Solving field equations

In this notebook I will be deriving the Schwarzschild solution to Einstein's field equations. In Evgeny's class we did this using Cartan's structure equations, but here we will stick to the tensor definitions that I have developed in the second chapter of my thesis.

The calculations for calculating the Ricci tensor's and so on are straight foward, but they are tedious. So instead I will use a mathematica package OGRe to do these calculations, instead I will focus on the physical reasoning rather than the mathematics (which wasn't discussed much in my GR class).

OGRe

In[ ]:=
```
ClearAll["Global'*"]
```

In[ ]:=
```
Get["OGRe.m", Path → NotebookDirectory[]]
```

OGRe:

OGRe: An Object-Oriented General Relativity Package for Mathematica

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GitHub repository: https://github.com/bshoshany/OGRe

- To view the full documentation for the package, type TDocs[].
- To list all available modules, type ?OGRe`*.
- To get help on a particular module, type ? followed by the module name.
- To enable parallelization, type TSetParallelization[True].
- OGRe`Private`UpdateMessage
  To disable automatic checks for updates at startup, type TSetAutoUpdates[False].

The form of the Schwarzschild metric

The general metric for space-time is

$$\begin{aligned} \mathrm{ds}^2 &= g_{\mu\nu}\,\mathrm{dx}^\mu\,\mathrm{dx}^\nu \\ &= g_{\mathrm{tt}}(t, x^i)\,\mathrm{dt}^2 + 2\,g_{\mathrm{ti}}(t, x^i)\,\mathrm{dx}^t\,\mathrm{dx}^i + g_{\mathrm{ij}}(t, x^i)\,\mathrm{dx}^i\,\mathrm{dx}^j \end{aligned} \tag{1}$$

Where greek letter μ,ν=0,1,2,3 are time and space inidices, while latin characters i,j=1,2,3 are just spatial indices.

In order to simplify find the line element, we want to exploit the symmetries of the system in order to write out the most simple form of equation (1) that still completely captures the possible metric.

Firstly, the space-time around a static spherically black hole is time independent i.e. is inveraiant under time translations
$t \to t-a \Rightarrow g_{\mu\nu}(t, x^i) = g_{\mu\nu}(x^i)$. Furthermore, the solutions should be invariant to time reversal t → -t, hence the cross terms involving dt must satisfy $\mathrm{dtdx}^i = d(-t)\,\mathrm{dx}^i = -\mathrm{dt}\,\mathrm{dx}^i \Rightarrow \mathrm{dt}\,\mathrm{dx}^i = 0$. This leaves us with the line element

$$\mathrm{ds}^2 = g_{\mathrm{tt}}(x^i)\,\mathrm{dt}^2 + g_{\mathrm{ij}}(x^i)\,\mathrm{dx}^i\,\mathrm{dx}^j \tag{2}$$

We now try to exloit the spatial symmetry to try to reduce the number of cross terms in the spatial components. Consider the set of concentric spherical shells surrounding the origin of the black hole, indexed by the continuous paraemter r (which is not actually the radius from the origin, since we are working in curved space-time).

Since the systen these spherical shells are obviously spherically symmetric (the black hole looks the same no matter how we rotate about the soherical shell) we choose to use spherical coordinates to parameterise this space. Giving the metric

$$ds^2 = g_{tt}(r,\ \theta,\ \phi)\,dt^2 + g_{rr}(r,\ \theta,\ \phi)\,dr^2 + 2\,g_{r\theta}(r,\ \theta,\ \phi)\,dr\,d\theta + 2\,g_{r\phi}(r,\ \theta,\ \phi)\,dr\,d\phi + 2\,g_{\theta\phi}(r,\ \theta,\ \phi)\,d\theta\,d\phi + d\Omega^2$$

On each of the spherical surfaces, dr=dt=0, so the metric becomes

$$ds^2 = 2\,g_{\theta\phi}(r,\ \theta,\ \phi)\,d\theta\,d\phi + d\Omega^2 \tag{3}$$

But on this hypersurface, by spherical symmetry, there is no curvature between points at same radius/time, so space-time is flat along this the sphere, hence the line element reduces to that of a sphere, where $ds^2 = r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) = d\Omega^2$, hence the $d\theta\,d\phi$ cross term disappears. This leaves the metric to be

$$ds^2 = g_{tt}(r,\ \theta,\ \phi)\,dt^2 + g_{rr}(r,\ \theta,\ \phi)\,dr^2 + 2\,g_{r\theta}(r,\ \theta,\ \phi)\,dr\,d\theta + 2\,g_{r\phi}(r,\ \theta,\ \phi)\,dr\,d\phi + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) \tag{4}$$

From here, we can get rid of the cross terms involving r by noting the symmetry $\theta \to -\theta$ and $\phi \to -\phi$, similarly removing the cross terms as with the time-reversal invariance. This leaves us with

$$ds^2 = g_{tt}(r,\ \theta,\ \phi)\,dt^2 + g_{rr}(r,\ \theta,\ \phi)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) \tag{5}$$

Finally, using spherical symmetry one last time, the metric will only depend on the distance from the orgin, leaving the final form

$$\begin{aligned} ds^2 &= g_{tt}(r)\,dt^2 + g_{rr}(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) \\ &= -A(r)\,dt^2 + B(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) \end{aligned} \tag{6}$$

So we only need to determine two functions, each depending on a single coordinate.

Solving Einsteins equation

- **Einsteins equation**

We now need to solve Einsteins field equations

$$G_{\mu\nu} + \Lambda\,g_{\mu\nu} = \kappa\,T_{\mu\nu} \tag{7}$$

Where

- $G_{\mu\nu}$ is the Einstein tensor, representing the curvature of space-time.
- $T_{\mu\nu}$ is the stress energy tensor, representing the density and flux of energy/momnetum in space-time.
- Λ is the cosmological constant, representing the expansion of the universe.
- $\kappa = \frac{8\pi G}{c^4}$ is a constant.
- $g_{\mu\nu}$ is the metric tensor, the function that allows for measurements of distance in space-time.

The Einstein tensor can be written as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu}$$

Where

- $R_{\mu\nu} = R^{\alpha}{}_{\nu\alpha\beta}$is the Ricci curvature tensor, which roughly represents how much the curved space-time deviates from flat space-time.

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu\,\Gamma^{\rho}_{\nu\sigma} - \partial_\nu\,\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\,\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\,\Gamma^{\lambda}_{\mu\sigma}. \tag{8}$$

- $R = R^{\alpha}{}_{\alpha}$is the Ricci scaler, the contraction of the Ricci tensor, a measre of curvature.

Since we are working outside the black hole, in a vacuum, the stress-energy tensor $T_{\mu\nu} = 0$, and since we are not working on cosmological scales we can approximate Λ=0. Leaving us with the equation

$$R_{\mu\nu} = 0$$

Solving for space-time metric

In[∘]:=
```
?TNewMetric
```

Out[∘]=

Symbol

TNewMetric[**metricID**, **coordinatesID**, **components**, **symbol**} creates a new tensor object representing a metric.

metricID is a string that will be used to identify the new object, and must be unique.

coordinatesID is the unique ID of a tensor object representing a coordinate system, created using TNewCoordinates[].

components is a square, symmetric, and invertible matrix representing the metric with two lower indices in that coordinate system.

symbol will be used to represent the metric in formulas. If not given, "g" will be used.

In[∘]:=
```
TSetReservedSymbols[{t, r, θ, ϕ}];
TNewCoordinates["Spherical", {t, r, θ, ϕ}];
TNewMetric["TrialSchwarzschild", "Spherical", DiagonalMatrix[{-A[r], B[r], r^2, r^2 Sin[θ]^2}]]
```

TMessage: A tensor with the ID "Spherical" already exists. Please rename it using TChangeID[] or delete it using TDelete[] first. Type TSetAllowOverwrite[True] to allow overwriting tensors.

Out[∘]= $Aborted

Out[∘]= TrialSchwarzschild

In[∘]:=
```
TShow["TrialSchwarzschild"]
```

OGRe: TrialSchwarzschild: $g_{\mu\nu}(t, r, \theta, \phi) = \begin{pmatrix} -A[r] & 0 & 0 & 0 \\ 0 & B[r] & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$

In[∘]:=
```
TList@TCalcChristoffel["TrialSchwarzschild"]
```

OGRe: TrialSchwarzschildChristoffel:

$\Gamma^{t}{}_{tr} = \Gamma^{t}{}_{rt} = \frac{\partial_r A[r]}{2A[r]}$

$\Gamma^{r}{}_{tt} = \frac{\partial_r A[r]}{2B[r]}$

$\Gamma^{r}{}_{rr} = \frac{\partial_r B[r]}{2B[r]}$

$\Gamma^{r}{}_{\theta\theta} = -\frac{r}{B[r]}$

$\Gamma^{r}{}_{\phi\phi} = -\frac{r\,\mathrm{Sin}[\theta]^2}{B[r]}$

$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{1}{r}$

$\Gamma^{\theta}{}_{\phi\phi} = -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$

$\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \mathrm{Cot}[\theta]$

In[∘]:= `TList@TCalcRicciTensor["TrialSchwarzschild"]`

TrialSchwarzschildRicciTensor:

OGRe:

$$R_{tt} = \frac{\frac{\partial_r A[r]}{r} - \frac{\partial_r A[r]^2}{4A[r]} + \frac{1}{2}\partial_{r^2}A[r]}{B[r]} - \frac{\partial_r A[r]\,\partial_r B[r]}{4B[r]^2}$$

$$R_{rr} = \frac{r B[r]\left(\partial_r A[r]^2 - 2A[r]\,\partial_{r^2}A[r]\right) + A[r]\,(4A[r] + r\,\partial_r A[r])\,\partial_r B[r]}{4 r A[r]^2 B[r]}$$

$$R_{\theta\theta} = \frac{1}{2}\left(2 - \frac{2 + \frac{r\,\partial_r A[r]}{A[r]}}{B[r]} + \frac{r\,\partial_r B[r]}{B[r]^2}\right)$$

$$R_{\phi\phi} = \frac{\mathrm{Sin}[\theta]^2\,(-r B[r]\,\partial_r A[r] + A[r]\,(2\,(-1+B[r])\,B[r] + r\,\partial_r B[r]))}{2A[r]\,B[r]^2}$$

In[∘]:= `TList@TCalcRicciScalar["TrialSchwarzschild"]`

TrialSchwarzschildRicciScalar:

OGRe:

$$R = \frac{1}{2r^2 A[r]^2 B[r]^2}\left(r^2 B[r]\,\partial_r A[r]^2 + 4A[r]^2\,((-1+B[r])\,B[r] + r\,\partial_r B[r]) + r A[r]\,(-2 r B[r]\,\partial_{r^2}A[r] + \partial_r A[r]\,(-4B[r] + r\,\partial_r B[r]))\right)$$

In[∘]:= `TList@TCalc["TrialSchwarzschildRicciTensor"["μν"] - 1/2 "TrialSchwarzschild"["μν"] . "TrialSchwarzschildRicciScalar"[""], "G"]`

Result:

OGRe:

$$G_{tt} = \frac{A[r]\,((-1+B[r])\,B[r] + r\,\partial_r B[r])}{r^2 B[r]^2}$$

$$G_{rr} = \frac{A[r] - A[r]\times B[r] + r\,\partial_r A[r]}{r^2 A[r]}$$

$$G_{\theta\theta} = \frac{r\left(-r B[r]\,\partial_r A[r]^2 - 2A[r]^2\,\partial_r B[r] + A[r]\,(2B[r]\,(\partial_r A[r] + r\,\partial_{r^2}A[r]) - r\,\partial_r A[r]\,\partial_r B[r])\right)}{4A[r]^2 B[r]^2}$$

$$G_{\phi\phi} = \frac{1}{4A[r]^2 B[r]^2}\, r\,\mathrm{Sin}[\theta]^2\left(-r B[r]\,\partial_r A[r]^2 - 2A[r]^2\,\partial_r B[r] + A[r]\,(2B[r]\,(\partial_r A[r] + r\,\partial_{r^2}A[r]) - r\,\partial_r A[r]\,\partial_r B[r])\right)$$

`(*Alternatively, we could have done*)`

In[∘]:= `TGetComponents@TCalcEinsteinTensor["TrialSchwarzschild"]`

TGetComponents: Using the default index configuration {−1, −1} and the default coordinate system "Spherical".

Out[∘]=

$$\left\{\left\{\frac{A[r]\,((-1+B[r])\,B[r] + r B'[r])}{r^2 B[r]^2}, 0, 0, 0\right\}, \left\{0, \frac{A[r] - A[r]\times B[r] + r A'[r]}{r^2 A[r]}, 0, 0\right\}, \left\{0, 0, \frac{1}{4A[r]^2 B[r]^2}\, r\left(-r B[r]\,A'[r]^2 - 2A[r]^2 B'[r] + A[r]\,(-r A'[r]\,B'[r] + 2B[r]\,(A'[r] + r A''[r]))\right), 0\right\}, \left\{0, 0, 0, \frac{1}{4A[r]^2 B[r]^2}\, r\,\mathrm{Sin}[\theta]^2\left(-r B[r]\,A'[r]^2 - 2A[r]^2 B'[r] + A[r]\,(-r A'[r]\,B'[r] + 2B[r]\,(A'[r] + r A''[r]))\right)\right\}\right\}$$

In[∘]:= `system = %`

Out[∘]=

$$\left\{\left\{\frac{A[r]\,((-1+B[r])\,B[r] + r B'[r])}{r^2 B[r]^2}, 0, 0, 0\right\}, \left\{0, \frac{A[r] - A[r]\times B[r] + r A'[r]}{r^2 A[r]}, 0, 0\right\}, \left\{0, 0, \frac{1}{4A[r]^2 B[r]^2}\, r\left(-r B[r]\,A'[r]^2 - 2A[r]^2 B'[r] + A[r]\,(-r A'[r]\,B'[r] + 2B[r]\,(A'[r] + r A''[r]))\right), 0\right\}, \left\{0, 0, 0, \frac{1}{4A[r]^2 B[r]^2}\, r\,\mathrm{Sin}[\theta]^2\left(-r B[r]\,A'[r]^2 - 2A[r]^2 B'[r] + A[r]\,(-r A'[r]\,B'[r] + 2B[r]\,(A'[r] + r A''[r]))\right)\right\}\right\}$$

In[ ]:= `system[[1]][[1]]`

Out[ ]= $$\frac{A[r]\,((-1+B[r])\,B[r]+r\,B'[r])}{r^2\,B[r]^2}$$

In[ ]:= `system[[2]][[2]]`

Out[ ]= $$\frac{A[r]-A[r]\times B[r]+r\,A'[r]}{r^2\,A[r]}$$

We can solve the first two equations in the system (the equations come from setting $G_{\mu\nu} = 0$. Giving us

$$r\,B' + B\,(B-1) = 0 \Rightarrow \frac{B'}{B\,(1-B)} = \frac{A}{r} \tag{9}$$

$$rA' + A\,(1-B) = 0 \Rightarrow \frac{A'}{A} = \frac{B-1}{r} \tag{10}$$

Plugging $1 - B = -\mathrm{rA}'/A$ (from equation (10)) into equation (9) gives us

$$\frac{B'}{B\,(-rA'/A)} = \frac{A}{r} \Rightarrow \frac{B'}{B} = -A' \Rightarrow (\ln B)' = -A' \Rightarrow \frac{1}{B} = -A + k \tag{11}$$

Or using mathematica DSolve

In[ ]:= `DSolve[{system[[1]][[1]] == 0, system[[2]][[2]] == 0}, {A[r], B[r]}, r]`

Out[ ]= $$\left\{\left\{B[r] \to \frac{r}{e^{c_1}+r},\ A[r] \to \frac{(e^{c_1}+r)\,c_2}{r}\right\}\right\}$$

Giving the final result (by writing $c_2=\alpha$, e^{c_1}=k)

$$A\,(r) = \alpha\left(1+\frac{k}{r}\right)$$

$$B\,(r) = \left(1+\frac{k}{r}\right)^{-1} \tag{12}$$

$$ds^2 = -\alpha\left(1+\frac{k}{r}\right)dt^2 + \left(1+\frac{k}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right)$$

We can determine the constants α and k by considering the limiting behavior of the line element.

As $r\to\infty$, we get $ds^2 = -\alpha\,dt^2 + dr^2 + r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right)$, which should match flat minkowski space-time, which it indeed does if $\alpha = c^2$=1.

To determine k, recall that the metric in the week field limit has Subscript[g, tt] $= 1 - 2\,\mathrm{GM}/r = 1 + \frac{k}{r} \Rightarrow k\,2\,\mathrm{GM}$. This gives us the final metric

$$ds^2 = -\left(1+\frac{2\,GM}{r}\right)dt^2 + \left(1+\frac{2\,GM}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right) \tag{13}$$