Abstract

B.A. Davey and H.A. Priestley: "Introduction to Lattices and Order", Cambridge UP, 2nd edition.

1 Chapter 1 "Ordered Sets"

1.1 Exercises from the text

Assertion 1.27.1. *$\downarrow Q$ is the smallest down-set containing Q.*

Proof. Assume Q' is a down-set with $Q \subseteq Q' \subset \downarrow Q$. Then $\exists z \in \downarrow Q$ s.t. $z \notin Q'$, and $\exists x \in Q$ s.t. $z \leq x$. Contradiction with assumption that Q' is a down-set. □

Assertion 1.27.2. *Q is a down-set if and only if $Q = \downarrow Q$.*

Proof. Definition of $\downarrow Q$ is logically equivalent to def. of down-set (?). □

Lemma 1.30. *Let P be an ordered set and $x, y \in P$. Then the following are equivalent:*

1. $x \leq y$
2. $\downarrow x \subseteq \downarrow y$
3. $(\forall Q \in \mathcal{O}(P)) y \in Q \Longrightarrow x \in Q$

Proof. *(1 ⇔ 2):* Assume $x \leq y$ and let $x' \in \downarrow x$. Then $x' \leq x$, and by transitivity of $\leq$, $x' \leq y$. Then $x' \in \downarrow y$.

(1 ⇔ 3): Assume $x \leq y$ and let $Q \in \mathcal{O}(P)$ s.t. $y \in Q$. Since $\downarrow y$ is the smallest down-set containing y, $\downarrow y \subseteq Q$. $x \in \downarrow x$, and by the above $\downarrow x \subseteq \downarrow y \subseteq Q$, so $x \in Q$. □

Assertion 1.31. *Q is a down-set of P if and only if $P \setminus Q$ is an up-set of P*

Proof. *(⇒)* Let Q down-set of P and let $Q' = P \setminus Q$. Let $x \in Q'$, $y \in P$ and $x \leq y$. Either $y \in Q$ or $y \in Q'$. Assume $y \in Q$. Then, by Def. of down-set, $x \in Q$, which contradicts the assumption that $x \in Q'$. So $y \in Q'$ and Q' is a down-set. *(⇐)* Analogous □

Assertion 1.36 (1). *Let $\varphi : P \to Q$ and $\psi : Q \to R$ be order-preserving maps. Then the composite map $\psi \circ \varphi$, given by $(\psi \circ \varphi)(x) = \psi(\varphi(x))$ for $x \in P$, is order-preserving. More generally the composite of a finite number of order-preserving maps is order-preserving.*

Proof. Let $x, y \in P$ s.t. $x \leq y$.

$$\begin{aligned} x \leq y &\Rightarrow \varphi(x) \leq \varphi(y) && \{\varphi \text{ is order-preserving }\} \\ &\Rightarrow \psi(\varphi(x)) \leq \psi(\varphi(y)) && \{\psi \text{ is order-preserving }\} \end{aligned}$$

□

Assertion 1.36 (2). *Let* $\psi : P \hookrightarrow Q$ *and let* $\psi(P)$ *(defined to be* $\{\psi(x) \mid x \in P\}$*) be the image of* ψ*. Then* $\psi(P) \cong P$*. This justifies the use of the term embedding.*

Proof. Let $x, y \in P$ s.t. $x \leq y$.

$$x \leq y \Leftrightarrow \psi(x) \leq \psi(y)$$

$\square$