# Calculus Notes

West Virginia Wesleyan College

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Go Seahawks.

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# Preface

This document was created to serve as a single source for my lecture notes for the calculus sequence at West Virginia Wesleyan College. As such these notes are divided into three self-explanatory parts.

- The first part, Differential Calculus, introduces the derivative and covers its important properties and applications. This content forms the majority of the Calculus I course.
- The next part, Integral Calculus, introduces the integral and goes over multiple methods for its calculation. This content forms the majority of the Calculus II course.
- The final part, Multivariable Calculus, generalizes the concepts of derivatives and integrals to two and three dimensions. This content forms basis of the Calculus III course.

This document is very much *in progress* and therefore typos and other errors are to be expected. If you find any, I would appreciate you letting me know by contacting me by email.

# Contents

# Part I

# Differential Calculus

# Chapter 1

# Functions

The primary object of study in calculus is the function. In this chapter, we'll review important types of functions that will appear in this course.

## 1.1 Function Review

This section reviews basic facts about functions. To appear in a later version.

## 1.2 Types of Functions

### Linear Functions

The most basic example of a function that we'll study in calculus is the *linear function*.

**Definition 1.2.1 Linear Functions.**

A **linear function** is a function $f(x)$ of the form

$$f(x) = mx + b$$

for constants $m$ and $b$.

As the name suggests, any (non-vertical) line is an example of a linear function. Such functions are completely determined by the slope of the corresponding line and the $y$-intercept.

Equivalently, any line is determined by knowing two distinct points on the line.

In particular, if $y = mx + b$ then $m$ is the slope and $b$ is the $y$-intercept. This is known as the *slope-intercept* equation of a line. Equations of lines are also often written in *point-slope form* as $y - y_0 = m(x - x_0)$.

**Example 1.2.2 Finding the Equation of a Line.**

Find a function $f(x)$ whose graph is a line passing through $(-5, 2)$ and $(1, 1)$.

## Polynomial Functions

After linear functions, we also study functions with higher powers of $x$.

**Definition 1.2.3 Polynomial Functions.**

A **polynomial function** is a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $n$ is a nonnegative whole number and the **coefficients** $a_0, a_1, \ldots, a_n$ are constants and $n$ is the **degree** of the polynomial.

If $f(x) = 0$, then we say that $f(x)$ has degree $-\infty$.

So linear functions are just polynomial functions of degree at most 1. And just as linear functions are determined by knowing two distinct points on the line, any polynomial functions is determined by knowing $n+1$ distinct points on the polynomial.

**Example 1.2.4 Composition of Polynomial Functions.**

Let $f(x) = 3x - x^2$ and $g(x) = 4x^3 + 5$. Find $f(g(x))$.

## Algebraic Functions

The operations of addition, subtraction, multiplication, division, taking whole number powers and taking whole number roots applied to polynomials give rise to *algebraic functions*. Two particularly important examples are *rational functions* and *root functions*.

**Definition 1.2.5 Rational and Root Functions.**

A **rational function** is a function of the form $\frac{p(x)}{q(x)}$ where $p$ and $q$ are polynomial functions. A **root function** is a function of the form $x^{1/n} = \sqrt[n]{x}$ for some natural number $n$.

As functions become more complicated, we have to worry more and more about their domains. For a polynomial function, the domain is the set of all real numbers $\mathbb{R}$ (if we ignore complex numbers). The domain of a rational function is the set of all numbers where the denominator is nonzero. The domain of a root function is the set of all nonnegative numbers.

**Example 1.2.6 Finding the Domain of an Algebraic Function.**

Find the domain of

$$\sqrt{x^2 - 3x + 2} + \frac{x-4}{3x+4}.$$

**Solution**. We need to find where both the radicand $x^2 - 3x + 2$ is nonnegative and where $3x + 4 \neq 0$. If we solve $x^2 - 3x + 2 \geq 0$, we see that $x$ must be in $(-\infty, 1] \cup [2, \infty)$. Likewise, $3x + 4 \neq 0$ if and only if

$x \neq -\frac{4}{3}$. Hence the domain is

$$(-\infty, -\frac{4}{3}) \cup (-\frac{4}{3}, 1] \cup [2, \infty).$$

We'll make use of the computer algebra system Sage to perform certain computations. As an example, we use it below to solve the inequality $x^2 - 3x + 2 \geq 0$:

```
# click the Sage button to evaluate this cell
solve(x^2 - 3*x + 2 >= 0, x)    # tells Sage the inequality
    to solve and the variable to solve for
```

## 1.3 Trigonometric Functions

### Angles and Terminal Points

Consider the unit circle in $\mathbb{R}^2$, which is given by $x^2 + y^2 = 1$. Each point $P$ on this circle makes an angle $\theta$ with the positive $x$-axis, and therefore $\theta$ must also determine the point completely. The angle $\theta$ is typically specified using either *radians* or *degrees*. Converting from one unit to the other can be done by noting that $\pi$ radians is precisely equal to 180 degrees.

Unless otherwise specified, we will use radians for angle measures in this course.

**Example 1.3.1 Converting from Degrees to Radians.**

Convert 270 degrees to radians and $\frac{\pi}{6}$ radians to degrees.

Just as points on the unit circle determine angles, angles also determine points. A point $P$ determined by an angle $\theta$ is also known as the *terminal point* for the angle. Terminal points for certain *reference angles* are useful to remember.

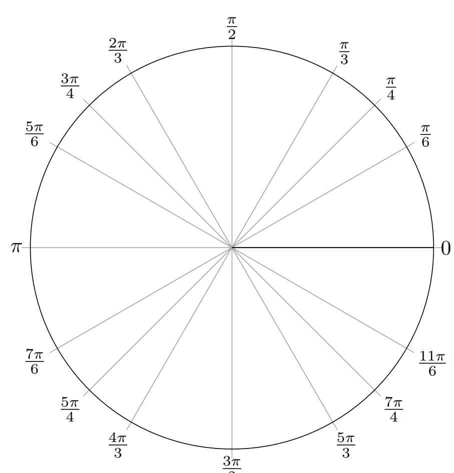



**Figure 1.3.2** The unit circle

**Example 1.3.3 Finding Terminal Points.**

Find the terminal points for the angles 210° and $\frac{5\pi}{3}$.

**Solution**. First, note that $210° = \frac{7\pi}{6}$. We therefore choose $\frac{\pi}{6}$ as our reference angle and obtain the terminal point $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$. The terminal point for $\frac{5\pi}{3}$ is likewise $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$.

## The Trigonometric Functions

Since angles determine terminal points on circles, the coordinates $(x, y)$ of each point on the circle can be viewed as functions of the angle $\theta$. These *coordinate functions*, $x(\theta)$ and $y(\theta)$, are the fundamental trigonometric functions *sine* and *cosine*, and can be used to define the other four trigonometric functions commonly used.

**Definition 1.3.4 Trigonometric Functions.**

Let $\theta \in \mathbb{R}$ and let $P = (x, y)$ denote the corresponding terminal point. The **cosine** function is the function $\cos\theta = x$, and the **sine** function is the function $\sin\theta = y$. The **secant, cosecant, tangent** and **cotangent** functions are defined as follows:

$$\sec\theta = \frac{1}{\cos\theta}, \csc\theta = \frac{1}{\sin\theta}, \tan\theta = \frac{\sin\theta}{\cos\theta} \text{ and } \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

The trigonometric functions satisfy important equalities known as *Pythagorean identities*.

**Theorem 1.3.5 Pythagorean Identities.**

*Let $\theta \in \mathbb{R}$. Then*

$$\sin^2\theta + \cos^2\theta = 1.$$

*If $\theta \neq \frac{\pi}{2} + k\pi$ for some $k \in \mathbb{Z}$, then*

$$\tan^2\theta + 1 = \sec^2\theta.$$

*If $\theta \neq k\pi$ for some $k \in \mathbb{Z}$, then*

$$\cot^2\theta + 1 = \csc^2\theta.$$

# Chapter 2

# Limits of Functions

At its heart, calculus is just the mathematics of change. In particular, calculus provides the tools necessary to describe how a function changes. Since a function can be used to represent many quantities that appear in real life, calculus therefore gives us a way to study how many different quantities (such as temperature, acceleration, energy of a signal, etc.) can change. But before we can start using calculus, we need to come up with the proper language to describe it. The language we will need to develop is that of the *limit of a function*.

## 2.1 The limit of a function

### Motivating limits

Imagine creating a mathematical valley out of the graph of $f(x) = x^2$, and in this valley walks a mathematical ant. The ant is walking towards the place on the hill directly above $x = -\frac{1}{2}$. At $x = -2$, the ant is 4 units above the ground. At $x = -1$, the ant is now 1 unit above the ground. As the ant moves towards $x = -\frac{1}{2}$, its height above the ground gets closer and closer to $\frac{1}{4}$. To say it more clearly, and because modern calculus is built on this idea, *as the ant approaches* the point above $x = -\frac{1}{2}$, its *height above the ground approaches* the value $\frac{1}{4}$. In other words, the *limit* of $f(x) = x^2$ as $x$ approaches $-\frac{1}{2}$ is $\frac{1}{4}$.

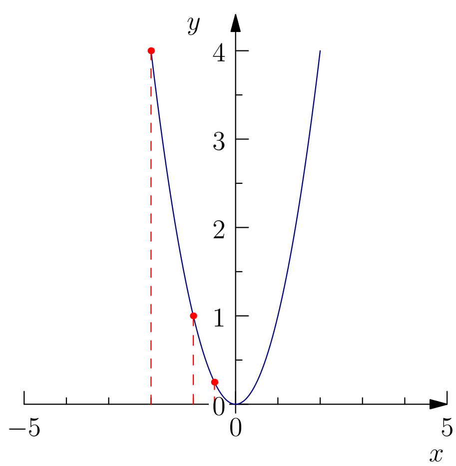



**Figure 2.1.1** A mathematical valley minus a mathematical ant

Remember that a function $f$ of the variable $x$ is just a rule that turns one number, $x$, into another number, $f(x)$. So the idea that the *limit of a function* is trying to express is what happens to the number $f(x)$ (the output) as the number $x$ (the input) approaches some particular value.

> Actually, functions are much more general than this. But for calculus, it won't hurt us to view functions in this way.

We're not quite ready to define the limit of a function precisely, but we can point one thing out right away: *the limit of a function requires two pieces of information: the function itself and the number that $x$ is approaching.* The limit should then be whatever number that $f(x)$ is approaching.

**Example 2.1.2 Estimating the limit of a trigonometric function.**

Let $f(x) = \sin x \cos x$. What is the limit of $f(x)$ as $x$ approaches the number $\pi$?

**Solution**. We don't have a lot of tools to find limits yet, so we'll try to estimate it instead. What we'll do is we'll plug numbers that are closer and closer to $\pi$ into $f(x)$. Let's list several values of $f(x)$ as $x$ gets closer to $\pi$ from the left:

**Table 2.1.3 Estimating** $\lim_{x\to\pi} f(x)$

| $x$ | $f(x)$ |
|---|---|
| 3 | $-.140$ |
| 3.1 | $-.042$ |
| 3.14 | $-.002$ |

We can even let $x$ approach $\pi$ from the other direction as well (i.e. "from the right") and $f(x)$ will still approach 0 as $x$ gets closer and closer to $\pi$. So it looks like the limit should be 0.

To keep ourselves from writing "the limit of $f(x)$ as $x$ approaches some number $a$" over and over, let's introduce some notation: $\lim\limits_{x\to a} f(x)$.

**Example 2.1.4 Limit of a piecewise function.**

Let
$$f(x) = \begin{cases} \sin x & \text{if } x < 0 \\ \cos x & \text{otherwise} \end{cases}.$$

Find $\lim\limits_{x\to 0} f(x)$.

**Solution**. If we graph this function, we see that at $x = 0$ there is a jump in the graph. In particular, if $x$ approaches 0 from the left then $f(x)$ approaches 0, whereas if $x$ approaches 0 from the right then $f(x)$ approaches 1. So this function does not appear to have an unambiguous limit as $x$ approaches 0. Another way to say this: $\lim\limits_{x\to 0} f(x)$ *does not exist.*

Example 2.1.4 shows us something very important about limits: they depend on the two different ways $x$ can approach a number. So we introduce two new pieces of notation: the **left-hand limit** $\lim_{x\to a^-} f(x)$ will stand for the value $f(x)$ approaches (if any) as $x$ approaches $a$ from the left (i.e. as $x$ increases to $a$), and **the right-hand limit** $\lim_{x\to a^+} f(x)$ will stand for the value $f(x)$ approaches (if any) as $x$ approaches $a$ from the right (i.e. as $x$ decreases to $a$). In Example 2.1.4, we would say that

$$\lim_{x\to 0^-} f(x) = 0 \text{ and } \lim_{x\to 0^+} f(x) = 1.$$

At this point, we can make a rough definition for the limit of a function.

**Definition 2.1.5 Limit of a function.**

Let $f(x)$ be a function. Suppose that both the left-hand limit $\lim_{x\to a^-} f(x)$ and the right-hand limit $\lim_{x\to a^+} f(x)$ exist and are equal to the same number $L$. Then we say that the limit of $f(x)$ as $x$ approaches $a$ exists and is equal to $L$. We denote this by writing $\lim_{x\to a} f(x) = L$.

**Example 2.1.6 Piecewise function again.**

Let $f(x)$ be given by

$$f(x) = \begin{cases} 0 & -3 \le x \le -1 \\ x^2 & -1 < x < \frac{1}{2} \\ \frac{1-x}{2} & \frac{1}{2} < x < 3. \end{cases}$$

Evaluate $\lim_{x\to -1} f(x)$ and $\lim_{x\to \frac{1}{2}} f(x)$.

**Solution.** If we graph $f(x)$, we get the following:

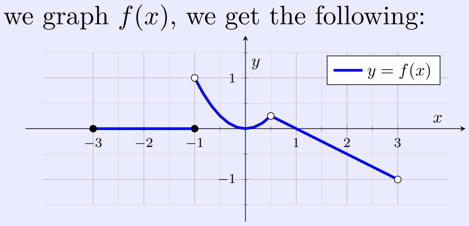



**Figure 2.1.7** Graphing $f(x)$

The graph shows us that $\lim_{x\to -1^-} f(x) = 0$, while $\lim_{x\to -1^+} f(x) = 1$. Therefore $\lim_{x\to -1} f(x)$ does not exist. On the other hand, $\lim_{x\to \frac{1}{2}} f(x)$ exists and is equal to $\frac{1}{4}$.

It's important to note that the value of a function at a point $x = a$ is in general *completely independent* of the value of $\lim_{x\to a} f(x)$, i.e., we can't always expect $\lim_{x\to a} f(x)$ to be equal to $f(a)$. Functions for which this is true, however, are known as *continuous functions* and will be very important in Section 2.3 and beyond.

## 2.2 Computing limits

We've got a handle on how to estimate limits from Section 2.1, but the process is very tedious. It requires either graphing the function in question or laboriously entering values into a calculator. So our first order of business now that we have the concept of a limit is to find an easier way to calculate it. This will be a running theme throughout the course.

## The limit laws

In many cases of interest, we can use knowledge of simpler limits to obtain more complicated limits. We do this via the **Limit Laws**. Before we get to them, we'll state two very simple (and hopefully very believable) limits.

**Proposition 2.2.1 Simple limits.**

*For any value of $a$, the following limits hold:*

$$\lim_{x\to a} c = c$$

*if $c$ is a constant and*

$$\lim_{x\to a} x = a.$$

**Theorem 2.2.2 The limit laws.**

*Let $c$ be a constant, let $n$ be a positive whole number and let $f(x)$ and $g(x)$ be functions. Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist for some number $a$. Then the following rules hold:*

***Table 2.2.3 The limit laws***

| | |
|---|---|
| 1. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$ | 2. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$ |
| 3. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\lim_{x\to a} g(x)$ | 4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *(if* $\lim_{x\to a} g(x)$ |
| 5. $\lim_{x\to a}[f(x)^n] = [\lim_{x\to a} f(x)]^n$ | 6. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ |

*Note that item six in the table above only holds (in this class...) if $n$ is odd or if $\lim_{x\to a} f(x) \geq 0$.*

Theorem 2.2.2 gives us the ability to compute a wide variety of limits.

**Example 2.2.4 Limit of a rational function.**

Let

$$f(x) = \frac{3 - x^5 + 5x}{2x - \sqrt[4]{x}}.$$

Evaluate $\lim_{x\to 16} f(x)$.

**Solution**. We can evaluate $\lim_{x\to 16} f(x)$ by making use of the appropriate Limit Laws and Proposition 2.2.1:

$$\begin{aligned}
\lim_{x\to 16} \frac{3 - x^5 + 5x}{2x - \sqrt[4]{x}} &= \frac{\lim_{x\to 16}(3 - x^5 + 5x)}{\lim_{x\to 16}(2x - \sqrt[4]{x})} && \text{by Limit Law 4}\\
&= \frac{\lim_{x\to 16} 3 - \lim_{x\to 16} x^5 + \lim_{x\to 16} 5x}{\lim_{x\to 16} 2x - \lim_{x\to 16} \sqrt[4]{x}} && \text{by Limit Law 1.}\\
&= \frac{3 - [\lim_{x\to 16} x]^5 + 5\lim_{x\to 16} x}{2\lim_{x\to 16} x - \sqrt[4]{\lim_{x\to 16} x}} && \text{by Limit Laws 2, 5, and 6.}\\
&= \frac{3 - 16^5 + 80}{30}
\end{aligned}$$

In particular, Limit Laws 1-5 give us the following: if $f(x)$ is a polynomial or rational function, then $\lim_{x\to a} f(x) = f(a)$ as long as $a$ is in the domain of $f(x)$. If $a$ is not in the domain, trickery may be required.

**Example 2.2.5 Trickery.**

Evaluate

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}.$$

**Solution**. First, note that we can't use the Limit Laws right away since the denominator is 0 at $x = 1$. What we need to do is use algebra to simplify the expression inside the limit:

$$\begin{aligned}\frac{\sqrt{x}-1}{x-1} &= \frac{\sqrt{x}-1}{x-1}\frac{\sqrt{x}+1}{\sqrt{x}+1}\\ &= \frac{x-1}{(x-1)(\sqrt{x}+1)}\\ &= \frac{1}{\sqrt{x}+1}.\end{aligned}$$

Now we can use the Limit Laws to find the limit as $x$ approaches 1, since we no longer have a divide-by-zero problem in the denominator:

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} &= \lim_{x\to 1}\frac{1}{\sqrt{x}+1}\\ &= \frac{1}{2}.\end{aligned}$$

## 2.3 Continuity

We saw in Section 2.2 that for a function like $f(x) = x+3x^2$, we could evaluate $\lim_{x\to a} f(x)$ by simply plugging in $x = a$. In other words, $\lim_{x\to a} f(x) = f(a)$. Functions that have this property are extremely important in mathematics, so we give them a name.

### Continuous functions

**Definition 2.3.1 Continuous Functions.**

Let $f(x)$ be a function and suppose that $a$ is in the domain of $f(x)$. Then we say that $f(x)$ is **continuous at** $a$ if

$$\lim_{x\to a} f(x) = f(a).$$

Otherwise, we say that $f(x)$ is **discontinuous at** $a$. We say that $f(x)$ is continuous on an interval if it is continuous at every point of an interval. Otherwise, we say that $f(x)$ is discontinuous on the interval.

Definition 2.3.1 says that it is extremely easy to evaluate limits of continuous functions: just plug the value that $x$ is approaching into the function $f(x)$. So the limit is then $f(a)$. If a function $f(x)$ is continuous on an interval, then this means that the graph of $f(x)$ has no "gaps" over this interval.

**Example 2.3.2 Determining if a function is continuous.**

Let $f(x) = \frac{1}{x}$. Is $f(x)$ continuous on $(-\infty, \infty)$?

**Solution**. If we graph $f(x)$, then we see that it is discontinuous at $x = 0$. Therefore $f(x)$ is discontinuous on the interval $(-\infty, \infty)$.

If we're dealing with a function on a (bounded) closed interval, we need to introduce some new terminology. We say that a function $f(x)$ is **continuous from the left** at $x = a$ if $\lim_{x\to a^-} f(x) = f(a)$. Similarly, we say that $f(x)$ is **continuous from the right** at $x = a$ if $\lim_{x\to a^+} f(x) = f(a)$. This is of course assuming that $a$ is in the domain of $f(x)$. Finally, we say that $f(x)$ is continuous on the closed interval $[a, b]$ if it is continuous on $(a, b)$, continuous from the right at $a$ and continuous from the left at $b$. The main idea is still that the graph contains no gaps over this interval.

**Example 2.3.3 Continuity over a closed interval.**

Let $f(x)$ be given by

$$f(x) = \begin{cases} 3x - 1 & 1 \le x < 2 \\ 9 - x^2 & 2 \le x \le 4 \end{cases}.$$

Is $f(x)$ continuous over $[1, 4]$?

**Solution**. If we graph $f(x)$ over this interval, we get the following:

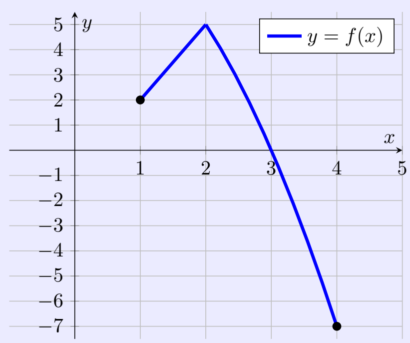



**Figure 2.3.4** Graphing $f(x)$ over $[1, 4]$

So from the graph it appears that $f(x)$ is continuous on this interval.

Remember that we said a function is continuous over an interval if its graph has no gaps over that interval. This is a very rough explanation of continuity, but it should make the following theorem plausible.

**Theorem 2.3.5 Continuous functions.**

*Polynomial, rational, root and trigonometric functions are continuous at every point of their domain.*

## Combining continuous functions

Although it doesn't directly mention piecewise functions, Theorem 2.3.5 is still useful for determining if they are continuous. If a piecewise function is defined using any of the functions from Theorem 2.3.5, then the only points we really need to check for continuity are the the places where the function "changes rules".

**Example 2.3.6 Another piecewise function.**

Over what intervals is the function $g(s)$ given by

$$g(s) = \begin{cases} s^2 & s < -1 \\ -5\cos(\pi s) & -1 \leq s \leq 1 \\ 5 & s > 1 \end{cases}$$

continuous?

**Solution**. We need to check continuity at $s = -1$ and $s = 1$. At $s = -1$, we need to make sure that $\lim_{s\to -1} g(s)$ exists and is equal to $g(-1)$. Since

$$\lim_{s\to -1^{-1}} g(s) = \lim_{s\to -1^{-1}} s^2 = 1$$

and

$$\lim_{s\to -1^{+}} g(s) = \lim_{s\to -1^{+}} \cos(\pi s) = -1,$$

it follows that $\lim_{s\to -1} g(s)$ does not exist. So $g(s)$ can't be continuous at $s = -1$.

On the other hand, since $\lim_{s\to 1^{-1}} g(s) = 5 = \lim_{s\to 1^{+}} g(s)$, it follows that $\lim_{s\to 1} g(s)$ exists and is equal to 5. Since $g(1)$ also equals 5, $g(s)$ is continuous at $s = 1$. So $g(s)$ must be continuous on $(-\infty, -1)\cup(-1, \infty)$.

We can also build more complicated continuous functions out of simpler ones.

**Theorem 2.3.7 Combining continuous functions.**

*Let $f(x)$ and $g(x)$ be continuous at a point $a$. Then the following statements are true:*

1. *$f(x) + g(x)$ is continuous at $a$.*
2. *$f(x)g(x)$ is continuous at $a$.*
3. *$\frac{f(x)}{g(x)}$ is continuous at $a$ if $g(a) \neq 0$.*
4. *$f(g(x))$ is continuous at $a$ if $g(a)$ is in the domain of $f(x)$.*

**Example 2.3.8 Determining where functions are continuous.**

Let $h(t) = \sqrt{t} + \frac{t}{t-4} - \frac{t+1}{t^2-1}$. On what intervals is $h(t)$ continuous?

**Solution**. By Theorem 2.3.7, $h(t)$ should be continuous wherever $\sqrt{t}$, $\frac{t}{t-4}$ and $\frac{t+1}{t^2-1}$ are all defined. Since $\sqrt{t}$ is defined for $t \geq 0$, $\frac{t}{t-4}$ is defined for $t \neq 4$ and $\frac{t+1}{t^2-1}$ is defined for $t \neq \pm 1$, it follows that $h(t)$ is continuous on $[0, 1) \cup (1, 4) \cup (4, \infty)$.

**Example 2.3.9 Using continuity to evaluate a limit.**

Evaluate $\lim_{x\to\pi}\cos(x+\sin x)$.

**Solution**. Since $x, \cos x$ and $\sin x$ are all continuous, this means that $\cos(x+\sin x)$ must be continuous as well. Therefore

$$\lim_{x\to\pi}\cos(x+\sin x)=\cos(\pi+\sin\pi)=-1.$$

We've seen that continuous functions are precisely those functions that make limits easy to compute. Since the two primary concepts in calculus, the derivative (Chapter 3) and the integral (Chapter 6), both depend on the concept of a limit, this means that continuous functions themselves will also be extremely important objects going forward.

## 2.4 Limits involving infinity

Limits are used to describe the behavior of a function $f(x)$ as $x$ approaches some quantity. In this section, we will see how limits can be used to describe singularities of functions (which appear as vertical asymptotes) and long-term behavior of functions (which usually appear as horizontal asymptotes).

### Limits involving vertical asymptotes

Consider the function $f(x)=\frac{1}{x}$. We know from algebra that this function has a vertical asymptote at $x=0$, and so in particular is undefined there. However, just because it's undefined at $x=0$ doesn't mean that we can't gather important information about the function near 0. This is because the function behaves in a very specific way as we let $x$ approach 0. For example, if we let $x$ approach 0 from the right, then $f(x)$ increases without bound. Similarly, $f(x)$ decreases without bound as $x$ approaches 0 from the left.

Even though $f(x)$ is not approaching a specific value as $x$ approaches 0 from either direction, this behavior shows up often enough and is important enough that we want to introduce notation to describe it. For this function, we would say $\lim_{x\to0^+}\frac{1}{x}=\infty$ and $\lim_{x\to0^-}\frac{1}{x}=-\infty$.

Now consider $g(x)=\frac{1}{(x-3)^2}$. Then $\lim_{x\to3^-}g(x)=\lim_{x\to3^+}g(x)=\infty$ since the function increases without bound when $x$ approaches 3 from both directions. In this case, we say that $\lim_{x\to3}g(x)=\infty$.

It's *extremely* important to remember that the symbol $\infty$ is not being used to represent a number or variable that we can perform algebra on. It's a symbol indicating how a particular function is behaving at a certain point.

If $f(x)$ is a function and $\lim_{x\to a}f(x)=\pm\infty$, $\lim_{x\to a^-}f(x)=\pm\infty$ or $\lim_{x\to a^+}f(x)=\pm\infty$, then this means that the function has a vertical asymptote at $x=a$. In this course, this basically corresponds to a divide-by-zero problem.

**Example 2.4.1 Infinite limit involving a rational function.**

Determine $\lim_{x\to 4^-} \frac{x-3}{2x-8}$.

**Solution**. If we try to plug in $x = 4$ into $\frac{x-3}{2x-8}$ we get $\frac{1}{0}$, which means we have run into a divide-by-zero problem. This is a good hint that the limit should be $\pm\infty$, we just need to figure out the correct sign. There are a couple ways we can do this. First, we could set up a sign chart for this function to see where it's positive and negative and then use that to see if it's increasing or decreasing without bound as $x \to 4^-$. Second, we could just plug in values of $x$ that are closer and closer to 4 and see how the function behaves. Either way, we see that it's negative for values of $x$ that are close to (but less than) 4. Hence $\lim_{x\to 4^-} \frac{x-3}{2x-8} = -\infty$.

## Limits at infinity

The previous subsection involved limits of functions whose values approached $\pm\infty$. Now we look at what can happen to a function if its input values approach $\pm\infty$. First, a definition of sorts.

**Definition 2.4.2 Limit at infinity.**

Let $f(x)$ be a function. We say that $\lim_{x\to\infty} f(x) = L$ if $f(x)$ gets (and stays) arbitrarily close to $L$ as $x$ is made arbitrarily large. Similarly, we say that $\lim_{x\to-\infty} f(x) = L$ if $f(x)$ gets (and stays) arbitrarily close to $L$ as $x$ is made arbitrarily small.

**Example 2.4.3 An important limit at infinity.**

Let $f(x) = \frac{1}{x}$. Determine $\lim_{x\to\infty} f(x)$.

**Solution**. As $x$ gets arbitrarily large, $\frac{1}{x}$ gets arbitrarily close to 0. Therefore $\lim_{x\to\infty} f(x) = 0$.

Example 2.4.3 holds for many other reciprocal powers of $x$. In particular, if $n > 0$ then $\lim_{x\to\infty} \frac{1}{x^n} = \lim_{x\to-\infty} \frac{1}{x^n} = 0$.

**Example 2.4.4 A limit at infinity involving cosine.**

Let $g(s) = \cos s^{-3}$. Compute $\lim_{s\to-\infty} g(s)$.

**Solution**. First, note that $g(s) = \cos\frac{1}{s^3}$. By the previous remark, we know that $\lim_{s\to-\infty} \frac{1}{s^3} = 0$. Therefore $\lim_{s\to-\infty} g(s) = \cos 0 = 1$.

The reason we were able to find the limit in Example 2.4.4 was because of the following fact: if $\lim_{x\to a} g(x)$ exists and if $f(x)$ is continuous at $\lim_{x\to a} g(x)$, then $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$. Basically, we can swap continuous functions with limits without causing any harm.

For limits at infinity involving powers of a variable, it is the highest power variables that determine the outcome.

**Example 2.4.5 Limit at infinity of a rational function.**

Let
$$f(t) = \frac{100t - 3 + t^3}{5t^2 + 1} \text{ and } g(t) = \frac{1 - 2t^4}{5t^4 + 3000t}.$$
Find $\lim_{t\to\infty} f(t)$ and $\lim_{t\to-\infty} g(t)$.

**Solution**. Let's start with $f(t)$. To figure out what this limit should be, we could try the following. As $t$ gets very large the $t^3$ term in the numerator should drown out everything else in the numerator. Similarly, the $5t^2$ term in the denominator should drown out everything else in the denominator. So for $t$ very large, $f(t) \approx \frac{t^3}{5t^2} = \frac{1}{5}t$. Hence $f(t)$ should probably go to $\infty$ as $t$ goes to $\infty$. To make this more precise, we'll just divide the numerator and denominator by the largest power of the denominator, $t^2$, and then take the limit:

$$\begin{aligned}\lim_{t\to\infty} f(t) &= \lim_{t\to\infty} \frac{100t - 3 + t^3}{5t^2 + 1}\\ &= \lim_{t\to\infty} \frac{\frac{100}{t} - \frac{3}{t^2} + t}{5 + \frac{1}{t^2}}\\ &= \infty.\end{aligned}$$

We can find $\lim_{t\to-\infty} g(t)$ using the same idea. Just divide by the highest power in the denominator and then take the limit:

$$\begin{aligned}\lim_{t\to-\infty} g(t) &= \lim_{t\to-\infty} \frac{\frac{1}{t^4} - 2}{5 + \frac{3000}{t^3}}\\ &= -\frac{2}{5}.\end{aligned}$$

These limits at infinity have a graphical meaning as well. If $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$ exists and is equal to $L$, then the line $y = L$ is a horizontal asymptote of the graph of $y = f(x)$.

**Example 2.4.6 Asymptotic equivalence.**

Two functions $f(x)$ and $g(x)$ are said to be **asymptotically equivalent**, written $f \sim g$, if the following is true:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

Show that $\sin \frac{1}{x} \sim \frac{1}{x}$.

**Solution**. All we need to do is compute $\lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}}$ :

$$\begin{aligned}\lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} &= \lim_{\theta\to0^+} \frac{\sin\theta}{\theta}\\ &= 1.\end{aligned}$$

Therefore $\sin \frac{1}{x} \sim \frac{1}{x}$.

Now that we've used limits to describe function behaviors that we are already familiar with, such as gaps in graphs and asymptotes, we will move on to using limits to define new concepts of importance. In Chapter 3 we will introduce the first such concept, and one of the most important concepts in calculus as a whole: that of the *derivative* of a function.

# Chapter 3

# Derivatives

There are two problems that inspired the creation of calculus. The first is the tangent line problem, or rate of change problem. This problem is concerned with determining how a quantity $f(x)$ changes as $x$ varies. We will focus on this problem and its solution, derivatives, for the next two chapters. Afterwards, we will focus on the area problem, the second problem that inspired the creation of calculus.

## 3.1 The definition of the derivative

### Tangent lines

Consider $f(x) = x^2$. If we graph this, we get a parabola. What we'd like to do is to find a way to describe how quickly this parabola is changing at a point, i.e. find the "slope" of the parabola. One way to try to deal with this is to use **secant lines**. Recall that the secant line through the points $(a, f(a))$ and $(b, f(b))$ has slope $\frac{f(b)-f(a)}{b-a}$, which is just the average rate of change of $f$ from $x = a$ to $x = b$. If $b$ is very close to $a$, then the slope of the secant line through these points should be a good approximation to the "slope" of $f(x) = x^2$ at the point $x = a$.

**Example 3.1.1 Secant lines on a parabola.**

Let $f(x) = x^2$. Find the slope of the secant line through $(2, f(2))$ and $(x, f(x))$.

**Solution.** Since the slope of the secant line is the average rate of change, we get that the slope must be equal to

$$\frac{f(x) - f(2)}{x - 2} = \frac{x^2 - 4}{x - 2} = x + 2.$$

What Example 3.1.1 is telling us is that if $x \approx 2$, then the slope of $f(x) = x^2$ at $x = 2$ should be very close to $x + 2$, the slope of the secant line. Now we'll do a trick that shows up everywhere in calculus, and is the entire reason we introduced limits in the first place. We'll make this approximation exact by taking a limit. In particular, we'll say that the slope of $f(x) = x^2$ should be equal to

$$\lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2}(x + 2) = 4.$$

This is the slope of the **tangent line** to $f(x) = x^2$ at $x = 2$. Instead of an average rate of change over an interval $[2, x]$, we now have an **instantaneous rate of change** at a point $x = 2$.

**Definition 3.1.2 Tangent lines.**

The tangent line to a curve $y = f(x)$ through a point $(a, f(a))$ is the line passing through $(a, f(a))$ with slope given by

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

assuming this limit exists. The slope of the tangent line represents the slope of the graph of $f(x)$ at $a$ and gives the instantaneous rate of change of $f(x)$ at $x = a$.

**Example 3.1.3 Tangent line to a root.**

Find the equation of the tangent line to $y = \sqrt{x}$ through the point $(4, 2)$.

**Solution.** We need two things to find the equation of a line: the slope of the line and a point on the line. Since we know the tangent line has to pass through $(4, 2)$, we just need to find the slope. The slope is given by

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{x - 4}{(x - 4)(\sqrt{x} + 2)} = \frac{1}{4}.$$

Hence the equation of the tangent line through $(4, 2)$ is

$$y - 2 = \frac{1}{4}(x - 4).$$

As a reminder, the slope of the tangent line represents the slope, or instantaneous rate of change, of the function.

**Example 3.1.4 Velocity from position.**

The displacement (i.e. position) of a particle moving in a straight line is described by the function $s(t) = 3t^2 - 5t$, where $t$ is in seconds and $s$ is in meters. Find the velocity, or instantaneous rate of change, of the particle at $t = 3$.

**Solution.** The velocity is just the slope of the tangent line to $s(t)$ at $t = 3$, which we can find as follows:

$$\begin{aligned} \lim_{t \to 3} \frac{s(t) - s(3)}{t - 3} &= \lim_{t \to 3} \frac{(3t^2 - 5t) - 12}{t - 3} \\ &= \lim_{t \to 3} \frac{3t^2 - 5t - 12}{t - 3}. \end{aligned}$$

At this step it's a little unclear where to go next, so we'll try long division. If we do so, we get

$$\frac{3t^2 - 5t - 12}{t - 3} = 3t + 4.$$

Hence the velocity must be

$$\lim_{t\to 3}(3t+4) = 13$$

meters per second.

Example 3.1.4 was a little tricky because we needed to compute $\lim_{t\to 3}\frac{3t^2-5t-12}{t-3}$, and it was unclear how to simplify this at first. This stemmed in large part from how we computed the velocity in the first place, using the formula

$$\lim_{t\to 3}\frac{s(t)-s(3)}{t-3}$$

or more generally

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$

We want to rewrite this formula to make it a little easier to work with in certain cases. We'll do this by making the denominator easier to handle. In particular, set $x-a=h$. Then

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

Either formula can be used to compute the slope of the tangent line.

**Example 3.1.5 Velocity revisited.**

Let $s(t)$ be given as in Example 3.1.4. Find the velocity at $t=3$.

**Solution.** We know that the velocity should be 13, but we'll try to find it again using our new formula. So the velocity should also be

$$\begin{aligned}\lim_{h\to 0}\frac{s(3+h)-s(3)}{h} &= \lim_{h\to 0}\frac{3(9+6h+h^2)-5(3+h)-12}{h}\\ &= \lim_{h\to 0}\frac{13h+3h^2}{h}\\ &= \lim_{h\to 0}(13+3h)\\ &= 13.\end{aligned}$$

Typically, if $f(x)-f(a)$ is easy to factor in terms of $x-a$ we'll want to use the first formula we had for computing rates of change. Otherwise, we'll stick with the new formula involving $h$.

## The derivative

Suppose we go back to Example 3.1.4 one more time, but now we want to find the velocity of $s(t)$ at an arbitrary number $a$. Then we could still use our limit formulas, which would give us

$$\begin{aligned}\lim_{h\to 0}\frac{s(a+h)-s(a)}{h} &= \lim_{h\to 0}\frac{3(a^2+2ah+h^2)-5(a+h)-3a^2+5a}{h}\\ &= \lim_{h\to 0}\frac{6ah+3h^2-5h}{h}\\ &= \lim_{h\to 0}(6a+3h-5)\end{aligned}$$

$$= 6a - 5.$$

So the velocity at $t = a$ of the particle modeled by $s(t)$ is given by $6a - 5$. So we can represent the velocity, or rate of change or slope of the tangent line, by a function. We call this the **derivative**.

**Definition 3.1.6 Definition of the derivative.**

Let $f(x)$ be a function. Then its derivative at $x = a$ is the number $f'(a)$ given by

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

assuming the limit exists. If this limit exists, we say that the function $f(x)$ is **differentiable** at $a$.

We could also define the derivative by

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

These two limits are equivalent.

The derivative of a function $f(x)$ at a point $a$ represents two things: the slope of the tangent line to $f(x)$ at $a$ and the instantaneous rate of change of $f(x)$ at $a$.

**Example 3.1.7 Slope of the sine function.**

Let $f(x) = \sin x$. Find its slope at 0.

**Solution**. The slope at 0 is exactly $f'(0)$, which is

$$\lim_{h\to 0} \frac{\sin(0+h) - \sin(0)}{h} = \lim_{h\to 0} \frac{\sin h}{h} = 1.$$

**Example 3.1.8 Tangent line of the sine function.**

Find the equation of the tangent line to $f(x) = \sin x$ at 0.

**Solution**. The tangent line must pass through $(0, 0)$ and must have slope $f'(0) = 1$, so its equation is

$$y - 0 = 1(x - 0)$$

or just $y = x$.

## 3.2 The derivative as a function

### The derivative function

There's no reason we can't look at an arbitrary value for $a$ in the definition of $f'(a)$ given in Definition 3.1.6. If we do this, we can define the *derivative function*.

**Definition 3.2.1 The derivative function.**

Let $f(x)$ be a function. The **derivative function**, or more simply **derivative**, of $f(x)$ is the function $f'(x)$ defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

assuming this limit exists. This is also often denoted by $\frac{d}{dx}(f)$ or $\frac{df}{dx}$. If this limit exists for all $x$ in some interval $I$, we say that $f$ is **differentiable on** $I$, or more simply **differentiable** if we do not wish to specify the interval.

**Example 3.2.2 Computing a derivative.**

Compute the derivative of $f(x) = x - 3x^2$.

**Solution**. Using Definition 3.2.1, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{[(x+h)-3(x+h)^2]-x+3x^2}{h} \\ &= \lim_{h\to 0} \frac{[x+h-3x^2-6xh-3h^2]-x+3x^2}{h} \\ &= \lim_{h\to 0} \frac{h-6xh-3h^2}{h} \\ &= \lim_{h\to 0} (1-6x-3h) \\ &= 1-6x. \end{aligned}$$

If $f(x)$ is a function, then its derivative $f'(x)$ (assuming it exists!) is a function that gives the rate of change of $f$ at $x$, or equivalently the slope of the tangent line to $f$ at $x$.

**Example 3.2.3 Sketching a derivative.**

Sketch $\frac{dg}{dt}$ where $g(t)$ is the function whose graph is given in Figure 3.2.4.

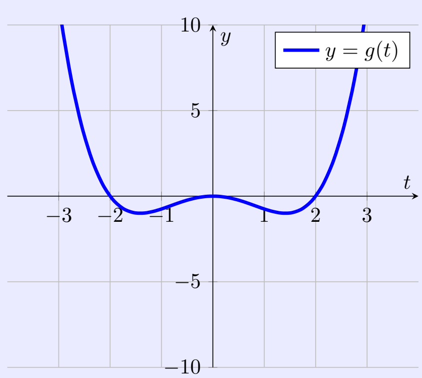



**Figure 3.2.4** Graph of $g(t)$

**Solution**. Remember that $\dfrac{dg}{dt}$ represents the slope of $g(t)$, so sketching $\dfrac{dg}{dt}$ amounts to sketching the different values that the slopes of $g(t)$ can take. We can eyeball these values from Figure 3.2.4. A rough sketch of $\dfrac{dg}{dt}$, added to the original graph, will look similar to Figure 3.2.5.

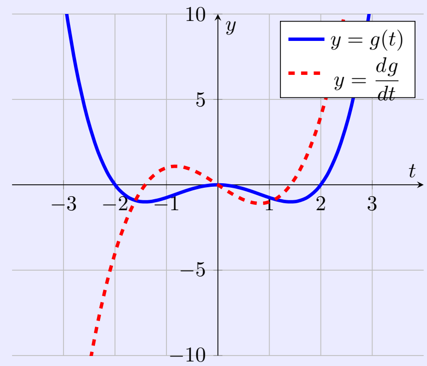



**Figure 3.2.5** Graph of $g'(t)$

We've mentioned before that continuous functions are functions whose graphs can be drawn without lifting your pencil off of the page. Likewise, differentiable functions are functions whose graphs can be drawn "smoothly", without any sudden movements or cusps, and without drawing a vertical tangent line. If we think about these two concepts, we may suspect that a differentiable function is also continuous. If we can draw a graph smoothly, we certainly can't lift our pencil off the page to draw it. The next theorem makes this precise.

**Theorem 3.2.6 Differentiable functions are continuous.**

*Let $f(x)$ be a function that is differentiable at $x = a$. Then $f(x)$ is continuous at $x = a$.*

*Proof.* We need to show that $\lim_{x\to a} f(x)$ exists and is equal to $f(a)$. To do this, we'll start by considering (somewhat counterintuitively) $\lim_{x\to a}[f(x) - f(a)]$:

$$\begin{aligned}\lim_{x\to a}[f(x) - f(a)] &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a}(x - a)\\ &= \lim_{x\to a} f'(a)(x - a)\\ &= 0.\end{aligned}$$

Note that we are using our alternate definition of the derivative here.

Now we can prove that $\lim_{x\to a} f(x) = f(a)$ as follows:

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}[f(x) - f(a) + f(a)]\\ &= \lim_{x\to a}[f(x) - f(a)] + \lim_{x\to a} f(a)\\ &= 0 + f(a).\end{aligned}$$

So $\lim_{x\to a} f(x) = f(a)$, which means that $f(x)$ is continuous at $a$. ■

At this point we might think that a continuous function should also be differentiable, but this is not the case.

Example 3.2.7 A continuous function that is not differentiable at a point.

Let $f(x) = |x|$. Show that $f$ is *not* differentiable at 0.

**Solution**. If we graph $f(x)$ it looks like it shouldn't be differentiable at 0 because of the cusp. We'll try to prove this mathematically by showing that the limit in Definition 3.2.1 doesn't exist if $x = 0$. First, we'll compute the left hand limit:

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1.$$

Now, the right hand limit:

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1.$$

Since these limits are different, $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ does not exist. Hence $f$ is not differentiable at 0.

You may think that a continuous function must at least be differentiable "almost everywhere" at this point. After all, how could it be possible to draw a graph without lifting your pencil off the paper that still has a cusp or a vertical tangent line *everywhere*? Most mathematicians before the 19$^{\text{th}}$ century thought this as well, until Weierstrass came up with a function, the Weierstrass function, that is continuous everywhere but differentiable *nowhere*.

## Higher order derivatives

Example 3.2.8 Acceleration from position.

The position of some particle moving in a line is given by $s(t) = 3t - 5t^3$, where $t$ is in seconds and $s$ is in meters. Find $a(t)$, the acceleration of the particle at time $t$.

**Solution**. Acceleration is the rate of change of velocity, and velocity is the rate of change of position. So we should probably find the velocity first! Let's call it $v(t)$. We have

$$\begin{aligned}
v(t) &= \lim_{h\to 0} \frac{s(t+h)-s(t)}{h} \\
&= \lim_{h\to 0} \frac{3(t+h)-5(t+h)^3-3t+5t^3}{h} \\
&= \lim_{h\to 0} \frac{3h+5[t^3-(t+h)^3]}{h} \\
&= \lim_{h\to 0} \frac{3h+5[(-h)(t^2+t(t+h)+(t+h)^2)]}{h} \\
&= \lim_{h\to 0} (3-5[t^2+t(t+h)+(t+h)^2]) \\
&= 3-15t^2.
\end{aligned}$$

Now we can get the acceleration as well:

$$a(t) = \lim_{h\to 0} \frac{v(t+h) - v(t)}{h} = -30t.$$

In Example 3.2.8, we had to take two derivatives of the original function $s(t)$ in order to get the acceleration $a(t)$. In other words, *acceleration is the second derivative of position.* So $a(t) = \frac{d}{dt}\frac{ds}{dt}$, which we also write as $\frac{d^2 s}{dt^2}$ or $s''(t)$. This is an example of a **second-order derivative**. In general, we have the following definition.

**Definition 3.2.9 $n^{\text{th}}$-order derivatives.**

Let $f(x)$ be a function. The **$n^{\text{th}}$-order derivative** of $f(x)$ is the function obtained by differentiating $f(x)$ $n$ times. This function is denoted by

$$f^{(n)}(x) \quad \text{or} \quad \frac{d^n f}{dx^n}.$$

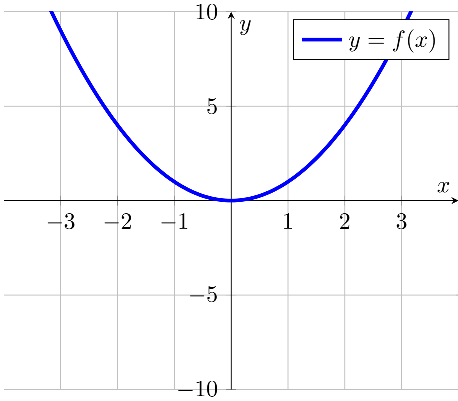
If $n = 1, 2$ or 3, we typically write $f'(x), f''(x)$, and $f'''(x)$ instead of $f^{(1)}(x), f^{(2)}(x)$ or $f^{(3)}(x)$.

Although it gets more difficult to assign a physical or geometric significance to higher order derivatives, we can still derive meaning from the second derivative. One interpretation of the second derivative is as acceleration, as shown in Example 3.2.8, and it turns out there's a nice geometric interpretation as well. Recall that if $f(x)$ is a function then $f'(x)$ represents the slope, or rate of change, of the graph of $f(x)$ at $x$. Therefore $f''(x)$ represents the rate of change of the slope, i.e. how quickly the slope is increasing or decreasing. If $f''(x) > 0$ then the slope of $f(x)$ should be increasing, leading to a u-shaped graph. Conversely, if $f''(x) < 0$ then the slope of $f(x)$ should be decreasing, leading to an upside down u-shaped graph. This leads to the following definition.

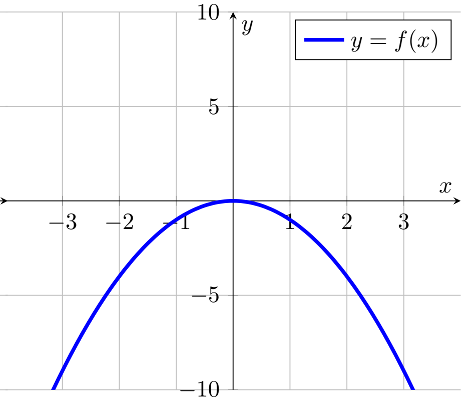
**Definition 3.2.10 Concavity.**

Let $f(x)$ be a function with second derivative $f''(x)$. We say that $f(x)$ is **concave up** (respectively, **concave down**) on an interval if $f''(x) > 0$ (respectively, $f''(x) < 0$) on that interval.

So functions that are concave up on an interval tend to be u-shaped on that interval, and functions that are concave down tend to be upside down u-shaped. See Figure 3.2.11.




(a) $f''(x) > 0$




(b) $f''(x) < 0$

**Figure 3.2.11** Concavity

## 3.3 Differentiation formulas

Now we start to find methods that allow us to compute derivatives without going back to Definition 3.2.1. Perhaps the easiest rule is the **constant rule**, which just says that the derivative of a constant is 0. We'll derive more complicated rules in this section and the next two.

### The power rule and trigonometric derivatives

Our first goal will be to determine a general formula for the derivative of $x^n$ for some power $n$. If $n \geq 0$ is a whole number, then we can find the derivative of $x^n$ without too much trouble. In fact, from the Definition 3.2.1 it's not too hard to show that the derivative of $x$ is 1, the derivative of $x^2$ is $2x$, the derivative of $x^3$ is $3x^2$, and so on. This suggests our first derivative rule, the **power rule**.

**Theorem 3.3.1 The power rule.**

*Let $f(x) = x^n$ where $n$ is some real number. Then $f'(x) = nx^{n-1}$.*

Note that Theorem 3.3.1 is actually quite general: it works for all powers of $x$! This includes negative powers and fractional powers.

**Example 3.3.2 Derivatives using the power rule.**

Find the derivatives of the following functions:

1. $f(x) = x^7$.
2. $g(x) = \frac{1}{\sqrt[5]{x^{11}}}$.
3. $h(x) = x^{\pi}$.

**Solution**. The derivative of $f(x)$ isn't too hard to find using the power rule, and we quickly get $f'(x) = 7x^6$. For $g(x)$, first rewrite it as $g(x) = x^{-\frac{11}{5}}$. Then $g'(x) = -\frac{11}{5}x^{-16}5$. Finally, $h'(x) = \pi x^{\pi-1}$.

With a little bit of geometry and the squeeze theorem, we can get the derivatives of the basic trigonometric functions $\sin x$ and $\cos x$.

**Theorem 3.3.3 Derivatives of sine and cosine.**

*Let $x$ be in radians. Then*

$$\frac{d}{dx}\sin x = \cos x \quad \text{and} \quad \frac{d}{dx}\cos x = -\sin x.$$

Note that if $x$ is in degrees instead of radians these formulas don't work. Instead, they become

$$\frac{d}{dx}\sin x = \frac{\pi}{180}\cos x \quad \text{and} \quad \frac{d}{dx}\cos x = -\frac{\pi}{180}\sin x.$$

**Example 3.3.4 Concavity of the sine function.**

On which intervals is $f(x) = \sin x$ concave up?

**Solution**. We need to find where $f''(x)$ is positive. Since $f''(x) = -\sin x$, this means we need to figure out where $\sin x$ is *negative*. If we go back to the unit circle definition of sine, then we can see that $\sin x < 0$ on the following intervals:

$$\ldots, (-3\pi, -2\pi), (-\pi, 0), (\pi, 2\pi), (3\pi, 4\pi), \ldots$$

So $\sin x$ is concave up on every open interval of the form $((2k-1)\pi, 2k\pi)$ where $k$ is some integer.

## Derivatives of sums and constant multiples

Now that we have derivative formulas for some basic functions, we want to extend these to more complicated functions. For this section we'll look at what happens when we multiply a function by a constant or add it to another function. In the next two sections we'll consider more advanced rules.

**Theorem 3.3.5 Constant multiple rule.**

*Let $f(x)$ be differentiable function and let $c$ be some constant. Then* $\frac{d}{dx}[cf(x)] = cf'(x)$.

*Proof.* To prove this, we go back to Definition 3.2.1:

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= \lim_{h\to 0}\frac{cf(x+h) - cf(x)}{h}\\ &= c\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}\\ &= cf'(x).\end{aligned}$$

Hence the derivative of $cf(x)$ is $cf'(x)$. ■

We can find the derivative of the sum of two functions just as easily.

**Theorem 3.3.6 Sum rule.**

*Let $f(x)$ and $g(x)$ be two differentiable functions. Then $[f(x)+g(x)]' = f'(x)+g'(x)$.*

## Product rule and quotient rule

Many functions can be written in the form $f(x)g(x)$, where $f(x), g(x)$ may each have previously known derivatives. What we want to do now is to find a way to get the derivative of $f(x)g(x)$ from the derivatives of $f(x)$ and $g(x)$. We do this using the **product rule**.

**Theorem 3.3.7 The product rule.**

*Let $f(x)$ and $g(x)$ be differentiable functions. Then*

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x).$$

*Proof.* We prove this using the definition of the derivative:

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)[g(x+h)-g(x)]+g(x)[f(x+h)-f(x)]}{h}\\
&= \lim_{h\to 0} f(x+h)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= f(x)g'(x)+g(x)f'(x).
\end{aligned}$$

■

**Example 3.3.8 Using the product rule.**

Let $J(v) = (v^{10} - 2\sin v)(\cos v + \frac{1}{\sqrt[3]{x^4}})$. Find $J'(v)$.

**Solution**. We could foil this out and take derivatives, but it will be easier to use the product rule.

## The quotient rule

Now that we know how to differentiate products, we move on to quotients.

**Theorem 3.3.9 The quotient rule.**

*Let $f(x)$ and $g(x)$ be differentiable functions. Then*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

*wherever $g(x) \neq 0$.*

**Example 3.3.10 Derivative of tangent.**

Let $f(x) = \tan x$. Find $f'(x)$.

**Solution.** Since $\tan x = \frac{\sin x}{\cos x}$, then we can apply the quotient rule to get the derivative of $\tan x$:

$$\frac{d}{dx} \tan x = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x.$$

The derivatives for $\sec x, \csc x$ and $\cot x$ may also be computed using the quotient rule and the facts that $\dfrac{d}{dx} \sin x = \cos x$ and $\dfrac{d}{dx} \cos x = -\sin x$.

## 3.4 The chain rule

At this point, we can take derivatives of sums, differences, products and quotients of functions. However, these rules aren't very useful for differentiating functions like $f(x) = (3x^2 + \sin x)^{50}$. We could technically evaluate $f'(x)$ using these rules but it would be an awful way to spend your weekend. But if we make the substitution $u = 3x^2 + \sin x$, then we can rewrite $f(x)$ as $f(u) = u^{50}$, which is *much* easier to differentiate: $\dfrac{df}{du} = 50u^{49}$. At this step we might be tempted to say that $f'(x) = 50u^{49} = 50(3x^2 + \sin x)^{49}$, but this isn't quite right. To get the actual derivative we need to consider how the new variable $u$ depends on $x$ as well. The **chain rule** is what we need.

**Theorem 3.4.1 The chain rule.**

*Let $y = f(u)$ and $u = g(x)$ be differentiable functions. Then*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

*Equivalently, if we set $F(x) = f(g(x))$ we have*

$$F'(x) = f'(g(x))g'(x).$$

**Example 3.4.2 Using the chain rule.**

Let $f(x) = (3x^2 + \sin x)^{50}$. Find $f'(x)$.

**Solution.** We'll try the same trick we used before and we'll set $u = 3x^2 + \sin x$. Then the chain rule says that

$$\begin{aligned} f'(x) &= \frac{df}{du}\frac{du}{dx} \\ &= (50u^{49})(6x + \cos x) \\ &= 50(3x^2 + \sin x)^{49}(6x + \cos x). \end{aligned}$$

As the last example highlighted, we can use the chain rule in combination with any of the other derivative rules we know if the function we're differentiating is complicated.

**Example 3.4.3 Combining rules.**

Let $f(s) = \sqrt{\frac{s^2-1}{s^2+1}}$. Find $f'(s)$.

**Solution**. We'll let $u = \frac{s^2-1}{s^2+1}$ stand in for the "inside function." Then we have $f(u) = \sqrt{u}$ and so

$$\begin{aligned}\frac{df}{ds} &= \frac{df}{du}\frac{du}{ds}\\ &= \frac{1}{2}u^{-\frac{1}{2}}\frac{2s(s^2+1)-2s(s^2-1)}{(s^2+1)^2}\\ &= \frac{1}{2}\left(\frac{s^2-1}{s^2+1}\right)^{-\frac{1}{2}}\frac{2s(s^2+1)-2s(s^2-1)}{(s^2+1)^2}.\end{aligned}$$

**Example 3.4.4 Chain rule within a chain rule.**

Find the slope of $g(x) = \sin^4(\cos^3 x)$ at $x = \pi$.

**Solution**. We need to compute $g'(\pi)$. First, note that $g(x) = [\sin(\cos^3 x)]^4$, so let $u = \sin(\cos^3 x)$. Now we could try to use the chain rule right here but this would require finding $\dfrac{du}{dx}$, and $u$ is itself a complicated function of $x$. So let $v = [\cos x]^3$, and finally let $w = \cos x$. Then we can say that

$$\begin{aligned}g'(x) &= \frac{dg}{du}\frac{du}{dv}\frac{dv}{dw}\frac{dw}{dx}\\ &= \left(\frac{d}{du}u^4\right)\left(\frac{d}{dv}\sin v\right)\left(\frac{d}{dw}w^3\right)\frac{d}{dx}\cos x\\ &= (4u^3)(\cos v)(3w^2)(-\sin x).\end{aligned}$$

We could plug in what $u, v, w$ are in terms of $x$ and then plug in $x = \pi$, but it's easier to just find $u, v, w$ at $x = \pi$ and enter these values into the above. At $x = \pi$ we have $u = \sin(-1), v = -1$ and $w = -1$, so

$$g'(\pi) = 4(\sin(-1))^3\cos(-1)(3)\cdot 0 = 0.$$

## 3.5 Implicit differentiation

**Example 3.5.1 Derivative of an implicit function.**

Consider the curve given by the equation $x^3 - y^2 + \sin y = 3$. Find the slope of this curve at the point $(\sqrt[3]{3}, 0)$.

**Solution**. We could try to solve for $y$ and then differentiate that to find the slope, but we have a slight problem: it's impossible, at least in terms of "elementary functions". However, we can still use the chain rule to find $y'$, at least in terms of $x$ and $y$. We just need to remember that $y$ is a function of $x$. If we differentiate $x^3 - y^2 + \sin y = 3$ with respect to $x$, we get

$$3x^2 - 2y\frac{dy}{dx} + \cos y\frac{dy}{dx} = 0$$

and so

$$\frac{dy}{dx} = -\frac{3x^2}{\cos y - 2y}.$$

So the slope of the curve at $(\sqrt[3]{3}, 0)$ is just $-3\sqrt[3]{3}^2$.

The method we used to get $\frac{dy}{dx}$ in Example 3.5.1 is called **implicit differentiation**. It's extremely useful if we want to solve for $\frac{dy}{dx}$ without first solving for $y$. Even if we can solve for $y$ without too much trouble, it's often easier to find $\frac{dy}{dx}$ implicitly as the next example shows.

Example 3.5.2 Implicit differentiation to save algebra.

Let $f(x) = \sqrt{1-x^2}$. Find $f'(x)$.

**Solution**. If we let $y = f(x)$, then $x^2 + y^2 = 1$. Then

$$2x + 2yy' = 0$$

which means that

$$y' = -\frac{x}{y} = -\frac{x}{\sqrt{1-x^2}}.$$

We must also be aware of when to use appropriate derivative rules when doing implicit differentiation.

Example 3.5.3 Chain and quotient rule.

Suppose $y$ is defined implicitly by $\tan(x-y) = \frac{y}{1+x^2}$. Find $y'$.

**Solution**. We start by taking the derivative with respect to $x$ of each side of the equation:

$$\frac{d}{dx}\tan(x-y) = (1-y')\sec^2(x-y)$$
$$\frac{d}{dx}\frac{y}{1+x^2} = \frac{y'(1+x^2) - 2xy}{(1+x^2)^2}.$$

Therefore

$$(1-y')\sec^2(x-y) = \frac{y'(1+x^2) - 2xy}{(1+x^2)^2},$$

and we can solve this for $y'$:

$$(1+x^2)^2\sec^2(x-y) + 2xy = y'(1+x^2) + (1+x^2)^2 y'\sec^2(x-y)$$

or just

$$y' = \frac{(1+x^2)^2\sec^2(x-y) + 2xy}{(1+x^2) + (1+x^2)^2\sec^2(x-y)}.$$

**Example 3.5.4 A differential equation.**

Let $P(t)$ denote the population of the United States, where $P$ is in millions and $t$ is the number of years after 1990. Then the growth of $P(t)$ can be modeled by the **differential equation**

$$P' = \frac{1}{23500} P(532 - P).$$

According to this model, it's not too hard to see that $P'$ should be positive given the current population of the US, so the model predicts the population to increase. Is this rate of growth increasing or decreasing?

**Solution**. We need to find $P''$, which is the rate of change of $P'$. This means we need to differentiate both sides of the differential equation $P' = \frac{1}{23500} P(532 - P)$ with respect to $t$:

$$P'' = \frac{1}{23500} P'(532 - P) - \frac{1}{23500} P(P').$$

The current population is about 323.1 million, so we can take $P = 323.1$, which also gives

$$P' = \frac{1}{23500}(323.1)(532 - 323.1) = 2.9,$$

and so

$$P'' = \frac{1}{23500}(2.9)(532 - 323.1) - \frac{1}{23500}(323.1)(2.9) < 0.$$

Hence it appears that the rate of population increase is itself decreasing, which implies that the population growth of the US is slowing down.

# Chapter 4

# Inverse Functions

Recall that a function $f(x)$ is **invertible** if it's one-to-one. In this case, there's an inverse function $f^{-1}(x)$ that undoes $f(x)$. In symbols, $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$.

## 4.1 Exponential functions

Many changing quantities can be modeled by using **exponential functions**, which are functions of the form $f(x) = a^x$ where $a > 0$. Exponential functions have several important properties which we list below.

**Theorem 4.1.1 Properties of exponential functions.**

*Let $a > 0$ and $x, y$ be real numbers. Then*

- $a^0 = 1$ *and* $a^1 = a$.
- $a^{-x} = \frac{1}{a^x}$.
- $a^{x+y} = a^x a^y$ *and* $a^{x-y} = \frac{a^x}{a^y}$.
- $a^{x/y} = \sqrt[y]{a^x}$ *is* $y > 0$.
- $(a^x)^y = a^{xy}$.

*Finally, if $b > 0$ as well, then $(ab)^x = a^x b^x$.*

**Example 4.1.2 Limit of an exponential function.**

Let $f(x) = 2^{-x}$. Find $\lim_{x\to\infty} f(x)$.

**Solution**. First, note that

$$f(x) = \frac{1}{2^x}.$$

So as $x \to \infty$, the denominator $2^x \to \infty$ as well. Hence the fraction as a whole should decrease to 0, and so

$$\lim_{x\to\infty} 2^{-x} = 0.$$

In general, we have the following:

$$
\begin{aligned}
a > 1 \Rightarrow \lim_{x\to\infty} a^x = \infty \quad &\text{and} \lim_{x\to-\infty} a^x = 0 \\
a < 0 \Rightarrow \lim_{x\to\infty} a^x = 0 \quad &\text{and} \lim_{x\to-\infty} a^x = \infty.
\end{aligned}
$$

Every exponential function $f(x) = a^x$ is continuous everywhere. In fact, they are differentiable everywhere, though we'll see more about that later. We can also say the following: if $a > 1$ then $a^x$ is an increasing function, and if $0 < a < 1$ then $a^x$ is a decreasing function. In neither case will $a^x$ have any local maxima or minima.

Of all the different exponential functions, the most important one is the so-called **natural exponential function**, denoted $e^x$ where

$$e = 2.71828....$$

This is the unique exponential function by the property that its derivative at 0 is 1. In fact, as we will see later $e^x$ is its own derivative.

**Example 4.1.3 Exponential derivative.**

Using the fact that $\frac{d}{dx}e^x = e^x$, compute $\frac{d}{dx}e^{5x-x^4+\cos x}$.

**Solution**. We'll have to use the chain rule. If we do so, we get

$$\frac{d}{dx}e^{5x-x^4+\cos x} = (5 - 4x^3 - \sin x)e^{5x-x^4+\cos x}.$$

Since $e^x$ is its own derivative, this makes computing derivatives of functions of the form $e^{f(x)}$ relatively straightforward:

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}.$$

Technically, we have worked with the natural exponential function before in this class. Recall from Example 3.5.4 that if $P(t)$ represents the population of the USA (in millions) $t$ years after 1990, then we said that $P$ satisfied the differential equation

$$\frac{dP}{dt} = \frac{1}{23500}P(532 - P).$$

If we say the population of the US in 1990 was 250 million (i.e. $P(0) = 250$), then

$$P(t) = \frac{133000}{250 + 282e^{-.023t}}.$$

Now, in Example 5.1.4 we used linear approximation to estimate the population of the US using this model in the year 2020, where we got 331.8 million people. If we use the above formula for $P(t)$, we get that the population of the US in 2020 should be about

$$P(30) = 339.8,$$

or 339.8 million people.

The Census Bureau predicted in 2014 that the population in 2020 would be 334.5 million. So our simple predictions before actually aren't too far off from what the Census Bureau is expecting. This suggests that the differential equation $P' - \frac{1}{23500}P(532 - P)$ is a reasonable model for the growth of the US.

## 4.2 Logarithms

### Logarithms

Since exponential functions are important in mathematics and its applications, their inverses are important as well. Each exponential function $f(x) = a^x$ $(a > 0)$ has an inverse function that we call the **logarithm with base** $a$, and denote by $f^{-1}(x) = \log_a x$. The domain of each logarithm is $(0, \infty)$ and the range is $(-\infty, \infty)$. The defining properties of the logarithm function are as follows:

$$\log_a a^x = x$$
$$a^{\log_a x} = x.$$

Essentially, logarithms and exponentials cancel each other out. To put this another way, $\log_a x$ is the *exponent* needed to turn $a$ into $x$.

We can say a few things about logarithms just from looking at graphs of exponentials. Suppose that $a > 1$. Then $\log_a x$ is both continuous and differentiable for all $x > 0$,

$$\log_a 1 = 0$$
$$\lim_{x \to 0^+} \log_a x = -\infty$$
$$\lim_{x \to \infty} \log_a x = \infty$$

and the derivative of $\log_a x$ approaches 0 as $x \to \infty$.

The properties of the exponential listed in Theorem 4.1.1 have corresponding properties for logarithms.

**Theorem 4.2.1 Properties of logarithm functions.**

*Let $a > 0$ with $a \neq 1$. Let $x, y$ be positive real numbers, and let $r$ be any real number. Then*

- $\log_a 1 = 0$ *and* $\log_a a = 1$.
- $\log_a(xy) = \log_a x + \log_a y$.
- $\log_a \frac{x}{y} = \log_a x - \log_a y$.
- $\log_a x^r = r \log_a x$.

An important takeaway from Theorem 4.2.1 is that logarithms turn the complicated operations of multiplication and division into the simpler operations of addition and subtraction.

Since every exponential function has a corresponding logarithm, the natural exponential function $e^x$ has a logarithm as well. We call this inverse the **natural logarithm** and denote it by $\ln x$. Note that $\frac{d}{dx} \ln x = \frac{1}{x}$ by Example 4.3.8.

Example 4.2.2 Simplifying an exponential.

Simplify $e^{-4\ln x}$ and $e^{x\ln 3}$.

**Solution**. We'll simplify by using the cancellation property $e^{\ln y} = y$. First, we need to put the entire exponent inside of the natural log:

$$e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}$$

and

$$e^{x\ln 3} = 3^x.$$

Example 4.2.2 shows us that *every* exponential function can be written in terms of the natural exponential: $a^x = e^{x\ln a}$. Similarly, every logarithm can be written in terms of the natural logarithm by using the change of base formula:

$$\log_a x = \frac{\ln x}{\ln a}.$$

## 4.3 Derivatives of exponential and logarithmic functions

### Inverse function review

Suppose we want to solve the equation $y = f(x)$ for $x$. Then we can only do so unambiguously if $f$ is an **invertible function**.

Definition 4.3.1 Invertible functions.

A function $f(x)$ is invertible if and only if it is a one-to-one function. That is, $f(x_1) = f(x_2)$ forces $x_1 = x_2$.

Since a function is one-to-one if and only if its graph passes the horizontal line test, the horizontal line test also gives us a useful way to check if a function is invertible.

Example 4.3.2

Let $f(x) = e^x$, the natural exponential function introduced in Section 4.1. Then $f(x)$ is an invertible function, since its graph passes the horizontal line test.

If a function $f$ is invertible, then there exists an **inverse function** $f^1$ satisfying the following equations:

$$f(f^{-1}(y)) = y$$
$$f^{-1}(f(x)) = x.$$

Essentially, $f^{-1}$ is the rule that undoes the transformation that $f$ applies to $x$. The domain of $f^{-1}$ is the range of $f$ and the range of $f^{-1}$ is the domain of $f$. Note that $f^{-1} \neq \frac{1}{f}$!.

The graph of an inverse function is related to the graph of the original function in a very nice way. To get the graph of $f^{-1}$ from $f$, just reflect the graph of $f$ across the line $y = x$.

**Example 4.3.3 Graphing the inverse of the natural exponential.**

Let $f(x) = e^x$. Graph $f^{-1}(x)$.

**Solution**. If we graph $e^x$ and then rotate the graph across the line $y = x$, we get the graph in Figure 4.3.4.

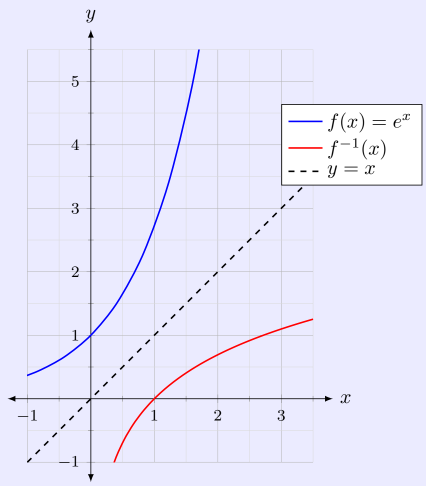



**Figure 4.3.4** The inverse of $e^x$.

## Calculus with inverse functions

There are two important things to notice about the graph of the inverse function in Example 4.3.2: it's continuous and differentiable. This is because the same was true of the original graph, and if we think about it this should be true in general as well since graphing the inverse doesn't add any new gaps or cusps to the graph.

**Theorem 4.3.5 Continuity and differentiability of inverses.**

*Let $f(x)$ be a continuous (respectively, differentiable) function. Suppose that $f(x)$ is invertible, and has inverse $f^{-1}(x)$. Then $f^{-1}(x)$ is continuous (respectively, differentiable).*

**Example 4.3.6 Finding the derivative of an inverse function.**

Let $f(x) = \sqrt{x-1}$ and note that $f$ is invertible. Find the derivative of $f^{-1}$.

**Solution**. One way to do this is just start by finding $f^{-1}$. So we'll set $y = \sqrt{x-1}$, solve for $x$ and then switch $x$ and $y$. If we do this, we get

$$y = x^2 + 1$$

so $f^{-1}(x) = x^2 + 1$ (and note that the domain of $f^{-1}$ is $[0, \infty)$ since this is the range of $f$!). Hence

$$\frac{d}{dx} f^{-1}(x) = 2x.$$

The method used in Example 4.3.6 will certainly work in simple cases. But what do we do if we can't (or don't want to) find an explicit formula for the inverse function? The following formula will help us to do this.

**Theorem 4.3.7 Derivative of an inverse function.**

*Let $f(x)$ be a differentiable function with inverse $f^{-1}(x)$. Assume that $f^{-1}$ is itself differentiable. Then*

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

*Proof.* First, let $y = f^{-1}(x)$, so that we need to find $y'$. Then we can say that $f(y) = x$. Now differentiate this equation implicitly to get

$$f'(y)y' = 1$$

or just

$$y' = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}.$$

■

**Example 4.3.8 Exponential inverse revisited.**

Using the fact that the derivative of $e^x$ is itself, find the derivative of its inverse.

**Solution**. Let $f(x) = e^x$. Then by Theorem 4.3.7 we have

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f(f^{-1}(x))} = \frac{1}{x}.$$

So the derivative of the inverse of $e^x$ is $\frac{1}{x}$.

**Example 4.3.9 Derivative at a point.**

Let $f(x) = x^3 + 3\sin x + 2\cos x$. Find $(f^{-1})'(2)$

**Solution**. By Theorem 4.3.7, we know that

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))}.$$

By inspection, $f(0) = 2$ which means $f^{-1}(2) = 0$. Since

$$f'(x) = 3x^2 + 3\cos x - 2\sin x,$$

this gives

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(0)} = \frac{1}{3}.$$

## Derivatives of exponentials

We've already mentioned that $\frac{d}{dx}e^x = e^x$, but we've said nothing about differentiating functions such as $3^x$ or $(\frac{1}{\pi})^{-x}$. However, it turns out that we can get these derivatives from the derivative of $e^x$ without too much trouble.

**Theorem 4.3.10 Derivatives of exponential functions.**

*Let $a > 0$ and set $f(x) = a^x$. Then $f'(x) = a^x \ln a$.*

*Proof.* To prove this, we need to use what's available to us: the derivative of $e^x$. From Example 4.2.2 we say that $e^{x\ln 3} = 3^x$. More generally, we can also say that $e^{x\ln a} = a^x$. Hence

$$\begin{aligned}\frac{d}{dx}a^x &= \frac{d}{dx}e^{x\ln a}\\ &= (\ln a)e^{x\ln a}\\ &= (\ln a)a^x.\end{aligned}$$

■

**Example 4.3.11 Differentiating an exponential.**

Let $g(t) = 2.6^{3t-\sin t}$. Find $g'(t)$.

**Solution**. If we set $u = 3t - \sin t$, then $g(t) = g(u) = 2.6^u$. By the chain rule, $g'(t) = g'(u)\dfrac{du}{dt} = (\ln 2.6)2.6^u(3 - \cos t) = (\ln 2.6)2.6^{3t-\sin t}(3 - \cos t)$.

Although we certainly want to know the derivative of $a^x$, or at least how to find it, most applications involving exponential functions use the natural exponential function $e^x$ instead. The derivative of $e^x$ is probably the most important derivative in this course.

**Example 4.3.12 Solutions of a differential equation.**

Show that $x(t) = Ae^{-t} + Bte^{-t}$ satisfies the differential equation $x'' + 2x' + x = 0$, where $A, B$ are arbitrary constants.

**Solution**. We need to show that if we compute $x'$ and $x''$ and plug these expressions into the differential equation, this will simplify out to 0. Since

$$\begin{aligned}x' &= -Ae^{-t} + Be^{-t} - Bte^{-t}\\ x'' &= Ae^{-t} - Be^{-t} - (Be^{-t} - Bte^{-t})\\ &= Ae^{-t} - 2Be^{-t} + Bte^{-t},\end{aligned}$$

it follows that

$$\begin{aligned}x'' + 2x' + x &= (Ae^{-t} - 2Be^{-t} + Bte^{-t}) + 2(-Ae^{-t} + Be^{-t} - Bte^{-r}) + Ae^{-t} + Bte^{-t}\\ &= 0.\end{aligned}$$

Exponential functions often appear in the solutions of differential equations, which are themselves often viewed as mathematical models of physical systems. Hence exponential functions play a significant role in predicting physical quantities, which goes a long way towards justifying their importance.

## Derivatives of logarithms

Just as we can get the derivative of every exponential function $a^x$ just by knowing the derivative of $e^x$, we can get the derivative of every logarithmic function $\log_a x$ just by knowing that $\dfrac{d}{dx} \ln x = \frac{1}{x}$.

**Theorem 4.3.13 Derivatives of logarithmic functions.**

*Let $a > 0, a \neq 1$ and set $f(x) = \log_a x$. Then*

$$f'(x) = \frac{1}{x \ln a}.$$

**Example 4.3.14 Differentiating nested logarithms.**

Let $f(x) = \log_2(\log_5 x)$. Find $f'(x)$.

**Solution**. By the chain rule, we have

$$f'(x) = \frac{1}{(\log_5 x)\ln 2} \frac{d}{dx} \log_5 x = \frac{1}{(\log_5 x)\ln 2} \frac{1}{x \ln 5},$$

Logarithms can be used to greatly simplify derivatives involving products, division or exponentiation using a technique known as **logarithmic differentiation**.

**Example 4.3.15 A simple fraction.**

Let $f(x) = \frac{e^{-x}\cos x}{x^2+4x+4}$. Find $f'(x)$.

**Solution**. We can find $f'(x)$ without resorting to logarithms, but this would require using the product, quotient and chain rules. The algebra would be awful. So we'll using logarithms instead! Set $y = f(x)$. Then

$$\ln y = \ln e^{-x} + \ln \cos x - \ln(x^2 + 4x + 4) = -x + \ln \cos x - 2\ln(x + 2).$$

Now we differentiate both sides implicitly to obtain

$$\frac{y'}{y} = -1 - \frac{\sin x}{\cos x} - \frac{2}{x + 2}.$$

Hence

$$y' = y\left(-1 - \tan x - \frac{2}{x+2}\right) = \frac{e^{-x}\cos x}{x^2 + 4x + 4}\left(-1 - \tan x - \frac{2}{x+2}\right).$$

Logarithmic differentiation is useful for finding derivatives of expressions containing complicated quotients, products or powers.

**Example 4.3.16 A simple exponent.**

Let $h(z) = z^z$. Find $h'(z)$.

**Solution**. We'll use logarithmic differentiation again to simplify $h$ and remove the exponent. Set $y = h(z)$, which gives

$$\ln y = z \ln z.$$

So

$$\frac{y'}{y} = \ln z + 1,$$

which means that

$$y' = y(\ln z + 1) = z^z(\ln z + 1).$$

## 4.4 Exponential Models

### Exponential growth

Suppose we want to predict the spread of some infectious disease through a city. A reasonable, though simplistic, assumption is that the disease will spread quicker if more people are infected. In other words, we'll assume that the rate at which people are infected is proportional to the number of people infected. If we let $N(t)$ denote the number of people infected at time $t$, then we're basically saying that

$$\frac{dN}{dt} = kN(t)$$

for some constant $k > 0$.

This differential equation is a **mathematical model** that allows us to predict the growth of $N(t)$. We can actually use it to get an expression for $N(t)$. This differential equation is saying that $N(t)$ is a function whose derivative looks quite a bit like $N(t)$. In fact, the solution of this differential equation is $N(t) = Ce^{kt}$, where $C = N(0)$ is a constant that represents the number of people infected at time $t = 0$. This is an example of **exponential growth**, and the proportionality constant $k$ is called the **relative growth rate**.

**Example 4.4.1 Modeling an outbreak.**

A game of Humans vs. Zombies breaks out at WVWC. When the game starts, there is just one zombie. After one hour, there are 3 zombies. How many zombies will there be after three hours assuming that the relative growth rate is constant?

**Solution**. Let $N(t)$ denote the number of zombies $t$ hours after the game starts. Then we're given that $N(0) = 1$ and $N(1) = 3$. We need to find $N(3)$.

Since the relative growth rate is assumed to be constant, we can say that $N' = kN$ for some constant $k$. As we saw before, the solution of this differential equation is given by $N(t) = N(0)e^{kt} = e^{kt}$. So if we want to find $N(3)$, then we need to figure out what $k$ is.

We can do this by using the fact that $N(1) = 3$. If we plug this into $N(t) = e^{kt}$, we get

$$3 = e^k \Rightarrow k = \ln 3.$$

So

$$N(t) = e^{(\ln 3)t} = 3^t.$$

Hence there will be $N(3) = 3^3 = 27$ zombies after three hours.

### Exponential decay

Closely related to the concept of exponential growth is that of **exponential decay**. A quantity $M(t)$ undergoes exponential decay if it satisfies the differential equation

$$\frac{dM}{dt} = kM,$$

where $k < 0$. Every solution of this differential equation looks like $M(t) = Ce^{kt} = M(0)e^{kt}$.

We often write the decay constant $k$ as positive, and then rewrite the ODE for $M(t)$ as $\frac{dM}{dt} = -kM$. The solution becomes $M(t) = M(0)e^{-kt}$. This has the effect of highlighting the negative growth rate inherent to this system.

Perhaps the most important example of exponential decay is that of radioactive decay. If $m(t)$ represents the mass of some radioactive substance, then experiments show that substance decays at a rate proportional to the amount of the substance remaining. In symbols, $\frac{dm}{dt} = km$ where $k < 0$. If we let $m_0 = m(0)$ denote the initial mass of the substance, then we can say that $m(t) = m_0 e^{kt}$. The rate of decay is often expressed in terms of **half-life**: the amount of time it will take for precisely half of the mass to decay.

Example 4.4.2 Decay from half-life.

A radioactive substance has a half-life of 28 days. If we start with a 50 mg sample, how much of the mass will remain after $t$ days?

**Solution**. If we let $m(t)$ denote the mass of the sample at time $t$, and let $t = 0$ denote the first day we have the sample, then

$$m(t) = m_0 e^{kt} = 50e^{kt}.$$

We still need to find $k$, but we can do this using the fact that the half-life is 28 days. This means that

$$m(28) = \frac{m(0)}{2} = 25.$$

So

$$25 = 50e^{28k} \Rightarrow \ln\frac{1}{2} = 28k \Rightarrow k = -\frac{\ln 2}{28}.$$

Hence

$$m(t) = 50e^{-\frac{\ln 2}{28}t} = 50 \cdot 2^{-\frac{t}{28}}.$$

## 4.5 Inverse trigonometric functions

Recall that $\sin x$ is not a one-to-one function since it fails the horizontal line test. Hence it has no inverse function. However, we can *restrict* the domain of $\sin x$ so that what's left over passes the horizontal line test. For example, if we define $f(x) = \sin x$ but restrict the domain to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then $f(x)$ *is* one-to-one and so has an inverse function. We call this function **inverse sine** or **arcsine**.

**Definition 4.5.1 Inverse sine.**

Let $f(x) = \sin x$ and have domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Then $f^{-1}(x)$ is called the inverse sine or arcsine of $x$, and is denoted by $\sin^{-1} x$ or $\arcsin x$. The domain is the interval $[-1, 1]$ and the range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

We can think of $\sin^{-1} x = \arcsin x$ as the angle required to turn sine into $x$.

Note that $\sin^{-1} x$ does *not* mean $\frac{1}{\sin x}$, which is $\csc x$. It's an unfortunate, though standard, choice of notation.

**Example 4.5.2 Finding inverse sine.**

Find $\sin^{-1}(\frac{\sqrt{3}}{2})$.

**Solution**. This is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ that turns sine into $\frac{\sqrt{3}}{2}$. So $\sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$.

**Example 4.5.3 Simplifying cosine and inverse sine.**

Simplify $\cos(\arcsin x)$, where $-1 \leq x \leq 1$.

**Solution**. By definition, $\sin(\arcsin x) = x$ as long as $-1 \leq x \leq 1$. So we want to try to rewrite $\cos(\arcsin x)$ to make use of this cancellation property. We can do this using the Pythagorean identity $\cos\theta = \pm\sqrt{1 - \sin^2\theta}$. Hence

$$\cos(\arcsin x) = \sqrt{1 - (\sin(\arcsin x))^2} = \sqrt{1 - x^2}$$

whenever $-1 \leq x \leq 1$.

Since $\sin x$ is differentiable, this means $\arcsin x$ is also differentiable.

**Theorem 4.5.4 Derivative of inverse sine.**

*Let* $-1 < x < 1$. *Then* $\dfrac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$.

*Proof.* If we set $y = \sin^{-1} x$, then we get $\sin y = x$. Differentiating this gives $y' \cos y = 1$ or just

$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}.$$

■

We can also define the **inverse cosine function** in much the same way as the inverse sine function. The idea once again is to restrict the domain of $\cos x$ to $[0, \pi]$ to get an invertible function. We call the inverse of this function the inverse cosine, or arccosine, function. We denote this function by $\cos^{-1} x$ or $\arccos x$.

**Example 4.5.5 Inverse cosine value.**

Find $\csc(\cos^{-1}(-\frac{1}{2}))$.

**Solution**. First, note that $\cos^{-1}(-\frac{1}{2}) = \frac{4\pi}{3}$. So

$$\csc(\cos^{-1}(-\frac{1}{2})) = \csc(\frac{4\pi}{3}) = \frac{2}{\sqrt{3}}.$$

Inverse cosine behaves in much the same way as inverse sine. However, this function won't be as useful to us as the inverse sine function or the next function we will look at: the inverse tangent.

If we restrict the domain of $\tan x$ to $(-\frac{\pi}{2}, \frac{\pi}{2})$, then we get an invertible function. We call the inverse of this function the **inverse tangent** or **arctangent**, and denote it by $\tan^{-1} x$ or $\arctan x$. The domain of $\arctan x$ is $(-\infty, \infty)$ and the range is $(-\frac{\pi}{2}, \frac{\pi}{2})$. $\tan^{-1} x$ can be thought of as the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ that makes tangent equal to $x$. It satisfies the following cancellation properties: $\tan(\tan^{-1} x) = x$ for all $x$ and $\tan^{-1}(\tan y) = y$ for all $y$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

**Example 4.5.6 Simplifying an inverse tangent.**

Simplify $\sin(2 \arctan x)$ using the formula $\sin 2x = 2 \sin x \cos x$.

**Solution**. If we use the double-angle formula given, we get

$$\sin(2 \arctan x) = 2 \sin(\arctan x) \cos(\arctan x).$$

So we just need to find $\sin(\arctan x)$ and $\cos(\arctan x)$.
Set $\theta = \arctan x$, so that $\tan\theta = x$. Then we can find $\sin\theta$ and $\cos\theta$ using triangles, which gives

$$\sin\theta = \frac{x}{\sqrt{1+x^2}}$$
$$\cos\theta = \frac{1}{\sqrt{1+x^2}}$$

Hence

$$\begin{aligned}\sin(2 \arctan x) &= \sin 2\theta \\ &= 2 \sin\theta \cos\theta \\ &= \frac{2x}{1+x^2}.\end{aligned}$$

**Example 4.5.7 Limit of inverse tangent.**

Determine $\lim_{x\to\infty} \tan^{-1} x$.

**Solution**. Recall that $\tan^{-1} x$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ for which tangent is equal to $x$. So finding $\lim_{x\to\infty} \tan^{-1} x$ is equivalent to finding what angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ we have to approach in order for tangent to blow up to $\infty$. Either from looking at a graph or by using the definition of tangent, we see that the angle we need to approach is exactly $\frac{\pi}{2}$. Hence $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$.

**Theorem 4.5.8 Derivative of inverse tangent.**

*The derivative of* $\tan^{-1} x$ *is* $\frac{1}{1+x^2}$.

*Proof.* We can prove this the same way we proved that $\dfrac{d}{dx} \arcsin x = \frac{1}{\sqrt{1+x^2}}$. All we need to do is to set $y = \tan^{-1} x$, and then find $y'$ by implicit differentiation. ■

**Example 4.5.9 Tangent half-angle substitution.**

An important substitution in integral calculus is the **tangent half-angle substitution** defined by $t = \tan \frac{x}{2}$. Use this equation to find $\dfrac{dx}{dt}$.

**Solution**. We can find $x'$ implicitly, but we can also solve for $x$ to get

$$x = 2\tan^{-1} t.$$

Therefore

$$\frac{dx}{dt} = 2\frac{d}{dt}\tan^{-1} t = \frac{2}{1+t^2}.$$

# Chapter 5

# Applications of Differentiation

If a function is differentiable, then its derivative gives us information about how that function is changing. This information is important in many different applications of calculus.

## 5.1 Linear approximations

Suppose we graph the tangent line to $y = \sqrt{x}$ at $x = 4$. The equation of this line will be

$$y - 2 = \frac{x-4}{4} \quad \text{or} \quad y = 2 + \frac{x-4}{4}.$$

If we look at the tangent line next to the graph, we'll see that the line is very close to the graph of $y = \sqrt{x}$ if $x$ is very close to 4. So we can use the equation of the tangent line to approximate $y = \sqrt{x}$ when $x$ is near 4. For example, we can say that

$$\sqrt{4.2} \approx 2 + \frac{4.2-4}{4} = 2.05.$$

We call this the **linear approximation** of $\sqrt{x}$ at 4.

**Definition 5.1.1 Linear approximation.**

Let $f(x)$ be a function that is differentiable at $x = a$. The linear approximation (or **linearization**) of $f$ at $a$ is the function $L(x)$ given by

$$L(x) = f(a) + f'(a)(x - a).$$

The formula of the linear approximation to a function $f$ at $a$ is nothing more than the equation of the tangent line to $f$ at $a$.

**Example 5.1.2 Linear approximation of sine.**

Find the linear approximation of $f(\theta) = \sin\theta$ at $\theta = 0$ and use this to estimate $\sin(-.02)$.

**Solution**. The linear approximation is given by

$$L(\theta) = f(0) + f'(0)(\theta - 0) = \sin 0 + \cos 0(\theta - 0),$$

so $L(\theta) = \theta$. Therefore

$$\sin(-.02) \approx -.02$$

since $-.02$ is very close to 0.

**Example 5.1.3 Estimating a tangent.**

Estimate $\tan(\pi + \frac{3}{49})$.

**Solution**. We want to estimate this using a linear approximation, and we want to make sure the linear approximation is easy to set up. This means we want to base the linear approximation at a value of $a$ that both tangent and its derivative are relatively easy to compute at, and is also close to $\pi + \frac{3}{49}$. So we'll pick $a = \pi$ and find the linear approximation to $\tan x$ at $a = \pi$: Set $f(x) = \tan x$. Then

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= \tan \pi + \sec^2(\pi)(x - \pi) \\ &= x - \pi. \end{aligned}$$

So

$$\tan(\pi + \frac{3}{49}) \approx (\pi + \frac{3}{49}) - \pi = \frac{3}{49}.$$

**Example 5.1.4 Estimating the population of the United States in 2020.**

Use the differential equation from Example 3.5.4 and the fact that the current population of the US is about 323.1 million to estimate the population of the US in 2020.

**Solution**. Recall that $P(t)$ in Example 3.5.4 represented the population of the US (in millions) $t$ years after 1990. To estimate the population in 2020, we want to find the linearization of $P$ at $t = 27$. This is given by

$$L(t) = P(27) + P'(27)(t - 27).$$

Since $P(27) = 323.1$ and

$$P'(27) = \frac{1}{23500} P(532 - P) = \frac{1}{23500}(323.1)(532 - 323.1) = 2.9,$$

we have

$$L(t) = 323.1 + 2.9(t - 27).$$

Hence the population of the US in 2020 should be

$$L(30) = 323.1 + 2.9 \cdot 3 = 331.8,$$

or about 331.8 million people.

## 5.2 Related rates

**Example 5.2.1 Changing volume on a sphere.**

The radius of a sphere is increasing at a rate of $4\,\frac{\mathrm{mm}}{\mathrm{s}}$. How fast is the volume increasing when the radius is 13 mm?

**Solution**. First, let's assign names to all of the changing quantities in this problem:

$$\begin{aligned} V &= \text{volume} \\ r &= \text{radius}. \end{aligned}$$

Note that we're considering $V$ and $r$ to be functions of time, i.e. $V = V(t)$ and $r = r(t)$, where $t$ is in seconds. Then we're given that $r' = 4$ and we need to find $V'$ when $r = 13$. To do this, we had better find some relationship between $V$ and $r$. We can find this by looking at the volume formula for a sphere:

$$V = \frac{4}{3}\pi r^3.$$

This equation relates $r$ and $V$, and if we take the derivatives of both sides with respect to time $t$ (using implicit differentiation) then we get an equation relating $V'$ and $r'$. So let's do that:

$$V' = 4\pi r^2 r'$$

Now we can plug in our given information to get

$$V' = 4\pi(13^2)(4) = 2704\pi.$$

So the volume is increasing at a rate of $2704\pi\,\frac{\mathrm{mm}^3}{\mathrm{s}}$.

Note that we *never* found what $V(t)$ and $r(t)$ were in Example 5.2.1, but we didn't need to. All we needed to find was a relationship between the two changing quantities, i.e. the two derivatives, to answer the question.

**Example 5.2.2 A tank problem.**

Water is leaking out of a conical tank at a rate of $10000\,\frac{\mathrm{cm}^3}{\mathrm{min}}$ while water is being pumped into the tank at some (unknown) constant rate. The tank has a height of 6 m and the diameter of the top is 4 m. If the water level is rising at a rate of $20\,\frac{\mathrm{cm}}{\mathrm{min}}$ when the height of the water is 2 m, what is the rate at which water is being poured into the tank?

**Solution**. We have a *lot* of information to process here, so we'll take things a step at a time. The changing quantities are

$$\begin{aligned} h(t) &= \text{height of water} \\ r(t) &= \text{radius of water} \\ V(t) &= \text{volume of water} \end{aligned}$$

where $t$ is in minutes. We need to find the rate at which water is entering the tank, so let's call this mystery number $C$. Then all we know about $C$ is that it's related to the rate that the volume is changing. In particular,

we should have $V' = C - 10000$. So to find $C$ we need to find $V'$, which means we need to set up a relationship between $V$ and the other changing quantities to determine how exactly $V$ is changing, using the fact that $h' = 20$ when $h = 200$ (after converting to centimeters).
Using the fact that the water is in a conical tank, we can use the formula for the volume of a cone to say that $V = \frac{1}{3}\pi r^2 h$. If we differentiate both sides with respect to $t$, then we get

$$V' = \frac{1}{3}\pi(2rr'h + r^2h').$$

Unfortunately, we don't have any information on $r$ or $r'$ that we can use, just information on $h$ and $h'$. So we need to get $r$ in terms of $h$. By similar triangles, we can say that $\frac{r}{h} = \frac{200}{600} = \frac{1}{3}$, and so $r = \frac{h}{3}$ and likewise $r' = \frac{h'}{3}$. Hence

$$V' = \frac{1}{3}\pi(2rr'h + r^2h') = \frac{1}{3}\pi\left(2\frac{h^2h'}{9} + \frac{h^2h'}{9}\right)$$

which boils down to

$$V' = \frac{h^2h'}{9}\pi = \frac{800000\pi}{9}.$$

Now, *finally*, we can answer the original question. The rate that water is flowing into the tank is

$$C = V' + 10000 = \frac{800000\pi}{9} + 10000,$$

or just $289252.68\ \frac{\text{cm}^3}{\text{min}}$. In terms of meters this is $.29\ \frac{\text{m}^3}{\text{min}}$, which perhaps looks a bit more reasonable.

## 5.3 L'Hospital's Rule

Consider the important limit $\lim_{x\to 0}\frac{\sin x}{x}$. If we try to plug in $x = 0$, we get $\frac{0}{0}$, which is undefined. However, we can prove using geometry that the limit is 1. As another example, consider $\lim_{x\to\infty}\frac{1-x^3}{4x^3}$. Once again, if we try to plug the limit in we get an expression of the form $\frac{-\infty}{\infty}$. However, the limit is just $-\frac{1}{4}$, which we can find using algebra.

Limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ are known as **indeterminate forms**. There is no restriction whatsoever on what value a limit involving an indeterminate form may take, or even that it has to exist at all.

**Example 5.3.1 Different indeterminate forms.**

Find $\frac{\infty}{\infty}$ indeterminate forms that, respectively, evaluate to $1, 0$ and do not exist.

**Solution**. Let $f(x) = \frac{1-x}{3-x}$, $g(x) = \frac{1+x^4}{1+x^3}$ and $h(x) = \frac{\ln x}{\cot x}$. Then $\lim_{x\to\infty} f(x), \lim_{x\to\infty} g(x)$ and $\lim_{x\to 0^+} h(x)$ are all $\frac{\infty}{\infty}$ indeterminate forms. The first evaluates to 1, the second does not exist (it's $\infty$) while the third appears to be equal to 0.

The goal of this section is to determine a method that can help us evaluate limits involving indeterminate forms. This method is called **L'Hospital's Rule**.

**Theorem 5.3.2 L'Hospital's Rule.**

*Let $f(x)$ and $g(x)$ be differentiable functions. If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is either one of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'x)}{g'(x)}$$

*if the limit on the right exists or is $\pm\infty$.*

It's important to note that L'Hospital's Rule does not necessarily ask you to use the quotient rule!

**Example 5.3.3 Using L'Hospital's Rule.**

Let $h(x) = \frac{\ln x}{\cot x}$. Find $\lim_{x\to 0^+} h(x)$.

**Solution**. We saw in Example 5.3.1 that this limit gives us the indeterminate form $\frac{\infty}{\infty}$, so L'Hospital's Rule applies. Hence

$$\begin{aligned}\lim_{x\to 0^+} \frac{\ln x}{\cot x} &= \lim_{x\to 0^+} \frac{1/x}{-\csc^2 x}\\ &= \lim_{x\to 0^+} \left(-\frac{\sin x}{x}\sin x\right)\\ &= 1\cdot 0\end{aligned}$$

and so the limit is indeed 0.

L'Hospital's Rule also applies for limits approaching $\pm\infty$.

**Example 5.3.4 Exponential and polynomial growth.**

Let $f(x) = ax^2 + bx + c$ and let $g(x) = e^x$, where $a, b, c$ are arbitrary constants. Find $\lim_{x\to\infty} \frac{ax^2+bx+c}{e^x}$.

**Solution**. This is another example of a $\frac{\infty}{\infty}$ indeterminate form, and so L'Hospital's Rule applies. If we use it, we get

$$\begin{aligned}\lim_{x\to\infty} \frac{ax^2+bx+c}{e^x} &= \lim_{x\to\infty} \frac{2ax+b}{e^x}\\ &= \lim_{x\to\infty} \frac{2a}{e^x}\\ &= 0.\end{aligned}$$

In other words, the exponential function grows faster than *any* quadratic function.

**Example 5.3.5 Another limit involving exponentials.**

Find $\lim_{t\to 1^+} \frac{9^{t-1}-4^{t-1}}{t-1}$.

**Solution**. This is a $\frac{0}{0}$ indeterminate form, so L'Hospital's Rule applies.

We get

$$\lim_{t\to 1^+} \frac{9^{t-1} - 4^{t-1}}{t-1} = \lim_{t\to 1^+} \frac{9^{t-1}\ln 9 - 4^{t-1}\ln 4}{1}$$
$$= \ln \frac{9}{4}.$$

L'Hospital's Rule only applies *directly* to the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$, but these are not the only problems when L'Hospital's Rule proves useful.

**Example 5.3.6 A different indeterminate form.**

Find $\lim_{x\to\infty} xe^{-3x-x^2}$.

**Solution**. If we try to evaluate the limit, we get the expression $\infty \cdot 0$. This is another indeterminate form, but is not one that we can apply L'Hospital's Rule to without doing some algebra first. We can write

$$\lim_{x\to\infty} xe^{-3x-x^2} = \lim_{x\to\infty} \frac{x}{e^{3x+x^2}}$$

which is a $\frac{\infty}{\infty}$ indeterminate form. We *can* use L'Hospital's on this expression!
If we do so, we get

$$\lim_{x\to\infty} \frac{x}{e^{3x+x^2}} = \lim_{x\to\infty} \frac{1}{(3+2x)e^{3x+x^2}}$$
$$= 0.$$

So $\lim_{x\to\infty} xe^{-3x-x^2} = 0$.

Example 5.3.6 shows that $0 \cdot \infty$ indeterminate forms can be dealt with by rewriting them in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then applying L'Hospital's Rule. It's trickier, but we can also deal with $\infty - \infty$ indeterminate forms by rewriting them in this way.

**Example 5.3.7 $\infty - \infty$ indeterminate form.**

Find $\lim_{x\to \frac{\pi}{2}^{-1}} (\sec x - \tan x)$.

**Solution**. As $x \to \frac{\pi}{2}^-$, $\sec x \to \infty$ and $\tan x \to \infty$. So this limit is the indeterminate form $\infty - \infty$. We can't sue L'Hospital's Rule yet, but we can try to rewrite this limit as a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ indeterminate form. We'll try to do this by replacing $\sec x, \tan x$ with $\sin x, \cos x$ to get

$$\lim_{x\to\frac{\pi}{2}^-} (\sec x - \tan x) = \lim_{x\to\frac{\pi}{2}^-} \frac{1-\sin x}{\cos x}$$
$$= \lim_{x\to\frac{\pi}{2}^-} \frac{-\cos x}{-\sin x}$$
$$= 0.$$

**Example 5.3.8 Limit involving radicals.**

Find $\lim_{x\to\infty}(\sqrt{2x^2-x}-\sqrt{2x^2+1})$.

**Solution**. This is another $\infty-\infty$ form, so we'll try to rewrite it into a form we can use L'Hospital's Rule on. The trick here is to factor an $x$ out so we can get a $0\cdot\infty$ form, which we've already seen how to handle:

$$\begin{aligned}\lim_{x\to\infty}(\sqrt{2x^2-x}-\sqrt{2x^2+1}) &= \lim_{x\to\infty}(x\sqrt{2-x^{-1}}-x\sqrt{2+x^{-2}})\\ &= \lim_{x\to\infty}\frac{\sqrt{2-x^{-1}}-\sqrt{2+x^{-2}}}{\frac{1}{x}}\\ &= \lim_{x\to\infty}\frac{\frac{1}{2}(2-x^{-1})^{-1/2}(x^{-2})-\frac{1}{2}(2+x^{-2})^{-1/2}(-2x^{-3})}{-x^{-2}}\\ &= \lim_{x\to\infty}\frac{\frac{1}{2}(2-x^{-1})^{-1/2}-\frac{1}{2}(2+x^{-2})^{-1/2}(-2x^{-1})}{-1}\\ &= -\frac{1}{2\sqrt{2}}.\end{aligned}$$

There are three other indeterminate forms that L'Hospital's Rule can help us with (after some algebra): $1^\infty, \infty^0$ and $0^0$. All of these can be found by first using logarithms.

**Example 5.3.9 A natural limit.**

Find $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

**Solution**. This limit is a $1^\infty$ indeterminate form. We'll try taking logarithms to rewrite it as a $0\cdot\infty$ form, so set $y=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$. Then

$$\ln y = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)$$

is a $0\cdot\infty$ form. We can use L'Hospital's by rewriting this limit at a $\frac{0}{0}$ form or $\frac{\infty}{\infty}$. Either way, we get $\ln y = 1$ and so the original limit is $y=e$.

## 5.4 Extreme values of functions

It's often of interest to determine how large or small some quantity can get.

**Definition 5.4.1 Absolute extrema.**

Let $f(x)$ be a function defined on some domain $D$. Let $c$ be in $D$. Then

- $f(c)$ is an **absolute maximum** of $f$ on $D$ if $f(c)\geq f(x)$ for all $x$ in $D$.
- $f(c)$ is an **absolute minimum** of $f$ on $D$ if $f(c)\leq f(x)$ for all $x$ in $D$.

These values, if they exist, are **extreme values**.

**Example 5.4.2**

List the extreme values, if any, of the following functions:

1. $f(x) = e^{-x}$.
2. $g(x) = x^2$.
3. $h(x) = \sin x$.

Some functions may not have extreme values, but they could have values that are smaller or larger than all other values of the function "nearby".

**Definition 5.4.3 Local extrema.**

Let $f(x)$ be a function and let $c$ be in the domain of $f$. Then

- $f(c)$ is a **local maximum** if $f(c) \geq f(x)$ for all $x$ near $c$.
- $f(c)$ is a **local minimum** if $f(c) \leq f(x)$ for all $x$ near $c$.

Roughly, local maxima correspond to "hilltops" whereas local minima correspond to "valleys" in the graph of $f(x)$.

In general, local extrema and absolute extrema can be different. However, the following theorem does provide a relationship between the two on *closed, bounded* intervals.

**Theorem 5.4.4 Extreme Value Theorem.**

*Let $f(x)$ be a continuous function defined on the interval $[a, b]$ Then $f(x)$ has both an absolute maximum and absolute minimum on $[a, b]$. Furthermore, these values occur at either an endpoint or a local extrema.*

What Theorem 5.4.4 tells us is that we can find the extreme values of any continuous function defined on a closed, bounded interval by just checking the function at the endpoint and at its local extrema.

**Example 5.4.5 Finding extreme values.**

Let $f(x) = (x^2 - 1)^2$. Given that $f$ has local extrema at $x = -1, 0, 1$, find the extreme values of $f$ on the interval $[\frac{1}{2}, 6]$.

**Solution**. Theorem 5.4.4 tells us that we can find the extreme values by checking local extrema and the endpoints of our interval. Since $x = 1$ is the only local extreme value inside of our interval, that's the only one we need to check. We have

$$f(\frac{1}{2}) = \frac{9}{16}, f(1) = 0 \quad \text{and} \quad f(6) = 35^2.$$

So the absolute minimum of $f$ on $[\frac{1}{2}, 6]$ is 0 and the absolute maximum is $35^2$.

So if we can find where a function has local extrema, then finding absolute extrema won't be too much more difficult. Thankfully, this is relatively straightforward if the function is differentiable.

**Theorem 5.4.6 Fermat's Theorem.**

*Let $f(x)$ be a function and let $c$ be a local extreme value of $f$. If $f'c)$ exists, then $f'c) = 0$.*

So finding local extrema of $f(x)$ often amounts to finding where $f'x) = 0$, i.e., where it has a root. However, we also need to worry about where $f'x)$ doesn't exist (just think about $y = |x|$). This leads to the following definition.

**Definition 5.4.7 Critical points.**

Let $f(x)$ be a differentiable function. The **critical points** of $f(x)$ are the points where $f'x) = 0$ or $f'x)$ does not exist.

Our method for finding extreme values on a closed interval will proceed as follows: find all the critical points, and then compare the values of our function at the critical points and the endpoints of the interval.

**Example 5.4.8 Extreme values of tangent.**

Let $f(\theta) = \frac{4}{3}\theta - \tan\theta$. Find the extreme values of $f$ on the interval $[-\frac{\pi}{3}, \frac{\pi}{3}]$.

**Solution.** We need to find the critical points first. We have

$$f'\theta) = \frac{4}{3} - \sec^2\theta.$$

This is 0 if $\cos\theta = \frac{\sqrt{3}}{2}$, which occurs in our interval only if $\theta = \pm\frac{\pi}{6}$. So we need to check $f$ at $\pm\frac{\pi}{3}, \pm\frac{\pi}{6}$.

**Example 5.4.9 Dealing with a cusp.**

Let $g(x) = x^{1/3} - x^{4/3}$. Find the absolute extrema of $g$ on $[-1, 1]$.

**Solution.** First, find the critical points of $g$:

$$g'(x) = \frac{1}{3}x^{-\frac{2}{3}} - \frac{4}{3}x^{\frac{1}{3}} = \frac{1}{3}x^{-\frac{2}{3}}(1 - 4x).$$

This is 0 at $x = \frac{1}{4}$ and undefined at 0, so our critical points are $x = 0, \frac{1}{4}$. So to find the absolute extrema we need to check $g$ at $x = -1, 0, \frac{1}{4}, 1$. Doing so, we see that the absolute minimum value is $-2$ at $x = -1$ and the absolute maximum is $(\frac{1}{4})^{1/3} - (\frac{1}{4})^{4/3}$ at $x = \frac{1}{4}$.

## 5.5 Optimization problems

Now that we can use derivatives to find local maxima and minima, we can solve **optimization problems**.

**Example 5.5.1 Minimizing a product.**

Find two numbers that differ by 100 and whose product is a minimum.

**Solution**. Call our numbers $x, y$ and suppose $x > y$. Then $x - y = 100$ and we need to choose $x, y$ in such a way that $xy$ is as small as possible. If we replace $x$ with $100 + y$, then this means we need to minimize

$$f(y) = 100y + y^2.$$

The critical point of $f(y)$ is $y = -50$, and since $f$ is a parabola we know that this must be the location of the absolute minimum value on the graph. So the numbers we need are $50, -50$.

**Example 5.5.2 Minimizing material costs.**

A box with an open top and a square base is to be built using material that costs \$5 per square meter. The box must have a volume of $32000\,\mathrm{cm}^3$. Find the dimensions of the box that will be cheapest to build to these specifications.

**Solution**. There are two things we need to do here:

1. Find the quantity to be optimize.

2. Solve the resulting optimization problem.

We need to minimize costs, which means we need to minimize how much material is used to construct this box. Hence we must minimize the surface area of the box. If we let $x$ denote the length of the base of the box and $h$ the height of the box, then the surface area is given by

$$x^2 + 4xh.$$

Now, we can't minimize this yet because we have too many variables. However, we can solve for $h$ in terms of $x$ by using the volume constraint. Since the volume of the box is $x^2 h = 32000$, this gives $h = \frac{32000}{x^2}$. So the quantity we need to minimize is

$$f(x) = x^2 + 4xh = x^2 + \frac{128000}{x}.$$

To minimize this, we find the critical points by solving $f'(x) = 0$. This gives, after some algebra, $x = 40$. Now we need to be careful here! We only have a critical point at this step. We need to make an argument now as to why this critical point should be an absolute minimum of $f(x)$. One way to do this is by setting up a sign chart for $f'(x)$. If we do this, then we see that $f' < 0$ for $x$ in $(0, 40)$, while $f' > 0$ for $x$ in $(40, \infty)$. If we think about what this tells us of the graph of $f(x)$, we see that $x = 40$ must minimize $f(x)$, at least for $x > 0$. So the dimensions that minimize the cost of building this box are $40 \times 40 \times 20$.

**Example 5.5.3 Distance on an ellipse.**

Find the point(s) on the ellipse $4x^2 + y^2 = 4$ that are farthest from the point $(1, 0)$.

**Solution**. This is an optimization problem since we're trying to maximize distance. There are two things we need to do:

1. Find the function we need to optimize.
2. Find the extrema.

The function we need to optimize is just the distance function. In particular, if $(x, y)$ is a point on the ellipse then we need to maximize

$$d = \sqrt{(x-1)^2 + (y-0)^2}.$$

Now, we have a constraint that $(x, y)$ must satisfy; namely, this must lie on the ellipse $4x^2 + y^2 = 4$. This means that $y^2 = 4 - 4x^2$, and if we plug this into our distance function we get

$$d = \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{(x-1)^2 + 4 - 4x^2}.$$

Now here's a trick we can use: maximizing $d$ is the same thing as maximizing $d^2$, but $d^2$ is *much* nicer to work with algebraically. So instead of maximizing $d$, we'll maximize the function

$$f(x) = d^2 = (x-1)^2 + 4 - 4x^2.$$

We'll start by finding it's local extrema, which means we need to find the critical points. These are the solutions of $f'(x) = 0$. Since

$$f'(x) = 2x - 2 - 8x = -6x - 2,$$

we see that the only critical point is $x = -\frac{1}{3}$. Since $f''(x) = -6 < 0$, this means that $x = -\frac{1}{3}$ gives us a local maximum of $f(x)$ by the second derivative test. In fact, we can go further: this must be an absolute maximum of $f(x)$, since $f(x)$ is always concave down (it's actually a parabola).
So the point on the ellipse $4x^2 + y^2 = 4$ farthest from $(1, 0)$ has $x$-coordinate equal to $-\frac{1}{3}$. This means that the corresponding $y$-coordinate is $y = \pm\frac{2\sqrt{5}}{3}$, so the points on the ellipse that are the farthest from $(1, 0)$ are the points $(-\frac{1}{3}, -\frac{2\sqrt{5}}{3}), (-\frac{1}{3}, \frac{2\sqrt{5}}{3})$.

## 5.6 The Mean Value Theorem

The linear approximations we came up with in Section 5.1 are useful for estimating complicated functions with simpler, linear models. In essence, we use the derivative of a function to tell us how much to change a given value of the function in order to estimate that function. There is one problem with this approach, at least currently. We have good reason to suspect that our approximations are close to the exact values in certain circumstances, but we don't know how close. The goal of this section is to derive an estimate for the derivative that can help us to find more precise approximations.

### Rolle's Theorem

We start with a theorem that is very reasonable, at least geometrically. Let $f(x)$ be a differentiable function on some interval $[a, b]$, and suppose $f(a) = f(b)$. Then this means the graph of $f$ must "turn around" at some point in

$[a, b]$, i.e., there is a local maximum or minimum contained within the interval. Combining this observation with Theorem 5.4.6 gives us **Rolle's Theorem.**

**Theorem 5.6.1 Rolle's Theorem.**

*Let $f(x)$ be a differentiable function on $[a, b]$ and suppose $f(a) = f(b)$. Then there exists some number $c$ in $(a, b)$ such that $f'(c) = 0$.*

Theorem 5.6.1 is an example of an **existence theorem**. It tells us nothing about how to find the number $c$, or how many such $c$ can exist. It only tells us that there is at least one number $c$ in $(a, b)$ for which $f'(c) = 0$. This may not seem very useful, but existence theorems can be quite powerful in mathematics.

**Example 5.6.2 Rolle's Theorem and traffic.**

A car enters a highway going at 45 $\frac{\text{mi}}{\text{h}}$ and leaves the highway going at the same speed. Was the car's acceleration ever 0?

**Solution**. The speed of the car can be represented by a velocity function $v(t)$. If we assume that the car entered the highway at time $t_{\text{in}}$ and left at some future time $t_{\text{out}}$, then we know that

$$v(t_{\text{in}}) = v(t_{\text{out}}) = 45.$$

By Theorem 5.6.1, there must be some time $t_1$ between $t_{\text{in}}$ and $t_{\text{out}}$ for which $v'(t_1) = 0$. Since $v'$ is exactly the acceleration, we know that the car had to stop accelerating somewhere on the highway.

## The Mean Value Theorem

Rolle's Theorem is powerful because it has very general conditions for its use. However, it does require the function in question to take on the same values at the endpoints of $[a, b]$, and this is a condition we'd like to try to relax. However, we'll try to be clever about this and use Rolle's Theorem to do most of the heavy lifting for us.

If we imagine graphing some differentiable function $f(x)$ on some interval $[a, b]$, but $f(a) \neq f(b)$, then we can't apply Rolle's Theorem. But maybe we can adjust it just a little bit so that we can? In particular, the only reason we can't use Theorem 5.6.1 is that $f(a) \neq f(b)$. But if we subtract the line through these points from $f(x)$, we should get a new function for which Theorem 5.6.1 applies.

The line through these points has equation

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

So define

$$g(x) = f(x) - y = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then

$$g(a) = g(b) = 0,$$

So Theorem 5.6.1 *does* apply to this function. Hence there exists some number $c$ between $a$ and $b$ for which $g'(c) = 0$.

But

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a},$$

and if this is 0 then we can solve $g'(c) = 0$ for $f'(c)$ to get

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This gives us the **Mean Value Theorem**.

**Theorem 5.6.3 Mean Value Theorem.**

*Let $f(x)$ be differentiable on some interval $[a, b]$. Then there exists a number $c$ in $(a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

*or equivalently*

$$f(b) - f(a) = f'(c)(b - a).$$

The Mean Value Theorem essentially says that there is at least one point inside of $(a, b)$ for which the slope $f'(c)$ at that point matches the "average slope" $\frac{f(b)-f(a)}{b-a}$. Like Rolle's Theorem, this is an existence theorem. However, it's slightly more general, and so is applicable in more situations. It's also useful in deriving error estimates.

**Example 5.6.4 Estimating error in a linear approximation.**

Let $f(x) = \sqrt{x}$ and let $L(x)$ denote the linear approximation to $f(x)$ at $a = 9$. Find the largest possible error between $f(x)$ and $L(x)$ on the interval $[9, 16]$.

**Solution**. First, we should find $L(x)$, which from Section 5.1 is just

$$L(x) = 3 + \frac{1}{6}(x - 9).$$

The error we need to estimate is $|f(x) - L(x)|$, for $x$ in $[9, 16]$. So we'll set $g(x) = f(x) - L(x)$ and let $x$ be some number in $[9, 16]$. By Theorem 5.6.3, there exists some number $c$ in $(9, x)$ such that

$$g(x) - g(9) = g'(c)(x - 9).$$

Now, $g(9) = 0$ and $g'(c) = \frac{1}{2\sqrt{c}} - \frac{1}{6}$. So the above equation becomes

$$f(x) - L(x) = \left(\frac{1}{2\sqrt{c}} - \frac{1}{6}\right)(x - 9).$$

Now, we don't know anything about $c$ aside from the fact that it lives in $(9, 16)$. However, we do have an expression for $|f(x) - L(x)|$ now:

$$\begin{aligned} |f(x) - L(x)| &= \left|\left(\frac{1}{2\sqrt{c}} - \frac{1}{6}\right)(x - 9)\right| \\ &\leq \left(\frac{1}{6} - \frac{1}{8}\right)7. \end{aligned}$$

So if $x$ is in $[9, 16]$, then the error $|f(x) - L(x)|$ is *at most* $\frac{7}{24}$. In other words,

$$L(x) - \frac{7}{24} \leq f(x) \leq L(x) + \frac{7}{24}.$$

Perhaps the most important example of Theorem 5.6.3 in this course will be the following theorem.

**Theorem 5.6.5 Zero Derivative Theorem.**

*Let $f(x)$ be differentiable on $(a,b)$. If $f'(x) = 0$ for all $x$ in $(a,b)$, then $f(x)$ has to be a constant.*

*Proof.* The idea of the proof is to use Theorem 5.6.3 to show that $f(x_2) - f(x_1) = 0$ for any points $x_1, x_2$ in $(a,b)$. Now, by Theorem 5.6.3 we have

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some $c$ in $(a,b)$. However, $f'(c) = 0$ by assumption, and so $f(x_2) - f(x_1) = 0$. In other words, $f(x_2) = f(x_1)$. Hence $f$ must be constant. ■

## 5.7 Derivatives and graphs

Recall that if $f(x)$ is a function and $f'(x)$ exists, then $f'(c) > 0$ for some $c$ means that $f$ is increasing at $c$, while $f'(c) < 0$ means that $f$ is decreasing at $c$. Now, suppose that $c$ is a critical point of $f(x)$. Then $f'(c) = 0$. If $f'$ changes sign from negative to positive, then $f'(c)$ is a local minimum. Conversely, if $f'$ changes sign from positive to negative, then $f'(c)$ is a local maximum. This is the **first derivative test**.

**Example 5.7.1 Local maxima and minima using the first derivative test.**

Let $f(x) = x^4 e^{-x}$. Find where $f$ is increasing, decreasing, and any local maxima or minima.

**Solution**. We can answer all of these questions by setting up a sign chart for

$$f'(x) = x^3 e^{-x}(4 - x).$$

The critical points are $x = 0, 4$, and $f' < 0$ on $(-\infty, 0) \cup (4, \infty)$ and $f' > 0$ on $(0, 4)$. So $f$ is decreasing on the first set of intervals, increasing on $(0, 4)$, $f$ has a local maximum at $x = 4$ and a local minimum at $x = 0$.

**Example 5.7.2 First derivative test and a discontinuous function.**

Find any local maxima or minima of $G(x) = \frac{x^2}{x-1}$.

**Solution**. We need to find the critical points, which means we need to find $G'(x)$:

$$G'(x) = \frac{(x-1)(2x) - x^2}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}.$$

So the critical points are $x = 0, 1, 2$. Note that $x = 1$ *cannot* be a local extreme value of $G$ since it's not in the domain of $G$. However, we still need to include it in our sign chart. If we do so, we find that $G$ has a local minimum at $x = 2$ and a local maximum at $x = 0$ by the first derivative test.

One benefit of the first derivative test is that we only need to compute first derivatives to use it. However, if a function has a second derivative, then it's often easier to use the concavity of the graph at a critical point to determine whether a critical point is a local maximum or minimum. In particular, if $f(x)$ is a function and $f''(x)$ is continuous near the critical point $c$ (so $f'(c) = 0$), then $f(c)$ is a local minimum if $f''(c) > 0$ and $f(c)$ is a local maximum if $f''(c) < 0$. This is the **second derivative test**.

**Example 5.7.3 Using the second derivative test.**

Find any local extrema of $h(x) = 5x^3 - 3x^5$.

**Solution**. First, we find the critical points:

$$h'x) = 15x^2 - 15x^4 = 0$$

forces $x = 0, \pm 1$. Now we check these critical points in $h''(x) = 30x - 60x^3$ :

$$\begin{aligned} h''(0) &= 0 \\ h''(-1) &= 30 \\ h''(1) &= -30. \end{aligned}$$

So $h$ has a local minimum at $-1$ and a local maximum at 1.

The point $x = 0$ in Example 5.7.3 is an example of an **inflection point** of $h(x)$: a place where the concavity of $h$ changes, or equivalently a point where the second derivative changes sign.

**Example 5.7.4 Finding inflection points.**

Find the inflection points of

$$f(x) = \frac{12x}{3 + x^2}.$$

**Solution**. We need to find the points where $f''$ changes sign, so our first job is to compute $f''$. We'll do so using technology:

```
# Define the function.
f(x) = 12*x / (3 + x^2)

# Use diff() to compute the second derivative.
diff(f(x), 2)
```

```
96*x^3/(x^2 + 3)^3 - 72*x/(x^2 + 3)^2
```

We need to find where this is zero, so we'll simplify and then factor:

```
diff(f(x), 2).full_simplify().factor()
```

```
24*(x + 3)*(x - 3)*x/(x^2 + 3)^3
```

Therefore

$$f''(x) = \frac{24\,(x+3)(x-3)x}{\left(x^2+3\right)^3},$$

and we can see that $f''(x)$ is zero when $x = -3, 3$ or $0$. Since $f''$ changes sign at each of these points (as can be seen from a sign chart), all of these are inflection points. This is also indicated on the graph below:

```
# Make sure the code cell containing f(x) has been run
    before running this cell!

# Plot y = f(x) from -5 to 5.
plot(f(x), (x, -5, 5))
```

The code used in Example 5.7.4 is the Sage programming language, a variant of Python that is designed for mathematical applications. The Sage computer algebra system (CAS) is very useful for quickly performing tedious algebraic calculations. We could also have used Sage to solve the equation $f''(x) = 0$ for us with the `solve()` command, like so:

```
# Make sure to define f(x) first!

# Solve f''(x) = 0.
solve(f(x).diff(2) == 0, x)
```

```
[x == -3, x == 3, x == 0]
```

However, it may be the case that some solutions of $f''(x) = 0$ are *not* inflection points. As a simple example, consider $f(x) = x^4$. From the graph of this function it's clearly concave up everywhere even though the $f''(0) = 0$. The problem here is that $f''(x) = 12x^2$ doesn't change sign at $x = 0$, and so this point can't be an inflection point.

## 5.8 Newton's Method

Consider a differentiable function $f(x)$, and suppose we want to find a root of $f(x)$, which is a number $x_0$ such that $f(x_0) = 0$. In some cases this is very easy (like $f(x) = x^2 + 2x + 1$), but in others this may be more complicated (such as $f(x) = x^5 - x - 1$). So we'd like to find a way to *approximate* the root $x_0$.

**Newton's Method** begins as follows: pick some starting point, or guess, $x_1$. Then draw the tangent line $L$ to the graph of $f(x)$ at $x_1$. Now, the $x$-intercept of this line may not be a root, but hopefully it'll be closer to the root. So call this point $x_2$. We can solve for $x_2$ (if $f'(x_1) \neq 0$) to get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Now, at this point $x_2$ is (hopefully!) a better approximation to the root $x_0$ than $x_1$, and we can run through the same procedure to get a new point $x_3$.

In general, we can get a **sequence** of approximations $x_1, x_2, \ldots, x_n$ to the root $x_0$ by using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

as long as $f'(x_n) \neq 0$.

**Example 5.8.1 Approximating a root.**

Find the third approximation given by Newton's Method for the root of $f(x) = x^5 - x - 1$, using $x_1 = -1$.

**Solution**. First, note that $f'(x) = 5x^4 - 1$. Then

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= -1 - \frac{-1}{4} \\ &= 3, \end{aligned}$$

so $x_2 = 3$. Now we run through this method again to get $x_3$:

$$\begin{aligned} x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\ &= 3 - \frac{239}{404} \\ &= \frac{973}{404}. \end{aligned}$$

So $x_3 = \frac{973}{404}$.

**Example 5.8.2 Approximating $\pi$.**

Approximate $\pi$ using Newton's method.

**Solution**. We need to find a function $f(x)$ for which $f(\pi) = 0$. Perhaps the easiest is $f(x) = \sin x$. Now, to use Newton's method we also need a starting guess. We'll pick $x_1 = 3$ since this is close to $\pi$. Then

$$x_2 = x_1 - \frac{\sin x_1}{\cos x_1} = 3 - \tan 3$$

so $x_2 \approx 3.142546543$. Similarly,

$$\begin{aligned} x_3 &= x_2 - \tan x_2 \approx 3.141592653 \\ x_4 &\approx 3.141592654 \end{aligned}$$

Since these approximations are so close, we estimate that $\pi \approx 3.14159265$.

**Example 5.8.3 A Babylonian problem.**

Use Newton's method to find an algorithm for computing $\sqrt{a}$.

**Solution**. To use Newton's method, we need to come up with a function $f(x)$ whose root is $\sqrt{a}$. A simple choice for this is $f(x) = x^2 - a$, since $f(\sqrt{a}) = 0$. Now, if we're getting some sequence $x_1, \ldots, x_n$ from Newton's method then the next term in the sequence is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n}.$$

After some algebra, we can rewrite this as

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

We can actually test this algorithm out. Say we want to approximate $\sqrt{5}$. A reasonable guess would be $x_1 = 2$, since this should be close to $\sqrt{5}$. Then we get

$$\begin{aligned}
x_2 &= \frac{1}{2}\left(2 + \frac{5}{2}\right) = 2.25 \\
x_3 &= \frac{1}{2}\left(2.25 + \frac{5}{2.25}\right) \approx 2.236111111 \\
x_4 &\approx 2.236067978 \\
x_5 &\approx 2.236067977
\end{aligned}$$

So it looks like $\sqrt{5} \approx 2.23606797$, and indeed this is quite close to the actual value of $\sqrt{5} = 2.236067977\ldots$.

We can go one step further for this algorithm. If $x_1, x_2, \ldots, x_n, \ldots$ is the sequence we obtain using Newton's method for approximating $\sqrt{a}$, then as we saw earlier

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Now, assume that $\lim_{n\to\infty} x_n$ exists and is nonzero, and call it $s$. Then

$$\lim_{n\to\infty} x_{n+1} = \frac{1}{2}\left(\lim_{n\to\infty} x_n + \frac{a}{\lim_{n\to\infty} x_n}\right)$$

becomes

$$s = \frac{1}{2}\left(s + \frac{a}{s}\right)$$

which we can rearrange to get

$$s^2 = \frac{1}{2}(s^2 + a),$$

or just $s^2 = a$. Hence $s = \pm\sqrt{a}$.

### Newton's Method for optimization

To be written.

## 5.9 Antiderivatives

Suppose we're tracking a moving object, and through experimentation we know its acceleration to be equal to $3\ \frac{\mathrm{m}}{\mathrm{s}^2}$ to the right. Say we want to find the velocity of the mass at time $t$. Then if we set $v(t)$ equal to the velocity and $a(t)$ equal to the acceleration, we know that

$$v'(t) = a(t) = 3.$$

Therefore $v(t)$ is a function whose derivative is 3. So one possible choice for $v(t)$ is $v(t) = 3t$. This leads us to the definition of an **antiderivative**.

**Definition 5.9.1 Antiderivatives.**

A function $F(x)$ is called an antiderivative of the function $f(x)$ if $F'x) = f(x)$ for all $x$ in an interval $I$.

**Example 5.9.2 Multiple antiderivatives.**

Find three different antiderivatives for $f(x) = \cos x$.

**Solution**. An antiderivative of $\cos x$ is any function whose derivative is $\cos x$. So three antiderivatives could be

$$F_1(x) = \sin x, F_2(x) = \sin x + 4, F_3(x) = \sin x - 100.$$

From Example 5.9.2 we see that any function of the form $\sin x + C$ is an antiderivative of $\cos x$. In fact, this describes all possible antiderivatives of $\cos x$. This suggests the following theorem.

**Theorem 5.9.3 General antiderivatives.**

*Let $f(x)$ be some function with antiderivative $F(x)$ on some interval $I$. Then the most general antiderivative of $f(x)$ is*

$$F(x) + C$$

*where $C$ is an arbitrary constant.*

*Proof.* It might not seem like this statement requires a proof, but it does! We can check very easily that every function of the form $F(x)+C$ is an antiderivative of $f(x)$, but how do we know every antiderivative takes this form? To prove this, let $G(x)$ be an arbitrary antiderivative of $f(x)$. Then we need to show that $G(x) = F(x) + C$, or equivalently that $G(x) - F(x) = C$. We can do this by taking the derivative of $G(x) - F(x)$ to get

$$G'(x) - F'x) = f(x) - f(x) = 0,$$

and so by Theorem 5.6.5 we know that $G(x) - F(x)$ must be constant on $I$. Hence $G(x) - F(x) = C$ for some constant $C$, or equivalently $G(x) = F(x) + C$. Therefore *every* antiderivative of $f(x)$ can be written as $F(x) + C$. ■

It might seem superfluous, but typically when dealing with models requiring us to find antiderivatives we'll want to find the most general antiderivative somewhere along the way. Moral of the story: *don't forget to add $C$*.

**Example 5.9.4 Finding antiderivatives.**

Find the most general antiderivatives of the following functions:

1. $x^3$.
2. $-\dfrac{1}{2}x^{\frac{5}{3}}$.
3. $\sqrt{x} - 3x^3 + x^{-2}$.

4. $\sin x + \cos(5x) + e^{4x}$.

**Solution**. We have the following:

1. The general antiderivative of $x^3$ is $\frac{1}{4}x^4 + C$.
2. The general antiderivative of $-\frac{1}{2}x^{\frac{5}{3}}$ is $-\frac{1}{2}\frac{3}{8}x^{\frac{8}{3}} + C$.
3. The general antiderivative of $\sqrt{x} - 3x^3 + x^{-2}$ is $\frac{2}{3}x^{\frac{3}{2}} - \frac{3}{4}x^4 - \frac{1}{x} + C$.
4. The general antiderivative of $\sin x + \cos(5x) + e^{4x}$ is $-\cos x + \frac{1}{5}\sin(5x) + \frac{1}{4}e^{4x} + C$.

**Example 5.9.5 The antiderivative of $\frac{1}{x}$.**

Find the general antiderivative of $f(x) = \frac{1}{x}$.

**Solution**. We have two cases we need to consider, since the domain of $f(x)$ consists of two intervals. First, suppose that $x$ is in $(0, \infty)$. Then we know that $\dfrac{d}{dx}\ln x = \frac{1}{x}$, so on this interval the most general antiderivative of $\frac{1}{x}$ is $\ln x + C_1$.

Now let $x$ be in $(-\infty, 0)$. Then $\ln x$ isn't defined here. However, we can write

$$\frac{1}{x} = -\frac{1}{-x},$$

and if $x < 0$ then $-x > 0$ and $\ln(-x)$ is defined, and in fact

$$\frac{d}{dx}\ln(-x) = -\frac{1}{(-x)} = \frac{1}{x}.$$

So the most general antiderivative of $\frac{1}{x}$ on $(-\infty, 0)$ is $\ln(-x) + C_2$. Putting this all together, we can say that the most general antiderivative of $f(x) = \frac{1}{x}$ is given by

$$F(x) = \begin{cases} \ln x + C_1 & x > 0 \\ \ln(-x) + C_2 & x < 0 \end{cases}.$$

If we know that we're only selecting values of $x$ with the same sign, then we can just say that the antiderivative of $\frac{1}{x}$ is $\ln|x| + C$.

**Example 5.9.6 Antiderivatives involving secant.**

Find the general antiderivatives of $\sec^2 t$ and $-2\sec 3t \tan 3t$.

**Solution**. The first antiderivative isn't too hard; it's just $\tan t + C$. The second is a little more complicated but still not too bad. Since $\dfrac{d}{dt}\sec t = \sec t \tan t$, a good guess for an antiderivative of $-2\sec 3t \tan 3t$ would be $-2\sec 3t$. However, the derivative of this is $-6\sec 3t \tan 3t$, so we're off by a factor of 3. So we just need to divide our guess by 3 to correct for this. Hence the general antiderivative of $-2\sec 3t \tan 3t$ is $-\frac{2}{3}\sec 3t$.

**Theorem 5.9.7 Antidifferentiation formulas.**

*Let $F(x)$ and $G(x)$ denote antiderivatives of $f(x)$ and $g(x)$, and let $c$ be a constant. Then we have the following:*

***Table 5.9.8 Particular antiderivatives of functions***

| *Function* | *Antiderivative* |
|---|---|
| $cf(x)$ | $cF(x)$ |
| $f(x)+g(x)$ | $F(x)+G(x)$ |
| $x^n, n \neq 1$ | $\frac{x^{n+1}}{n+1}$ |
| $\frac{1}{x}$ | $\ln\lvert x\rvert$ |
| $e^{cx}, c \neq 0$ | $\frac{1}{c}e^{cx}$ |
| $\sin cx, \cos cx, c \neq 0$ | $-\frac{1}{c}\cos cx, \frac{1}{c}\sin cx$ |

*These are by no means the only antiderivatives that you will need to deal with, but they are probably the most common.*

**Example 5.9.9 Falling objects.**

An object is dropped from a height of 100 m above sea level and falls with downward acceleration equal to $9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$. Find the height $h(t)$ of the object $t$ seconds after it's dropped.

**Solution**. Take downward to be the negative direction and sea level to be $h = 0$. Since acceleration is the second derivative of position, then the position (i.e. height) of the object should be the second *antiderivative* of acceleration. At this step, it's tempting to say that since $h''(t) = -9.8$, we have

$$h't) = -9.8t \quad \text{and} \quad h(t) = -4.9t^2.$$

And indeed, $-4.9t^2$ is a function whose second derivative is $-9.8$. However, we have a slight problem here. We know that $h(0) = 100$ and $h'0) = 0$. If $h(t) = -4.9t^2$, then it's impossible for $h(0)$ to equal 100. The problem here is we forgot about the arbitrary constants.

To get an accurate expression for $h(t)$, we go back to $h''(t) = -9.8$. Then $h't) = -9.8t + C_1$ for some constant $C_1$. Since $h'0) = 0$, this forces $C_1 = 0$. So we have $h't) = -9.8t$. Now we antidifferentiate one more time to get $h(t) = -4.9t^2 + C_2$. Since $h(0) = 100$, this forces $C_2 = 100$. So

$$h(t) = -4.9t^2 + 100.$$

Some functions that do not have an obvious antiderivative can be simplified through algebra into a form that is perhaps more helpful.

**Example 5.9.10 A tricky antiderivative.**

Find the most general antiderivative of

$$f(x) = \frac{2+x^2}{1+x^2}.$$

**Solution**. It's tough to think of a function whose derivative is $f(x)$, though since the denominator is $1 + x^2$ it seems likely that this antiderivative will involve $\tan^{-1} x$ in some way. In order to actually find the antiderivative, we'll rewrite $f(x)$ into a more convenient form. First, note that the numerator is very close to the denominator, which means we can almost cancel it out. So we'll split up the numerator as follows:

$$\frac{2+x^2}{1+x^2} = \frac{1+(1+x^2)}{1+x^2} = \frac{1}{1+x^2} + 1.$$

We can find the antiderivative of this term by term, so the most general antiderivative of $f(x)$ is

$$F(x) = \tan^{-1} x + x + C.$$

At this point, it's natural to think about what an antiderivative means geometrically since the derivative of a function has such a nice interpretation. One approach to this is to think about **net change** of the antiderivative. Let $F(x)$ be an antiderivative of $f(x)$ on some interval containing $a, b$. The net change of $F(x)$ from $x = a$ to $x = b$ is given by

$$F(b) - F(a).$$

Now, $F'x) = f(x)$ represents instantaneous rate of change, not net change. However, if we can somehow add these instantaneous changes from $x = a$ to $x = b$, they should accumulate until they give the net change of $F$ from $a$ to $b$. Graphically, adding these instantaneous rates of change looks like it's giving us the *area* under $f(x)$, which suggests the following geometric interpretation: *the area under of $f(x)$ from $x = a$ to $x = b$ is just the net change of an antiderivative $F(x)$ from $x = a$ to $x = b$.* Much of the next chapter is devoted to making this idea precise.

# Part II

# Integral Calculus

# Chapter 6

# Integrals

By now we have a pretty good idea of how to calculate derivatives and what they represent. In particular, we've seen that $f'(x)$ tells us how quickly $f(x)$ is changing at a given point. However, we don't yet have a good intuition for the meaning of the *antiderivative* of a function aside from a rough interpretation as the inverse of differentiation.

To help develop some intuition for what the antiderivative represents, consider a continuous function $f(x)$ defined over some interval $[a, b]$. Now let $A(t)$ denote the area between $f(x)$ and the $x$-axis from $x = a$ to $x = t$, where $a \leq t \leq b$:

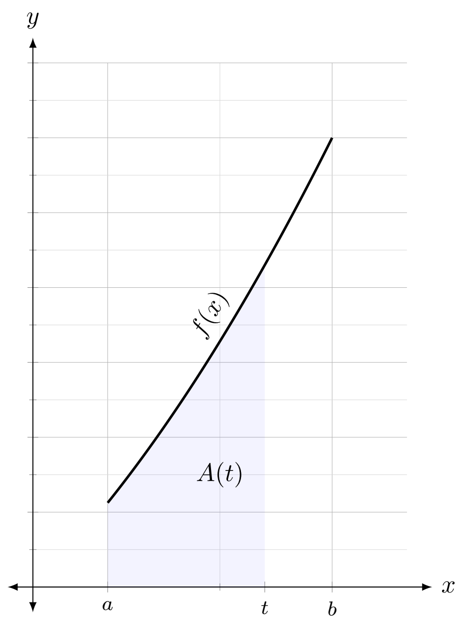



**Figure 6.0.1** A graph of $f(x)$ and its corresponding area function.

Then it appears that

$$A(x + h) - A(x) \approx hf(x)$$

for small values of $h$. If we let $h \to 0$, then the above equation suggests that $A'(x) = f(x)$. In other words, *the area function is an antiderivative of* $f(x)$!

Going forward this will be our primary geometric interpretation of an antiderivative, but we need to address two issues to make the above intuition legitimate:

- What exactly *is* the area under a general curve $y = f(x)$? We can find the areas of simple shapes such as rectangles, triangles and circles without issue, but we haven't even defined what the area of a more complicated region should be.
- What is the precise relationship between antiderivatives and areas? Our intuition above suggests that antiderivatives should be related to areas, but as we saw in Theorem 5.9.3 a function can have infinitely many antiderivatives. Which one represents the area under the curve?

The first issue will be addressed in Section 6.2, where we use the concept of a Riemann sum to give a precise meaning to the notion of area under a curve, which we identify as a definite integral. The second issue will be addressed in Section **??** and forms the content of Fundamental Theorem of Calculus.

## 6.1 Areas Under Curves

We know how to find areas of simple shapes, such as rectangles, triangles and circles. But how can we find the area of a more complicated shape? Say, for example, the area under $y = x^3$ from $x = 0$ to $x = 1$? Well, if we don't know it exactly then we can at least approximate it using a shape we do know. For example, using rectangles.

A simple way to divide the graph of $y = x^3$ using rectangles is to pick several equally spaced points in the interval $[0, 1]$, say

$$x_0 = 1, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, \quad \text{and} \quad x_3 = 1,$$

and then draw rectangles using these points to determine the base of each rectangle and the height of the graph above the points to determine the height of each rectangle. In particular, we will have three rectangles; the first with base $[0, \frac{1}{3}]$, the second with base $[\frac{1}{3}, \frac{2}{3}]$ and the third with base $[\frac{2}{3}, 1]$. Now, the height of each rectangle comes from a point on the graph above the corresponding interval. To make these specific (and relatively straightforward), we'll use **right endpoints** to determine the height of each rectangle. So the height of each respective rectangle is given by $(\frac{1}{3})^3$, $(\frac{2}{3})^3$ and by $1^3$.

Now, if we add up the areas of these rectangles we get an approximation for the area under the graph. Call these areas $A_1, A_2, A_3$. So the area under the graph of $y = x^3$ from $x = 0$ to $x = 1$ is about

$$A_1 + A_2 + A_3 = \frac{4}{9}.$$

There's nothing stopping us from going further here. For example, we pick $n$ rectangles, each with base length $\frac{1}{n}$ and heights determined by right endpoints as above, then the area $R_n$ of all of the rectangles will be an approximation (in fact, an overestimate) of the area of $y = x^3$. The bases of these rectangles are given by

$$[0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}], \ldots, [\frac{n-1}{n}, \frac{n}{n}].$$

So the areas of these rectangles are

$$\frac{1}{n}\left(\frac{1}{n}\right)^3, \frac{1}{n}\left(\frac{2}{n}\right)^3, \ldots, \frac{1}{n}\left(\frac{n-1}{n}\right)^3, \frac{1}{n}\left(\frac{n}{n}\right)^3.$$

Adding these up, we get

$$R_n = \frac{1}{n}\frac{1^3 + 2^3 + 3^3 + \cdots + n^3}{n^3}$$

or just

$$R_n = \frac{1^3 + 2^3 + \cdots + n^3}{n^4}.$$

So if we sent $n$ to $\infty$, we should get the exact area. In other words, the area under $y = x^3$ from $x = 0$ to $x = 1$ is given by

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{1^3 + 2^3 + \cdots + n^3}{n^4}.$$

Actually finding this limit requires some trickery. It's not at all obvious, but we can write

$$1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Hence

$$\begin{aligned}
\lim_{n\to\infty} R_n &= \lim_{n\to\infty} \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2 \\
&= \lim_{n\to\infty} \frac{1}{4}\frac{n^2(n^2+2n+1)}{n^4} \\
&= \lim_{n\to\infty} \frac{1}{4}\frac{n^4+2n^3+n^2}{n^4} \\
&= \frac{1}{4}.
\end{aligned}$$

So the exact area under $y = x^3$ from $x = 0$ to $x = 1$ is $\frac{1}{4}$. Note that this matches up with our intuition from the last chapter: the area should be equal to the net change of an antiderivative. In this case, an antiderivative of $x^3$ is $\frac{1}{4}x^4$, and the net change of $\frac{1}{4}x^4$ from $x = 0$ to $x = 1$ is exactly $\frac{1}{4}$.

The process we used above for $y = x^3$ can be carried out for any continuous function. If we have a function defined on the interval $[a, b]$, we can approximate the area under the graph by using rectangles. First, we split $[a, b]$ into $n$ different subintervals $[x_0, x_1], \ldots, [x_{n-1}, x_n]$, each of width $\Delta x = \frac{b-a}{n}$ and where

$$\begin{aligned}
x_0 &= a \\
x_1 &= a + \Delta x = a + \frac{b-a}{n} \\
x_2 &= a + 2\Delta x = a + 2\frac{b-a}{n} \\
&\vdots \\
x_n a &+ n\Delta x = b
\end{aligned}$$

The approximate area under the graph of $y = f(x)$ from $x = a$ to $x = b$ is then

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

We can now define the area under a graph of a continuous function. Again, the idea is to approximate the graph with more and more rectangles.

**Definition 6.1.1 Area.**

The **area** under the graph of a continuous function $y = f(x)$ is the limit of the sum of the areas of the approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + \cdots + f(x_n)\Delta x].$$

Although the previous definition used right endpoints to find the height of each rectangle, this doesn't have to be the case. We can use left endpoints, midpoints or any other **sample points** $x_1^*, x_2^*, \ldots, x_n^*$, where each $x_i^*$ is a point in the interval $[x_{i-1}, x_i]$.

Since finding areas in this way requires writing complicated sums, it seems like a good idea at this point to introduce notation for writing large summations. We call this **sigma notation**. Sigma notation takes the following form:

$$\sum_{i=\text{start}}^{\text{finish}} (\text{terms involving } i).$$

To evaluate such an expression, just plug in the different values for $i$ from start to finish and add up what you get.

**Example 6.1.2 Evaluating a sum.**

Determine $\sum_{i=-2}^{3}(4-3i)$.

**Solution**. We start at $i = -2$ and finish at $i = 3$. Plugging all of these values in for $i$, we get the following table:

**Table 6.1.3 Table of values for $\sum_{i=-2}^{3}(4-3i)$**

| $i$ | $4-3i$ |
|---|---|
| $-2$ | 10 |
| $-1$ | 7 |
| 0 | 4 |
| 1 | 1 |
| 2 | $-2$ |
| 3 | $-5$ |

So we can say that

$$\begin{aligned}\sum_{i=-2}^{3}(4-3i) &= 10+7+4+1-2-5\\ &= 15.\end{aligned}$$

**Example 6.1.4 Using sigma notation.**

Use sigma notation to write the sum of all squares from 31 to 245.

**Solution**. We need to start at $i = 31$ and stop at $i = 245$. So the first component of this sum using sigma notation looks like $\sum_{i=31}^{245}$. Now we just need to figure out what goes inside of it. Since we're adding squares of numbers, we'll just put $i^2$ inside of this sum. Hence the sum of all squares from 31 to 245 can be denoted

$$\sum_{i=31}^{245} i^2.$$

Sigma notation has some useful properties, since it's just another way to write sums.

**Theorem 6.1.5 Properties of Sigma Notation.**

*Let* $X = \sum_{i=a}^{b} x_i$ *and let* $Y = \sum_{i=a}^{b} y_i$*. Let* $c$ *be a constant. Then the following are true:*

$$\sum_{i=a}^{b} c = c + \cdots + c = (b - a + 1)c$$

$$\sum_{i=a}^{b} cx_i = c \sum_{i=a}^{b} x_i = cX$$

$$\sum_{i=a}^{b} (x_i + y_i) = \sum_{i=a}^{b} x_i + \sum_{i=a}^{b} y_i = X + Y$$

$$\sum_{i=a}^{b} (x_i - y_i) = \sum_{i=a}^{b} x_i - \sum_{i=a}^{b} y_i = X - Y$$

$$\sum_{i=a}^{k} a_i + \sum_{i=k+1}^{b} a_i = \sum_{i=a}^{b} a_i.$$

*In other words, we can break sigma notation up over addition and subtraction, and we can move constants outside of it.*

**Example 6.1.6 Areas using sigma notation.**

Write down a limit that gives the area under $f(x) = \sin x$ from $x = 1$ to $x = 3$ using sigma notation.

**Solution**. First, note that we can rewrite Definition 6.1.1 using sigma notation as

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x.$$

We know what $f(x)$ is, so we just need to identify $x_i$ and $\Delta x$. $\Delta x$ isn't too bad here, it's just

$$\Delta x = \frac{3-1}{n} = \frac{2}{n}.$$

The $x_i$ are a little more complicated, but we'll still use right endpoints to find them. So we have

$$x_0 = 1$$

$$x_1 = 1 + \frac{2}{n} = \frac{n+2}{n}$$

$$x_2 = 1 + 2\frac{2}{n} = \frac{n+4}{n}$$

$$x_3 = 1 + 3\frac{2}{n} = \frac{n+6}{n}$$
$$\vdots$$
$$x_n = 3$$

In general,

$$x_i = 1 + i\frac{2}{n} = \frac{n+2i}{n}$$

for $1 \le i \le n$.

Now we have everything we need to write our limit. Hence the area under $\sin x$ from 1 to 3 is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{n+2i}{n}\right)\frac{2}{n}.$$

Just as the problem of finding the slope of a tangent line can be viewed as a rate of change problem, the problem of finding areas can be viewed as an *accumulation problem*.

Example 6.1.7 Finding work done.

In physics, the work done by a constant force $F$ on some particle that moves a displacement $d$ is given by the formula

$$W = Fd.$$

How should we define the work done by a variable force?

**Solution**. The main idea here is to treat the variable force as the combination of several constant forces. To be specific, let $F(x)$ denote our force (which depends on position $x$) and suppose it acts on a particle moving from $x = a$ to $x = b$. $F(x)$ itself may not be constant, but if we break $[a, b]$ up into $n$ subintervals of the form $[x_{i-1}, x_i]$ each of length $\Delta x = \frac{b-a}{n}$, then $F(x)$ should be nearly constant on each subinterval. So we can pick points $x_i^*$ from each subinterval and say that $F(x) \approx F(x_i^*)$ for all $x$ in $[x_{i-1}, x_i]$.

Now we can approximate the work done by $F$ as the particle moves through the subinterval $[x_{i-1}, x_i]$. The work done should be about equal to

$$F(x_i^*)\Delta x.$$

Now, if we add up these estimates for each subinterval, we see that the total work done should be approximated by

$$\sum_{i=1}^{n} F(x_i^*)\Delta x,$$

which suggests that we should define the exact work done to be

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*)\Delta x.$$

Note that Example 6.1.7 was not written in terms of finding areas, but still involved the limit from Definition 6.1.1. This tells us that the ideas in this section have a wider use than in just finding areas under curves.

## 6.2 The Definite Integral

The process used in Section 6.1 can be generalized quite a bit. Start with a function $f(x)$ defined on some interval $[a, b]$, and divide $[a, b]$ into $n$ subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$$

where $x_0 = a, x_n = b$ and each subinterval has length $\Delta x_k$ (i.e. $\Delta x_k = x_k - x_{k-1}$). This is known as a **partition** of $[a, b]$. Then we can choose sample points $x_1^*$ from $[x_0, x_1]$, $x_2^*$ from $[x_1, x_2]$, all the way up to $x_n^*$ from $[x_{n-1}, x_n]$. Note that the only restriction we're placing on the sample point $x_k^*$ is that is must come from the interval $[x_{k-1}, x_k]$, but it may be *any* point in this interval.

The partition and sample points selected are now used to determine a sum that represents an approximation to the area under the curve:

$$\sum_{i=1}^{n} f(x_k^*)\Delta x_k.$$

Geometrically, this sum represents the areas of multiple rectangles added together. Since these sums are essential for what follows, we will give them a name.

**Definition 6.2.1 Riemann Sum.**

Let $a, b \in \mathbb{R}$ with $a < b$ and let $f(x)$ be a function defined on the interval $[a, b]$. Let $P = \{x_k\}_{k=0}^{n}$ be a partition of $[a, b]$ and choose corresponding sample points $x_k^* \in [x_{k-1}, x_k]$ for $1 \le k \le n$. Then a **Riemann sum** is a sum of the form

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

where $\Delta x_k = x_k - x_{k-1}$.

Although this can be more complicated than our definition in Definition 6.1.1, it still approximates the area under $f(x)$ from $x = a$ to $x = b$. And as each subinterval in this partition gets smaller (i.e. as $\Delta x_k \to 0$ or equivalently as $n \to \infty$), then this approximation should become exact, at least if $f(x)$ is "nice enough" (see Theorem 6.2.3 for a more precise statement).

**Definition 6.2.2 The Definite Integral.**

Let $f(x)$ be a function defined on $[a, b]$. Then the **definite integral of** $f(x)$ **from** $a$ **to** $b$ is the number

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_k^*)\Delta x_k,$$

provided that the limit exists (and that $\Delta x_i \to 0$). If this limit exists, we say that $f(x)$ is **integrable** on $[a, b]$.

We can use Definition 6.2.2 to rewrite our area definition: the area under $y = f(x)$ from $x = a$ to $x = b$ is defined to be the definite integral of $f$ on $[a, b]$, assuming it exists. For a function to be integrable, the particular partition of $[a, b]$ that we use *cannot affect the limit* in Definition 6.2.2. This gives us a great deal of freedom in approximating the definite integral/area under a curve: when choosing sample points, we can use left endpoints; right endpoints; midpoints and more. However, if $f$ is integrable then we lose nothing by just using right endpoints and subintervals of equal length, as in Definition 6.1.1.

**Theorem 6.2.3 Integrability of Piecewise Continuous Functions.**

*Let $f(x)$ be a bounded, piecewise continuous function on $[a, b]$ with finitely many jump discontinuities. Then $f$ is integrable on $[a, b]$. In particular, continuous functions are always integrable on bounded, closed intervals.*

All of the functions that we will work with will be piecewise continuous, and many important quantities in mathematics and its applications can be represented by bounded, piecewise continuous functions. Definite integrals of such quantities therefore exist and are often used as a measure of "accumulation." In particular, Example 6.1.7 can be generalized to provide the following definition of work done by a force.

**Definition 6.2.4 Work.**

Let $F(x)$ denote the force acting on a particle over $[a, b]$. The **work** done by the force $F(x)$ on the particle as it moves from $x = a$ to $x = b$ is defined to be

$$W = \int_a^b F(x)\, dx\ .$$

**Example 6.2.5 Expressing work done.**

A particle moving from $x = -1$ to $x = 3$ is acted upon by the force $F(x) = 3x$. Find the work done by the force on this particle.

**Solution**. Using Definition 6.2.4, we see that the work done should be

$$W = \int_{-1}^{3} 3x\, dx.$$

Since the definite integral represents area, and the definite integral of $3x$ exists (since this is continuous), then we can see from Figure 6.2.6 that

$$W = \int_{-1}^{3} 3x\, dx = \frac{1}{2}(1)(-3) + \frac{1}{2}(3)(9) = 12.$$

Note that regions underneath the $x$-axis correspond to negative areas!

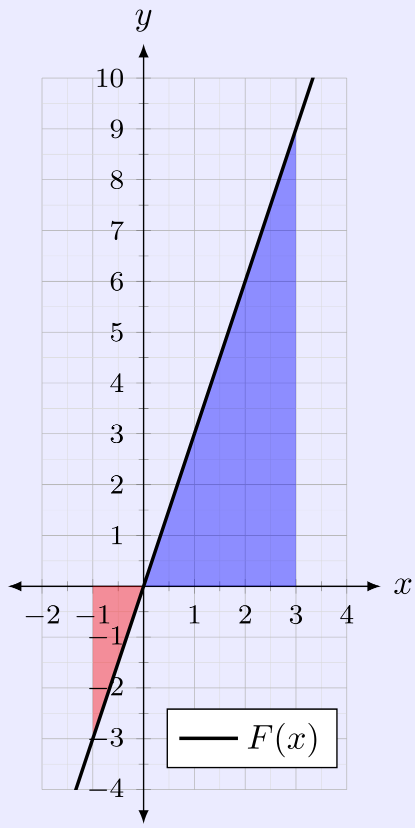



**Figure 6.2.6** A graph of the force $F(x) = 3x$ and shaded areas corresponding to work.

In Example 6.2.5 we used the connection between definite integrals and areas to find the work done. For the moment, this is our only method for calculating definite integrals. We will develop other methods starting in Section **??**, but the connection between definite integrals and areas is an important one that we will make use of frequently.

**Example 6.2.7 A radical integral.**

Determine $\int_0^2 \sqrt{4-x^2}\,dx$.

**Solution**. We won't be able to solve this integral algebraically until Calculus 2, but we can solve it geometrically instead. If we graph $y = \sqrt{4-x^2}$, we get the upper half of the circle of radius 2 centered at the origin, as seen in Figure 6.2.8. So the integral given must be exactly one-fourth the area of this circle, so

$$\int_0^2 \sqrt{4-x^2}\,dx = \frac{1}{4}\pi(2^2) = \pi.$$

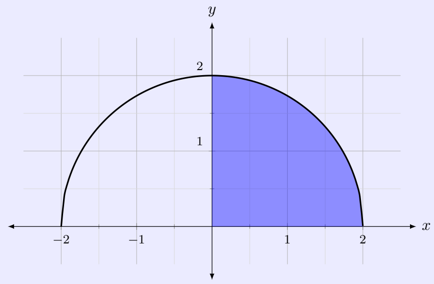



**Figure 6.2.8** The graph of $y = \sqrt{4-x^2}$.

Since the definite integral is defined as a limit of sums, it shares many properties with sums. In particular, we have the following version of Theorem 6.1.5

**Theorem 6.2.9 Properties of the Definite Integral.**

*Let $f(x)$ and $g(x)$ be integrable on $[a,b]$ and let $c$ be a constant. Let $t$ be any number in $[a,b]$. Then the following are true:*

$$\int_a^b c\,dx = c(b-a)$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$\int_a^t f(x)\,dx + \int_t^b f(x)\,dx = \int_a^b f(x)\,dx.$$

*In other words, we can break definite integrals up over addition and subtraction, and we can move constants outside of it.*

Computing integrals in practice often involves setting up and approximating Riemann sums. These sums depend on how we choose the sample points $x_k^*$ from each interval $[x_{k-1}, x_k]$. We've mostly worked with $x_k^* = x_k$, or right endpoints. We could also choose $x_k^* = x_{k-1}$, using left endpoints. However, it's often more accurate to use *midpoints*: $x_k^* = \frac{x_{k-1}+x_k}{2}$. The reason for this is that if we choose midpoints of intervals as our sample points, these tend to be less affected than either endpoint if the function is increasing or decreasing. In particular, we have the **midpoint rule**.

**Definition 6.2.10 Midpoint Rule.**

Let $f(x)$ be a continuous function. Then

$$\int_a^b f(x)\,dx \approx M_n = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

where

$$x_k^* = \frac{x_{k-1}+x_k}{2}$$

$$\Delta x = \frac{b-a}{n}.$$

Note that this rule calls for evenly dividing $[a,b]$ into a partition $[x_{k-1}, x_k]$ where each subinterval has equal length $\Delta x$.

Replacing the midpoints in Definition 6.2.10 with corresponding left endpoints (respectively, right endpoints) gives the **left endpoint rule** (respectively, the **right endpoint rule**). Together with the midpoint rule, these approximations are

$$\begin{aligned} L_n &= \sum_{k=1}^{n} f(x_{k-1})\Delta x \\ M_n &= \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)\Delta x \\ R_n &= \sum_{k=1}^{n} f(x_k)\Delta x. \end{aligned}$$

Careful! In general, $M_n \neq \frac{L_n+R_n}{2}$.

**Example 6.2.11 Approximating an integral.**

Approximate $\int_1^3 \frac{1}{x}\,dx$ using Riemann sums with $n=4$.

**Solution**. For this integral, we have

$$\begin{aligned} \Delta x &= \frac{3-1}{4} = \frac{1}{2} \\ x_0 &= 1 \\ x_1 &= \frac{3}{2} \\ x_2 &= 2 \\ x_3 &= \frac{5}{2} \\ x_4 &= 3 \end{aligned}$$

We will have different estimates using left endpoints, right endpoints and midpoints. Using left endpoints, we have

$$\int_1^3 \frac{1}{x}\,dx \approx L_4 = \frac{1}{2}\left(1+\frac{2}{3}+\frac{1}{2}+\frac{2}{5}\right) = 1.283\ldots.$$

Using midpoints, we have

$$\int_1^3 \frac{1}{x}\,dx \approx M_4 = \frac{1}{2}\left(\frac{4}{5}+\frac{4}{7}+\frac{4}{9}+\frac{4}{11}\right) \approx 1.09.$$

Using right endpoints, we have

$$\int_1^3 \frac{1}{x}\,dx \approx R_4 = \frac{1}{2}\left(\frac{2}{3}+\frac{1}{2}+\frac{2}{5}+\frac{1}{3}\right) = 0.95.$$