Calculus Notes

West Virginia Wesleyan College

Calculus Notes

West Virginia Wesleyan College

Jesse Oldroyd
West Virginia Wesleyan College

February 4, 2026

Go Seahawks.

Edition: Annual Edition 2025

Website: j-oldroyd.github.io/wvwc-calculus/

Preface

This document was created to serve as a single source for my lecture notes for the calculus sequence at West Virginia Wesleyan College. As such these notes are divided into three self-explanatory parts.

- The first part, Differential Calculus, introduces the derivative and covers its important properties and applications. This content forms the majority of the Calculus I course.
- The next part, Integral Calculus, introduces the integral and goes over multiple methods for its calculation. This content forms the majority of the Calculus II course.
- The final part, Multivariable Calculus, generalizes the concepts of derivatives and integrals to two and three dimensions. This content forms basis of the Calculus III course.

This document is very much *in progress* and therefore typos and other errors are to be expected. If you find any, I would appreciate you letting me know by contacting me by email.

Contents

Part I

Differential Calculus

Chapter 1

Functions

The primary object of study in calculus is the function. In this chapter, we'll review important types of functions that will appear in this course.

1.1 Function Review

This section reviews basic facts about functions. To appear in a later version.

1.2 Types of Functions

Linear Functions

The most basic example of a function that we'll study in calculus is the *linear function*.

Definition 1.2.1 Linear Functions.

A **linear function** is a function $f(x)$ of the form

$$f(x) = mx + b$$

for constants m and b.

As the name suggests, any (non-vertical) line is an example of a linear function. Such functions are completely determined by the slope of the corresponding line and the y-intercept.

Equivalently, any line is determined by knowing two distinct points on the line.

In particular, if $y = mx + b$ then m is the slope and b is the y-intercept. This is known as the *slope-intercept* equation of a line. Equations of lines are also often written in *point-slope form* as $y - y_0 = m(x - x_0)$.

Example 1.2.2 Finding the Equation of a Line.

Find a function $f(x)$ whose graph is a line passing through $(-5, 2)$ and $(1, 1)$.

Polynomial Functions

After linear functions, we also study functions with higher powers of x.

Definition 1.2.3 Polynomial Functions.

A **polynomial function** is a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where n is a nonnegative whole number and the **coefficients** $a_0, a_1, \ldots, a_n$ are constants and n is the **degree** of the polynomial.

If $f(x) = 0$, then we say that $f(x)$ has degree $-\infty$.

So linear functions are just polynomial functions of degree at most 1. And just as linear functions are determined by knowing two distinct points on the line, any polynomial functions is determined by knowing $n+1$ distinct points on the polynomial.

Example 1.2.4 Composition of Polynomial Functions.

Let $f(x) = 3x - x^2$ and $g(x) = 4x^3 + 5$. Find $f(g(x))$.

Algebraic Functions

The operations of addition, subtraction, multiplication, division, taking whole number powers and taking whole number roots applied to polynomials give rise to *algebraic functions*. Two particularly important examples are *rational functions* and *root functions*.

Definition 1.2.5 Rational and Root Functions.

A **rational function** is a function of the form $\frac{p(x)}{q(x)}$ where p and q are polynomial functions. A **root function** is a function of the form $x^{1/n} = \sqrt[n]{x}$ for some natural number n.

As functions become more complicated, we have to worry more and more about their domains. For a polynomial function, the domain is the set of all real numbers $\mathbb{R}$ (if we ignore complex numbers). The domain of a rational function is the set of all numbers where the denominator is nonzero. The domain of a root function is the set of all nonnegative numbers.

Example 1.2.6 Finding the Domain of an Algebraic Function.

Find the domain of

$$\sqrt{x^2 - 3x + 2} + \frac{x-4}{3x+4}.$$

Solution. We need to find where both the radicand $x^2 - 3x + 2$ is nonnegative and where $3x + 4 \neq 0$. If we solve $x^2 - 3x + 2 \geq 0$, we see that x must be in $(-\infty, 1] \cup [2, \infty)$. Likewise, $3x + 4 \neq 0$ if and only if

$x \neq -\frac{4}{3}$. Hence the domain is

$$(-\infty, -\frac{4}{3}) \cup (-\frac{4}{3}, 1] \cup [2, \infty).$$

We'll make use of the computer algebra system Sage to perform certain computations. As an example, we use it below to solve the inequality $x^2 - 3x + 2 \geq 0$:

```
# click the Sage button to evaluate this cell
solve(x^2 - 3*x + 2 >= 0, x)    # tells Sage the inequality
    to solve and the variable to solve for
```

1.3 Trigonometric Functions

Angles and Terminal Points

Consider the unit circle in $\mathbb{R}^2$, which is given by $x^2 + y^2 = 1$. Each point P on this circle makes an angle θ with the positive x-axis, and therefore θ must also determine the point completely. The angle θ is typically specified using either *radians* or *degrees*. Converting from one unit to the other can be done by noting that π radians is precisely equal to 180 degrees.

Unless otherwise specified, we will use radians for angle measures in this course.

Example 1.3.1 Converting from Degrees to Radians.

Convert 270 degrees to radians and $\frac{\pi}{6}$ radians to degrees.

Just as points on the unit circle determine angles, angles also determine points. A point P determined by an angle θ is also known as the *terminal point* for the angle. Terminal points for certain *reference angles* are useful to remember.

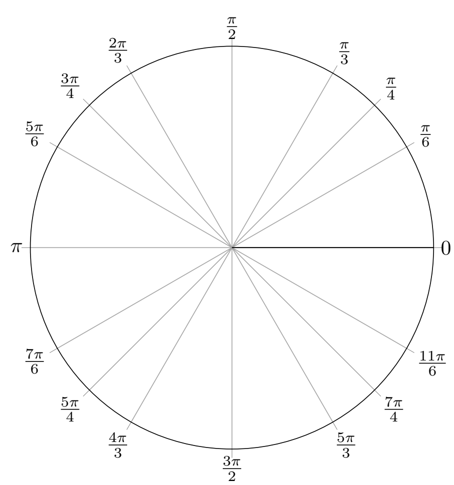

Figure 1.3.2 The unit circle

Example 1.3.3 Finding Terminal Points.

Find the terminal points for the angles 210° and $\frac{5\pi}{3}$.

Solution. First, note that $210° = \frac{7\pi}{6}$. We therefore choose $\frac{\pi}{6}$ as our reference angle and obtain the terminal point $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$. The terminal point for $\frac{5\pi}{3}$ is likewise $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$.

The Trigonometric Functions

Since angles determine terminal points on circles, the coordinates (x, y) of each point on the circle can be viewed as functions of the angle θ. These *coordinate functions*, $x(\theta)$ and $y(\theta)$, are the fundamental trigonometric functions *sine* and *cosine*, and can be used to define the other four trigonometric functions commonly used.

Definition 1.3.4 Trigonometric Functions.

Let $\theta \in \mathbb{R}$ and let $P = (x, y)$ denote the corresponding terminal point. The **cosine** function is the function $\cos\theta = x$, and the **sine** function is the function $\sin\theta = y$. The **secant, cosecant, tangent** and **cotangent** functions are defined as follows:

$$\sec\theta = \frac{1}{\cos\theta}, \csc\theta = \frac{1}{\sin\theta}, \tan\theta = \frac{\sin\theta}{\cos\theta} \text{ and } \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

The trigonometric functions satisfy important equalities known as *Pythagorean identities*.

Theorem 1.3.5 Pythagorean Identities.

Let $\theta \in \mathbb{R}$. Then

$$\sin^2\theta + \cos^2\theta = 1.$$

If $\theta \neq \frac{\pi}{2} + k\pi$ for some $k \in \mathbb{Z}$, then

$$\tan^2\theta + 1 = \sec^2\theta.$$

If $\theta \neq k\pi$ for some $k \in \mathbb{Z}$, then

$$\cot^2\theta + 1 = \csc^2\theta.$$

Chapter 2

Limits of Functions

At its heart, calculus is just the mathematics of change. In particular, calculus provides the tools necessary to describe how a function changes. Since a function can be used to represent many quantities that appear in real life, calculus therefore gives us a way to study how many different quantities (such as temperature, acceleration, energy of a signal, etc.) can change. But before we can start using calculus, we need to come up with the proper language to describe it. The language we will need to develop is that of the *limit of a function*.

2.1 The limit of a function

Motivating limits

Imagine creating a mathematical valley out of the graph of $f(x) = x^2$, and in this valley walks a mathematical ant. The ant is walking towards the place on the hill directly above $x = -\frac{1}{2}$. At $x = -2$, the ant is 4 units above the ground. At $x = -1$, the ant is now 1 unit above the ground. As the ant moves towards $x = -\frac{1}{2}$, its height above the ground gets closer and closer to $\frac{1}{4}$. To say it more clearly, and because modern calculus is built on this idea, *as the ant approaches* the point above $x = -\frac{1}{2}$, its *height above the ground approaches* the value $\frac{1}{4}$. In other words, the *limit* of $f(x) = x^2$ as x approaches $-\frac{1}{2}$ is $\frac{1}{4}$.

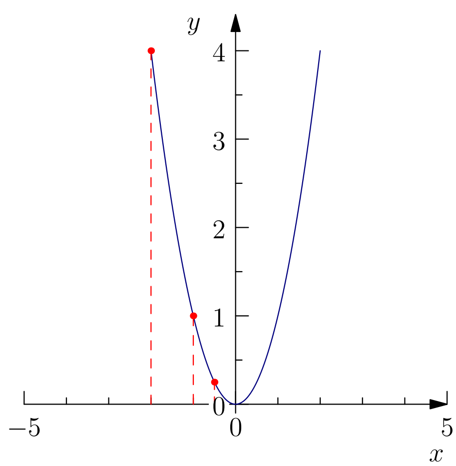

Figure 2.1.1 A mathematical valley minus a mathematical ant

Remember that a function f of the variable x is just a rule that turns one number, x, into another number, $f(x)$. So the idea that the *limit of a function* is trying to express is what happens to the number $f(x)$ (the output) as the number x (the input) approaches some particular value.

> Actually, functions are much more general than this. But for calculus, it won't hurt us to view functions in this way.

We're not quite ready to define the limit of a function precisely, but we can point one thing out right away: *the limit of a function requires two pieces of information: the function itself and the number that x is approaching.* The limit should then be whatever number that $f(x)$ is approaching.

Example 2.1.2 Estimating the limit of a trigonometric function.

Let $f(x) = \sin x \cos x$. What is the limit of $f(x)$ as x approaches the number π?

Solution. We don't have a lot of tools to find limits yet, so we'll try to estimate it instead. What we'll do is we'll plug numbers that are closer and closer to π into $f(x)$. Let's list several values of $f(x)$ as x gets closer to π from the left:

Table 2.1.3 Estimating $\lim_{x\to\pi} f(x)$

x	$f(x)$
3	$-.140$
3.1	$-.042$
3.14	$-.002$

We can even let x approach π from the other direction as well (i.e. "from the right") and $f(x)$ will still approach 0 as x gets closer and closer to π. So it looks like the limit should be 0.

To keep ourselves from writing "the limit of $f(x)$ as x approaches some number a" over and over, let's introduce some notation: $\lim\limits_{x\to a} f(x)$.

Example 2.1.4 Limit of a piecewise function.

Let

$$f(x) = \begin{cases} \sin x & \text{if } x < 0 \\ \cos x & \text{otherwise} \end{cases}.$$

Find $\lim\limits_{x\to 0} f(x)$.

Solution. If we graph this function, we see that at $x = 0$ there is a jump in the graph. In particular, if x approaches 0 from the left then $f(x)$ approaches 0, whereas if x approaches 0 from the right then $f(x)$ approaches 1. So this function does not appear to have an unambiguous limit as x approaches 0. Another way to say this: $\lim\limits_{x\to 0} f(x)$ *does not exist.*

Example 2.1.4 shows us something very important about limits: they depend on the two different ways x can approach a number. So we introduce two new pieces of notation: the **left-hand limit** $\lim_{x\to a^-} f(x)$ will stand for the value $f(x)$ approaches (if any) as x approaches a from the left (i.e. as x increases to a), and **the right-hand limit** $\lim_{x\to a^+} f(x)$ will stand for the value $f(x)$ approaches (if any) as x approaches a from the right (i.e. as x decreases to a). In Example 2.1.4, we would say that

$$\lim_{x\to 0^-} f(x) = 0 \text{ and } \lim_{x\to 0^+} f(x) = 1.$$

At this point, we can make a rough definition for the limit of a function.

Definition 2.1.5 Limit of a function.

Let $f(x)$ be a function. Suppose that both the left-hand limit $\lim_{x\to a^-} f(x)$ and the right-hand limit $\lim_{x\to a^+} f(x)$ exist and are equal to the same number L. Then we say that the limit of $f(x)$ as x approaches a exists and is equal to L. We denote this by writing $\lim_{x\to a} f(x) = L$.

Example 2.1.6 Piecewise function again.

Let $f(x)$ be given by

$$f(x) = \begin{cases} 0 & -3 \le x \le -1 \\ x^2 & -1 < x < \frac{1}{2} \\ \frac{1-x}{2} & \frac{1}{2} < x < 3. \end{cases}$$

Evaluate $\lim_{x\to -1} f(x)$ and $\lim_{x\to \frac{1}{2}} f(x)$.

Solution. If we graph $f(x)$, we get the following:

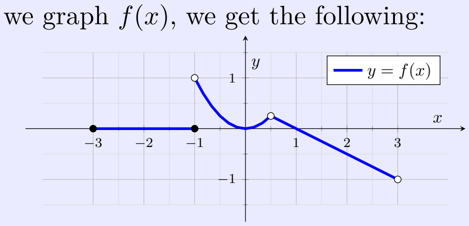

Figure 2.1.7 Graphing $f(x)$

The graph shows us that $\lim_{x\to -1^-} f(x) = 0$, while $\lim_{x\to -1^+} f(x) = 1$. Therefore $\lim_{x\to -1} f(x)$ does not exist. On the other hand, $\lim_{x\to \frac{1}{2}} f(x)$ exists and is equal to $\frac{1}{4}$.

It's important to note that the value of a function at a point $x = a$ is in general *completely independent* of the value of $\lim_{x\to a} f(x)$, i.e., we can't always expect $\lim_{x\to a} f(x)$ to be equal to $f(a)$. Functions for which this is true, however, are known as *continuous functions* and will be very important in Section 2.3 and beyond.

2.2 Computing limits

We've got a handle on how to estimate limits from Section 2.1, but the process is very tedious. It requires either graphing the function in question or laboriously entering values into a calculator. So our first order of business now that we have the concept of a limit is to find an easier way to calculate it. This will be a running theme throughout the course.

The limit laws

In many cases of interest, we can use knowledge of simpler limits to obtain more complicated limits. We do this via the **Limit Laws**. Before we get to them, we'll state two very simple (and hopefully very believable) limits.

Proposition 2.2.1 Simple limits.

For any value of a, the following limits hold:

$$\lim_{x\to a} c = c$$

if c is a constant and

$$\lim_{x\to a} x = a.$$

Theorem 2.2.2 The limit laws.

Let c be a constant, let n be a positive whole number and let $f(x)$ and $g(x)$ be functions. Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist for some number a. Then the following rules hold:

Table 2.2.3 The limit laws

1. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$	2. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$
3. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\lim_{x\to a} g(x)$	4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ (*if* $\lim_{x\to a} g(x)$
5. $\lim_{x\to a}[f(x)^n] = [\lim_{x\to a} f(x)]^n$	6. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

Note that item six in the table above only holds (in this class...) if n is odd or if $\lim_{x\to a} f(x) \geq 0$.

Theorem 2.2.2 gives us the ability to compute a wide variety of limits.

Example 2.2.4 Limit of a rational function.

Let

$$f(x) = \frac{3 - x^5 + 5x}{2x - \sqrt[4]{x}}.$$

Evaluate $\lim_{x\to 16} f(x)$.

Solution. We can evaluate $\lim_{x\to 16} f(x)$ by making use of the appropriate Limit Laws and Proposition 2.2.1:

$$\begin{aligned}
\lim_{x\to 16} \frac{3 - x^5 + 5x}{2x - \sqrt[4]{x}} &= \frac{\lim_{x\to 16}(3 - x^5 + 5x)}{\lim_{x\to 16}(2x - \sqrt[4]{x})} && \text{by Limit Law 4} \\
&= \frac{\lim_{x\to 16} 3 - \lim_{x\to 16} x^5 + \lim_{x\to 16} 5x}{\lim_{x\to 16} 2x - \lim_{x\to 16} \sqrt[4]{x}} && \text{by Limit Law 1.} \\
&= \frac{3 - [\lim_{x\to 16} x]^5 + 5\lim_{x\to 16} x}{2\lim_{x\to 16} x - \sqrt[4]{\lim_{x\to 16} x}} && \text{by Limit Laws 2, 5, and 6.} \\
&= \frac{3 - 16^5 + 80}{30}
\end{aligned}$$

In particular, Limit Laws 1-5 give us the following: if $f(x)$ is a polynomial or rational function, then $\lim_{x\to a} f(x) = f(a)$ as long as a is in the domain of $f(x)$. If a is not in the domain, trickery may be required.

Example 2.2.5 Trickery.

Evaluate

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}.$$

Solution. First, note that we can't use the Limit Laws right away since the denominator is 0 at $x = 1$. What we need to do is use algebra to simplify the expression inside the limit:

$$\begin{aligned}\frac{\sqrt{x}-1}{x-1} &= \frac{\sqrt{x}-1}{x-1}\,\frac{\sqrt{x}+1}{\sqrt{x}+1}\\ &= \frac{x-1}{(x-1)(\sqrt{x}+1)}\\ &= \frac{1}{\sqrt{x}+1}.\end{aligned}$$

Now we can use the Limit Laws to find the limit as x approaches 1, since we no longer have a divide-by-zero problem in the denominator:

$$\begin{aligned}\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} &= \lim_{x\to 1} \frac{1}{\sqrt{x}+1}\\ &= \frac{1}{2}.\end{aligned}$$

2.3 Continuity

We saw in Section 2.2 that for a function like $f(x) = x+3x^2$, we could evaluate $\lim_{x\to a} f(x)$ by simply plugging in $x = a$. In other words, $\lim_{x\to a} f(x) = f(a)$. Functions that have this property are extremely important in mathematics, so we give them a name.

Continuous functions

Definition 2.3.1 Continuous Functions.

Let $f(x)$ be a function and suppose that a is in the domain of $f(x)$. Then we say that $f(x)$ is **continuous at** a if

$$\lim_{x\to a} f(x) = f(a).$$

Otherwise, we say that $f(x)$ is **discontinuous at** a. We say that $f(x)$ is continuous on an interval if it is continuous at every point of an interval. Otherwise, we say that $f(x)$ is discontinuous on the interval.

Definition 2.3.1 says that it is extremely easy to evaluate limits of continuous functions: just plug the value that x is approaching into the function $f(x)$. So the limit is then $f(a)$. If a function $f(x)$ is continuous on an interval, then this means that the graph of $f(x)$ has no "gaps" over this interval.