Differential Equations Lecture Notes

West Virginia Wesleyan College

Differential Equations Lecture Notes

West Virginia Wesleyan College

Jesse Oldroyd
West Virginia Wesleyan College

October 16, 2023

Go Seahawks.

Edition: Annual Edition 2023

Website: j-oldroyd.github.io/wvwc-differential-equations

Preface

This document was created to serve as lecture notes for the Differential Equations course at West Virginia Wesleyan College. These notes are divided into two parts.

- The first part, Ordinary Differential Equations, introduces differential equations in one variable along with methods for their solution and several applications.
- The next part, Partial Differential Equations, introduces Fourier series and differential equations in several variables, building up to solving the heat and wave equations.

You can find a PDF version of these notes here[1].

This document is very much *in progress* and therefore typos and other errors are to be expected. If you find any, I would appreciate you letting me know by contacting me by email.

[1] j-oldroyd.github.io/wvwc-differential-equations/output/print/wvwc-differential-equations.pdf

Contents

Part I

Ordinary Differential Equations

Chapter 1

Introduction to Ordinary Differential Equations

In the sciences, many quantities of interest are varying quantities. Some changing quantities, like the position of a baseball dropped from a height of 100 ft, are relatively simple to determine, but others are more complicated. Think of a wildlife population, which varies from season to season and year to year. How can one accurately estimate the size of such a population?

The field of **differential equations** attempts to answer such questions by providing a model for the quantity of interest using information about its rate of change, which is sometimes easier to obtain than information about the quantity itself. Once we know enough about how a quantity changes, we hope to make a decent estimate for how the quantity actually behaves. There are actually two different types of differential equations that we will look at in this course. The first is the ordinary differential equation.

1.1 Ordinary Differential Equations

This section introduces basic concepts from the field of differential equations.

Basic Concepts

First, a definition of the primary concept in this course: the differential equation.

Definition 1.1.1 Differential Equations.

A **differential equation** is an equation relating some function with its derivatives. A differential equation that involves a function of only one independent variable is called an **ordinary differential equation**, or ODE. A differential equation that involves a function of more than one independent variable (which you see a lot of in Calculus 3) is called a **partial differential equation**, or PDE, and will be studied in more detail in Chapter **??**. The **order** of a differential equation is the highest derivative that appears in the equation.

Examples of ODEs:

- $\dfrac{d^2y}{dx^2} + y = 0$; this is a second order ODE relating the unknown function y with its second derivative.
- $5t^2x^{'''} - e^x = 3t$; this is a seventh order ODE involving the derivatives of the unknown function x. Note that in this ODE t is the independent variable whereas x serves as the dependent variable.

Just as with equations in algebra, we can sometimes solve a differential equation.

Definition 1.1.2 Solution of a Differential Equation.

A function is a **solution** of a differential equation if it satisfies the differential equation.

It is straightforward to check if a function is a solution of some given differential equation, but *finding* solutions will make up the bulk of this course.

Example 1.1.3 Verifying solutions.

Is $y = 5e^{x^2}$ a solution of the ODE $y' = 2xy$?

Solution. At this point we don't know how to solve differential equations, but that doesn't mean we can't *check* solutions of differential equations. To do so, we just plug $5e^{x^2}$ wherever y shows up in the ODE and see if the resulting equation is true. So we have

$$\begin{aligned} y' &= 2xy \\ (5e^{x^2})' &= 2x(5e^{x^2}) \text{ after substituting } y = 5e^{x^2} \\ 10xe^{x^2} &= 10xe^{x^2} \text{ after simplifying} \end{aligned}$$

This is a true statement, so $y = 5e^{x^2}$ satisfies the ODE. Hence $5e^{x^2}$ is a solution of the ODE.

In Example 1.1.3, $y = 5e^{x^2}$ is not the only solution of $y' = 2xy$. You can check that $y = 3e^{x^2}$ and $y = -10e^{x^2}$ are also solutions. In fact, *any* function of the form Ce^{x^2} where C is a constant is a solution of $y' = 2xy$. See Figure 1.1.4.

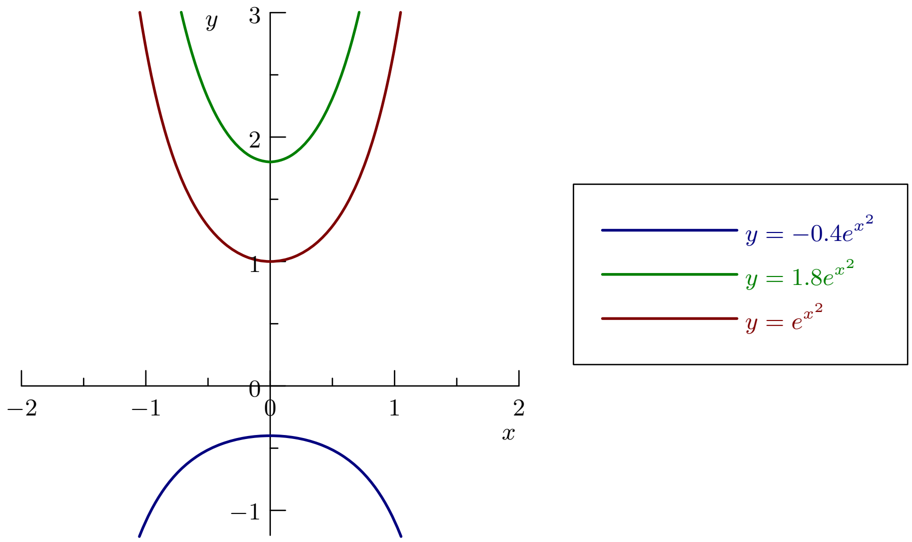

Figure 1.1.4 A family of solutions of $y' = 2xy$.

Solutions of ODEs that depend on arbitrary constants, such as $y = Ce^{x^2}$ above, are called **general solutions**. Solutions of ODEs that do not depend on arbitrary constants, such as $y = 5e^{x^2}$, are called **particular solutions**.

Example 1.1.5 A trigonometric solution.

Show that $x = A\cos(3t) + B\sin(3t)$, where A and B are arbitrary constants, is a general solution of $x'' + 9x = 0$. Then find a particular solution.

Solution. To check that $x = A\cos(3t)+B\sin(3t)$ is a general solution of $x'' + 9x = 0$, we just need to plug it into the ODE and show that it satisfies it. Since

$$x'' = -9(A\cos(3t) + B\sin(3t)) = -9x,$$

this follows very quickly.
To find a particular solution, all we need to do in this case is to pick specific values for A and B (any values will work here). So one particular solution of $x'' + 9x = 0$ is given by $x = \frac{3}{2}\cos(3t) - \sin(3t)$, among infinitely many others.

In Example 1.1.5, to find a particular solution we just needed to plug in specific values for the arbitrary constants. In general, the particular solutions we'll be interested in are chosen to satisfy a given condition, which we call an **initial condition**. These conditions often take the form $y(x_0) = y_0$. Geometrically, we are picking a specific point (x_0, y_0) in the xy-plane that the solution must pass through.

An ODE together with an initial condition is called an **initial value problem (IVP)**. Although ODEs by themselves typically have infinitely many solutions, specifying an initial condition that the solution must satisfy is often enough to get a unique particular solution instead of a general solution.

Example 1.1.6 Solving an IVP.

Solve the IVP $\frac{dy}{dx} = -xe^{-x^2}, y(0) = 1$.

Solution. We need to find a function y that satisfies two different constraints: $y' = -xe^{-x^2}$ and $y(0) = 1$. We'll start with the first one, which we actually know how to do from Calculus I. If $y' = -xe^{-x^2}$, then

$$y = \int -xe^{-x^2}\,dx = \frac{1}{2}e^{-x^2} + C.$$

Now we need to make sure that y is equal to 1 if $x = 0$. We do this by setting $y = 1$, $x = 0$ and choosing the right value for C to make the resulting equation true:

$$1 = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}.$$

So the solution of this IVP is the function $y = \frac{e^{-x^2}+1}{2}$.

Two important things to keep in mind before we move to the next topic:

1. ODEs by themselves have general solutions, whereas IVPs have particular solutions.

2. When solving IVPs, it's important to keep track of any arbitrary constants that appear. Neglecting arbitrary constants usually makes it impossible to find the right particular solution.

Mathematical Models

Differential equations are useful because they can provide a mathematical model of a physical quantity. Analyzing the model allows us to infer something meaningful about the quantity in question. A relatively simple model comes from **Newton's Law of Cooling**, which relates the temperature of an object with the temperature of the surrounding medium (such as air or water). In particular, Newton's Law of Cooling states that the time rate of change of the temperature of an object is proportional to the difference of the temperature of the object with that of the temperature of the surrounding medium.

Example 1.1.7 Newton's Law of Cooling.

Restate Newton's Law of Cooling as a differential equation.

Solution. It may not be obvious that Newton's Law of Cooling can be restated as a differential equation, but the phrase "rate of change" that appears in the statement of the law is a good clue that this can be done. To do this, first we need to give the relevant quantities (the temperature of the object and the temperature of the surrounding medium) names. Let $T(t)$ denote the temperature of the object at time t and let $A(t)$ denote the temperature of the surrounding medium at time t. Then Newton's Law of Cooling says that

$$\frac{dT}{dt} = k(T - A)$$

where k is some constant.

Although we can't determine k precisely (this would require experimentation and depends on the object and medium in question), we can still say something useful about it. In particular, k must be negative. To see why, consider what the object does if $T > A$ and $T < A$. If $T > A$, then the object must be cooling since the surrounding medium is cooler than the object. If the object is cooling, then $\frac{dT}{dt} < 0$. On the other hand, if $T < A$ then $\frac{dT}{dt} > 0$ since the object would be heating up in this case. The only way for this to occur is if $k < 0$.

Most of the mathematical models we'll look at will take the form of an IVP.

Example 1.1.8 An IVP modeling a falling object.

A ball weighing 0.5 kg is dropped from a height of 100 m and is acted upon by gravity and air resistance. Assuming that the force of air resistance is proportional to the velocity of the ball, what is an IVP that models the movement of the ball?

Solution. What we need to do is to translate this physical situation into mathematics, and to do that we need to start assigning names to the quantities of interest. The quantities that matter in this problem are the movement of the ball, the force of gravity and the force of air resistance. We'll name them as follows:

$$\text{height of the ball } t \text{ seconds after its released} = y(t)$$
$$\text{force of gravity} = F_G = mg = -4.9$$
$$\text{force of air resistance} = F_R = k\frac{dy}{dt}$$

where C is negative (since air resistance should act *against* velocity). To get a differential equation out of all this, we'll use Newton's Second Law (which is actually a second-order ODE in disguise). This says that the net force on the ball should be equal to its mass times acceleration: $F = ma$. So we have

$$F_G + F_R = 0.5\frac{d^2y}{dt^2}$$

or just

$$-4.9 + k\frac{dy}{dt} = 0.5\frac{d^2y}{dt^2}.$$

The initial condition in this case would be $y(0) = 100$. So our IVP is

$$-4.9 + k\frac{dy}{dt} = 0.5\frac{d^2y}{dt^2}, y(0) = 100.$$

One quick note about the IVP in Example 1.1.8 The differential equation was second-order, but there was only one corresponding initial condition. As we'll see later, this is not enough to find a unique solution to this IVP. For most IVPs we'll solve, we'll need as many initial conditions as the order of the ODE. Something to look forward to.

Now that we have a rough idea of what an ODE and an IVP actually are, we can move on to solving them. In the next section, we'll look at a method that we can use to visualize ODEs and their solutions and another method that can be used to approximate solutions of ODEs.

1.2 Direction Fields

This section corresponds to Section 1.2 from the textbook.

Direction Fields and Solution Curves

Suppose we wanted to solve the ODE $\frac{dy}{dx} = x$. Then we can do so using the tools already available to us, since the mystery function $y = y(x)$ must have derivative given by x. So

$$y = \int x\, dx = \frac{1}{2}x^2 + C.$$

Any choice for C yields another solution of the ODE $\frac{dy}{dx} = x$.

> Again, it's very important in this course not to forget the arbitrary constant C!

Now let's make things a little more interesting. Suppose we wanted to solve the ODE $\frac{dy}{dx} = e^{-x^2}$. Then this is *impossible* to do in a single "closed-form".

> By closed-form we basically mean in terms of the usual exponential, trigonometric and polynomial functions, as well as their inverses.

This is because solving this ODE requires integrating e^{-x^2}, which as you may remember from Calculus 2 cannot be done without resorting to something like power series. Even if we can't solve the ODE, or if we can't solve it easily, we still want to be able to obtain some information from it. One way to do this is by using **direction fields**, which is a graphical representation of the ODE.

To construct a direction field for an ODE of the form

$$\frac{dy}{dx} = f(x, y)$$

perform the following steps:

1. Pick a point (a, b) in the xy-plane.
2. Plug the point (a, b) into $f(x, y)$ to obtain the number $f(a, b)$.
3. Plot a short line segment with slope $f(a, b)$ at the point (a, b).
4. Repeat at several other points in the xy-plane until you develop a satisfactory picture of the behavior of the ODE.

The resulting graph is called the direction field for the ODE.

> Direction fields are often called slope fields.

Example 1.2.1 Plotting a direction field by hand.

Fill in the direction field for $y' = t - y$ at the indicated points:

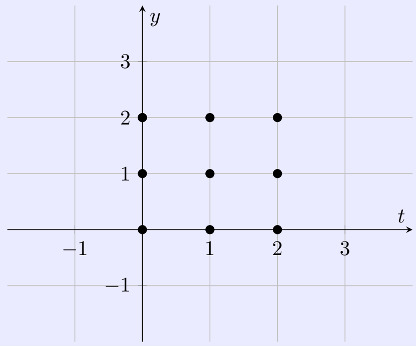

Figure 1.2.2 Graph for hand plot of direction field.

Solution. To plot the direction field, remember that we're basically plotting *slopes*. So we first need to figure out $y' = t - y$ at the indicated points. The following table lists values for y' at some of these points:

Table 1.2.3 Slopes of $t - y$

(t, y)	$y' = t - y$
$(0, 0)$	$0 - 0 = 0$
$(2, 2)$	$2 - 2 = 0$
$(1, 2)$	$1 - 2 = -1$
$(0, 1)$	$0 - 1 = -1$

If we fill out the remaining values of y' and plot the corresponding slopes, we should get something like this:

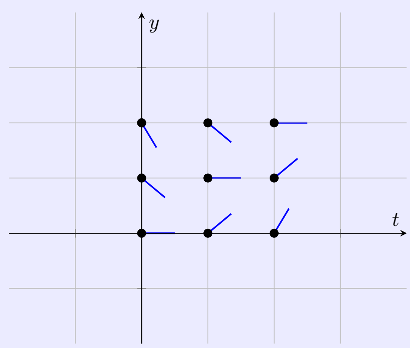

Figure 1.2.4 Direction field of $y' = t - y$

Example 1.2.5 Plotting a direction field with a CAS.

Plot the direction field for the differential equation $x' = x(1-x)$, where $x = x(t)$.

Solution. We can easily do this with a computer system (such as Sage!). For example, try running the cell below this example. If we do so, we get something like the following diagram:

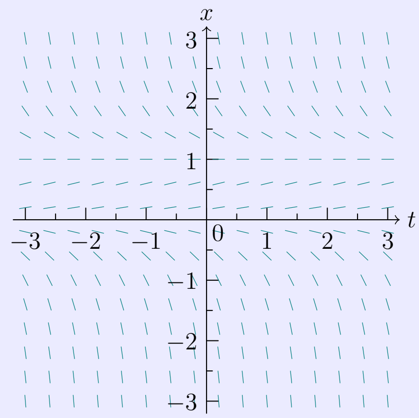

Figure 1.2.6 Direction field for $x' = x(1-x)$

```
var('t,␣x')    # tells Sage what variables we're using
f = x * (1 - x)
plot_slope_field(f, (t, -10, 10), (x, -5, 5), color = "blue")
```

Direction fields are useful because they provide a means to obtain information about a differential equation (and the corresponding model) without actually having to solve the differential equation. One way to do so is to create a *streamline plot*. This can be done easily in Sage, like so:

```
var('t,␣x')
f = x * (1 - x)
streamline_plot(f, (t, -5, 5), (x, -1, 2))
```

This can also be created by hand from a slope field without too much trouble.

If we only graph a single curve in the direction field we get what's known as a **solution curve**, which represents a solution of an initial value problem corresponding to the ODE the direction field is drawn from.

Example 1.2.7 Information from a solution curve.

Let $x(t)$ represent the solution of the initial value problem

$$\begin{aligned} x' &= x(1-x) \\ x(0) &= \frac{1}{2}. \end{aligned}$$

Determine $\lim_{t\to\infty} x(t)$.

Solution. Since we don't know how to solve this IVP yet, we'll make use of the direction field from Example 1.2.5 to find an approximate solution curve. Since the initial condition is $x(0) = \frac{1}{2}$, this means that the solution must pass through the point $(0, \frac{1}{2})$. So if we start at this point and trace a curve that flows with the direction field, we get the following solution curve:

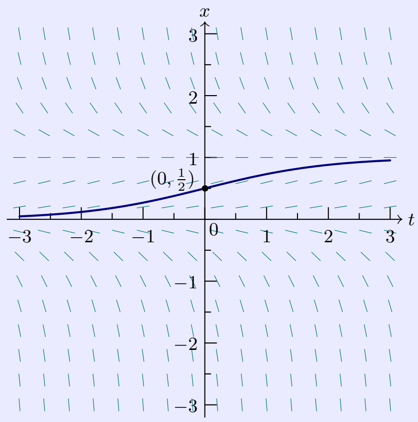

Figure 1.2.8 The solution curve corresponding to the initial condition $x(0) = \frac{1}{2}$.

So it appears that $\lim_{t\to\infty} x(t) = 1$.

1.3 Separable ODEs and Substitution

This section corresponds to Section 1.3 from the text.

Separable ODEs

The simplest ODEs to solve are the first-order ODEs of the form $\dfrac{dy}{dx} = f(x)$. The Fundamental Theorem of Calculus guarantees that the solution y is given by $y = \int f(x)\,dx$.

Rather, the Fundamental Theorem of Calculus guarantees that the solution will be $y = \int f(x)\,dx$ as long as f is a continuous function.

Another type of ODE that is relatively straightforward to solve is the **separable ODE**, which is a first-order ODE that can be written in the form

$$\frac{dy}{dx} = f(x)g(y).$$

These ODEs can be solved by integration as well, but only after some rearranging.

Example 1.3.1 Solving a separable ODE.

Solve the IVP given by $y' = x + 4xy^2, y(0) = 1$.

Solution. The first step to solving this IVP is to solve the ODE $y' = x + 4xy^2$. It may not look like it at first, but this ODE is separable since we can rewrite it as $y' = x(1 + 4y^2)$. To solve this ODE, we need to move the y terms to the left hand side of the equation and the x terms to the right hand side. We'll abuse notation a little bit to do so by rewriting $y' = \dfrac{dy}{dx}$ and treating $\dfrac{dy}{dx}$ as a fraction, but it won't get us into too much trouble here:

$$\begin{aligned}\frac{dy}{dx} = x(1+4y^2) \Rightarrow \frac{dy}{1+4y^2} &= x\,dx \\ \Rightarrow \int \frac{dy}{1+4y^2} &= \int x\,dx \\ \Rightarrow \frac{1}{2}\tan^{-1}(2y) &= \frac{1}{2}x^2 + C \\ \Rightarrow \tan^{-1}(2y) &= x^2 + C_1\end{aligned}$$

At this step we can either leave the solution as is (in **implicit form**) or solve for y to get an **explicit form**. We'll leave this in implicit form and then plug in the initial condition to get

$$\tan^{-1}(2) = C_1.$$

So the implicit solution of this IVP is given by

$$\tan^{-1}(2y) = x^2 + \tan^{-1}(2).$$

Example 1.3.2 Newton's Law of Cooling again.

A metal plate is removed from an oven and placed in a room. The temperature of the plate is 100° Celsius and the temperature of the room is fixed at 15° Celsius. After 20 minutes the temperature of the plate drops to 90° Celsius. How hot is the plate after five hours?

Solution. Let $T(t)$ denote the temperature of the plate t minutes after being removed from the oven and let $A(t)$ denote the temperature of the room t minutes after the plate is removed from the oven. Then $T(0) = 100$ and Newton's Law of Cooling says that

$$\frac{dT}{dt} = k(T-15).$$

To answer this question we need to find T, and although we don't know k at the moment we can still make some progress just by remembering that it's a constant. This ODE is separable, so we'll separate variables and integrate both sides to get

$$\ln(T-15) = kt + C$$

which simplifies to $T - 15 = C_1 e^{kt}$ or just $T = 15 + C_1 e^{kt}$.
Now we need to find C_1 and k. To find C_1, we just use the initial condition to get $C_1 = 85$. The only piece of information that we have left to find k is the fact that the temperature of the plate drops to 90 after 20 minutes. In other words, $T(20) = 90$. Therefore

$$90 = 15 + 85e^{20k}$$

which becomes

$$\ln\frac{75}{85} = 20k.$$

Therefore $k = \frac{1}{20}\ln\frac{15}{17} \approx -.006$.
So, finally, $T(t) = 15 + 85e^{-.006t}$ and the temperature after five hours is $T(300) \approx 28$.

Substitution Methods

At this point we can only solve a couple types of differential equations. An ODE that isn't of the form $\frac{dy}{dx} = f(x)$ or separable may prove troublesome. However, there are certain cases where we can rewrite an ODE into one of these forms by using the right substitution.

Example 1.3.3 Substitution to solve an ODE.

Find the general solution of $y' = (x + y + 3)^2$.

Solution. This ODE is not separable and we can't just integrate it (since the right hand side depends on the dependent variable). However, the form of the right hand side suggests a substitution: $u = x + y + 3$. This would simplify things quite a bit, leaving us with $y' = u^2$. The only problem with this is that y' depends on x, not u. We must rewrite y' in terms of the new variable u, which isn't too bad. Since $u = x+y+3$, we get $y' = u' - 1$. Therefore the ODE becomes

$$u' - 1 = u^2 \text{ or just } \frac{du}{dx} = 1 + u^2.$$

This new ODE *is* separable, and so we separate variables and integrate to get $\tan^{-1} u = x + C$. If we don't care about finding an explicit solution, then we can just replace u to get the equation back in terms of y. So our (implicit) general solution is $\tan^{-1}(x + y + 3) = x + C$.

Example 1.3.4 A less obvious substitution.

Find an explicit solution of $xy' = x + y$.

Solution. It's tough to see what to do right away, so we'll try simplifying the ODE first. In particular, we'll solve for y' to get

$$y' = 1 + \frac{y}{x}.$$

If we stare at this for a while, we might convince ourselves that the right hand side really just depends on $\frac{y}{x}$, so we'll try replacing that with u. Then the ODE becomes $y' = 1 + u$.
Once again, this is much simpler but we need to rewrite y' in terms of u. Since $u = \frac{y}{x}$, this means that $y = ux$ and so $y' = u + u'x$. Then the ODE becomes

$$u + x\frac{du}{dx} = 1 + u \text{ or just } x\frac{du}{dx} = 1.$$

This new ODE can be rearranged to get $\frac{du}{dx} = \frac{1}{x}$, and so $u = \ln|x| + C$. Getting back in terms of y, we have $\frac{y}{x} = \ln|x| + C$ or just $y = x\ln|x| + Cx$.

1.4 First-Order Linear ODEs

In this section we introduce a new type of ODE that we can solve, in addition to separable ODEs and ”simple” ODEs of the form $y' = f(x)$. The ODEs that we'll consider in this section are **first-order linear ODEs**.

Definition 1.4.1 First-Order Linear ODEs.

A first-order ODE is said to be linear if it can be written in the following form:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

We've actually seen such an ODE all the way back in Example 1.1.8. The ODE that we came up with in that problem can be rewritten as a first-order linear ODE with the right substitution (say, $u = y'$). Our first goal in this section is then to figure out how to solve these ODEs.

Note that this section corresponds to Section 1.5 from the text.

Integrating Factors

To get a sense of how to solve first-order linear ODEs, we'll try some relatively simple examples first.

Example 1.4.2 Solving a first-order linear ODE.

Find the general solution of $\frac{dy}{dx} + 2y = x$.

Solution. First, note that the ODE is indeed a first-order linear ODE since it takes the form given in Definition 1.4.1. If we stare at the ODE for a bit, we might think that the left hand side looks like something we'd get after using the product rule. Just compare $(fg)' = fg' + f'g$ with $\frac{dy}{dx} + 2y$, and it appears that the unknown function y is taking the place of g in the product rule formula. If we could just figure out what the function f is supposed to be, then we could drastically simplify the left hand side of the ODE.

Unfortunately, there is no such function f that works here. If there were, we'd have to have $f' = 2$ and $f = 2x$, and clearly we aren't multiplying $\frac{dy}{dx}$ by $2x$ in the ODE. But we can pull a dirty trick here! We'll multiply through the ODE by e^{2x} to get the new ODE

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = xe^{2x}.$$

It might not be all that obvious why this helps us out, but now the left hand side can be simplified by the product rule:

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = \frac{d}{dx}[e^{2x}y].$$

So we can rewrite the entire ODE as

$$\frac{d}{dx}[e^{2x}y] = xe^{2x}.$$

We can integrate this on both sides to get $e^{2x}y = \int xe^{2x}\,dx$, or just

$$e^{2x}y = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C.$$

The explicit solution would be $y = \frac{1}{2}x - \frac{1}{4} + Ce^{-2x}$.

The function e^{2x} that we used in Example 1.4.2 is called an **integrating factor**. Integrating factors are our primary tool in solving first-order linear ODEs. In general, to solve a first-order linear ODE $y' + P(x)y = Q(x)$ the first thing you must do is to multiply through it by the integrating factor $e^{\int P(x)\,dx}$.

Example 1.4.3 Solving a first-order linear ODE in disguise.

Solve the second-order ODE given by

$$tx'' + \frac{t}{1+t^2}x' = \frac{2t^2}{(1+t^2)^2}e^{-\tan^{-1} t}$$

with initial conditions $x(0) = x'(0) = -1$.

Solution. Even though this is a second-order ODE, we can rewrite it as a first-order ODE using the substitution $u = x'$. Then the ODE becomes

$$tu + \frac{t}{1+t^2}u = \frac{2t^2}{(1+t^2)^2}e^{-\tan^{-1} t}.$$

If we divide through by t, we get

$$u + \frac{1}{1+t^2}u = \frac{2t}{(1+t^2)^2}e^{-\tan^{-1} t}.$$

This can be solved by integrating factors since it takes the form given in Definition 1.4.1. The integrating factor we need is given by

$$e^{\int \frac{1}{1+t^2}\,dt} = e^{\tan^{-1} t}.$$

Now we multiply through the ODE by this integrating factor and rewrite the left hand side using the product rule to get

$$\frac{d}{dt}[e^{\tan^{-1} t}u] = \frac{2t}{(1+t^2)^2}.$$

At this step we can integrate both sides to get

$$e^{\tan^{-1} t}u = -\frac{1}{1+t^2} + C,$$

which becomes

$$x' = -\frac{e^{-\tan^{-1} t}}{1+t^2} + Ce^{-\tan^{-1} t}.$$

If we plug in the initial condition $x'(0) = -1$, this forces $C = 0$. Hence

$$x' = -\frac{e^{-\tan^{-1} t}}{1+t^2}.$$

Now we integrate one last time to get x:

$$x = e^{-\tan^{-1} t} + D.$$

If we use the last initial condition $x(0) = -1$, we see that $D = -2$. Hence the solution of this IVP is

$$x = e^{-\tan^{-1} t} - 2.$$

Applications

A common application of first-order linear ODEs is in modeling "mixture" problems. Suppose we have a tank which contains a solution (mixture of solute and solvent, such as salt and water). Some amount of solution is also flowing into and out of the tank. We want to measure the amount of solute in the tank at time t, call this amount $x(t)$. Then $x(t)$ will change depending on how the solute flows into and out of the tank, making it a prime target for a differential equation.

If we set

$$\begin{aligned} c_i &= \text{concentration of solute flowing in} \\ c_o &= \text{concentration of solute flowing out} \\ r_i &= \text{rate solution is flowing in} \\ r_o &= \text{rate solution is flowing out} \end{aligned}$$

then we can say that

$$\frac{dx}{dt} = r_i c_i - r_o c_o = r_i c_i - r_0 \frac{x(t)}{V(t)}$$

where $V(t)$ is the volume of solution in the tank at time t.

We assume that c_i, r_i, r_o are all constant.

Furthermore, if we let $V_0 = V(0)$ denote the initial volume of the solution in the tank then we can say that $V(t) = V_0 + (r_i - r_0)t$. Hence the amount of solute $x(t)$ obeys the first-order linear ODE

$$\frac{dx}{dt} = r_i c_i - \frac{r_o}{V_0 + (r_i - r_o)t} x.$$

Example 1.4.4 Salt in a tank.

A tank contains 100 L of a solution consisting of 50 kg of salt dissolved in water. Solution containing $1\,\frac{\text{kg}}{\text{L}}$ of salt is pumped into the tank at a rate of $2\,\frac{\text{L}}{\text{min}}$ and the well-mixed solution is pumped out at the rate of $3\,\frac{\text{L}}{\text{min}}$. How much salt will be in the tank after t minutes?

Solution. Let $x(t)$ denote the amount of salt in the tank after t minutes, so $x(0) = 50$. Then

$$\frac{dx}{dt} = r_i c_i - r_o c_o = 2 \cdot 1 - 3\frac{x}{100 - t}.$$

We can rearrange this to get

$$\frac{dx}{dt} + \frac{3}{100 - t} x = 2.$$

This ODE is linear and has integrating factor $(100 - t)^{-3}$. Multiplying through the ODE by the integrating factor and rewriting it using the product rule then gives us

$$\frac{d}{dt}[(100 - t)^{-3} x] = 2(100 - t)^{-3}.$$

Now we can integrate both sides to get

$$(100-t)^{-3}x = (100-t)^{-2} + C$$

or just $x = 100 - t + C(100-t)^3$. Finally, the initial condition can be used to show that $C = -\frac{1}{20000}$, so $x = 100 - t - \frac{1}{20000}(100-t)^3$.

1.5 Existence and Uniqueness of Solutions

This section corresponds to Section 1.7 from the text.

Existence and Uniqueness Theorem

There are two important questions we need to consider when developing a mathematical model using differential equations (i.e. IVPs):

1. Does the initial value problem have a solution? (Existence).
2. If it has a solution, is the solution unique? (Uniqueness).

Ideally, the answer to both of these questions will be yes.

Example 1.5.1 The answer is no.

Does $x\dfrac{dy}{dx} = \sqrt[3]{y}, y(1) = 0$ have a unique solution?

Solution. We can find a solution to this IVP by treating the ODE as separable. If we do so, we find that $y = \left(\frac{2}{3}\ln x\right)^{\frac{3}{2}}$. On the other hand, we can also eyeball a second solution: $y = 0$. So this IVP has *two* different solutions: $y_1 = \left(\frac{2}{3}\ln x\right)^{\frac{3}{2}}$ and $y_2 = 0$.

Clearly, IVPs don't always have unique solutions. Sometimes it's difficult to determine precisely when an IVP can have a unique solution, but most of the cases we'll care about in this class will fall under the following theorem.

Theorem 1.5.2 Existence and Uniqueness Theorem.

Consider the IVP given by $\dfrac{dy}{dx} = f(x,y), y(x_0) = y_0$. *If* $f(x,y)$ *is bounded and continuous within some rectangle in the plane containing* (x_0, y_0)*, then the IVP has at least one solution. If in addition* $f_y(x,y)$ *is also bounded and continuous within some rectangle containing* (x_0, y_0)*, then the IVP has a unique solution.*

If we go back to Example 1.5.1, then we see that Theorem 1.5.2 has something to say about the IVP in that example. In that example, we had $f(x,y) = \sqrt[3]{y}, f_y(x,y) = \frac{1}{3}y^{-\frac{2}{3}}$ and $(x_0, y_0) = (1, 0)$. Let's draw a rectangle around this point:

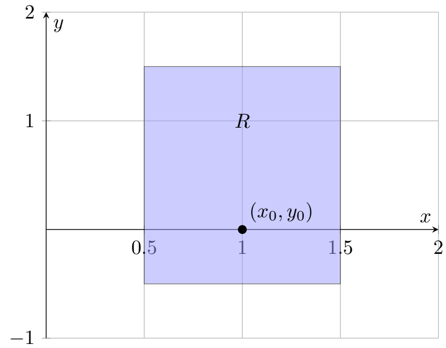

Figure 1.5.3

Now f is continuous and $-\frac{1}{2} \leq y \leq \frac{3}{2}$ within this rectangle, so

$$-\sqrt[3]{\frac{1}{2}} \leq f(x, y) \leq \sqrt[3]{\frac{3}{2}}$$

everywhere inside of this triangle. So Theorem 1.5.2 guarantees *at least one* solution of the IVP, and indeed there is at least one solution. However, the problem with uniqueness stems from the fact that f_y has a divide-by-zero problem inside this rectangle. Furthermore, it's impossible to draw a rectangle around $(1, 0)$ that avoids this divide-by-zero problem. Hence there is no guarantee of uniqueness.

On the other hand, if we changed the initial condition to $y(1) = 0.000001$ then we would be guaranteed a unique solution. Moving that initial condition off of the x-axis is all we need to do to guarantee uniqueness.

Picard Iteration

Now that we know of a way to determine whether or not certain ODEs have solutions, we'd like a method for actually finding these solutions. We've seen a few different methods for solving specific first-order ODEs, but what we'll do now is discuss a method that works for a large class of first-order ODEs. The only catch is that it may take us an infinite amount of time to get the solution.

Consider the IVP $\dfrac{dy}{dx} = f(x, y), y(x_0) = y_0$. We can rewrite this differential equation as an **integral equation**:

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$

It looks quite a bit different, but solutions of this integral equation are also solutions of the corresponding differential equation. Our goal now is to approximate a solution to this integral equation.

To start, let's make a guess as to what the solution of our IVP should be. To keep things simple we'll start with a constant function, say $\phi_0(x) = y_0$ so that we at least satisfy the initial condition. Now this guess may not be a good match for the solution of the IVP away from the initial condition, so we'll adjust the guess using the integral equation to get the new function ϕ_1:

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0)\, dt.$$

Now ϕ_1 may not be a great approximation either away from the initial condition, but we can adjust it using the integral equation just like we did to ϕ_0.

The method described in the previous paragraph is **Picard's Method**. In general, the n^{th} iteration of Picard's Method is given by

$$\phi_n(x) = y_0 + \int_{x_0}^{x} f(t, \phi_{n-1}(t))\, dt,$$

and the first iterate is the constant function $\phi_0 = y_0$. It may seem strange to consider these functions as approximate solutions of the IVP in question, but each iterate actually solves an IVP very similar to the one that we care about, $\dfrac{dy}{dx} = f(x, y), y(x_0) = y_0$. In particular,

$$\frac{d\phi_n}{dx} = f(x, \phi_{n-1}), \phi_n(x_0) = y_0$$

for all $n \geq 1$. This method doesn't always work, but if $f(x, y)$ satisfies the conditions given in Theorem 1.5.2 then this method will (after potentially infinitely many steps) provide a solution to the IVP. Since the computations involved are quite tedious, it's best to use a CAS if possible.

Example 1.5.4 Using Sage to Compute Picard Iterates.

Consider the initial value problem given by

$$y' = x + y^2, y(1) = -1.$$

We want to approximate the solution by using Picard iteration (note that the differential equation is neither separable nor linear!). We do this with the Sage cell below:

```
# We need to tell Sage that y is a variable.
var('t,y')

# Now we define f(x,y) and our initial conditions.
f(x, y) = x + y^2
x0, y0 = 1, -1

# We will also decide how many Picard iterates we want.
# The below computes 3 iterates, but you're welcome
# to change n. However, things get very complicated
# very quickly for this example!
n = 3

# Define iterates.
yn(x) = y0
for k in range(n):
yn(x) = y0 + integral(f(t, yn(x=t)), t, x0, x)

# Display n^th iterate.
show(yn(x))
```

```
1/4400*x^11 + 1/400*x^10 + 1/720*x^9 - 9/160*x^8 -
    1/56*x^7 + 77/100*x^6 - 191/200*x^5 - 399/80*x^4 +
    4741/240*x^3 - 497/16*x^2 + 10201/400*x -
    5517209/554400
```

Euler's Method

The Picard iteration approach can be useful for finding series approximations of the solution of an ODE. If a more numerical approach is desired, then *Euler's method* might be useful. Euler's method can be thought of as an algorithmic version of tracing a solution curve through a direction field (see Section 1.2).

Algorithm 1.5.5 Euler's Method.

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0.$$

The equations defining Euler's method for a given step size h are

$$\begin{aligned} x_n &= x_{n-1} + h \\ y_n &= y_{n-1} + hf(x_{n-1}, y_{n-1}). \end{aligned}$$

For Euler's method, the general rule of thumb is as follows: the smaller h is, the better the approximation. However, one should expect degrading performance as the method moves farther from the initial condition (x_0, y_0).

Example 1.5.6 Euler's Method Applied to Nonlinear ODE.

Let $y(x)$ denote the solution of the IVP given by

$$y' = x + y^2, y(0) = 1.$$

Estimate $y(2)$ using Euler's method with a step size of $h = \frac{1}{2}$.

Solution. First, note that we are justified in saying the solution exists at all by Theorem 1.5.2. Euler's method now produces the following approximations:

n	x_n	y_n
0	0	1
1	0.5	1.5
2	1	2.875
3	1.5	7.508
4	2	36.441

Such computations are best performed using a CAS, such as the Sage code below:

```
# Define y to be a variable in case this hasn't
# been done.
var('y')

# Define starting parameters and f(x,y).
xn, yn, h = 0, 1, 0.5
f(x,y) = x + y^2

# To get to x = 2 we need n = 4 steps.
n = 4

# Compute Euler's method approximations.
for k in range(n):
xn, yn = xn + h, yn + h*f(xn, yn)

show((xn, yn))
```

```
(2.00000000000000, 36.4414367675781)
```

1.6 Population Models and Autonomous Equations

Population Equations

Suppose we're monitoring the population of some species, and let's denote the population at time t by $P(t)$. An obvious question to consider is how that population will change over time. Mathematically, this means we want to obtain information on $\frac{dP}{dt}$ and then use it to estimate $P(t)$.

A simple model for $\frac{dP}{dt}$ is to assume it depends only on the birth rate β and death rate δ of the species in question. Then we can write

$$\frac{dP}{dt} = (\beta - \delta)P. \tag{1.1}$$

If we assume that β, δ are constants, then this equation is separable and we can solve it to obtain

$$P(t) = P_0 e^{(\beta-\delta)t},$$

where P_0 represents the "initial population", or population at time $t = 0$. We call (1.1) the **natural growth equation**.

The natural growth equation is simple, but it's probably too simple to be useful expect in certain scenarios (such as measuring half-life). To get a more flexible model, we can generalize (1.1) by assuming that the birth and death rates are actually functions of time. This gives us the **general population equation**.

Definition 1.6.1 General Population Equation.

The general population equation for a population $P(t)$ is given by

$$\frac{dP}{dt} = (\beta(t) - \delta(t))P.$$

Example 1.6.2 Population Explosion.

A population has 100 members at time $t = 0$ years with a death rate of $.25P$ and a birth rate of $.5P$, where $P(t)$ denotes the population after t years. Find $P(t)$ and determine if this is a reasonable population model.

Solution. If we assume that the population obeys the general growth equation, then we get

$$P' = .25P^2, P(0) = 100.$$

This ODE is separable, and we can therefore solve it to get

$$P(t) = \frac{100}{1 - 25t}.$$

So we have a solution, and furthermore Theorem 1.5.2 guarantees that the solution is unique. But if you stare at this for a bit, you might see that it has a divide-by-zero problem. In particular,

$$\lim_{t \to (\frac{1}{25})^+} P(t) = \infty.$$

In other words, the population becomes infinite in about two weeks!

The Logistic Equation

Example 1.6.2 shows that we need to be more careful with our assumptions on population growth. One relatively simple assumption we can make is to assume that the birth rate $\beta(t)$ decreases as population P increases. This makes sense in the physical world as well: as population increases, existing and finite resources (such as food) must be shared between more and more members of the population. Since there's less to go around, we should expect growth to slow down. In particular, let's assume that

$$\begin{aligned} \beta(t) &= \beta_0 - \beta_1 P \\ \delta(t) &= \delta_0 \end{aligned}$$

where β_0, β_1 and δ_0 are all positive constants. Then the population equation for this scenario becomes

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta_0)P.$$

With a little algebra, we get the **logistic equation**:

$$\frac{dP}{dt} = kP(M - P)$$

for constants k and M. This equation is separable, and can be solved using partial fractions to obtain

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}},$$

where $P_0 = P(0)$. In order to verify the reasonableness of our logistic model, let's see what happens to the population as time increases.

Example 1.6.3 Long-Term Behavior of Logistic Growth.

What is the long-term population of a species that grows according to the logistic equation $\frac{dP}{dt} = kP(M-P)$?

Solution. Using the fact that

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}},$$

we have

$$\lim_{t\to\infty} P(t) = M.$$

So the population should eventually level out at M.

In the logistic equation $P' = kP(M-P)$, the value M is the **carrying capacity**, and denotes the maximum sustainable population according to the model.

Example 1.6.4 Population Growth in the USA.

In millions, the population of the USA in 1990 was 250 and was growing at a rate of 3.1 per year. In 2012, the population was 314 and was growing at a rate of 2.3 per year. Assuming that the population of the USA grows logistically, estimate the population of the USA in 2017 and compare it to the current estimate of 325.7.

Solution. Let $P(t)$ denote the population of the USA (in millions), where t is the number of years after 1990. Then

$$\frac{dP}{dt} = kP(M-P)$$

and

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}.$$

So we need to find k and M.
When $t = 0$, we have $P' = 3.1$ and $P = 250$. Similarly, when $t = 22$ we have $P' = 2.3$ and $P = 314$. Therefore

$$\begin{aligned} 3.1 &= 250k(M - 250) \\ 2.3 &= 314k(M - 314) \end{aligned}$$

Solving this system gives us $k \approx .00008$ and $M \approx 406.4$. Hence

$$P(t) = \frac{101600}{250 + 156.4e^{-.03t}}.$$

This model estimates the population in 2017 to be

$$P(27) = 317.9,$$

which is about a 2% error. Note also that this model predicts the carrying capacity of the USA to be 406.4.

Stability of Solutions

The logistic equation

$$\frac{dP}{dt} = kP(M - P)$$

is a particularly nice separable ODE since the right hand side depends only on the unknown function P. So we can write $P' = f(P)$, where $f(P) = kP(M - P)$. ODEs like this (where the independent variable does not appear explicitly) are called **autonomous ODEs**.

Autonomous ODEs like $\dfrac{dx}{dt} = f(x)$ are useful because the behavior of their solutions can be determined *qualitatively*, without actually solving the ODE. This is done by looking for the constant solutions of the ODE, that is, solutions of the form $x = c$. For any such solution, we must have $f(c) = 0$ as well. These solutions (i.e., the solutions of $f(x) = 0$) are called the **critical points** or **equilibrium solutions** of the ODE. These solutions completely determine the long-term behavior of *every other solution*.

Example 1.6.5 Finding Equilibrium Solutions.

Find the equilibrium solutions of $\dfrac{dx}{dt} = -x^2 + 7x - 10$.

Solution. We need to solve the equation $-x^2 + 7x - 10 = 0$. Thankfully, we can factor this to get $(2 - x)(x - 5) = 0$, and so the equilibrium solutions are $x = 2, 5$.

Definition 1.6.6 Stability of Solutions.

A critical point is **stable** if solutions that start "near" the point stay near it. A critical point is **unstable** if solutions that start "near" the point can diverge away from it.

Example 1.6.7 Determining the Stability of Solutions.

What are the stable critical points of $\dfrac{dx}{dt} = -x^2 + 7x - 10$?

Solution. We already know that the critical points are $x = 2, 5$. We can determine their stability by making use of a **phase diagram**, which is essentially a sign chart for $f(x) = -x^2 + 7x - 10$:

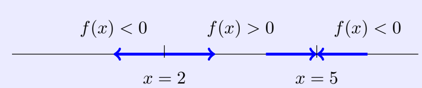

Figure 1.6.8 The phase diagram for $x' = f(x)$.

This shows us that solutions that begin near $x = 2$ tend to move away from $x = 2$, which solutions near $x = 5$ tend to move towards $x = 5$. So $x = 2$ is unstable and $x = 5$ is stable.

Example 1.6.9 Determining a Sustainable Population.

Consider a population of fish that obeys the logistic equation

$$\frac{dP}{dt} = 2P(30 - P)$$

where $P(t)$ is the population of fish (in thousands) after t years. Suppose that the population is also harvested at some rate h (in thousands per year). What is the maximum sustainable rate of harvesting?

Solution. To account for the harvesting, we need to modify the ODE:

$$\frac{dP}{dt} = 2P(30 - P) - h.$$

The harvesting will be sustainable as long as the population does not become extinct. To determine this long term behavior, we'll find the critical points and set up a phase diagram.
The critical points are given by

$$P = 15 \pm \sqrt{3600 - 8h}$$

by the quadratic formula. We now have three cases to consider: $3600 - 8h < 0, 3600 - 8h = 0, 3600 - 8h > 0$. In terms of h, these reduce to $h < 450, h = 450, h > 450$.

1. In the first case, if $h < 450$ then we have two positive, real critical points:
   $$0 < c_1 < 15 < c_2 < 75.$$
   The phase diagram for this situation is

   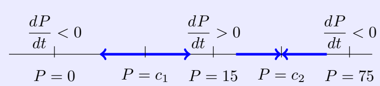

   Figure 1.6.10

   So we see that c_1 is unstable while c_2 is stable. In particular, as long as $P \geq c_1 = 15 - \sqrt{3600 - 8h}$, then the rate of harvesting is sustainable.

2. Now assume that $h = 450$. Then we have only one equilibrium solution: $c = 15$. The corresponding phase diagram is

   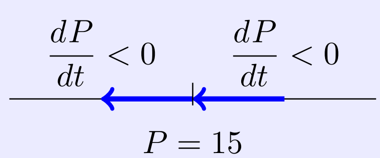

   Figure 1.6.11

   We interpret the phase diagram as follows: if P is less than 15,000then the population will collapse to extinction. Otherwise, the population will stabilize at $15,000$. This type of critical point is often called **semi-stable.**

3. Finally, consider the case $h > 450$. Then we have no (real) critical points. Since imaginary populations don't make sense in this

model, there is no sustainable population. No matter how large the initial population, it will eventually go extinct if harvested at a rate greater than 450.

By the above, the largest sustainable harvesting rate is $h = 450$, as long as $P_0 \geq 15$.

Linear Stability Analysis

Given the autonomous ODE $\frac{dx}{dt} = f(x)$, we saw above that we can qualify the behavior of equilibrium solutions by setting up a phase diagram. We can go a step further and actually qualify the growth of solutions that are "near" equilibrium solutions. In particular, we have the following theorem.

Theorem 1.6.12 Linear Stability Analysis.

Suppose $\frac{dx}{dt} = f(x)$ where $f(x)$ is continuously differentiable, and let x^* denote a critical point/equilibrium solution of the ODE. If $f'(x^*) < 0$, then x^* is stable and solutions near x^* will move exponentially towards x^*. If $f'(x^*) > 0$, then x^* is unstable and solutions near x^* will move exponentially away from x^*. If $f'(x^*) = 0$, then more advanced methods are required.

Example 1.6.13 Classifying the Critical Points of the Logistic Equation.

Classify the critical points of the logistic equation as stable or unstable.

Solution. Recall that the logistic equation is given by $P' = kP(M - P) = f(P)$ for (we'll assume) positive constants k, M. From here, we clearly see that the critical points are $P = 0$ and $P = M$ (which makes sense from a population standpoint!). We could set up a phase diagram to determine stability, but we'll use Theorem 1.6.12 instead.
Since $f'(P) = k(M - P) - kP$, we see that

$$f'(0) = kM > 0$$
$$f'(M) = -kM < 0$$

Hence $P = 0$ is unstable, while $P = M$ is stable.

Chapter 2

Linear ODEs with Constant Coefficients

Now we move on to solving linear ODEs with constant coefficients. We'll start with solving second-order ODEs.

2.1 Second-Order Linear ODEs

Recall that a second-order ODE is an ODE whose highest derivative is the second derivative. In this section, we'll look at how to solve second-order ODEs of a special type. Our method of solution here will be generalized to many, many other ODEs.

Types of Linear ODEs

A first-order ODE is linear if it can be written in the form $y' + P(x)y = Q(x)$ (see Definition 1.4.1). We have a similar definition for second-order ODEs.

Definition 2.1.1 Second-Order Linear ODEs.

A second-order ODE is **linear** if it can be written in the form

$$y'' + P(x)y' + Q(x)y = R(x).$$

A second-order linear ODE is **homogeneous** if $R(x) = 0$ and **nonhomogeneous** if $R(x) \neq 0$.

Example 2.1.2 Types of second-order ODEs.

Consider the following ODEs:

1. $y'' = y' - \sin^2 x$ is linear but nonhomogeneous.
2. $\dfrac{d^2x}{dt^2} - x^3 = 0$ is nonlinear.
3. $\sqrt{x}e^{\tan^{-1} x}y'' = x^2y$ is linear and homogeneous.

Newton's Second Law is a great source of linear second-order ODEs, as Example 2.1.4 shows. However, we will first need to state **Hooke's Law**.

Theorem 2.1.3 Hooke's Law.

Consider an object attached to a spring. The force exerted by the spring on the object is directly proportional to the displacement of the object from the spring's equilibrium, or at rest, position.

Example 2.1.4 A second-order model.

An object of mass 4 kg is attached to a horizontal, frictionless spring. Let $x = 0$ denote the equilibrium position of the spring and let $x(t)$ denote the displacement of the mass from the spring's equilibrium position. The only force acting on the mass is the force of the spring itself. Find a mathematical model for $x(t)$.

Solution. Let F_S denote the force of the spring on the mass. Then by Theorem 2.1.3 we must have $F_S = -kx$ for some $k > 0$. Now, by Newton's Second Law we must also have $F_S = 4x''$. Hence

$$4x'' = -kx \quad \text{or} \quad 4x'' + kx = 0.$$

So the motion of the mass is modeled by a linear, second-order homogeneous ODE.

The reason we take $F_S = -kx$ in Hooke's Law is because we want to emphasize that the spring force is a *restoring force*, since it acts against the displacement of the mass.

The general trend that we will see for mathematical models using linear ODEs is that they will be homogeneous if we assume that there is no external force, and nonhomogeneous if we assume there is an external force.

Solutions of Second-Order Linear ODEs

The reason we restrict ourselves to linear ODEs is because their solutions behave very nicely. In particular, we have the Theorem 2.1.6, which is an important principle for homogeneous ODEs. First, some terminology.

Definition 2.1.5 Linear Combinations.

Let f and g denote functions of x. A **linear combination** of f and g is a function of the form $c_1 f + c_2 g$ where c_1, c_2 are constants.

Theorem 2.1.6 The Superposition Principle.

Let y_1 and y_2 denote two (possibly different) solutions of the ODE $y'' + P(x)y' + Q(x)y = 0$. Then any linear combination of these solutions is itself another solution of the same ODE.

Proof. We need to show that $y = c_1y_1 + c_2y_2$ is another solution of the same ODE, where c_1, c_2 are arbitrary constants. To do this, we'll just plug the linear combination into the ODE and simplify:

$$\begin{aligned} y'' + P(x)y' + Q(x)y &= (c_1y_1 + c_2y_2)'' + P(x)(c_1y_1 + c_2y_2) + Q(x)(c_1y_1 + c_2y_2)y \\ &= c_1y_1'' + c_2y_2'' + c_1P(x)y_1' + c_2P(x)y_2' + c_1Q(x)y_1 + c_2Q(x)y_2 \\ &= c_1(y_1'' + P(x)y_1' + Q(x)y_1) + c_2(y_2'' + P(x)y_2' + Q(x)y_2) \\ &= 0. \end{aligned}$$

■

Theorem 2.1.6 is important because it tells us how to construct new solutions of homogeneous ODEs out of known solutions. The next example demonstrates this.

Example 2.1.7 Solving a second-order IVP.

Using the fact that $y_1 = \cosh \frac{x}{2}$ and $y_2 = \sinh \frac{x}{2}$ are both solutions of $4y'' - y = 0$, solve the IVP given by

$$4y'' - y = 0 \quad \text{and} \quad y(0) = 3, y'(0) = 1.$$

Solution. Note that we have *two* initial conditions. In general, we'll need as many initial conditions as the order of the ODE.
To solve the IVP, we'll use the superposition principle to give us as much leeway as possible in constructing a solution of out y_1 and y_2. So we'll guess the solution takes the form $y = c_1 \cosh \frac{x}{2} + c_2 \sinh \frac{x}{2}$. By the superposition principle we're guaranteed that this is a solution of the ODE $4y'' - y = 0$, so we just need to pick the constants c_1, c_2 in order to satisfy the initial conditions. Let's start with the first initial condition, $y(0) = 3$. This gives us the equation

$$3 = c_1 \cosh 0 + c_2 \sinh 0 = 3.$$

So $c_1 = 3$. Similarly, $y'(0) = 1$ gives us

$$1 = \frac{3}{2} \sinh 0 + \frac{c_2}{2} \cosh 0 = \frac{c_2}{2}.$$

Hence $c_2 = 2$, and the solution of the IVP is

$$y = 3 \cosh \frac{x}{2} + 2 \sinh \frac{x}{2}.$$

The reason we were able to solve the IVP in Example 2.1.7 was because the individual solutions $\cosh \frac{x}{2}$ and $\sinh \frac{x}{2}$ gave us enough to build the particular solution of the IVP. It turns out that *every* solution of $4y'' - y = 0$ can be written as a linear combination of these two functions, so knowing these two functions is enough to solve every IVP involving this ODE.

In general, our goal will be to describe a ”basis” of solutions for a given homogeneous ODE, a finite set of solutions that can be used to describe all possible solutions. This can be done if the functions in the basis aren't too ”similar”, in the following sense.

Definition 2.1.8 Linear Independence of Functions.

Two functions f and g are said to be **linearly independent** on an interval I if the linear combination $c_1 f + c_2 g$ is equal to 0 if and only if

$c_1 = c_2 = 0$. Otherwise, we say that they are **linearly dependent**.

The main idea behind Definition 2.1.8 is that f and g are linearly independent if they are not like terms (i.e. they don't cancel each other out). Note that f, g are linearly dependent if and only if $f = \alpha g$ for some $\alpha \neq 0$. Equivalently, they are linearly dependent if and only if $\frac{f}{g} = \alpha$ for some $\alpha \neq 0$.

Example 2.1.9 Linear independence of sine and cosine.

Show that $\sin x$ and $\cos x$ are linearly independent.

Solution. Since $\frac{\sin x}{\cos x} = \tan x$ is *not* constant, this means that $\sin x$ and $\cos x$ must be linearly independent.

Although it wasn't too hard to see that the functions in Example 2.1.9 were linearly independent, in other cases it can be much trickier (especially when we move to higher order ODEs). For example, suppose we set $f(t) = -5(t-15)^2 + 450$ and $g(t) = 12t(30-t)$. Then it's not obvious at all that these two functions are linearly dependent! In fact,

$$\frac{2}{5}f(t) - \frac{1}{6}g(t) = 0.$$

So generally when we try to determine if two functions are linearly independent, we'll make use of the **Wronskian**.

Definition 2.1.10 Wronskian of Two Functions.

Let f and g denote two differentiable functions. The Wronskian of f and g, denoted $W(f, g)$, is given by

$$W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g.$$

Theorem 2.1.11 Linear Independence and the Wronskian.

Let f and g be two functions that are differentiable on some interval I. Then f and g are linearly independent on I if $W(f, g) \neq 0$ somewhere *I. Conversely, if $W(f, g) = 0$* and *f and g can both be represented by power series on I, then f and g are linearly dependent.*

Example 2.1.12 Using the Wronskian.

If we let $f(t) = -5(t-15)^2 + 450$ and $g(t) = 12t(30-t)$ and compute their Wronskian, we obtain (after a fair bit of algebra) $W(f, g) = 0$. Since f and g can clearly be represented by power series on any interval I (since, being polynomials, they already *are* power series), this means that the two functions are linearly independent.

We mentioned earlier that we'll try to find a basis of solutions for linear ODEs. Now that we have the concept of linear independence and the Wronskian for checking if two functions are linearly independent, we can make the following definition.

Definition 2.1.13 Basis of Solutions.

Let y_1 and y_2 denote solutions of some second-order linear homogeneous ODE. We call $\{y_1, y_2\}$ a **basis** if y_1 and y_2 are also linearly independent.

Once we have a basis for a second-order linear homogeneous ODE, we can solve any IVP that we wish involving that ODE. In particular, if $\{y_1, y_2\}$ is a basis for such an ODE, then *every* solution of the ODE can be written as a linear combination of y_1 and y_2.

Example 2.1.14 Linear independence using the Wronskian.

Let

$$x_1 = t\cos\ln t \quad \text{and} \quad t\sin\ln t.$$

Given that these functions are both solutions of

$$t^2x'' - tx' + 2x = 0,$$

solve the corresponding IVP with initial conditions $x(1) = 1, x'(1) = -1$.

Solution. We need to start by showing that $\{x_1, x_2\}$ is a basis for the ODE. First, we compute $W(x_1, x_2)$:

$$W(x_1, x_2) = (t\cos\ln t)(\sin\ln t + \cos\ln t) - (t\sin\ln t)(\cos\ln t - \sin\ln t) = t(\cos^2\ln t + \sin^2\ln t) = t.$$

So $W(x_1, x_2) = t$, which is clearly nonzero on the interval $(0, \infty)$. Hence x_1 and x_2 are linearly independent, and therefore $\{x_1, x_2\}$ is a basis for the ODE.
To actually find the solution (call it x), we'll set $x = c_1x_1 + c_2x_2$ and use the initial conditions to find c_1 and c_2. Doing so gives $c_1 = 1$ and $c_2 = -2$, and so the solution of the IVP is

$$x = \cos\ln t - 2\sin\ln t.$$

2.2 Homogeneous ODEs with Constant Coefficients

Now we'll move on to solving homogeneous ODEs, at least with constant coefficients.

Example 2.2.1 Solving a homogeneous ODE.

Suppose we wanted to solve $y'' - y' - 6y = 0$, where $y = y(x)$. If we stare at this for a bit, we may realize the following: the only way for a function y to be a solution of this ODE is for y and its derivatives y', y'' to cancel each other out. In other words, y and its derivatives *should be like terms*. This is a huge hint that y should look like an exponential function. So we'll guess that $y = e^{rx}$ for some real number r, and see if we can't pick r in just the right way to get a solution to the ODE. If we plug e^{rx} into the ODE, we get

$$y'' - y' - 6y = r^2e^{rx} - re^{rx} - 6e^{rx}$$

$$= e^{rx}(r^2 - r - 6).$$

So we need to set $e^{rx}(r^2 - r - 6)$ equal to 0 and solve for r, which gives

$$r = -2, 3.$$

Therefore two solutions of $y'' - y' - 6y = 0$ are given by

$$y_1 = e^{-2x}$$

and $y_2 = e^{3x}$. Since $W(y_1, y_2) \neq 0$, this means that $c_1e^{-2x} + c_2e^{3x}$ is actually the general solution of the ODE.

The process in Example 2.2.1 will work for *every* second-order homogeneous ODE with constant coefficients. So solving such an ODE is even easier than integrating: all we need to do is to find the roots of a particular polynomial.

Definition 2.2.2 The Characteristic Polynomial.

Let a, b and c be constants. Given the ODE $ay'' + by' + cy = 0$, the **characteristic equation** of this ODE is the polynomial

$$ar^2 + br + c = 0.$$

We can now state the following result for finding solutions of homogeneous ODEs with constant coefficients.

Theorem 2.2.3 Characteristic Equations with Distinct Roots.

Let a, b and c be constants. Suppose that the characteristic equation of $ay'' + by' + cy = 0$ has distinct roots r_1 and r_2. Then the general solution of the ODE is given by

$$c_1e^{r_1x} + c_2e^{r_2x}.$$

Example 2.2.4 Spring-mass system revisited.

An object of mass 4 kg is attached to a horizontal, frictionless spring. Suppose the spring constant is given by $k = 16$. The mass is held 3 m to the right of the spring's equilibrium position, and is then released at time $t = 0$ where t is in seconds. Find the displacement $x(t)$ of the mass.

Solution. We know from Example 2.1.4 that the second-order ODE given by

$$4x'' + kx = 0 \quad \text{or} \quad x'' + 4x = 0$$

provides a model for $x(t)$, but now we are in a position to solve it. The characteristic equation of this ODE is $r^2 + 4 = 0$, which has roots $r = \pm 2i$. The imaginary roots *are not a problem*, and in fact provide significant information about the motion of the mass, as we'll soon see. The general solution of the ODE is

$$x = c_1e^{-2it} + c_2e^{2it}.$$

The initial conditions are $x(0) = 3$ and $x'(0) = 0$, which give the equations

$$
\begin{aligned}
3 &= c_1 + c_2 \\
0 &= -2ic_1 + 2ic_2
\end{aligned}
$$

The second equation implies that $c_1 = c_2$, and applying this to the first equation now gives $c_1 = c_2 = \frac{3}{2}$. Hence the displacement of the mass is given by

$$x = \frac{3}{2}e^{-2it} + \frac{3}{2}e^{2it}.$$

The appearance of the "imaginary" solution in Example 2.2.4 may seem strange, but they're actually quite natural. In fact, we can use **Euler's formula** to write the solution in terms of a more familiar function.

Theorem 2.2.5 Euler's Formula.

For all x, the following equations hold:

$$
\begin{aligned}
e^{ix} &= \cos x + i \sin x \\
\cos x &= \frac{e^{ix} + e^{-ix}}{2} \\
\sin x &= \frac{e^{ix} - e^{-ix}}{2i}
\end{aligned}
$$

So using Euler's Formula on the solution from Example 2.2.4 gives

$$
\begin{aligned}
x &= \frac{3}{2}e^{-2it} + \frac{3}{2}e^{2it} \\
&= 3\frac{e^{-2it} + e^{2it}}{2} \\
&= 3\cos 2t.
\end{aligned}
$$

So the imaginary roots $\pm 2i$ from Example 2.2.4 actually relate to the *frequency* of the spring-mass system in that problem. This is a trend we will see often in this course: imaginary numbers corresponding to oscillating quantities.

Now we'll take a look at how to solve second-order homogeneous ODEs whose characteristic equations have repeated roots.

Example 2.2.6 Repeated roots in the characteristic equation.

Find the general solution of $y'' + 2y' + y = 0$ where $y = y(x)$.

Solution. We begin by solving the characteristic equation, which for this ODE is

$$r^2 + 2r + 1 = 0.$$

The only solution of this equation is $r = -1$, which is a repeated root. We can still get one solution of the ODE using this root, namely $y_1 = e^{-x}$, but we need two linearly independent solutions in order to find the general solution.

To get the other solution, we'll make another guess: whatever it happens to be, it should look quite a bit like the first solution e^{-x}, since it still needs to cancel out with its derivatives the same way that e^{-x} does. The easiest way to get a function that looks like e^{-x} but is still linearly independent from e^{-x} is to just multiply by x. In other words, we'll guess (and check!) that $y_2 = xe^{-x}$ is another solution of $y''+2y'+y=0$. If we plug y_2 into the ODE, we get

$$\begin{aligned} y_2'' + 2y_2' + y_2 &= \underbrace{-2e^{-x} + xe^{-x}}_{y_2''} + \underbrace{2e^{-x} - 2xe^{-x}}_{y_2'} + \underbrace{xe^{-x}}_{y_2} \\ &= 0, \end{aligned}$$

which shows that y_2 is indeed a solution of the ODE. Since it's also linearly independent (which we can check via the Wronskian), this means that the general solution of $y'' + 2y' + y = 0$ is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{-x} + c_2 x e^{-x}.$$

The method used in Example 2.2.6 also works for other homogeneous ODEs with constant coefficients whose characteristic equations have repeated roots. Hence the roots of the characteristic equation *completely* determine the general solutions of such ODEs. We summarize this in the following table.

Table 2.2.7

Roots	General solution
$r_1 \neq r_2$	$c_1 e^{r_1 x} + c_2 e^{r_2 x}$
$r_1 = r_2$	$c_1 e^{r_1 x} + c_2 x e^{r_1 x}$

Remember that it's not a problem if the characteristic equation has imaginary roots, and in fact we must account for these in order to completely describe the corresponding physical system. If we have imaginary roots, then we can simply use Euler's Formula to rewrite the solutions in terms of sine and cosine.

2.3 Spring-Mass Models

In this section we examine a common application of second-order ODEs: modeling movement in a spring-mass system. We will look at two different types of motion: undamped and damped. Note that this section corresponds to Section 2.4 of the text.

Free Undamped Motion

Suppose we have a mass m attached to a spring as in the following diagram:

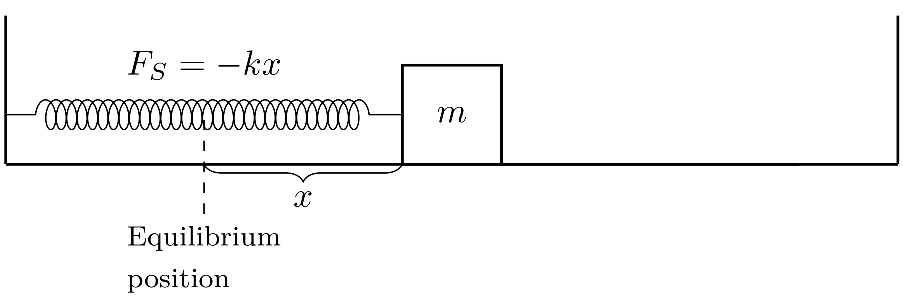

Figure 2.3.1 A spring-mass system.

If we let $x(t)$ denote the displacement of the mass from the spring's equilibrium position and let $F_S = -kx$ denote the force of the spring on the mass (see Theorem 2.1.3, and assume that no other force is acting on the mass, then we know from Example 2.1.4 that $x(t)$ satisfies the second-order ODE given by

$$mx'' + kx = 0.$$

If we set $\omega_0 = \sqrt{\frac{k}{m}}$, we can rewrite this to get

$$x'' + \omega_0^2 x = 0.$$

One thing we can notice right away about solutions of $x'' + \omega_0^2 x = 0$ is that they should all be periodic (see Definition **??**), which, of course, makes sense!. To see why, note that the roots of the characteristic equation are $\pm i\omega_0$, which means that the solutions may be written in the form

$$x = c_1 e^{-i\omega_0 t} + c_2 e^{i\omega_0 t},$$

which we can rewrite as

$$x = A\cos\omega_0 t + B\sin\omega_0 t$$

using Euler's Formula. Hence every solution of $x'' + \omega_0^2 x = 0$ is a sinusoidal wave.

We can go even further by making use of some clever algebra and the appropriate trigonometric formulas if we assume that $x \neq 0$ (i.e. A and B are not *both* 0):

$$\begin{aligned} x &= A\cos\omega_0 t + B\sin\omega_0 t \\ &= \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos\omega_0 t + \frac{B}{\sqrt{A^2+B^2}}\sin\omega_0 t\right) \end{aligned}$$

Now set $C = \sqrt{A^2+B^2}$ and define φ implicitly by $\cos\varphi = \frac{A}{C}$ and $\sin\varphi = \frac{B}{C}$.

> There are infinitely many choices for φ here, but we can select a unique one if we make the extra restriction that $\varphi \in [0, 2\pi)$.

Then we get

$$x = C(\cos\varphi\cos\omega_0 t + \sin\varphi\sin\omega_0 t) = C\cos(\omega_0 t - \varphi).$$

C is the **amplitude** of the wave x, ω_0 is the **circular frequency**, $T = \frac{2\pi}{\omega_0}$ is the **period** of motion and φ is the **phase**.

Example 2.3.2 Another spring-mass system.

An object with a mass of 5 kg is fixed to a spring and a force of 10 N holds the mass 5 m to the left of the spring's equilibrium position. If the object is released, how long will it take for the mass to return to its original position? And what is the position $x(t)$ of the mass?

Solution. Let $x(t)$ denote the position of the mass t seconds after being released, so that $x(0) = -5$ and $x'(0) = 0$. We could find the time it takes for the mass to return to its starting position by first finding $x(t)$, but a quicker way is to just find the period T of x. To do this, we must find the circular frequency $\omega_0 = \sqrt{\frac{k}{m}}$. Thankfully, half of the work is done for us (we know $m = 5$), so we only need the value of k which itself comes from Hooke's Law.

Call the spring force F_S and recall that $F_S = -kx$. We know that it takes a force of 10 N to hold the mass still at $x = -5$. The spring force must *precisely* counterbalance this force in order for the mass to remain still as it's held, which means that $F_S = 10$ since F_S pulls the mass to the right. Therefore $-k(-5) = 10$ and so $k = 2$, which gives $\omega_0 = \sqrt{\frac{2}{5}}$. This means that the period of motion is

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{5}{2}}.$$

To find $x(t)$, we'll use the fact that it can be written as $x = C\cos(\omega_0 t - \varphi)$. Since $C = 5$ (because the mass can never go more than five meters from the equilibrium position) and we already know that $\omega_0 = \sqrt{\frac{2}{5}}$, we just need to find φ. We can just make use of the initial condition $x(0) = -5$ to get this:

$$-5 = C\cos\varphi = 5\cos\varphi$$

and so we can choose $\varphi = \pi$. Hence

$$x(t) = 5\cos(\sqrt{\frac{2}{5}}t - \pi).$$

You may be troubled by the fact that we only explicitly used the first initial condition in this example. However, the second condition $x'(0) = 0$ was actually used implicitly: it allowed us to assume that the amplitude was 5 as opposed to another number. If $x'(0) \neq 0$, finding C would have been a little trickier.

Free Damped Motion

Now we look at how the motion of a mass attached to a spring is altered if the motion is *damped*, say, by a dashpot. See Figure 2.3.3.

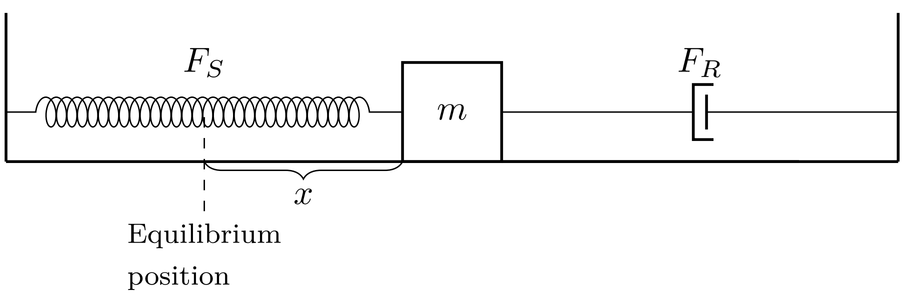

Figure 2.3.3 A damped spring-mass system.

Now in addition to the spring force F_S, we must also worry about the force F_R of the dashpot on the mass. F_R is always going to act against the velocity of the mass, so for simplicity we assume that $F_R = -cx'$ for some $c > 0$. Using $F_S = -kx$ as usual in conjunction with Newton's Second Law gives us the second-order ODE

$$-cx' - kx = mx'' \quad \text{or} \quad mx'' + cx + kx = 0.$$

This ODE is homogeneous and has constant coefficients so we may solve it using the method of characteristic equations. The characteristic equation of this ODE is

$$mr^2 + cr + k = 0 \tag{2.1}$$

The roots of this equation are

$$r_1 = \frac{-c + \sqrt{c^2 - 4km}}{2m} \quad \text{and} \quad r_2 = \frac{-c - \sqrt{c^2 - 4km}}{2m}. \tag{2.2}$$

The behavior of this system therefore depends on the quantity $c^2 - 4km$, and so we have three cases to consider.

1: $c^2 - 4km > 0$.

In this case, the characteristic equation (2.1) has two real roots, and so the solution $x(t)$ has the form

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Now, r_1 and r_2 are both negative since $c^2 - 4km < c^2$ (remember that we're assuming that k and m are positive!). This means that $x(t) \to 0$ as $t \to \infty$. There is no oscillation present in the motion of the mass in this case since the mass never passes through $x = 0$, so we call this **overdamped motion**.

2: $c^2 - 4km = 0$.

In this case, the characteristic equation (2.1) has a repeated (real) root, and so the solution $x(t)$ takes the form

$$x(t) = e^{r_1 t}(c_1 + c_2 t).$$

This mass can pass through $x = 0$ only once, at $t = -\frac{c_1}{c_2}$. Once it does, the mass will ”turn around” soon afterwards and beginning moving back to 0, as in the first case. We call this type of motion **critically damped**, since it's right on the border between overdamped motion and oscillating motion.

3: $c^2 - 4km < 0$.

In this case the characteristic equation (2.1) has two distinct complex roots of the form $r = a \pm bi$, where

$$a = -\frac{c}{2m} \quad \text{and} \quad b = \frac{\sqrt{4km - c^2}}{2m}.$$

In particular, the real part a of these roots is always negative. The solution $x(t)$ in this case takes the form

$$x(t) = e^{at}(A \cos bt + B \sin bt)$$

after applying Euler's formula. As in the previous two cases, $x(t) \to 0$ as $t \to \infty$. However, oscillation is now present in the system for all time! We call this motion **underdamped**, since the damping term e^{at} is not strong enough to cancel out the oscillation present in the system. Note that the real part a contributes to the ”amplitude” e^{at} of the motion, while the imaginary part b represents the angular frequency of the motion. The ordinary frequency of the motion is given by $\frac{b}{2\pi}$.

Example 2.3.4 A Spring-Dashpot System.

Suppose that an object of mass $m = 25$ is attached to both a spring and a dashpot. The mass is held 1 meter to the left of the spring's equilibrium position $x = 0$. The force of the spring on the mass is $F_S = -226x$ and the force of the dashpot on the mass is $F_R = -10x'$, where x is the displacement of the mass. At time $t = 0$, the mass is released. Find $x(t)$.

Solution. The ODE that models the motion of this mass is

$$25x'' + 10x' + 226x = 0,$$

and the roots of the corresponding characteristic equation are

$$r_1 = -\frac{1}{5} \pm \frac{1}{5}\sqrt{-225} = -\frac{1}{5} \pm 3i.$$

We can already see that the motion must be underdamped since we have complex roots, and the position $x(t)$ itself is given by

$$x(t) = e^{-\frac{t}{5}}(A\cos 3t + B\sin 3t).$$

Now we can use the initial conditions $x(0) = -1$ and $x'(0) = 0$ to find A and B. Doing so, we quickly get $A = -1$ and $B = -\frac{1}{15}$. Hence

$$x(t) = -e^{-\frac{t}{5}}\left(\cos 3t + \frac{1}{15}\sin 3t\right).$$

As mentioned previously, the exponential term in $x(t)$ from Example 2.3.4 serves to dampen the motion of the spring. This is illustrated in Figure 2.3.5.

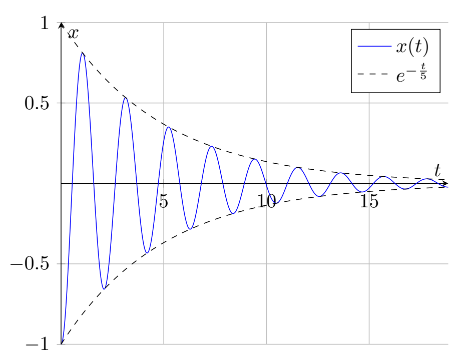

Figure 2.3.5 An exponential term damping motion.

2.4 Solutions of Non-homogeneous Equations

In this section we'll deal with non-homogeneous equations and finding their solutions. This will help us to model systems involving an external force.

This section corresponds to Section 2.7 of the text.

The Method of Undetermined Coefficients

Consider the non-homogeneous linear ODE with constant coefficients given by

$$ay'' + by' + cy = f(x). \tag{2.3}$$

If $f(x)$ were zero then we could solve this by finding the roots of the characteristic equations and using them to determine the appropriate form of the solution. Although that's no longer enough if $f(x) \neq 0$, our method for solving homogeneous equations still plays an important role.

Theorem 2.4.1 Solution of Non-homogeneous Equations.

*Consider the ODE given in (2.3). Let y_c (the **complementary solution**) denote the solution of the **associated homogeneous equation***

$$ay'' + by' + cy = 0$$

*and let y_p (the **particular solution**) denote a single solution of (2.3). Then the general solution of (2.3) is given by $y = y_c + y_p$.*

Theorem 2.4.1 shows that to solve (2.3), we only need to solve the associated homogeneous equation (which we're quite used to by now!) and find a *single* solution of (2.3). The method we will use is the **method of undetermined coefficients**, which we'll demonstrate by example.

Example 2.4.2 Using the Method of Undetermined Coefficients.

Find the general solution of the ODE given by

$$y'' - y' - 2y = 3x + 4.$$

Solution. To find the general solution we need to do two things: find the complementary solution y_c of the associated homogeneous equation and find a single particular solution y_p of the given ODE. We already know how to find y_c,

> The characteristic equation of the ODE is $(r + 1)(r - 2)$.

which is just

$$y_c = c_1 e^{x} + c_2 e^{-2x}.$$

Once we can find a single particular solution y_p we'll be finished.
If we stare that the ODE, then we see that y and its derivatives must cancel each other and leave a polynomial. So it's reasonable to guess that a polynomial might be a solution of the ODE, or equivalently, y_p *should be a polynomial.* Since the degree

> Recall that the degree of a polynomial is just the highest power of the variable in the polynomial.

of the right hand side is 1, then y_p should probably be degree 1 as well.
This means that $y_p = A_1 x + A_2$ for some constants A_1, A_2.
To find these constants (**undetermined coefficients**), we plug our guess into the ODE to get

$$0 - A_1 - 2A_1 x - 2A_2 = 3x + 4.$$

The only way for this equation to be true is for

$$\begin{aligned} -A_1 - 2A_2 &= 4 \\ -2A_1 &= 3 \end{aligned}$$

So $A_1 = -\frac{3}{2}$, $A_2 = \frac{5}{4}$ and $y_p = -\frac{3}{2}x + \frac{5}{4}$. Hence the general solution of the ODE is

$$y = y_c + y_p = c_1 e^x + c_2 e^{-2x} - \frac{3}{2}x + \frac{5}{4}.$$

Note that in Example 2.4.2, we didn't need any initial conditions to find y_p. This means that if a non-homogeneous ODE like the one in Example 2.4.2 represents some physical system, then the initial configuration of that system *has no effect on* y_p. We will see soon that particular solutions correspond to external forces on a system, like gravity, whereas complementary solutions correspond to internal forces in a system, such as the spring force.

Example 2.4.3

Find the general solution of

$$y'' - 2y' + y = x - \sin(2x).$$

Solution. The general solution will take the form $y = y_c + y_p$. Once again, we find y_c by solving the characteristic equation to get

$$y_c = c_1 x + c_2 x e^x.$$

Now we can make a guess as to what y_p should be, once again based on the right hand side of the ODE. If we want to differentiate y_p and obtain $x - \sin(2x)$, then y_p should include both an x term and a $\sin(2x)$ term. If we make the guess that $y_p = A_1 x + A_2 \sin(2x)$, then we get

$$\begin{aligned} A_1 x - 2A_1 &= x \\ -4A_2 \cos(2x) - 3A_2 \sin(2x) &= -\sin(2x) \end{aligned}$$

This forces $A_1 = 1$ *and* $A_1 = 0$, as well as $A_2 = \frac{1}{3}$ *and* $A_2 = 0$. Obviously, this is a problem!
What happened here is we didn't give our guess for y_p enough flexibility. We know we want y_p to involve x and $\sin(2x)$, but as soon as we plug this into the ODE and start differentiating constant terms and cosine terms will begin to appear, and *we need to account for these as well.* So we'll update our guess for y_p, and assume

$$y_p = A_1 x + A_2 + A_3 \cos(2x) + A_4 \sin(2x).$$

Plugging this into the ODE and collecting like terms gives

$$\begin{aligned} A_1 &= 1 \\ -2A_1 + A_2 &= 0 \\ -3A_3 - 4A_4 &= 0 \\ 4A_3 - 3A_4 &= -1 \end{aligned}$$

Hence

$$A_1 = 1$$
$$A_2 = 2$$
$$C = -\frac{4}{25}$$
$$D = \frac{3}{25}$$

and the solution of our ODE is

$$c_1e^x + c_2xe^x + x + 2 - \frac{4}{25}\cos(2x) + \frac{3}{25}\sin(2x).$$

Example 2.4.4 Method of Undetermined Coefficients with Overlap.

Find the solution $y(x)$m> of the IVP

$$y'' + 9y = 3\cos(3x), \quad y(0) = 1, y'(0) = -1$$

Solution. We can start this example the same way we've done the previous ones. First, we find y_c by solving $y'' + 9y = 0$. So

$$y_c = c_1\cos(3x) + c_2\sin(3x).$$

Now we find y_p. Since the RHS of the ODE is $3\cos(3x)$, we'll guess that $y_p = A\cos(3x) + B\sin(3x)$. However, this will cause us problems! Since $\cos(3x), \sin(3x)$ are both solutions of the corresponding homogeneous ODE, then plugging y_p into $y'' + 9y$ will just give 0 again, instead of $3\cos(3x)$.

The problem here is that our guess for y_p overlaps with y_c. To fix this, we'll multiply our guess for y_p by the *smallest* power of x that removes the overlap. In this case, we'll just multiply by x to get

$$y_p = x[A\cos(3x) + B\sin(3x)]$$

Now, we'll plug our modified guess into the ODE and set it equal to $3\cos(3x)$:

$$\begin{aligned} 3\cos(3x) &= y_p'' + 9y \\ &= -6A\sin(3x) + 6B\cos(3x) \end{aligned}$$

So we need $-6A = 0$ and $6B = 3$, or just $A = 0, B = \frac{1}{2}$. Hence

$$y_p = x\left[\frac{1}{2}\sin(3x)\right],$$

and the general solution is then

$$y = y_c + y_p = c_1\cos(3x) + c_2\sin(3x) + \frac{x}{2}\sin(3x).$$

To find the solution of the IVP, we just plug in the initial conditions. Since $y(0) = 1$, this gives

$$1 = c_1.$$

And since

$$y' = -3c_1\sin(3x) + 3c_2\cos(3x) + \frac{1}{2}\sin(3x) + \frac{3}{2}x\cos(3x),$$

$y'(0) = -1$ gives

$$-1 = 3c_2 \Rightarrow c_2 = -\frac{1}{3}.$$

So

$$y = \cos(3x) - \frac{1}{3}\sin(3x) + \frac{x}{2}\sin(3x).$$

What we did in Example 2.4.4 will work in general: if y_c and the initial guess for y_p overlap, i.e. contain linearly dependent terms, then we multiply y_p by the *smallest* power of x (or the appropriate independent variable) that removes the overlap.

Example 2.4.5 Determining the Appropriate Form of the Particular Solution.

Consider the ODE $x'' + 4x' + 4 = 3t^3 - t + t^2\sin(2t) + 3e^{-2t}$. Find the correct guess for x_p.

Solution. Before we can do anything with x_p, we need to find the complementary solution x_c. This is given by $x_c = c_1e^{-2t} + c_2te^{-2t}$. Now we can try to guess an appropriate form for x_p using the right hand side of the ODE. Each "component" of the right hand side contributes to our guess for x_p as follows:

Table 2.4.6

component	contribution to x_p
$3t^3 - t$	$At^3 + Bt^2 + Ct + D$
$t^2\sin(2t)$	$(Et^2 + Ft + G)\cos(2t) + (Ht^2 + It + J)\sin(2t)$
$3e^{-2t}$	Ke^{-2t}

But now we have a problem, since Ke^{-2t} overlaps with x_c. So we multiply by t^2 to remove the overlap, and hence $x_p = At^3 + Bt^2 + Ct + D + (Et^2 + Ft + G)\cos(2t) + (Ht^2 + It + J)\sin(2t) + 3e^{-2t} + Kt^2e^{-2t}$.

2.5 Forced Oscillations and Resonance

In this section we will develop models for certain systems under the influence of a periodic, external force. The presence of an external force leads to non-homogeneous models, and we will use the techniques we developed in Section 2.4 to deal with these systems.

This section corresponds to Section 2.8 of the text.

Undamped Systems

Consider the spring-mass system set up as in Figure 2.3.1. Then we know that the displacement $x(t)$ satisfies $x'' + \omega_0^2x = 0$, where $\omega_0 = \sqrt{\frac{k}{m}}$.

Now suppose that an external force $F_E = F_0 \cos(\omega t)$ also acts on the mass. Then the ODE that models the displacement $x(t)$ is

$$x'' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t).$$

The solution of this ODE is then $x = x_c + x_p$, where

$$x_c = A\cos(\omega - 0t) + B\sin(\omega_0 t)$$

$$x_p = \begin{cases} \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t) & \text{if } \omega \neq \omega_0 \\ \frac{F_0}{2m\omega_0} t\sin(\omega_0 t) & \text{if } \omega = \omega_0 \end{cases}$$

Systems where the **external frequency** ω is equal to the **internal frequency** ω_0 are said to be in **resonance**. Without a damping force, the mass in such systems will move wildly out of control $|x_p(t)|$ gets arbitrarily large as $t \to \infty$.

Example 2.5.1 Determining Resonance.

An object with mass 2 kg is attached to a spring and is held 1 m to the right of the spring's equilibrium position by a force of 8 N At time $t = 0$ seconds the mass is set in motion with an initial velocity of $2\,\frac{\text{m}}{\text{s}}$ to the left. Suppose an external force $F_E = 3\cos(2t)$ acts on the mass as well. Will the spring eventually break?

Solution. We can answer this question by determining if resonance is present in this system. The external frequency is $\omega = 2$, and the internal frequency is $\omega_0 = \sqrt{\frac{k}{m}}$. Since $k = 8$ and $m = 2$, we have $\omega_0 = \sqrt{\frac{8}{2}} = 2$, and so the frequencies match. Hence the system is in resonance, and we can expect the spring to eventually break.

It's not too hard to solve for $x(t)$ exactly here to get

$$x(t) = \frac{1}{4}\cos(2t) - \sin(2t) + \frac{3}{8}t\sin(2t).$$

Graphing this, we get the figure produced in Figure 2.5.2.

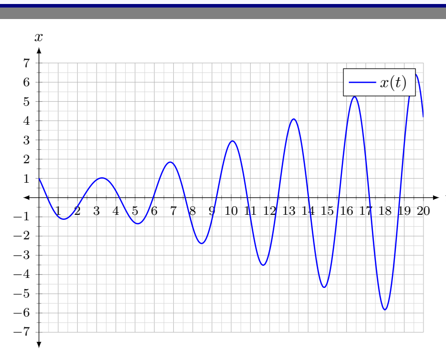

Figure 2.5.2 A plot of the motion of the mass in Example 2.5.1.

Damped Systems

Now we'll take a look at forced, damped systems. Suppose a mass m is fixed to a spring with spring force $F_S = -kx$, and is acted upon by a dashpot with force $F_R = -cx'$, where $c, k > 0$ and x represents the displacement of the mass at time t. If the mass is still acted upon by an external force $F_E = F_0 \cos(\omega t)$, then by Newton's Second Law the displacement x must satisfy

$$mx'' + cx' + kx = F_0 \cos(\omega t).$$

The solution is given by $x = x_c + x_p$, where x_c is found as in Free Damped Motion and goes to 0 as $t \to \infty$. With a little help from a computer algebra system such as Sage (see below), we see that

$$x_p = \frac{m(\omega_0^2 - \omega^2)F_0}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2} \cos(\omega t) + \frac{\omega c F_0}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2} \sin(\omega t),$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ as usual.

Since x_c will always approach 0 in these situations, as time goes on the position is determined increasingly by x_p. We call x_c the **transient solution** and x_p the **steady-state solution**. Note that resonance is impossible in this system since x_c and x_p can never overlap (assuming the external force is still a sinusoid). Therefore the smallest amount of damping prevents the mass from going out of control.

```
# Set variables
t = var('t')
m, c, k, F0, omega, omega_0 = var('m␣c␣k␣F0␣omega␣omega_0')

# Make assumption to help Sage/Maxima with computation; this
    corresponds to underdamped motion
assume(4*k*m - c^2 > 0)

# Define x as a function of t
x = function('x')(t)

# Set up and solve the ODE
P = desolve(m*diff(x, t, 2) + c*diff(x,t) + k*x ==
    F0*cos(omega*t), x, ivar=t)

# Make "nice" output for solution
pretty_print(P.substitute(k == m*(omega_0)^2))
```

```
# Sage output without pretty printing.
(_K2*cos(1/2*sqrt(4*omega_0^2 - c^2/m^2)*t) +
    _K1*sin(1/2*sqrt(4*omega_0^2 -
    c^2/m^2)*t))*e^(-1/2*c*t/m) + (F0*c*omega*sin(omega*t) -
    (F0*m*omega^2 -
    F0*m*omega_0^2)*cos(omega*t))/(m^2*omega^4 +
    m^2*omega_0^4 - (2*m^2*omega_0^2 - c^2)*omega^2)
```

Example 2.5.3 Steady-State Approximation.

An object of mass 3 kg is fixed to both a spring and a dashpot with respective forces $F_S = -9x$ and $F_R = -2x'$, where x is the displacement of the mass in meters and $x = 0$ is the equilibrium position. An external force $F_E = 5\cos(\sqrt{3}t)$ is also applied to the mass, where t is in seconds. The mass was set in motion with an unknown speed and unknown velocity approximately 7 s ago. What will be the approximate position of the mass in 40 s?

Solution. We know that the position will look like $x = x_c + x_p$ where $x_c \to 0$ as $t \to \infty$, but we can't find the exact form of the transient solution without knowing the initial conditions. So we'll assume that we can estimate the position of the mass using the steady-state solution x_p. Since

$$\omega = \omega_0 = \sqrt{3}, \quad F_0 = 5 \quad \text{and} \quad c = 2,$$

we get

$$x_p = \frac{5}{6}\sqrt{3}\sin(\sqrt{3}t).$$

So after 40 s more seconds the mass should be around $x_p(47)$, or about -0.388 m.

In fact, the actual initial conditions used in Example 2.5.3 were $x(0) = 1$ and $x'(0) = -2$. The corresponding exact solution is

$$x = e^{-\frac{t}{3}}\left[\cos\frac{\sqrt{26}}{3}t - \frac{25}{52}\sin\frac{\sqrt{26}}{3}t\right] + \frac{5}{6}\sqrt{3}\sin(\sqrt{3}t).$$

The exact value of $x(47)$ is within several *millionths* of the approximation $x_p(47)$.

Chapter 3

Higher Order Linear ODEs

This chapter applies many of the ideas developed in Chapter 2 to higher order linear ODEs. We'll begin by transferring many of the concepts and terms introduced in Sections 2.1–2.2.

3.1 Homogeneous Linear ODEs with Constant Coefficients

Each tool that we used for solving second order linear ODEs with constant coefficients, in other words those ODEs of the form $ay'' + by' + cy = f(x)$, can be extended to solving more general n^{th} order linear ODEs with constant coefficients, which take the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x) \tag{3.1}$$

where $a_i, 0 \leq i \leq n$ are constants and $a_n \neq 0$. In this section, we'll focus on solutions of *homogeneous* n^{th} order linear ODEs with constant coefficients. These are the ODEs where $f(x) = 0$ in (3.1).

Recall that in Chapter 2, the general solution of $ay'' + by' + cy = 0$ could be obtained by finding *two* linearly independent solutions y_1, y_2. The general solution was then $y = c_1 y_1 + c_2 y_2$ (which is guaranteed to be a solution by Theorem 2.1.6 where c_1, c_2 are arbitrary constants. Similarly, the general solution of (3.1) is found by obtaining n linearly independent solutions $y_1, \dots, y_n$. The general solution in this case is now $y = \sum_{i=1}^{n} c_i y_i$, where $c_i, 1 \leq i \leq n$ are arbitrary constants. Our main tool for showing that a collection $\{y_1, \dots, y_n\}$ of solutions is in fact linearly independent is again the *Wronskian*.

Definition 3.1.1 The Wronskian of Several Functions.

Let $\{y_1, \dots, y_n\}$ be a collection of functions. Then the **Wronskian** of $\{y_1, \dots, y_n\}$, denoted by $W(y_1, \dots, y_n)$, is defined by

$$W(y_1, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

Just as before, we have the following connection between the Wronskian and linear independence. This theorem is a restatement of Theorem 2.1.11 for collections involving more than two functions.

Theorem 3.1.2 Linear Independence and the Wronskian (Several Functions).

Let I be some open interval (we often take I to be $(-\infty, \infty)$, but it doesn't have to be so) and let $\{y_1, \ldots, y_n\}$ be solutions of (3.1). If $W(y_1, \ldots, y_n) \neq 0$ for some point $x_0 \in I$, then $\{y_1, \ldots, y_n\}$ is linearly independent.

Now that we have a tool for determining linear independence of several functions, we can also define a *basis of solutions* for higher order ODEs.

Definition 3.1.3 Basis of Solutions (Higher Order ODEs).

Let $y_1, y_2, \ldots,$ and y_n denote solutions of some n^{th} order linear homogeneous ODE. We call $\{y_1, y_2, \ldots, y_n\}$ a **basis** if this set is also linearly independent.

Bases are used to determine general solutions of linear ODEs.

Example 3.1.4 Finding a Basis Set of Solutions.

Find the general solution of

$$y^{(3)} - 6y'' + 11y' - 6y = 0$$

where y is a function of x.

Solution. Just as we did for second order ODEs, we'll solve this by finding the characteristic equation. To get the characteristic equation, we replace derivatives of y with powers of r to get

$$r^3 - 6r^2 + 11r - 6 = 0.$$

Now we need to solve this equation. It can factor (notice that $r = 1$ is a solution and then divide $r^3 - 6r^2 + 11r - 6$ by $r - 1$) as

$$(r-1)(r-2)(r-3) = 0$$

and so $r = 1, 2, 3$. This means that the functions

$$y_1 = e^x, y_2 = e^{2x}, y_3 = e^{3x}$$

are all solutions of the original ODE. If we can show that they're also linearly independent, then this will imply that the general solution of the ODE is given by

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

To do this, we just compute the Wronskian of these functions:

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$
\begin{aligned}
&= e^x(18e^{5x} - 12e^{5x}) - e^{2x}(9e^{4x} - 3e^{4x}) + e^{3x}(4e^{3x} - 2e^{3x}) \\
&= 2e^{6x} \\
&\neq 0.
\end{aligned}
$$

Since the Wronskian is nonzero, y_1, y_2 and y_3 are in fact linearly independent, and so the general solution of this ODE is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

It can get tedious to try to compute the Wronskian every time when solving linear ODEs with constant coefficients, so it's good to note that $\{e^{r_1 x}, \ldots, e^{r_n x}\}$ is guaranteed to be linearly independent as long as each of the r_i are distinct from the others.

Example 3.1.5

Find the general solution of $y^{(4)} - y = 0$, where y is a function of x.

Solution. We begin by finding the characteristic equation, which is

$$r^4 - 1 = 0 \quad \text{or} \quad (r^2 - 1)(r^2 + 1) = 0.$$

This has solutions $r = \pm 1, \pm i$. This means that $y_1 = e^x, y_2 = e^{-x}, y_3 = e^{ix}$ and $y = e^{-ix}$ are all solutions of the ODE. Since the roots of the characteristic equation are all distinct, this means that these solutions are linearly independent from each other. Since we have four linearly independent solutions, we can then construct the general solution of this ODE:

$$y = c_1 e^x + c_2 e^{-x} + c_3 e^{ix} + c_4 e^{-ix},$$

which we can also rewrite using Euler's Formula $e^{i\theta} = \cos\theta + i \sin\theta$ to get

$$y = c_1 e^x + c_2 e^{-x} + A\cos x + B \sin x.$$

In general, any root of the characteristic equation of the form $r = a \pm bi$ contributes the term $e^{ax}\left[A\cos bx + B\sin bx\right]$ to the general solution. As we saw in Chapter 2, it's possible for some characteristic equations to have repeated roots. In this case, we initially weren't able to get enough linearly independent solutions, so we had to adjust our method a bit. The same adjustment will work here.

Example 3.1.6 Characteristic Equation with Repeated Roots.

Find the general solution of

$$x^{(3)} + 4x'' + 4x' = 0$$

where x is a function of t.

Solution. The characteristic equation is

$$r^3 + 4r^2 + 4r = 0 \quad \text{or} \quad r(r+2)^2 = 0$$

so $r = 0, -2, -2$. One solution of the ODE will be $x_1 = e^{0t} = 1$, and a second solution will be $x_2 = e^{-2t}$. But since $r = -2$ is a repeated root, it does not provide a third linearly independent solution x_3. So we'll use the same trick we used before and multiply by t to get a third solution: $x_3 = te^{-2t}$. It can be verified that x_3 is in fact a solution of the ODE, and is also linearly independent from x_1, x_2. Therefore the general solution of the ODE is

$$x = c_1 + c_2e^{-2t} + c_3te^{-2t}.$$

Solutions of linear, homogeneous ODEs with constant coefficients depend *entirely* on the roots of the corresponding characteristic equation. If we write the independent variable as x, and if r_k denotes a single root of the characteristic equation, then the general solution of the ODE will contain $e^{r_kx}, xe^{r_kx}, x^2e^{r_kx}, \ldots$, with the number of exponentials contributed by r_k being equal to its multiplicity. That is, the number of times r_k appears as a solution of the characteristic equation.

Example 3.1.7

A linear, homogeneous ODE with constant coefficients (and independent variable t) has characteristic equation given by

$$r(r-1)(r+1)^2(2r-3)^3 = 0.$$

What is the general solution of the ODE?

Solution. We'll set up a table listing each root, its multiplicity and its contribution to the general solution:

Table 3.1.8

Root	Multiplicity	Contribution
0	1	1
1	1	e^t
-1	2	e^{-t}, te^{-t}
$\frac{3}{2}$	3	$e^{3t/2}, te^{3t/2}, t^2e^{3t/2}$

So the general solution of this ODE is

$$c_1 + c_2e^t + c_3e^{-t} + c_4te^{-t} + c_5e^{3t/2} + c_6te^{3t/2} + c_7t^2e^{3t/2}.$$

Example 3.1.9

An ODE (with independent variable x) has characteristic equation given by

$$r^2(r-3)(r^2+4)(r^2+6r+13)^3 = 0.$$

Find the general solution.

Solution. We'll set up another table help us determine the general solution:

Table 3.1.10

Root	Multiplicity	Contribution
0	2	$1, x$
3	1	e^{3x}
$\pm 2i$	1	$\cos 2x, \sin 2x$
$-3 \pm 2i$	3	$e^{-3x}\cos 2x, e^{-3x}\sin 2x, xe^{-3x}\cos 2x, xe^{-3x}\sin 2x, x^2e^{-3x}\cos 2x, x^2e^{-3x}\sin 2x$

So the general solution is

$$c_1+c_2x+c_3e^{3x}+A\cos 2x+B\sin 2x+e^{-3x}\left[(A_1+A_2x+A_3x^2)\cos 2x+(B_1+B_2x+B_3x^2)\sin 2x\right].$$

3.2 Non-homogeneous Linear ODEs with Constant Coefficients

For second order ODEs that were nonhomogeneous, linear and had constant coefficients, we found their general solution by first finding the complementary solution y_c and then a corresponding particular solution y_p. The general solution was then $y = y_c + y_p$. y_c was found by solving the corresponding homogeneous equation and we used the method of undetermined coefficients to find y_p. Although we are now looking at higher order ODEs, the method of undetermined coefficients remains unchanged.

Example 3.2.1

Find the general solution of

$$y^{(3)} - y' = \sinh 2x.$$

Solution. The general solution takes the form $y = y_c + y_p$, where y_c is a solution of the associated homogeneous equation $y^{(3)} - y' = 0$ and y_p is a single solution of the original ODE $y^{(3)} - y' = \sinh 2x$. Since the characteristic equation of $y^{(3)} - y' = 0$ is $r^3 - r = 0$, we get

$$y_c = c_1 + c_2e^x + c_3e^{-x} = c_1 + A\cosh x + B\sinh x.$$

Now we'll find y_p. Since the right hand side of the ODE is $\sinh 2x$, a good initial guess would be $y_p = C\sinh 2x$. However, when we take this guess and plug it into the ODE, we'll start seeing terms involving $\cosh 2x$ as well (since $\frac{d}{dx}\sinh 2x = 2\cosh 2x$) so this means we'll want to include $\cosh 2x$ into our guess for y_p also. So we'll modify our guess to be $y_p = C\cosh 2x + D\sinh 2x$.
Since our guess for y_p doesn't overlap with y_c, we can proceed with plugging our guess into the original ODE $y^{(3)} - y' = \sinh 2x$ and equating coefficients, just as we did before.

$$\begin{aligned}\sinh 2x &= y_p^{(3)} - y_p' \\ &= (8C\sinh 2x + 8D\cosh 2x) - (2C\sinh 2x + 2D\cosh 2x) \\ &= 6C\sinh 2x + 6D\cosh 2x.\end{aligned}$$

So we get $C = \frac{1}{6}$ and $D = 0$, which means that $y_p = \frac{1}{6}\cosh 2x$. When solving for y_p, always remember to plug the values you find back into the guess for y_p that you used! So the general solution of the ODE is

$$y = c_1 + A\cosh x + B\sinh x + \frac{1}{6}\cosh 2x.$$

Just as before, we also need to worry about overlaps.

Example 3.2.2

Find the appropriate form of x_p for the ODE

$$\frac{d^7x}{dt^7} + 8\frac{d^5x}{dt^5} + 16\frac{d^3x}{dt^3} = 5t - 3t^2 + e^{-t}\cos 2t - 3t\sin 2t.$$

Solution. We need to find x_c first since x_p will change depending on x_c. Since the characteristic equation of the associated homogeneous ODE $\frac{d^7x}{dt^7} + 8\frac{d^5x}{dt^5} + 16\frac{d^3x}{dt^3} = 0$ is

$$r^7 + 8r^5 + 16r^3 = 0 \quad \text{or} \quad r^3(r^2+4)^2 = 0,$$

we get

$$x_c = c_1 + c_2t + c_3t^2 + A\cos 2t + B\sin 2t + t\left[C\cos 2t + D\sin 2t\right].$$

Now we come up with a guess for x_p using the right hand side of the original ODE and dividing it into ``components:''

Table 3.2.3

Component	Contribution to x_p
$2t - 3t^2$	$C_1t^2 + C_2t + C_3$
$e^{-t}\cos 2t$	$e^{-t}\left[C_4\cos 2t + C_5\sin 2t\right]$
$-3t\sin 2t$	$(C_6t + C_7)\cos 2t + (C_8t + C_9)\sin 2t$

However, we now have a problem with overlaps between x_p and x_c. The guess corresponding to the first component overlaps with x_c, so we need to multiply it by t^3 to remove the overlap. Similarly, the guess corresponding to the third component overlaps, so we must multiply it by t^2. Therefore, our guess for x_p should be

$$x_p = C_1t^5 + C_2t^4 + C_3t^3 + e^{-t}\left[C_4\cos 2t + C_5\sin 2t\right] + (C_6t^3 + C_7t^2)\cos 2t + (C_8t^3 + C_9t^2)\sin 2t.$$

The method of undetermined coefficients applied to the ODE

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = f(x)$$

can be summarized by the following table. Note that x^m is the *smallest* power of x required to remove any overlaps with y_c.

Table 3.2.4

Component of $f(x)$	Contribution to y_p
$c_kx^k + \cdots + c_1x + c_0$	$x^m(C_kx^k + C_{k-1}x^{k-1} + \cdots + C_1x + C_0)$
$(c_kx^k + \cdots + c_1x + c_0)e^{ax}\sin bx$	$x^me^{ax}\left[(C_kx^k + \cdots + C_1x + C_0)\cos bx + (D_kx^k + \cdots + D_1x + D_0\right.$
$(c_kx^k + \cdots + c_1x + c_0)e^{ax}\cos bx$	$x^me^{ax}\left[(C_kx^k + \cdots + C_1x + C_0)\cos bx + (D_kx^k + \cdots + D_1x + D_0\right.$
$(c_kx^k + \cdots + c_1x + c_0)e^{ax}\sinh bx$	$x^me^{ax}\left[(C_kx^k + \cdots + C_1x + C_0)\cosh bx + (D_kx^k + \cdots + D_1x + D\right.$
$(c_kx^k + \cdots + c_1x + c_0)e^{ax}\cosh bx$	$x^me^{ax}\left[(C_kx^k + \cdots + C_1x + C_0)\cosh bx + (D_kx^k + \cdots + D_1x + D\right.$

Example 3.2.5

Find the general solution of

$$y^{(3)} - 2y'' + 3y' = x + e^{-3x}\sin x$$

Solution. We begin by finding y_c. Since the characteristic equation of the corresponding homogeneous ODE is $r^3 - 2r^2 + 3r = 0$, we get $r = -0, \frac{2\pm\sqrt{4-12}}{2}$ or just $r = 0, 1 \pm i\sqrt{2}$. So

$$y_c = c_1 + e^x(A\cos\sqrt{2}x + B\sin\sqrt{2}x).$$

Now we can set up y_p:

Table 3.2.6

Component	Contribution to y_p
x	$x(C_1x + C_2)$
$e^{-3x}\sin x$	$e^{-3x}\left[C_3\cos x + C_4\sin x\right]$

So our initial guess for y_p is given by $y_p = C_1x^2 + C_2x + C_3e^{-3x}\cos x + C_4e^{-3x}\sin x$. Plugging this into the ODE into a CAS such as Maple or Sage gives

$$\begin{aligned} x + e^{-3x}\sin x &= y_p^{(3)} - 2y_p'' + 3y_p' \\ &= 6C_1x + (3C_2 - 4C_1) \\ &\quad + (-43C_3 + 41C_4)\,e^{-3x}\cos x + (-41C_3 - 43C_4)\,e^{-3x}\sin x \end{aligned}$$

This gives us the system of equations

$$\begin{aligned} 6C_1 &= 1 \\ 3C_2 - 4C_1 &= 0 \\ -43C_3 + 41C_4 &= 0 \\ -41C_3 - 43C_4 &= 1 \end{aligned}$$

which we can solve using Sage to get

$$C_1 = \frac{1}{6}, C_2 = \frac{2}{9}, C_3 = -\frac{41}{3530}, C_4 = -\frac{43}{3530}.$$

So the general solution of the ODE is

$$y = y_c + y_p = c_1 + e^x(A\cos\sqrt{2}x + B\sin\sqrt{2}x) + \frac{1}{6}x^2 + \frac{2}{9}x - \frac{e^{-3x}}{3530}\left[41\cos x + 43\sin x\right].$$

```
# Set variables
c1, c2, c3, c4 = var('c1␣c2␣c3␣c4')

# Set up equations as variables for clarity
eqn1 = 6*c1
eqn2 = 3*c2 - 4*c1
eqn3 = -43*c3 + 41*c4
eqn4 = -41*c3 - 43*c4

solve([eqn1 == 1, eqn2 == 0, eqn3 == 0, eqn4 == 1], c1, c2,
```

```
    c3, c4)
```

```
[[c1 == (1/6), c2 == (2/9), c3 == (-41/3530), c4 ==
    (-43/3530)]]
```

Chapter 4

Systems of Ordinary Differential Equations

4.1 Systems of ODEs as Models

Interdependent quantities can often be represented mathematically by a system of equations. If we have information about the rates of change of these quantities, then we may be able to develop a model using a system of differential equations.

Definition 4.1.1

A **first order system** of ODEs is a system of differential equations involving some collection of functions and their first derivatives.

We are still only dealing with ordinary differential equations which means that we will only ever have one independent variable. However, when dealing with systems of ODEs we will be working with several *dependent* variables.

The systems of ODEs that we will consider will typically look like the following:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_2 \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned}$$

where a_{ij} are constants and y_i are functions of t.

Example 4.1.2

Two brine tanks are set up as in Figure 4.1.3. Fresh water flows into the tank at a rate of r_1, well-mixed solution flows from Tank 1 to Tank 2 at a rate of r_2 and well-mixed solution flows out of Tank 2 at a rate of r_3. Suppose that r_1, r_2 and r_3 are $5\,\frac{\text{gal}}{\text{min}}$, the volume of solution in Tank 1 is $10\,\text{gal}$ and the volume of solution in Tank 2 is $7\,\text{gal}$. Suppose Tank 1 has $5\,\text{lb}$ of salt at time $t = 0$ and Tank 2 has $2\,\text{lb}$ of salt at time $t = 0$. Set up a first-order system that describes the amount of salt in each tank at time t.

Solution. Let $x_1(t)$ denote the amount of salt in Tank 1 at time t, and $x_2(t)$ denote the amount of salt in Tank 2 at time t. Using the mixture ODE $\dfrac{dx}{dt} = r_i c_i - r_o \frac{x}{V(t)}$ developed in Section 1.4, we can write

$$\begin{aligned}\frac{dx_1}{dt} &= 5 \cdot 0 - 5\frac{x_1}{10} \\ \frac{dx_2}{dt} &= 5\frac{x_1}{10} - 5\frac{x_2}{7}\end{aligned}$$

or just

$$\begin{aligned}x_1' &= -\frac{1}{2}x_1 \\ x_2' &= \frac{1}{2}x_1 - \frac{5}{7}x_2,\end{aligned}$$

with initial conditions $x_1(0) = 5$ and $x_2(0) = 2$.

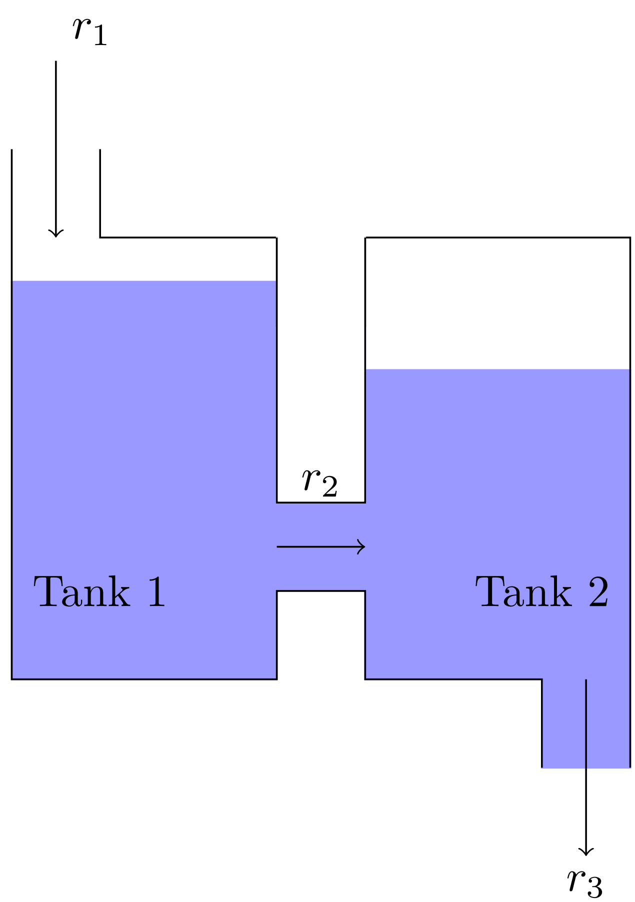

Figure 4.1.3 The two interconnected tanks from Example 4.1.2.

To actually solve systems of ODEs, we'll use *matrices* to rewrite these systems as *matrix ODEs*.

Definition 4.1.4

An $m \times n$ **matrix** is an array of m rows and n columns. $m \times 1$ matrices are called (**column**) **vectors**. Matrices are typically denoted with capital italic letters (such as A, M) and vectors are often denoted with lower case bold letters (such as $\mathbf{v}, \mathbf{x}$). A **zero matrix** will be denoted using $\mathbf{0}$.

As a brief example, let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Then A is a 2×3 matrix and $\mathbf{y}$ is a 3×1 vector.

Definition 4.1.5 Matrix-Vector Product.

Let A be the 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Then their **product** $A\mathbf{v}$ is the vector defined to be

$$A\mathbf{v} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{bmatrix}.$$

The 2×2 **identity matrix** is the matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. A **scalar** is just a constant. To multiply a scalar c with a matrix A, just multiply every element of A with c.

If A is any 2×2 matrix and $\mathbf{v}$ and 2×1 vector, then $AI = IA = A$ and $I\mathbf{v} = \mathbf{v}$.

Example 4.1.6

Let $A = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$. Compute $A\mathbf{v}_1$ and $A\mathbf{v}_2$.

Solution. By definition,

$$A\mathbf{v}_1 = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 1 \\ -3 \cdot 1 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad A\mathbf{v}_2 = \begin{bmatrix} 1 \cdot 0 + 0 \cdot 5 \\ -3 \cdot 0 + 2 \cdot 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}.$$

In Example 4.1.6, notice that $A\mathbf{v}_2 = 2\mathbf{v}_2$. This means that A didn't really do all that much to $\mathbf{v}_2$ except to stretch it by a factor of 2. Vectors with this property will turn out to be the key to solving our systems of ODEs.

Any linear system can be written as an equivalent first-order system or matrix ODE.

Example 4.1.7 Interconnected spring-mass system.

Consider a spring mass system with two masses arranged as follows:

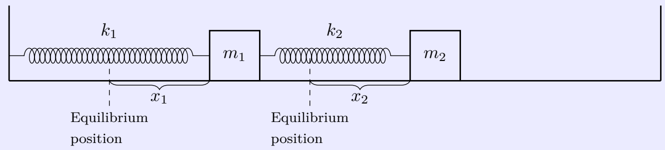

Figure 4.1.8 A spring-mass system with interconnected masses

Determine a first-order system that the displacements x_1 and x_2 must satisfy.

Solution. From Section 2.3 we know how to model a spring-mass system with a single mass using Theorem 2.1.3 and Newton's Second Law. We will apply this same analysis to the displacements x_1 and x_2 individually.

To begin, we will analyze the forces acting on the first mass. Here there are two forces to consider: the force caused by the motion of m_1 and the force caused by the motion of m_2. Likewise, the second mass is also influenced by two forces. We arrange these in the following table:

Table 4.1.9 Forces acting on an interconnected spring-mass system

mass	forces
m_1	$-k_1x_1, -k_2x_1, k_2x_2$
m_2	$k_2x_1, -k_2x_2$

Now we can apply Newton's Second Law to get the *second*-order system

$$\begin{aligned} m_1x_1'' &= -k_1x_1 - k_2x_1 + k_2x_2 \\ m_2x_2'' &= k_2x_1 - k_2x_2. \end{aligned}$$

At this point we can introduce new dependent variables u_1, u_2, u_3 and u_4 to get an equivalent first-order system.

Although this type of system is new, the solutions behave as expected. In particular, both x_1 and x_2 display periodic motion as can be seen by the Sage example below.

```
# Define our independent variable
var('t')

# Define the dependent variables as functions of t
x1 = function('x_1')(t)
x2 = function('x_2')(t)

# System parameters
m1, m2 = 1, 1
k1, k2 = 4, 4

# Define the equations in our system
# If you wish to solve the corresponding first-order
    system,
# you will need to specify four equations.
de1 = m1*diff(x1, t, 2) == -(k1+k2)*x1 + k2*x2
de2 = m2*diff(x1, t, 2) == k2*x1 - k2*x2

# Display the (general) solution of this system
# Note that the solution depends on initial conditions
    which
# were NOT provided. In particular, D_0(x)(0)
    represents an
# iniital condition of the form x'(0)
show(desolve_system([de1, de2], [x1, x2]))
```

Definition 4.1.10 Eigenvectors and eigenvalues.

Let A be a matrix. A nonzero vector $\mathbf{v}$ is an **eigenvector** of A if $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ. We call λ an **eigenvalue** of A corresponding to the eigenvector $\mathbf{v}$.

Example 4.1.11

Determine if $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$.

Solution. To do this, we just need to compute $A\mathbf{v}$:

$$A\mathbf{v} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = -\mathbf{v}.$$

So $\mathbf{v}$ is an eigenvector of A with corresponding eigenvalue $\lambda = -1$.

Since we will be looking at systems of ODEs which involve functions, we will need to define vector-valued functions. These objects will represent the solutions of our systems.

Definition 4.1.12 Vector-Valued Functions.

A **vector-valued function** is a vector whose elements are functions. If each of the functions in a vector $\mathbf{x}$ depends on the variable t, we often write $\mathbf{x}(t)$ to denote this. The derivative of a vector-valued function

$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ is the new vector-valued function $\mathbf{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$.

We now have all of the tools we need to rewrite a first-order system as a matrix ODE. Let

$$\begin{aligned} x' &= a_{11}x + a_{12}y \\ y' &= a_{21}x + a_{22}y \end{aligned}$$

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, then

$$A\mathbf{x} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{x}'.$$

In other words, we may rewrite the system as the matrix ODE

$$\mathbf{x}' = A\mathbf{x}.$$

Example 4.1.13

Write the system

$$\begin{aligned} x_1' &= -\frac{1}{2}x_1 \\ x_2' &= \frac{1}{2}x_1 - \frac{5}{7}x_2 \end{aligned}$$

as a matrix ODE.

Solution. We need to find a matrix A and vector $\mathbf{x}$ to let us rewrite this system. The matrix A is formed from the coefficients of x_1, x_2 on the right hand side of the system:

$$A = \begin{bmatrix} -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{5}{7} \end{bmatrix}.$$

The vector $\mathbf{x}$ is just made up of the dependent variables x_1, x_2:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

With these terms, the original system of ODEs is equivalent to the single matrix ODE

$$\mathbf{x}' = A\mathbf{x}.$$

Example 4.1.14

Show that $e^{-t}\mathbf{x}_0$ where $\mathbf{x}_0 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a solution of the system

$$\mathbf{x}' = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}\mathbf{x}.$$

Solution. We'll check that $\mathbf{x}_0$ is a solution of the matrix ODE just as we check solutions for normal ODEs: plug the potential solution into the ODE and check both sides. If we do so, we get

$$\frac{d}{dt}\,[*]\,e^{-t}\mathbf{x}_0 = -e^{-t}\begin{bmatrix}-2\\1\end{bmatrix} = e^{-t}\begin{bmatrix}2\\-1\end{bmatrix}$$

and

$$\begin{bmatrix}1 & 4\\1 & 1\end{bmatrix}e^{-t}\mathbf{x}_0 = e^{-t}\begin{bmatrix}1 & 4\\1 & 1\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix} = e^{-t}\begin{bmatrix}2\\-1\end{bmatrix}.$$

Since these expressions match, this means that $e^{-t}\mathbf{x}_0$ is a solution of the ODE.

One thing to note about the previous example is that $\begin{bmatrix}-2\\1\end{bmatrix}$ was an eigenvector of $\begin{bmatrix}1 & 4\\1 & 1\end{bmatrix}$ with corresponding eigenvalue $\lambda = -1$. See Example 4.1.11. This suggests that solutions of the matrix ODE $\mathbf{x}' = A\mathbf{x}$ take the form $\mathbf{x} = e^{\lambda t}\mathbf{x}_0$, where λ is an eigenvalue of A with corresponding eigenvector $\mathbf{x}_0$. One last concept we need is that of linear independence of vectors.

Definition 4.1.15 Linear Independence of Vectors.

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ denote a collection of vectors. We say that the vectors are **linearly independent** if the equality

$$\sum_{i=1}^{n} c_i\mathbf{x}_i = \mathbf{0}$$

is possible if and only if $c_1 = \cdots = c_n = 0$. Otherwise, we say that the vectors are **linearly dependent**.

Just as before, our primary tool for showing if a collection is linearly independent is the Wronskian.

Definition 4.1.16

The **Wronskian** of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is the number $W(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ defined by

$$W(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \begin{vmatrix}\mathbf{x}_1 & \ldots & \mathbf{x}_n\end{vmatrix}.$$

The vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent if and only if their Wronskian is nonzero.

4.2 Constant Coefficient Systems

The content in this section represents a higher-dimensional analog of the content in Section 2.1.

Solutions of $\mathbf{x}' = A\mathbf{x}$

A system of ODEs involving only constant coefficients can be rewritten as a matrix ODE of the form $\mathbf{x}' = A\mathbf{x}$ where A is a constant matrix. Such a system can be solved using exponentials.

Theorem 4.2.1 Solutions of Systems.

Let A be an $n \times n$ constant matrix, and suppose that A has n linearly independent eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_1$. Then the general solution of $\mathbf{x}' = A\mathbf{x}$ is given by

$$\mathbf{x} = \sum_{i=1}^{n} c_i e^{\lambda_i t} \mathbf{x}_i.$$

Example 4.2.2

Find the general solution of the system

$$\begin{aligned} x_1' &= x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 \\ x_2' &= -\frac{1}{2}x_1 + x_2 - \frac{1}{2}x_3 \\ x_3' &= -\frac{1}{2}x_1 - \frac{1}{2}x_2 + x_3 \end{aligned}$$

given that

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

are eigenvectors of the matrix

$$G = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

with corresponding eigenvalues $\lambda_1 = 0, \lambda_2 = \lambda_3 = \frac{3}{2}$.

Solution. First, note that the system we need to solve is equivalent to the matrix ODE $\mathbf{x}' = G\mathbf{x}$. If we can show that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent, then we can use Theorem 4.2.1 to find the general solution of the system. So we'll compute their Wronskian:

$$\begin{aligned} W(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= \begin{vmatrix} 1 & -1 & -\frac{1}{2} \\ 1 & 1 & -\frac{1}{2} \\ 1 & 0 & 1 \end{vmatrix} \\ &= 1 + \frac{3}{2} + \frac{1}{2} \\ &= 3 \end{aligned}$$

Since the Wronskian is nonzero, these eigenvectors are linearly independent. Therefore the general solution of the system is given by

$$\begin{aligned} \mathbf{x} &= c_1 e^{0t}\mathbf{x}_1 + c_2 e^{3t/2}\mathbf{x}_2 + c_3 e^{3t/2}\mathbf{x}_3 \\ &= \begin{bmatrix} c_1 - c_2 e^{3t/2} - \frac{1}{2}c_3 e^{3t/2} \\ c_1 + c_2 e^{3t/2} - \frac{1}{2}c_3 e^{3t/2} \\ c_1 + c_3 e^{3t/2} \end{bmatrix} \end{aligned}$$

or just

$$
\begin{aligned}
x_1 &= c_1 - \left(c_2 - \frac{1}{2}c_3\right) e^{3t/2} \\
x_2 &= c_1 + \left(c_2 - \frac{1}{2}c_3\right) e^{3t/2} \\
x_3 &= c_1 + c_3 e^{3t/2}
\end{aligned}
$$

Finding Eigenvalues and Eigenvectors

Theorem 4.2.1 shows that solving systems of first-order ODEs comes down to finding eigenvalues and eigenvectors of the corresponding matrix ODE. So it's important for us to know how to find these.

Let A be an $n \times n$ matrix and suppose that $\mathbf{v}$ is an eigenvector with corresponding eigenvalue λ. Then

$$A\mathbf{v} = \lambda\mathbf{v}.$$

We can rearrange this to get

$$A\mathbf{v} - \lambda\mathbf{v} = (A - \lambda I)\mathbf{v} = \mathbf{0}$$

where I is the identity matrix. Since $\mathbf{v} \neq \mathbf{0}$ (since it's an eigenvector!), linear algebra tells us that $\det(A - \lambda I) = 0$. This gives us the following theorem.

Theorem 4.2.3

The eigenvalues of a square matrix A are the solutions of the equation $\det(A - \lambda I) = 0$.

Definition 4.2.4

$\det(A - \lambda I) = 0$ is called the **characteristic equation** of the matrix A.

Example 4.2.5 Finding eigenvalues of a 2×2 matrix.

Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$.

Solution. First, we need to set up the characteristic equation of A. Since

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{bmatrix},$$

we get

$$\det(A - \lambda I) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda + 3$$

so the characteristic equation of A is

$$\lambda^2 - 2\lambda + 3 = 0$$

which has solutions $\lambda_1 = -1, \lambda_2 = 3$. So the eigenvalues of A are $-1, 3$.

These computations are easily verified by Sage or MATLAB/Octave. Sage can provide exact answers but the code is somewhat cumbersome. MATLAB on the other hand is designed for performing matrix computations and therefore its code for finding eigenvalues is simpler. We only need to use the `eig` command:

```
% Define the matrix
A = [1, 1; 1, 4];

% Compute the eigenvalues of A
eig(A)
```

```
ans =

    3.0000
   -1.0000
```

A useful fact to remember is that the eigenvalues of a "triangular" matrix are just the diagonal entries.

Example 4.2.6 Finding the eigenvalues of a triangular matrix.

Let

$$A = \begin{bmatrix} 1 & 3 & -4 & 5 \\ 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 10^{50\pi-300} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find the eigenvalues of A.

Solution. A is a triangular matrix since everything below the main diagonal is 0. Therefore the eigenvalues of A are $1, 3, 1, 0$.

Once we have the eigenvalues of a matrix, we can find their corresponding eigenvectors.

Example 4.2.7 Finding eigenvectors for a 2×2 matrix.

Find eigenvectors of $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$ corresponding to the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 3$.

Solution. Suppose that $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is an eigenvector corresponding to λ. Then we know that

$$\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \mathbf{v} = \lambda \mathbf{v} \Rightarrow \begin{bmatrix} v_1 + 4v_2 \\ v_1 + v_2 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix}.$$

This tells us that if $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is an eigenvector for λ, then its entries need to satisfy

$$\begin{aligned} v_1 + 4v_2 &= \lambda v_1 \\ v_1 + v_2 &= \lambda v_2 \end{aligned}$$

which boils down to

$$\begin{aligned} (1-\lambda)v_1 + 4v_2 &= 0 \\ v_1 + (1-\lambda)v_2 &= 0. \end{aligned}$$

Now set $\lambda = -1$ to get the system

$$\begin{aligned} 2v_1 + 4v_2 &= 0 \\ v_1 + 2v_2 &= 0 \end{aligned}$$

and so $v_1 = -2v_2$. We don't really care about what the entries of $\mathbf{v}$ look like so long as $\mathbf{v}$ is an eigenvector, so we can pick v_1, v_2 however we want, just so long as they satisfy this relation (and are not both 0!). So pick $v_2 = 1$, which forces $v_1 = -2$. Then

$$\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = -1$.
To find an eigenvector for $\lambda_2 = 3$ we just set $\lambda = 3$ and run through the same process:

$$\begin{aligned} -2v_1 + 4v_2 &= 0 \\ v_1 - 2v_2 &= 0 \end{aligned}$$

The second equation simplifies to $v_1 = 2v_2$, so one eigenvector for λ_2 is

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Of course, all of this can be done in Sage or MATLAB/Octave as well. If we use MATLAB/Octave, then the `eig` command once again does the heavy lifting for us. Each column of the matrix U produced below is an eigenvector of A.

```
# Define A
A = [1, 4; 1, 1];

# Compute eigenvectors
[U, V] = eig(A)
```

```
U =

    0.8944   -0.8944
    0.4472    0.4472


V =

    3.0000         0
         0   -1.0000
```

Looking forward to Theorem 4.2.8, note that the eigenvectors we found in Example 4.2.7 are linearly indepednent. This can be verified by computing the Wronskian as done using Sage below:

You may have noticed that the matrix constructed in the Sage cell here is actually "flipped": the eigenvectors are appearing as the rows instead of the columns. It turns out that this causes no problems for us since turning rows into columns (or columns into rows) has no affect on the determinant. Therefore the Wronskian is unchanged.

```
# Define our vectors
v1 = vector([-2, 1])
v2 = vector([2, 1])

# Define our matrix
M = matrix([v1, v2])

# Compute the determinant, i.e., the Wronskian
M.det()
```

```
-4
```

Solving Matrix ODEs

We now have the tools we need to begin solving matrix ODEs. Recall that if A is an $n \times n$ matrix with constant entries, and if $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are n linearly independent solutions of the matrix ODE $\mathbf{x}' = A\mathbf{x}$, then the general solution of the matrix ODE is

$$\mathbf{x} = \sum_{k=1}^{n} c_k \mathbf{x}_k.$$

Furthermore, if λ is an eigenvalue of A with eigenvector $\mathbf{v}$, then $e^{\lambda t}\mathbf{v}$ is a solution of $\mathbf{x}' = A\mathbf{x}$. So solving the matrix ODE $\mathbf{x}' = A\mathbf{x}$ requires finding enough eigenvectors and eigenvalues. A useful theorem is the following:

Theorem 4.2.8

Let A be an $n \times n$ matrix with constant entries. If the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are distinct (that is, none are repeated) then eigenvectors associated with different eigenvalues are linearly independent. That is, if $\mathbf{v}_i$ is an eigenvector corresponding to λ_i then the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

Example 4.2.9 Solving a matrix ODE.

Solve the matrix ODE given by $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}.$$

Solution. We already have everything we need. We know that the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 3$ from Example 4.2.5, and likewise some corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

thanks to Example 4.2.7. Since the eigenvalues are distinct it follows that these eigenvectors are linearly independent (we could also check this using the Wronskian). We can therefore build two linearly independent solutions to the matrix ODE:

$$\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1 = e^{-t}\begin{bmatrix}-2\\1\end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{\lambda_2 t}\mathbf{v}_2 = e^{3t}\begin{bmatrix}2\\1\end{bmatrix}.$$

So the general solution of the matrix ODE is

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1 e^{-t}\begin{bmatrix}-2\\1\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}2\\1\end{bmatrix}.$$

Note that the choice of eigenvector *doesn't matter*. We only need to find enough linearly independent eigenvectors for each distinct eigenvalue.

Example 4.2.10 Solving a first-order system with two equations.

Solve the first-order system given by

$$\begin{aligned} x_1' &= x_1 - 5x_2 \\ x_2' &= x_1 - x_2 \end{aligned}$$

where x_1 and x_2 are functions of t.

Solution. First, note that this system is equivalent to the matrix ODE $\mathbf{x}' = A\mathbf{x}$ where

$$\mathbf{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix}1 & -5\\1 & -1\end{bmatrix}.$$

To solve this system we need to find the eigenvalues and eigenvectors of A, and then use these to build our general solution.

1. Find the eigenvalues.

   We find the eigenvalues of A by solving the characteristic equation $\det(A-\lambda I) = 0$ for λ. Since $\det(A-\lambda I) = \lambda^2+4$, we see that the eigenvalues of A are $\lambda_1 = -2i$ and $\lambda_2 = 2i$. The fact that these eigenvalues are complex is *not* a problem. They're still distinct, so our method will work.

2. Find corresponding eigenvectors.

   Set $\mathbf{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$. Then $A\mathbf{v} = \lambda\mathbf{v}$ implies that

   $$\begin{aligned} v_1 - 5v_2 &= \lambda v_1 \\ v_1 - v_2 &= \lambda v_2 \end{aligned}$$

   or just

   $$\begin{aligned} (1-\lambda)v_1 - 5v_2 &= 0 \\ v_1 - (1+\lambda)v_2 &= 0. \end{aligned}$$

   Setting $\lambda = -2i$ in the second equation gives $v_1 = (1-2i)v_2$, so an eigenvector of A corresponding to $\lambda_1 = -2i$ is

   $$\mathbf{v} = \begin{bmatrix}1-2i\\1\end{bmatrix}.$$

Similarly, an eigenvector corresponding to $\lambda_2 = 2i$ is

$$\mathbf{v}_2 = \begin{bmatrix} 1+2i \\ 1 \end{bmatrix}.$$

3. Find the general solution.

   At this step it is easy to construct the solution of the matrix ODE. It's just

$$\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 = \begin{bmatrix} c_1 e^{-2it}(1-2i) + c_2 e^{2it}(1+2i) \\ c_1 e^{-2it} + c_2 e^{2it} \end{bmatrix}.$$

Example 4.2.11 Solving a system of three differential equations.

Solve the first-order system

$$\begin{aligned} \frac{dx_1}{dt} &= 3x_1 + x_3 \\ \frac{dx_2}{dt} &= 9x_1 - x_2 + 2x_3 \\ \frac{dx_3}{dt} &= -9x_1 + 4x_2 - x_3 \end{aligned}$$

Solution. As long as this system has distinct eigenvalues the above method will work. Once again we rewrite the system as a matrix ODE; in this case, the matrix ODE we must solve is

$$\mathbf{x}' = \begin{bmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{bmatrix}\mathbf{x} = A\mathbf{x}.$$

To find the eigenvalues we must solve the characteristic equation $\det(A - \lambda I) = 0$. However, we can also use Sage to perform this task.

We could also use MATLAB/Octave, but the resulting eigenvectors wouldn't look as nice as the output provided by Sage. This is because MATLAB's `eig` command produces **normalized** output which often involves dividing entries by square roots.

```
# Define our matrix
M = matrix([ [3, 0, 1],
[9, -1, 2],
[-9, 4, -1]] )

# Finds eigenvectors, corresponding eigenvalues and
    "algebraic multiplicty".
M.eigenvectors_right()
```

```
[(3,
  [
  (1, 9/4, 0)
  ],
  1),
 (-1 - 1*I, [(1, 2 + 1*I, -4 - 1*I)], 1),
 (-1 + 1*I, [(1, 2 - 1*I, -4 + 1*I)], 1)]
```

This produces a list containing the eigenvalues of A as well as the corresponding eigenvectors. So we see that the eigenvalues are given by

$$\lambda_1 = 3, \quad \lambda_2 = -1 - i \quad \text{and} \quad \lambda_3 = -1 + i,$$

while the corresponding eigenvectors are given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \frac{9}{4} \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2+i \\ -4-i \end{bmatrix} \quad \text{and} \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2-i \\ -4+i \end{bmatrix}.$$

We now have everything we need for the general solution of the matrix ODE. It's just

$$\mathbf{x} = c_1 e^{3t} \begin{bmatrix} 1 \\ \frac{9}{4} \\ 0 \end{bmatrix} + c_2 e^{(-1-i)t} \begin{bmatrix} 1 \\ 2+i \\ -4-i \end{bmatrix} + c_3 e^{(-1+i)t} \begin{bmatrix} 1 \\ 2-i \\ -4+i \end{bmatrix}.$$

Applications of Matrix ODEs

Now we use matrix ODEs to model physical systems. The methods we've developed for solving matrix ODEs will then let us come up with descriptions for such systems. Recall that we introduced systems of ODEs (and then matrix ODEs) to model quantities that depended on time (an independent variable) and each other (dependent variables). The physical systems we will consider will be ones where the quantities of interest depend on each other in some way.

Example 4.2.12 Determining salt concentration in connected tank system.

Two brine tanks are set up as in Figure 4.1.3. Fresh water flows into the tank at a rate of r_1, well-mixed solution flows from Tank 1 to Tank 2 at a rate of r_2 and well-mixed solution flows out of Tank 2 at a rate of r_3. Suppose that r_1, r_2 and r_3 are $5\,\frac{\text{gal}}{\text{min}}$, the volume of solution in Tank 1 is $10\,\text{gal}$ and the volume of solution in Tank 2 is $7\,\text{gal}$. Suppose Tank 1 has $5\,\text{lb}$ of salt at time $t = 0$ and Tank 2 has $2\,\text{lb}$ of salt at time $t = 0$. How much salt is in each tank at time t?

Solution. To start, let $x_1(t)$ denote the amount of salt in Tank 1 at time t and $x_2(t)$ denote the amount of salt in Tank 2 at time t, where t is in minutes. Then from Section 4.1, we know that

$$\begin{aligned} x_1' &= -\frac{1}{2}x_1 \\ x_2' &= \frac{1}{2}x_1 - \frac{5}{7}x_2 \end{aligned}$$

If we set

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{5}{7} \end{bmatrix}$$

then this system is equivalent to the matrix ODE $\mathbf{x}' = A\mathbf{x}$.
To solve this, we find the eigenvalues and corresponding eigenvectors. To find the eigenvalues, we could solve the characteristic equation $\det(A - \lambda I) = 0$ or use technology, but it's easier to note that A is a triangular matrix. So the eigenvalues are just $\lambda_1 = -\frac{1}{2}, \lambda_2 = -\frac{5}{7}$.
Now we find corresponding eigenvectors. So let

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

If $\mathbf{v}$ is an eigenvector for λ, then we know $A\mathbf{v} = \lambda\mathbf{v}$, which gives the system

$$\begin{aligned} (-\frac{1}{2} - \lambda)v_1 &= 0 \\ \frac{1}{2}v_1 + (-\frac{5}{7} - \lambda)v_2 &= 0 \end{aligned}$$

If we set $\lambda = -\frac{1}{2}$, then we just get $v_1 = \frac{3}{14}v_2$. So an eigenvector corresponding to $\lambda_1 = -\frac{1}{2}$ is

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 14 \end{bmatrix}.$$

Similarly, if we set $\lambda = -\frac{5}{7}$ we get $v_1 = 0$, but no restrictions on v_2. So an eigenvector corresponding to $\lambda_2 = -\frac{5}{7}$ is

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We can now write down the general solution of the matrix ODE:

$$\mathbf{x} = c_1 e^{-\frac{5t}{7}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 e^{-\frac{t}{2}} \begin{bmatrix} 3 \\ 14 \end{bmatrix} = \begin{bmatrix} 3c_2 e^{-\frac{t}{2}} \\ c_1 e^{-\frac{5t}{7}} + 14c_2 e^{-\frac{t}{2}} \end{bmatrix}.$$

But we're not done yet, since we have the initial conditions $x_1(0) = 5$ and $x_2(0) = 2$, or in terms of our matrix ODE

$$\mathbf{x}(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

We can use this to find c_1 and c_2. If we set $t = 0$ then we get

$$\begin{bmatrix} 3c_2 \\ c_1 + 14c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix},$$

so $c_2 = \frac{5}{3}$ and $c_1 = 2 - 14\frac{5}{3} = -\frac{64}{3}$.
So the solution of the matrix ODE (and hence the original system) is

$$\mathbf{x} = \begin{bmatrix} 5e^{-\frac{t}{2}} \\ -\frac{64}{3}e^{-\frac{5t}{7}} + \frac{70}{3}e^{-\frac{t}{2}} \end{bmatrix}.$$

The amount of salt in the first tank, x_1, is given by the top entry and the amount of salt in the second tank, x_2, is given by the bottom entry.

4.3 Phase Portraits and Critical Points

The techniques used in Section 1.6 to study long-term behavior of solutions near critical points can be adapted to higher dimensional systems as well. The main difference now is that we must consider the *phase plane* instead of a simple one-dimensional number line. This also allows for different types of behavior at critical points for solutions, or *trajectories*, near the critical point.

The Phase Plane

Just as we were able to plot direction fields for first-order autonomous ODEs, we can do something similar for autonomous first-order systems with two equations and constant coefficients. These are precisely the systems that can be written as a matrix ODE of the form

$$\mathbf{x}' = A\mathbf{x}$$

where A is a 2×2 matrix.

Consider the first-order system

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 \\ x_2' &= a_{21}x_1 + a_{22}x_2 \end{aligned} \tag{4.1}$$

or

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

The solution of this system looks like $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

As t varies, $\mathbf{x}(t)$ will trace out a curve in the x_1x_2-plane, which we call a **trajectory**. The x_1x_2-plane is called the **phase plane**, and the collection of all trajectories of the system (4.1) is called the **phase portrait** of the system. The phase portrait of a system provides us with a way to study the behavior of solutions of (4.1) without actually solving the system.

Example 4.3.1 Sketching a phase portrait.

Sketch a phase portrait for the system

$$\begin{aligned} \frac{dy_1}{dt} &= 2y_1 - 3y_2 \\ \frac{dy_2}{dt} &= -2y_1 + y_2. \end{aligned}$$

Solution. First, note that we can rewrite the system as $\mathbf{y}' = A\mathbf{y}$ using

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix}.$$

Now, we can view $\mathbf{y}$ as corresponding to a point in the phase plane. Hence $\mathbf{y}'$ corresponds to a *tangent* of a trajectory passing through the point $\mathbf{y}$.

For example, let's find the tangent at the point $\mathbf{y} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. The tangent is given by the corresponding $\mathbf{y}'$ at this point which is just $A\mathbf{y}$:

$$\mathbf{y}' = A\mathbf{y} = \begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$$

So at the point $(2, 2)$ in the phase plane, the trajectory should be heading in the same direction as that of the point $(-2, -2)$ relative to the origin. In other words, the tangent vector would point two units left and two units down from the point $(2, 2)$.

Similarly, if we let $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then we get

$$\mathbf{y}' = A\mathbf{y} = \begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

So the trajectory going through $(0, 1)$ in the phase plane should be heading in the direction of $(-3, 1)$ viewed from the origin.

Plotting other points in the phase plane like this, we get Figure 4.3.2. One thing we can see from this is that trajectories that lie on the line, equivalently, those with initial conditions $y_1(0) = y_2(0)$, appear to approach the origin while all others move away from the origin. We can see why this is by looking at the general solution of the original system, which is

$$\mathbf{y} = c_1 e^{4t} \begin{bmatrix} -3 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

If $\mathbf{y}$ lies on the line $y_2 = y_1$, then c_1 has to equal 0, which follows from the fact that $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are linearly independent. So trajectories that lie on the line $y_2 = y_1$ must take the form $\mathbf{y} = c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and every solution of this form goes to $\mathbf{0}$ as $t \to \infty$. *Every other trajectory* will move away from the origin as $t \to \infty$, although the trajectories that lie on the line $y_2 = -\frac{2}{3} y_1$ will travel to the origin as $t \to -\infty$ (i.e. "backwards in time"):

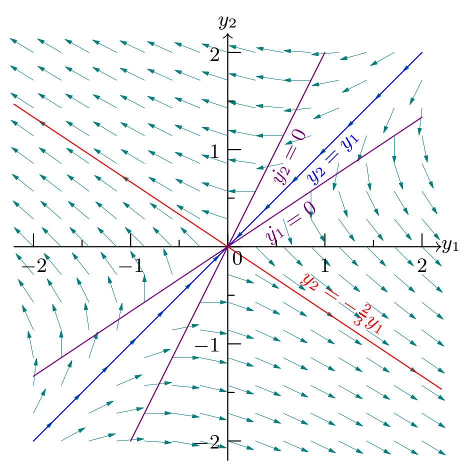

Figure 4.3.2 The phase portrait from Example 4.3.1. The blue line represents the two incoming trajectories at $\mathbf{0}$ and the red line represents two outgoing trajectories at $\mathbf{0}$.

Vector fields can also be plotted easily using SageMath. The code cell below demonstrates the use of the `plot_vector_field` command to sketch the phase portrait from Example 4.3.1.

```
# Define variables
var('y1␣y2')

# Create phase portrait for system
VF=plot_vector_field([2*y1 - 3*y2,-2*y1 +
    y2],[y1,-2,2],[y2,-2,2])

# Display the plot
show(VF)
```

Note that $\mathbf{x} = \mathbf{0}$ is *always* a solution of $\mathbf{x}' = A\mathbf{x}$. This is because $\mathbf{0}' = A\mathbf{0} = \mathbf{0}$. We call $\mathbf{x} = \mathbf{0}$ the **equilibrium solution** or **critical point** of the system $\mathbf{x}' = A\mathbf{x}$. Later in this section, we will be concerned with the behavior of trajectories of the system $\mathbf{x}' = A\mathbf{x}$ near the equilibrium solution $\mathbf{x} = \mathbf{0}$. One thing we will see is that the behavior is determined in large part by the eigenvalues of the matrix A.

We will separate the behavior of trajectories at the critical point $\mathbf{x} = \mathbf{0}$ into five different cases:

Table 4.3.3 Types of critical points

Classification	Behavior at $\mathbf{0}$
Improper node	Every trajectory except two has the same limiting tangent at $\mathbf{0}$
Proper node	For every direction $\mathbf{d}$ there exists trajectory with limiting tangent $\mathbf{d}$
Saddle point	Two incoming trajectories, two outgoing trajectories; all others bypass $\mathbf{0}$
Center	$\mathbf{0}$ is enclosed by infinitely many closed (repeating) trajectories
Spiral point	Trajectories spiral inwards or outwards from $\mathbf{0}$

$\mathbf{0}$ was a saddle point in Example 4.3.1 since there were incoming trajectories on the line $y_2 = y_1$ and outgoing trajectories on the line $y_2 = -\frac{2}{3}y_1$ as indicated in Figure 4.3.2.

Example 4.3.4 Classifying a critical point using a phase portrait.

Using a phase portrait, determine the type of critical point that $\mathbf{y} = \mathbf{0}$ is for the matrix ODE $\mathbf{y}' = A\mathbf{y}$ where

$$\mathbf{y} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}.$$

Solution. As seen in Figure 4.3.5, every (nonzero) trajectory will spiral outward from $\mathbf{y} = \mathbf{0}$ as $t \to \infty$, so $\mathbf{0}$ is a spiral point of this system. To see why, we only need to look at the eigenvalues of A, which we find to be

$$\lambda_1 = 3 + 4i, \lambda_2 = 3 - 4i.$$

This means that the general solution of $\mathbf{y}' = A\mathbf{y}$ must look like

$$\begin{aligned} \mathbf{y} &= c_1 e^{(3+4i)t}\mathbf{y}_1 + c_2 e^{(3-4i)t}\mathbf{y}_2 \\ &= e^{3t}\left[c_1 e^{4it}\mathbf{y}_1 + c_2 e^{-4it}\mathbf{y}_2\right] \\ &= e^{3t}\left[(\cos 4t)\mathbf{x}_1 + (\sin 4t)\mathbf{x}_2\right]. \end{aligned}$$

The real part of the eigenvalues leads to the "growth term" of e^{3t} appearing in the solution, which causes the trajectories to diverge as $t \to \infty$. The imaginary part of the eigenvalues leads to the "oscillating terms" of $\cos 4t, \sin 4t$ appearing in the solution, which gives the trajectories their spiral motion.

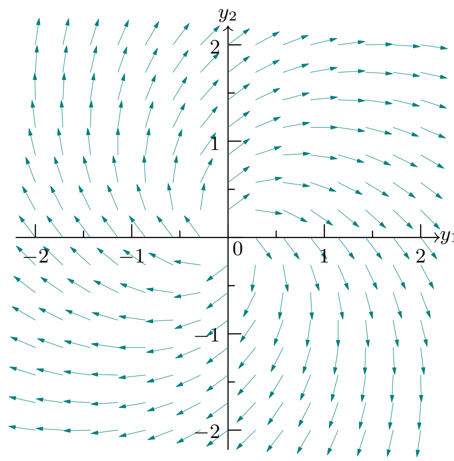

Figure 4.3.5 The phase portrait for Example 4.3.4.

In general, the eigenvalues of the matrix A in the system $\mathbf{y}' = A\mathbf{y}$ will determine the type of critical point that $\mathbf{0}$ is for the system $\mathbf{y}' = A\mathbf{y}$.

Example 4.3.6 Classifying trajectories algebraically.

What kind of critical point is $\mathbf{y} = \mathbf{0}$ for the system

$$\begin{aligned} y_1' &= -4y_2 \\ y_2' &= 4y_1 \end{aligned}$$

where $y_i = y_i(t)$?

Solution. We could sketch the phase portrait for this system, but we can also determine the behavior of the trajectories $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ if we can find a relationship between y_1 and y_2. To do so, we "cross-multiply" the system to get

$$4y_1y_1' = -4y_2y_2' \quad \text{or} \quad 4y_1\,dy_1 = -4y_2\,dy_2\ .$$

So we can integrate this to get

$$2y_1^2 = -2y_2^2 + C \quad \text{or} \quad y_1^2 + y_2^2 = C_1.$$

This is the equation of a circle of radius $\sqrt{C_1}$, and so every trajectory $\mathbf{y}$ for this system will be a circle centered at $\mathbf{0}$. Hence $\mathbf{0}$ is a center.

Eigenvalue Criteria for Stability

Consider the matrix ODE $\mathbf{y}' = A\mathbf{y}$. Let λ_1, λ_2 denote the eigenvalues of the 2×2 matrix A. Then $\mathbf{0}$ is a:

Table 4.3.7 Eigenvalue conditions for stability.

Name	Conditions on λ_1, λ_2
Node	Real, same sign
Saddle point	Real, opposite sign
Center	Pure imaginary
Spiral point	Complex, not pure imaginary

The rule of thumb is this: the real parts of the eigenvalues determine whether a trajectory moves towards or away from the origin, and the imaginary part determines if the trajectory has a periodic/oscillating nature to it.

We say that the origin is a **stable** critical point of $\mathbf{y}' = A\mathbf{y}$ if all trajectories that start "close" to $\mathbf{0}$ remain close at all future times. Equivalently, it's stable if each trajectory will eventually be contained within some circle centered at the origin as $t \to \infty$. Otherwise, we say that $\mathbf{0}$ is **unstable**. If it so happens that every trajectory that starts close to $\mathbf{0}$ tends to $\mathbf{0}$ as $t \to \infty$, we then say that $\mathbf{0}$ is a **stable and attractive** (or **asymptotically stable**) critical point. Equivalently, $\mathbf{0}$ is asymptotically stable if *every* trajectory goes to $\mathbf{0}$ as $t \to \infty$.

Example 4.3.8 Eigenvalue conditions for asymptotic stability.

Let $\mathbf{y}' = A\mathbf{y}$ denote a matrix ODE where A is a constant 2×2 matrix. What conditions on the eigenvalues of A will give an asymptotically stable critical point at $\mathbf{y} = \mathbf{0}$?

Solution. Let $\mathbf{y} = \mathbf{y}(t)$ denote a nonzero solution of the matrix ODE (and therefore a trajectory). Then in order for $\mathbf{0}$ to be asymptotically stable, we need $\mathbf{y} \to \mathbf{0}$ as $t \to \infty$. Let λ_1, λ_2 denote the eigenvalues of A. Then $\mathbf{y}$ will have the form

$$\mathbf{y} = c_1 e^{\lambda_1 t}\mathbf{y}_1 + c_2 e^{\lambda_2 t}\mathbf{y}_2.$$

The previous paragraph shows that $\mathbf{y}$ must go to $\mathbf{0}$ as $t \to \infty$ if either $c_1 = c_2 = 0$ or if each exponential goes to 0 as $t \to \infty$. Since we assume $\mathbf{y} \neq \mathbf{0}$, this means we need $e^{\lambda_i t} \to 0$ for $i = 1, 2$ as $t \to \infty$. This implies that the *real part* of each eigenvalue must be negative, because the real part of each eigenvalue is what determines the growth of $e^{\lambda_i t}$: if $\lambda_i = a + bi$, then

$$e^{\lambda_i t} = e^{at}[A\cos bt + B\sin bt].$$

So $\mathbf{0}$ is asymptotically stable if the real parts of *both* eigenvalues are negative.

By a similar argument to that used in Example 4.3.8, we can say that $\mathbf{0}$ is stable as long as the real part of each eigenvalue is no greater than 0. Likewise, $\mathbf{0}$ is unstable if the real part of *any* eigenvalue is positive.

Example 4.3.9 Long term behavior of a system of interconnected tanks.

Two tanks T_1 and T_2 containing 200 gal each of a water-salt mixture are set up as follows:

- Tank 1: Pure water flows in at 12 $\frac{\text{gal}}{\text{min}}$ and solution from Tank 2 flows in at $4\,\frac{\text{gal}}{\text{min}}$; solution also flows out of Tank 1 and into Tank 2 at $16\,\frac{\text{gal}}{\text{min}}$.

- Solution from Tank 1 flows in at $16\,\frac{\text{gal}}{\text{min}}$; solution flows out of Tank 2 and into Tank 1 at $4\,\frac{\text{gal}}{\text{min}}$, and solution is emptied from Tank 2 at an addition rate of $12\,\frac{\text{gal}}{\text{min}}$

Will the salt eventually empty from both tanks?

Solution. Let $y_1(t)$ denote the amount of salt (in pounds) in Tank 1 at time t (in minutes), and let $y_2(t)$ do the same for Tank 2. Then

$$\begin{aligned} y_1' &= 4\frac{y_2}{200} - 16\frac{y_1}{200} \\ y_2' &= 16\frac{y_1}{200} - 16\frac{y_2}{200}. \end{aligned}$$

This system is equivalent to the matrix ODE $\mathbf{y}' = A\mathbf{y}$ where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} -\frac{2}{25} & \frac{1}{50} \\ \frac{2}{25} & -\frac{2}{25} \end{bmatrix}.$$

We need to determine the long-term behavior of solutions of this ODE, which is itself determined by the eigenvalues of A.
The eigenvalues of A are

$$\lambda_1 = -\frac{1}{25} \quad \text{and} \quad \lambda_2 = -\frac{3}{25}.$$

Since both eigenvalues have negative real part, it follows that $\mathbf{0}$ is an asymptotically stable critical point of $\mathbf{y}' = A\mathbf{y}$. Therefore *every* trajectory $\mathbf{y} \to \mathbf{0}$ as $t \to \infty$. So no matter how much salt is initially in the tanks, the amount of salt will always go to 0.

4.4 Nonlinear Systems

Now we apply phase plane methods to study **nonlinear autonomous systems**, which for systems involving two ODEs take the form

$$\begin{aligned} y_1' &= f_1(y_1, y_2) \\ y_2' &= f_2(y_1, y_2). \end{aligned}$$

where $y_i = y_i(t)$.

Autonomous just means we can write the system without explicitly referring to the independent variable t.

We can also write such a system as a vector equation:

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}). \tag{4.2}$$

although not as a matrix ODE (if the functions f_i are nonlinear).

Just as in the previous sections, the **phase plane** is still the y_1y_2-plane, **trajectories** are still the solutions $\mathbf{y}$ of (4.2) (represented as curves in the phase plane), and the **phase portrait** of (4.2) is the set of all trajectories in the phase plane.

We call a point $P = (y_1, y_2)$ in the phase plane a **critical point** of (4.2) if

$$f_1(y_1, y_2) = 0 \quad \text{and} \quad f_2(y_1, y_2) = 0.$$

In other words, P is a critical point of $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ if $\mathbf{f}(P) = \mathbf{0}$. Just as before, critical points represent solutions of the system that are in equilibrium.

Example 4.4.1 The pendulum equation.

Express the *pendulum equation* $\theta'' + \frac{g}{l}\sin\theta = 0$, where $\theta = \theta(t)$ represents the angular displacement of a pendulum from the vertical, as a nonlinear system $\boldsymbol{\theta}' = \mathbf{f}(\boldsymbol{\theta})$ and then find its critical points.

Solution. First, we have to rewrite the pendulum ODE as a first order system. We can do this without too much trouble as follows: set

$$\theta_1 = \theta \quad \text{and} \quad \theta_2 = \theta_1' = \theta'.$$

Then the ODE $\theta'' + \frac{g}{l}\sin\theta = 0$ turns into the system

$$\begin{aligned} \theta_1' &= \theta_2 \\ \theta_2' &= -\frac{g}{l}\sin\theta_1, \end{aligned}$$

which we can also write as $\boldsymbol{\theta}' = \mathbf{f}(\boldsymbol{\theta})$ using

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \quad \text{and} \quad \mathbf{f}(\boldsymbol{\theta}) = \begin{bmatrix} \theta_2 \\ -\frac{g}{l}\sin\theta_2 \end{bmatrix}.$$

Now we need to find the critical points $\boldsymbol{\theta}$ in the $\theta_1\theta_2$-plane that make $\mathbf{f}(\boldsymbol{\theta}) = \mathbf{0}$. This requires $\theta_2 = 0$ and $\theta_1 = \pm k\pi$ for $k = 0, \pm1, \pm2, \ldots$, and so the critical points of this system are all points of the form $(\pm k\pi, 0)$.

Classification of Critical Points and Linearization

Critical points of systems are important because they can represent long-term behavior of a system. For example, if we have a first-order system representing the population of two species, and it turns out the the origin is asymptotically stable, then this suggests that both species could be driven to extinction. So we want to classify critical points for nonlinear systems in addition to what we have already for linear systems; unfortunately, nonlinear systems are often difficult, if not outright impossible, to solve exactly.

Thankfully, in many cases we can approximate a nonlinear system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ with critical points P_i by a suitably chosen linear system $\mathbf{y}' = A\mathbf{y}$ at each critical point P_i; we call such a system the **linearization** at P_i.

Definition 4.4.2 The Jacobian of a Nonlinear System.

Let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{and} \quad \mathbf{f}(\mathbf{y}) = \begin{bmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{bmatrix}.$$

The **Jacobian** of $\mathbf{f}$ is the matrix $J_{\mathbf{f}}(y_1, y_2)$ given by

$$J_{\mathbf{f}}(y_1, y_2) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix}.$$

If $\mathbf{f}$ is understood from context, we often write J instead of $J_{\mathbf{f}}$ for the Jacobian.

The Jacobian is important since it allows us to *linearize* a nonlinear system. More precisely, the **linearization** of $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ at the point $P = (p_1, p_2)$ is the linear system $\mathbf{y}' = A\mathbf{y}$, where

$$A = J(p_1, p_2).$$

Example 4.4.3 Linearizing the pendulum system.

Find the linearization of the pendulum system $\boldsymbol{\theta}' = \mathbf{f}(\boldsymbol{\theta})$ at the critical point $(0, 0)$.

Solution. For this system, we have $f_1(\theta_1, \theta_2) = \theta_2$ and $f_2(\theta_1, \theta_2) = -\frac{g}{l}\sin\theta_1$. The Jacobian is then given by

$$J(\theta_1, \theta_2) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos\theta_1 & 0 \end{bmatrix}.$$

So to get the linearization we need to set

$$A = J(0, 0) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}.$$

The linearization of a nonlinear system isn't just useful for approximating the nonlinear system. It's also incredibly useful for classifying the critical points of a nonlinear system; for the most part, the eigenvalues of the matrix A from the linearization also classify the critical points of the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$.

Example 4.4.4 Classifying critical points using linearization.

Find and classify the critical points of the nonlinear system

$$\frac{dx}{dt} = x - y - x^2 + xy$$
$$\frac{dy}{dt} = -x^2 - y.$$

This example taken from here.[2]

Solution. The critical points occur at intersections between the nullclines $\dot{x} = 0$ and $\dot{y} = 0$. The equations of the nullclines for $\dot{x} = 0$ are

$$x = 1$$
$$y = x,$$

while the equation of the nullcline $\dot{y} = 0$ is

$$y = -x^2.$$

Hence there are three critical points for this system as seen in Figure 4.4.5. In particular, these points are $(0,0), (-1,-1)$ and $(1,-1)$.

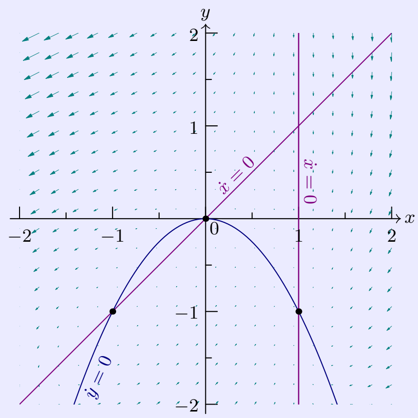

Figure 4.4.5 The phase portrait and nullclines for the system

To determine the behavior of solutions at these critical points, we'll find the Jacobian at each point. First, we have

$$J(x,y) = \begin{bmatrix} 1-2x+y & x-1 \\ -2x & -1 \end{bmatrix}.$$

At $(0,0)$, we get

$$J(0,0) = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}.$$

The eigenvalues are 1 and -1, meaning that this critical point is a saddle point.

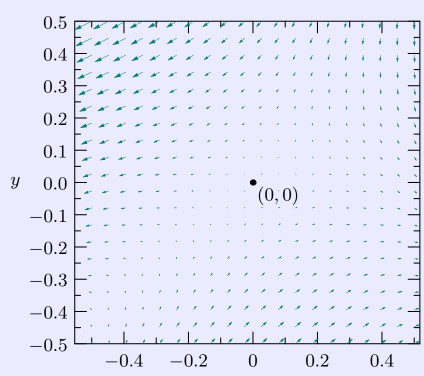

Figure 4.4.6 The phase portrait at $(0,0)$

At $(-1,-1)$ we get

$$J(-1,-1) = \begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix}$$

which has eigenvalues $\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{7}i$. Hence $(-1,-1)$ should be a spiral point.

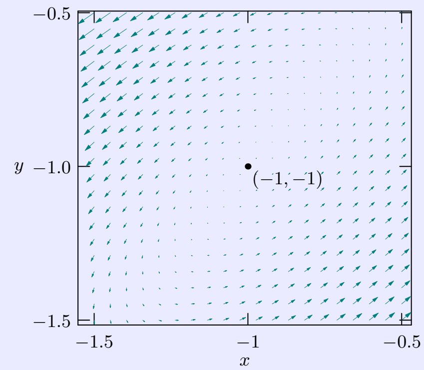

Figure 4.4.7 The phase portrait at $(-1,-1)$

Finally, at $(1,-1)$ we get

$$J(1,-1)=\begin{bmatrix}-2 & 0\\ -2 & -1\end{bmatrix},$$

which has eigenvalues $-2,-1$. Hence $(1,-1)$ is an asymptotically stable node.

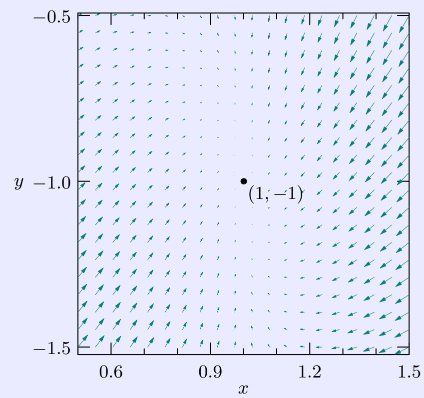

Figure 4.4.8 The phase portrait at $(1,-1)$

Linearization works well to classify the behavior of systems at certain types of critical points. A critical point P of a system is **hyperbolic** if the Jacobian $J(P)$ has eigenvalues with nonzero real part. Unfortunately, linearization is not guaranteed to give an accurate description of the behavior of non-hyperbolic critical points.

Example 4.4.9 The Lotka-Volterra population model.

Predator-prey populations can be modeled using the *Lotka-Volterra model*. Let $y_1(t)$ denote the population of a prey species at time t and let $y_2(t)$ denote the population of a predator species at time t. Then the **Lotka-Volterra model** says that

$$\begin{aligned} y_1' &= ay_1 - by_1y_2\\ y_2' &= ky_1y_2 - ly_2, \end{aligned}$$

where $a,b,k,l>0$. Find and classify the critical points of this system.

[2] www.math.uci.edu/~ndonalds/math3d/nonlinear.pdf

Solution. The critical points are the points (y_1, y_2) that satisfy the equations

$$ay_1 - by_1y_2 = 0 \quad \text{and} \quad ky_1y_2 - ly_2 = 0.$$

Equivalently, we need

$$y_1(a - by_2) = 0 \quad \text{and} \quad y_2(ky_1 - l) = 0.$$

This has solutions $y_1 = y_2 = 0$ and $y_1 = \frac{l}{k}, y_2 = \frac{a}{b}$, which shows that the critical points are $(0, 0)$ and $(\frac{l}{k}, \frac{a}{b})$.
To classify the critical points of this system we will linearize the system. The Jacobian of

$$\mathbf{f}(\mathbf{y}) = \begin{bmatrix} ay_1 - by_1y_2 \\ ky_1y_2 - ly_2 \end{bmatrix}$$

is

$$J(y_1, y_2) = \begin{bmatrix} a - by_2 & -by_1 \\ ky_2 & ky_1 - l \end{bmatrix}.$$

Now we will examine the Jacobian at each critical point.
At $(0, 0)$, we get

$$J(0, 0) = \begin{bmatrix} a & 0 \\ 0 & -l \end{bmatrix},$$

which has eigenvalues $\lambda = a, -l$ which indicate a saddle point. Since these eigenvalues have nonzero real part, the origin is a hyperbolic critical point of the system and so we know that it also behaves as a saddle point in the original system. In particular, there exist trajectories heading into the origin, so it's possible for both species to go extinct in this model.
Now we'll classify the second critical point $(\frac{l}{k}, \frac{a}{b})$. The Jacobian at this point gives us the matrix

$$A = J(\frac{l}{k}, \frac{a}{b}) = \begin{bmatrix} 0 & -\frac{bl}{k} \\ \frac{ak}{b} & 0 \end{bmatrix}.$$

This matrix has characteristic equation $\lambda^2 + al = 0$, and so has eigenvalues $\lambda = \pm i\sqrt{al}$. Since the eigenvalues are pure imaginary, this suggests that $(\frac{l}{k}, \frac{a}{b})$ is a center, which is indeed the case. In particular, trajectories near this critical point *must be periodic*. Unfortunately we can't quite justify this conclusion. This is because both eigenvalues of the Jacobian at this critical point have zero real part, which means that this critical point is not hyperbolic and so the behavior of the linearization is not guaranteed to match the behavior of the original system at this critical point. That said, a more detailed analysis (or simply a sketch of the phase portrait) does indeed confirm that $(\frac{l}{k}, \frac{a}{b})$ is a center of the original system.

Chapter 5

Series Solutions of ODEs

In Chapter 2 we developed a general method for solving linear ODEs with constant coefficients. There were ODEs of the form

$$ay'' + by' + cy = 0$$

for some constants a, b, c. We saw that the solution of such ODEs looked like exponentials and then determined which exponentials would provide us with a general solution.

Now we move on to more complicated (but still linear) ODEs of the form

$$y'' + P(x)y' + Q(x)y = 0. \tag{5.1}$$

Some extremely important ODEs of this form include Bessel's equation. However, the method of characteristic equations is not flexible enough to solve come ODEs of this form.

To address this, we will change our choice of *ansatz* from exponential solutions $y = e^{rx}$ to *power series solutions* $y = \sum_{k=0}^{\infty} a_k x^k$. This change will allow us more flexibility in finding solutions to equations of the form given in (5.1).

> Note that $e^{rx} = \sum_{k=0}^{\infty} \frac{r^k}{k!} x^k$, and so the exponential solutions found in Chapter 2 were a special case of the more general series solution. In fact, the same is true of the other solution methods we've discussed for ODEs.

5.1 Power Series Method

Since power series form the basis of our solution strategy in this chapter, we begin by reviewing some important concepts related to power series and their convergence.

Review of Power Series

In calculus, it's important to know how to differentiate and integrate functions. For some functions (say, $x - 1, x^2, 3 - x^5$) it can be very straightforward, but for others (such as e^{-x^2}) it can be impossible.

At least, it can be impossible to integrate certain functions in terms of the "everyday", or *elementary* functions that we're used to.

Power series were introduced in calculus to allow us to write complicated functions $f(x)$ in terms of simpler functions $1, x, x^2, \ldots$. In particular, our goal is to write $f(x)$ in the form

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = \sum_{k=0}^{\infty} c_k x^k \tag{5.2}$$

where the coefficients c_k are all constants.

Definition 5.1.1 Power series.

A **power series (centered at 0)** is a series (that is, an infinite sum) of the form $\sum_{k=0}^{\infty} c_k x^k$. The power series is said to **converge** on some interval I if the sum exists for each x in I.

A power series doesn't have to start at $k = 0$, but *it may not contain any negative powers of x.*

The question, now, is to determine the values of the coefficients c_k to make (5.2) true. If we look at the equation we see that we can solve for c_0 very easily. All that we need to do is to set $x = 0$ in (5.2) to make all of the other terms disappear:

$$f(0) = c_0 + c_1 \cdot 0 + \cdots = c_0.$$

We can use a similar approach to solve for c_1 by plugging in $x = 0$, but we need to get rid of the power of x attached to it. This is done by taking the derivative of $f(x)$ and then setting $x = 0$:

$$\begin{aligned} f'(x) &= c_1 + 2c_2 x + 3c_3 x^2 + \ldots \\ f'(0) &= c_1. \end{aligned}$$

The same trick works for c_2:

$$\begin{aligned} f''(x) &= 2 \cdot 1 c_2 + 3 \cdot 2 c_3 x + \ldots \\ f''(0) &= 2 \cdot 1 c_2 \end{aligned}$$

so $c_2 = \frac{f''(0)}{2 \cdot 1}$. Let's try this one more time to get c_3:

$$\begin{aligned} f^{(3)}(x) &= 3 \cdot 2 \cdot 1 c_3 + \ldots \\ f^{(3)}(0) &= 3 \cdot 2 \cdot 1 c_3 \end{aligned}$$

and so $c_3 = \frac{f^{(3)}(0)}{3 \cdot 2 \cdot 1}$.

In general, to get the coefficient c_k of x^k in the power series of $f(x)$, we have the following equation:

$$c_k = \frac{f^{(k)}(0)}{k \cdot (k-1) \cdot \cdots \cdot 2 \cdot 1} = \frac{f^{(k)}(0)}{k!}. \tag{5.3}$$