Chapter 1

Fourier Analysis

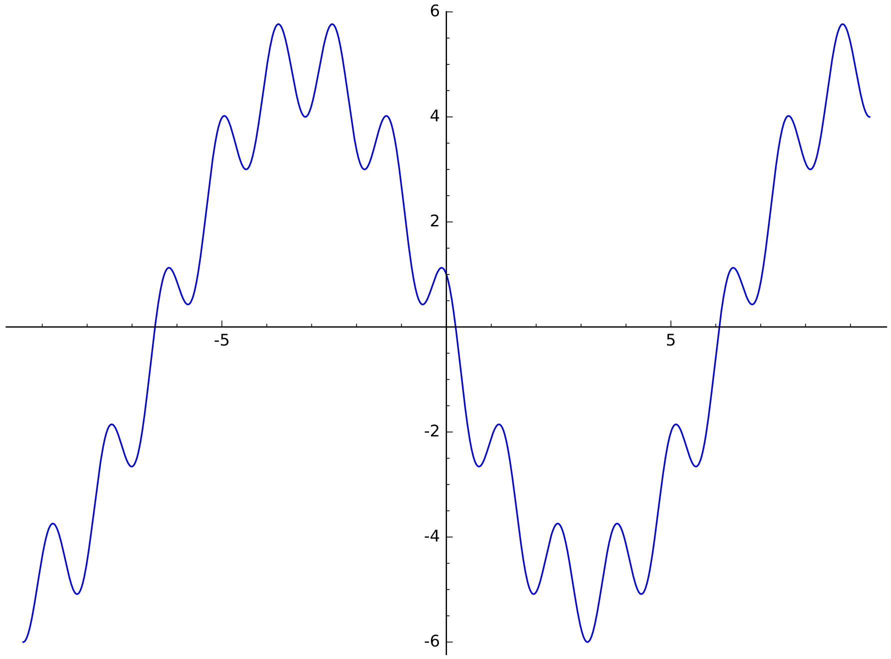
Our goal now is to move to solving *partial* differential equations (PDEs), which are differential equations that involve multiple independent variables. Such equations are particularly useful for modeling quantities that depend on position x and time t. It turns out that these partial differential equations often involve periodic functions. So our first step to solving PDEs will be to find useful descriptions of periodic (and in some cases, non-periodic) functions.

1.1 Fourier Series

The main idea behind Fourier series, and the field of harmonic analysis in general, is to represent more complicated objects in terms of simpler objects. A fundamental example of this idea comes from the field of linear algebra in the form of *orthonormal bases*. Knowing an orthonormal basis for a vector space V can greatly simplify linear algebra in that vector space. In this section, we'll do something similar with periodic functions.

1.1.1 Periodic Functions

Consider the function $f(x)$ given by the following graph:



A

periodic function.

This function can be plotted using the following code:

```
reset

import matplotlib.pyplot as plt
import numpy as np

# The following is here to adjust text on the graph... not
    a huge deal.
from matplotlib import rc
rc('font',**{'family':'sans-serif','sans-serif':['Helvetica']})
rc('text', usetex=True)

# This is setting up our graph.
x = np.linspace(-3*np.pi,3*np.pi,200)
y1 = np.cos(5*x)
y2 = np.sin(.5*x)

# This actually plots the graph.
p = plt.plot(x,y1-5*y2,label=r'$f(x)$')
plt.legend()
```

If we look at the graph, we see that it repeats itself if we wait long enough. Functions that have this property are called **periodic functions**.

Definition 1.1.1 (Periodic Functions)**.** Let $f(x)$ be a real function defined for all x. We say that f is a periodic function if there exists a positive number p such that

$$f(x+p) = f(x)$$

for all x.

Let n be any integer. Then the functions $\sin nx$ and $\cos nx$ are both periodic and have period 2π since

$$\sin[n(x+2\pi)] = \sin nx \cos 2\pi + \cos nx \sin 2\pi = \sin nx$$
$$\cos[n(x+2\pi)] = \cos nx \cos 2\pi - \sin nx \sin 2\pi = \cos nx$$

The periodic nature of these functions can also be seen from their graphs:

```
# This is setting up our graph.
x = np.linspace(-3*np.pi,3*np.pi,200)
y1 = np.cos(5*x)
y2 = np.sin(2*x)

# This actually plots the graphs.
plt.subplot(2,1,1)
plt.plot(x,y1,label=r'$\cos5x$')
plt.legend()

plt.subplot(2,1,2)
plt.plot(x,y2,label=r'$\sin2x$')
plt.legend()
plt.show()
```

If you go back to the first plot, you may notice from the code used to generate the function $f(x)$ that $f(x)$ is just the sum of two basic trigonometric functions:

$$f(x) = \cos 5x - 5\sin\frac{x}{2}.$$

In other words, the (finite) sum of functions of the form $\sin nx, \cos mx$ where n, m are real numbers is also periodic, and also has period 2π. One of the greatest accomplishments in mathematics was the realization that many other periodic functions can be written in this way, if we allow *infinite* sums of this form, which we call **trigonometric series**.

1.1.2 Trigonometric Series and Fourier Series

Definition 1.1.2 (Trigonometric Series)**.** A trigonometric series is a series of the form

$$\sum_{k=0}^{\infty}(a_k \cos kx + b_k \sin kx) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

where a_k, b_k are constants, called the **coefficients** of the series.

Our primary goal in this section is to take a function $f(x)$ of period 2π and express it as a trigonometric series. To see how, we'll suppose that we have the trigonometric series we want, i.e. that

$$f(x) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx),$$

and we'll look at what the coefficients of the series need to be to make this equation true. To do this, we'll need the so-called **orthogonality relations** for $\sin nx, \cos mx$

Theorem 1.1.3 (Orthogonality Relations)**.** *Let m, n be whole numbers with $m, n > 0$. Then*

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$

and

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} \pi & m = n \\ 0 & m \neq n \end{cases} \quad \text{and} \quad \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} \pi & m = n \\ 0 & m \neq n \end{cases}$$

We can verify Theorem 1.1.3 using a computer algebra system. Proving it is a little bit more work, but can be done using trigonometric identities or Euler's formula.

```
# Declare variables.
var('x,m,n')
assume(m,n,'integer')

# Perform the integrations.
I1 = integral(sin(m*x)*cos(n*x),x,-pi,pi)
I2 = integral(sin(m*x)*sin(n*x),x,-pi,pi)
I3 = integral(cos(m*x)*cos(n*x),x,-pi,pi)
I4 = integral(sin(m*x)*sin(m*x),x,-pi,pi)
I5 = integral(cos(n*x)*cos(n*x),x,-pi,pi)

# List the results.
I1,I2,I3,I4,I5
```

Theorem 1.1.3 will be our primary tool for expressing a function $f(x)$ as a trigonometric series. To see how, suppose that we have

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

If this equation were true, then we should be able to integrate both sides of it and get another true equation. Since Theorem 1.1.3 suggests that integrals involving $\sin nx, \cos nx$ simplify very nicely, we'll try to integrate both sides of the equation against $\sin nx, \cos nx$ from $x = -\pi$ to $x = \pi$ for some $n > 0$. If we do this, we get

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \sin nx \, dx &= a_0 \int_{-\pi}^{\pi} \sin nx \, dx + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos kx \sin nx \, dx + b_k \int_{-\pi}^{\pi} \sin kx \sin nx \, dx \right) \\ &= b_n \int_{-\pi}^{\pi} \sin nx \sin nx \, dx \\ &= \pi b_n \end{aligned}$$

This lets us solve for b_n! We have

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \text{ for } n \geq 1.$$

Similarly,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \text{ for } n \geq 1. \\ a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx \end{aligned}$$

Example 1.1.4 (The Fourier series of x^3)**.** Define $f(x) = x^3$ for $-\pi \leq x \leq \pi$. To find its Fourier series, we can just use the previous formulas to find the values of the coefficients a_0, a_k, b_k for $k \geq 1$. We know that

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^3 \, dx \\
a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^3 \cos kx \, dx \\
b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^3 \sin kx \, dx.
\end{aligned}
$$

As nasty as these are, the first two are actually very easy to compute. Here's why: x^3 and $x^3 \cos kx$ are both *odd* functions, and the integral of any odd function in an interval that is symmetric about 0 is always 0 (since the areas cancel out). So $a_0 = a_k = 0$ for all $k \geq 0$. The last term is a bit more complicated, but we can use integration by parts (and I definitely recommend using a computer here) to show that

$$\int x^3 \sin kx \, dx = -\frac{\left(k^3x^3 - 6\,kx\right)\cos(kx) - 3\left(k^2x^2 - 2\right)\sin(kx)}{k^4} + C.$$

If we plug in the limits of integration and simplify (again, computers are handy for this!), we get

$$b_k = \frac{2\left(6\,\pi - \pi^3 k^2\right)(-1)^k}{\pi k^3}.$$

So the Fourier series for $f(x) = x^3$ is given by

$$
\begin{aligned}
a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) &= \sum_{k=1}^{\infty} b_k \sin kx \\
&= \sum_{k=1}^{\infty} \frac{2\left(6\,\pi - \pi^3 k^2\right)(-1)^k}{\pi k^3} \sin kx.
\end{aligned}
$$

A very good question at this point is, what relationship does the Fourier series that we found in the previous example have with the original function $f(x)$? Are they actually equal? If we use the following code (adapted from here) to compare the partial sums

$$\sum_{k=1}^{n} \frac{2\left(6\,\pi - \pi^3 k^2\right)(-1)^k}{\pi k^3} \sin kx$$

of the Fourier series with $f(x) = x^3$, then it looks like the partial sums get closer and closer if we choose larger values of n.

```
var('x,k,i')

# Defines the function to determine Fourier series of.
    Feel free to play around with this.
f(x) = x**3

# Defines the Fourier coefficients.
def a(k):
    coeff = integral(f(x)*cos(k*x), (x,-pi,pi))/pi
    return coeff

def b(k):
    coeff = integral(f(x)*sin(k*x), (x,-pi,pi))/pi
```

```
    return coeff

# Defines the corresponding (partial) Fourier series.
def FS(k):
    return a(0)/2 + sum(a(i)*cos(i*x) + b(i)*sin(i*x), i,
        1, k)

n = 3
p = plot(FS(n),(x,-pi,pi), color='red') +
    plot(f(x),(x,-pi,pi))
p
```

In general, the question of whether or not a given Fourier series makes sense is a difficult one to answer. However, for many of the functions we care about in this course we have the following theorem.

Theorem 1.1.5 (Fourier Series of Piecewise Continuous Functions)**.** *Let $f(x)$ be a piecewise continuous function on the interval $-\pi \leq x \leq \pi$, and suppose that it's also periodic with period 2π, and is differentiable everywhere that it's continuous. Then the Fourier series of $f(x)$ converges to $f(x)$ except at the points where $f(x)$ is continuous.*

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