acceleration}} = \Sigma_i \vec{F}_i - \frac{1}{\rho}\nabla \underbrace{\left(p + \frac{B^2}{8\pi}\right)}_{\text{Effective pressure}} + \underbrace{\frac{(\vec{B} \cdot \nabla)\vec{B}}{4\pi\rho}}_{\text{Effective tension}} + \underbrace{\frac{1}{\rho c}\vec{J} \times \vec{B}}_{\text{Lorentz force}} + \nu \nabla^2 \vec{v} \tag{6.23}$$

This equation introduces two new phenomenon, one is magnetic pressure, given by $\frac{B^2}{8\pi}$; and the other is magnetic tension, given by

[2]Fill this out with more detail later

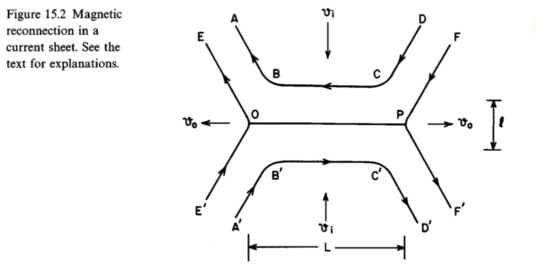


Figure 6.2: Illustration of magnetic tension, which acts to straighten magnetic field lines. κ is the reciprocal of the radius of curvature. This was taken by the wikipedia article on magnetic tension, which is decent.

$$\vec{f_T} = \frac{(\vec{B}\cdot\nabla)\vec{B}}{4\pi} \tag{6.24}$$

6.6.4 Magnetic pressure

Magnetic pressure is $B^2/8\pi$. If there is a magnetic field gradient, so region with stronger magnetic field has a greater pressure then neighbouring region, there will be a net force on the particles pushing the plasma from the high magnetic field region to the lower magnetic field region.

To see where this comes from, imagine charged particles in plasma moving along a magnetic field line, in helical motion. If the magnetic field strength is weaker to one side, as the ion spirals towards the lower magnetic field, the Lorentz force back towards the higher B field will be weaker than before, hence the particle will experience a force toward the region with a lower magnetic field – acting exactly like a pressure gradient in a fluid.

Magnetic tension

Magnetic tension is a restoring force that acts to oppose curvature in magnetic field lines. It has a magnitude $\frac{B^2}{4\pi}$, and its direction is given by the directional derivative of the magnetic field $\vec{B}$ in the direction of $\vec{B}$, hence it points to the centre of curvature of the magnetic field lines[3], as shown in figure **??** .

Energy equation

From the Maxwell equation **??**, for the electric field in the moving reference frame of a plasma with velocity $\vec{v}$ given by

$$\vec{E} = -\vec{v}\times\vec{B} \tag{6.25}$$

We can arrive at the induction equation

$$\frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{v}\times\vec{B}) + \lambda\nabla^2\vec{B} \tag{6.26}$$

[3] This is the same as how the acceleration vector of a car passing through a round about points towards the centre of the round about, vector diagram is helpful.

Where $\lambda = \frac{c^2}{4\pi\sigma}$ is the magnetic diffusivity. When the conductivity σ is very high (infinite in ideal case), the diffusion term is negligible (this is the case for most plasma's). However, this diffusion leads to damping of magnetic fields in relatively low conducting plasma regions, which is important for magnetic reconnection. This magnetic damping becomes important when the magnetic Reynolds number is sufficiently small.

Magnetic Reynolds number

The magnetic Reynolds is a dimensionless number defined in terms of the length scale of the system L, typical velocity scale V, typical magnetic field B, and diffusivity λ. It can be found by making equation **??** dimensionless by scaling the variables. Rewriting all quantities in terms of dimensionless form, $\vec{B} = B\vec{B}'$, $\vec{v} = V\vec{v}'$, and $\nabla = \nabla'$, we get

And thus we identify the dimensionless quantity

$$R_M = \frac{VB/L}{\lambda B/L^2} = \frac{LV}{\lambda} \tag{6.27}$$

The magnetic Reynolds number is analogous the the Reynolds number from fluid dynamics. In fluid dynamics, the Reynolds number is the same for fluid flows around geometrically similar objects, so experiments can be done at smaller scales to determine dynamics. The Reynolds number also determines the relative importance of the nonlinear behavior, and hence characterises whether a flow will be laminar (simple) or turbulent (complicated and hard to predict).

For small R_m, the field will diffuse away, smoothing out across space. For large R_m, the field will remain frozen into the fluid, moving along with the plasma flow.

6.6.5 Alfven's theorem and flux freezing

A very important concept in MHD is Alfven's theorem. Which states that in ideal MHD (i.e. where the fluids are a perfect electrical conductor, such that resistance is infinite and there are no electric fields) magnetic fields in the conducting fluid are constrained to move with the fluid, and the fluid is constrained to move with the magnetic field. This means the magnetic fields are "frozen in" to the fluid.

This theorem only applies in ideal MHD, but is relevant for high magnetic Reynolds numbers R_m. (Explain what the magnetic Reynolds number is, bring up analogy with vorticity and Kelvins circulation theorem, etc).

Intuitively, Alfven's theorem works as follows: Plasma is a bunch of electrons and ions, so when a magnetic field line moves charged particles experience a force, forcing plasma to move with the moving magnetic field (by Lenz law, charges are opposing change in magnetic flux, hence will maintain their position). Alfen's theorem only holds in the non-relativistic limit, since when magnet moves at v c there is a non-negligable delay time before the charged particles respond, meaning the particles do not perfectly flow with the field.

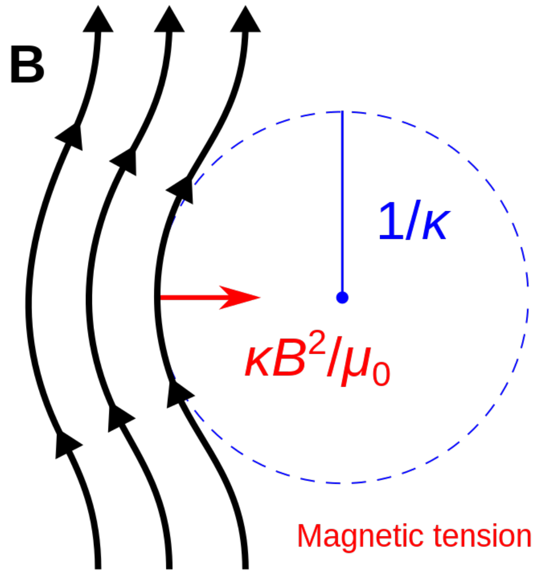


Figure 6.3: Magnetic reconnection figure, from textbook.

6.6.6 Alfven waves

Magnetic tension gives rise to Alfven waves. Alfven waves are analogous to sound waves, but instead of a pertubraiton in pressure travelling, Alfven waves result from a perturbation of the magnetic field in the fluid, and are propagated thanks to the restoring force provided by magnetic tension.

6.6.7 Magnetic reconnection

Thanks to Alfven's theorem, the magnetic topology of an ideal magnetofluid ($R_m = \infty$) is fixed. However, magnetic reconnection relies on the fact that for a region finite resistivity there can be a change in the topology, allowing magnetic field lines to combine, changing the structure of the magnetic field and hence redirecting plasma in potentially highly energetic bursts.

Magnetic field lines combine in locations where the magnetic field gradient is large, and hence overpowering the small value of λ in equation **??** for high conductivity, allowing for "cutting and pasting" of field lines. These regions of large magnetic field gradients must be associated with high currents, and hence are known as current sheets. Outside these current sheets, we can assume Alfven's theorem holds, and hence magnetic topologies are preserved everywhere except for small regions known as current sheets.

A current sheet is shown in figure **??**, where the magnetic field rapidly decreases approaching OP from below, and reaches zero at OP before changing direction and rapdily increasing extending above OP. Due to magnetic pressure being $B^2/8\pi$, the pressure above and below the central region is much greater, this will result in plasma from above and below OP to be sucked into the centre, dragging the frozen in flux with it. This magnetic field will then decay according to the induction equation, since in the high magnetic gradient at the centre means the diffusion term in the induction equation become important, resulting in the magnetic field going to zero along the line OP. By conservation of mass, the sucked in plasma must be squeezed out to the left and right, and at this point the magnetic field lines of the originally separated field line travelling from A' to B' will connect with the line going from A to B, and likewise C'D' will connect with CD, due to the fact that the horizontal field lines B'C' and BC will have decayed to zero, effectively cutting the horizontal field lines, and pasting the colliding vertical field lines together.

TODO: Condense the plasma discussion, focussing more on physical reasoning than mathematical derivations. Talk more about high energy astro phenomena and applications.

Chapter 7

Orbits around black holes

7.1 Schwarzschild solution

7.1.1 Task list

Starting From Schwarzschild metric:

1. Calculate gravitational potential for metric - equation (**??**).
2. Calculate orbits for massive and massless particles - section (**??**).
3. Show innermost stable circular orbit for each and compare them - section (**??**)
4. Make some plots to show how the gravitational potential behaves, and stable orbits in that potential - figure (**??**)
5. Diagram to show how the location of the ISCO relative to event horizon
6. Short discussion of implications for an astrophysical black hole - why is having the ISCO different for massive/mass-less particles significant? - section (**??**)

After this, extend to Kerr black hole, explain the differences - section (**??**).

Separate the Physics from the maths, what is actually physical?

Look at Perihelion shift, should be very high for Kerr.

Kerr event horizon is smaller, only relative to an external observer - a observer falling who does not see space-time rapping around in circles will see an event horizon of the Schwarzschild metric. It is just that due to the curvature of spacetime the radius looks smaller.

7.2 Governing equations

7.2.1 Einstein's field equations

We need to solve EFE's, given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{7.1}$$

Where

- $G_{\mu\nu}$ is the Einstein tensor, representing the curvature of space-time.
- $T_{\mu\nu}$ is the stress energy tensor, representing the density and flux of energy/momentum in space-time.
- Λ is the cosmological constant, representing the expansion of the universe.
- $\kappa = \frac{8\pi G}{c^4}$ is a constant.
- $g_{\alpha\beta} = \boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta$ is the metric tensor, given by the dot product of the basis vectors, and $g^{\alpha\beta}$ is the inverse metric tensor. The metric tensor allows for measurements of distance in space-time .

The Einstein tensor can be written as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{7.2}$$

Where the R tensor are contractions of the Ricci tensor defined by equation

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\lambda_{\nu\sigma}\Gamma^\rho_{\mu\lambda} - \Gamma^\lambda_{\mu\sigma}\Gamma^\rho_{\nu\lambda} \tag{7.3}$$

- $R_{\mu\nu} = R^\alpha_{\nu\alpha\beta}$ is the Ricci curvature tensor, which roughly represents how much the curved space-time deviates from flat space-time.
- R $= R^\alpha_\alpha$ is the Ricci scalar, the contraction of the Ricci tensor, a measure of curvature.

And gamma coefficients $\Gamma^\mu_{\nu\sigma}$ describes the Christoffel symbols (or affine connection), which itself is given by the formula

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\alpha\mu}(\partial_\nu g_{\alpha\sigma} + \partial_\sigma g_{\nu\alpha} - \partial_\alpha g_{\nu\sigma}) \tag{7.4}$$

Since we are dealing with space-time outside a black hole, assuming a vacuum and covering a distance small enough that expansion of the universe is negligible, we can set $T_{\mu\nu} = 0$ (vacuum) and $\Lambda = 0$ (ignore expansion). So we are only solving $G_{\mu\nu} = 0$. But if we take the trace of equation (**??**), by contracting with inverse metric tensor $g^{\mu\nu}$, we get $R = 0$, hence we only need to solve EFE in a vacuum

$$R_{\mu\nu} = 0 \tag{7.5}$$

This is 16 coupled non-linear partial differential equations[1], involving the calculation of the Christoffel symbols according to equation (**??**), which themselves depend on the derivatives of the metric components. Due to this high complexity, EFE are only analytically solvable in a few highly symmetric cases, notably the Schwarzschild and the Kerr solutions.

[1]The Ricci tensor is symmetric, so it is actually just 10 PDEs.

7.2.2 Schwarzschild metric

The distance between two points in space-time is given by the invariant line element ds^2. In Schwarzschild space-time, the line is given by equation (**??**) (given in units c=1).

$$ds^2 = -d\tau^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{7.6}$$

I derived the Schwarzschild metric in schwarzschild_solution.nb, a Mathematica notebook based on the OGRe Mathematica package **?** for GR calculations.

7.2.3 Kerr metric

The metric for a Kerr black hole (spherically symmetric, stationary, black hole of mass M rotating at constant angular momentum J) is given by (in units G=c=1)

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}d\phi dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin\theta^2 d\phi^2 \tag{7.7}$$

Where $a := \frac{J}{M}$, $\rho^2 := r^2 + a^2\cos\theta$, and $\Delta := r^2 - 2Mr + a^2$ **?**.

It is easy to see that setting $a = 0$ leaves us with the Schwarzschild solution (the case of zero spin is the same as Schwarzschild solution in equation (**??**)), and in the limit that $r \gg a$ we get the weak field metric (in terms of gravitational potential $\Phi(r) = -\frac{GM}{r}$)

$$ds^2 = -(1 + 2\Phi(r))dt^2 + (1 - 2\Phi(r))dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{7.8}$$

Which as $r \to \infty$ approaches flat (Minkowski) spacetime

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{7.9}$$

7.2.4 Geodesic equation

The paths followed by particles in general relativity correspond to shortest paths through (curved) space-time. These shortest paths are known as geodesics, and the formula for determining these paths is know as the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \tag{7.10}$$

This equation is derived by minimizing the action S defined by

$$S = \int \mathcal{L}(x_\mu, \dot{x}_\mu)d\lambda = \int \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu d\lambda \tag{7.11}$$

Using the Euler Lagrange equations

$$\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}_i} - \frac{\partial\mathcal{L}}{\partial x_i} = 0 \tag{7.12}$$

We can physically motivate this definition by considering that this is equivalent to extremizing the *proper time* τ between two events[2]. The proper time is the time a clock carried by a particle will experience, while a distant observer (at a relative velocity and different gravitational field) will measure a different time between two events. So just find the extremal value of

$$\tau_{AB} = \int \sqrt{-ds^2} = \int_0^1 \sqrt{-g_{\mu\nu}(x(\lambda))\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda \tag{7.13}$$

Where λ is the parametrization of the curve in space. Hence finding the maxima of equation **??** is equivalent to finding the minima of the action given in equation (**??**), as required.

Another way to view equation (**??**) is as a path that is locally straight in curved spacetime. The particles four velocity

$$\boldsymbol{u} := \frac{dx^\mu}{d\tau} \tag{7.14}$$

Is always tangent to its path in space, the four velocity will therefore be straight if the four vector $\boldsymbol{u}$ does not change after an infinitesimal displacement.

7.2.5 Effective potential

Calculating the gravitational potential in general relativity is not as straight forward as it is in Newtonian gravity. There is not a well defined definition for potential in curved spacetime, so instead we define the potential in some region of spacetime to be such that it fits the definition of potential energy that arises from the law of conservation of energy. There are several approaches to be taken, using both the Lagrangian and Hamiltonian definition. I believe the following general procedure, illustrated in later sections, is the following:

1. Calculate the Lagrangian defined by equation (**??**).
2. Use the Euler-Lagrange equation (**??**) to identify conserved quantities, and equations of motion for geodesic.
3. Use conserved quantities, along with the identity in equation (**??**) - which depends on whether you are interested in massive or massless orbits - to simplify the Lagrangian in terms of conserved quantities.
4. Rewrite the Lagrangian equation to clearly identify the effective kinetic energy ($\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2$), the effective total energy (some function of the conserved quantity E), and the effective potential energy (whatever remains).

7.2.6 Event horizon

The event horizon of a black hole is an imaginary boundary in space surrounding a black hole, such that anything (including light) within that boundary will be unable to escape the black hole, so the

[2] In special relativity, i.e. flat spacetime, time dilation means that the worldine of free particles moving between two events is the worldline of largest proper time between those events. General relativity says that this principle is extended to curved spacetime.

future of that particle is the singularity at the origin (r=0) of the black hole. In the Schwarzschild solution, the event horizon coincides with the point where a photon experiences infinite time-dilation, and hence infinite redshift according to an observer at infinity. Hence, the event horizon occurs when $g_{tt} = 0$, from equation (**??**) this occurs when $1 - \frac{2GM}{r} = 0$, i.e. the Schwarzschild radius is $r = \frac{2GM}{r}$.

The problem of finding the event horizon is more complicated in less idealized black holes, such as in the Kerr metric, where a photon can pass the surface of infinite redshift, and still return to a larger radius. The surface between the event horizon and the infinite redshift surface is known as the *ergoregion*. The surface of infinite redshift can be found by setting $g_{tt} = 0$, so from equation (**??**) we find

$$\begin{aligned} & 1 - \frac{2Mr}{r^2 + a^2 \cos\theta} = 0 \\ \Rightarrow \qquad & 2Mr = r^2 - a^2 \cos\theta \\ \Rightarrow \qquad & r_{\pm} = M \pm \sqrt{M^2 + a^2 \cos\theta} \end{aligned} \tag{7.15}$$

The event horizon can be found by finding when g_{rr} diverges in equation (**??**), i.e. when $\Delta = 0$, giving the event horizon radius

$$r_{\text{eh},\pm} = M \pm \sqrt{M^2 - a^2} \tag{7.16}$$

So we can see the two regions overlap at the poles (as expected, since there is locally no difference to a Schwarzschild black hole at region where there is no rotation), but the sphere of infinite redshift extends further in all other regions. The region between these two surfaces is called the *ergosphere*.

Note that particles can escape a Kerr black hole near its equator relatively easily compared to a Schwarzschild black hole, i.e. the event horizon is somehow "squished" along the equator due to the black holes spin. This is a result of frame dragging, the process whereby spacetime in the vicinity of a spinning mass co-rotates with the spin of the mass. This curls up spacetime as shown in figure (**??**), increasing the effective radius from the event horizon to the center of the black holes. Note, the Kerr event horizon is smaller only relative to an external observer - a observer falling into the black hole (and hence co-rotating with the spin due to frame dragging) will see an event horizon of the Schwarzschild metric. See notes on gravitomagnetism for a more in depth discussion on Frame dragging[3].

7.2.7 Constants of motion

Solving the Geodesic equation directly would be extremely difficult, if not impossible. Calculations in GR almost always require the use of constants of motion in order to be able to solve even the simplest problems.

We will exploit what is essentially conservation of energy and conservation of angular momentum.

Plugging in $\mu = t$ into equation (**??**), using equation (**??**) to calculate the required Christoffel symbols, we can show that the following quantity (corresponding to the energy per unit mass of a test

[3] Add in link to gravitomagnetism notes.

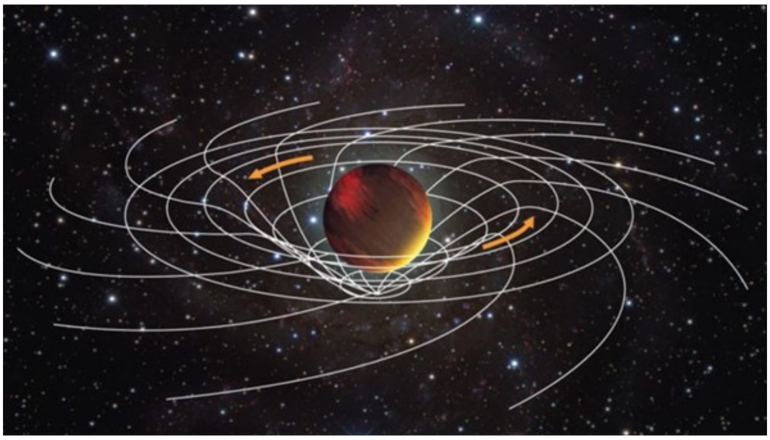


Figure 7.1: Illustration of frame dragging. Illustration by Annie Rosen.

particle at infinity) is conserved

$$E = (1 - \frac{2GM}{r})\frac{dt}{d\tau} \tag{7.17}$$

This energy symmetry is a result of time symmetry, specifically the invariance of the metric with respect to time $\partial_t g_{\mu\nu} = 0$[4]. So this symmetry won't exist in more general space-time (it will in Kerr and static Schwarzschild spacetimes, but not for binary orbits).

Another conserved quantity can be found[5] by substituting $x = \phi$ into equation (**??**), yielding the conserved quantity

$$r^2 \sin^2\theta \frac{d\phi}{d\tau} = L \tag{7.18}$$

Corresponding to the angular momentum per unit mass (if observed at infinity). This corresponds to rotational symmetry (won't exist in binary orbit).

The final symmetry to solve the geodesic equation comes from the quantity

$$d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu \implies -g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \begin{cases} 1, & \text{if } d\tau > 0 \\ 0, & \text{if } d\tau = 0 \end{cases} \tag{7.19}$$

But this is just conservation of (four) momentum $P^\mu := m\frac{dx^\mu}{d\tau}$. Indeed the contraction $P_\mu P^\mu$ is a scalar, so it must be the same in all inertial reference frames. Hence by considering a particle at $r = \infty$ in a frame such that the particle is at rest, we see all velocity components are zero, leaving the only non-zero component to be $P^0 = m\frac{dt}{d\tau} = m$, we can see $P_\mu P^\mu = \eta_{\mu\nu}P^\mu P^\nu = -m^2$ for non-zero mass. Since this is a scalar quantity, it will be the same in all reference frames, and will be a constant. Hence we get the conservation of four momentum to be

[4]This is a result of Noether's theorem, symmetries result in conserved quantities.

[5]These conserved quantities will also pop out of the Euler Lagrange equations applied to the Lagrangian described earlier.

$$g_{\mu\nu}P^{\mu}P^{\nu} = -m^2 \tag{7.20}$$
$$\Rightarrow g_{\mu\nu}m\frac{dx^{\mu}}{d\tau}m\frac{dx^{\nu}}{d\tau} = -m^2 \tag{7.21}$$
$$\Rightarrow \quad g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = -1 \tag{7.22}$$

Which implies equation (**??**) for the case of massive particles, and similar reasoning can be used to justify the $d\tau = 0$ case.

P

Explain the physical meaning of this conservation.

Since we have two different situations for massive ($d\tau > 0$) and massless ($d\tau = 0$, e.g. photons) particles, so we can rewrite equation (**??**) as $-g_{\mu\nu}dx^{\mu}dx^{\nu} = \sigma$, where $\sigma = 0$ for massless particles, and $\sigma = 1$ for massive particles.

This gives us the formula

$$-\sigma = -(1 - \frac{2GM}{r})(\frac{dt}{d\tau})^2 + (1 - \frac{2GM}{r})^{-1}(\frac{dt}{dr})^2 + r^2((\frac{d\theta}{dt})^2 + \sin^2\theta(\frac{d\phi}{dt})^2) \tag{7.23}$$

Using equation (**??**), along with conservation of energy and angular momentum to eliminate $\frac{dt}{d\tau}$ and $\frac{d\phi}{d\tau}$, and impose $\theta = \pi/2$ (which we can do without loss of generality, due to spherical symmetry and the fact that orbits in Schwarzschild space-time lie in a fixed plane) we can arrive at

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V(r) = \frac{1}{2}E^2 \tag{7.24}$$

Where we have defined the effective potential energy

$$V(r) := \frac{1}{2}\sigma - \sigma\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \tag{7.25}$$

So we can plot the effective potential for different L/M ratios.

7.2.8 Potential for massive and massless orbits

A circular orbit will occur at the point where the derivative of the effective potential is zero. If these potential correspond to a maxima, they are unstable, if it corresponds to a minima, it is stable. Looking at figure (**??**) we can immediately see the mass-less particles only form unstable circular orbits, with the radius increasing with the L/M ratio.

Massive particles, as seen in the diagram, there are either zero circular orbits, or one stable and one unstable circular orbit (when L/M ratio gets high enough). It is easy to prove all of this by differentiating equation (**??**) with respect to r, and solving the equation for r.

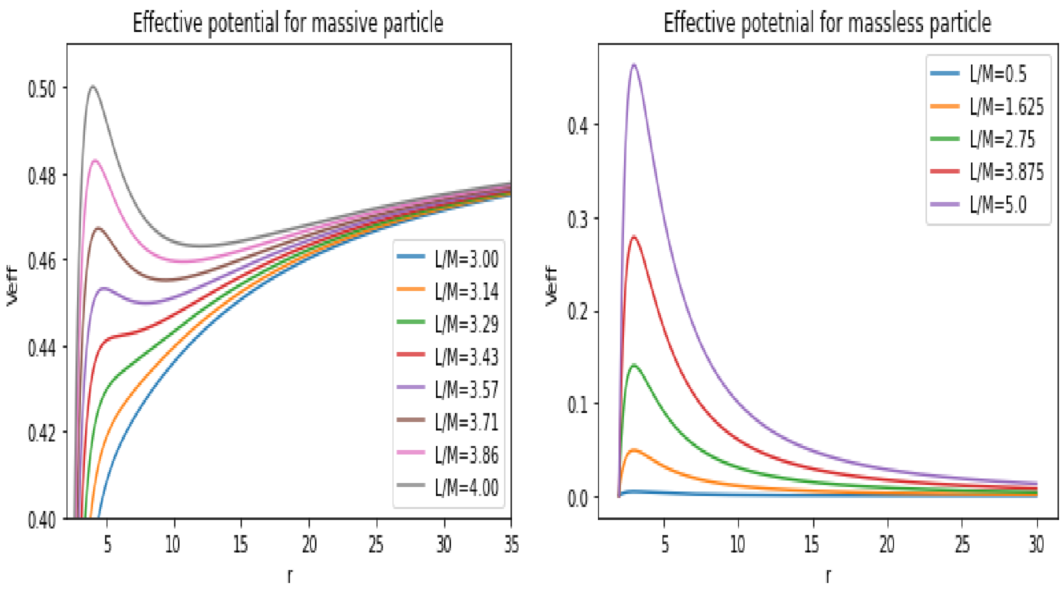


Figure 7.2: Schwarzschild effective potential, varying angular momentum to mass ratio. Note the Schwarzschild radius is a constant $r_s = 2GM$, and since we have set $GM = 1$ in this diagram, this corresponds to the point $r = 2$ - can see the potential is zero at the event horizon.

$$V'(r) = 0$$

$$\Rightarrow \quad \frac{\sigma GM}{r^2} - \frac{L^2}{r^3} + \frac{3GML^2}{r^4} = 0 \tag{7.26}$$

$$\Rightarrow \quad r_{c,\pm} = \frac{L^2 \pm \sqrt{L^4 - 12\sigma G^2 M^2 L^2}}{2\sigma GM} \tag{7.27}$$

7.2.9 ISCO

So there exists solutions to this equation for $\sigma = 1$ (massive particles) if the value inside the square root of equation (**??**) is non-negative, i.e. $L^2 < 12(GM)^2$. We can see geometrically from the plots that for such solutions, the negative corresponds to the unstable circular orbit, while the positive corresponds to the outer stable orbit. These two orbits appear when $L = 2\sqrt{3}GM \Rightarrow \quad r_{c,+} = r_{c,-} = 6(GM)$[6], and in the limit as $L \longrightarrow \infty$ is $r_{c,-} = 0$ from (**??**), and equation (**??**) implies that $r_{c,+} \rightarrow GM$. Since the inner orbit is unstable, any perturbation will radially inward will cause the particle to spiral into the event horizon at $r = 2GM$, and any perturbation radially outward will cause the particle to move to the stable outer orbit.

Hence, massive particles have stable circular orbits for

$$6GM \leq r \leq \infty \iff 2\sqrt{3} \leq \frac{L}{M} \leq \infty \tag{7.28}$$

[6] $r_c = 6GM$ is the "inner-most stable circular orbit.

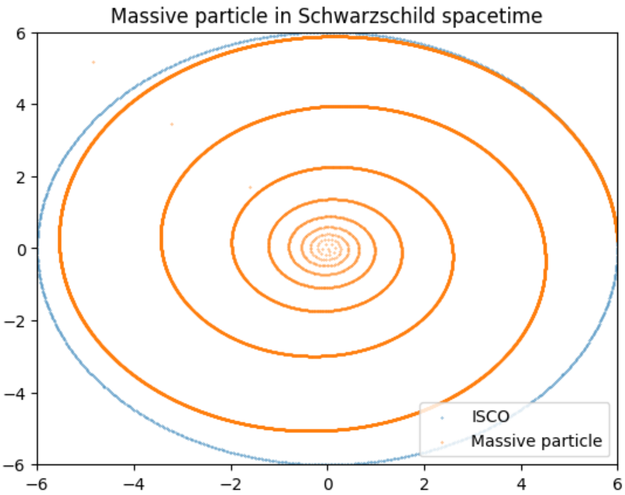


Figure 7.3: This is an orbital diagram from a numerical scheme that was wrong. Likely didn't conserve energy.

7.2.10 Massless particle

The mass-less orbits can be found by setting $\sigma = 0$, so from equation (**??**) we immediately see that there is only one solution, $r_c = 3GM$, independent of any other factor. We also see graphically, that it is an unstable orbit (it corresponds to a maxima, could do second derivative test). It should be mentioned that mass-less particles do not have a well defined angular momentum L, L is only a physical value when the ratio $\frac{L}{E}$ is taken.

$\frac{L}{E} = 3\sqrt{3}GM$ is critical value, the impact parameter shows whether the light ray is absorbed by black hole. This can be seen by noting that a particle is bound to an orbit if $E \leq V_{\text{eff}}$, since $V_{\text{eff}}(r_c = 3GM)\frac{E^2}{2} \Rightarrow \frac{E^2}{2}\frac{L^2}{54GM}$.

The reason that there is no *stable* circular orbit for light rays is that light travels in a straight path, and since the curvature towards the BH will be monotonically increasing towards the black hole, any perturbation outward/inward from a circular orbit will result in the photon continuing out from the straight line orbit around the black hole. Massive particles have some additional properties that make them behave differently, namely their mass and angular momentum.

Shapiro delay

7.2.11 Particle orbits

Equation **??**, we have

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \sigma\left(1 - 2\frac{GM}{r}\right) - \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \tag{7.29}$$

Can manipulate this into a second order DE to give radius as a function of angle phi. This is not exactly solvable, but we can either expand as a perturbation of the exact solvable Newtonian case (conic sections), or solve numerically for given boundary conditions.

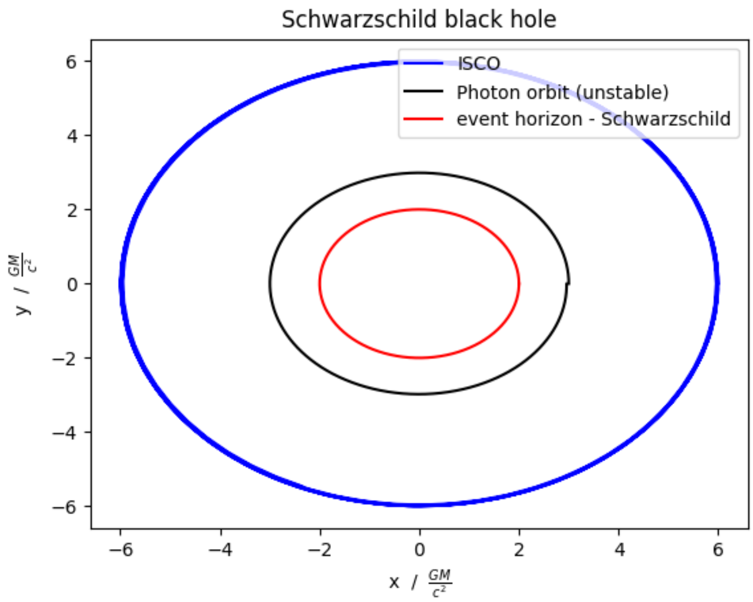


Figure 7.4: Diagram showing difference between massive and massless particle orbits in Schwarzschild spacetime. Can see the Photon orbit is close to the black hole, and is unstable. These orbits can be compared to the event horizon

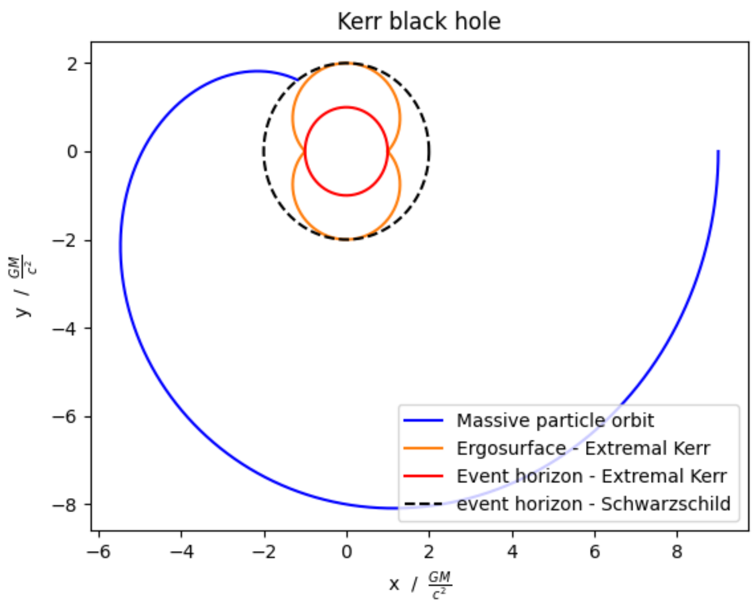


Figure 7.5: Kerr massive particle circular orbit for extremal black hole (a=1). It does not work, don't know why.

7.3 Kerr black hole

7.3.1 Form of the metric

The general form of a space-time metric is

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= g_{tt}(t, x^i) dt^2 + 2g_{ti}(t, x^i) dx^t dx^i + g_{ij}(t, x^i) dx^i dx^j \end{aligned} \tag{7.30}$$

Where the Greek indices $\mu, \nu = 0, 1, 2, 3$ run over both time and space indices (we are using spherical polar coordinates, so $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$), while Latin indices $i, j = 1, 2, 3$ run only over spatial indices. We now go through the procedure to find the symmetries of the system such that the metric takes the simplest form, giving us a chance to solve the Einstein field equations (EFE) analytically. Note the following symmetries

1. Time translation $t \to t - a \Rightarrow g_{\mu\nu}(t, x^i) = g_{\mu\nu}(x^i)$.
2. Time reversal $t \to -t$.
3. Axial symmetry $\phi \to \phi + a \Rightarrow g_{\mu\nu}(x^i) = g_{\mu\nu}(r, \theta)$.
4. $\theta \to -\theta \Rightarrow$ cross terms involving θ are zero.

The metric tensor transforms as any (2,0) tensor, i.e. for a coordinate transform $x^\alpha \to x^{\alpha'}$, the metric transforms $g_{\mu\nu} \to g'_{\mu\nu}$ according to the transformation law

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{\nu'}} g_{\alpha\beta} \tag{7.31}$$

Time reversal symmetry, **which we do not have for a Kerr black hole**, would mean that $g_{\mu\nu}(t, x^i) = g'_{\mu\nu}(t, x^i)$ for a coordinate transformation $t = x^0 \to x^{0'} = -t$. So plugging this into equation (**??**) we get

$$\begin{aligned} \frac{\partial x^0}{\partial x^{0'}} &= -1 \\ \frac{\partial x^i}{\partial x^{i'}} &= +1 \\ \Rightarrow \quad g'_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{\nu'}} g_{\alpha\beta} \\ \Rightarrow \quad g'_{\mu t} &= \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{t'}} g_{\alpha\beta} \\ &= \frac{\partial x^\alpha}{\partial x^{\mu'}} (-1)\, g_{\alpha t} \\ &= -g_{\mu t} \end{aligned} \tag{7.32}$$

But, $g_{\mu t} = g'_{\mu t}$, hence $g_{\mu t} = 0, \forall \mu \neq t$. This is what eliminated the cross terms in the Schwarzschild metric, but in the Kerr metric there is still the cross term $d\phi dt$, since space-time is different for objects rotating with the spin of the black hole, and objects orbiting in opposite direction to the black hole (frame dragging). However, cross terms like *drdt* must be zero, because stepping outward a radial distance in forward time dt should be an equivalent distance to stepping inward dr at negative time -dt.

7.3.2 Symmetries

To go through the same procedure to find particle orbits as we did for the Schwarzschild solution, we need to go through the same step of finding integrals of motion. We will approach this slightly more rigorously in our second attempt, by introducing the concept of *Killing vectors*. See the appendix.

The Kerr solution is stationary (does not depend on time), hence $(t, r, \theta, \phi) \rightarrow (t + a, r, \theta, \phi)$ is a symmetry, corresponding to the Killing vector with components $\xi^\alpha = (1, 0, 0, 0)$. The solution is also independent of ϕ since $(t, r, \theta, \phi) \rightarrow (t + a, r, \theta, \phi)$ has no effect on the system, hence another Killing components is $\eta^\alpha = (0, 0, 0, 1)$. These correspond to Killing vectors

$$\begin{aligned}\boldsymbol{\xi} &= \xi^\alpha \partial_\mu = \partial_t \\ \boldsymbol{\eta} &= \eta^\alpha \partial_\alpha = \partial_\phi\end{aligned}$$

Killing vectors give us conserved quantities according to

$$\boldsymbol{K_\mu} \frac{dx^\mu}{d\tau} = const. \tag{7.33}$$

So when $p^\mu = m\frac{dx^\mu}{d\tau}$ we also have $\boldsymbol{K_\mu} p^\mu = const.$ as well. So for $\boldsymbol{K} = \boldsymbol{\eta}$ we get

$$\begin{aligned}e &= -g_{\mu\nu}\xi^\mu u^\nu \\ &= -g_{00}\xi^0 u^0 - g_{03}\xi^0 u^3 \\ &= \left(1 - \frac{2Mr}{\rho^2}\right)\frac{dt}{d\tau} + \left(\frac{4MrA^2\sin^2\theta}{\rho^2}\right)\frac{d\phi}{d\tau} \end{aligned}\tag{7.34}$$

$$\begin{aligned}l &= g_{\mu\nu}\eta^\mu u^\nu \\ &= g_{30}\eta^3 u^0 + g_{33}\eta^3 u^3 \\ &= -\left(\frac{4MrA^2\sin^2\theta}{\rho^2}\right)\frac{dt}{d\tau} + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\frac{d\phi}{d\tau}\end{aligned}\tag{7.35}$$

Combining equations (**??**) and (**??**) we can solve for $\frac{d\phi}{d\tau}$ and $\frac{dt}{d\tau}$.

$$\frac{dt}{d\tau} = \frac{1}{\Delta}\left[(r^2 + a^2 + \frac{2Ma^2}{r})e - \frac{2Ma}{r}l\right] \tag{7.36}$$

$$\frac{d\phi}{d\tau} = \frac{1}{\Delta}\left[\left(1 - \frac{2M}{r}\right)l + \frac{2Ma}{r}e\right] \tag{7.37}$$

Then plugging these into the relation from equation (**??**), and (for now) assuming a equatorial orbit (so that we can set $\frac{d\theta}{d\tau} = 0$), we arrive at the equation for a massive particle

$$\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 + V_{\text{eff}}(r,e,l) \tag{7.38}$$

$$V_{\text{eff}}(r,e,l) := -\frac{M}{r} + \frac{l^2 - a^2(e^2-1)}{2r^2} - M\frac{(l-ae)^2}{r^3} \tag{7.39}$$

As well as for a massless particle, which must be parameterised by a different parameter, λ, since massless particles have constant proper time τ

$$\frac{1}{l^2}\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - W_{\text{eff}}(r,b,l) \tag{7.40}$$

$$W_{\text{eff}}(r,b,l) := \frac{1}{b^2} - \frac{1}{r^2}\left[1 - \frac{a^2}{b^2} - \frac{2M}{r}\left(1 - \text{sign(l)}\frac{\text{a}}{\text{b}}\right)^2\right] \tag{7.41}$$

$p^0 = E$ a constant (conservation of energy), and for $\boldsymbol{K} = \boldsymbol{\xi}$ we get $p^3 = m\frac{d\phi}{d\tau} = L$ a constant (conservation of angular momentum)

See schwarzschild_orbits.nb for more exploration.

Most Kerr black holes will be approaching the a=M ($J = M^2$) spin limit of the Kerr black hole, as angular momentum from accreting matter increases spin of BH. So in our diagram, a=1 is of most interest.

7.4 Physical relevance of equations

The equations that we have discussed produce non-physical situations when parameters reach certain values. We need to determine the validity range of all the parameters in the model, otherwise the model will produce garbage. These include hard boundaries, (no energy creation), and soft boundaries (particles have real mass).

7.4.1 Hard boundaries

Schwarzschild black hole

The four momentum can become imaginary for certain parameter values.To see one example of this, consider a radially infalling particle of mass $m > 0$ in Schwarzschild spacetime. Can show (done in notebook - 24/03/2023) that for

$$v^2 = 1 - \left(\frac{E}{1 - \frac{2M}{r}}\right) \tag{7.42}$$

This equation implies velocity becomes imaginary when

$$r < \frac{2M}{1-E} \tag{7.43}$$

For a massless particle in the equatorial plane, it is shown in notebook,

$$\frac{dr}{dt} = -\left(1 - \frac{2M}{r}\right)\sqrt{1 - \frac{b^2}{r^2}\left(1 - \frac{2M}{r}\right)} \tag{7.44}$$

This expression becomes imaginary if

$$b > \frac{r}{\sqrt{1 - \frac{2m}{r}}} \tag{7.45}$$

This can be inverted to find the allowable radius, see notebook (it is a pretty ugly solution to a cubic equation).

Interestingly, using the four momentum conservation equation (**??**) for massless particles, the range where $\frac{dr}{d\tau}$ becomes imaginary only exists if the condition $b = 3\sqrt{3}GM$ is satisfied, i.e. if the particle is not bounded by the black hole.

Kerr black hole - soft boundaries

For a particle with no angular momentum ($l = 0$) and initial energy $e = 1$ the shape of the orbit can be determined from equations (**??**) and (**??**), giving

$$\frac{d\phi}{dr} = -\frac{2Ma}{r\Delta}\left[\frac{2M}{r}\left(1 - \frac{a^2}{r^2}\right)\right] \tag{7.46}$$

For equation (**??**) to yield real values, we must have $r > a = J/M$.

Note also, from the discussion in section (**??**, coordinate singularities occur when $\rho = 0$ or $\Delta = 0$. The $\rho = 0$ singularity occurs if

$$r^2 + a^2 \cos\theta = 0 \Rightarrow \theta = \cos^{-1}\left(-\frac{r^2}{a^2}\right) \tag{7.47}$$

This singularity indeed occurs, since $r \leq a$. The singularity corresponding to $\Delta = 0$ occurs if

$$r^2 - 2Mr + a^2 = 0 \Rightarrow (r - M)^2 + (a^2 - M^2) = 0$$

$\Rightarrow r = M \pm \sqrt{M^2 - a^2}$(7.48)

Hence, we require $a < M \Rightarrow J < M^2$, i.e. the spin of the Kerr black hole is limited.

7.5 Soft boundaries

Penrose process? Blandford-Znajek mechanism? They aren't producing energy from nothing though.

7.6 Double Schwarzschild

7.7 Astrophysical significance

The ISCO for massive particles will result in many particles orbiting the BH in this region. Interactions between the particles, resulting in energy transfer will result in particles accreting into the BH. The gravitational binding energy of the particles as they fall onto the BH will be released in the form of electromagnetic radiation, probably high energy x-rays, which can be observed.

In the case of Kerr black holes, the angular momentum of the accreting matter will be transferred to the BH, resulting in an increase in spin - this will push the BH to reach the maximal spin $J = M^2$.

The circular photon orbits correspond to the photon ring around the BH.The fact that the orbits are unstable means that photons will not stay in orbit forever, and will thus escape the black whole, resulting in a faintly visible ring.

The fact that orbits

7.8 GR and EM analogy

Blandford-Znajek mechanism

Gravito-magnetism

Magnetic re-connection and sunspots

7.8.1 Magnetic reconnection

When magnetic field lines pointing in opposite directions cross, changing magnetic topology, releasing lots of energy

7.9 Appendix

7.9.1 Geometric units

Throughout most of this document we will be using geometric units

$$G = c = 1\,(= M) \tag{7.49}$$

If we have M represent the mass of the sun, then we get the following SI unit relations

$$\begin{aligned} 1kg &= \frac{1}{1.989 \cdot 10^{30}} \\ 1m &= \frac{1}{1477} \\ 1s &= 2.030 \cdot 10^5 \end{aligned}$$

[b]0.3

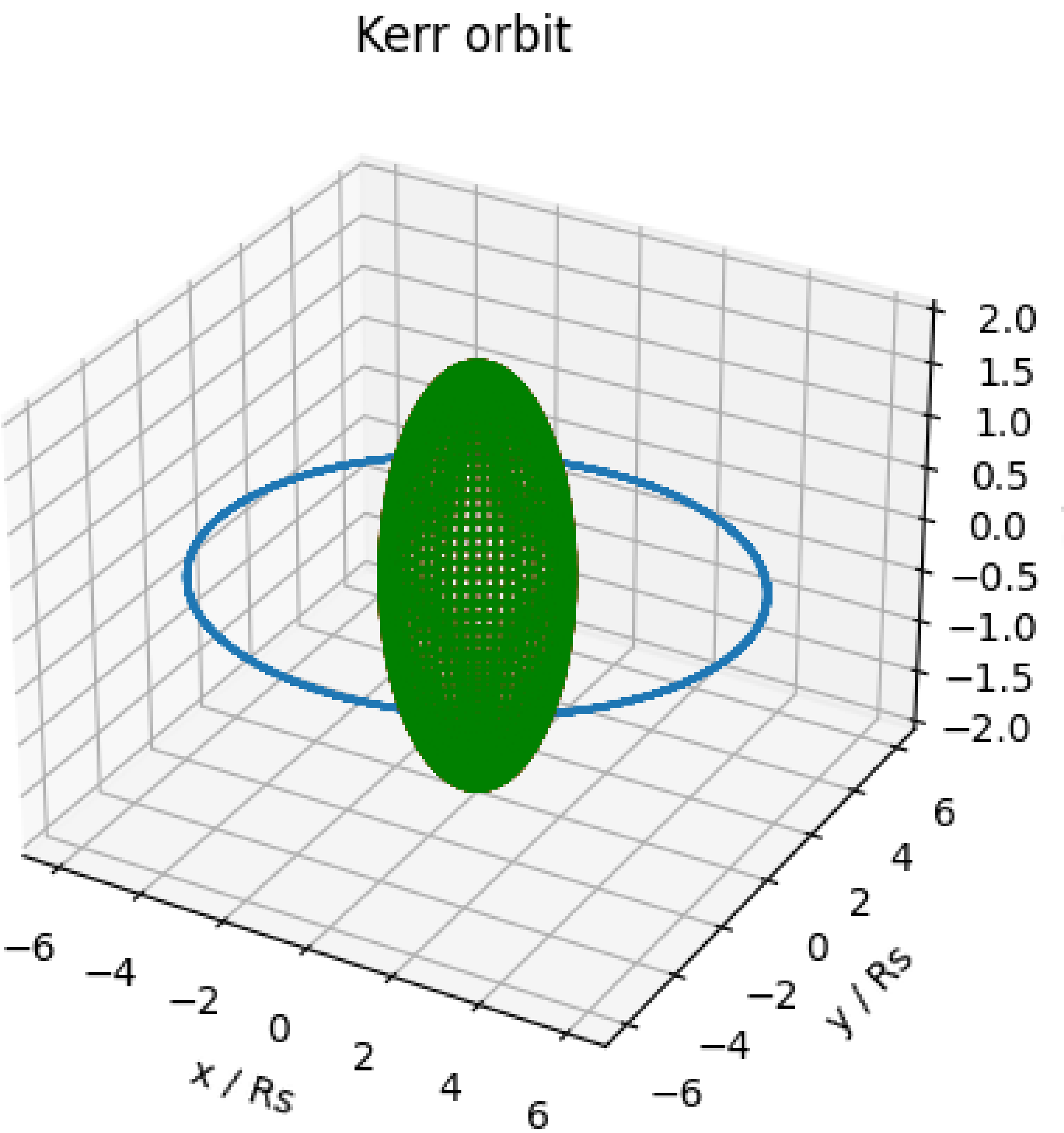
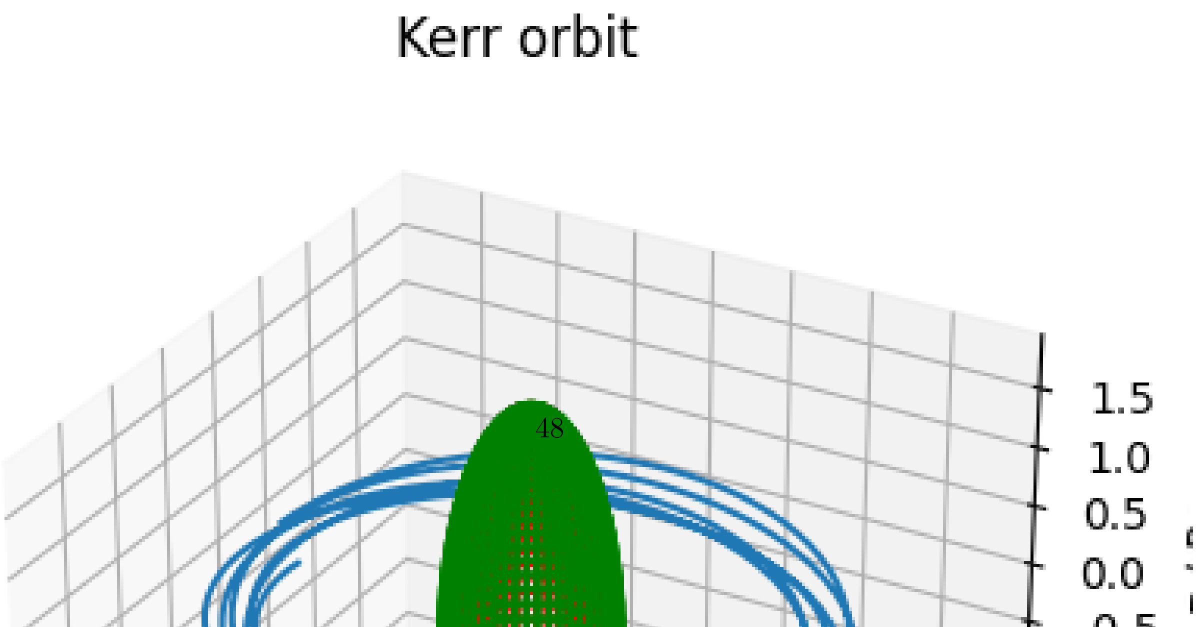


Figure 7.6: a=0.0

[b]0.3



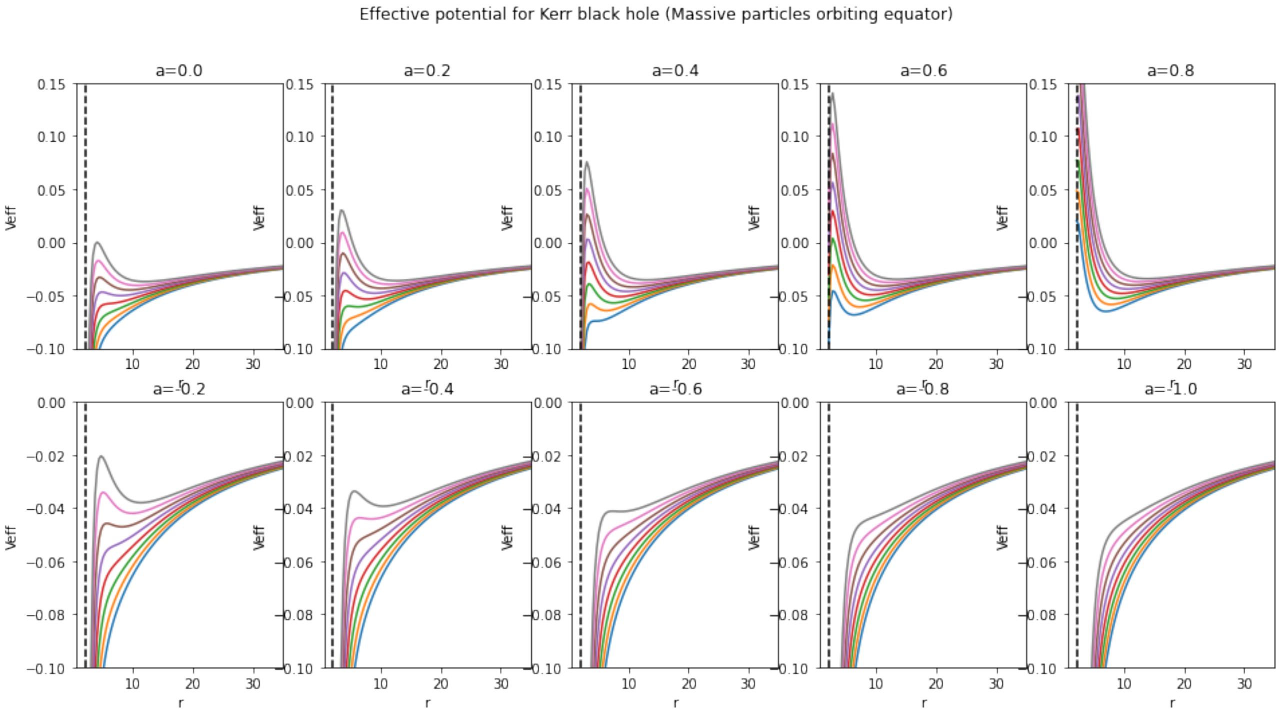


Figure 7.10: Kerr BH massive particle potential around equator. Clearly all unstable, in contrast to massive orbit potentials.

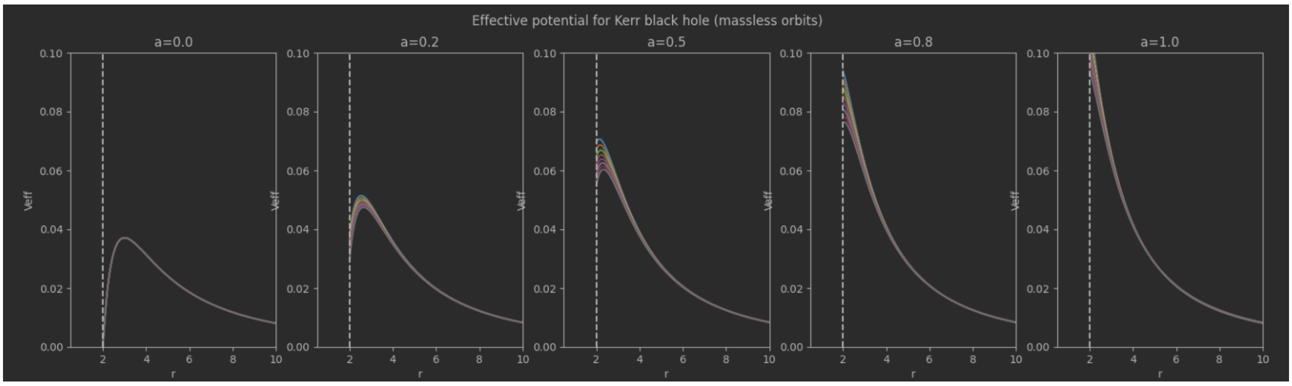


Figure 7.11: Potential for massless particle orbits, around Kerr black hole

Chapter 8

Thesis proposal

8.1 Aims and significance

The aim of the thesis is to study the gravitational field within orbiting compact binary objects near the point of merger. The primary objective is to explore a poorly studied phenomenon, gravitational reconnection. This concept is analogous to its magnetic counterpart, which leads to phenomena such as solar flares and coronal mass ejections. Similarly, gravitational reconnection could result in significant mass ejection or acceleration in the context of merging compact binary objects. By investigating the conditions under which gravitational reconnection can occur, the project aims to gain a deeper understanding of the astrophysical processes involved in the final stages of compact binary mergers. If gravitational reconnection is found to occur in this context, this could have important implications for our understanding of the formation and evolution of astrophysical jets in the universe.

The potential discovery of gravitational reconnection leading to jet formation would imply an electromagnetic counterpart to black hole mergers, much like the post-merger electromagnetic counterparts of neutron stars. This would open up exciting opportunities for further research into the behavior of compact binary mergers and the physics of extreme astrophysical environments. Additionally, this would inform future efforts to perform electromagnetic follow-ups after gravitational wave detections between binary black holes. By combining observations of gravitational waves with electromagnetic radiation at different wavelengths, we can obtain a more complete picture of the physical processes involved in the most extreme environments in the universe, and gain insights into the fundamental laws that govern the behavior of matter and energy on a cosmic scale. Furthermore, these investigations could reveal exotic shapes of the event horizon during these extreme events, providing a new perspective on the dynamical nature of spacetime in the vicinity of merging compact objects.

8.2 Status (literature review)

Since Einstein published his general theory of relativity, it has become evident that gravitation and electromagnetism are intricately intertwined. In the weak field limit of general relativity, the Einstein field equations, governing gravity, simplify to the gravitomagnetic equations, which bear a striking resemblance to Maxwell's equations. Consequently, this remarkable similarity between gravity and electromagnetism has given rise to various analogies between gravitational and electromagnetic phenomena.

In recent years, the field of astronomy has witnessed a significant transformation, empowering researchers to explore general relativistic phenomena with unprecedented precision. Particularly, multimessenger astronomy has emerged as a powerful approach, combining data from diverse detectors such as gravitational wave detectors, telescopes operating across different wavelengths, and neutrino detectors

Nonetheless, for the situation of interest in our study - the merger of binary black holes - it is currently unknown if there exists an electromagnetic counterpart **?**. However, an intriguing possibility arises if gravitational field lines were to reconnect at a stable Lagrange point of orbiting Kerr black holes. In such a scenario, any matter trapped at that Lagrange point - which would be subjected to high pressure - would briefly become liberated from the field, potentially resulting in the ejection of material in the form of a jet. This mechanism could give rise to an electromagnetic counterpart and offer plausible explanations for jet phenomena with currently unknown origins.

The possibility of electromagnetic counterparts to binary black hole mergers has been proposed, with potential signals already detected **?**. Various theories have been put forth to explain the mechanisms behind such signals **?**. To the best of my knowledge, investigation into gravitational reconnection has not been undertaken. Nevertheless, recent studies on magnetic reconnection in the vicinity of black holes have emerged **??**, led by researchers such as Felipe A. Asenjo and Luca Comisso.

Thus far, the project has primarily focused on acquiring an understanding of the relevant background material. Subjects that have been studied (including references to some of the texts I have used) include general relativity **??**, high-energy astrophysics **?**, and plasma physics **?**. Additionally, I have undertaken simpler simulations as part of practice exercises. For instance, a Python program was developed to simulate the trajectory of both massive and massless particles around a Kerr black hole, showcasing phenomena such as the innermost stable circular orbits and frame dragging.

8.3 Research Methods and Plan

My research will involve both theoretical calculations, along with computer simulations (numerical relativity). For the theoretical calculations, these will primarily be conducted via traditional 'pen and paper' methods, supplemented with computational tools such as Mathematica and general relativity packages like the one cited in reference **?**. The computer simulations, if we decide to go down the numerical relativity route, will likely involve the Einstein toolkit, which has associated Python packages. The next step in the project is to begin simulating spacetime near a stable Lagrange point of orbiting black hole pairs.