Differential Equations Lecture Notes

West Virginia Wesleyan College

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Go Seahawks.

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Preface

This document was created to serve as lecture notes for the Differential Equations course at West Virginia Wesleyan College. These notes are divided into two parts.

- The first part, Ordinary Differential Equations, introduces differential equations in one variable along with methods for their solution and several applications.
- The next part, Partial Differential Equations, introduces Fourier series and differential equations in several variables, building up to solving the heat and wave equations.

You can find a PDF version of these notes here[1].

This document is very much *in progress* and therefore typos and other errors are to be expected. If you find any, I would appreciate you letting me know by contacting me by email.

[1] `j-oldroyd.github.io/wvwc-differential-equations/output/print/wvwc-differential-equations.pdf`

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Chapter 1

Introduction to Ordinary Differential Equations

In the sciences, many quantities of interest are varying quantities. Some changing quantities, like the position of a baseball dropped from a height of 100 ft, are relatively simple to determine, but others are more complicated. Think of a wildlife population, which varies from season to season and year to year. How can one accurately estimate the size of such a population?

The field of **differential equations** attempts to answer such questions by providing a model for the quantity of interest using information about its rate of change, which is sometimes easier to obtain than information about the quantity itself. Once we know enough about how a quantity changes, we hope to make a decent estimate for how the quantity actually behaves. There are actually two different types of differential equations that we will look at in this course. The first is the ordinary differential equation.

1.1 Ordinary differential equations

This section introduces basic concepts from the field of differential equations.

Basic concepts

First, a definition of the primary concept in this course: the differential equation.

Definition 1.1.1 Differential equations.

A **differential equation** is an equation relating some function with its derivatives. A differential equation that involves a function of only one independent variable is called an **ordinary differential equation**, or ODE. A differential equation that involves a function of more than one independent variable (which you see a lot of in Calculus 3) is called a **partial differential equation**, or PDE, and will be studied in more detail in Chapter 8. The **order** of a differential equation is the highest derivative that appears in the equation.

Examples of ODEs:

- $\dfrac{d^2y}{dx^2} + y = 0$; this is a second order ODE relating the unknown function y with its second derivative.
- $5t^2x''' - e^x = 3t$; this is a seventh order ODE involving the derivatives of the unknown function x. Note that in this ODE t is the independent variable whereas x serves as the dependent variable.

Just as with equations in algebra, we can sometimes solve a differential equation.

Definition 1.1.2 Solution of a differential equation.

A function is a **solution** of a differential equation if it satisfies the differential equation.

It is straightforward to check if a function is a solution of some given differential equation, but *finding* solutions will make up the bulk of this course.

Example 1.1.3 Verifying solutions.

Is $y = 5e^{x^2}$ a solution of the ODE $y' = 2xy$?

Solution. At this point we don't know how to solve differential equations, but that doesn't mean we can't *check* solutions of differential equations. To do so, we just plug $5e^{x^2}$ wherever y shows up in the ODE and see if the resulting equation is true. So we have

$$\begin{aligned} y' &= 2xy \\ (5e^{x^2})' &= 2x(5e^{x^2}) \text{ after substituting } y = 5e^{x^2} \\ 10xe^{x^2} &= 10xe^{x^2} \text{ after simplifying} \end{aligned}$$

This is a true statement, so $y = 5e^{x^2}$ satisfies the ODE. Hence $5e^{x^2}$ is a solution of the ODE.

In Example 1.1.3, $y = 5e^{x^2}$ is not the only solution of $y' = 2xy$. You can check that $y = 3e^{x^2}$ and $y = -10e^{x^2}$ are also solutions. In fact, *any* function of the form Ce^{x^2} where C is a constant is a solution of $y' = 2xy$. See Figure 1.1.4.

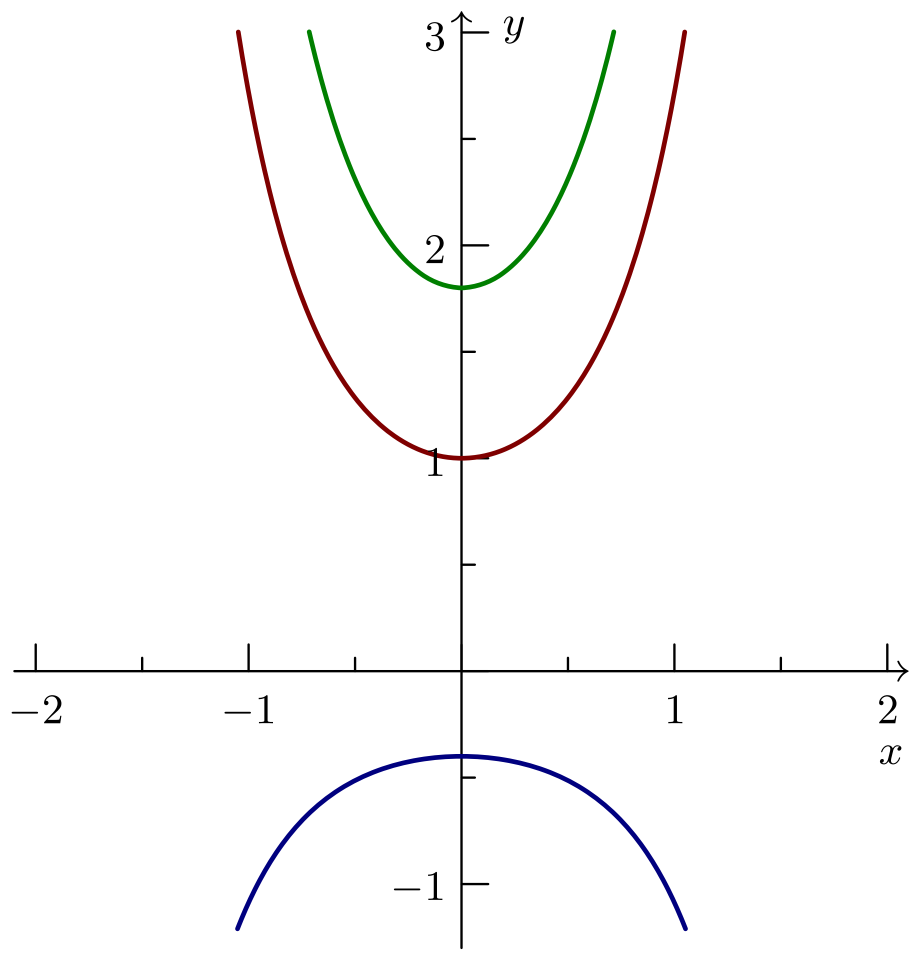

Figure 1.1.4 A family of solutions of $y' = 2xy$.

Solutions of ODEs that depend on arbitrary constants, such as $y = Ce^{x^2}$ above, are called **general solutions**. Solutions of ODEs that do not depend on arbitrary constants, such as $y = 5e^{x^2}$, are called **particular solutions**.

Example 1.1.5 A trigonometric solution.

Show that $x = A\cos(3t) + B\sin(3t)$, where A and B are arbitrary constants, is a general solution of $x'' + 9x = 0$. Then find a particular solution.

Solution. To check that $x = A\cos(3t)+B\sin(3t)$ is a general solution of $x'' + 9x = 0$, we just need to plug it into the ODE and show that it satisfies it. Since

$$x'' = -9(A\cos(3t) + B\sin(3t)) = -9x,$$

this follows very quickly.
To find a particular solution, all we need to do in this case is to pick specific values for A and B (any values will work here). So one particular solution of $x'' + 9x = 0$ is given by $x = \frac{3}{2}\cos(3t) - \sin(3t)$, among infinitely many others.

In Example 1.1.5, to find a particular solution we just needed to plug in specific values for the arbitrary constants. In general, the particular solutions we'll be interested in are chosen to satisfy a given condition, which we call an **initial condition**. These conditions often take the form $y(x_0) = y_0$. Geometrically, we are picking a specific point (x_0, y_0) in the xy-plane that the solution must pass through.

An ODE together with an initial condition is called an **initial value problem (IVP)**. Although ODEs by themselves typically have infinitely many solutions, specifying an initial condition that the solution must satisfy is often enough to get a unique particular solution instead of a general solution.

Example 1.1.6 Solving an IVP.

Solve the IVP $\dfrac{dy}{dx} = -xe^{-x^2}, y(0) = 1$.

Solution. We need to find a function y that satisfies two different constraints: $y' = -xe^{-x^2}$ and $y(0) = 1$. We'll start with the first one, which we actually know how to do from Calculus I. If $y' = -xe^{-x^2}$, then

$$y = \int -xe^{-x^2}\,dx = \frac{1}{2}e^{-x^2} + C.$$

Now we need to make sure that y is equal to 1 if $x = 0$. We do this by setting $y = 1$, $x = 0$ and choosing the right value for C to make the resulting equation true:

$$1 = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}.$$

So the solution of this IVP is the function $y = \frac{e^{-x^2}+1}{2}$.

Two important things to keep in mind before we move to the next topic:

1. ODEs by themselves have general solutions, whereas IVPs have particular solutions.

2. When solving IVPs, it's important to keep track of any arbitrary constants that appear. Neglecting arbitrary constants usually makes it impossible to find the right particular solution.

Mathematical models

Differential equations are useful because they can provide a mathematical model of a physical quantity. Analyzing the model allows us to infer something meaningful about the quantity in question. A relatively simple model comes from **Newton's Law of Cooling**, which relates the temperature of an object with the temperature of the surrounding medium (such as air or water). In particular, Newton's Law of Cooling states that the time rate of change of the temperature of an object is proportional to the difference of the temperature of the object with that of the temperature of the surrounding medium.

Example 1.1.7 Newton's Law of Cooling.

Restate Newton's Law of Cooling as a differential equation.

Solution. It may not be obvious that Newton's Law of Cooling can be restated as a differential equation, but the phrase "rate of change" that appears in the statement of the law is a good clue that this can be done.

To do this, first we need to give the relevant quantities (the temperature of the object and the temperature of the surrounding medium) names. Let $T(t)$ denote the temperature of the object at time t and let $A(t)$ denote the temperature of the surrounding medium at time t. Then Newton's Law of Cooling says that

$$\frac{dT}{dt} = k(T - A)$$

where k is some constant.
Although we can't determine k precisely (this would require experimentation and depends on the object and medium in question), we can still say something useful about it. In particular, k must be negative. To see why, consider what the object does if $T > A$ and $T < A$. If $T > A$, then the object must be cooling since the surrounding medium is cooler than the object. If the object is cooling, then $\frac{dT}{dt} < 0$. On the other hand, if $T < A$ then $\frac{dT}{dt} > 0$ since the object would be heating up in this case. The only way for this to occur is if $k < 0$.

Most of the mathematical models we'll look at will take the form of an IVP.

Example 1.1.8 An IVP modeling a falling object.

A ball weighing 0.5 kg is dropped from a height of 100 m and is acted upon by gravity and air resistance. Assuming that the force of air resistance is proportional to the velocity of the ball, what is an IVP that models the movement of the ball?

Solution. What we need to do is to translate this physical situation into mathematics, and to do that we need to start assigning names to the quantities of interest. The quantities that matter in this problem are the movement of the ball, the force of gravity and the force of air resistance. We'll name them as follows:

$$\begin{aligned} \text{height of the ball } t \text{ seconds after its released} &= y(t) \\ \text{force of gravity} &= F_G = mg = -4.9 \\ \text{force of air resistance} &= F_R = k\frac{dy}{dt} \end{aligned}$$

where C is negative (since air resistance should act *against* velocity). To get a differential equation out of all this, we'll use Newton's Second Law (which is actually a second-order ODE in disguise). This says that the net force on the ball should be equal to its mass times acceleration: $F = ma$. So we have

$$F_G + F_R = 0.5\frac{d^2y}{dt^2}$$

or just

$$-4.9 + k\frac{dy}{dt} = 0.5\frac{d^2y}{dt^2}.$$

The initial condition in this case would be $y(0) = 100$. So our IVP is

$$-4.9 + k\frac{dy}{dt} = 0.5\frac{d^2y}{dt^2}, y(0) = 100.$$

One quick note about the IVP in Example 1.1.8 The differential equation was second-order, but there was only one corresponding initial condition. As we'll see later, this is not enough to find a unique solution to this IVP. For most IVPs we'll solve, we'll need as many initial conditions as the order of the ODE. Something to look forward to.

Now that we have a rough idea of what an ODE and an IVP actually are, we can move on to solving them. In the next section, we'll look at a method that we can use to visualize ODEs and their solutions and another method that can be used to approximate solutions of ODEs.

1.2 Direction fields

Differential equations can often be quite difficult to solve, if not impossible. In this section we will discuss a method to visualize the behavior of certain first-order differential equations without first solving them. This approach, the *direction field* or *slope field*, is one that we will use again in Chapter 4.

Direction fields and solution curves

Suppose we wanted to solve the ODE $\frac{dy}{dx} = x$. Then we can do so using the tools already available to us, since the mystery function $y = y(x)$ must have derivative given by x. So

$$y = \int x\,dx = \frac{1}{2}x^2 + C.$$

Any choice for C yields another solution of the ODE $\frac{dy}{dx} = x$.

> Again, it's very important in this course not to forget the arbitrary constant C!

Now let's make things a little more interesting. Suppose we wanted to solve the ODE $\frac{dy}{dx} = e^{-x^2}$. Then this is *impossible* to do in a single "closed-form".

> By closed-form we basically mean in terms of the usual exponential, trigonometric and polynomial functions, as well as their inverses.

This is because solving this ODE requires integrating e^{-x^2}, which as you may remember from Calculus 2 cannot be done without resorting to something like power series. Even if we can't solve the ODE, or if we can't solve it easily, we still want to be able to obtain some information from it. One way to do this is by using **direction fields**, which is a graphical representation of the ODE.

To construct a direction field for an ODE of the form

$$\frac{dy}{dx} = f(x, y)$$

perform the following steps:

1. Pick a point (a, b) in the xy-plane.

2. Plug the point (a, b) into $f(x, y)$ to obtain the number $f(a, b)$.
3. Plot a short line segment with slope $f(a, b)$ at the point (a, b).
4. Repeat at several other points in the xy-plane until you develop a satisfactory picture of the behavior of the ODE.

The resulting graph is called the direction field for the ODE.

Direction fields are often called slope fields.

Example 1.2.1 Plotting a direction field by hand.

Fill in the direction field for $y' = t - y$ at the indicated points:

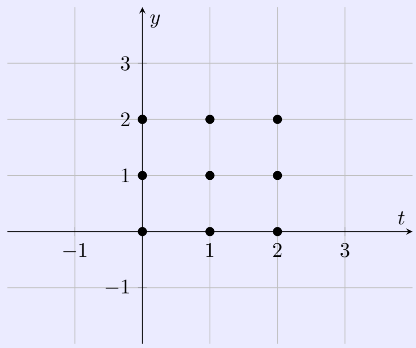

Figure 1.2.2 Graph for hand plot of direction field.

Solution. To plot the direction field, remember that we're basically plotting *slopes*. So we first need to figure out $y' = t - y$ at the indicated points. The following table lists values for y' at some of these points:

Table 1.2.3 Slopes of $t - y$

(t, y)	$y' = t - y$
$(0, 0)$	$0 - 0 = 0$
$(2, 2)$	$2 - 2 = 0$
$(1, 2)$	$1 - 2 = -1$
$(0, 1)$	$0 - 1 = -1$

If we fill out the remaining values of y' and plot the corresponding slopes, we should get something like this:

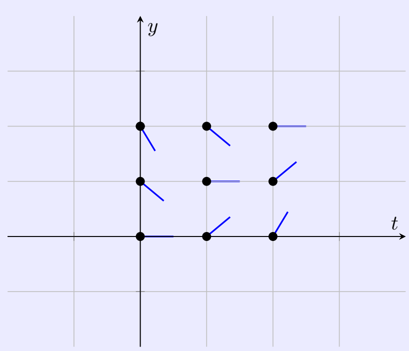

Figure 1.2.4 Direction field of $y' = t - y$

Example 1.2.5 Plotting a direction field with a CAS.

Plot the direction field for the differential equation $x' = x(1-x)$, where $x = x(t)$.

Solution. We can easily do this with a computer system (such as Sage!). For example, try running the cell below this example. If we do so, we get something like the following diagram:

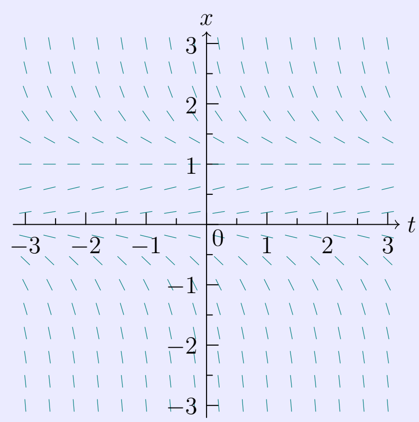

Figure 1.2.6 Direction field for $x' = x(1-x)$

```
var('t,␣x')    # tells Sage what variables we're using
f = x * (1 - x)
plot_slope_field(f, (t, -10, 10), (x, -5, 5), color = "blue")
```

Direction fields are useful because they provide a means to obtain information about a differential equation (and the corresponding model) without actually having to solve the differential equation. One way to do so is to create a *streamline plot*. This can be done easily in Sage, like so:

```
var('t,␣x')
f = x * (1 - x)
streamline_plot(f, (t, -5, 5), (x, -1, 2))
```

This can also be created by hand from a slope field without too much trouble.

If we only graph a single curve in the direction field we get what's known as a **solution curve**, which represents a solution of an initial value problem corresponding to the ODE the direction field is drawn from.

Example 1.2.7 Information from a solution curve.

Let $x(t)$ represent the solution of the initial value problem

$$x' = x(1-x)$$
$$x(0) = \frac{1}{2}.$$

Determine $\lim_{t\to\infty} x(t)$.

Solution. Since we don't know how to solve this IVP yet, we'll make use of the direction field from Example 1.2.5 to find an approximate solution curve. Since the initial condition is $x(0) = \frac{1}{2}$, this means that the solution must pass through the point $(0, \frac{1}{2})$. So if we start at this point and trace a curve that flows with the direction field, we get the following solution curve:

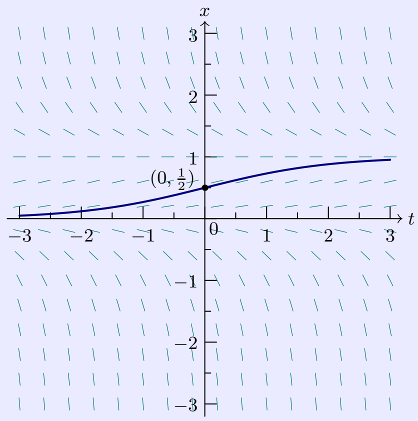

Figure 1.2.8 The solution curve corresponding to the initial condition $x(0) = \frac{1}{2}$

So it appears that $\lim_{t\to\infty} x(t) = 1$.

1.3 Separable ODEs and substitution

This section corresponds to Section 1.3 from the text.

Separable ODEs

The simplest ODEs to solve are the first-order ODEs of the form $\dfrac{dy}{dx} = f(x)$. The Fundamental Theorem of Calculus guarantees that the solution y is given by $y = \int f(x)\,dx$.

Rather, the Fundamental Theorem of Calculus guarantees that the solution will be $y = \int f(x)\,dx$ as long as f is a continuous function.

Another type of ODE that is relatively straightforward to solve is the **separable ODE**, which is a first-order ODE that can be written in the form

$$\frac{dy}{dx} = f(x)g(y).$$

These ODEs can be solved by integration as well, but only after some rearranging.

Example 1.3.1 Solving a separable ODE.

Solve the IVP given by $y' = x + 4xy^2, y(0) = 1$.

Solution. The first step to solving this IVP is to solve the ODE $y' = x + 4xy^2$. It may not look like it at first, but this ODE is separable since we can rewrite it as $y' = x(1 + 4y^2)$. To solve this ODE, we need to move the y terms to the left hand side of the equation and the x

terms to the right hand side. We'll abuse notation a little bit to do so by rewriting $y' = \frac{dy}{dx}$ and treating $\frac{dy}{dx}$ as a fraction, but it won't get us into too much trouble here:

$$\begin{aligned}\frac{dy}{dx} = x(1+4y^2) &\Rightarrow \frac{dy}{1+4y^2} = x\,dx \\ &\Rightarrow \int \frac{dy}{1+4y^2} = \int x\,dx \\ &\Rightarrow \frac{1}{2}\tan^{-1}(2y) = \frac{1}{2}x^2 + C \\ &\Rightarrow \tan^{-1}(2y) = x^2 + C_1.\end{aligned}$$

At this step we can either leave the solution as is (in **implicit form**) or solve for y to get an **explicit form**. We'll leave this in implicit form and then plug in the initial condition to get

$$\tan^{-1}(2) = C_1.$$

So the implicit solution of this IVP is given by

$$\tan^{-1}(2y) = x^2 + \tan^{-1}(2).$$

Example 1.3.2 Newton's Law of Cooling again.

A metal plate is removed from an oven and placed in a room. The temperature of the plate is 100° Celsius and the temperature of the room is fixed at 15° Celsius. After 20 minutes the temperature of the plate drops to 90° Celsius. How hot is the plate after five hours?

Solution. Let $T(t)$ denote the temperature of the plate t minutes after being removed from the oven and let $A(t)$ denote the temperature of the room t minutes after the plate is removed from the oven. Then $T(0) = 100$ and Newton's Law of Cooling says that

$$\frac{dT}{dt} = k(T - 15).$$

To answer this question we need to find T, and although we don't know k at the moment we can still make some progress just by remembering that it's a constant. This ODE is separable, so we'll separate variables and integrate both sides to get

$$\ln(T - 15) = kt + C$$

which simplifies to $T - 15 = C_1 e^{kt}$ or just $T = 15 + C_1 e^{kt}$.
Now we need to find C_1 and k. To find C_1, we just use the initial condition to get $C_1 = 85$. The only piece of information that we have left to find k is the fact that the temperature of the plate drops to 90 after 20 minutes. In other words, $T(20) = 90$. Therefore

$$90 = 15 + 85e^{20k}$$

which becomes

$$\ln\frac{75}{85} = 20k.$$

Therefore $k = \frac{1}{20} \ln \frac{15}{17} \approx -.006$.
So, finally, $T(t) = 15 + 85e^{-.006t}$ and the temperature after five hours is $T(300) \approx 28$.

Substitution methods

At this point we can only solve a couple types of differential equations. An ODE that isn't of the form $\dfrac{dy}{dx} = f(x)$ or separable may prove troublesome. However, there are certain cases where we can rewrite an ODE into one of these forms by using the right substitution.

Example 1.3.3 Substitution to solve an ODE.

Find the general solution of $y' = (x + y + 3)^2$.

Solution. This ODE is not separable and we can't just integrate it (since the right hand side depends on the dependent variable). However, the form of the right hand side suggests a substitution: $u = x + y + 3$. This would simplify things quite a bit, leaving us with $y' = u^2$. The only problem with this is that y' depends on x, not u. We must rewrite y' in terms of the new variable u, which isn't too bad. Since $u = x+y+3$, we get $y' = u' - 1$. Therefore the ODE becomes

$$u' - 1 = u^2 \text{ or just } \frac{du}{dx} = 1 + u^2.$$

This new ODE *is* separable, and so we separate variables and integrate to get $\tan^{-1} u = x + C$. If we don't care about finding an explicit solution, then we can just replace u to get the equation back in terms of y. So our (implicit) general solution is $\tan^{-1}(x + y + 3) = x + C$.

Example 1.3.4 A less obvious substitution.

Find an explicit solution of $xy' = x + y$.

Solution. It's tough to see what to do right away, so we'll try simplifying the ODE first. In particular, we'll solve for y' to get

$$y' = 1 + \frac{y}{x}.$$

If we stare at this for a while, we might convince ourselves that the right hand side really just depends on $\frac{y}{x}$, so we'll try replacing that with u. Then the ODE becomes $y' = 1 + u$.
Once again, this is much simpler but we need to rewrite y' in terms of u. Since $u = \frac{y}{x}$, this means that $y = ux$ and so $y' = u + u'x$. Then the ODE becomes

$$u + x\frac{du}{dx} = 1 + u \text{ or just } x\frac{du}{dx} = 1.$$

This new ODE can be rearranged to get $\dfrac{du}{dx} = \frac{1}{x}$, and so $u = \ln|x| + C$. Getting back in terms of y, we have $\frac{y}{x} = \ln|x| + C$ or just $y = x \ln|x| + Cx$.

1.4 First-order linear ODEs

In this section we introduce a new type of ODE that we can solve, in addition to separable ODEs and ”simple” ODEs of the form $y' = f(x)$. The ODEs that we'll consider in this section are **first-order linear ODEs**.

Definition 1.4.1 First-order linear ODEs.

A first-order ODE is said to be linear if it can be written in the following form:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

We've actually seen such an ODE all the way back in Example 1.1.8. The ODE that we came up with in that problem can be rewritten as a first-order linear ODE with the right substitution (say, $u = y'$). Our first goal in this section is then to figure out how to solve these ODEs.

Note that this section corresponds to Section 1.5 from the text.

Integrating factors

To get a sense of how to solve first-order linear ODEs, we'll try some relatively simple examples first.

Example 1.4.2 Solving a first-order linear ODE.

Find the general solution of $\dfrac{dy}{dx} + 2y = x$.

Solution. First, note that the ODE is indeed a first-order linear ODE since it takes the form given in Definition 1.4.1. If we stare at the ODE for a bit, we might think that the left hand side looks like something we'd get after using the product rule. Just compare $(fg)' = fg' + f'g$ with $\dfrac{dy}{dx} + 2y$, and it appears that the unknown function y is taking the place of g in the product rule formula. If we could just figure out what the function f is supposed to be, then we could drastically simplify the left hand side of the ODE.

Unfortunately, there is no such function f that works here. If there were, we'd have to have $f' = 2$ and $f = 2x$, and clearly we aren't multiplying $\dfrac{dy}{dx}$ by $2x$ in the ODE. But we can pull a dirty trick here! We'll multiply through the ODE by e^{2x} to get the new ODE

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = xe^{2x}.$$

It might not be all that obvious why this helps us out, but now the left hand side can be simplified by the product rule:

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = \frac{d}{dx}[e^{2x}y].$$

So we can rewrite the entire ODE as

$$\frac{d}{dx}[e^{2x}y] = xe^{2x}.$$

We can integrate this on both sides to get $e^{2x}y = \int xe^{2x}\,dx$, or just

$$e^{2x}y = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C.$$

The explicit solution would be $y = \frac{1}{2}x - \frac{1}{4} + Ce^{-2x}$.

The function e^{2x} that we used in Example 1.4.2 is called an **integrating factor**. Integrating factors are our primary tool in solving first-order linear ODEs. In general, to solve a first-order linear ODE $y' + P(x)y = Q(x)$ the first thing you must do is to multiply through it by the integrating factor $e^{\int P(x)\,dx}$.

Example 1.4.3 Solving a first-order linear ODE in disguise.

Solve the second-order ODE given by

$$tx'' + \frac{t}{1+t^2}x' = \frac{2t^2}{(1+t^2)^2}e^{-\tan^{-1} t}$$

with initial conditions $x(0) = x'(0) = -1$.

Solution. Even though this is a second-order ODE, we can rewrite it as a first-order ODE using the substitution $u = x'$. Then the ODE becomes

$$tu + \frac{t}{1+t^2}u = \frac{2t^2}{(1+t^2)^2}e^{-\tan^{-1} t}.$$

If we divide through by t, we get

$$u + \frac{1}{1+t^2}u = \frac{2t}{(1+t^2)^2}e^{-\tan^{-1} t}.$$

This can be solved by integrating factors since it takes the form given in Definition 1.4.1. The integrating factor we need is given by

$$e^{\int \frac{1}{1+t^2}\,dt} = e^{\tan^{-1} t}.$$

Now we multiply through the ODE by this integrating factor and rewrite the left hand side using the product rule to get

$$\frac{d}{dt}[e^{\tan^{-1} t}u] = \frac{2t}{(1+t^2)^2}.$$

At this step we can integrate both sides to get

$$e^{\tan^{-1} t}u = -\frac{1}{1+t^2} + C,$$

which becomes

$$x' = -\frac{e^{-\tan^{-1} t}}{1+t^2} + Ce^{-\tan^{-1} t}.$$

If we plug in the initial condition $x'(0) = -1$, this forces $C = 0$. Hence

$$x' = -\frac{e^{-\tan^{-1} t}}{1+t^2}.$$

Now we integrate one last time to get x:

$$x = e^{-\tan^{-1} t} + D.$$

If we use the last initial condition $x(0) = -1$, we see that $D = -2$. Hence the solution of this IVP is

$$x = e^{-\tan^{-1} t} - 2.$$

Applications

A common application of first-order linear ODEs is in modeling "mixture" problems. Suppose we have a tank which contains a solution (mixture of solute and solvent, such as salt and water). Some amount of solution is also flowing into and out of the tank. We want to measure the amount of solute in the tank at time t, call this amount $x(t)$. Then $x(t)$ will change depending on how the solute flows into and out of the tank, making it a prime target for a differential equation.

If we set

$$\begin{aligned} c_i &= \text{concentration of solute flowing in} \\ c_o &= \text{concentration of solute flowing out} \\ r_i &= \text{rate solution is flowing in} \\ r_o &= \text{rate solution is flowing out} \end{aligned}$$

then we can say that

$$\frac{dx}{dt} = r_i c_i - r_o c_o = r_i c_i - r_0 \frac{x(t)}{V(t)}$$

where $V(t)$ is the volume of solution in the tank at time t. Furthermore, if we let $V_0 = V(0)$ denote the initial volume of the solution in the tank then we can say that $V(t) = V_0 + (r_i - r_0)t$. Hence the amount of solute $x(t)$ obeys the first-order linear ODE

$$\frac{dx}{dt} = r_i c_i - \frac{r_o}{V_0 + (r_i - r_o)t} x.$$

We assume that c_i, r_i, r_o are all constant.

Example 1.4.4 Salt in a tank.

A tank contains 100 L of a solution consisting of 50 kg of salt dissolved in water. Solution containing $1\,\frac{\text{kg}}{\text{L}}$ of salt is pumped into the tank at a rate of $2\,\frac{\text{L}}{\text{min}}$ and the well-mixed solution is pumped out at the rate of $3\,\frac{\text{L}}{\text{min}}$. How much salt will be in the tank after t minutes?

Solution. Let $x(t)$ denote the amount of salt in the tank after t minutes, so $x(0) = 50$. Then

$$\frac{dx}{dt} = r_i c_i - r_o c_o = 2 \cdot 1 - 3\frac{x}{100 - t}.$$

We can rearrange this to get

$$\frac{dx}{dt} + \frac{3}{100 - t} x = 2.$$

This ODE is linear and has integrating factor $(100 - t)^{-3}$. Multiplying through the ODE by the integrating factor and rewriting it using the product rule then gives us

$$\frac{d}{dt}[(100 - t)^{-3} x] = 2(100 - t)^{-3}.$$

Now we can integrate both sides to get

$$(100 - t)^{-3} x = (100 - t)^{-2} + C$$

or just $x = 100 - t + C(100 - t)^3$. Finally, the initial condition can be used to show that $C = -\frac{1}{20000}$, so $x = 100 - t - \frac{1}{20000}(100 - t)^3$.

1.5 Existence and uniqueness of solutions

Now that we have seen how to solve several different types of differential equations, we will move on to the more general problem of determining when a given differential equation is actually solvable. This section is, therefore, a little unusual in that we don't actually care about solving the differential equations presented. Despite this, many important differential equations in practice have no "closed-form" solutions and so the best we can often do is approximate solutions. Hence, it will be useful to know when exactly a solution exists in order to justify our approximations. At the end of the section we will examine two approaches to approximating solutions of differential equations.

Existence and uniqueness for linear differential equations

Recall that a first-order linear ordinary differential equation is one that can be written in the form Definition 1.4.1. For such equations, we have the following relatively simple test for existence and uniqueness.

Theorem 1.5.1

Consider the initial value problem given by

$$y' + P(x)y = Q(x), y(x_0) = y_0$$

and let I be an open interval on which $P(x)$ and $Q(x)$ are both continuous, with $b \in I$. Then the initial value problem has a unique solution.

Existence and uniqueness theorem

There are two important questions we need to consider when developing a mathematical model using differential equations (i.e. IVPs):

1. Does the initial value problem have a solution? (Existence).
2. If it has a solution, is the solution unique? (Uniqueness).

Ideally, the answer to both of these questions will be yes.

Example 1.5.2 The answer is no.

Does $x\dfrac{dy}{dx} = \sqrt[3]{y}, y(1) = 0$ have a unique solution?

Solution. We can find a solution to this IVP by treating the ODE as separable. If we do so, we find that $y = \left(\frac{2}{3}\ln x\right)^{\frac{3}{2}}$. On the other hand, we can also eyeball a second solution: $y = 0$. So this IVP has *two* different solutions: $y_1 = \left(\frac{2}{3}\ln x\right)^{\frac{3}{2}}$ and $y_2 = 0$.

Clearly, IVPs don't always have unique solutions. Sometimes it's difficult to determine precisely when an IVP can have a unique solution, but most of the cases we'll care about in this class will fall under the following theorem.

Theorem 1.5.3 Existence and uniqueness theorem.

Consider the IVP given by $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$. If $f(x,y)$ is bounded and continuous within some rectangle in the plane containing (x_0,y_0), then the IVP has at least one solution. If in addition $f_y(x,y)$ is also bounded and continuous within some rectangle containing (x_0,y_0), then the IVP has a unique solution.

If we go back to Example 1.5.2, then we see that Theorem 1.5.3 has something to say about the IVP in that example. In that example, we had $f(x,y) = \sqrt[3]{y}$, $f_y(x,y) = \frac{1}{3}y^{-\frac{2}{3}}$ and $(x_0,y_0) = (1,0)$. Let's draw a rectangle around this point:

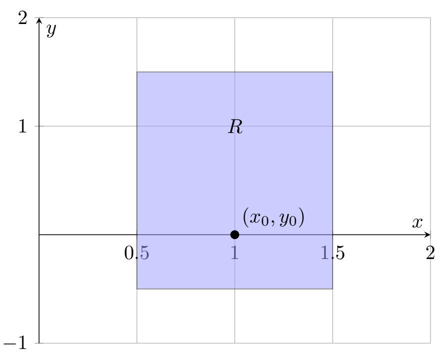

Figure 1.5.4 A rectangle around (x_0, y_0)

Now f is continuous and $-\frac{1}{2} \leq y \leq \frac{3}{2}$ within this rectangle, so

$$-\sqrt[3]{\frac{1}{2}} \leq f(x,y) \leq \sqrt[3]{\frac{3}{2}}$$

everywhere inside of this triangle. So Theorem 1.5.3 guarantees *at least one* solution of the IVP, and indeed there is at least one solution. However, the problem with uniqueness stems from the fact that f_y has a divide-by-zero problem inside this rectangle. Furthermore, it's impossible to draw a rectangle around $(1,0)$ that avoids this divide-by-zero problem. Hence there is no guarantee of uniqueness.

On the other hand, if we changed the initial condition to $y(1) = 0.000001$ then we would be guaranteed a unique solution. Moving that initial condition off of the x-axis is all we need to do to guarantee uniqueness.

Picard iteration

Now that we know of a way to determine whether or not certain ODEs have solutions, we'd like a method for actually finding these solutions. We've seen a few different methods for solving specific first-order ODEs, but what we'll do now is discuss a method that works for a large class of first-order ODEs. The only catch is that it may take us an infinite amount of time to get the solution.

Consider the IVP $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$. We can rewrite this differential equation as an **integral equation**:

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$

It looks quite a bit different, but solutions of this integral equation are also solutions of the corresponding differential equation. Our goal now is to approximate a solution to this integral equation.

To start, let's make a guess as to what the solution of our IVP should be. To keep things simple we'll start with a constant function, say $\phi_0(x) = y_0$ so that we at least satisfy the initial condition. Now this guess may not be a good match for the solution of the IVP away from the initial condition, so we'll adjust the guess using the integral equation to get the new function ϕ_1:

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0)\, dt.$$

Now ϕ_1 may not be a great approximation either away from the initial condition, but we can adjust it using the integral equation just like we did to ϕ_0.

The method described in the previous paragraph is **Picard's Method**. In general, the n^{th} iteration of Picard's Method is given by

$$\phi_n(x) = y_0 + \int_{x_0}^{x} f(t, \phi_{n-1}(t))\, dt,$$

and the first iterate is the constant function $\phi_0 = y_0$. It may seem strange to consider these functions as approximate solutions of the IVP in question, but each iterate actually solves an IVP very similar to the one that we care about, $\dfrac{dy}{dx} = f(x, y), y(x_0) = y_0$. In particular,

$$\frac{d\phi_n}{dx} = f(x, \phi_{n-1}), \phi_n(x_0) = y_0$$

for all $n \geq 1$. This method doesn't always work, but if $f(x, y)$ satisfies the conditions given in Theorem 1.5.3 then this method will (after potentially infinitely many steps) provide a solution to the IVP. Since the computations involved are quite tedious, it's best to use a CAS if possible.

Example 1.5.5 Using Sage to compute Picard iterates.

Consider the initial value problem given by

$$y' = x + y^2, y(1) = -1.$$

We want to approximate the solution by using Picard iteration (note that the differential equation is neither separable nor linear!). We do this with the Sage cell below:

```
# We need to tell Sage that y is a variable.
var('t,y')

# Now we define f(x,y) and our initial conditions.
f(x, y) = x + y^2
x0, y0 = 1, -1

# We will also decide how many Picard iterates we want.
# The below computes 3 iterates, but you're welcome
# to change n. However, things get very complicated
# very quickly for this example!
n = 3

# Define iterates.
yn(x) = y0
for k in range(n):
  yn(x) = y0 + integral(f(t, yn(x=t)), t, x0, x)

# Display n^th iterate.
show(yn(x))
```

```
1/4400*x^11 + 1/400*x^10 + 1/720*x^9 - 9/160*x^8 -
    1/56*x^7 + 77/100*x^6 - 191/200*x^5 - 399/80*x^4 +
    4741/240*x^3 - 497/16*x^2 + 10201/400*x -
    5517209/554400
```

Euler's method

The Picard iteration approach can be useful for finding series approximations of the solution of an ODE. If a more numerical approach is desired, then *Euler's method* might be useful. Euler's method can be thought of as an algorithmic version of tracing a solution curve through a direction field (see Section 1.2).

Algorithm 1.5.6 Euler's method.

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0.$$

The equations defining Euler's method for a given step size h are

$$\begin{aligned} x_n &= x_{n-1} + h \\ y_n &= y_{n-1} + hf(x_{n-1}, y_{n-1}). \end{aligned}$$

For Euler's method, the general rule of thumb is as follows: the smaller h is, the better the approximation. However, one should expect degrading performance as the method moves farther from the initial condition (x_0, y_0).

Example 1.5.7 Euler's method applied to a nonlinear ODE.

Let $y(x)$ denote the solution of the IVP given by

$$y' = x + y^2, y(0) = 1.$$

Estimate $y(2)$ using Euler's method with a step size of $h = \frac{1}{2}$.

Solution. First, note that we are justified in saying the solution exists at all by Theorem 1.5.3. Euler's method now produces the following approximations:

n	x_n	y_n
0	0	1
1	0.5	1.5
2	1	2.875
3	1.5	7.508
4	2	36.441

Such computations are best performed using a CAS, such as the Sage code below:

```
# Define y to be a variable in case this hasn't
# been done.
var('y')

# Define starting parameters and f(x,y).
xn, yn, h = 0, 1, 0.5
f(x,y) = x + y^2

# To get to x = 2 we need n = 4 steps.
n = 4

# Compute Euler's method approximations.
for k in range(n):
  xn, yn = xn + h, yn + h*f(xn, yn)

show((xn, yn))
```

```
(2.00000000000000, 36.4414367675781)
```

1.6 Population models and autonomous equations

Population models

Suppose we're monitoring the population of some species, and let's denote the population at time t by $P(t)$. An obvious question to consider is how that population will change over time. Mathematically, this means we want to obtain information on $\dfrac{dP}{dt}$ and then use it to estimate $P(t)$.

A simple model for $\dfrac{dP}{dt}$ is to assume it depends only on the birth rate β and death rate δ of the species in question. Then we can write

$$\frac{dP}{dt} = (\beta - \delta)P. \tag{1.1}$$

If we assume that β, δ are constants, then this equation is separable and we can solve it to obtain

$$P(t) = P_0 e^{(\beta-\delta)t},$$

where P_0 represents the "initial population", or population at time $t = 0$. We call (1.1) the **natural growth equation**.

The natural growth equation is simple, but it's probably too simple to be useful expect in certain scenarios (such as measuring half-life). To get a more flexible model, we can generalize (1.1) by assuming that the birth and death rates are actually functions of time. This gives us the **general population equation**.

Definition 1.6.1 General population equation.

The general population equation for a population $P(t)$ is given by

$$\frac{dP}{dt} = (\beta(t) - \delta(t))P.$$

Example 1.6.2 Population explosion.

A population has 100 members at time $t = 0$ years with a death rate of $.25P$ and a birth rate of $.5P$, where $P(t)$ denotes the population after t years. Find $P(t)$ and determine if this is a reasonable population model.

Solution. If we assume that the population obeys the general growth equation, then we get

$$P' = .25P^2, P(0) = 100.$$

This ODE is separable, and we can therefore solve it to get

$$P(t) = \frac{100}{1 - 25t}.$$

So we have a solution, and furthermore Theorem 1.5.3 guarantees that the solution is unique. But if you stare at this for a bit, you might see that it has a divide-by-zero problem. In particular,

$$\lim_{t \to (\frac{1}{25})^+} P(t) = \infty.$$

In other words, the population becomes infinite in about two weeks!

The logistic equation

Example 1.6.2 shows that we need to be more careful with our assumptions on population growth. One relatively simple assumption we can make is to assume that the birth rate $\beta(t)$ decreases as population P increases. This makes sense in the physical world as well: as population increases, existing and finite resources (such as food) must be shared between more and more members of the population. Since there's less to go around, we should expect growth to slow down. In particular, let's assume that

$$\beta(t) = \beta_0 - \beta_1 P$$
$$\delta(t) = \delta_0$$

where β_0, β_1 and δ_0 are all positive constants. Then the population equation for this scenario becomes

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta_0)P.$$

With a little algebra, we get the **logistic equation**:

$$\frac{dP}{dt} = kP(M-P)$$

for constants k and M. This equation is separable, and can be solved using partial fractions to obtain

$$P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}},$$

where $P_0 = P(0)$. In order to verify the reasonableness of our logistic model, let's see what happens to the population as time increases.

Example 1.6.3 Long-term behavior of logistic growth.

What is the long-term population of a species that grows according to the logistic equation $\frac{dP}{dt} = kP(M-P)$?

Solution. Using the fact that

$$P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}},$$

we have

$$\lim_{t\to\infty} P(t) = M.$$

So the population should eventually level out at M.

In the logistic equation $P' = kP(M-P)$, the value M is the **carrying capacity**, and denotes the maximum sustainable population according to the model.

Example 1.6.4 Population growth in the USA.

In millions, the population of the USA in 1990 was 250 and was growing at a rate of 3.1 per year. In 2012, the population was 314 and was growing at a rate of 2.3 per year. Assuming that the population of the USA grows logistically, estimate the population of the USA in 2017 and compare it to the current estimate of 325.7.

Solution. Let $P(t)$ denote the population of the USA (in millions), where t is the number of years after 1990. Then

$$\frac{dP}{dt} = kP(M-P)$$

and

$$P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}}.$$

So we need to find k and M.
When $t = 0$, we have $P' = 3.1$ and $P = 250$. Similarly, when $t = 22$ we have $P' = 2.3$ and $P = 314$. Therefore

$$3.1 = 250k(M-250)$$
$$2.3 = 314k(M-314)$$

Solving this system gives us $k \approx .00008$ and $M \approx 406.4$. Hence

$$P(t) = \frac{101600}{250 + 156.4e^{-.03t}}.$$

This model estimates the population in 2017 to be

$$P(27) = 317.9,$$

which is about a 2% error. Note also that this model predicts the carrying capacity of the USA to be 406.4.

Stability of solutions

The logistic equation

$$\frac{dP}{dt} = kP(M - P)$$

is a particularly nice separable ODE since the right hand side depends only on the unknown function P. So we can write $P' = f(P)$, where $f(P) = kP(M - P)$. ODEs like this (where the independent variable does not appear explicitly) are called **autonomous ODEs**.

Autonomous ODEs like $\frac{dx}{dt} = f(x)$ are useful because the behavior of their solutions can be determined *qualitatively*, without actually solving the ODE. This is done by looking for the constant solutions of the ODE, that is, solutions of the form $x = c$. For any such solution, we must have $f(c) = 0$ as well. These solutions (i.e., the solutions of $f(x) = 0$) are called the **critical points** or **equilibrium solutions** of the ODE. These solutions completely determine the long-term behavior of *every other solution*.

Example 1.6.5 Finding equilibrium solutions.

Find the equilibrium solutions of $\frac{dx}{dt} = -x^2 + 7x - 10$.

Solution. We need to solve the equation $-x^2 + 7x - 10 = 0$. Thankfully, we can factor this to get $(2-x)(x-5) = 0$, and so the equilibrium solutions are $x = 2, 5$.

Definition 1.6.6 Stability of solutions.

A critical point is **stable** if solutions that start "near" the point stay near it. A critical point is **unstable** if solutions that start "near" the point can diverge away from it.

Example 1.6.7 Determining the stability of solutions.

What are the stable critical points of $\frac{dx}{dt} = -x^2 + 7x - 10$?

Solution. We already know that the critical points are $x = 2, 5$. We can determine their stability by making use of a **phase diagram**, which is essentially a sign chart for $f(x) = -x^2 + 7x - 10$:

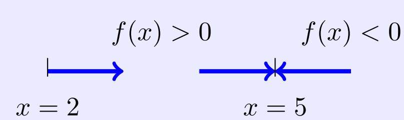

Figure 1.6.8 The phase diagram for $x' = f(x)$.

This shows us that solutions that begin near $x = 2$ tend to move away from $x = 2$, which solutions near $x = 5$ tend to move towards $x = 5$. So $x = 2$ is unstable and $x = 5$ is stable.

Example 1.6.9 Determining a sustainable population.

Consider a population of fish that obeys the logistic equation

$$\frac{dP}{dt} = 2P(30 - P)$$

where $P(t)$ is the population of fish (in thousands) after t years. Suppose that the population is also harvested at some rate h (in thousands per year). What is the maximum sustainable rate of harvesting?

Solution. To account for the harvesting, we need to modify the ODE:

$$\frac{dP}{dt} = 2P(30 - P) - h.$$

The harvesting will be sustainable as long as the population does not become extinct. To determine this long term behavior, we'll find the critical points and set up a phase diagram.
The critical points are given by

$$P = 15 \pm \sqrt{3600 - 8h}$$

by the quadratic formula. We now have three cases to consider: $3600 - 8h < 0, 3600 - 8h = 0, 3600 - 8h > 0$. In terms of h, these reduce to $h < 450, h = 450, h > 450$.

1. In the first case, if $h < 450$ then we have two positive, real critical points:
$$0 < c_1 < 15 < c_2 < 75.$$
The phase diagram for this situation is

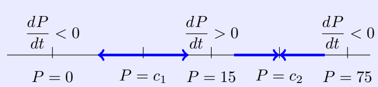

Figure 1.6.10 Phase diagram for $h < 450$

So we see that c_1 is unstable while c_2 is stable. In particular, as long as $P \geq c_1 = 15 - \sqrt{3600 - 8h}$, then the rate of harvesting is sustainable.

2. Now assume that $h = 450$. Then we have only one equilibrium solution: $c = 15$. We interpret the phase diagram in Figure 1.6.11 as follows: if P is less than $15,000$ then the population will collapse to extinction. Otherwise, the population will stabilize at $15,000$. This type of critical point is often called **semi-stable.**

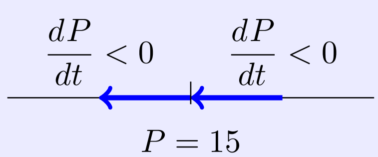

Figure 1.6.11 Phase diagram for $h = 450$

3. Finally, consider the case $h > 450$. Then we have no (real) critical points. Since imaginary populations don't make sense in this model, there is no sustainable population. No matter how large the initial population, it will eventually go extinct if harvested at a rate greater than 450.

By the above, the largest sustainable harvesting rate is $h = 450$, as long as $P_0 \geq 15$.

Linear stability analysis

Given the autonomous ODE $\frac{dx}{dt} = f(x)$, we saw above that we can qualify the behavior of equilibrium solutions by setting up a phase diagram. We can go a step further and actually qualify the growth of solutions that are "near" equilibrium solutions. In particular, we have the following theorem.

Theorem 1.6.12 Linear stability analysis.

Suppose $\frac{dx}{dt} = f(x)$ where $f(x)$ is continuously differentiable, and let x^* denote a critical point/equilibrium solution of the ODE. If $f'(x^*) < 0$, then x^* is stable and solutions near x^* will move exponentially towards x^*. If $f'(x^*) > 0$, then x^* is unstable and solutions near x^* will move exponentially away from x^*. If $f'(x^*) = 0$, then more advanced methods are required.

Example 1.6.13 Classifying the critical points of the logistic equation.

Classify the critical points of the logistic equation as stable or unstable.

Solution. Recall that the logistic equation is given by $P' = kP(M - P) = f(P)$ for (we'll assume) positive constants k, M. From here, we clearly see that the critical points are $P = 0$ and $P = M$ (which makes sense from a population standpoint!). We could set up a phase diagram to determine stability, but we'll use Theorem 1.6.12 instead.
Since $f'(P) = k(M - P) - kP$, we see that

$$\begin{gathered} f'(0) = kM > 0 \\ f'(M) = -kM < 0 \end{gathered}$$

Hence $P = 0$ is unstable, while $P = M$ is stable.

Chapter 2

Linear ODEs with Constant Coefficients

Now we move on to solving linear ODEs with constant coefficients. We'll start with solving second-order ODEs since we already know how to solve first-order linear ODEs.

2.1 Second-order linear ODEs

Recall that a second-order ODE is an ODE whose highest derivative is the second derivative. In this section, we'll look at how to solve second-order ODEs of a special type. Our method of solution here will be generalized to many, many other ODEs.

Types of linear ODEs

A first-order ODE is linear if it can be written in the form $y' + P(x)y = Q(x)$ (see Definition 1.4.1). We have a similar definition for second-order ODEs.

Definition 2.1.1 Second-order linear ODEs.

A second-order ODE is **linear** if it can be written in the form

$$y'' + P(x)y' + Q(x)y = R(x).$$

A second-order linear ODE is **homogeneous** if $R(x) = 0$ and **nonhomogeneous** if $R(x) \neq 0$.

Example 2.1.2 Types of second-order ODEs.

Consider the following ODEs:

1. $y'' = y' - \sin^2 x$ is linear but nonhomogeneous.
2. $\dfrac{d^2x}{dt^2} - x^3 = 0$ is nonlinear.
3. $\sqrt{x}e^{\tan^{-1} x}y'' = x^2 y$ is linear and homogeneous.

Newton's Second Law is a great source of linear second-order ODEs, as Example 2.1.4 shows. However, we will first need to state **Hooke's law**.

Theorem 2.1.3 Hooke's law.

Consider an object attached to a spring. The force exerted by the spring on the object is directly proportional to the displacement of the object from the spring's equilibrium, or at rest, position.

Example 2.1.4 A second-order model.

An object of mass 4 kg is attached to a horizontal, frictionless spring. Let $x = 0$ denote the equilibrium position of the spring and let $x(t)$ denote the displacement of the mass from the spring's equilibrium position. The only force acting on the mass is the force of the spring itself. Find a mathematical model for $x(t)$.

Solution. Let F_S denote the force of the spring on the mass. Then by Theorem 2.1.3 we must have $F_S = -kx$ for some $k > 0$. Now, by Newton's Second Law we must also have $F_S = 4x''$. Hence

$$4x'' = -kx \quad \text{or} \quad 4x'' + kx = 0.$$

So the motion of the mass is modeled by a linear, second-order homogeneous ODE.

The reason we take $F_S = -kx$ in Hooke's law is because we want to emphasize that the spring force is a *restoring force*, since it acts against the displacement of the mass.

The general trend that we will see for mathematical models using linear ODEs is that they will be homogeneous if we assume that there is no external force, and nonhomogeneous if we assume there is an external force.

Solutions of second-order linear ODEs

The reason we restrict ourselves to linear ODEs is because their solutions behave very nicely. In particular, we have the Theorem 2.1.6, which is an important principle for homogeneous ODEs. First, some terminology.

Definition 2.1.5 Linear combinations.

Let f and g denote functions of x. A **linear combination** of f and g is a function of the form $c_1 f + c_2 g$ where c_1, c_2 are constants.

Theorem 2.1.6 The superposition principle.

Let y_1 and y_2 denote two (possibly different) solutions of the ODE $y'' + P(x)y' + Q(x)y = 0$. Then any linear combination of these solutions is itself another solution of the same ODE.

Proof. We need to show that $y = c_1 y_1 + c_2 y_2$ is another solution of the same ODE, where c_1, c_2 are arbitrary constants. To do this, we'll just plug the linear combination into the ODE and simplify:

$$\begin{aligned} y'' + P(x)y' + Q(x)y &= (c_1y_1 + c_2y_2)'' + P(x)(c_1y_1 + c_2y_2) + Q(x)(c_1y_1 + c_2y_2)y \\ &= c_1y_1'' + c_2y_2'' + c_1P(x)y_1' + c_2P(x)y_2' + c_1Q(x)y_1 + c_2Q(x)y_2 \\ &= c_1(y_1'' + P(x)y_1' + Q(x)y_1) + c_2(y_2'' + P(x)y_2' + Q(x)y_2) \\ &= 0. \end{aligned}$$

■

Theorem 2.1.6 is important because it tells us how to construct new solutions of homogeneous ODEs out of known solutions. The next example demonstrates this.

Example 2.1.7 Solving a second-order IVP.

Using the fact that $y_1 = \cosh\frac{x}{2}$ and $y_2 = \sinh\frac{x}{2}$ are both solutions of $4y'' - y = 0$, solve the IVP given by

$$4y'' - y = 0 \quad \text{and} \quad y(0) = 3, y'(0) = 1.$$

Solution. Note that we have *two* initial conditions. In general, we'll need as many initial conditions as the order of the ODE.
To solve the IVP, we'll use the superposition principle to give us as much leeway as possible in constructing a solution of out y_1 and y_2. So we'll guess the solution takes the form $y = c_1\cosh\frac{x}{2} + c_2\sinh\frac{x}{2}$. By the superposition principle we're guaranteed that this is a solution of the ODE $4y'' - y = 0$, so we just need to pick the constants c_1, c_2 in order to satisfy the initial conditions. Let's start with the first initial condition, $y(0) = 3$. This gives us the equation

$$3 = c_1\cosh 0 + c_2\sinh 0 = 3.$$

So $c_1 = 3$. Similarly, $y'(0) = 1$ gives us

$$1 = \frac{3}{2}\sinh 0 + \frac{c_2}{2}\cosh 0 = \frac{c_2}{2}.$$

Hence $c_2 = 2$, and the solution of the IVP is

$$y = 3\cosh\frac{x}{2} + 2\sinh\frac{x}{2}.$$

The reason we were able to solve the IVP in Example 2.1.7 was because the individual solutions $\cosh\frac{x}{2}$ and $\sinh\frac{x}{2}$ gave us enough to build the particular solution of the IVP. It turns out that *every* solution of $4y'' - y = 0$ can be written as a linear combination of these two functions, so knowing these two functions is enough to solve every IVP involving this ODE.

In general, our goal will be to describe a "basis" of solutions for a given homogeneous ODE, a finite set of solutions that can be used to describe all possible solutions. This can be done if the functions in the basis aren't too "similar", in the following sense.

Definition 2.1.8 Linear independence of functions.

Two functions f and g are said to be **linearly independent** on an interval I if the linear combination $c_1f + c_2g$ is equal to 0 if and only if

$c_1 = c_2 = 0$. Otherwise, we say that they are **linearly dependent**.

The main idea behind Definition 2.1.8 is that f and g are linearly independent if they are not like terms (i.e. they don't cancel each other out). Note that f, g are linearly dependent if and only if $f = \alpha g$ for some $\alpha \neq 0$. Equivalently, they are linearly dependent if and only if $\frac{f}{g} = \alpha$ for some $\alpha \neq 0$.

Example 2.1.9 Linear independence of sine and cosine.

Show that $\sin x$ and $\cos x$ are linearly independent.

Solution. Since $\frac{\sin x}{\cos x} = \tan x$ is *not* constant, this means that $\sin x$ and $\cos x$ must be linearly independent.

Although it wasn't too hard to see that the functions in Example 2.1.9 were linearly independent, in other cases it can be much trickier (especially when we move to higher order ODEs). For example, suppose we set $f(t) = -5(t-15)^2 + 450$ and $g(t) = 12t(30-t)$. Then it's not obvious at all that these two functions are linearly dependent! In fact,

$$\frac{2}{5}f(t) - \frac{1}{6}g(t) = 0.$$

So generally when we try to determine if two functions are linearly independent, we'll make use of the **Wronskian**.

Definition 2.1.10 Wronskian of two functions.

Let f and g denote two differentiable functions. The Wronskian of f and g, denoted $W(f, g)$, is given by

$$W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g.$$

Theorem 2.1.11 Linear independence and the Wronskian.

Let f and g be two functions that are differentiable on some interval I. Then f and g are linearly independent on I if $W(f, g) \neq 0$ somewhere I. Conversely, if $W(f, g) = 0$ and f and g can both be represented by power series on I, then f and g are linearly dependent.

Example 2.1.12 Using the Wronskian.

If we let $f(t) = -5(t-15)^2 + 450$ and $g(t) = 12t(30-t)$ and compute their Wronskian, we obtain (after a fair bit of algebra) $W(f, g) = 0$. Since f and g can clearly be represented by power series on any interval I (since, being polynomials, they already *are* power series), this means that the two functions are linearly independent.

We mentioned earlier that we'll try to find a basis of solutions for linear ODEs. Now that we have the concept of linear independence and the Wronskian for checking if two functions are linearly independent, we can make the following definition.

Definition 2.1.13 Basis of solutions.

Let y_1 and y_2 denote solutions of some second-order linear homogeneous ODE. We call $\{y_1, y_2\}$ a **basis** if y_1 and y_2 are also linearly independent.

Once we have a basis for a second-order linear homogeneous ODE, we can solve any IVP that we wish involving that ODE. In particular, if $\{y_1, y_2\}$ is a basis for such an ODE, then *every* solution of the ODE can be written as a linear combination of y_1 and y_2.

Example 2.1.14 Linear independence using the Wronskian.

Let

$$x_1 = t\cos\ln t \quad \text{and} \quad t\sin\ln t.$$

Given that these functions are both solutions of

$$t^2x'' - tx' + 2x = 0,$$

solve the corresponding IVP with initial conditions $x(1) = 1, x'(1) = -1$.

Solution. We need to start by showing that $\{x_1, x_2\}$ is a basis for the ODE. First, we compute $W(x_1, x_2)$:

$$W(x_1,x_2) = (t\cos\ln t)(\sin\ln t+\cos\ln t)-(t\sin\ln t)(\cos\ln t-\sin\ln t) = t(\cos^2\ln t+\sin^2\ln t) = t.$$

So $W(x_1, x_2) = t$, which is clearly nonzero on the interval $(0, \infty)$. Hence x_1 and x_2 are linearly independent, and therefore $\{x_1, x_2\}$ is a basis for the ODE.
To actually find the solution (call it x), we'll set $x = c_1x_1 + c_2x_2$ and use the initial conditions to find c_1 and c_2. Doing so gives $c_1 = 1$ and $c_2 = -2$, and so the solution of the IVP is

$$x = \cos\ln t - 2\sin\ln t.$$

2.2 Homogeneous ODEs with constant coefficients

Now we'll move on to solving homogeneous ODEs, at least with constant coefficients.

Example 2.2.1 Solving a homogeneous ODE.

Suppose we wanted to solve $y'' - y' - 6y = 0$, where $y = y(x)$. If we stare at this for a bit, we may realize the following: the only way for a function y to be a solution of this ODE is for y and its derivatives y', y'' to cancel each other out. In other words, y and its derivatives *should be like terms*. This is a huge hint that y should look like an exponential function. So we'll guess that $y = e^{rx}$ for some real number r, and see if we can't pick r in just the right way to get a solution to the ODE. If we plug e^{rx} into the ODE, we get

$$\begin{aligned} y'' - y' - 6y &= r^2e^{rx} - re^{rx} - 6e^{rx} \\ &= e^{rx}(r^2 - r - 6). \end{aligned}$$

So we need to set $e^{rx}(r^2 - r - 6)$ equal to 0 and solve for r, which gives

$$r = -2, 3.$$

Therefore two solutions of $y'' - y' - 6y = 0$ are given by

$$y_1 = e^{-2x}$$

and $y_2 = e^{3x}$. Since $W(y_1, y_2) \neq 0$, this means that $c_1 e^{-2x} + c_2 e^{3x}$ is actually the general solution of the ODE.

The process in Example 2.2.1 will work for *every* second-order homogeneous ODE with constant coefficients. So solving such an ODE is even easier than integrating: all we need to do is to find the roots of a particular polynomial.

Definition 2.2.2 The characteristic polynomial.

Let a, b and c be constants. Given the ODE $ay'' + by' + cy = 0$, the **characteristic equation** of this ODE is the polynomial

$$ar^2 + br + c = 0.$$

We can now state the following result for finding solutions of homogeneous ODEs with constant coefficients.

Theorem 2.2.3 Characteristic equations with distinct roots.

Let a, b and c be constants. Suppose that the characteristic equation of $ay'' + by' + cy = 0$ has distinct roots r_1 and r_2. Then the general solution of the ODE is given by

$$c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

Example 2.2.4 Spring-mass system revisited.

An object of mass 4 kg is attached to a horizontal, frictionless spring. Suppose the spring constant is given by $k = 16$. The mass is held 3 m to the right of the spring's equilibrium position, and is then released at time $t = 0$ where t is in seconds. Find the displacement $x(t)$ of the mass.

Solution. We know from Example 2.1.4 that the second-order ODE given by

$$4x'' + kx = 0 \quad \text{or} \quad x'' + 4x = 0$$

provides a model for $x(t)$, but now we are in a position to solve it. The characteristic equation of this ODE is $r^2 + 4 = 0$, which has roots $r = \pm 2i$. The imaginary roots *are not a problem*, and in fact provide significant information about the motion of the mass, as we'll soon see. The general solution of the ODE is

$$x = c_1 e^{-2it} + c_2 e^{2it}.$$

The initial conditions are $x(0) = 3$ and $x'(0) = 0$, which give the equations

$$3 = c_1 + c_2$$

$$0 = -2ic_1 + 2ic_2.$$

The second equation implies that $c_1 = c_2$, and applying this to the first equation now gives $c_1 = c_2 = \frac{3}{2}$. Hence the displacement of the mass is given by

$$x = \frac{3}{2}e^{-2it} + \frac{3}{2}e^{2it}.$$

The appearance of the "imaginary" solution in Example 2.2.4 may seem strange, but they're actually quite natural. In fact, we can use **Euler's formula** to write the solution in terms of a more familiar function.

Theorem 2.2.5 Euler's formula.

For all x, the following equations hold:

$$\begin{aligned} e^{ix} &= \cos x + i \sin x \\ \cos x &= \frac{e^{ix} + e^{-ix}}{2} \\ \sin x &= \frac{e^{ix} - e^{-ix}}{2i}. \end{aligned}$$

So using Euler's formula on the solution from Example 2.2.4 gives

$$\begin{aligned} x &= \frac{3}{2}e^{-2it} + \frac{3}{2}e^{2it} \\ &= 3\frac{e^{-2it} + e^{2it}}{2} \\ &= 3\cos 2t. \end{aligned}$$

So the imaginary roots $\pm 2i$ from Example 2.2.4 actually relate to the *frequency* of the spring-mass system in that problem. This is a trend we will see often in this course: imaginary numbers corresponding to oscillating quantities.

Now we'll take a look at how to solve second-order homogeneous ODEs whose characteristic equations have repeated roots. This can only happen if the characteristic equation takes the form $(r - r_1)^2 = 0$, which means that the original ODE still has $y = e^{r_1 x}$ as one solution. To get the general solution we need a second, linearly independent, solution. We can guess that this second solution must also involve the exponential $e^{r_1 x}$, but it can't be a scalar multiple since we need a linearly independent solution. Therefore, we will try $xe^{r_1 x}$ as our second solution.

Example 2.2.6 Repeated roots in the characteristic equation.

Find the general solution of $y'' + 2y' + y = 0$ where $y = y(x)$.

Solution. We begin by solving the characteristic equation, which for this ODE is

$$r^2 + 2r + 1 = 0.$$

The only solution of this equation is $r = -1$, which is a repeated root. We can still get one solution of the ODE using this root, namely $y_1 = e^{-x}$, but we need two linearly independent solutions in order to find the general solution. We'll guess (and check!) that $y_2 = xe^{-x}$ is another solution of $y'' + 2y' + y = 0$. If we plug y_2 into the ODE, we get

$$\begin{aligned} y_2'' + 2y_2' + y_2 &= \underbrace{-2e^{-x} + xe^{-x}}_{y_2''} + \underbrace{2e^{-x} - 2xe^{-x}}_{y_2'} + \underbrace{xe^{-x}}_{y_2} \\ &= 0 \end{aligned}$$

which shows that y_2 is indeed a solution of the ODE. Since $\{y_1, y_2\}$ is a linearly independent set of solutions (which we can check via the Wronskian), this means that the general solution of $y'' + 2y' + y = 0$ is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{-x} + c_2 xe^{-x}.$$

The method used in Example 2.2.6 also works for other homogeneous ODEs with constant coefficients whose characteristic equations have repeated roots. Hence the roots of the characteristic equation *completely* determine the general solutions of such ODEs. We summarize this in the following table.

Table 2.2.7 Types of solutions from roots

Roots	General solution
$r_1 \neq r_2$	$c_1 e^{r_1 x} + c_2 e^{r_2 x}$
$r_1 = r_2$	$c_1 e^{r_1 x} + c_2 x e^{r_1 x}$

Remember that it's not a problem if the characteristic equation has imaginary roots, and in fact we must account for these in order to completely describe the corresponding physical system. If we have imaginary roots, then we can simply use Euler's formula to rewrite the solutions in terms of sine and cosine.

2.3 Spring-mass models

In this section we examine a common application of second-order ODEs: modeling movement in a spring-mass system. We will look at two different types of motion: undamped and damped. Damped motion will result from forces internal to the system and lead to homogeneous ODEs. External forces lead to non-homogeneous models, which we will consider in Section 2.5.

Free undamped motion

Suppose we have a mass m attached to a spring as in the following diagram:

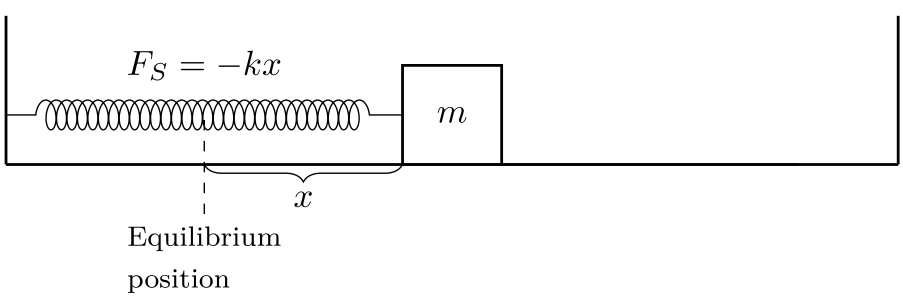

Figure 2.3.1 A spring-mass system

If we let $x(t)$ denote the displacement of the mass from the spring's equilibrium position and let $F_S = -kx$ denote the force of the spring on the mass (see Hooke's law), and assume that no other force is acting on the mass, then we know from Example 2.1.4 that $x(t)$ satisfies the second-order ODE given by

$$mx'' + kx = 0.$$

Now set $\omega_0 = \sqrt{\frac{k}{m}}$ so that we can rewrite this to get

$$x'' + \omega_0^2 x = 0.$$

One thing we can notice right away about solutions of $x'' + \omega_0^2 x = 0$ is that they should all be periodic (see Definition 7.1.2), which, of course, makes sense! To see why, note that the roots of the characteristic equation are $\pm i\omega_0$, which means that the solutions may be written in the form

$$x = c_1 e^{-i\omega_0 t} + c_2 e^{i\omega_0 t},$$

which we can rewrite as

$$x = A\cos\omega_0 t + B\sin\omega_0 t$$

using Euler's formula. Hence every solution of $x'' + \omega_0^2 x = 0$ is a sinusoidal wave.

We can go even further by making use of some clever algebra and the appropriate trigonometric formulas if we assume that $x \neq 0$ (i.e. A and B are not *both* 0). In this case, we may write

$$\begin{aligned} x &= A\cos\omega_0 t + B\sin\omega_0 t \\ &= \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos\omega_0 t + \frac{B}{\sqrt{A^2+B^2}}\sin\omega_0 t\right). \end{aligned}$$

Now set $C = \sqrt{A^2+B^2}$ and define φ implicitly by $\cos\varphi = \frac{A}{C}$ and $\sin\varphi = \frac{B}{C}$. Then we get

$$x = C(\cos\varphi\cos\omega_0 t + \sin\varphi\sin\omega_0 t) = C\cos(\omega_0 t - \varphi).$$

C is the **amplitude** of the wave x, ω_0 is the **circular frequency**, $T = \frac{2\pi}{\omega_0}$ is the **period** of motion and φ is the **phase**.

There are infinitely many choices for φ here, but we can select a unique one if we make the extra restriction that $\varphi \in [0, 2\pi)$. Computer systems often include a function named `atan2` to handle the computation of φ, although this usually returns an angle in $(-\pi, \pi)$.

Example 2.3.2 Another spring-mass system.

An object with a mass of 5 kg is fixed to a spring and a force of 10 N holds the mass 5 m to the left of the spring's equilibrium position. If the object is released, how long will it take for the mass to return to its original position? And what is the position $x(t)$ of the mass?

Solution. Let $x(t)$ denote the position of the mass t seconds after being released, so that $x(0) = -5$ and $x'(0) = 0$. We could find the time it takes for the mass to return to its starting position by first finding $x(t)$, but a quicker way is to just find the period T of x. To do this, we must find the circular frequency $\omega_0 = \sqrt{\frac{k}{m}}$. Thankfully, half

of the work is done for us (we know $m = 5$), so we only need the value of k which itself comes from Hooke's law.
Call the spring force F_S and recall that $F_S = -kx$. We know that it takes a force of 10 N to hold the mass still at $x = -5$. The spring force must *precisely* counterbalance this force in order for the mass to remain still as it's held, which means that $F_S = 10$ since F_S pulls the mass to the right. Therefore $-k(-5) = 10$ and so $k = 2$, which gives $\omega_0 = \sqrt{\frac{2}{5}}$. This means that the period of motion is

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{5}{2}}.$$

To find $x(t)$, we'll use the fact that it can be written as $x = C\cos(\omega_0 t - \varphi)$. Since $C = 5$ (because the mass can never go more than five meters from the equilibrium position) and we already know that $\omega_0 = \sqrt{\frac{2}{5}}$, we just need to find φ. We can just make use of the initial condition $x(0) = -5$ to get this:

$$-5 = C\cos\varphi = 5\cos\varphi$$

and so we can choose $\varphi = \pi$. Hence

$$x(t) = 5\cos(\sqrt{\frac{2}{5}}t - \pi).$$

You may be troubled by the fact that we only explicitly used the first initial condition in this example. However, the second condition $x'(0) = 0$ was actually used implicitly: it allowed us to assume that the amplitude was 5 as opposed to another number. If $x'(0) \neq 0$, finding C would have been a little trickier.

Free damped motion

Now we look at how the motion of a mass attached to a spring is altered if the motion is *damped*, say, by a dashpot. See Figure 2.3.3.

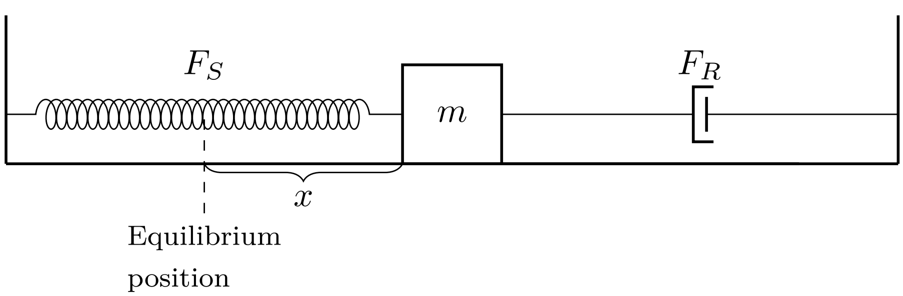

Figure 2.3.3 A damped spring-mass system

Now in addition to the spring force F_S, we must also worry about the force F_R of the dashpot on the mass. F_R is always going to act against the velocity of the mass, so for simplicity we assume that $F_R = -cx'$ for some $c > 0$. Using $F_S = -kx$ as usual in conjunction with Newton's Second Law gives us the second-order ODE

$$-cx' - kx = mx'' \quad \text{or} \quad mx'' + cx + kx = 0.$$

This ODE is homogeneous and has constant coefficients so we may solve it using the method of characteristic equations. The characteristic equation of this ODE is

$$mr^2 + cr + k = 0 \tag{2.1}$$

The roots of this equation are

$$r_1 = \frac{-c + \sqrt{c^2 - 4km}}{2m} \quad \text{and} \quad r_2 = \frac{-c - \sqrt{c^2 - 4km}}{2m}. \tag{2.2}$$

The behavior of this system therefore depends on the quantity $c^2 - 4km$, and so we have three cases to consider:

1: $c^2 - 4km > 0$.

In this case, the characteristic equation (2.1) has two real roots, and so the solution $x(t)$ has the form

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Now, r_1 and r_2 are both negative since $c^2 - 4km < c^2$ (remember that we're assuming that k and m are positive!). This means that $x(t) \to 0$ as $t \to \infty$. There is no oscillation present in the motion of the mass in this case since the mass never passes through $x = 0$, so we call this **overdamped motion**.

2: $c^2 - 4km = 0$.

In this case, the characteristic equation (2.1) has a repeated (real) root, and so the solution $x(t)$ takes the form

$$x(t) = e^{r_1 t}(c_1 + c_2 t).$$

This mass can pass through $x = 0$ only once, at $t = -\frac{c_1}{c_2}$. Once it does, the mass will "turn around" soon afterwards and beginning moving back to 0, as in the first case. We call this type of motion **critically damped**, since it's right on the border between overdamped motion and oscillating motion.

3: $c^2 - 4km < 0$.

In this case the characteristic equation (2.1) has two distinct complex roots of the form $r = a \pm bi$, where

$$a = -\frac{c}{2m} \quad \text{and} \quad b = \frac{\sqrt{4km - c^2}}{2m}.$$

In particular, the real part a of these roots is always negative. The solution $x(t)$ in this case takes the form

$$x(t) = e^{at}(A \cos bt + B \sin bt)$$

after applying Euler's formula. As in the previous two cases, $x(t) \to 0$ as $t \to \infty$. However, oscillation is now present in the system for all time! We call this motion **underdamped**, since the damping term e^{at} is not strong enough to cancel out the oscillation present in the system. Note that the real part a contributes to the "amplitude" e^{at} of the motion, while the imaginary part b represents the angular frequency of the motion. The ordinary frequency of the motion is given by $\frac{b}{2\pi}$.

Example 2.3.4 A spring-dashpot system.

Suppose that an object of mass $m = 25$ is attached to both a spring and a dashpot. The mass is held 1 meter to the left of the spring's equilibrium position $x = 0$. The force of the spring on the mass is $F_S = -226x$ and the force of the dashpot on the mass is $F_R = -10x'$, where x is the displacement of the mass. At time $t = 0$, the mass is released. Find $x(t)$.

Solution. The ODE that models the motion of this mass is

$$25x'' + 10x' + 226x = 0,$$

and the roots of the corresponding characteristic equation are

$$r_1 = -\frac{1}{5} \pm \frac{1}{5}\sqrt{-225} = -\frac{1}{5} \pm 3i.$$

We can already see that the motion must be underdamped since we have complex roots, and the position $x(t)$ itself is given by

$$x(t) = e^{-\frac{t}{5}}(A\cos 3t + B\sin 3t).$$

Now we can use the initial conditions $x(0) = -1$ and $x'(0) = 0$ to find A and B. Doing so, we quickly get $A = -1$ and $B = -\frac{1}{15}$. Hence

$$x(t) = -e^{-\frac{t}{5}}\left(\cos 3t + \frac{1}{15}\sin 3t\right).$$

As mentioned previously, the exponential term in $x(t)$ from Example 2.3.4 serves to dampen the motion of the spring. This is illustrated in Figure 2.3.5.

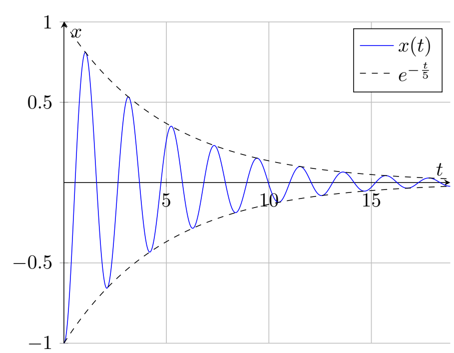

Figure 2.3.5 An exponential term damping motion

2.4 Solutions of non-homogeneous equations

In this section we'll deal with non-homogeneous equations and finding their solutions. This will help us to model systems involving an external force.

This section corresponds to Section 2.7 of the text.

The method of undetermined coefficients

Consider the non-homogeneous linear ODE with constant coefficients given by

$$ay'' + by' + cy = f(x). \tag{2.3}$$

If $f(x)$ were zero then we could solve this by finding the roots of the characteristic equations and using them to determine the appropriate form of the solution. Although that's no longer enough if $f(x) \neq 0$, our method for solving homogeneous equations still plays an important role.

Theorem 2.4.1 Solution of non-homogeneous equations.

Consider the ODE given in (2.3). Let y_c (the ***complementary solution****) denote the solution of the* ***associated homogeneous equation***

$$ay'' + by' + cy = 0$$

and let y_p (the ***particular solution****) denote a single solution of (2.3). Then the general solution of (2.3) is given by $y = y_c + y_p$.*

Theorem 2.4.1 shows that to solve (2.3), we only need to solve the associated homogeneous equation (which we're quite used to by now!) and find a *single* solution of (2.3). The method we will use is the **method of undetermined coefficients**, which we'll demonstrate by example.

Example 2.4.2 Using the method of undetermined coefficients.

Find the general solution of the ODE given by

$$y'' - y' - 2y = 3x + 4.$$

Solution. To find the general solution we need to do two things: find the complementary solution y_c of the associated homogeneous equation and find a single particular solution y_p of the given ODE. We already know how to find y_c, which is just

$$y_c = c_1 e^{x} + c_2 e^{-2x}.$$

Once we can find a single particular solution y_p we'll be finished.

> The characteristic equation of the ODE is $(r + 1)(r - 2)$.

If we stare that the ODE, then we see that y and its derivatives must cancel each other and leave a polynomial. So it's reasonable to guess that a polynomial might be a solution of the ODE, or equivalently, y_p *should be a polynomial.* Since the degree of the right hand side is 1, then y_p should probably be degree 1 as well. This means that $y_p = A_1 x + A_2$ for some constants A_1, A_2.

> Recall that the degree of a polynomial is just the highest power of the variable in the polynomial.

To find these constants (**undetermined coefficients**), we plug our guess into the ODE to get

$$0 - A_1 - 2A_1 x - 2A_2 = 3x + 4.$$

The only way for this equation to be true is for

$$\begin{aligned} -A_1 - 2A_2 &= 4 \\ -2A_1 &= 3. \end{aligned}$$

So $A_1 = -\frac{3}{2}$, $A_2 = \frac{5}{4}$ and $y_p = -\frac{3}{2}x + \frac{5}{4}$. Hence the general solution of the ODE is

$$y = y_c + y_p = c_1 e^x + c_2 e^{-2x} - \frac{3}{2}x + \frac{5}{4}.$$

Note that in Example 2.4.2, we didn't need any initial conditions to find y_p. This means that if a non-homogeneous ODE like the one in Example 2.4.2 represents some physical system, then the initial configuration of that system *has no effect on* y_p. We will see soon that particular solutions correspond to external forces on a system, like gravity, whereas complementary solutions correspond to internal forces in a system, such as the spring force.

Example 2.4.3

Find the general solution of

$$y'' - 2y' + y = x - \sin(2x).$$

Solution. The general solution will take the form $y = y_c + y_p$. Once again, we find y_c by solving the characteristic equation to get

$$y_c = c_1 x + c_2 x e^x.$$

Now we can make a guess as to what y_p should be, once again based on the right hand side of the ODE. If we want to differentiate y_p and obtain $x - \sin(2x)$, then y_p should include both an x term and a $\sin(2x)$ term. If we make the guess that $y_p = A_1 x + A_2 \sin(2x)$, then we get

$$\begin{aligned} A_1 x - 2A_1 &= x \\ -4A_2 \cos(2x) - 3A_2 \sin(2x) &= -\sin(2x) \end{aligned}$$

This forces $A_1 = 1$ *and* $A_1 = 0$, as well as $A_2 = \frac{1}{3}$ *and* $A_2 = 0$. Obviously, this is a problem!
What happened here is we didn't give our guess for y_p enough flexibility. We know we want y_p to involve x and $\sin(2x)$, but as soon as we plug this into the ODE and start differentiating constant terms and cosine terms will begin to appear, and *we need to account for these as well.* So we'll update our guess for y_p, and assume

$$y_p = A_1 x + A_2 + A_3 \cos(2x) + A_4 \sin(2x).$$

Plugging this into the ODE and collecting like terms gives

$$\begin{aligned} A_1 &= 1 \\ -2A_1 + A_2 &= 0 \\ -3A_3 - 4A_4 &= 0 \\ 4A_3 - 3A_4 &= -1 \end{aligned}$$

Hence

$$\begin{aligned} A_1 &= 1 \\ A_2 &= 2 \\ C &= -\frac{4}{25} \\ D &= \frac{3}{25} \end{aligned}$$

and the solution of our ODE is

$$c_1 e^x + c_2 x e^x + x + 2 - \frac{4}{25}\cos(2x) + \frac{3}{25}\sin(2x).$$

Example 2.4.4 Method of undetermined coefficients with overlap.

Find the solution $y(x)$ of the IVP

$$y'' + 9y = 3\cos(3x), \quad y(0) = 1, y'(0) = -1$$

Solution. We can start this example the same way we've done the previous ones. First, we find y_c by solving $y'' + 9y = 0$. So

$$y_c = c_1 \cos(3x) + c_2 \sin(3x).$$

Now we find y_p. Since the RHS of the ODE is $3\cos(3x)$, we'll guess that $y_p = A\cos(3x) + B\sin(3x)$. However, this will cause us problems! Since $\cos(3x), \sin(3x)$ are both solutions of the corresponding homogeneous ODE, then plugging y_p into $y'' + 9y$ will just give 0 again, instead of $3\cos(3x)$.
The problem here is that our guess for y_p overlaps with y_c. To fix this, we'll multiply our guess for y_p by the *smallest* power of x that removes the overlap. In this case, we'll just multiply by x to get

$$y_p = x[A\cos(3x) + B\sin(3x)]$$

Now, we'll plug our modified guess into the ODE and set it equal to $3\cos(3x)$:

$$\begin{aligned} 3\cos(3x) &= y_p'' + 9y \\ &= -6A\sin(3x) + 6B\cos(3x) \end{aligned}$$

So we need $-6A = 0$ and $6B = 3$, or just $A = 0, B = \frac{1}{2}$. Hence

$$y_p = x\left[\frac{1}{2}\sin(3x)\right],$$

and the general solution is then

$$y = y_c + y_p = c_1\cos(3x) + c_2\sin(3x) + \frac{x}{2}\sin(3x).$$

To find the solution of the IVP, we just plug in the initial conditions. Since $y(0) = 1$, this gives

$$1 = c_1.$$

And since

$$y' = -3c_1 \sin(3x) + 3c_2 \cos(3x) + \frac{1}{2}\sin(3x) + \frac{3}{2}x\cos(3x),$$

$y'(0) = -1$ gives

$$-1 = 3c_2 \Rightarrow c_2 = -\frac{1}{3}.$$

So

$$y = \cos(3x) - \frac{1}{3}\sin(3x) + \frac{x}{2}\sin(3x).$$

What we did in Example 2.4.4 will work in general: if y_c and the initial guess for y_p overlap, i.e. contain linearly dependent terms, then we multiply y_p by the *smallest* power of x (or the appropriate independent variable) that removes the overlap.

Example 2.4.5 Determining the appropriate form of the particular solution.

Consider the ODE $x'' + 4x' + 4 = 3t^3 - t + t^2\sin(2t) + 3e^{-2t}$. Find the correct guess for x_p.

Solution. Before we can do anything with x_p, we need to find the complementary solution x_c. This is given by $x_c = c_1e^{-2t} + c_2te^{-2t}$. Now we can try to guess an appropriate form for x_p using the right hand side of the ODE. Each "component" of the right hand side contributes to our guess for x_p as shown in Table 2.4.6. But now we have a problem, since Ke^{-2t} overlaps with x_c. So we multiply by t^2 to remove the overlap, and hence $x_p = At^3 + Bt^2 + Ct + D + (Et^2 + Ft + G)\cos(2t) + (Ht^2 + It + J)\sin(2t) + 3e^{-2t} + Kt^2e^{-2t}$.

Table 2.4.6 Determining the appropriate form of a particular solution

component	contribution to x_p
$3t^3 - t$	$At^3 + Bt^2 + Ct + D$
$t^2\sin(2t)$	$(Et^2 + Ft + G)\cos(2t) + (Ht^2 + It + J)\sin(2t)$
$3e^{-2t}$	Ke^{-2t}

2.5 Forced oscillations and resonance

In this section we will develop models for certain systems under the influence of a periodic, external force. The presence of an external force leads to non-homogeneous models, and we will use the techniques we developed in Section 2.4 to deal with these systems.

Undamped systems

Consider the spring-mass system set up as in Figure 2.3.1. Then we know that the displacement $x(t)$ satisfies $x'' + \omega_0^2 x = 0$, where $\omega_0 = \sqrt{\frac{k}{m}}$.

Now suppose that an external force $F_E = F_0\cos(\omega t)$ also acts on the mass.

Then the ODE that models the displacement $x(t)$ is

$$x'' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t).$$

The solution of this ODE is then $x = x_c + x_p$, where

$$x_c = A\cos(\omega - 0t) + B\sin(\omega_0 t)$$
$$x_p = \begin{cases} \frac{F_0}{m(\omega_0^2-\omega^2)}\cos(\omega t) & \text{if } \omega \neq \omega_0 \\ \frac{F_0}{2m\omega_0} t\sin(\omega_0 t) & \text{if } \omega = \omega_0 \end{cases}$$

Systems where the **external frequency** ω is equal to the **internal frequency** ω_0 are said to be in **resonance**. Without a damping force, the mass in such systems will move wildly out of control $|x_p(t)|$ gets arbitrarily large as $t \to \infty$.

Example 2.5.1 Determining resonance.

An object with mass 2 kg is attached to a spring and is held 1 m to the right of the spring's equilibrium position by a force of 8 N At time $t = 0$ seconds the mass is set in motion with an initial velocity of $2\,\frac{\mathrm{m}}{\mathrm{s}}$ to the left. Suppose an external force $F_E = 3\cos(2t)$ acts on the mass as well. Will the spring eventually break?

Solution. We can answer this question by determining if resonance is present in this system. The external frequency is $\omega = 2$, and the internal frequency is $\omega_0 = \sqrt{\frac{k}{m}}$. Since $k = 8$ and $m = 2$, we have $\omega_0 = \sqrt{\frac{8}{2}} = 2$, and so the frequencies match. Hence the system is in resonance, and we can expect the spring to eventually break.

It's not too hard to solve for $x(t)$ exactly here to get

$$x(t) = \frac{1}{4}\cos(2t) - \sin(2t) + \frac{3}{8}t\sin(2t).$$

Graphing this, we get the figure produced in Figure 2.5.2.

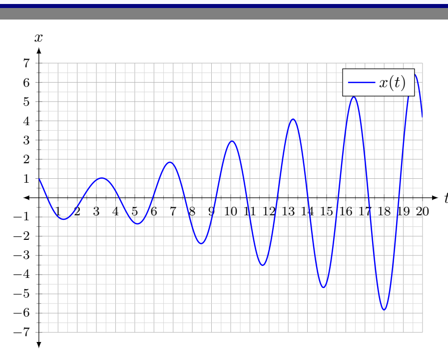

Figure 2.5.2 A plot of the motion of the mass in Example 2.5.1

Damped systems

Now we'll take a look at forced, damped systems. Suppose a mass m is fixed to a spring with spring force $F_S = -kx$, and is acted upon by a dashpot with force $F_R = -cx'$, where $c, k > 0$ and x represents the displacement of the mass at time t. If the mass is still acted upon by an external force $F_E = F_0 \cos(\omega t)$, then by Newton's Second Law the displacement x must satisfy

$$mx'' + cx' + kx = F_0 \cos(\omega t).$$

The solution is given by $x = x_c + x_p$, where x_c is found as in Free damped motion and goes to 0 as $t \to \infty$. With a little help from a computer algebra system such as Sage (see below), we see that

$$x_p = \frac{m(\omega_0^2 - \omega^2)F_0}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2} \cos(\omega t) + \frac{\omega c F_0}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2} \sin(\omega t),$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ as usual.

Since x_c will always approach 0 in these situations, as time goes on the position is determined increasingly by x_p. We call x_c the **transient solution** and x_p the **steady-state solution**. Note that resonance is impossible in this system since x_c and x_p can never overlap (assuming the external force is still a sinusoid). Therefore the smallest amount of damping prevents the mass from going out of control.

```
# Set variables
t = var('t')
m, c, k, F0, omega, omega_0 = var('m␣c␣k␣F0␣omega␣omega_0')
# Make assumption to help Sage/Maxima with computation; this
    corresponds to underdamped motion
assume(4*k*m - c^2 > 0)
# Define x as a function of t
x = function('x')(t)
# Set up and solve the ODE
P = desolve(m*diff(x, t, 2) + c*diff(x,t) + k*x ==
    F0*cos(omega*t), x, ivar=t)
# Make "nice" output for solution
pretty_print(P.substitute(k == m*(omega_0)^2))
```

```
# Sage output without pretty printing.
(_K2*cos(1/2*sqrt(4*omega_0^2 - c^2/m^2)*t) +
    _K1*sin(1/2*sqrt(4*omega_0^2 -
    c^2/m^2)*t))*e^(-1/2*c*t/m) + (F0*c*omega*sin(omega*t) -
    (F0*m*omega^2 -
    F0*m*omega_0^2)*cos(omega*t))/(m^2*omega^4 +
    m^2*omega_0^4 - (2*m^2*omega_0^2 - c^2)*omega^2)
```

Example 2.5.3 Steady-state approximation.

An object of mass 3 kg is fixed to both a spring and a dashpot with respective forces $F_S = -9x$ and $F_R = -2x'$, where x is the displacement of the mass in meters and $x = 0$ is the equilibrium position. An external force $F_E = 5\cos(\sqrt{3}t)$ is also applied to the mass, where t is in seconds. The mass was set in motion with an unknown speed and unknown velocity approximately 7 s ago. What will be the approximate position of the mass in 40 s?

Solution. We know that the position will look like $x = x_c + x_p$ where $x_c \to 0$ as $t \to \infty$, but we can't find the exact form of the transient solution without knowing the initial conditions. So we'll assume that we can estimate the position of the mass using the steady-state solution

x_p. Since

$$\omega = \omega_0 = \sqrt{3}, \quad F_0 = 5 \quad \text{and} \quad c = 2,$$

we get

$$x_p = \frac{5}{6}\sqrt{3}\sin(\sqrt{3}t).$$

So after 40 s more seconds the mass should be around $x_p(47)$, or about -0.388 m.

In fact, the actual initial conditions used in Example 2.5.3 were $x(0) = 1$ and $x'(0) = -2$. The corresponding exact solution is

$$x = e^{-\frac{t}{3}}\left[\cos\frac{\sqrt{26}}{3}t - \frac{25}{52}\sin\frac{\sqrt{26}}{3}t\right] + \frac{5}{6}\sqrt{3}\sin(\sqrt{3}t).$$

The exact value of $x(47)$ is within several *millionths* of the approximation $x_p(47)$.

Chapter 3

Higher Order Linear ODEs

This chapter applies many of the ideas developed in Chapter 2 to higher order linear ODEs. We'll begin by transferring many of the concepts and terms introduced in Sections 2.1–2.2.

3.1 Homogeneous Linear ODEs with Constant Coefficients

Each tool that we used for solving second order linear ODEs with constant coefficients, in other words those ODEs of the form $ay'' + by' + cy = f(x)$, can be extended to solving more general n^{th} order linear ODEs with constant coefficients, which take the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x) \tag{3.1}$$

where $a_i, 0 \leq i \leq n$ are constants and $a_n \neq 0$. In this section, we'll focus on solutions of *homogeneous* n^{th} order linear ODEs with constant coefficients. These are the ODEs where $f(x) = 0$ in (3.1).

Recall that in Chapter 2, the general solution of $ay'' + by' + cy = 0$ could be obtained by finding *two* linearly independent solutions y_1, y_2. The general solution was then $y = c_1 y_1 + c_2 y_2$ (which is guaranteed to be a solution by Theorem 2.1.6 where c_1, c_2 are arbitrary constants. Similarly, the general solution of (3.1) is found by obtaining n linearly independent solutions $y_1, \ldots, y_n$. The general solution in this case is now $y = \sum_{i=1}^{n} c_i y_i$, where $c_i, 1 \leq i \leq n$ are arbitrary constants. Our main tool for showing that a collection $\{y_1, \ldots, y_n\}$ of solutions is in fact linearly independent is again the *Wronskian*.

Definition 3.1.1 The Wronskian of Several Functions.

Let $\{y_1, \ldots, y_n\}$ be a collection of functions. Then the **Wronskian** of $\{y_1, \ldots, y_n\}$, denoted by $W(y_1, \ldots, y_n)$, is defined by

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

Just as before, we have the following connection between the Wronskian and linear independence. This theorem is a restatement of Theorem 2.1.11 for collections involving more than two functions.

Theorem 3.1.2 Linear Independence and the Wronskian (Several Functions).

Let I be some open interval (we often take I to be $(-\infty, \infty)$, but it doesn't have to be so) and let $\{y_1, \ldots, y_n\}$ be solutions of (3.1). If $W(y_1, \ldots, y_n) \neq 0$ for some point $x_0 \in I$, then $\{y_1, \ldots, y_n\}$ is linearly independent.

Now that we have a tool for determining linear independence of several functions, we can also define a *basis of solutions* for higher order ODEs.

Definition 3.1.3 Basis of Solutions (Higher Order ODEs).

Let $y_1, y_2, \ldots,$ and y_n denote solutions of some n^{th} order linear homogeneous ODE. We call $\{y_1, y_2, \ldots, y_n\}$ a **basis** if this set is also linearly independent.

Bases are used to determine general solutions of linear ODEs.

Example 3.1.4 Finding a Basis Set of Solutions.

Find the general solution of

$$y^{(3)} - 6y'' + 11y' - 6y = 0$$

where y is a function of x.

Solution. Just as we did for second order ODEs, we'll solve this by finding the characteristic equation. To get the characteristic equation, we replace derivatives of y with powers of r to get

$$r^3 - 6r^2 + 11r - 6 = 0.$$

Now we need to solve this equation. It can factor (notice that $r = 1$ is a solution and then divide $r^3 - 6r^2 + 11r - 6$ by $r - 1$) as

$$(r-1)(r-2)(r-3) = 0$$

and so $r = 1, 2, 3$. This means that the functions

$$y_1 = e^x, y_2 = e^{2x}, y_3 = e^{3x}$$

are all solutions of the original ODE. If we can show that they're also linearly independent, then this will imply that the general solution of the ODE is given by

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

To do this, we just compute the Wronskian of these functions:

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$
\begin{aligned}
&= e^x(18e^{5x} - 12e^{5x}) - e^{2x}(9e^{4x} - 3e^{4x}) + e^{3x}(4e^{3x} - 2e^{3x}) \\
&= 2e^{6x} \\
&\neq 0.
\end{aligned}
$$

Since the Wronskian is nonzero, y_1, y_2 and y_3 are in fact linearly independent, and so the general solution of this ODE is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

It can get tedious to try to compute the Wronskian every time when solving linear ODEs with constant coefficients, so it's good to note that $\{e^{r_1 x}, \ldots, e^{r_n x}\}$ is guaranteed to be linearly independent as long as each of the r_i are distinct from the others.

Example 3.1.5

Find the general solution of $y^{(4)} - y = 0$, where y is a function of x.

Solution. We begin by finding the characteristic equation, which is

$$r^4 - 1 = 0 \quad \text{or} \quad (r^2 - 1)(r^2 + 1) = 0.$$

This has solutions $r = \pm 1, \pm i$. This means that $y_1 = e^x, y_2 = e^{-x}, y_3 = e^{ix}$ and $y = e^{-ix}$ are all solutions of the ODE. Since the roots of the characteristic equation are all distinct, this means that these solutions are linearly independent from each other. Since we have four linearly independent solutions, we can then construct the general solution of this ODE:

$$y = c_1 e^x + c_2 e^{-x} + c_3 e^{ix} + c_4 e^{-ix},$$

which we can also rewrite using Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$ to get

$$y = c_1 e^x + c_2 e^{-x} + A\cos x + B\sin x.$$

In general, any root of the characteristic equation of the form $r = a \pm bi$ contributes the term $e^{ax}\left[A\cos bx + B\sin bx\right]$ to the general solution. As we saw in Chapter 2, it's possible for some characteristic equations to have repeated roots. In this case, we initially weren't able to get enough linearly independent solutions, so we had to adjust our method a bit. The same adjustment will work here.

Example 3.1.6 Characteristic Equation with Repeated Roots.

Find the general solution of

$$x^{(3)} + 4x'' + 4x' = 0$$

where x is a function of t.

Solution. The characteristic equation is

$$r^3 + 4r^2 + 4r = 0 \quad \text{or} \quad r(r+2)^2 = 0$$

so $r = 0, -2, -2$. One solution of the ODE will be $x_1 = e^{0t} = 1$, and a second solution will be $x_2 = e^{-2t}$. But since $r = -2$ is a repeated root, it does not provide a third linearly independent solution x_3. So we'll use the same trick we used before and multiply by t to get a third solution: $x_3 = te^{-2t}$. It can be verified that x_3 is in fact a solution of the ODE, and is also linearly independent from x_1, x_2. Therefore the general solution of the ODE is

$$x = c_1 + c_2e^{-2t} + c_3te^{-2t}.$$

Solutions of linear, homogeneous ODEs with constant coefficients depend *entirely* on the roots of the corresponding characteristic equation. If we write the independent variable as x, and if r_k denotes a single root of the characteristic equation, then the general solution of the ODE will contain $e^{r_kx}, xe^{r_kx}, x^2e^{r_kx}, \ldots$, with the number of exponentials contributed by r_k being equal to its multiplicity. That is, the number of times r_k appears as a solution of the characteristic equation.

Example 3.1.7

A linear, homogeneous ODE with constant coefficients (and independent variable t) has characteristic equation given by

$$r(r-1)(r+1)^2(2r-3)^3 = 0.$$

What is the general solution of the ODE?

Solution. Table 3.1.8 gives the roots of the characterisitc equation and their contributions to the general solution. So the general solution of this ODE is

$$c_1 + c_2e^t + c_3e^{-t} + c_4te^{-t} + c_5e^{3t/2} + c_6te^{3t/2} + c_7t^2e^{3t/2}.$$

Table 3.1.8 Table of roots

Root	Multiplicity	Contribution
0	1	1
1	1	e^t
-1	2	e^{-t}, te^{-t}
$\frac{3}{2}$	3	$e^{3t/2}, te^{3t/2}, t^2e^{3t/2}$

Example 3.1.9

An ODE (with independent variable x) has characteristic equation given by

$$r^2(r-3)(r^2+4)(r^2+6r+13)^3 = 0.$$

Find the general solution.

Solution. We'll set up another table help us determine the general solution:

Table 3.1.10

Root	Multiplicity	Contribution
0	2	$1, x$
3	1	e^{3x}
$\pm 2i$	1	$\cos 2x, \sin 2x$
$-3 \pm 2i$	3	$e^{-3x}\cos 2x, e^{-3x}\sin 2x, xe^{-3x}\cos 2x, xe^{-3x}\sin 2x, x^2e^{-3x}\cos 2x, x^2e^{-3x}\sin 2x$

So the general solution is

$$c_1+c_2x+c_3e^{3x}+A\cos 2x+B\sin 2x+e^{-3x}\left[(A_1+A_2x+A_3x^2)\cos 2x+(B_1+B_2x+B_3x^2)\sin 2x\right].$$

3.2 Non-homogeneous Linear ODEs with Constant Coefficients

For second order ODEs that were nonhomogeneous, linear and had constant coefficients, we found their general solution by first finding the complementary solution y_c and then a corresponding particular solution y_p. The general solution was then $y = y_c + y_p$. y_c was found by solving the corresponding homogeneous equation and we used the method of undetermined coefficients to find y_p. Although we are now looking at higher order ODEs, the method of undetermined coefficients remains unchanged.

Example 3.2.1

Find the general solution of

$$y^{(3)} - y' = \sinh 2x.$$

Solution. The general solution takes the form $y = y_c + y_p$, where y_c is a solution of the associated homogeneous equation $y^{(3)} - y' = 0$ and y_p is a single solution of the original ODE $y^{(3)} - y' = \sinh 2x$. Since the characteristic equation of $y^{(3)} - y' = 0$ is $r^3 - r = 0$, we get

$$y_c = c_1 + c_2e^x + c_3e^{-x} = c_1 + A\cosh x + B\sinh x.$$

Now we'll find y_p. Since the right hand side of the ODE is $\sinh 2x$, a good initial guess would be $y_p = C\sinh 2x$. However, when we take this guess and plug it into the ODE, we'll start seeing terms involving $\cosh 2x$ as well (since $\frac{d}{dx}\sinh 2x = 2\cosh 2x$) so this means we'll want to include $\cosh 2x$ into our guess for y_p also. So we'll modify our guess to be $y_p = C\cosh 2x + D\sinh 2x$.
Since our guess for y_p doesn't overlap with y_c, we can proceed with plugging our guess into the original ODE $y^{(3)} - y' = \sinh 2x$ and equating coefficients, just as we did before.

$$\begin{aligned}\sinh 2x &= y_p^{(3)} - y_p' \\ &= (8C\sinh 2x + 8D\cosh 2x) - (2C\sinh 2x + 2D\cosh 2x) \\ &= 6C\sinh 2x + 6D\cosh 2x.\end{aligned}$$

So we get $C = \frac{1}{6}$ and $D = 0$, which means that $y_p = \frac{1}{6}\cosh 2x$. When solving for y_p, always remember to plug the values you find back into the guess for y_p that you used! So the general solution of the ODE is

$$y = c_1 + A\cosh x + B\sinh x + \frac{1}{6}\cosh 2x.$$

Just as before, we also need to worry about overlaps.

Example 3.2.2

Find the appropriate form of x_p for the ODE

$$\frac{d^7x}{dt^7} + 8\frac{d^5x}{dt^5} + 16\frac{d^3x}{dt^3} = 5t - 3t^2 + e^{-t}\cos 2t - 3t\sin 2t.$$

Solution. We need to find x_c first since x_p will change depending on x_c. Since the characteristic equation of the associated homogeneous ODE $\frac{d^7x}{dt^7} + 8\frac{d^5x}{dt^5} + 16\frac{d^3x}{dt^3} = 0$ is

$$r^7 + 8r^5 + 16r^3 = 0 \quad \text{or} \quad r^3(r^2+4)^2 = 0,$$

we get

$$x_c = c_1 + c_2t + c_3t^2 + A\cos 2t + B\sin 2t + t\left[C\cos 2t + D\sin 2t\right].$$

Now we come up with a guess for x_p using the right hand side of the original ODE and dividing it into ``components:''

Table 3.2.3

Component	Contribution to x_p
$2t - 3t^2$	$C_1t^2 + C_2t + C_3$
$e^{-t}\cos 2t$	$e^{-t}\left[C_4\cos 2t + C_5\sin 2t\right]$
$-3t\sin 2t$	$(C_6t + C_7)\cos 2t + (C_8t + C_9)\sin 2t$

However, we now have a problem with overlaps between x_p and x_c. The guess corresponding to the first component overlaps with x_c, so we need to multiply it by t^3 to remove the overlap. Similarly, the guess corresponding to the third component overlaps, so we must multiply it by t^2. Therefore, our guess for x_p should be

$$x_p = C_1t^5 + C_2t^4 + C_3t^3 + e^{-t}\left[C_4\cos 2t + C_5\sin 2t\right] + (C_6t^3 + C_7t^2)\cos 2t + (C_8t^3 + C_9t^2)\sin 2t.$$

The method of undetermined coefficients applied to the ODE

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = f(x)$$

can be summarized by the following table. Note that x^m is the *smallest* power of x required to remove any overlaps with y_c.

Table 3.2.4

Component of $f(x)$	Contribution to y_p
$c_kx^k + \cdots + c_1x + c_0$	$x^m(C_kx^k + C_{k-1}x^{k-1} + \cdots + C_1x + C_0)$
$(c_kx^k + \cdots + c_1x + c_0)e^{ax}\sin bx$	$x^me^{ax}\left[(C_kx^k + \cdots + C_1x + C_0)\cos bx + (D_kx^k + \cdots + D_1x + D_0\right.$
$(c_kx^k + \cdots + c_1x + c_0)e^{ax}\cos bx$	$x^me^{ax}\left[(C_kx^k + \cdots + C_1x + C_0)\cos bx + (D_kx^k + \cdots + D_1x + D_0\right.$
$(c_kx^k + \cdots + c_1x + c_0)e^{ax}\sinh bx$	$x^me^{ax}\left[(C_kx^k + \cdots + C_1x + C_0)\cosh bx + (D_kx^k + \cdots + D_1x + D\right.$
$(c_kx^k + \cdots + c_1x + c_0)e^{ax}\cosh bx$	$x^me^{ax}\left[(C_kx^k + \cdots + C_1x + C_0)\cosh bx + (D_kx^k + \cdots + D_1x + D\right.$

Example 3.2.5

Find the general solution of

$$y^{(3)} - 2y'' + 3y' = x + e^{-3x} \sin x$$

Solution. We begin by finding y_c. Since the characteristic equation of the corresponding homogeneous ODE is $r^3 - 2r^2 + 3r = 0$, we get $r = -0, \frac{2\pm\sqrt{4-12}}{2}$ or just $r = 0, 1 \pm i\sqrt{2}$. So

$$y_c = c_1 + e^x(A\cos\sqrt{2}x + B\sin\sqrt{2}x).$$

Now we can set up y_p:

Table 3.2.6

Component	Contribution to y_p
x	$x(C_1x + C_2)$
$e^{-3x}\sin x$	$e^{-3x}\left[C_3\cos x + C_4\sin x\right]$

So our initial guess for y_p is given by $y_p = C_1x^2 + C_2x + C_3e^{-3x}\cos x + C_4e^{-3x}\sin x$. Plugging this into the ODE into a CAS such as Maple or Sage gives

$$\begin{aligned} x + e^{-3x}\sin x &= y_p^{(3)} - 2y_p'' + 3y_p' \\ &= 6C_1x + (3C_2 - 4C_1) \\ &\quad + (-43C_3 + 41C_4)\,e^{-3x}\cos x + (-41C_3 - 43C_4)\,e^{-3x}\sin x \end{aligned}$$

This gives us the system of equations

$$\begin{aligned} 6C_1 &= 1 \\ 3C_2 - 4C_1 &= 0 \\ -43C_3 + 41C_4 &= 0 \\ -41C_3 - 43C_4 &= 1 \end{aligned}$$

which we can solve using Sage to get

$$C_1 = \frac{1}{6}, C_2 = \frac{2}{9}, C_3 = -\frac{41}{3530}, C_4 = -\frac{43}{3530}.$$

So the general solution of the ODE is

$$y = y_c + y_p = c_1 + e^x(A\cos\sqrt{2}x + B\sin\sqrt{2}x) + \frac{1}{6}x^2 + \frac{2}{9}x - \frac{e^{-3x}}{3530}\left[41\cos x + 43\sin x\right].$$

```
# Set variables
c1, c2, c3, c4 = var('c1␣c2␣c3␣c4')

# Set up equations as variables for clarity
eqn1 = 6*c1
eqn2 = 3*c2 - 4*c1
eqn3 = -43*c3 + 41*c4
eqn4 = -41*c3 - 43*c4

solve([eqn1 == 1, eqn2 == 0, eqn3 == 0, eqn4 == 1], c1, c2,
```

```
    c3, c4)
```

```
[[c1 == (1/6), c2 == (2/9), c3 == (-41/3530), c4 ==
    (-43/3530)]]
```

Chapter 4

Systems of Ordinary Differential Equations

4.1 Systems of ODEs as Models

Interdependent quantities can often be represented mathematically by a system of equations. If we have information about the rates of change of these quantities, then we may be able to develop a model using a system of differential equations.

Definition 4.1.1

A **first order system** of ODEs is a system of differential equations involving some collection of functions and their first derivatives.

We are still only dealing with ordinary differential equations which means that we will only ever have one independent variable. However, when dealing with systems of ODEs we will be working with several *dependent* variables.

The systems of ODEs that we will consider will typically look like the following:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_2 \\ &\ \vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned}$$

where a_{ij} are constants and y_i are functions of t.

Example 4.1.2

Two brine tanks are set up as in Figure 4.1.3. Fresh water flows into the tank at a rate of r_1, well-mixed solution flows from Tank 1 to Tank 2 at a rate of r_2 and well-mixed solution flows out of Tank 2 at a rate of r_3. Suppose that r_1, r_2 and r_3 are $5\,\frac{\text{gal}}{\text{min}}$, the volume of solution in Tank 1 is $10\,\text{gal}$ and the volume of solution in Tank 2 is $7\,\text{gal}$. Suppose Tank 1 has $5\,\text{lb}$ of salt at time $t = 0$ and Tank 2 has $2\,\text{lb}$ of salt at time $t = 0$. Set up a first-order system that describes the amount of salt in each tank at time t.

Solution. Let $x_1(t)$ denote the amount of salt in Tank 1 at time t, and $x_2(t)$ denote the amount of salt in Tank 2 at time t. Using the mixture ODE $\dfrac{dx}{dt} = r_i c_i - r_o \frac{x}{V(t)}$ developed in Section 1.4, we can write

$$\frac{dx_1}{dt} = 5 \cdot 0 - 5\frac{x_1}{10}$$
$$\frac{dx_2}{dt} = 5\frac{x_1}{10} - 5\frac{x_2}{7}$$

or just

$$x_1' = -\frac{1}{2}x_1$$
$$x_2' = \frac{1}{2}x_1 - \frac{5}{7}x_2,$$

with initial conditions $x_1(0) = 5$ and $x_2(0) = 2$.

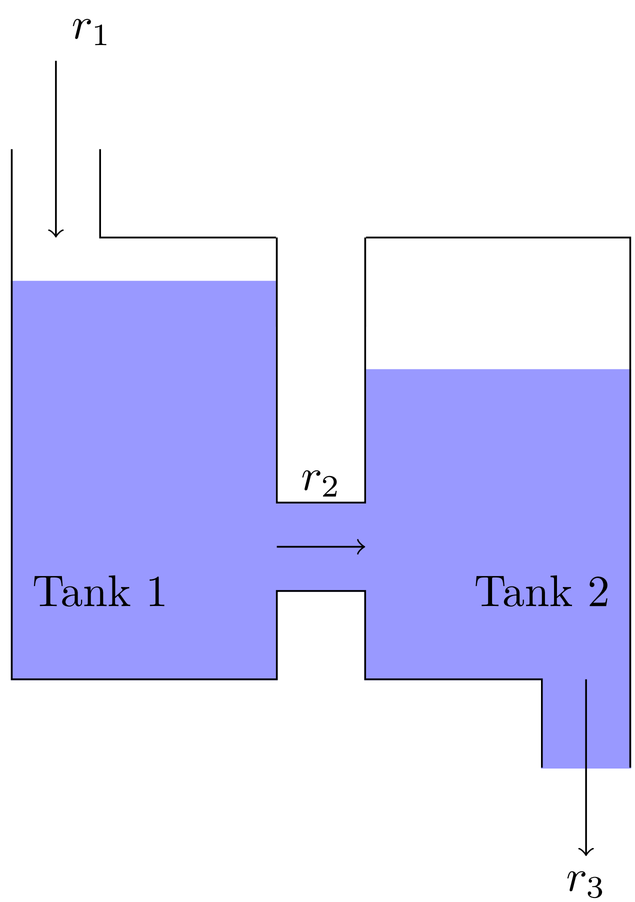

Figure 4.1.3 The two interconnected tanks from Example 4.1.2.

To actually solve systems of ODEs, we'll use *matrices* to rewrite these systems as *matrix ODEs*.

Definition 4.1.4

An $m \times n$ **matrix** is an array of m rows and n columns. $m \times 1$ matrices are called (**column**) **vectors**. Matrices are typically denoted with capital italic letters (such as A, M) and vectors are often denoted with lower case bold letters (such as $\mathbf{v}, \mathbf{x}$). A **zero matrix** will be denoted using $\mathbf{0}$.

As a brief example, let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Then A is a 2×3 matrix and $\mathbf{y}$ is a 3×1 vector.

Definition 4.1.5 Matrix-Vector Product.

Let A be the 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Then their **product** $A\mathbf{v}$ is the vector defined to be

$$A\mathbf{v} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{bmatrix}.$$

The 2×2 **identity matrix** is the matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. A **scalar** is just a constant. To multiply a scalar c with a matrix A, just multiply every element of A with c.

If A is any 2×2 matrix and $\mathbf{v}$ and 2×1 vector, then $AI = IA = A$ and $I\mathbf{v} = \mathbf{v}$.

Example 4.1.6

Let $A = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$. Compute $A\mathbf{v}_1$ and $A\mathbf{v}_2$.

Solution. By definition,

$$A\mathbf{v}_1 = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 1 \\ -3 \cdot 1 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad A\mathbf{v}_2 = \begin{bmatrix} 1 \cdot 0 + 0 \cdot 5 \\ -3 \cdot 0 + 2 \cdot 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}.$$

In Example 4.1.6, notice that $A\mathbf{v}_2 = 2\mathbf{v}_2$. This means that A didn't really do all that much to $\mathbf{v}_2$ except to stretch it by a factor of 2. Vectors with this property will turn out to be the key to solving our systems of ODEs.

Any linear system can be written as an equivalent first-order system or matrix ODE.

Example 4.1.7 Interconnected spring-mass system.

Consider a spring mass system with two masses arranged as follows:

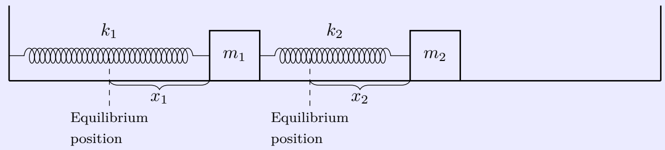

Figure 4.1.8 A spring-mass system with interconnected masses

Determine a first-order system that the displacements x_1 and x_2 must satisfy.

Solution. From Section 2.3 we know how to model a spring-mass system with a single mass using Theorem 2.1.3 and Newton's Second Law. We will apply this same analysis to the displacements x_1 and x_2 individually.

To begin, we will analyze the forces acting on the first mass. Here there are two forces to consider: the force caused by the motion of m_1 and the force caused by the motion of m_2. Likewise, the second mass is also influenced by two forces. We arrange these in the following table:

Table 4.1.9 Forces acting on an interconnected spring-mass system

mass	forces
m_1	$-k_1x_1, -k_2x_1, k_2x_2$
m_2	$k_2x_1, -k_2x_2$

Now we can apply Newton's Second Law to get the *second*-order system

$$\begin{aligned} m_1x_1'' &= -k_1x_1 - k_2x_1 + k_2x_2 \\ m_2x_2'' &= k_2x_1 - k_2x_2. \end{aligned}$$

At this point we can introduce new dependent variables u_1, u_2, u_3 and u_4 to get an equivalent first-order system.

Although this type of system is new, the solutions behave as expected. In particular, both x_1 and x_2 display periodic motion as can be seen by the Sage example below.

```
# Define our independent variable
var('t')

# Define the dependent variables as functions of t
x1 = function('x_1')(t)
x2 = function('x_2')(t)

# System parameters
m1, m2 = 1, 1
k1, k2 = 4, 4

# Define the equations in our system
# If you wish to solve the corresponding first-order
    system,
# you will need to specify four equations.
de1 = m1*diff(x1, t, 2) == -(k1+k2)*x1 + k2*x2
de2 = m2*diff(x1, t, 2) == k2*x1 - k2*x2

# Display the (general) solution of this system
# Note that the solution depends on initial conditions
    which
# were NOT provided. In particular, D_0(x)(0)
    represents an
# iniital condition of the form x'(0)
show(desolve_system([de1, de2], [x1, x2]))
```

Definition 4.1.10 Eigenvectors and eigenvalues.

Let A be a matrix. A nonzero vector $\mathbf{v}$ is an **eigenvector** of A if $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ. We call λ an **eigenvalue** of A corresponding to the eigenvector $\mathbf{v}$.

Example 4.1.11

Determine if $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$.

Solution. To do this, we just need to compute $A\mathbf{v}$:

$$A\mathbf{v} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = -\mathbf{v}.$$

So $\mathbf{v}$ is an eigenvector of A with corresponding eigenvalue $\lambda = -1$.

Since we will be looking at systems of ODEs which involve functions, we will need to define vector-valued functions. These objects will represent the solutions of our systems.

Definition 4.1.12 Vector-Valued Functions.

A **vector-valued function** is a vector whose elements are functions. If each of the functions in a vector $\mathbf{x}$ depends on the variable t, we often write $\mathbf{x}(t)$ to denote this. The derivative of a vector-valued function

$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ is the new vector-valued function $\mathbf{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$.

We now have all of the tools we need to rewrite a first-order system as a matrix ODE. Let

$$\begin{aligned} x' &= a_{11}x + a_{12}y \\ y' &= a_{21}x + a_{22}y \end{aligned}$$

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, then

$$A\mathbf{x} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{x}'.$$

In other words, we may rewrite the system as the matrix ODE

$$\mathbf{x}' = A\mathbf{x}.$$

Example 4.1.13

Write the system

$$\begin{aligned} x_1' &= -\frac{1}{2}x_1 \\ x_2' &= \frac{1}{2}x_1 - \frac{5}{7}x_2 \end{aligned}$$

as a matrix ODE.

Solution. We need to find a matrix A and vector $\mathbf{x}$ to let us rewrite this system. The matrix A is formed from the coefficients of x_1, x_2 on the right hand side of the system:

$$A = \begin{bmatrix} -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{5}{7} \end{bmatrix}.$$

The vector $\mathbf{x}$ is just made up of the dependent variables x_1, x_2:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

With these terms, the original system of ODEs is equivalent to the single matrix ODE

$$\mathbf{x}' = A\mathbf{x}.$$

Example 4.1.14

Show that $e^{-t}\mathbf{x}_0$ where $\mathbf{x}_0 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a solution of the system

$$\mathbf{x}' = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}\mathbf{x}.$$

Solution. We'll check that $\mathbf{x}_0$ is a solution of the matrix ODE just as we check solutions for normal ODEs: plug the potential solution into the ODE and check both sides. If we do so, we get

$$\frac{d}{dt}[*]\, e^{-t}\mathbf{x}_0 = -e^{-t}\begin{bmatrix}-2\\1\end{bmatrix} = e^{-t}\begin{bmatrix}2\\-1\end{bmatrix}$$

and

$$\begin{bmatrix}1 & 4\\1 & 1\end{bmatrix} e^{-t}\mathbf{x}_0 = e^{-t}\begin{bmatrix}1 & 4\\1 & 1\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix} = e^{-t}\begin{bmatrix}2\\-1\end{bmatrix}.$$

Since these expressions match, this means that $e^{-t}\mathbf{x}_0$ is a solution of the ODE.

One thing to note about the previous example is that $\begin{bmatrix}-2\\1\end{bmatrix}$ was an eigenvector of $\begin{bmatrix}1 & 4\\1 & 1\end{bmatrix}$ with corresponding eigenvalue $\lambda = -1$. See Example 4.1.11. This suggests that solutions of the matrix ODE $\mathbf{x}' = A\mathbf{x}$ take the form $\mathbf{x} = e^{\lambda t}\mathbf{x}_0$, where λ is an eigenvalue of A with corresponding eigenvector $\mathbf{x}_0$. One last concept we need is that of linear independence of vectors.

Definition 4.1.15 Linear Independence of Vectors.

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ denote a collection of vectors. We say that the vectors are **linearly independent** if the equality

$$\sum_{i=1}^{n} c_i\mathbf{x}_i = \mathbf{0}$$

is possible if and only if $c_1 = \cdots = c_n = 0$. Otherwise, we say that the vectors are **linearly dependent**.

Just as before, our primary tool for showing if a collection is linearly independent is the Wronskian.

Definition 4.1.16

The **Wronskian** of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is the number $W(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ defined by

$$W(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \begin{vmatrix}\mathbf{x}_1 & \ldots & \mathbf{x}_n\end{vmatrix}.$$

The vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent if and only if their Wronskian is nonzero.

4.2 Constant Coefficient Systems

The content in this section represents a higher-dimensional analog of the content in Section 2.1.

Solutions of $\mathbf{x}' = A\mathbf{x}$

A system of ODEs involving only constant coefficients can be rewritten as a matrix ODE of the form $\mathbf{x}' = A\mathbf{x}$ where A is a constant matrix. Such a system can be solved using exponentials.

Theorem 4.2.1 Solutions of Systems.

Let A be an $n \times n$ constant matrix, and suppose that A has n linearly independent eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_1$. Then the general solution of $\mathbf{x}' = A\mathbf{x}$ is given by

$$\mathbf{x} = \sum_{i=1}^{n} c_i e^{\lambda_i t} \mathbf{x}_i.$$

Example 4.2.2

Find the general solution of the system

$$\begin{aligned} x_1' &= x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 \\ x_2' &= -\frac{1}{2}x_1 + x_2 - \frac{1}{2}x_3 \\ x_3' &= -\frac{1}{2}x_1 - \frac{1}{2}x_2 + x_3 \end{aligned}$$

given that

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

are eigenvectors of the matrix

$$G = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

with corresponding eigenvalues $\lambda_1 = 0, \lambda_2 = \lambda_3 = \frac{3}{2}$.

Solution. First, note that the system we need to solve is equivalent to the matrix ODE $\mathbf{x}' = G\mathbf{x}$. If we can show that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent, then we can use Theorem 4.2.1 to find the general solution of the system. So we'll compute their Wronskian:

$$\begin{aligned} W(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= \begin{vmatrix} 1 & -1 & -\frac{1}{2} \\ 1 & 1 & -\frac{1}{2} \\ 1 & 0 & 1 \end{vmatrix} \\ &= 1 + \frac{3}{2} + \frac{1}{2} \\ &= 3 \end{aligned}$$

Since the Wronskian is nonzero, these eigenvectors are linearly independent. Therefore the general solution of the system is given by

$$\begin{aligned} \mathbf{x} &= c_1 e^{0t}\mathbf{x}_1 + c_2 e^{3t/2}\mathbf{x}_2 + c_3 e^{3t/2}\mathbf{x}_3 \\ &= \begin{bmatrix} c_1 - c_2 e^{3t/2} - \frac{1}{2}c_3 e^{3t/2} \\ c_1 + c_2 e^{3t/2} - \frac{1}{2}c_3 e^{3t/2} \\ c_1 + c_3 e^{3t/2} \end{bmatrix} \end{aligned}$$

or just

$$x_1 = c_1 - \left(c_2 - \frac{1}{2}c_3\right)e^{3t/2}$$
$$x_2 = c_1 + \left(c_2 - \frac{1}{2}c_3\right)e^{3t/2}$$
$$x_3 = c_1 + c_3 e^{3t/2}$$

Finding Eigenvalues and Eigenvectors

Theorem 4.2.1 shows that solving systems of first-order ODEs comes down to finding eigenvalues and eigenvectors of the corresponding matrix ODE. So it's important for us to know how to find these.

Let A be an $n \times n$ matrix and suppose that $\mathbf{v}$ is an eigenvector with corresponding eigenvalue λ. Then

$$A\mathbf{v} = \lambda\mathbf{v}.$$

We can rearrange this to get

$$A\mathbf{v} - \lambda\mathbf{v} = (A - \lambda I)\mathbf{v} = \mathbf{0}$$

where I is the identity matrix. Since $\mathbf{v} \neq \mathbf{0}$ (since it's an eigenvector!), linear algebra tells us that $\det(A - \lambda I) = 0$. This gives us the following theorem.

Theorem 4.2.3

The eigenvalues of a square matrix A are the solutions of the equation $\det(A - \lambda I) = 0$.

Definition 4.2.4

$\det(A - \lambda I) = 0$ is called the **characteristic equation** of the matrix A.

Example 4.2.5 Finding eigenvalues of a 2×2 matrix.

Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$.

Solution. First, we need to set up the characteristic equation of A. Since

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{bmatrix},$$

we get

$$\det(A - \lambda I) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda + 3$$

so the characteristic equation of A is

$$\lambda^2 - 2\lambda + 3 = 0$$

which has solutions $\lambda_1 = -1, \lambda_2 = 3$. So the eigenvalues of A are $-1, 3$.

These computations are easily verified by Sage or MATLAB/Octave. Sage can provide exact answers but the code is somewhat cumbersome. MATLAB on the other hand is designed for performing matrix computations and therefore its code for finding eigenvalues is simpler. We only need to use the `eig` command:

```
% Define the matrix
A = [1, 1; 1, 4];

% Compute the eigenvalues of A
eig(A)
```

```
ans =

    3.0000
   -1.0000
```

A useful fact to remember is that the eigenvalues of a "triangular" matrix are just the diagonal entries.

Example 4.2.6 Finding the eigenvalues of a triangular matrix.

Let

$$A = \begin{bmatrix} 1 & 3 & -4 & 5 \\ 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 10^{50\pi-300} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find the eigenvalues of A.

Solution. A is a triangular matrix since everything below the main diagonal is 0. Therefore the eigenvalues of A are $1, 3, 1, 0$.

Once we have the eigenvalues of a matrix, we can find their corresponding eigenvectors.

Example 4.2.7 Finding eigenvectors for a 2×2 matrix.

Find eigenvectors of $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$ corresponding to the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 3$.

Solution. Suppose that $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is an eigenvector corresponding to λ. Then we know that

$$\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \mathbf{v} = \lambda \mathbf{v} \Rightarrow \begin{bmatrix} v_1 + 4v_2 \\ v_1 + v_2 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix}.$$

This tells us that if $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is an eigenvector for λ, then its entries need to satisfy

$$\begin{aligned} v_1 + 4v_2 &= \lambda v_1 \\ v_1 + v_2 &= \lambda v_2 \end{aligned}$$

which boils down to

$$\begin{aligned} (1-\lambda)v_1 + 4v_2 &= 0 \\ v_1 + (1-\lambda)v_2 &= 0. \end{aligned}$$

Now set $\lambda = -1$ to get the system

$$\begin{aligned} 2v_1 + 4v_2 &= 0 \\ v_1 + 2v_2 &= 0 \end{aligned}$$

and so $v_1 = -2v_2$. We don't really care about what the entries of $\mathbf{v}$ look like so long as $\mathbf{v}$ is an eigenvector, so we can pick v_1, v_2 however we want, just so long as they satisfy this relation (and are not both 0!). So pick $v_2 = 1$, which forces $v_1 = -2$. Then

$$\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = -1$.
To find an eigenvector for $\lambda_2 = 3$ we just set $\lambda = 3$ and run through the same process:

$$\begin{aligned} -2v_1 + 4v_2 &= 0 \\ v_1 - 2v_2 &= 0 \end{aligned}$$

The second equation simplifies to $v_1 = 2v_2$, so one eigenvector for λ_2 is

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Of course, all of this can be done in Sage or MATLAB/Octave as well. If we use MATLAB/Octave, then the `eig` command once again does the heavy lifting for us. Each column of the matrix U produced below is an eigenvector of A.

```
# Define A
A = [1, 4; 1, 1];

# Compute eigenvectors
[U, V] = eig(A)
```

```
U =

    0.8944   -0.8944
    0.4472    0.4472


V =

    3.0000         0
         0   -1.0000
```

Looking forward to Theorem 4.2.8, note that the eigenvectors we found in Example 4.2.7 are linearly indepednent. This can be verified by computing the Wronskian as done using Sage below:

You may have noticed that the matrix constructed in the Sage cell here is actually "flipped": the eigenvectors are appearing as the rows instead of the columns. It turns out that this causes no problems for us since turning rows into columns (or columns into rows) has no affect on the determinant. Therefore the Wronskian is unchanged.

```
# Define our vectors
v1 = vector([-2, 1])
v2 = vector([2, 1])

# Define our matrix
M = matrix([v1, v2])

# Compute the determinant, i.e., the Wronskian
M.det()
```

```
-4
```

Solving Matrix ODEs

We now have the tools we need to begin solving matrix ODEs. Recall that if A is an $n \times n$ matrix with constant entries, and if $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are n linearly independent solutions of the matrix ODE $\mathbf{x}' = A\mathbf{x}$, then the general solution of the matrix ODE is

$$\mathbf{x} = \sum_{k=1}^{n} c_k \mathbf{x}_k.$$

Furthermore, if λ is an eigenvalue of A with eigenvector $\mathbf{v}$, then $e^{\lambda t}\mathbf{v}$ is a solution of $\mathbf{x}' = A\mathbf{x}$. So solving the matrix ODE $\mathbf{x}' = A\mathbf{x}$ requires finding enough eigenvectors and eigenvalues. A useful theorem is the following:

Theorem 4.2.8

Let A be an $n \times n$ matrix with constant entries. If the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are distinct (that is, none are repeated) then eigenvectors associated with different eigenvalues are linearly independent. That is, if $\mathbf{v}_i$ is an eigenvector corresponding to λ_i then the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

Example 4.2.9 Solving a matrix ODE.

Solve the matrix ODE given by $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}.$$

Solution. We already have everything we need. We know that the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 3$ from Example 4.2.5, and likewise some corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

thanks to Example 4.2.7. Since the eigenvalues are distinct it follows that these eigenvectors are linearly independent (we could also check this using the Wronskian). We can therefore build two linearly independent solutions to the matrix ODE:

$$\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1 = e^{-t}\begin{bmatrix}-2\\1\end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{\lambda_2 t}\mathbf{v}_2 = e^{3t}\begin{bmatrix}2\\1\end{bmatrix}.$$

So the general solution of the matrix ODE is

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1 e^{-t}\begin{bmatrix}-2\\1\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}2\\1\end{bmatrix}.$$

Note that the choice of eigenvector *doesn't matter*. We only need to find enough linearly independent eigenvectors for each distinct eigenvalue.

Example 4.2.10 Solving a first-order system with two equations.

Solve the first-order system given by

$$\begin{aligned} x_1' &= x_1 - 5x_2 \\ x_2' &= x_1 - x_2 \end{aligned}$$

where x_1 and x_2 are functions of t.

Solution. First, note that this system is equivalent to the matrix ODE $\mathbf{x}' = A\mathbf{x}$ where

$$\mathbf{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix}1 & -5\\1 & -1\end{bmatrix}.$$

To solve this system we need to find the eigenvalues and eigenvectors of A, and then use these to build our general solution.

1. Find the eigenvalues.

   We find the eigenvalues of A by solving the characteristic equation $\det(A-\lambda I) = 0$ for λ. Since $\det(A-\lambda I) = \lambda^2+4$, we see that the eigenvalues of A are $\lambda_1 = -2i$ and $\lambda_2 = 2i$. The fact that these eigenvalues are complex is *not* a problem. They're still distinct, so our method will work.

2. Find corresponding eigenvectors.

   Set $\mathbf{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$. Then $A\mathbf{v} = \lambda\mathbf{v}$ implies that

   $$\begin{aligned} v_1 - 5v_2 &= \lambda v_1 \\ v_1 - v_2 &= \lambda v_2 \end{aligned}$$

   or just

   $$\begin{aligned} (1-\lambda)v_1 - 5v_2 &= 0 \\ v_1 - (1+\lambda)v_2 &= 0. \end{aligned}$$

   Setting $\lambda = -2i$ in the second equation gives $v_1 = (1-2i)v_2$, so an eigenvector of A corresponding to $\lambda_1 = -2i$ is

   $$\mathbf{v} = \begin{bmatrix}1-2i\\1\end{bmatrix}.$$

Similarly, an eigenvector corresponding to $\lambda_2 = 2i$ is

$$\mathbf{v}_2 = \begin{bmatrix} 1+2i \\ 1 \end{bmatrix}.$$

3. Find the general solution.

   At this step it is easy to construct the solution of the matrix ODE. It's just

$$\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 = \begin{bmatrix} c_1 e^{-2it}(1-2i) + c_2 e^{2it}(1+2i) \\ c_1 e^{-2it} + c_2 e^{2it} \end{bmatrix}.$$

Example 4.2.11 Solving a system of three differential equations.

Solve the first-order system

$$\begin{aligned} \frac{dx_1}{dt} &= 3x_1 + x_3 \\ \frac{dx_2}{dt} &= 9x_1 - x_2 + 2x_3 \\ \frac{dx_3}{dt} &= -9x_1 + 4x_2 - x_3 \end{aligned}$$

Solution. As long as this system has distinct eigenvalues the above method will work. Once again we rewrite the system as a matrix ODE; in this case, the matrix ODE we must solve is

$$\mathbf{x}' = \begin{bmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{bmatrix} \mathbf{x} = A\mathbf{x}.$$

To find the eigenvalues we must solve the characteristic equation $\det(A - \lambda I) = 0$. However, we can also use Sage to perform this task.

We could also use MATLAB/Octave, but the resulting eigenvectors wouldn't look as nice as the output provided by Sage. This is because MATLAB's `eig` command produces **normalized** output which often involves dividing entries by square roots.

```
# Define our matrix
M = matrix([ [3, 0, 1],
[9, -1, 2],
[-9, 4, -1]] )

# Finds eigenvectors, corresponding eigenvalues and
    "algebraic multiplicty".
M.eigenvectors_right()
```

```
[(3,
  [
  (1, 9/4, 0)
  ],
  1),
 (-1 - 1*I, [(1, 2 + 1*I, -4 - 1*I)], 1),
 (-1 + 1*I, [(1, 2 - 1*I, -4 + 1*I)], 1)]
```

This produces a list containing the eigenvalues of A as well as the corresponding eigenvectors. So we see that the eigenvalues are given by

$$\lambda_1 = 3, \quad \lambda_2 = -1 - i \quad \text{and} \quad \lambda_3 = -1 + i,$$

while the corresponding eigenvectors are given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \frac{9}{4} \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2+i \\ -4-i \end{bmatrix} \quad \text{and} \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2-i \\ -4+i \end{bmatrix}.$$

We now have everything we need for the general solution of the matrix ODE. It's just

$$\mathbf{x} = c_1 e^{3t} \begin{bmatrix} 1 \\ \frac{9}{4} \\ 0 \end{bmatrix} + c_2 e^{(-1-i)t} \begin{bmatrix} 1 \\ 2+i \\ -4-i \end{bmatrix} + c_3 e^{(-1+i)t} \begin{bmatrix} 1 \\ 2-i \\ -4+i \end{bmatrix}.$$

Applications of Matrix ODEs

Now we use matrix ODEs to model physical systems. The methods we've developed for solving matrix ODEs will then let us come up with descriptions for such systems. Recall that we introduced systems of ODEs (and then matrix ODEs) to model quantities that depended on time (an independent variable) and each other (dependent variables). The physical systems we will consider will be ones where the quantities of interest depend on each other in some way.

Example 4.2.12 Determining salt concentration in connected tank system.

Two brine tanks are set up as in Figure 4.1.3. Fresh water flows into the tank at a rate of r_1, well-mixed solution flows from Tank 1 to Tank 2 at a rate of r_2 and well-mixed solution flows out of Tank 2 at a rate of r_3. Suppose that r_1, r_2 and r_3 are $5\,\frac{\text{gal}}{\text{min}}$, the volume of solution in Tank 1 is $10\,\text{gal}$ and the volume of solution in Tank 2 is $7\,\text{gal}$. Suppose Tank 1 has $5\,\text{lb}$ of salt at time $t = 0$ and Tank 2 has $2\,\text{lb}$ of salt at time $t = 0$. How much salt is in each tank at time t?

Solution. To start, let $x_1(t)$ denote the amount of salt in Tank 1 at time t and $x_2(t)$ denote the amount of salt in Tank 2 at time t, where t is in minutes. Then from Section 4.1, we know that

$$\begin{aligned} x_1' &= -\frac{1}{2}x_1 \\ x_2' &= \frac{1}{2}x_1 - \frac{5}{7}x_2 \end{aligned}$$

If we set

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{5}{7} \end{bmatrix}$$

then this system is equivalent to the matrix ODE $\mathbf{x}' = A\mathbf{x}$.
To solve this, we find the eigenvalues and corresponding eigenvectors. To find the eigenvalues, we could solve the characteristic equation $\det(A - \lambda I) = 0$ or use technology, but it's easier to note that A is a triangular matrix. So the eigenvalues are just $\lambda_1 = -\frac{1}{2}, \lambda_2 = -\frac{5}{7}$.
Now we find corresponding eigenvectors. So let

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

If $\mathbf{v}$ is an eigenvector for λ, then we know $A\mathbf{v} = \lambda\mathbf{v}$, which gives the system

$$\begin{aligned} (-\frac{1}{2} - \lambda)v_1 &= 0 \\ \frac{1}{2}v_1 + (-\frac{5}{7} - \lambda)v_2 &= 0 \end{aligned}$$

If we set $\lambda = -\frac{1}{2}$, then we just get $v_1 = \frac{3}{14}v_2$. So an eigenvector corresponding to $\lambda_1 = -\frac{1}{2}$ is

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 14 \end{bmatrix}.$$

Similarly, if we set $\lambda = -\frac{5}{7}$ we get $v_1 = 0$, but no restrictions on v_2. So an eigenvector corresponding to $\lambda_2 = -\frac{5}{7}$ is

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We can now write down the general solution of the matrix ODE:

$$\mathbf{x} = c_1 e^{-\frac{5t}{7}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 e^{-\frac{t}{2}} \begin{bmatrix} 3 \\ 14 \end{bmatrix} = \begin{bmatrix} 3c_2 e^{-\frac{t}{2}} \\ c_1 e^{-\frac{5t}{7}} + 14 c_2 e^{-\frac{t}{2}} \end{bmatrix}.$$

But we're not done yet, since we have the initial conditions $x_1(0) = 5$ and $x_2(0) = 2$, or in terms of our matrix ODE

$$\mathbf{x}(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

We can use this to find c_1 and c_2. If we set $t = 0$ then we get

$$\begin{bmatrix} 3c_2 \\ c_1 + 14c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix},$$

so $c_2 = \frac{5}{3}$ and $c_1 = 2 - 14\frac{5}{3} = -\frac{64}{3}$.
So the solution of the matrix ODE (and hence the original system) is

$$\mathbf{x} = \begin{bmatrix} 5e^{-\frac{t}{2}} \\ -\frac{64}{3}e^{-\frac{5t}{7}} + \frac{70}{3}e^{-\frac{t}{2}} \end{bmatrix}.$$

The amount of salt in the first tank, x_1, is given by the top entry and the amount of salt in the second tank, x_2, is given by the bottom entry.

4.3 Phase Portraits and Critical Points

The techniques used in Section 1.6 to study long-term behavior of solutions near critical points can be adapted to higher dimensional systems as well. The main difference now is that we must consider the *phase plane* instead of a simple one-dimensional number line. This also allows for different types of behavior at critical points for solutions, or *trajectories*, near the critical point.

The Phase Plane

Just as we were able to plot direction fields for first-order autonomous ODEs, we can do something similar for autonomous first-order systems with two equations and constant coefficients. These are precisely the systems that can be written as a matrix ODE of the form

$$\mathbf{x}' = A\mathbf{x}$$

where A is a 2×2 matrix.

Consider the first-order system

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 \\ x_2' &= a_{21}x_1 + a_{22}x_2 \end{aligned} \tag{4.1}$$

or

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

The solution of this system looks like $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

As t varies, $\mathbf{x}(t)$ will trace out a curve in the x_1x_2-plane, which we call a **trajectory**. The x_1x_2-plane is called the **phase plane**, and the collection of all trajectories of the system (4.1) is called the **phase portrait** of the system. The phase portrait of a system provides us with a way to study the behavior of solutions of (4.1) without actually solving the system.

Example 4.3.1 Sketching a phase portrait.

Sketch a phase portrait for the system

$$\begin{aligned} \frac{dy_1}{dt} &= 2y_1 - 3y_2 \\ \frac{dy_2}{dt} &= -2y_1 + y_2. \end{aligned}$$

Solution. First, note that we can rewrite the system as $\mathbf{y}' = A\mathbf{y}$ using

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix}.$$

Now, we can view $\mathbf{y}$ as corresponding to a point in the phase plane. Hence $\mathbf{y}'$ corresponds to a *tangent* of a trajectory passing through the point $\mathbf{y}$.

For example, let's find the tangent at the point $\mathbf{y} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. The tangent is given by the corresponding $\mathbf{y}'$ at this point which is just $A\mathbf{y}$:

$$\mathbf{y}' = A\mathbf{y} = \begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$$

So at the point $(2, 2)$ in the phase plane, the trajectory should be heading in the same direction as that of the point $(-2, -2)$ relative to the origin. In other words, the tangent vector would point two units left and two units down from the point $(2, 2)$.

Similarly, if we let $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then we get

$$\mathbf{y}' = A\mathbf{y} = \begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

So the trajectory going through $(0, 1)$ in the phase plane should be heading in the direction of $(-3, 1)$ viewed from the origin.

Plotting other points in the phase plane like this, we get Figure 4.3.2. One thing we can see from this is that trajectories that lie on the line, equivalently, those with initial conditions $y_1(0) = y_2(0)$, appear to approach the origin while all others move away from the origin. We can see why this is by looking at the general solution of the original system, which is

$$\mathbf{y} = c_1 e^{4t} \begin{bmatrix} -3 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

If $\mathbf{y}$ lies on the line $y_2 = y_1$, then c_1 has to equal 0, which follows from the fact that $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are linearly independent. So trajectories that lie on the line $y_2 = y_1$ must take the form $\mathbf{y} = c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and every solution of this form goes to $\mathbf{0}$ as $t \to \infty$. *Every other trajectory* will move away from the origin as $t \to \infty$, although the trajectories that lie on the line $y_2 = -\frac{2}{3}y_1$ will travel to the origin as $t \to -\infty$ (i.e. "backwards in time"):

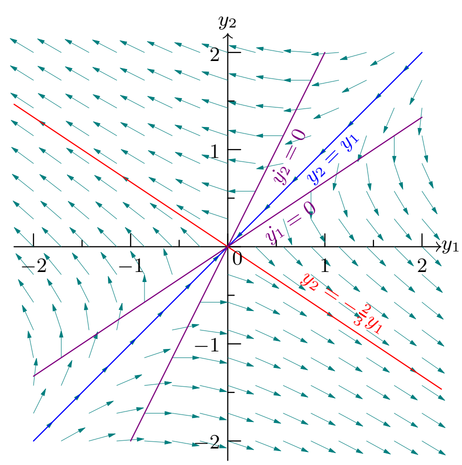

Figure 4.3.2 The phase portrait from Example 4.3.1. The blue line represents the two incoming trajectories at $\mathbf{0}$ and the red line represents two outgoing trajectories at $\mathbf{0}$.

Vector fields can also be plotted easily using SageMath. The code cell below demonstrates the use of the `plot_vector_field` command to sketch the phase portrait from Example 4.3.1.

```
# Define variables
var('y1␣y2')

# Create phase portrait for system
VF=plot_vector_field([2*y1 - 3*y2,-2*y1 +
    y2],[y1,-2,2],[y2,-2,2])

# Display the plot
show(VF)
```

Note that $\mathbf{x} = \mathbf{0}$ is *always* a solution of $\mathbf{x}' = A\mathbf{x}$. This is because $\mathbf{0}' = A\mathbf{0} = \mathbf{0}$. We call $\mathbf{x} = \mathbf{0}$ the **equilibrium solution** or **critical point** of the system $\mathbf{x}' = A\mathbf{x}$. Later in this section, we will be concerned with the behavior of trajectories of the system $\mathbf{x}' = A\mathbf{x}$ near the equilibrium solution $\mathbf{x} = \mathbf{0}$. One thing we will see is that the behavior is determined in large part by the eigenvalues of the matrix A.

We will separate the behavior of trajectories at the critical point $\mathbf{x} = \mathbf{0}$ into five different cases:

Table 4.3.3 Types of critical points

Classification	Behavior at $\mathbf{0}$
Improper node	Every trajectory except two has the same limiting tangent at $\mathbf{0}$
Proper node	For every direction $\mathbf{d}$ there exists trajectory with limiting tangent $\mathbf{d}$
Saddle point	Two incoming trajectories, two outgoing trajectories; all others bypass $\mathbf{0}$
Center	$\mathbf{0}$ is enclosed by infinitely many closed (repeating) trajectories
Spiral point	Trajectories spiral inwards or outwards from $\mathbf{0}$

$\mathbf{0}$ was a saddle point in Example 4.3.1 since there were incoming trajectories on the line $y_2 = y_1$ and outgoing trajectories on the line $y_2 = -\frac{2}{3}y_1$ as indicated in Figure 4.3.2.

Example 4.3.4 Classifying a critical point using a phase portrait.

Using a phase portrait, determine the type of critical point that $\mathbf{y} = \mathbf{0}$ is for the matrix ODE $\mathbf{y}' = A\mathbf{y}$ where

$$\mathbf{y} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}.$$

Solution. As seen in Figure 4.3.5, every (nonzero) trajectory will spiral outward from $\mathbf{y} = \mathbf{0}$ as $t \to \infty$, so $\mathbf{0}$ is a spiral point of this system. To see why, we only need to look at the eigenvalues of A, which we find to be

$$\lambda_1 = 3 + 4i, \lambda_2 = 3 - 4i.$$

This means that the general solution of $\mathbf{y}' = A\mathbf{y}$ must look like

$$\begin{aligned} \mathbf{y} &= c_1 e^{(3+4i)t}\mathbf{y}_1 + c_2 e^{(3-4i)t}\mathbf{y}_2 \\ &= e^{3t}\left[c_1 e^{4it}\mathbf{y}_1 + c_2 e^{-4it}\mathbf{y}_2\right] \\ &= e^{3t}\left[(\cos 4t)\mathbf{x}_1 + (\sin 4t)\mathbf{x}_2\right]. \end{aligned}$$

The real part of the eigenvalues leads to the "growth term" of e^{3t} appearing in the solution, which causes the trajectories to diverge as $t \to \infty$. The imaginary part of the eigenvalues leads to the "oscillating terms" of $\cos 4t, \sin 4t$ appearing in the solution, which gives the trajectories their spiral motion.

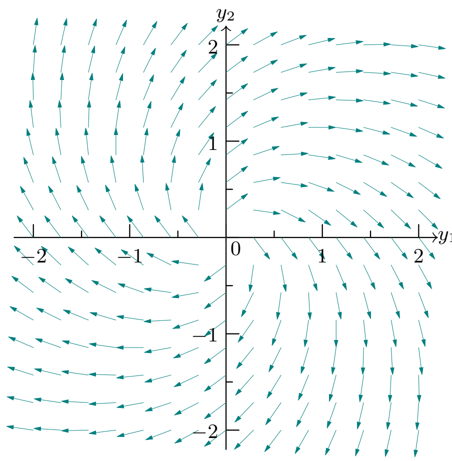

Figure 4.3.5 The phase portrait for Example 4.3.4.

In general, the eigenvalues of the matrix A in the system $\mathbf{y}' = A\mathbf{y}$ will determine the type of critical point that $\mathbf{0}$ is for the system $\mathbf{y}' = A\mathbf{y}$.

Example 4.3.6 Classifying trajectories algebraically.

What kind of critical point is $\mathbf{y} = \mathbf{0}$ for the system

$$\begin{aligned} y_1' &= -4y_2 \\ y_2' &= 4y_1 \end{aligned}$$

where $y_i = y_i(t)$?

Solution. We could sketch the phase portrait for this system, but we can also determine the behavior of the trajectories $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ if we can find a relationship between y_1 and y_2. To do so, we "cross-multiply" the system to get

$$4y_1y_1' = -4y_2y_2' \quad \text{or} \quad 4y_1\,dy_1 = -4y_2\,dy_2 .$$

So we can integrate this to get

$$2y_1^2 = -2y_2^2 + C \quad \text{or} \quad y_1^2 + y_2^2 = C_1.$$

This is the equation of a circle of radius $\sqrt{C_1}$, and so every trajectory $\mathbf{y}$ for this system will be a circle centered at $\mathbf{0}$. Hence $\mathbf{0}$ is a center.

Eigenvalue Criteria for Stability

Consider the matrix ODE $\mathbf{y}' = A\mathbf{y}$. Let λ_1, λ_2 denote the eigenvalues of the 2×2 matrix A. Then $\mathbf{0}$ is a:

Table 4.3.7 Eigenvalue conditions for stability.

Name	Conditions on λ_1, λ_2
Node	Real, same sign
Saddle point	Real, opposite sign
Center	Pure imaginary
Spiral point	Complex, not pure imaginary

The rule of thumb is this: the real parts of the eigenvalues determine whether a trajectory moves towards or away from the origin, and the imaginary part determines if the trajectory has a periodic/oscillating nature to it.

We say that the origin is a **stable** critical point of $\mathbf{y}' = A\mathbf{y}$ if all trajectories that start "close" to $\mathbf{0}$ remain close at all future times. Equivalently, it's stable if each trajectory will eventually be contained within some circle centered at the origin as $t \to \infty$. Otherwise, we say that $\mathbf{0}$ is **unstable**. If it so happens that every trajectory that starts close to $\mathbf{0}$ tends to $\mathbf{0}$ as $t \to \infty$, we then say that $\mathbf{0}$ is a **stable and attractive** (or **asymptotically stable**) critical point. Equivalently, $\mathbf{0}$ is asymptotically stable if *every* trajectory goes to $\mathbf{0}$ as $t \to \infty$.

Example 4.3.8 Eigenvalue conditions for asymptotic stability.

Let $\mathbf{y}' = A\mathbf{y}$ denote a matrix ODE where A is a constant 2×2 matrix. What conditions on the eigenvalues of A will give an asymptotically stable critical point at $\mathbf{y} = \mathbf{0}$?

Solution. Let $\mathbf{y} = \mathbf{y}(t)$ denote a nonzero solution of the matrix ODE (and therefore a trajectory). Then in order for $\mathbf{0}$ to be asymptotically stable, we need $\mathbf{y} \to \mathbf{0}$ as $t \to \infty$. Let λ_1, λ_2 denote the eigenvalues of A. Then $\mathbf{y}$ will have the form

$$\mathbf{y} = c_1 e^{\lambda_1 t}\mathbf{y}_1 + c_2 e^{\lambda_2 t}\mathbf{y}_2.$$

The previous paragraph shows that $\mathbf{y}$ must go to $\mathbf{0}$ as $t \to \infty$ if either $c_1 = c_2 = 0$ or if each exponential goes to 0 as $t \to \infty$. Since we assume $\mathbf{y} \neq \mathbf{0}$, this means we need $e^{\lambda_i t} \to 0$ for $i = 1, 2$ as $t \to \infty$. This implies that the *real part* of each eigenvalue must be negative, because the real part of each eigenvalue is what determines the growth of $e^{\lambda_i t}$: if $\lambda_i = a + bi$, then

$$e^{\lambda_i t} = e^{at}[A \cos bt + B \sin bt].$$

So $\mathbf{0}$ is asymptotically stable if the real parts of *both* eigenvalues are negative.

By a similar argument to that used in Example 4.3.8, we can say that $\mathbf{0}$ is stable as long as the real part of each eigenvalue is no greater than 0. Likewise, $\mathbf{0}$ is unstable if the real part of *any* eigenvalue is positive.

Example 4.3.9 Long term behavior of a system of interconnected tanks.

Two tanks T_1 and T_2 containing 200 gal each of a water-salt mixture are set up as follows:

- Tank 1: Pure water flows in at $12\,\frac{\text{gal}}{\text{min}}$ and solution from Tank 2 flows in at 4 $\frac{\text{gal}}{\text{min}}$; solution also flows out of Tank 1 and into Tank 2 at 16 $\frac{\text{gal}}{\text{min}}$.

- Solution from Tank 1 flows in at 16 $\frac{\text{gal}}{\text{min}}$; solution flows out of Tank 2 and into Tank 1 at 4 $\frac{\text{gal}}{\text{min}}$, and solution is emptied from Tank 2 at an addition rate of 12 $\frac{\text{gal}}{\text{min}}$

Will the salt eventually empty from both tanks?

Solution. Let $y_1(t)$ denote the amount of salt (in pounds) in Tank 1 at time t (in minutes), and let $y_2(t)$ do the same for Tank 2. Then

$$\begin{aligned} y_1' &= 4\frac{y_2}{200} - 16\frac{y_1}{200} \\ y_2' &= 16\frac{y_1}{200} - 16\frac{y_2}{200}. \end{aligned}$$

This system is equivalent to the matrix ODE $\mathbf{y}' = A\mathbf{y}$ where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} -\frac{2}{25} & \frac{1}{50} \\ \frac{2}{25} & -\frac{2}{25} \end{bmatrix}.$$

We need to determine the long-term behavior of solutions of this ODE, which is itself determined by the eigenvalues of A.
The eigenvalues of A are

$$\lambda_1 = -\frac{1}{25} \quad \text{and} \quad \lambda_2 = -\frac{3}{25}.$$

Since both eigenvalues have negative real part, it follows that $\mathbf{0}$ is an asymptotically stable critical point of $\mathbf{y}' = A\mathbf{y}$. Therefore *every* trajectory $\mathbf{y} \to \mathbf{0}$ as $t \to \infty$. So no matter how much salt is initially in the tanks, the amount of salt will always go to 0.

4.4 Nonlinear Systems

Now we apply phase plane methods to study **nonlinear autonomous systems**, which for systems involving two ODEs take the form

$$\begin{aligned} y_1' &= f_1(y_1, y_2) \\ y_2' &= f_2(y_1, y_2). \end{aligned}$$

where $y_i = y_i(t)$.

Autonomous just means we can write the system without explicitly referring to the independent variable t.

We can also write such a system as a vector equation:

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}). \tag{4.2}$$

although not as a matrix ODE (if the functions f_i are nonlinear).

Just as in the previous sections, the **phase plane** is still the y_1y_2-plane, **trajectories** are still the solutions $\mathbf{y}$ of (4.2) (represented as curves in the phase plane), and the **phase portrait** of (4.2) is the set of all trajectories in the phase plane.

We call a point $P = (y_1, y_2)$ in the phase plane a **critical point** of (4.2) if

$$f_1(y_1, y_2) = 0 \quad \text{and} \quad f_2(y_1, y_2) = 0.$$

In other words, P is a critical point of $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ if $\mathbf{f}(P) = \mathbf{0}$. Just as before, critical points represent solutions of the system that are in equilibrium.

Example 4.4.1 The pendulum equation.

Express the *pendulum equation* $\theta'' + \frac{g}{l}\sin\theta = 0$, where $\theta = \theta(t)$ represents the angular displacement of a pendulum from the vertical, as a nonlinear system $\boldsymbol{\theta}' = \mathbf{f}(\boldsymbol{\theta})$ and then find its critical points.

Solution. First, we have to rewrite the pendulum ODE as a first order system. We can do this without too much trouble as follows: set

$$\theta_1 = \theta \quad \text{and} \quad \theta_2 = \theta_1' = \theta'.$$

Then the ODE $\theta'' + \frac{g}{l}\sin\theta = 0$ turns into the system

$$\begin{aligned} \theta_1' &= \theta_2 \\ \theta_2' &= -\frac{g}{l}\sin\theta_1, \end{aligned}$$

which we can also write as $\boldsymbol{\theta}' = \mathbf{f}(\boldsymbol{\theta})$ using

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \quad \text{and} \quad \mathbf{f}(\boldsymbol{\theta}) = \begin{bmatrix} \theta_2 \\ -\frac{g}{l}\sin\theta_2 \end{bmatrix}.$$

Now we need to find the critical points $\boldsymbol{\theta}$ in the $\theta_1\theta_2$-plane that make $\mathbf{f}(\boldsymbol{\theta}) = \mathbf{0}$. This requires $\theta_2 = 0$ and $\theta_1 = \pm k\pi$ for $k = 0, \pm 1, \pm 2, \ldots$, and so the critical points of this system are all points of the form $(\pm k\pi, 0)$.

Classification of Critical Points and Linearization

Critical points of systems are important because they can represent long-term behavior of a system. For example, if we have a first-order system representing the population of two species, and it turns out the the origin is asymptotically stable, then this suggests that both species could be driven to extinction. So we want to classify critical points for nonlinear systems in addition to what we have already for linear systems; unfortunately, nonlinear systems are often difficult, if not outright impossible, to solve exactly.

Thankfully, in many cases we can approximate a nonlinear system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ with critical points P_i by a suitably chosen linear system $\mathbf{y}' = A\mathbf{y}$ at each critical point P_i; we call such a system the **linearization** at P_i.

Definition 4.4.2 The Jacobian of a Nonlinear System.

Let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{and} \quad \mathbf{f}(\mathbf{y}) = \begin{bmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{bmatrix}.$$

The **Jacobian** of $\mathbf{f}$ is the matrix $J_{\mathbf{f}}(y_1, y_2)$ given by

$$J_{\mathbf{f}}(y_1, y_2) = \begin{bmatrix} \dfrac{\partial f_1}{\partial y_1} & \dfrac{\partial f_1}{\partial y_2} \\ \dfrac{\partial f_2}{\partial y_1} & \dfrac{\partial f_2}{\partial y_2} \end{bmatrix}.$$

If $\mathbf{f}$ is understood from context, we often write J instead of $J_{\mathbf{f}}$ for the Jacobian.

The Jacobian is important since it allows us to *linearize* a nonlinear system. More precisely, the **linearization** of $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ at the point $P = (p_1, p_2)$ is the linear system $\mathbf{y}' = A\mathbf{y}$, where

$$A = J(p_1, p_2).$$

Example 4.4.3 Linearizing the pendulum system.

Find the linearization of the pendulum system $\boldsymbol{\theta}' = \mathbf{f}(\boldsymbol{\theta})$ at the critical point $(0, 0)$.

Solution. For this system, we have $f_1(\theta_1, \theta_2) = \theta_2$ and $f_2(\theta_1, \theta_2) = -\frac{g}{l}\sin\theta_1$. The Jacobian is then given by

$$J(\theta_1, \theta_2) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos\theta_1 & 0 \end{bmatrix}.$$

So to get the linearization we need to set

$$A = J(0, 0) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}.$$

The linearization of a nonlinear system isn't just useful for approximating the nonlinear system. It's also incredibly useful for classifying the critical points of a nonlinear system; for the most part, the eigenvalues of the matrix A from the linearization also classify the critical points of the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$.

Example 4.4.4 Classifying critical points using linearization.

Find and classify the critical points of the nonlinear system

$$\begin{aligned} \frac{dx}{dt} &= x - y - x^2 + xy \\ \frac{dy}{dt} &= -x^2 - y. \end{aligned}$$

This example taken from here.[2]

Solution. The critical points occur at intersections between the nullclines $\dot{x} = 0$ and $\dot{y} = 0$. The equations of the nullclines for $\dot{x} = 0$ are

$$\begin{aligned} x &= 1 \\ y &= x, \end{aligned}$$

while the equation of the nullcline $\dot{y} = 0$ is

$$y = -x^2.$$

Hence there are three critical points for this system as seen in Figure 4.4.5. In particular, these points are $(0,0), (-1,-1)$ and $(1,-1)$.

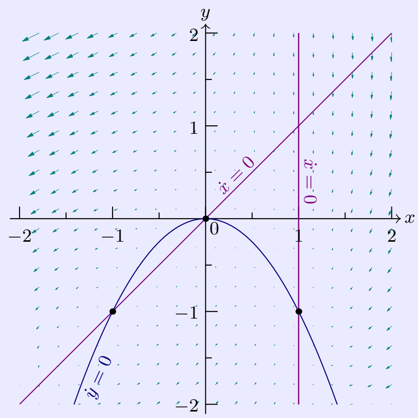

Figure 4.4.5 The phase portrait and nullclines for the system

To determine the behavior of solutions at these critical points, we'll find the Jacobian at each point. First, we have

$$J(x,y) = \begin{bmatrix} 1-2x+y & x-1 \\ -2x & -1 \end{bmatrix}.$$

At $(0,0)$, we get

$$J(0,0) = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}.$$

The eigenvalues are 1 and -1, meaning that this critical point is a saddle point.

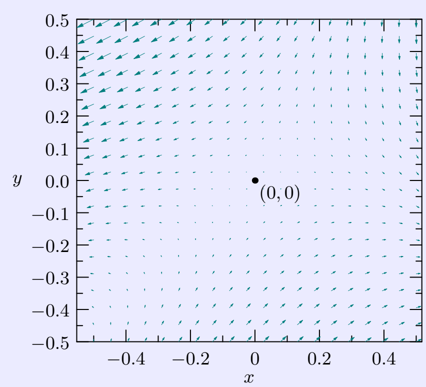

Figure 4.4.6 The phase portrait at $(0,0)$

At $(-1,-1)$ we get

$$J(-1,-1) = \begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix}$$

which has eigenvalues $\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{7}i$. Hence $(-1,-1)$ should be a spiral point.

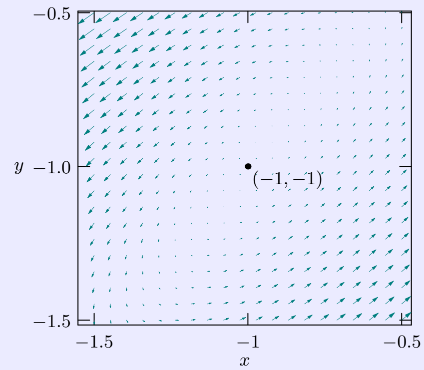

Figure 4.4.7 The phase portrait at $(-1,-1)$

Finally, at $(1,-1)$ we get

$$J(1,-1) = \begin{bmatrix} -2 & 0 \\ -2 & -1 \end{bmatrix},$$

which has eigenvalues $-2, -1$. Hence $(1,-1)$ is an asymptotically stable node.

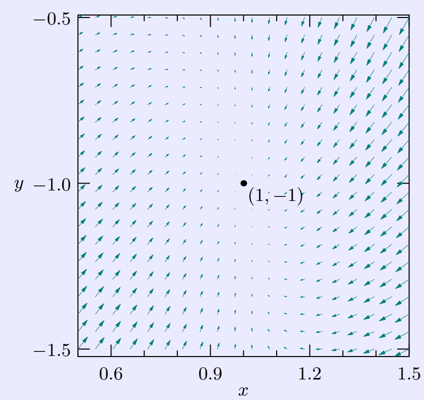

Figure 4.4.8 The phase portrait at $(1,-1)$

Linearization works well to classify the behavior of systems at certain types of critical points. A critical point P of a system is **hyperbolic** if the Jacobian $J(P)$ has eigenvalues with nonzero real part. Unfortunately, linearization is not guaranteed to give an accurate description of the behavior of non-hyperbolic critical points.

Example 4.4.9 The Lotka-Volterra population model.

Predator-prey populations can be modeled using the *Lotka-Volterra model*. Let $y_1(t)$ denote the population of a prey species at time t and let $y_2(t)$ denote the population of a predator species at time t. Then the **Lotka-Volterra model** says that

$$\begin{aligned} y_1' &= ay_1 - by_1y_2 \\ y_2' &= ky_1y_2 - ly_2, \end{aligned}$$

where $a, b, k, l > 0$. Find and classify the critical points of this system.

[2] www.math.uci.edu/~ndonalds/math3d/nonlinear.pdf

Solution. The critical points are the points (y_1, y_2) that satisfy the equations

$$ay_1 - by_1y_2 = 0 \quad \text{and} \quad ky_1y_2 - ly_2 = 0.$$

Equivalently, we need

$$y_1(a - by_2) = 0 \quad \text{and} \quad y_2(ky_1 - l) = 0.$$

This has solutions $y_1 = y_2 = 0$ and $y_1 = \frac{l}{k}, y_2 = \frac{a}{b}$, which shows that the critical points are $(0, 0)$ and $(\frac{l}{k}, \frac{a}{b})$.
To classify the critical points of this system we will linearize the system. The Jacobian of

$$\mathbf{f}(\mathbf{y}) = \begin{bmatrix} ay_1 - by_1y_2 \\ ky_1y_2 - ly_2 \end{bmatrix}$$

is

$$J(y_1, y_2) = \begin{bmatrix} a - by_2 & -by_1 \\ ky_2 & ky_1 - l \end{bmatrix}.$$

Now we will examine the Jacobian at each critical point.
At $(0, 0)$, we get

$$J(0, 0) = \begin{bmatrix} a & 0 \\ 0 & -l \end{bmatrix},$$

which has eigenvalues $\lambda = a, -l$ which indicate a saddle point. Since these eigenvalues have nonzero real part, the origin is a hyperbolic critical point of the system and so we know that it also behaves as a saddle point in the original system. In particular, there exist trajectories heading into the origin, so it's possible for both species to go extinct in this model.
Now we'll classify the second critical point $(\frac{l}{k}, \frac{a}{b})$. The Jacobian at this point gives us the matrix

$$A = J(\frac{l}{k}, \frac{a}{b}) = \begin{bmatrix} 0 & -\frac{bl}{k} \\ \frac{ak}{b} & 0 \end{bmatrix}.$$

This matrix has characteristic equation $\lambda^2 + al = 0$, and so has eigenvalues $\lambda = \pm i\sqrt{al}$. Since the eigenvalues are pure imaginary, this suggests that $(\frac{l}{k}, \frac{a}{b})$ is a center, which is indeed the case. In particular, trajectories near this critical point *must be periodic*. Unfortunately we can't quite justify this conclusion. This is because both eigenvalues of the Jacobian at this critical point have zero real part, which means that this critical point is not hyperbolic and so the behavior of the linearization is not guaranteed to match the behavior of the original system at this critical point. That said, a more detailed analysis (or simply a sketch of the phase portrait) does indeed confirm that $(\frac{l}{k}, \frac{a}{b})$ is a center of the original system.

Chapter 5

Series Solutions of ODEs

In Chapter 2 we developed a general method for solving linear ODEs with constant coefficients. There were ODEs of the form

$$ay'' + by' + cy = 0$$

for some constants a, b, c. We saw that the solution of such ODEs looked like exponentials and then determined which exponentials would provide us with a general solution.

Now we move on to more complicated (but still linear) ODEs of the form

$$y'' + P(x)y' + Q(x)y = R(x), \tag{5.1}$$

with $R(x) = 0$ typically. Some extremely important ODEs of this form include Bessel's equation. However, the method of characteristic equations is not flexible enough to solve come ODEs of this form.

To address this, we will change our choice of *ansatz* from exponential solutions $y = e^{rx}$ to *power series solutions* $y = \sum_{k=0}^{\infty} a_k x^k$. This change will allow us more flexibility in finding solutions to equations of the form given in (5.1).

> Note that $e^{rx} = \sum_{k=0}^{\infty} \frac{r^k}{k!} x^k$, and so the exponential solutions found in Chapter 2 were a special case of the more general series solution. In fact, the same is true of the other solution methods we've discussed for ODEs.

5.1 Power Series Method

Since power series form the basis of our solution strategy in this chapter, we begin by reviewing some important concepts related to power series and their convergence.

Review of Power Series

In calculus, it's important to know how to differentiate and integrate functions. For some functions (say, $x - 1, x^2, 3 - x^5$) it can be very straightforward, but for others (such as e^{-x^2}) it can be impossible.

At least, it can be impossible to integrate certain functions in terms of the "everyday", or *elementary* functions that we're used to.

Power series were introduced in calculus to allow us to write complicated functions $f(x)$ in terms of simpler functions $1, x, x^2, \ldots$. In particular, our goal is to write $f(x)$ in the form

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = \sum_{k=0}^{\infty} c_k x^k \tag{5.2}$$

where the coefficients c_k are all constants.

Definition 5.1.1 Power series.

A **power series (centered at** 0**)** is a series (that is, an infinite sum) of the form $\sum_{k=0}^{\infty} c_k x^k$. The power series is said to **converge** on some interval I if the sum exists for each x in I.

A power series doesn't have to start at $k = 0$, but *it may not contain any negative powers of* x.

The question, now, is to determine the values of the coefficients c_k to make (5.2) true. If we look at the equation we see that we can solve for c_0 very easily. All that we need to do is to set $x = 0$ in (5.2) to make all of the other terms disappear:

$$f(0) = c_0 + c_1 \cdot 0 + \cdots = c_0.$$

We can use a similar approach to solve for c_1 by plugging in $x = 0$, but we need to get rid of the power of x attached to it. This is done by taking the derivative of $f(x)$ and then setting $x = 0$:

$$\begin{aligned} f'(x) &= c_1 + 2c_2 x + 3c_3 x^2 + \ldots \\ f'(0) &= c_1. \end{aligned}$$

The same trick works for c_2:

$$\begin{aligned} f''(x) &= 2 \cdot 1 c_2 + 3 \cdot 2 c_3 x + \ldots \\ f''(0) &= 2 \cdot 1 c_2 \end{aligned}$$

so $c_2 = \frac{f''(0)}{2\cdot 1}$. Let's try this one more time to get c_3:

$$\begin{aligned} f^{(3)}(x) &= 3 \cdot 2 \cdot 1 c_3 + \ldots \\ f^{(3)}(0) &= 3 \cdot 2 \cdot 1 c_3 \end{aligned}$$

and so $c_3 = \frac{f^{(3)}(0)}{3\cdot 2\cdot 1}$.

In general, to get the coefficient c_k of x^k in the power series of $f(x)$, we have the following equation:

$$c_k = \frac{f^{(k)}(0)}{k \cdot (k-1) \cdot \cdots \cdot 2 \cdot 1} = \frac{f^{(k)}(0)}{k!}. \tag{5.3}$$

Example 5.1.2 Power series of e^x.

Find a power series for the exponential function e^x.

Solution. Any power series for $f(x) = e^x$ looks like $\sum_{k=0}^{\infty} c_k x^k$, where

$$c_k = \frac{f^{(k)}(0)}{k!}.$$

Since e^x is its own derivative, $f^{(k)}(x) = e^x$ for all choices of k. So

$$c_k = \frac{e^0}{k!} = \frac{1}{k!}$$

and the power series for e^x is

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

It turns out the $f(x) = e^x$ equals its power series for *all* values of x.

The above power series was written in terms of powers of x, but this doesn't have to be the case. We can also write power series in terms of powers of $x - a$, where a is some constant. A power series of the form

$$\sum_{k=0}^{\infty} c_k (x-a)^k$$

is said to be **centered at** a. For such a series, the formula for the c_k is given by

$$c_k = \frac{f^{(k)}(a)}{k!}. \tag{5.4}$$

Example 5.1.3 Power series of $\sin(t)$ at $a = \frac{\pi}{2}$.

Find the power series for $g(t) = \sin t$ centered at $a = \frac{\pi}{2}$.

Solution. A power series centered at $a = \frac{\pi}{2}$ will look like

$$\sum_{k=0}^{\infty} c_k \left(t - \frac{\pi}{2}\right)^k$$

where

$$c_k = \frac{g^{(k)}(\frac{\pi}{2})}{k!}.$$

To find these values, we need to compute the derivatives of $g(t)$ and evaluate them at $\frac{\pi}{2}$:

$$\begin{aligned} g^{(0)}(t) &= \sin t & &\Rightarrow g^{(0)}\left(\frac{\pi}{2}\right) = 1 \\ g'(t) &= \cos t & &\Rightarrow g'\left(\frac{\pi}{2}\right) = 0 \\ g''(t) &= -\sin t & &\Rightarrow g''\left(\frac{\pi}{2}\right) = -1 \\ g^{(3)}(t) &= -\cos t & &\Rightarrow g^{(3)}\left(\frac{\pi}{2}\right) = 0. \end{aligned}$$

So the power series centered at $\frac{\pi}{2}$ is

$$1 - \frac{1}{2!}(t - \frac{\pi}{2})^2 + \frac{1}{4!}(t - \frac{\pi}{2})^4 + \cdots = \sum_{k=0}^{\infty}(-1)^k \frac{(t - \frac{\pi}{2})^{2k}}{(2k)!}.$$

Just as with e^x, $\sin t$ is equal to its power series everywhere.

The following power series are used quite often:

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n &&= 1 + x + x^2 + x^3 + \cdots \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} &&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} &&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots
\end{aligned}$$

Viewing a function as a power series can be extremely beneficial; if you have a power series expression for some function, it (usually) makes the related calculus operations such as differentiation and integration trivial to perform.

Example 5.1.4 Integrating a series.

Find $\displaystyle\int_0^1 e^{x^2}\, dx$.

Solution. We can't integrate e^{x^2} using elementary functions but this is straightforward to integrate using power series:

$$\begin{aligned}
\int_0^1 e^{x^2}\, dx &= \int_0^1 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\, dx \\
&= \sum_{n=0}^{\infty} \int_0^1 \frac{x^{2n}}{n!}\, dx \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)n!}.
\end{aligned}$$

We can also write the integral of e^{x^2} in terms of the error function[3]:

$$\int_0^1 e^{x^2}\, dx = -\frac{1}{2} i \sqrt{\pi}\, \mathrm{erf}(i).$$

Calculations involving power series are only valid where the series converges. The following theorem can be used to determine when a power series converges.

[3] en.wikipedia.org/wiki/Error_function

Theorem 5.1.5

Given the power series $\sum_{n=0}^{\infty} c_n x^n$, define the number ρ by the limit

$$\rho = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|.$$

Suppose the limit exists or is infinite. Then ρ is the radius of convergence of the series: if $\rho \neq 0$ then the series converges for $|x| < \rho$ and diverges for $|x| > \rho$. If $\rho = 0$ then the series converges only at $x = 0$.

> *If $0 < \rho < \infty$ then convergence at the endpoints of the interval of convergence is not guaranteed and must be checked separately using some appropriate convergence test.*

It must be noted that convergence of the power series of some function $f(x)$ does not guarantee that $f(x)$ can be represented by its power series. For instance, consider the function $f(x)$ given by

$$f(x) = \begin{cases} e^{-x^{-2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

This has the power series representation 0 at $x = 0$ since $f^{(k)}(0) = 0$ for all integers k with $k \geq 0$. However, $f(x) \neq 0$ except at $x = 0$, so this power series representation is not very useful for representing $f(x)$ despite the fact that it converges for all x.

In general, if $f(x)$ has the power series representation $\sum_{k=0}^{\infty} a_k (x-a)^k$ *and* if this series converges to $f(x)$ for all x in some interval centered at $x = a$, then we say that the function $f(x)$ is **analytic** at $x = a$. The previous paragraph shows that

$$f(x) = \begin{cases} e^{-x^{-2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not analytic at $x = 0$. Thankfully, most of the functions we'll consider are in fact analytic.

Solving Differential Equations with Power Series

We now turn to the main topic of this chapter: solving differential equations with power series. This power series method is quite general and can theoretically be used whenever the functions involved in the differential equation are analytic, but we will primarily consider second-order linear ODEs that are homogeneous and have polynomial coefficients.

> Note that polynomials are automatically analytic since they're already power series. Hence a polynomial is its *own* power series representation.

Example 5.1.6 Solving a first-order differential equation with series.

Solve the ODE given by

$$\frac{dy}{dx} = 4y.$$

Solution. We could easily solve this using methods from Chapter 1, but we'll use power series instead to see how this method works. To start, we assume that the solution y can be written as a power series:

$$y = \sum_{n=0}^{\infty} c_n x^n.$$

See Theorem 5.1.10 for the justification behind this step.
The next step is to plug the ansatz into the ODE. Since

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n x^n\right) \\ &= \sum_{n=1}^{\infty} n c_n x^{n-1}\end{aligned}$$

we get the equation

$$\sum_{n=1}^{\infty} n c_n x^{n-1} = 4\sum_{n=0}^{\infty} c_n x^n. \tag{5.5}$$

We need to find the values of the coefficients c_n; we will do this by equating coefficients on both sides of (5.5). We want to write both series in terms of x^n so that we can equate coefficients, so we need to shift the summation on the left: we replace n with $n+1$ inside the sum and decrease the limit of summation $n = 1$ to $n = 0$ to get

$$\sum_{n=0}^{\infty} (n+1)c_{n+1}x^n = \sum_{n=0}^{\infty} 4c_n x^n.$$

Now we can equate coefficients: for $n \geq 0$, we have

$$(n+1)c_{n+1} = 4c_n \quad \text{or} \quad c_{n+1} = \frac{4}{n+1}c_n. \tag{5.6}$$

The equation (5.6) is a **recurrence relation** for the coefficients c_n. It describes the coefficients in terms of the previous ones, and can be used to determine explicitly what each c_n looks like. To see how, we plug several values for n into this recurrence relation to try to determine a pattern:

$$\begin{aligned}c_1 &= \frac{4}{2}c_0 \\ c_2 &= \frac{4}{3}c_1 = \frac{4^2}{3\cdot 2}c_0 \\ c_3 &= \frac{4}{4}c_2 = \frac{4^3}{4\cdot 3\cdot 2}c_0\end{aligned}$$

and in general it appears that

$$c_n = \frac{4^n}{n!}c_0 \tag{5.7}$$

for each n.

We can't use either (5.6) or (5.7) to find the initial constant c_0. This must be given by an initial condition of some kind.

Now that we've determined the form of the coefficients c_n, we can write down the solution y:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=0}^{\infty} \frac{4^n}{n!} c_0 x^n \\ &= c_0 \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} \\ &= c_0 e^{4x}. \end{aligned}$$

The power series method generally works as we utilized it in Example 5.1.6. In general, the power series method to solve ODEs consists of the following procedure:

1. Write $y = \sum_{n=0}^{\infty} c_n x^n$.
2. Use the ODE to build a recurrence relation for the coefficients c_n.
3. Find an explicit description of the coefficients.
4. Identify y as the power series of some function.

Example 5.1.7 Solving a first-order ODE with variable coefficients.

Use power series to solve the ODE $y' + 2xy = 0$.

Solution. We will solve this using the steps listed above. First, assume $y = \sum_{k=0}^{\infty} c_k x^k$. Now plug this guess for y into the ODE to get

$$\sum_{k=1}^{\infty} k c_k x^{k-1} = -\sum_{k=0}^{\infty} 2 c_k x^{k+1}.$$

As in Example 5.1.6 we want to equate coefficients to build a recurrence relation, so we need to rewrite these sums so that the same power of x appears on both sides. We do this by shifting the sums, but we need to remember to shift the limits of each sum as well:

Table 5.1.8 Changing limits of summation

Sum	Index	Limit
$\sum_{k=1}^{\infty} k c_k x^{k-1}$	$k-1 \to n$	$k=1 \to n=0$
$-\sum_{k=0}^{\infty} 2 c_k x^{k+1}$	$k+1 \to n$	$k=0 \to n=1$

So we get

$$\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n = \sum_{n=1}^{\infty} (-2) c_{n-1} x^n. \tag{5.8}$$

Hence a recurrence relation for c_n is

$$c_{n+1} = -\frac{2}{n+1} c_{n-1}$$

which is valid for $n \geq 1$.
Since this **two-step recurrence relation** is only valid for $n \geq 1$, it places no restrictions on c_0 or c_1. However, the original ODE was first-order! So we should only expect to have one arbitrary constant in our solution, which suggests that one of c_0 or c_1 must be zero. To determine which constant must vanish, we write out the first couple terms of the sums in (5.8) and equate coefficients:

$$c_1 + 2c_2x + \cdots = -2c_0x - 2c_1x^2 + \cdots.$$

This tells us that $c_1 = 0$. Again, we can't get this information from the recurrence relation!
Now we try to find an explicit formula for c_n. Because this is a two-step recurrence, we will write out the coefficients in two columns, one for odd n and one for even n:

$$\begin{aligned} c_2 &= -\frac{2}{2}c_0 & c_3 &= -\frac{2}{3}c_1 = 0 \\ c_4 &= -\frac{2}{4}c_2 = \frac{1}{2}\frac{1}{1}c_0 & c_5 &= -\frac{2}{5}c_3 = 0 \\ c_6 &= -\frac{2}{6}c_4 = -\frac{1}{3}\frac{1}{2}\frac{1}{1}c_0 & c_7 &= -\frac{2}{7}c_5 = 0. \end{aligned}$$

So it appears that

$$c_{2k} = \frac{(-1)^k}{k!}c_0 \quad \text{and} \quad c_{2k+1} = 0$$

for every k.
Now we plug this into our power series for y to get

$$\begin{aligned} y &= \sum_{k=0}^{\infty} c_k x^k \\ &= \sum_{k=0}^{\infty} c_{2k} x^{2k} \\ &= \sum_{k=0}^{\infty} c_0 \frac{(-1)^k}{k!} x^{2k} \\ &= c_0 \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} \\ &= c_0 e^{-x^2}. \end{aligned}$$

Let's now move on to an example where none of our previous methods are easily applicable. We will also demonstrate how the method applies to solving IVPs.

Example 5.1.9 Power series method with an IVP.

Let $y(x)$ denote the solution of

$$y'' + xy' + y = 0, y(0) = 1, y'(0) = 0.$$

Find y up to the x^6 term and determine the values of $y''(0)$ and $y^{(3)}(0)$.

Solution. It can be shown that $y = e^{-x^2/2}$.

Now that we have an idea of how to solve differential equations using power series, it can be useful to know when this method is actually valid, i.e., when power series solutions exist. We will be particularly concerned with solutions of second-order linear ODEs of the form (5.1).

Theorem 5.1.10 Existence of Series Solutions.

Consider the differential equation given by (5.1). If $P(x), Q(x)$ and $R(x)$ are analytic at a point x_0, then every solution of (5.1) is also analytic at x_0.

Points that satisfy the conditions in Theorem 5.1.10 are also called **ordinary points** of (5.1).

Example 5.1.11 A Legendre Equation.

Show that
$$(1-x^2)y'' - 2xy' + 2y = 0$$
has a series solution centered at 0 and then find the solution up to the coefficient of x^6.

Solution. First, note that the equation can be rewritten
$$y'' - \frac{2x}{1-x^2}y' + \frac{2}{1-x^2}y = 0,$$
so we are guaranteed a series solution centered at $x = 0$. Furthermore, this solution has radius of convergence at least $R = 1$.

To find the solution, we return to the original equation and substitute $y = \sum_{n=0}^{\infty} c_n x^n$ to get
$$\sum_{n\geq 0} n(n-1)c_n x^{n-2} + \sum_{n\geq 0} \left[-n(n-1) - 2n + 2\right] c_n x^n = 0$$
which becomes
$$\sum_{n\geq 0} \left[(n+2)(n+1)c_{n+2} - n(n-1)c_n - 2nc_n + 2c_n\right] x^n = 0.$$
After a little algebra, we get the recurrence relation
$$c_{n+2} = \frac{n-1}{n+1}c_n.$$
This recurrence is valid for $n \geq 0$.

Now we can use the recurrence to list the first several terms of the solution:
$$y = c_1 x + c_0(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots)$$
In fact,
$$y = c_1 x - \sum_{n\geq 0} \frac{1}{2n-1} c_0 x^{2n} = c_1 x + C_0 x \ln\left(\frac{1+x}{1-x}\right).$$

In the last equation, C_0 has been substituted for $-\frac{c_0}{2}$.

5.2 Legendre's Equation and Legendre Polynomials

An important differential equation in applications is the **Legendre equation** given by

$$(1-x^2)y'' - 2xy' + k(k+1)y = 0. \tag{5.9}$$

Our first example of this equation (with $n = 1$) was examined in Example 5.1.11. By this example, we see that (5.9) has a series solution centered at $x = 0$ with radius of convergence at least 1. Therefore the power series method is appropriate.

Solving the Legendre Equation

We'll proceed as we did in Example 5.1.11, altering the last sum as necessary to get

$$\sum_{n\geq 0}(n+2)(n+1)c_{n+2}x^n - \sum_{n\geq 0} n(n-1)c_n x^n - \sum_{n\geq 0} 2nc_n x^n + \sum_{n\geq 0} k(k+1)c_n x^n = 0$$

which gives (after a bit of algebra, once again)

$$c_{n+2} = -\frac{(k-n)(k+n+1)}{(n+2)(n+1)}c_n.$$

This recurrence is valid for $n \geq 0$, and allows us to write out the solution y in terms of the parameter k and the arbitrary constants c_0 and c_1:

$$y = c_0 y_1(x) + c_1 y_2(x)$$

where

$$y_1(x) = 1 - \frac{k(k+1)}{2!}x^2 + \frac{(k-2)k(k+1)(k+3)}{4!}x^4 - \cdots \tag{5.10}$$

$$y_2(x) = x - \frac{(k-1)(k+2)}{3!}x^3 + \frac{(k-3)(k-1)(k+2)(k+4)}{5!}x^5 - \cdots. \tag{5.11}$$

Note that y_1 and y_2 form a basis of solutions (Definition 2.1.13) of the Legendre equation, which means that $y = c_0y_1 + c_1y_2$ must also be the general solution.

Legendre Polynomials

Our solution of (5.9) simplifies greatly if k happens to be an integer. In particular, if k is a nonnegative integer then

$$c_{k+2} = c_{k+4} = \cdots = 0.$$

If k is even then the solution y_1 given in (5.10) becomes a polynomial:

$$y_1 = 1 - \frac{k(k+1)}{2!}x^2 + \cdots + (-1)^{k/2}\frac{[k-(k-2)][k-(k-4)]\cdots[k+(k-3)][k+(k-1)]}{k!}x^k.$$

Likewise, if k is odd then y_2 given in (5.11) becomes a polynomial instead:

$$y_2 = x - \frac{(k-1)(k+2)}{3!}x^3 + \cdots + (-1)^{\frac{k-1}{2}}\frac{[k-(k-2)][k-(k-4)]\cdots[k+(k-3)][k+(k-1)]}{k!}x^k.$$

By choosing c_0 and c_1 judiciously, we can guarantee that the polynomials c_0y_1 (if k is even) or c_1y_2 (if k is odd) are precisely equal to 1 at $x = 1$. Doing so gives us the **Legendre polynomials** $P_n(x)$, defined more precisely in (5.12):

$$P_n(x) = \begin{cases} \sum_{j=0}^{n/2}(-1)^j \frac{(2n-2j)!}{2^n j!(n-j)!(n-2j)!}x^{n-2j} & \text{if } n \text{ is even} \\ \sum_{j=0}^{(n-1)/2}(-1)^j \frac{(2n-2j)!}{2^n j!(n-j)!(n-2j)!}x^{n-2j} & \text{if } n \text{ is odd} \end{cases}. \tag{5.12}$$

These polynomials satisfy several nice properties, but one of the most important characteristics they have is that $\{P_n(x)\}$ forms an *orthogonal set* of polynomials on the interval $[-1, 1]$. This means that

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$$

if $m \neq n$. It can also be shown that

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{2}{2n+1}$$

if $m = n$.

This property allows us to express *any* polynomial as a finite sum of Legendre polynomials in a computationally efficient manner. Furthermore, if we allow infinite series then we can use Legendre polynomials to express any continuous function defined on $[-1, 1]$. In particular, if $f(x)$ is continuous on $[-1, 1]$ then

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

where

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx\ .$$

This is demonstrated in Figure 5.2.1 for the function

$$f(x) = \begin{cases} e^{-x^{-2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

This approximation is particularly interesting since we've already seen that $f(x)$ is not analytic. Hence $f(x)$ has no power series representation at $x = 0$ but it still has a Legendre series.

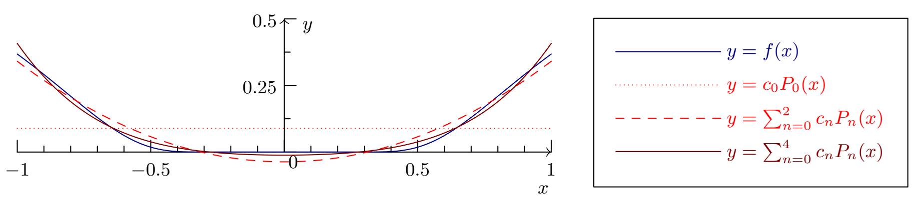

Figure 5.2.1 Legendre series approximation for $f(x)$

Note that we don't need to include the odd Legendre polynomials in Figure 5.2.1. Since $f(x)$ is an even function, its integral against any odd function over $[-1, 1]$ must be 0. Hence the odd degree polynomials contribute nothing to the corresponding Legendre series.

For actually computing Legendre polynomials, instead of using (5.12) we often use *Rodrigues' formula*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n] \tag{5.13}$$

or *Bonnet's recurrence*

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x). \tag{5.14}$$

Either recurrence is simple to program into a CAS, as seen in the Sage cell below:

```
def rodrigues(n):
Cn = (1 / (2^n * factorial(n)))
pn = (x^2 - 1)^n

pretty_print((Cn * diff(pn, n)).full_simplify())

def bonnet(n):
k = 1
P0 = 1
P1 = x
while k < n:
P0, P1 = P1, ((2*k + 1) / (k + 1)) * x * P1 - (k / (k + 1))
    * P0
k += 1

pretty_print(P1.full_simplify())

n = 9
rodrigues(n)
bonnet(n)
```

```
231/16*x^6 - 315/16*x^4 + 105/16*x^2 - 5/16
```

5.3 The Method of Frobenius

The power series method is guaranteed to work if the coefficient functions in (5.1) are analytic. However, there are important examples of ODEs where this property fails (see (5.18), which is used in the study of vibrating membranes). To solve ODEs where analyticity fails, we can sometimes use the *method of Frobenius*.

The main idea behind this method is to replace the ansatz $y = \sum_{k=0}^{\infty} c_k x^k$ with the modified guess

$$y = x^r \sum_{k=0}^{\infty} c_k x^k. \tag{5.15}$$

The value r here is chosen in such a way so as to guarantee a solution of the ODE in (5.1) of the form given in (5.15). However, the value of r used might not be a nonnegative whole number and could produce negative or fractional powers of x in the solution. Therefore the resulting solution (5.15) is typically *not* a power series.

Ordinary Points and Singular Points

Recall that a homogeneous, linear second order ODE has the form

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

We can rewrite this in the form

$$y'' + P(x)y' + Q(x)y = 0.$$

As we saw at the end of Section 5.1, the efficacy of the power series method depends on the behavior of $P(x)$ and $Q(x)$ at the point we're centering our series solution at.

Definition 5.3.1 Ordinary Points and Singular Points.

A point $x = a$ is called an **ordinary point** of the homogeneous form of (5.1) if $P(x)$ and $Q(x)$ both have power series expansions at $x = a$. If $x = a$ is not an ordinary point we call it a **singular point**.

Ordinary points of an ODE are precisely the points where the power series method is guaranteed to produce a valid solution.

Theorem 5.3.2 Existence of Series Solution.

Suppose that a is an ordinary point of the differential equation $y'' + P(x)y' + Q(x)y = 0$. Then the ODE has two linearly independent solutions of the form

$$y(x) = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

The radius of convergence of the resulting solution is at least as large as the distance from a to the nearest singular point of the ODE.

Example 5.3.3

Show that the ODE $(x^2+2)y'' + 4xy' + 2y = 0$ has a power series solution and estimate its radius of convergence. Then solve the ODE.

Solution. The first thing we will do is make sure that the ODE actually has a power series solution at $x = 0$. To do this, we need to show that $x = 0$ is an ordinary point of the ODE, which requires finding the appropriate $P(x)$ and $Q(x)$ from Theorem 5.3.2. If we divide through the ODE by $x^2 + 2$ we obtain

$$y'' + \frac{4x}{x^2+2}y' + \frac{2}{x^2+2}y = 0,$$

and so

$$P(x) = \frac{4x}{x^2+2}$$
$$Q(x) = \frac{2}{x^2+2}.$$

Since both of these functions are analytic at $x = 0$ (i.e., they have power series representations centered at $x = 0$), it follows that $x = 0$ is an ordinary point of the ODE. Therefore the ODE has a power series solution at $x = 0$. Since $P(x)$ and $Q(x)$ both have singular points at $x = \pm i$, it follows that the radius of convergence of the power series solution is at least $|i - 0| = 1$.

> Here, we're using the formula $|a + bi| = \sqrt{a^2 + b^2}$. The radius of convergence can also be visualized in the complex plane as marking out a circle of radius 1 centered at the origin.

We can now find our solution just as we did in Section 5.1, we assume the ODE has a series solution of the form $y = \sum_{k=0}^{\infty} c_k x^k$ (which is justified by the above!). We want to return to the original form of the ODE to solve it; if we didn't do so, we would need to expand $P(x)$ and $Q(x)$ using their own power series, and this would *greatly* complicate the algebra. So we will plug $y = \sum_{k=0}^{\infty} c_k x^k$ into

$$(x^2 + 2)y'' + 4xy' + 2y = 0$$

and then equate coefficients to get a recurrence relation for c_k. This can be done with a little help from Sage:

```
# Code cell demonstrating series calculations in Sage
# We start by guessing y = sum(c_k * x^k), which
    requires
# creating a list of variables c0, c1, ...,cn for the
# coefficients ck.

# Specify what term we want to stop at
n = 5
# Create a list c0, c1, ..., cn of our coefficients
coeffs = var([f"c{k}" for k in srange(n+1)])
# Set up the sum y = c0 + c1*x + c2*x^2 + ... + cn*x^n
y = sum([coeffs[k]*x^k for k in srange(n+1)])

# Now that we have set up our sum for y, we take this
# and plug it into the differential equation.
deqn = (x^2 + 2)*y.diff(x, 2) + 4*x*y.diff(x) + 2*y

# The last step is to collect terms so that we can
    begin
# equating coefficients. Note that the higher degree
    terms
# should not be used to equate coefficients here!
show(deqn.collect(x))
```

```
42*c5*x^5 + 30*c4*x^4 + 20*(c3 + 2*c5)*x^3 + 12*(c2 +
    2*c4)*x^2 + 6*(c1 + 2*c3)*x + 2*c0 + 4*c2
```

After equating coefficients, our computations with Sage suggest that $c_k + 2c_{k+2} = 0$, or more simply

$$c_{k+2} = -\frac{1}{2}c_k \text{ for } k \geq 0.$$

This can of course be verified algebraically as we've done several times already, but we'll trust Sage for now.

The next step now that we have a recurrence relation is to find a pattern for the coefficients. Since this is a two-step relation, we'll set up two columns: one column for even k and one column for odd k:

$$\begin{aligned} c_2 &= -\frac{1}{2}c_0 & c_3 &= -\frac{1}{2}c_1 \\ c_4 &= -\frac{1}{2}c_2 = \frac{1}{2^2}c_0 & c_5 &= -\frac{1}{2}c_3 = \frac{1}{2^2}c_1. \end{aligned}$$

So it appears that

$$c_{2k} = (-1)^k \frac{1}{2^k}c_0 \quad \text{and} \quad c_{2k+1} = (-1)^k \frac{1}{2^k}c_1.$$

Therefore the general solution of the ODE is

$$\begin{aligned} y &= \sum_{k\geq 0} c_k x^k \\ &= \sum_{k\geq 0} c_{2k}x^{2k} + \sum_{k\geq 0} c_{2k+1}x^{2k+1} \\ &= c_0 \sum_{k\geq 0} (-1)^k \frac{x^{2k}}{2^k} + c_1 \sum_{k\geq 0} (-1)^k \frac{x^{2k+1}}{2^k} \\ &= c_0 \sum_{k\geq 0} \left(-\frac{x^2}{2}\right)^k + c_1 x \sum_{k\geq 0} \left(-\frac{x^2}{2}\right)^k \\ &= \frac{c_0}{1+\frac{x^2}{2}} + \frac{c_1 x}{1+\frac{x^2}{2}} \end{aligned}$$

Solutions at Singular Points and Indicial Equations

We've seen several examples showing the effectiveness of the power series method at ordinary points, but the situation becomes more complicated at singular points. At these points, we may not be guaranteed a power series solution.

Example 5.3.4

Attempt to solve the ODE $x^2y'' + x^2y' + y = 0$.

Solution. We start, just as we did before, by assuming the solution is a power series: $y = \sum_{n\geq 0} c_n x^n$. We will once again use Sage to handle the algebra for us:

```
# Code cell demonstrating series calculations in Sage
# We start by guessing y = sum(c_k * x^k), which
    requires
# creating a list of variables c0, c1, ...,cn for the
# coefficients ck.

# Specify what term we want to stop at
n = 5
# Create a list c0, c1, ..., cn of our coefficients
coeffs = var([f"c{k}" for k in srange(n+1)])
# Set up the sum y = c0 + c1*x + c2*x^2 + ... + cn*x^n
y = sum([coeffs[k]*x^k for k in srange(n+1)])

# Now that we have set up our sum for y, we take this
# and plug it into the differential equation.
deqn = (x^2)*y.diff(x, 2) + x^2*y.diff(x) + y

# The last step is to collect terms so that we can
    begin
# equating coefficients. Note that the higher degree
    terms
# should not be used to equate coefficients here!
show(deqn.collect(x))
```

```
5*c5*x^6 + (4*c4 + 21*c5)*x^5 + (3*c3 + 13*c4)*x^4 +
    (2*c2 + 7*c3)*x^3 + (c1 + 3*c2)*x^2 + c1*x + c0
```

Now let's start equating coefficients. First, we immediately get that

$$c_0 = 0 \text{ and } c_1 = 0.$$

However, this forces $c_2 = 0$ since $c_1 + 3c_2 = 0$. And this in turn forces $c_3 = 0$, and so on.
Therefore our series solution is just

$$y = 0 + 0x + 0x^2 + \cdots = 0.$$

This is indeed a solution of the original ODE, but it's *not* a general solution. Our work in this example shows that the general solution of

$$x^2y'' + x^2y' + y = 0$$

cannot be written as a power series.

The reason we couldn't find a solution of the form $y = \sum_{n\geq 0} c_n x^n$ was because $x = 0$ is a singular point of the ODE. If we divide through by x^2 we get

$$y'' + y' + \frac{1}{x^2}y = 0$$

and it's obvious that the coefficients have a divide by 0 problem at $x = 0$.

Our goal is to find a way of dealing with situations where $x = 0$ is a singular point of an ODE of the form

$$x^2y'' + xp(x)y' + q(x)y = 0. \tag{5.16}$$

We know, in general, that we won't be able to find a power series solution $\sum_{n\geq 0} c_n x^n$; intuitively, a power series solution is too "nice" to be a solution of this ODE if $x = 0$ is a singular point.

To fix this, we change our guess for y to $y = x^r \sum_{n\geq 0} c_n x^n$ or, equivalently,

$$y = \sum_{n\geq 0} c_n x^{n+r}.$$

Here, r can be any number (real or complex!), so in general the solution y produced by this method *will not be a power series*. We lose a little bit by no longer assuming that y is a power series, but this expression may be flexible enough to lead to a solution of the ODE at a singular point.

> Recall that a power series, by definition, has only nonnegative whole number powers of x.

Our goal now is to find the value of r based on the ODE and the coefficient functions $p(x)$ and $q(x)$. To do so, we will plug $y = \sum_{n\geq 0} c_n x^{n+r}$ into the ODE

$$x^2 y'' + xp(x)y' + q(x)y = 0$$

and attempt to get some conditions on r. First, note that

$$y' = \sum_{n\geq 0}(n+r)c_n x^{n+r-1}$$

$$y'' = \sum_{n\geq 0}(n+r)(n+r-1)c_n x^{n+r-2}$$

so when we plug these into the ODE we get

$$\sum_{n\geq 0}(n+r)(n+r-1)c_n x^{n+r} + \sum_{n\geq 0} p(x)(n+r)c_n x^{n+r} + \sum_{n\geq 0} q(x)c_n x^{n+r} = 0.$$

Now combine everything into one sum to get

$$\sum_{n\geq 0} \left[(n+r)(n+r-1) + p(x)(n+r) + q(x)\right] c_n x^{n+r} = 0.$$

So for this equation to be true, we need to have

$$\left[(n+r)(n+r-1) + p(x)(n+r) + q(x)\right] c_n = 0$$

for every n and every x. Since we are trying to find r, we'll pick values for n and x. In particular, if we assume that $p(x)$ and $q(x)$ exist at $x = 0$ we can pick $n = x = 0$ to get

$$r(r-1) + p(0)r + q(0) = 0 \tag{5.17}$$

(we can assume that $c_0 \neq 0$). This equation tells us how to find r.

Definition 5.3.5 Indicial Equation.

Suppose that $x = 0$ is a singular point of the ODE in (5.16) but that $p(x)$ and $q(x)$ have well-defined power series at $x = 0$ (i.e., $p(0)$ and $q(0)$ make sense). Then (5.17) is called the **indicial equation** of (5.16).

What we've shown is that if $y = x^r \sum_{n\geq 0} c_n x^n$ is a solution of (5.16), then r must be a root of the indicial equation. In fact, we can say more.

Theorem 5.3.6 Method of Frobenius.

Consider the ODE

$$x^2y'' + xp(x)y' + q(x)y = 0.$$

Suppose that $r_1 \geq r_2$ *are (real) roots of the indicial equation* $r(r-1) + p(0)r + q(0) = 0$*. Then the following statements are true:*

1. *There is a solution of the ODE of the form* $y_1(x) = x^{r_1}\sum_{n\geq 0} c_n x^n$.
2. *If* $r_1 - r_2$ *is* not *equal to an integer, then there exists a second linearly independent solution of the form* $y_2(x) = x^{r_2}\sum_{n\geq 0} d_n x^n$.
3. *If* $r_1 = r_2$*, there exists a second linearly independent solution of the form* $y_2(x) = y_1(x)\ln x + x^{r_1}(C_0 + C_1x + \cdots)$.
4. *If* $r_1 - r_2$ *is a nonzero integer, there exists a second linearly independent solution of the form* $y_2(x) = ky_1(x)\ln x + x^{r_2}(C_0 + C_1x + \cdots)$.

Example 5.3.7 Using the Method of Frobenius.

Find a series solution centered at 0 of the ODE

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0.$$

Solution. If we divide through the ODE by x^2 we get a divide-by-zero problem at $x = 0$, so $x = 0$ is a singular point. Therefore we will use Theorem 5.3.6 to determine the appropriate form of a series solution for this ODE.
We have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x^2 - \frac{1}{4}, \end{aligned}$$

so $p(x)$ and $q(x)$ both have power series representations centered at 0 (namely, themselves!). This means we can use the method of Frobenius to find a solution of the form $y = x^r\sum_{n=0}^{\infty} c_n x^n$.
The first step is to set up and solve the indicial equation, which in this case is given by

$$r(r-1) + r - \frac{1}{4} = 0.$$

We solve this algebraically for r to get the roots $r_1 = \frac{1}{2}$ and $r_2 = -\frac{1}{2}$. Since $r_1 - r_2 = 1$ is an integer, we are guaranteed a solution based on $r_1 = \frac{1}{2}$ and a second solution based on $r_2 = -\frac{1}{2}$ and the natural logarithm.
To continue, we make the guess

$$y = x^{\frac{1}{2}}\sum_{n\geq 0} c_n x^n = \sum_{n\geq 0} c_n x^{n+\frac{1}{2}}.$$

Now we plug this into the ODE to get

$$\sum_{n\geq 0} c_n \left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)x^{n+\frac{1}{2}}+\sum_{n\geq 0} c_n \left(n+\frac{1}{2}\right)x^{n+\frac{1}{2}}+\sum_{n\geq 0} c_n x^{n+\frac{5}{2}}-\sum_{n\geq 0}\frac{1}{4}c_n x^{n+\frac{1}{2}}$$

or just

$$\sum_{n\geq 0}\left[\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)+\left(n+\frac{1}{2}\right)-\frac{1}{4}\right]c_n x^{n+\frac{1}{2}}+\sum_{n\geq 2} c_{n-2}x^{n+\frac{1}{2}}=0.$$

which simplifies to

$$\sum_{n\geq 0} n(n+1)c_n x^{n+\frac{1}{2}}+\sum_{n\geq 2} c_{n-2}x^{n+\frac{1}{2}}=0.$$

So the recurrence relation the coefficients c_n need to satisfy is

$$c_n=-\frac{1}{n(n+1)}c_{n-2}\quad\text{for}\quad n\geq 2.$$

The recurrence relation will tell us *nothing* about c_0 and c_1, so to see if there are any restrictions on these coefficients we separate the $n=0$ and $n=1$ terms from the summation to get

$$0c_0x^{\frac{1}{2}}+2c_1x^{\frac{3}{2}}+\sum_{n\geq 2}\left[n(n+1)c_n+c_{n-2}\right]x^{n+\frac{1}{2}}=0.$$

This equation places no restrictions at all on c_0, but it does force $c_1=0$ since we need the $x^{\frac{3}{2}}$ term to disappear to make this equation true. This tells us that we can ignore the coefficients c_n with odd index, since they will all disappear.
Now we'll try to find a pattern in the remaining coefficients:

$$\begin{aligned}
c_2&=-\frac{1}{2\cdot 3}c_0\\
c_4&=-\frac{1}{4\cdot 5}c_2=\frac{(-1)^2}{5!}c_0\\
c_6&=-\frac{1}{6\cdot 7}c_4=\frac{(-1)^3}{7!}c_0
\end{aligned}$$

and in general

$$c_{2n}=\frac{(-1)^n}{(2n+1)!}c_0.$$

Therefore a solution (but not the general solution!) of this ODE is given by

$$y=\sum_{n\geq 0} c_{2n}x^{2n+\frac{1}{2}}=\sum_{n\geq 0}\frac{(-1)^n}{(2n+1)!}c_0x^{2n+\frac{1}{2}},$$

which is actually just

$$y=c_0\frac{\sin x}{\sqrt{x}}.$$

Technically, this isn't the general solution of the ODE as we still need a second linearly independent solution to construct it. However, we know from Theorem 5.3.6 that the second solution must be of the form

$$y_2=k\frac{\sin(x)}{\sqrt{x}}\ln(x)+x^{-\frac{1}{2}}\sum_{n\geq 0} C_n x^n.$$

We can find appropriate values for k and the coefficients C_n by plugging this guess into the ODE and proceeding much as we did above. We will once again let Sage do the heavy lifting:

```
# Our guess for y2 will be based on y1 and r2
y1 = sin(x)/sqrt(x)
r2 = -1/2

# Specify what term we want to stop at
n = 5

# Create coefficients for y2 guess
var('k')
coeffs = var([f"C{k}" for k in srange(n+1)])

# Set up the sum y = k*y1*ln(x) + x^r2*sum(C_k * x^k)
y = k*y1*ln(x) + x^r2*sum([coeffs[k]*x^k for k in
    srange(n+1)])

# Now that we have set up our sum for y=y2, we take
    this
# and plug it into the differential equation.
deqn = (x^2)*y.diff(x, 2) + x*y.diff(x) + (x^2 - 1/4)*y

# The last step is to collect terms so that we can
    begin
# equating coefficients. Note that the higher degree
    terms
# should not be used to equate coefficients here!
show(deqn.collect(x))
```

```
c5*x^7 + c4*x^6 + 1/4*(4*c3 + 99*c5)*x^5 + 1/4*(4*c2 +
    63*c4)*x^4 + 1/4*(4*c1 + 35*c3)*x^3 + 1/4*(4*c0 +
    15*c2)*x^2 + 2*k*sqrt(x)*cos(x) + 3/4*c1*x -
    k*sin(x)/sqrt(x) - 1/4*c0
```

Plugging this into the original ODE (and using a computer algebra system such as Sage), we get

$$(C_3 + 20\,C_5)x^{\frac{9}{2}} + (C_2 + 12\,C_4)x^{\frac{7}{2}} + (C_1 + 6\,C_3)x^{\frac{5}{2}} + (C_0 + 2\,C_2)x^{\frac{3}{2}} + 2\,k\sqrt{x}\cos(x) - \frac{k\sin(x)}{\sqrt{x}} = 0$$

after truncating the expansion up to the $n = 5$ term.
This allows us (theoretically) to solve for k and the coefficients C_n. In fact, we get

$$k = 0$$
$$C_{2n} = \frac{(-1)^n}{(2n)!}C_0$$
$$C_{2n+1} = \frac{(-1)^n}{(2n+1)!}C_1$$

and so

$$y_2 = x^{-1/2}\left[C_0\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} + C_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}\right].$$

Since the second series corresponds to a multiple of $y_1 = \frac{\sin(x)}{\sqrt{x}}$, we can safely set $C_1 = 0$ and get $y_2 = C_0 \frac{\cos(x)}{\sqrt{x}}$. Therefore the general solution of the ODE is

$$y = A_1 \frac{\sin(x)}{\sqrt{x}} + A_2 \frac{\cos(x)}{\sqrt{x}}.$$

5.4 Bessel's Equation

As with Legendre's Equation (5.9), another important differential equation in applications is **Bessel's equation**:

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0, \tag{5.18}$$

where $\nu \geq 0$. By Theorem 5.3.6, this equation has a series solution at $x = 0$ of the form

$$y = x^r \sum_{k \geq 0} c_k x^k$$

where r is a solution of the indicial equation

$$r(r-1) + r - \nu^2 = 0,$$

or just

$$r = \pm\nu.$$

In particular, there we're guaranteed a series solution by setting $r = \nu$, since this is the larger root. Note that Example 5.3.7 is actually a Bessel equation with parameter $\nu = \frac{1}{2}$.

Let

$$y = x^\nu \sum_{k \geq 0} c_k x^k.$$

Then we can plug this into (5.18) to obtain

$$\sum_{k \geq 0} (k+\nu)(k+\nu-1) c_k x^{k+\nu} + \sum_{k \geq 0} (k+\nu) c_k x^{k+\nu} + \sum_{k \geq 2} c_{k-2} x^{k+\nu} - \sum_{k \geq 0} \nu^2 c_k x^{k+\nu} = 0, \tag{5.19}$$

which gives

$$c_k = -\frac{c_{k-2}}{k(k+2\nu)} \quad \text{for} \quad k \geq 2.$$

Since this only gives us data about $c_k, k \geq 2$, we should go back to (5.19) to see if we can say anything about c_0 or c_1. In fact, we get

$$(2\nu + 1)c_1 = 0 \implies c_1 = 0.$$

Hence our series solution only contains even-indexed coefficients. Rewriting the recurrence to reflect this, we get

$$c_{2k} = -\frac{c_{2k-2}}{2^2 k(k+\nu)} = \frac{(-1)^k c_0}{2^{2k} k!(\nu+1)(\nu+2)\cdots(\nu+k)} \quad \text{for} \quad k \geq 2. \tag{5.20}$$

Bessel Functions for Integer ν

Now we consider what happens to solutions given by (5.20) if $\nu = n$ is a nonnegative integer. To simplify matters (somewhat...), we add the restriction that $c_0 = \frac{1}{2^n n!}$. This allows us to write (5.20) more simply as

$$c_{2k} = \frac{(-1)^k}{2^{2k+n}(n+k)!}. \tag{5.21}$$

The resulting series

$$J_n(x) = x^n \sum_{k \geq 0} \frac{(-1)^k}{2^{2k+n} k!(n+k)!} x^{2k} \tag{5.22}$$

is known as the **Bessel function of the first kind** of order n.

Example 5.4.1 Finding J_0 and J_1.

Find the zeroth order and first order Bessel functions of the first kind.

Solution. Using (5.22), we get

$$J_0(x) = \sum_{k\geq 0} \frac{(-1)^k}{2^{2k}(k!)^2} x^{2k} = 1 - \frac{1}{4}x^2 + \frac{1}{16(4)}x^4 - \cdots$$

$$J_1(x) = x \sum_{k\geq 0} \frac{(-1)^k}{2^{2k+1}k!(1+k)!} x^{2k} = \frac{1}{2}x - \frac{1}{8(2)}x^3 + \cdots.$$

These functions are important enough that they are built-in to most computer algebra systems. In Sage, these functions are implemented as `bessel_J(n, x)`:

```
J0 = bessel_J(0, x)
J1 = bessel_J(1, x)

P = plot(J0, (x, -1, 50), legend_label = "$J_{0}(x)$")
P += plot(J1, (x, -1, 50), color = 'green', legend_label =
    "$J_{1}(x)$")
P.show()
```

As we can see, these functions oscillate and tend towards 0. A useful (asymptotic) approximation is given by

$$\sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right), \tag{5.23}$$

as shown below.

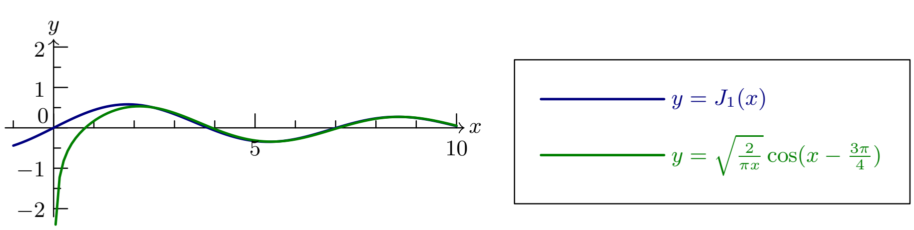

Figure 5.4.2 Approximating a Bessel function

Bessel Functions of the First Kind for Nonnegative Order

Now we try to find a formula for $J_\nu(x)$ assuming $\nu \geq 0$. To do so, we need to make sense of expressions like $(\nu+k)!$ for noninteger ν. Thankfully, we can do so using the *Gamma function*.

Definition 5.4.3 Gamma Function.

The **Gamma function** is the function $\Gamma(x)$ given by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

An important property of the Gamma function is the following:

$$\Gamma(x+1) = x\Gamma(x).$$

If we replace x with an integer $n \geq 0$, we get

$$\Gamma(n+1) = n!.$$

It turns out that we can replace $(\nu+k)!$ in (5.22) with $\Gamma(\nu+k+1)$, giving

$$J_\nu(x) = \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu}. \tag{5.24}$$

Note that the asymptotic expansion in (5.23) holds for noninteger ν as well.

General Solution of Bessel's Equation

Since (5.18) is second-order, we need a second linearly independent solution in combination with $J_{\nu(x)}$ to get the general solution. If ν is not an integer then we can find the second solution very quickly: $J_{-\nu}(x)$. However, if ν is an integer then it turns out that $J_{-\nu}(x) = (-1)^n J_\nu(x)$, and so fails to be linearly independent from J_ν.

It turns out that a second, linearly independent solution Y_ν is given as follows:

$$Y_\nu(x) = \frac{1}{\sin(\nu\pi)}[J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)] \tag{5.25}$$

$$Y_n(x) = \lim_{\nu\to n} Y_\nu(x). \tag{5.26}$$

The functions defined in (5.25)–(5.26) are called **Bessel functions of the second kind**. Sage implements these functions as well using the `bessel_Y(n,x)` command.

Chapter 6

Laplace Transforms

The power series method introduced in Chapter 5 is one example of a more general problem-solving strategy in mathematics. We started with a problem in one domain, a differential equation to be solved, and transformed it via power series into a problem about recurrence relations. This change of view was generally in our favor as the resulting algebra problems involved in solving recurrence relations was often easier than the calculus problem of solving the corresponding differential equation.

The *Laplace transform* is another instance of this same idea. We will take a problem in one domain, our differential equation to be solved, and transform it into an equivalent problem that can be solved by algebraic methods. This is similar to the techniques developed in Section 2.2, but the Laplace transform method is far more general. We can even use it in certain problems where the power series method does not apply.

6.1 The Laplace Transform

We start with an example motivating the need for the Laplace transform.

Example 6.1.1 Motivating the Laplace transform.

A mass of 1 kg is attached to a spring that is held 1 m to the right of its equilibrium position by a force of 4 N. Beginning at time $t = 0$, a machine is turned on and applies an external force of $\cos 2t$ to the mass. At time $t = 2\pi$ the machine is turned off and the external force disappears. Let $x(t)$ be the displacement of the mass at time t. What is an ODE that models the motion of the mass?

Solution. We can set this model up as we did in Section 2.5. By Hooke's Law and Newton's Second Law, we have

$$mx'' + kx = F_E(t)$$

where F_E is the external force at time t. Since $m = 1$, $k = 4$ and

$$F_E(t) = \begin{cases} \cos 2t & \text{if } 0 \le t < 2\pi \\ 0 & \text{if } t \ge 2\pi \end{cases}$$

the motion of the mass satisfies the ODE

$$x'' + 4x = F_E(t).$$

For the above example we could try to solve the resulting equation by using techniques that we developed in Chapter 2. This approach requires finding the complementary solution x_c (which isn't a problem) and then by finding the particular solution x_p corresponding to F_E. However, finding x_p will not be possible using our previous methods since the function F_E is not differentiable everywhere (nor continuous everywhere). The power series method will not be as effective here for the same reason: that only works if we're dealing with analytic functions.

We would like to develop a method that lets us solve ODEs that involve discontinuous quantities. The approach that we'll take is to introduce a "smoothing operator" in the form of an integral.

> We will see in Section 6.5 that integrals can be used to smooth out cusps and jumps in the graph of a function. This is analogous to smoothing edges in an image by applying a blur filter.

Definition and Basic Properties

Definition 6.1.2 The Laplace Transform.

Let $f(t)$ be defined for $t \geq 0$. The **Laplace transform** of f is the function $F(s)$, or $\mathcal{L}\{f(t)\}$, defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

for all values of s such that $F(s)$ exists. We often say that $f(t)$ is in the **time domain** while $F(s)$ is in the **frequency domain**.

> The time domain and frequency domain terminology employed in Definition 6.1.2 is a little unusual at the moment, but will hopefully make more sense in Section 7.5. Essentially, if s is a complex parameter then the exponential that appears in the Laplace transform introduces sinusoidal waves by Euler's formula. The parameter s then determines the frequency of these waves.

Derivatives of discontinuous functions are not as well behaved as integrals of discontinuous functions. Therefore the Laplace transform should prove to be a useful tool in dealing with differential equations that have discontinuous terms. This is something we will begin to look at in Section 6.2. For now, we just want to get some practice performing computations with this new formula in the next couple of examples.

Example 6.1.3 Computing a Laplace transform.

Compute $\mathcal{L}\{1\}$.

Example 6.1.4 Computing the Laplace transform of t.

Compute the Laplace transform of $f(t) = t$.

Solution. We compute the Laplace transform $F(s)$ using the definition:

$$\begin{aligned} F(s) &= \int_0^\infty te^{-st}\,dt \\ &= \left[-\frac{t}{s}e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt \quad \text{using integration by parts} \\ &= -\frac{1}{s^2}\left[e^{-st}\right]_0^\infty \\ &= \frac{1}{s^2}. \end{aligned}$$

In general,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

if $n \geq 0$ is a whole number.

The Laplace transform is also an example of a **linear transformation**. In particular, we have the following useful theorem for simplifying certain transforms.

Theorem 6.1.5 Linearity of the Laplace transform.

Let a and b be constants and suppose $f(t)$ and $g(t)$ are functions with respective Laplace transforms $F(s)$ and $G(s)$. Then

$$\mathcal{L}\{af + bg\} = aF(s) + bG(s).$$

In combination with the formula $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, we can now compute the Laplace transform of any polynomial function with relative ease.

Example 6.1.6 Computing the transform of a polynomial.

Compute $\mathcal{L}\left\{(t-1)^3\right\}$.

Solution. We could use the definition once again, but here's an easier way using the linearity of the Laplace transform. First, note that

$$(t-1)^3 = t^3 - 3t^2 + 3t - 1,$$

and so

$$\begin{aligned} \mathcal{L}\left\{(t-1)^3\right\} &= \mathcal{L}\left\{t^3 - 3t^2 + 3t - 1\right\} \\ &= \mathcal{L}\left\{t^3\right\} - 3\mathcal{L}\left\{t^2\right\} + 3\mathcal{L}\{t\} - \mathcal{L}\{1\} \\ &= \frac{3!}{s^4} - 3\frac{2!}{s^3} + 3\frac{1!}{s^2} - \frac{0!}{s} \\ &= \frac{6}{s^4} - \frac{6}{s^3} + \frac{3}{s^2} - \frac{1}{s}. \end{aligned}$$

As mentioned previously, the Laplace transform works well with some discontinuous functions due to its definition as an integral. The exponential *kernel* e^{-st} acts to smooth out discontinuities present in a function.

Example 6.1.7 The Unit Step Function.

Let $u(t)$ be the **unit step function**, which is defined by

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}.$$

Compute $\mathcal{L}\left\{u(t-3)\right\}$.

Solution. Note that we are computing a *translation* of the usual unit step function. Since we don't have a formula for the transform of a (translated) unit step, we will need to compute $\mathcal{L}\left\{u(t-3)\right\}$ using the definition of the Laplace transform:

$$\begin{aligned} \mathcal{L}\left\{u(t)\right\} &= \int_0^\infty u(t)e^{-st}\,dt \\ &= \int_3^\infty e^{-st}\,dt && \text{since } u(t-3) \text{ is 0 for values of } t < 3 \\ &= -\frac{1}{s}\left[e^{-st}\right]_3^\infty \\ &= \frac{e^{-3s}}{s}. \end{aligned}$$

We can verify the result of Example 6.1.7 using Sage as below:

```
var('s,␣t')

# define function to take the Laplace transform of
f = piecewise([((-oo,3),0), ((3,oo),1)])

# take the Laplace transform, going from t domain to s domain
show(f.laplace(t,s))
```

Some other important Laplace transforms are given below:

$$\begin{aligned} \mathcal{L}\left\{e^{at}\right\} &= \frac{1}{s-a} \\ \mathcal{L}\left\{\sin(k\,t)\right\} &= \frac{k}{s^2+k^2} \\ \mathcal{L}\left\{\cos(k\,t)\right\} &= \frac{s}{s^2+k^2}. \end{aligned}$$

Formulas such as these can be proved using Definition 6.1.2, which often requires (repeated) applications of integration by parts. A trickier way to prove some transform formulas makes use of power series.

Example 6.1.8 Finding a transform using power series.

Find $\mathcal{L}\left\{\sin(k\,t)\right\}$ where k is a constant.

Solution. The power series for $\sin(k\,t)$ centered at 0 is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(k\,t)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n k^{2n+1}}{(2n+1)!} t^{2n+1}.$$

This power series converges everywhere and we can take its Laplace transform. By linearity, we get

$$\begin{aligned}\mathcal{L}\left\{\sin(kt)\right\} &= \sum_{n=0}^{\infty} \frac{(-1)^n k^{2n+1}}{(2n+1)!}\mathcal{L}\left\{t^{2n+1}\right\} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n k^{2n+1}}{s^{2n+2}} \\ &= \frac{k}{s^2}\sum_{n=0}^{\infty}\left(-\frac{k^2}{s^2}\right) \\ &= \frac{k}{s^2+k^2}.\end{aligned}$$

Therefore $\mathcal{L}\left\{\sin(kt)\right\} = \frac{k}{s^2+k^2}$.

The Inverse Laplace Transform; s-shifting

The Laplace transform method that we will see in Section 6.2 involves transforming a time domain problem into a frequency domain problem that can be solved with algebra. The crucial next step is transforming the frequency domain solution of our problem back into the time domain. The next result justifies this last step.

Theorem 6.1.9 Uniqueness of Laplace Transforms.

Suppose $f(t)$ and $g(t)$ have respective Laplace transforms $F(s)$ and $G(s)$. If $F(s) = G(s)$ for all $s > c$ (for some constant c) and $f(t)$ and $g(t)$ are piecewise continuous, then $f(t) = g(t)$ on the interval $[0, \infty)$.

Theorem 6.1.9 says that the Laplace transform is unique for continuous functions: two different continuous functions will have two different Laplace transforms. This allows us to talk about taking *inverse* Laplace transforms.

Definition 6.1.10 Inverse Laplace Transform.

The **inverse Laplace transform** of a function $F(s)$, denoted $\mathcal{L}^{-1}\left\{F(s)\right\}$, is a function $f(t)$ such that $\mathcal{L}\left\{f(t)\right\} = F(s)$.

There is an integral formula for the inverse Laplace transform as well. In particular,

$$\mathcal{L}^{-1}\left\{F(s)\right\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}\,dt$$

where γ is a large enough real number. We aren't quite equipped to deal integrals such as this, so we will focus on using a table of Laplace transforms as in Table B.0.1 to compute inverse transforms.

Simple inverse transforms can be found by reversing known transforms. For example,

$$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$$

for $n = 0, 1, 2, \ldots$. The linearity of the inverse transform is also helpful in computing certain transformations.

Example 6.1.11 Finding an inverse transform.

A continuous function $f(t)$ has Laplace transform

$$F(s) = \frac{3s}{s^2+5} + \frac{7}{s^3}.$$

Find $f(t)$.

Solution. We can find $f(t)$ by taking the inverse Laplace transform of each term in $F(s)$:

$$\mathcal{L}^{-1}\left\{\frac{3s}{s^2+5}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+5}\right\} = 3\cos(\sqrt{5}t)$$
$$\mathcal{L}^{-1}\left\{\frac{7}{s^3}\right\} = \frac{7}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} = \frac{7}{2}t^2.$$

So

$$f(t) = 3\cos(\sqrt{5}t) + \frac{7}{2}t^2.$$

Technically, we can only say that we know $f(t)$ for $t \geq 0$. We have no knowledge whatsoever about $f(t)$ for $t < 0$. However, this will cause no trouble for us.

Now we will move on to more general formulas for computing Laplace transformations and their inverses. The first such formula shows a correspondence between *time scaling* and *frequency shifting*.

Theorem 6.1.12 s-shifting.

Suppose that $f(t)$ has Laplace transform $F(s)$, defined for $s > k$ for some k. Then

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a) \quad \textit{or equivalently} \quad \mathcal{L}^{-1}\left\{F(s-a)\right\} = e^{at}f(t).$$

In other words, multiplication by an exponential e^{at} in the time domain *corresponds to translation by a in the* frequency domain.

Theorem 6.1.12 allows to find a wide range of inverse transformations. In combination with partial fractions, we can now (theoretically) find the inverse transform of any rational function of s.

Example 6.1.13 Inverse Laplace with frequency shifting.

A function $f(t)$ has Laplace transform

$$F(s) = \frac{4}{s^2+2s+3}.$$

Find $f(t)$.

Solution. We'll start by completing the square on the denominator of $F(s)$ to see if we can make it look like $(s-a)^2+\omega^2$ for some a, ω:

$$F(s) = \frac{4}{(s^2+2s+1)+3-1} = \frac{4}{(s+1)^2+2}.$$

This looks an *awful* lot like the transform of $\sin(\sqrt{2}t)$. Since we have $(s+1)^2$ instead of s^2, this tells us that

$$f(t) = \frac{4}{\sqrt{2}}e^{-t}\sin(\sqrt{2}t).$$

The following Sage cell verifies the computation in Example 6.1.13.

```
var('s␣t') # specify variables
show(inverse_laplace(4/(s^2 + 2*s + 3), s, t))
```

```
2*sqrt(2)*e^(-t)*sin(sqrt(2)*t)
```

Now that we've had some practice in evaluating Laplace transforms, we'll move on to using them to solve differential equations.

6.2 Transformation of Initial Value Problems

In this section we begin using the Laplace transform to solve initial value problems. The basic outline of our approach is this:

1. Transform an IVP into a frequency domain equation.
2. Solve the transformed equation in the frequency domain to get a function $X(s)$.
3. Take the inverse transform of $X(s)$ to get a time solution $x(t)$ of the IVP.

The result that makes all of this work is Theorem 6.2.1 below.

Theorem 6.2.1 Laplace Transforms of Derivatives.

Suppose that the function $f(t)$ is piecewise smooth and that $F(s) = \mathcal{L}\{f(t)\}$ exists. Then $\mathcal{L}\{f'(t)\}$ exists and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0).$$

The above theorem says that differentiation in the "time domain" corresponds to multiplication by s in the "frequency domain." This will be the most useful property of the Laplace transform for us: it will turn differential equations in the time domain into algebraic equations in the frequency domain.

Solving IVPs with the Laplace Transform

We will mostly deal with second-order ODEs, so it will be useful to find how the Laplace transform affects a second derivative. This will be a quick application of Theorem 6.2.1.

Example 6.2.2 Laplace Transform of a second derivative.

Let $f(t)$ be piecewise smooth with Laplace transform $F(s)$. Compute $\mathcal{L}\{f''(t)\}$ using Theorem 6.2.1.

Solution. By Theorem 6.2.1 we know that

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

We can apply Theorem 6.2.1 to $f'(t)$ as well, since it must also be piecewise smooth:

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\} - f'(0)\\ &= s(sF(s) - f(0)) - f'(0)\\ &= s^2F(s) - sf(0) - f'(0).\end{aligned}$$

This process can be continued indefinitely. In general,

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

This gives us everything we need to start solving ODEs using Laplace transforms.

As seen in Example 6.2.2, the order of a differential equation in the time domain corresponds to a polynomial of matching degree in the frequency domain. Hence, we are transforming a calculus problem into an algebraic one. We will demonstrate the Laplace transform method with an example.

Example 6.2.3 Solving a second-order IVP.

Solve the IVP

$$x'' + 8x' + 15x = 0 \qquad x(0) = 2, x'(0) = -3$$

where x is a function of t.

Solution. We will solve this by using Laplace transforms. If we set $X(s) = \mathcal{L}\{x(t)\}$, then

$$\begin{aligned}\mathcal{L}\{x'(t)\} &= sX(s) - 2\\ \mathcal{L}\{x''(t)\} &= s^2X(s) - 2s + 3.\end{aligned}$$

So if we take the Laplace transform of the entire ODE, we get

$$\underbrace{s^2X(s) - 2s + 3}_{\mathcal{L}\{x''\}} + \underbrace{8sX(s) - 16}_{\mathcal{L}\{8x'\}} + 15X(s) = 0$$

which simplifies to

$$s^2X(s) + 8sX(s) + 15X(s) = 2s + 13,$$

and so

$$X(s) = \frac{2s+13}{s^2+8s+15}.$$

To solve the ODE we need to find $x(t) = \mathcal{L}^{-1}\{X(s)\}$. Now, $X(s)$ doesn't look like the Laplace transform of anything we know at the moment, but we can simplify the right hand side using partial fractions. If we do so, we get

$$X(s) = \frac{\frac{7}{2}}{s+3} - \frac{\frac{3}{2}}{s+5}.$$

Therefore,

$$\begin{aligned} x(t) &= \frac{7}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} \\ &= \frac{7}{2}e^{-3t} - \frac{3}{2}e^{-5t}. \end{aligned}$$

Note that we could have solved the ODE in Example 6.2.3 without resorting to Laplace transforms by using characteristic equations. However, the Laplace transform method did make dealing with the initial conditions more straightforward. Furthermore, the method remains the same even if applied to a linear nonhomogeneous IVP.

Example 6.2.4 Nonhomogeneous IVPs with Laplace Transforms.

Solve the IVP given by

$$y'' + 9y = 10e^{-t}, \quad y(0) = y'(0) = 0.$$

Solution. We begin by taking the Laplace transform of both sides to get

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{10}{s+1}$$

or just

$$s^2Y(s) + 9Y(s) = \frac{10}{s+1}.$$

So

$$Y(s) = \frac{10}{(s+1)(s^2+9)}.$$

Now we use partial fractions to help us find $\mathcal{L}^{-1}\{Y(s)\}$. In particular,

$$Y(s) = \frac{1-s}{s^2+9} + \frac{1}{s+1} = \frac{1}{s^2+9} - \frac{s}{s^2+9} + \frac{1}{s+1},$$

so the solution of the ODE is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \frac{1}{3}\sin 3t - \cos 3t + e^{-t}. \end{aligned}$$

```
Y(s) = 10/((s+1) * (s^2+9))
show(Y(s).partial_fraction())
```

One restriction of the Laplace transform approach as we've developed it is that it requires initial conditions at $t = 0$. If initial conditions are instead given at some other $t = t_0$, then we need to adjust our approach in order to continue to use Theorem 6.2.1 and its counterpart for second derivatives. We'll demonstrate this by example.

Example 6.2.5 Solving an IVP with Shifted Initial Conditions.

Solve the IVP given by

$$y'' - 2y' - 3y = 0 \text{ where } y(4) = -3, y'(4) = -17.$$

Solution. We would like to take the Laplace transform of both sides of this ODE, but since the initial conditions are not specified at 0 we can't do so right away. The trick here is to shift the initial conditions *back* to 0 as follows. First, we define a new function $\widetilde{y}$ by

$$\widetilde{y}(t) = y(t+4)$$

where $y(t)$ is the solution we seek. Then it follows that

$$\widetilde{y}'' - 2\widetilde{y}' - 3\widetilde{y} = 0$$

and $\widetilde{y}(0) = y(4) = -3$ and $\widetilde{y}'(0) = y'(4) = -17$. So the Laplace transform method applies to solving for $\widetilde{y}$ in this modified IVP.
Now that we can take the Laplace transform of both sides of the modified IVP, we do so and obtain the Laplace transform $\widetilde{Y}(s)$ of $\widetilde{y}(t)$. Using our initial conditions and a bit of algebra, we get

$$\widetilde{Y}(s) = \frac{-3s - 11}{s^2 - 2s - 3}.$$

At this point we can take the inverse Laplace transform (either using partial fractions or using technology) to get

$$\widetilde{y}(t) = -5e^{3t} + 2e^{-t}.$$

The final step is to convert our answer from $\widetilde{y}$ to y by making the substitution $t \mapsto t - 4$ in $\widetilde{y}$. This gives

$$y(t) = -5e^{3(t-4)} + 2e^{-(t-4)}$$

as the solution of our IVP.

```
var('s,t')

# modified Laplace transform from above example
Y(s) = (-3*s-11)/(s^2 - 2*s - 3)

# solution of modified ODE
f(t) = inverse_laplace(Y(s), s, t)

# translate to solution of original ODE
show(f(t-4))
```

6.3 Unit Step Functions and t-Shifting

Recall that Theorem 6.1.12 tells us how to deal with translation in the frequency domain:

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a) \quad \text{or} \quad \mathcal{L}^{-1}\left\{F(s-a)\right\} = e^{at}f(t).$$

A similar result is true for t-shifting, but we must examine another function first.

The **unit step function** or **Heaviside function** is the function $u(t)$ defined by

$$u(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}.$$

Note also that

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$

for any a. We examined the Laplace transform of a specific translate in Example 6.1.7, and the work done in that example can be used to prove the following general formula:

$$\mathcal{L}\left\{u(t-a)\right\} = \frac{e^{-as}}{s}.$$

This is the only transform equation we have up to this point that involves translation in the time domain. This suggests that expressions involving translations in the time domain might involve exponential terms in the frequency domain.

Time Shifting

Our general formula for dealing with translated functions in the time domain is the next theorem.

Theorem 6.3.1 Time Shifting of Laplace Transforms.

Let $f(t)$ denote a piecewise continuous function with Laplace transform $F(s)$. Let $a \geq 0$. Then

$$\mathcal{L}\left\{u(t-a)f(t-a)\right\} = e^{-as}F(s),$$

or equivalently

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = u(t-a)f(t-a).$$

Proof. To compute this we need to rely on the definition of the Laplace transform:

$$\begin{aligned} \mathcal{L}\left\{u(t-a)f(t-a)\right\} &= \int_0^\infty u(t-a)f(t-a)e^{-st}\,dt \\ &= \int_a^\infty f(t-a)e^{-st}\,dt \\ &= \int_0^\infty f(\tau)e^{-s(\tau+a)}\,d\tau \\ &= e^{-sa}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau \\ &= e^{-as}F(s). \end{aligned}$$

■

Theorem 6.3.1 allows us to take transforms of time-shifted functions and inverse transforms of expressions involving exponentials. This will become particularly important once we introduce the Dirac delta function in Section 6.4.

Example 6.3.2 Computing the transform of time-shifted functions.

Compute Laplace transforms of the following functions:

1. $f(t) = u(t-4)\sin(t-4)$.
2. $g(t) = u(t-2)t^2$.

Solution. We will use Theorem 6.3.1 to compute both of these transforms. The first transform is relatively easy to find once we recognize that $a = 4$, and so

$$F(s) = e^{-4s}\mathcal{L}\{\sin(t)\} = e^{-4s}\frac{1}{s^2+1}.$$

The next transform requires a bit more care since $u(t-2)t^2$ must fit the form $u(t-a)h(t-a)$ before we can use Theorem 6.3.1. In particular, we need

$$h(t-2) = t^2$$

which means that $h(t) = (t+2)^2$. Therefore

$$\begin{aligned} G(s) &= \mathcal{L}\{u(t-2)h(t-2)\} \\ &= e^{-2s}\mathcal{L}\{h(t)\} \\ &= e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right). \end{aligned}$$

Example 6.3.3 Inverse transform involving an exponential.

Compute the inverse Laplace transform of

$$G(s) = \frac{e^{-7s}}{s^2+2s+1}.$$

Solution. The exponential term e^{-7s} tells us that our inverse transform should involve the unit step $u(t-7)$. To find the rest of our inverse transform, we must identify $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+1}\right\}$. By factoring, we get

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}.$$

This involves frequency shifting and so we apply Theorem 6.1.12 to compute the inverse transform:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = te^{-t}.$$

Therefore,

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\{G(s)\} \\ &= u(t-7)f(t-7) \\ &= u(t-7)(t-7)e^{-(t-7)}. \end{aligned}$$

It's important to emphasize that that inverse transform of $e^{-as}F(s)$ is *not* $f(t)$. Rather, it is a *translate* of $f(t)$ given by shifting $f(t)$ exactly a units to the right. The figure below demonstrates this for Example 6.3.3.

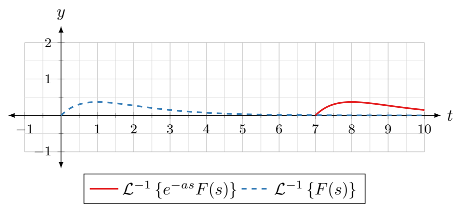

Figure 6.3.4 Inverse transform of $F(s)$ and $e^{-as}F(s)$

Transforms of Piecewise Functions

The Heaviside function is useful for describing forces that turn on or off at specified times. In particular, we can now solve the IVP given at the start of this chapter in Example 6.1.1.

Example 6.3.5 IVP with Discontinuous Forcing Function.

A mass of 1 kg is attached to a spring that is held 1 m to the right of its equilibrium position by a force of 4 N. Beginning at time $t = 0$, a machine is turned on and applies an external force of $\cos 2t$ to the mass. At time $t = 2\pi$ the machine is turned off and the external force disappears. Let $x(t)$ be the displacement of the mass at time t. What is an ODE that models the motion of the mass?

Solution. By Hooke's Law and Newton's Second Law, we have

$$mx'' + kx = F_E(t)$$

where F_E is the external force at time t. Since $m = 1$, $k = 4$ and

$$F_E(t) = \begin{cases} \cos 3t & \text{if } 0 \le t < 2\pi \\ 0 & \text{if } t \ge 2\pi \end{cases}$$

the motion of the mass satisfies the IVP

$$x'' + 4x = F_E(t), x(0) = 1, x'(0) = 0.$$

We can rewrite $F_E(t)$ as follows:

$$F_E(t) = [u(t-0) - u(t-2\pi)]\cos 3t = u(t)\cos 3t - u(t-2\pi)\cos 3t.$$

So the IVP we need to solve is

$$x'' + 4x = u(t)\cos 3t - u(t-2\pi)\cos 3t, x(0) = 1, x'(0) = 0.$$

If we take Laplace transforms, this becomes

$$s^2X(s) - s + 4X(s) = \mathcal{L}\{F_E(t)\},$$

where

$$\begin{aligned} \mathcal{L}\{F_E(t)\} &= \mathcal{L}\{u(t)\cos 3t\} - \mathcal{L}\{u(t-2\pi)\cos 3t\} \\ &= \frac{s}{s^2+9} - \mathcal{L}\{u(t-2\pi)\cos(3(t-2\pi)+6\pi)\} \\ &= \frac{s}{s^2+9} - \mathcal{L}\{u(t-2\pi)\cos(3(t-2\pi))\} \end{aligned}$$

$$= \frac{s}{s^2+9} - e^{-2\pi s}\frac{s}{s^2+9}.$$

So

$$s^2X(s) - s + 4X(s) = \frac{s}{s^2+9} - e^{-2\pi s}\frac{s}{s^2+9}.$$

If we solve this for $X(s)$, we get

$$X(s) = \frac{s}{s^2+4} + \frac{s}{(s^2+9)(s^2+4)} - e^{-2\pi s}\frac{s}{(s^2+9)(s^2+4)},$$

and if we simplify this using partial fractions this becomes

$$X(s) = \frac{s}{s^2+4} + \frac{1}{5}\left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right]\left(1 - e^{-2\pi s}\right).$$

So the solution of the IVP is

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4} + \frac{1}{5}\left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right]\left(1 - e^{-2\pi s}\right)\right\} \\ &= \cos 2t + \frac{1}{5}\left(\cos 2t - \cos 3t\right) - \frac{1}{5}\left(u(t-2\pi)\cos[2(t-2\pi)] - u(t-2\pi)\cos[3(t-2\pi)]\right). \end{aligned}$$

Although it's important to know how to deal with Laplace transforms of basic functions by hand, if only to understand the behavior of the transform itself in solving differential equations, computing transforms of more complicated functions or piecewise functions like the function in Example 6.3.5 are perhaps better left to computer systems. The code cell below demonstrates how Sage can compute such a transform. ***Be careful to place matching brackets and parentheses as appropriate*** when using the `piecewise` command in Sage to construct a piecewise function.

```
# declare meaningful variables for Laplace transform
var('s␣t')

# define piecewise function using piecewise()
# piecewise() command uses a list of cases separated by
    commas to construct its function
# add more cases if needed
case1 = ((0, 2*pi), cos(3*t))
case2 = ((2*pi, oo), 0)
F_E = piecewise([case1, case2])

# now we can compute the Laplace transform using the
    .laplace() method
# the arguments (t,s) tell Sage to change from t to s after
    computing the transform
F_E.laplace(t,s)
```

6.4 Dirac Delta Functions

Impulses

Forces that act over very short time intervals may be complicated to describe exactly, but it we can approximate such a force if we treat it as instantaneous.

Our goal now is describe a meaningful mathematical interpretation of an instantaneous force.

To be specific, let $f(t)$ be a force that acts only from $t = a$ to $t = b$ (and is otherwise 0). Then the **impulse** of the force $f(t)$ over the interval $[a, b]$ is given by

$$p = \int_a^b f(t)\,dt\ .$$

We view the impulse as essentially describing how the force acts over a short time interval, so we can switch from modeling instantaneous forces to instantaneous impulses. And since the impulse is a number, all we really need to do is model an instantaneous *unit impulse*; any other impulse we can get by multiplication by a constant.

So this is our goal: find some function $d(t)$ that has an instantaneous unit impulse at the point $t = 0$. In other words, we want to find a function $d(t)$ such that

$$\int_0^0 d(t)\,dt = 1$$

But this is impossible for any function, since

$$\int_0^0 d(t)\,dt\ = 0.$$

However, we can approximate the *idea* of an instantaneous unit impulse by defining

$$d_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon} & \text{if } -\frac{\varepsilon}{2} \le t \le \frac{\varepsilon}{2} \\ 0 & \text{otherwise} \end{cases}.$$

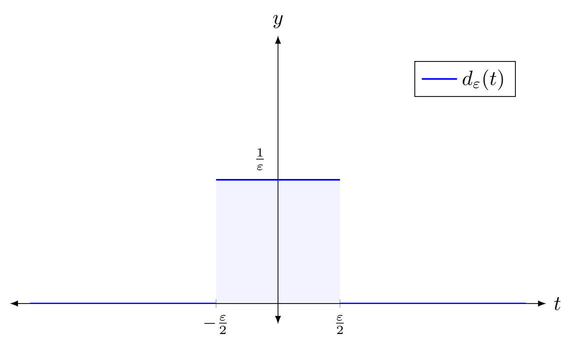

Figure 6.4.1 Approximating the instantaneous unit impulse.

As indicated in Figure 6.4.1, this function is defined so that the area under the graph is 1 regardless of the value of ε. Therefore

$$\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} d_\varepsilon(t)\,dt\ = 1$$

for all $\varepsilon > 0$. Although the instantaneous unit impulse we tried to define earlier can't actually be a function, $d_\varepsilon(t)$ is a completely valid function for all positive ε. Furthermore, if we send $\varepsilon \to 0$ then it becomes a better and better approximation to an ideal instantaneous unit impulse. Despite the fact that this limit does not exist (at least in the usual sense), we use it as a definition.

Definition 6.4.2 The Dirac Delta Function.

The **Dirac delta function**, denoted by the symbol $\delta(t)$, is "defined" by the equation

$$\delta(t) = \lim_{\varepsilon \to 0} d_\varepsilon(t).$$

The Dirac delta function is, of course, not an actual function, but it's still useful as a mathematical formulation of an instantaneous force with unit impulse at $t = 0$. And this expression is often perfectly valid to work with inside of integrals due to the sampling property discussed below.

The Dirac delta is actually an example of a **generalized function**, or **tempered distribution**.

We can also take translations of the Dirac delta, which we view as a "function" $\delta(t-a)$ defined piecewise by

$$\delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{if } t \neq a \end{cases}.$$

We can represent this graphically as an arrow (see Figure 6.4.3).

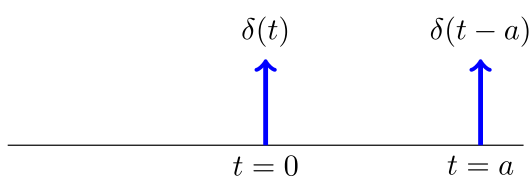

Figure 6.4.3 A plot of the Dirac delta function.

The most important property of the Dirac delta is that

$$\int_\alpha^\beta \delta(t-a)\,dt = \begin{cases} 1 & \text{if } \alpha \leq a \leq \beta \\ 0 & \text{otherwise.} \end{cases}$$

This property can be generalized to products involving the Dirac delta and continuous functions. Very roughly, if $\delta(t-a)$ represents an arrow with an "area" of 1 concentrated at $t = a$, then $\delta(t-a)g(t)$ represents an arrow with area $g(a)$ concentrated at $t = a$. This **sampling property** is stated more precisely as Theorem 6.4.4.

Theorem 6.4.4 Sampling Property of the Dirac Delta.

Let $g(t)$ be a continuous function and let $\alpha \leq a \leq \beta$. Then

$$\int_\alpha^\beta g(t)\delta(t-a)\,dt = g(a)$$

Proof. We treat $\delta(t)$ as a function and perform the above integration:

$$\begin{aligned} \int_\alpha^\beta g(t)\delta(t-a)\,dt &= \int_a^a g(t)\delta(t-a)\,dt \\ &= g(a)\int_a^a \delta(t-a)\,dt \end{aligned}$$

$$= g(a)$$

■

The above theorem gives us another interpretation of the Dirac delta: it's a "sampling function." When integrated against another function $g(t)$ over an interval containing a, $\delta(t-a)$ will pick out the value $g(a)$. We can use this to quickly find the Laplace transform of the Dirac delta.

Example 6.4.5 Laplace Transform of the Dirac Delta Function.

Compute $\mathcal{L}\left\{\delta(t-a)\right\}$, where $a \geq 0$.

Solution. We use the definition of the Laplace transform:

$$\begin{aligned}\mathcal{L}\left\{\delta(t-a)\right\} &= \int_0^\infty \delta(t-a)e^{-st}\,dt \\ &= e^{-as}.\end{aligned}$$

In particular, $\mathcal{L}\left\{\delta(t)\right\} = 1$.

Dirac Delta Models

We will primarily use the Dirac delta to model instantaneous forces, such as sudden kicks or jolts. Although this is not strictly realistic since such forces are still imparted over some interval of time, treating the force as instantaneous often simplifies computations.

Example 6.4.6 IVP with Impulse.

An object of mass $m = 1$, at rest, is attached to a spring with spring constant $k = 9$. At time $t = 0$, the a hammer strikes the mass providing an impulse of 3 and setting the mass in motion. What is the displacement $x(t)$?

Solution. The displacement $x(t)$ satisfies the ODE

$$mx'' + kx = f(t)$$

where $f(t)$ is the external force. Since the hammer strikes quickly, we can model it as an instantaneous force of the form $c\delta(t)$. And since it provides an impulse of 3, we can pick $f(t) = 3\delta(t)$. As the mass is initially at rest, x satisfies the IVP

$$x'' + 9x = 3\delta(t) \quad \text{where} \quad x(0) = x'(0) = 0.$$

To solve this, we take the Laplace transform of the IVP to get

$$s^2X(s) + 9X(s) = 3$$

or just

$$X(s) = \frac{3}{s^2+9}.$$

So $x(t) = \sin 3t$.

Note that the above solution does not appear to satisfy our initial conditions. However, since we assumed the impulse acted instantaneously at time $t = 0$, this is really the same as assuming that the mass had an initial velocity. Now we look at what happens if we delay the hammer strike.

Example 6.4.7 Time-delayed Strike.

Consider the spring-mass system above, but suppose now that the hammer hits the mass at time $t = 4$. What is the displacement $x(t)$?

Solution. This time, the IVP we must solve is

$$x'' + 9x = 3\delta(t-4) \quad \text{where} \quad x(0) = x'(0) = 0.$$

If we take Laplace transforms and solve for $X(s)$ we get

$$X(s) = \frac{3e^{-4s}}{s^2+9}$$

and so

$$x(t) = u(t-4)\sin[3(t-4)].$$

This function is plotted in Figure 6.4.8 below. Note the clear appearance of the sine wave in the solution once the hammer strikes the mass at $t = 4$.

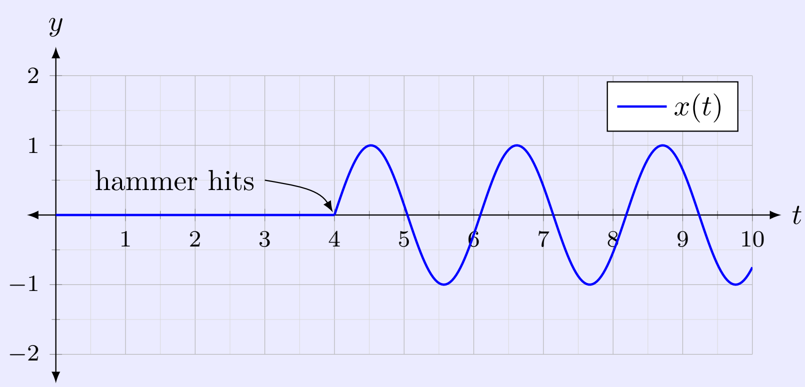

Figure 6.4.8 Displacement influenced by a time-delayed hammer strike.

Example 6.4.9 Resonance with an Impulse Train.

Once again we consider the spring-mass system used above in Example 6.4.7, but now we suppose that the mass is struck with the hammer once every $\frac{2\pi}{3}$ seconds, starting at $t = 0$. Find $x(t)$.

Solution. The IVP we need to solve now is

$$\begin{aligned} x'' + 9x &= 3\delta(t) + 3\delta(t - \frac{2\pi}{3}) + 3\delta(t - \frac{4\pi}{3}) + \cdots \\ &= 3\sum_{n=0}^{\infty} \delta(t - \frac{2\pi n}{3}) \end{aligned}$$

where $x(0) = x'(0) = 0$. So once more we take Laplace transforms to get

$$s^2X(s) + 9X(s) = 3\sum_{n=0}^{\infty} \mathcal{L}\left\{\delta(t - \frac{2\pi n}{3})\right\} = 3\sum_{n=0}^{\infty} e^{-\frac{2\pi n}{3}s}$$

and so

$$X(s) = \frac{3}{s^2+9}\sum_{n=0}^{\infty} e^{-\frac{2\pi n}{3}s}.$$

The displacement is then given by

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1}\{X(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\cdot\sum_{n=0}^{\infty} e^{-\frac{2\pi n}{3}s}\right\} \\
&= \mathcal{L}^{-1}\left\{\sum_{n=0}^{\infty} e^{-\frac{2\pi n}{3}}\frac{3}{s^2+9}\right\} \\
&= \sum_{n=0}^{\infty} u(t-\frac{2\pi n}{3})\sin[3(t-\frac{2\pi n}{3})] \\
&= \sum_{n=0}^{\infty} u(t-\frac{2\pi n}{3})\sin 3t.
\end{aligned}$$

Each time the hammer strikes the mass, a factor of $\sin 3t$ is added to the displacement. The repeated hammer strikes are in tune with the natural frequency of the mass, so they create resonance. This is clearly demonstrated in the figure below.

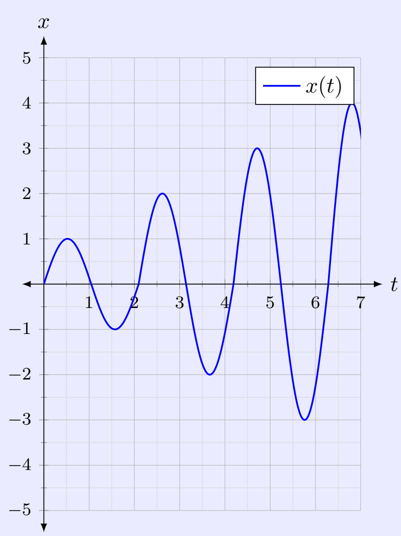

Figure 6.4.10 The displacement $x(t)$ and resonance with the impulse train.

The Dirac delta function is also important mathematically in determining the behavior of certain systems. For example, consider the system determined by the IVP

$$x'' + 9x = f(t) \text{ where } x(0) = x'(0) = 0.$$

We can imagine that the ODE is a mathematical "machine" that converts the input $f(t)$ into an output, or response, $x(t)$. Using Laplace transforms it's not difficult to determine $X(s)$:

$$X(s) = \frac{1}{s^2+9}F(s).$$

So we see that the Laplace transform of the response is related to the Laplace transform of the input by $G(s) = \frac{1}{s^2+9}$. This function is the **transfer function** of the system and determines all possible responses.

Now here's the connection with the Dirac delta. If we replace $f(t)$ with $\delta(t)$ to get the system $x'' + 9x = \delta(t)$ and take Laplace transforms, we get

$$X(s) = \frac{1}{s^2+9}.$$

In other words, the transfer function is just the Laplace transform of the solution of $x'' + 9x = \delta(t)$. We call the solution the **impulse reponse** of the system. In this case, the impulse response is $g(t) = \frac{1}{3}\sin(3t)$. Since the corresponding transfer function determines the form of $X(s)$ for all responses $x(t)$, the impulse reponse must determine the form of all responses in the time domain. We will see precisely how in the next section by introducing the concept of a convolution.

6.5 Convolution Products

Consider the function $F(s)$ in the frequency domain s defined by

$$F(s) = \frac{1}{s(s^2+4)}.$$

This does not match a Laplace transform on our table. If we wanted to find the inverse transform $f(t)$, we would have to use partial fractions to find it. This is certainly feasible, but more than a little tedious.

However, $F(s)$ is easily seen to be the *product* of two recognizable transforms:

$$F(s) = \frac{1}{s}\frac{1}{s^2+4} = \frac{1}{2}\mathcal{L}\left\{1\right\}\mathcal{L}\left\{\sin 2t\right\}.$$

What we would like to do is find a way to determine inverse transforms of products of transforms. To do this, we need to define the *convolution* of two functions, which is the time domain operation that corresponds to multiplication in the frequency domain.

Convolutions

Definition 6.5.1 Convolution of Functions.

Let f and g be piecewise continuous functions. The **convolution** of f and g, denoted by $(f*g)(t)$, $f(t)*g(t)$ or just $f*g$, is defined for $t \geq 0$ by

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Note that $f*g$ lies in the time domain instead of the frequency domain. In general, we will only take convolutions of functions in the time domain.

It's difficult to build intuition for precisely what convolutions do based on the formula alone, but one reason that convolutions are important is because they act as "smoothing operators". If you have a "rough" (i.e. non-differentiable) function, then taking the convolution of it with a properly chosen smooth function may give you a smooth approximation. We will see another interpretation of the convolution at the end of this section.

Example 6.5.2 Convolution with unit step.

Compute $u * t$, where $u(t)$ is the unit step function.

Solution. By definition,

$$\begin{aligned} u(t) * t &= \int_0^t u(\tau)(t-\tau)\, d\tau \\ &= \int_0^t (t-\tau)\, d\tau \\ &= \left[t\tau - \frac{\tau^2}{2} \right]_{\tau=0}^{\tau=t} \\ &= \frac{t^2}{2} \end{aligned}$$

Convolutions can be thought of as a peculiar kind of multiplication for functions. They are *commutative*, that is, $f * g = g * f$ for any piecewise continuous functions f and g. They are also *distributive*:

$$f * (g+h) = f * g + f * h.$$

Convolutions and Transform Problems

An important property of convolutions is that they tend to work well with integral transforms. In particular, we have the following theorem.

Theorem 6.5.3 Convolution Theorem.

The Laplace transform distributes over convolution. In other words, if f and g are piecewise continuous functions, then

$$\mathcal{L}\left\{f(t) * g(t)\right\} = \mathcal{L}\left\{f(t)\right\} \mathcal{L}\left\{g(t)\right\}.$$

Equivalently, if we write $F(s) = \mathcal{L}\left\{f(t)\right\}$ and $G(s) = \mathcal{L}\left\{g(t)\right\}$, then

$$\mathcal{L}^{-1}\left\{F(s)G(s)\right\} = f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\, dt\ .$$

One way to phrase the above result is that Laplace transforms turns convolution in the time domain into multiplication in the frequency domain. Let's return to the example we started with.

Example 6.5.4 Using the Convolution Theorem.

Let

$$F(s) = \frac{1}{s(s^2+4)}.$$

Find $f(t) = \mathcal{L}^{-1}\left\{F(s)\right\}$.

Solution. We will use Theorem 6.5.3 to express the inverse transform:

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\}$$

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+4}\right\} \\
&= 1 * \frac{1}{2}\sin 2t \\
&= \frac{1}{2}\int_0^t \sin 2\tau \, d\tau \\
&= -\frac{1}{4}\left[\cos 2\tau\right]_{\tau=0}^{\tau=t} \\
&= \frac{1-\cos 2t}{4}.
\end{aligned}
$$

Example 6.5.5 Solving IVPs with the Convolution Theorem.

Let x be a function of t. Solve the IVP

$$x'' + 4x' + 13x = f(t) \quad \text{where} \quad x(0) = x'(0) = 0$$

for $x(t)$ in terms of the function $f(t)$.

Solution. We're trying to find the solution $x(t)$ for arbitrary $f(t)$, which is something we definitely would not have been able to do in Chapter 3. We will do so using Laplace transforms and Theorem 6.5.3. We start by taking the Laplace transform of the ODE to get

$$s^2X(s) + 4sX(s) + 13X(s) = F(s)$$

where $X(s) = \mathcal{L}\{x(t)\}$ and $F(s) = \mathcal{L}\{f(t)\}$. Now we solve for $X(s)$ to get

$$X(s) = \frac{F(s)}{s^2+4s+13} = F(s)G(s)$$

where

$$G(s) = \frac{1}{s^2+4s+13}.$$

Theorem 6.5.3 tells us then that

$$x(t) = \mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t)$$

where

$$
\begin{aligned}
g(t) &= \mathcal{L}^{-1}\{G(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+13}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+9}\right\} && \text{complete the square} \\
&= \frac{1}{3}e^{-2t}\sin 3t && \text{use a translation formula}
\end{aligned}
$$

Therefore the solution of the ODE in terms of the function $f(t)$ is given by

$$
\begin{aligned}
x(t) &= f(t) * g(t) \\
&= \int_0^t f(\tau)g(t-\tau)\, d\tau \\
&= \int_0^t f(\tau)\frac{e^{-2(t-\tau)}\sin[3(t-\tau)]}{3}\, d\tau.
\end{aligned}
$$

There are a couple of interesting things happening in the last example. In particular, we were able to write the solution $x(t)$ in terms of $f(t)$ as

$$x(t) = g(t) * f(t).$$

In these terms, we can recognize $g(t)$ as the impulse response of the system discussed at the end of Section 6.4. Therefore the response of this system to the input $f(t)$ is related to the impulse response by a convolution:

$$x(t) = \int_0^t f(\tau)g(t-\tau)\, d\tau \ .$$

Essentially, this convolution integral tells us how to construct the response $x(t)$ using only information about the input $f(t)$ and the impulse response $g(t)$ from the "past and present." Recall that this relationship is even easier to write in the frequency domain: the frequency input $F(s)$ is turned into the frequency output $X(s)$ by means of the transfer function $G(s) = \mathcal{L}\{g(t)\}$:

$$X(s) = G(s)F(s).$$

There are multiple ways to find the transfer function, assuming that all initial conditions are 0. First, if $F(s)$ is the input and $X(s)$ is some measured output (once again, in the frequency domain), then the transfer function $G(s)$ satisfies

$$G(s) = \frac{X(s)}{F(s)},$$

and this quantity is independent of the particular choice of $F(s)$. We can also replace the time domain input $f(t)$ with $\delta(t)$ as discussed in Section 6.4 to determine the impulse response directly. In this case, the corresponding output in the frequency domain is

$$X(s) = G(s)\mathcal{L}\{\delta(t)\} = G(s).$$

Part II

Partial Differential Equations

Chapter 7

Fourier Analysis

Our goal now is to move to solving *partial* differential equations (PDEs), which are differential equations that involve multiple independent variables. Such equations are particularly useful for modeling quantities that depend on position x and time t. It turns out that some important partial differential equations (such as the wave equation and the heat equation) often have simple solutions in terms involve periodic functions, and general solutions can be constructed using these simpler periodic ones. So our first step to solving PDEs will be to find useful descriptions of periodic (and in some cases, non-periodic) functions.

7.1 Fourier Series

The main idea behind Fourier series, and the field of harmonic analysis in general, is to represent more complicated objects in terms of simpler objects. A fundamental example of this idea comes from the field of linear algebra in the form of *orthonormal bases*. Knowing an orthonormal basis for a vector space V can greatly simplify linear algebra in that vector space. In this section we'll do something similar with **periodic functions**, which are functions whose values repeat themselves.

Periodic Functions

Consider the function $f(x)$ given by the following graph:

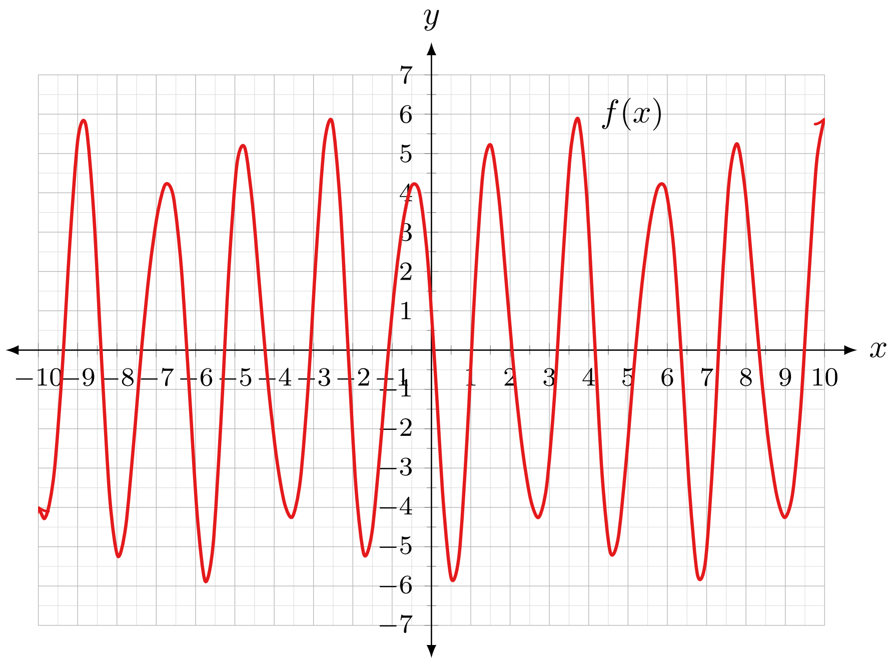

Figure 7.1.1 A periodic function

If we look at the graph, we see that it repeats itself if we wait long enough (approximately every six units). Functions that have this property are called **periodic functions**.

Definition 7.1.2 Periodic Functions.

Let $f(x)$ be a real function defined for all x. We say that f is a periodic function if there exists a positive number T such that

$$f(x+T) = f(x)$$

for all x. In this case we say that $f(x)$ is T**-periodic**. The **(fundamental) period** of $f(x)$ is the smallest positive value of T for which f is T-periodic, assuming this value exists.

Constant functions are examples of periodic functions with *no* fundamental period.

Example 7.1.3 Periods of Sine and Cosine.

Let n be any positive integer. Then the functions $\sin(nx)$ and $\cos(nx)$ are both 2π-periodic which follows from the corresponding addition formulas

$$\begin{aligned}\sin[n(x+2\pi)] &= \sin nx \cos 2\pi + \cos nx \sin 2\pi = \sin nx\\ \cos[n(x+2\pi)] &= \cos nx \cos 2\pi - \sin nx \sin 2\pi = \cos nx.\end{aligned}$$

The period, in particular, is $T = \frac{2\pi}{n}$. The periodic nature of these functions can also be seen from their graphs:

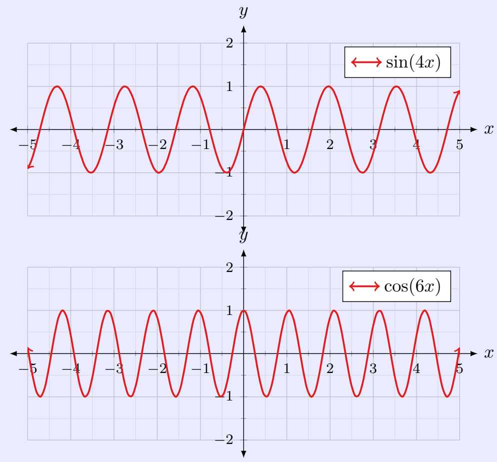

Figure 7.1.4 Graphs of $\sin(nx)$ and $\cos(nx)$

The graph in Figure 7.1.1 was produced by graphing

$$f(x) = \cos 5x - 5\sin 3x.$$

In general, the (finite) sum of functions of the form $\sin nx, \cos mx$ where n, m are integers is also 2π-periodic. In particular, we have the following result.

Theorem 7.1.5 Periods of Sums of Sinusoids.

Let $f(x) = \sin(mx)$ and $g(x) = \cos(nx)$ where $m, n > 0$. Suppose that $\frac{m}{n} = \frac{a}{b} \in \mathbb{Q}$ where $\frac{a}{b}$ represents the reduced fraction of $\frac{m}{n}$. Then $f(x) + g(x)$ has period given by

$$T = \frac{2\pi}{\left(\frac{m}{a}\right)} = \frac{2\pi}{\left(\frac{n}{b}\right)}.$$

Example 7.1.6 Finding Periods of Sums of Sinusoids.

Find the periods of $\sin(4x) + \cos(6x)$ and $\cos(5x) - 5\sin(3x)$.

Solution. For $\sin(4x)+\cos(6x)$ we have $\frac{m}{n} = \frac{4}{6}$, which in lowest terms is $\frac{2}{3}$. Therefore the period is $T = \frac{2\pi}{\frac{4}{2}} = \pi$. For $\cos(5x) - 5\sin(3x)$, we have $\frac{m}{n} = \frac{5}{3}$ which is already in lowest terms. Therefore its period is $\frac{2\pi}{\frac{5}{5}} = 2\pi$.

Trigonometric Series and Fourier Series

One of the greatest accomplishments in mathematics was the realization that many other periodic functions can be written as a sum of sinusoids using *trigonometric polynomials* and *trigonometric series*.

Definition 7.1.7 Trigonometric Polynomials and Series.

A **trigonometric polynomial** is a finite sum of the form

$$\sum_{k=0}^{n}(a_k \cos kx + b_k \sin kx) = a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

for some natural number n. A **trigonometric series** is a series of the form

$$\sum_{k=0}^{\infty}(a_k \cos kx + b_k \sin kx) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx).$$

For both sums the values a_k, b_k are constants called the **coefficients**.

Our primary goal in this section is to take a function $f(x)$ of period 2π and express it as a trigonometric series. To see how, we'll suppose that we have the trigonometric series we want, i.e. that

$$f(x) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx),$$

and we'll look at what the coefficients of the series need to be to make this equation true. To do this, we'll need the so-called **orthogonality relations** for $\sin nx, \cos mx$.

Theorem 7.1.8 Orthogonality Relations.

Let m, n be whole numbers with $m, n > 0$. Then

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0.$$

Furthermore,

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} \pi & m = n \\ 0 & m \neq n \end{cases}$$

and

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} \pi & m = n \\ 0 & m \neq n \end{cases}.$$

We can verify Theorem 7.1.8 using a computer algebra system as below. Proving it is a little bit more work, but can be done using trigonometric identities or Euler's formula.

```
# Declare variables for use in computations.
# By default, Sage considers x to be a variable unless
# it has been redefined by some equation.
var('m,n')

# We also want to tell Sage that m,n represent integers.
# This will allow Sage to simplify the integral
# computations below.
assume(m,n,'integer')

# Perform the integrations.
```

```
I1 = integral(sin(m*x)*cos(n*x),x,-pi,pi)
I2 = integral(sin(m*x)*sin(n*x),x,-pi,pi)
I3 = integral(cos(m*x)*cos(n*x),x,-pi,pi)
I4 = integral(sin(m*x)*sin(m*x),x,-pi,pi)
I5 = integral(cos(n*x)*cos(n*x),x,-pi,pi)

# List the results.
I1,I2,I3,I4,I5
```

Theorem 7.1.8 will be our primary tool for expressing a function $f(x)$ as a trigonometric series. To see how, suppose that we have

$$f(x) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx).$$

If this equation were true, then we should be able to integrate both sides of it and get another true equation. Since Theorem 7.1.8 suggests that integrals involving $\sin nx, \cos nx$ simplify very nicely, we'll try to integrate both sides of the equation against $\sin nx, \cos nx$ from $x = -\pi$ to $x = \pi$ for some $n > 0$. If we do this, we get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin nx\,dx &= a_0 \int_{-\pi}^{\pi} \sin nx\,dx + \sum_{k=1}^{\infty}\left(a_k \int_{-\pi}^{\pi} \cos kx \sin nx\,dx + b_k \int_{-\pi}^{\pi} \sin kx \sin nx\,dx\right)\\ &= b_n \int_{-\pi}^{\pi} \sin nx \sin nx\,dx\\ &= \pi b_n\end{aligned}$$

This lets us solve for b_n! We have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \text{ for } n \geq 1.$$

Similarly,

$$\begin{aligned}a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \text{ for } n \geq 1.\\ a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx\end{aligned}$$

The trigonometric series resulting from these coefficients is known as the **Fourier series** of $f(x)$. These formulas are useful enough that we'll place them together in a theorem.

Theorem 7.1.9 Fourier Series Coefficients.

Let $f(x)$ be a periodic function with period 2π. Then the Fourier coefficients of $f(x)$ are given by

$$\begin{aligned}a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx\\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \textit{ for } n \geq 1.\\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \textit{ for } n \geq 1.\end{aligned}$$

Note that the formulas in Theorem 7.1.9 tell us what the coefficients of a Fourier series representation of $f(x)$ must be if such a representation existed, but as yet there is no guarantee that a function actually equals its Fourier series. Also, since $f(x)$ is assumed to be 2π-periodic we can also integrate over $[0, 2\pi]$ instead without changing the values of the coefficients.

Example 7.1.10 The Fourier series of x^3.

Define $f(x) = x^3$ for $-\pi \le x \le \pi$. To find its Fourier series, we can just use the previous formulas to find the values of the coefficients a_0, a_k, b_k for $k \ge 1$. We know that

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^3\,dx \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \cos kx\,dx \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin kx\,dx. \end{aligned}$$

As nasty as these are, the first two are actually very easy to compute. Here's why: x^3 and $x^3 \cos kx$ are both *odd* functions, and the integral of any odd function in an interval that is symmetric about 0 is always 0 (since the areas cancel out). So $a_0 = a_k = 0$ for all $k \ge 0$.
The last term is a bit more complicated, but we can use integration by parts (and I definitely recommend using a computer here) to show that

$$\int x^3 \sin kx\,dx = -\frac{\left(k^3x^3 - 6\,kx\right)\cos(kx) - 3\left(k^2x^2 - 2\right)\sin(kx)}{k^4} + C.$$

If we plug in the limits of integration and simplify (again, computers are handy for this!), we get

$$b_k = \frac{2\left(6\,\pi - \pi^3k^2\right)(-1)^k}{\pi k^3}.$$

So the Fourier series for $f(x) = x^3$ is given by

$$\begin{aligned} a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) &= \sum_{k=1}^{\infty} b_k \sin kx \\ &= \sum_{k=1}^{\infty} \frac{2\left(6\,\pi - \pi^3k^2\right)(-1)^k}{\pi k^3} \sin kx. \end{aligned}$$

A very good question at this point is, what relationship does the Fourier series that we found in the previous example have with the original function $f(x)$? Are they actually equal? If we use the following code (adapted from here[4]) to compare the partial sums

$$\sum_{k=1}^{n} \frac{2\left(6\,\pi - \pi^3k^2\right)(-1)^k}{\pi k^3} \sin kx$$

of the Fourier series with $f(x) = x^3$, then it looks like the partial sums get closer and closer if we choose larger values of n. However, the graphs of the

[4] doxdrum.wordpress.com/2011/01/31/sage-tip-fourier-series-approximation/

partial sums always seem to vary wildly at the endpoints of this curve. This is typical of Fourier series that are used to represent a discontinuous periodic function.

```
# Declare variables for use in computations.
var('k,i')

# Define the function to determine Fourier series of.
# Feel free to play around with this.
f(x) = x^3

# Make this larger to include more terms
# in your Fourier series.
n = 3

# Define the Fourier coefficients.
def a(k):
  coeff = integral(f(x)*cos(k*x), (x,-pi,pi))/pi
  return coeff

def b(k):
  coeff = integral(f(x)*sin(k*x), (x,-pi,pi))/pi
  return coeff

# Define the corresponding (partial) Fourier sum.
# Note that we need to divide a0 by 2 manually!
def FS(k):
  return a(0)/2 + sum(a(i)*cos(i*x) + b(i)*sin(i*x), i, 1, k)

# Now we plot both our function and its
# corresponding Fourier series.
P1 = plot(f(x), (x, -pi, pi), color='blue',
          legend_label=f'${latex(f(x))}$')
P2 = plot(FS(n), (x, -pi, pi), color='red',
          legend_label=f'Fourier series with {n} terms')

# Now combine the plots and display the result.
p = P1 + P2
p.show()
```

In general, the question of whether or not a given Fourier series makes sense is a difficult one to answer.[5] However, for many of the functions we care about in this course we have the following theorem.

Theorem 7.1.11 Fourier Series of Piecewise Continuous Functions.

Let $f(x)$ be a piecewise continuous function on the interval $-\pi \leq x \leq \pi$, and suppose that it's also periodic with period 2π, and is differentiable everywhere that it's continuous. Then the Fourier series of $f(x)$ converges to $f(x)$ except at the points where $f(x)$ is discontinuous.

[5]In fact, the convergence of Fourier series for what one might consider to be the more ”well-behaved” functions in mathematics was an open question until the 1960s. See Carleson's Theorem.

7.2 Functions of Arbitrary Period; Even and Odd Extensions

Now that we know how to find Fourier series of periodic functions with period 2π, but not every periodic function has period π. Important examples of such functions include the **sawtooth wave** and the **square wave**, which have applications in signal processing. These are shown in Figure 7.2.1.

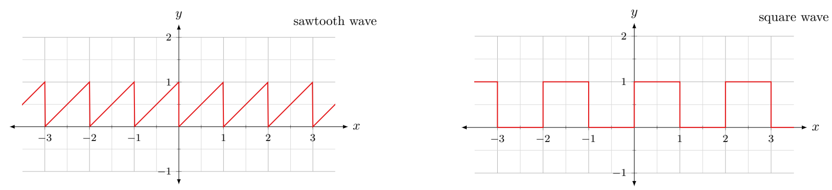

Figure 7.2.1 Sawtooth and square wave of period $p = 1$

Fourier Series of Functions of Arbitrary Period

We know how to find the Fourier series of a function of period 2π by using Theorem 7.1.9. So we'd like to adapt this to functions that have period $p = 2L$ instead of 2π. This actually won't be too hard to do, since any function of period $2L$ can be scaled into a function with period 2π.

To see how, let $f(x)$ denote a function of period $2L$. Then we want to find a constant c so that $f(cx)$ has period 2π, that is, so that

$$f(c(x + 2\pi)) = f(cx).$$

For this to be true, we need $2\pi c = 2L$, or in other words $c = \frac{L}{\pi}$. So if $f(x)$ has period $2L$, then $f(\frac{L}{\pi}x)$ has period 2π, and furthermore $f(\frac{L}{\pi}x)$ has Fourier coefficients given by

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}x\right)\,dx \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}x\right)\cos kx\,dx \text{ for } k \geq 1. \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}x\right)\sin kx\,dx \text{ for } k \geq 1.
\end{aligned}$$

Now we need to get everything back in terms of $f(x)$, the function we started with. We can do this by making the substitution $u = \frac{L}{\pi}x$. This gives us the following theorem after a little algebra.

Theorem 7.2.2 Fourier Series of Functions with Arbitrary Period.

Let $f(x)$ be a function with period $p = 2L$. Then the Fourier coefficients of $f(x)$ are given by

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\
a_k &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi}{L}x\right)\,dx \text{ for } k \geq 1
\end{aligned}$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi}{L}x\right)\,dx \text{ for } k \geq 1$$

and the corresponding Fourier series is

$$a_0 + \sum_{k=1}^{\infty}\left(a_k\cos\left(\frac{k\pi}{L}x\right) + b_k\sin\left(\frac{k\pi}{L}x\right)\right).$$

With this theorem we can now find Fourier expansions of general periodic functions, including those given in Figure 7.2.1.

Example 7.2.3 Fourier series of a periodic extension of $f(x) = x^2$.

Let $f(x) = x^2$ for $-1 \leq x \leq 1$ and have period $p = 2$. We can find its Fourier series using Theorem 7.2.2. If we do so, we get

$$a_0 = \frac{1}{3}$$
$$a_k = \frac{4(-1)^k}{\pi^2 k^2}$$
$$b_k = 0$$

So the Fourier series of $f(x)$ is given by

$$\frac{1}{3} + \sum_{k=1}^{\infty}\frac{4(-1)^k}{\pi^2 k^2}\cos(k\pi x).$$

```
# Tell Sage what we're using as variables and give specific
# assumptions on them.
var('k,i')
assume(k,i,'integer')

# Define the function to determine Fourier series of, and
# define L.
f(x) = x^2
L = 1

# Define the Fourier coefficients.
def a(k):
  coeff = integral(f(x)*cos(k*pi*x/L), (x,-L,L))/L
  return coeff

def b(k):
  coeff = integral(f(x)*sin(k*pi*x/L), (x,-L,L))/L
  return coeff

# Define the corresponding (partial) Fourier series.
def FS(k):
  return a(0)/2 + sum(a(i)*cos(i*pi*x/L) +
      b(i)*sin(i*pi*x/L), i, 1, k)

n = 3
p = plot(FS(n),(x,-L,L), color='red') + plot(f(x),(x,-L,L))
```

p

Even and Odd Extensions

If you look at Examples Example 7.1.10 and Example 7.2.3, then you'll notice that a lot of the Fourier coefficients were 0. In particular, $a_k = 0$ for $k \geq 0$ for the first example, and $b_k = 0$ for the second. The reason for this has to do with **even and odd functions**.

Definition 7.2.4 Even and Odd Functions.

Let $f(x)$ be a function. We say that $f(x)$ is

$$\begin{aligned} \text{even if } f(-x) &= f(x) \\ \text{odd if } f(-x) &= -f(x). \end{aligned}$$

In other words, a function $f(x)$ is even if and only if its graph is symmetric about the y-axis, and is odd if and only if its graph is symmetric about the origin.

In Example 7.1.10 and Example 7.2.3 we began with functions that typically aren't thought of as periodic and found their Fourier series. Essentially what we did was we restricted our view of each function to a limited interval[6] and then viewed that segment as defining a periodic function. This is the idea behind **even and odd extensions** of functions.

Definition 7.2.5 Even and Odd Extensions.

Let $f(x)$ be a function defined on $[0, L]$. The even extension of $f(x)$ is the even periodic function defined by extending the graph of $f(x)$ on $[0, L]$ to the rest of the real numbers in such a way that the resulting function is even and has period $2L$. The odd extension of $f(x)$ is defined similarly.

Computing Fourier series for even and odd functions is simpler than the general case.

Theorem 7.2.6 Fourier Coefficients of Even and Odd Functions.

Let $f(x)$ be periodic with period $2L$. If f is even, then the Fourier coefficients of $f(x)$ satisfy

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, a_k = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{k\pi}{L}x\right)\,dx \text{ and } b_k = 0$$

for $k \geq 1$. If f is odd, then the Fourier coefficients of $f(x)$ are

$$b_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi}{L}x\right)\,dx \text{ and } a_k = 0.$$

[6] $[-\pi, \pi]$ in the case of x^3 and $[-1, 1]$ in the case of x^2

Example 7.2.7 Even Extension of a Piecewise Function.

Define the piecewise function $f(x)$ by

$$f(x) = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \frac{\pi}{2} & \frac{\pi}{2} \le x \le \pi \end{cases}.$$

Then the even extension of $f(x)$ is the new function $f_1(x)$ given by

$$f_1(x) = \begin{cases} |x| & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \frac{\pi}{2} & \frac{\pi}{2} \le |x| \le \pi \end{cases}.$$

We can use Theorem 7.2.6 to help us find the Fourier series for $f_1(x)$. With a little bit of help, we get $a_0 = \frac{3\pi}{8}$ and $a_k = \frac{2\cos(\frac{\pi k}{2})-2}{\pi k^2}$, and so the Fourier series of $f_1(x)$ is

$$\frac{3\pi}{8} + \sum_{k=1}^{\infty} \frac{(-1)^k - 1}{2\pi k^2} \cos(2kx).$$

```
var('x,k,i')
assume(k,i,'integer')
L = pi

# Defines the Fourier coefficients.
def a(k):
  coeff1 = (2/L)*integral(x*cos(k*pi*x/L), (x,0,pi/2))
  coeff2 = (2/L)*integral((pi/2)*cos(k*pi*x/L), (x,pi/2,pi))
  return coeff1+coeff2

# Defines the corresponding (partial) Fourier series.
def FS(k):
  return a(0)/2 + sum(a(i)*cos(i*pi*x/L), i, 1, k)

# Define the even extension.
f_e = piecewise([((-pi/2,pi/2), abs(x)), ((-oo,-pi/2),
    pi/2), ((pi/2, oo), pi/2)])

n = 25
p = plot(FS(n),(x,-L,L), color='red') + plot(f_e, (x, -L,L))
p
```

7.3 Complex Fourier Series and Parseval's Identity

Although we have a decent formula for Fourier series (see Theorem 7.2.2), it's a little unwieldy due to the different expressions for a_k, b_k and a_0. We can fix this, perhaps surprisingly, by using complex exponentials and Euler's formula.

Complex Fourier Series

First, recall Euler's formula 2.2.5, which allows us to rewrite complex exponentials in terms of sine and cosine.

We can use Theorem 2.2.5 to rewrite the Fourier series in Theorem 7.2.2. Our goal now is to find a **complex Fourier series**

$$\sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi}{L}x}$$

for functions $f(x)$ with period $p = 2L$. We will also include the statement of Theorem 7.1.11 in this new context.

Theorem 7.3.1 Complex Fourier Series.

Let $f(x)$ be a piecewise smooth function with period $p = 2L$. Then the complex Fourier series of $f(x)$ is given by

$$\sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi}{L}x},$$

where

$$c_k = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{k\pi}{L}x}\,dx.$$

This Fourier series converges to $f(x)$ wherever $f(x)$ is continuous.

Proof. We need to use another orthogonality relation like we had in the real case, except now it will be written in terms of complex exponentials instead of sine and cosine. In particular, the relation we will use is the following:

$$\int_{-L}^{L} e^{i\frac{(m+n)\pi}{L}x}\,dx = \begin{cases} 2L & m = -n \\ 0 & \text{otherwise.} \end{cases}$$

So if we set $f(x)$ equal to a complex Fourier series and integrate both sides against $e^{-i\frac{n\pi}{L}x}$ for x from $-L$ to L, we get

$$\begin{aligned}\int_{-L}^{L} f(x)e^{-i\frac{n\pi}{L}x}\,dx &= \int_{-L}^{L}\sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi}{L}x}e^{-i\frac{n\pi}{L}x}\,dx \\ &= \sum_{k=-\infty}^{\infty} c_k \int_{-L}^{L} e^{i\frac{(k-n)\pi}{L}x}\,dx \\ &= 2Lc_n\end{aligned}$$

where the last equality follows from the orthogonality relation we just proved. Therefore

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi}{L}x}\,dx.$$

■

Example 7.3.2 Complex Fourier Series of Exponential Function.

Let $f(x) = e^{2x}$ on $-\pi < x < \pi$ and suppose that $f(x)$ is periodic with period 2π. We want to find the complex Fourier series for $f(x)$. We can do this by finding the correct coefficients c_k:

$$c_k = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{k\pi}{L}x}\,dx$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2x}e^{-ikx}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(2-ik)x}\,dx \\
&= \frac{(-1)^k \sinh 2\pi}{\pi(2-ik)} \\
&= \frac{(-1)^k \sinh 2\pi(2+ik)}{\pi(4+k^2)}
\end{aligned}
$$

So we have

$$f(x) = \sum_{k=-\infty}^{\infty} \frac{(-1)^k(2+ik)\sinh 2\pi}{\pi(4+k^2)} e^{ikx}$$

for $x \neq k\pi$, since this is where $f(x)$ has discontinuities.

Although the complex Fourier series can be easier to compute in some cases, there may be cases where we'd like to go back to the real Fourier series. The following formula lets us do so.

Theorem 7.3.3 Real Fourier Series from Complex Fourier Series.

Suppose $f(x)$ has the complex Fourier series

$$\sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi}{L}x}.$$

Then the corresponding coefficients a_k and b_k for the real Fourier series

$$a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\left(\frac{k\pi}{L}x\right) + b_k \sin\left(\frac{k\pi}{L}x\right)\right)$$

are given by

$$
\begin{aligned}
a_0 &= c_0 \\
a_k &= c_k + c_{-k} \\
b_k &= i(c_k - c_{-k})
\end{aligned}
$$

The real Fourier series corresponding to the complex Fourier series for $f(x)$ from Example 7.3.2 has coefficients

$$
\begin{aligned}
a_0 &= c_0 && = \frac{\sinh 2\pi}{2\pi} \\
a_k &= c_k + c_{-k} && = \frac{4(-1)^k \sinh 2\pi}{\pi(4+k^2)} \\
b_k &= i(c_k - c_{-k}) && = -\frac{2k(-1)^k \sinh 2\pi}{\pi(4+k^2)}.
\end{aligned}
$$

Either way, we get the following Fourier series.

```
# Defines our function.
f = lambda x:e**(2*x)
L = pi
f = piecewise([[(-L,L),f]])
```

```
# Partial sums of the Fourier series
def FS(k):
  fourier = f.fourier_series_partial_sum(k,L)
  return fourier

n = 15
p = plot(FS(n),(x,-L,L),color='red') + plot(f,(x,-L,L))
p
```

Parseval's Identity

One of the most important identities in mathematics is **Parseval's identity**, which we state next.

Theorem 7.3.4 Parseval's Identity.

Let $f(x)$ denote a piecewise-differentiable (real-valued) function on $[-L, L]$ with real Fourier coefficients a_k, b_k and complex Fourier coefficients c_k. If $\int_{-L}^{L} f(x)^2\,dx$ exists and is finite, then

$$\frac{1}{2L}\int_{-L}^{L} f(x)^2\,dx = a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2) = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

One of the great strengths of this identity is that it allows potentially complicated sums to be computed using integrals instead.

Example 7.3.5 The Basel Problem.

In the early 18th century, one of the most renowned problems in mathematics was the Basel problem, which asked for the value of

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Euler was the first person to show that the sum is actually $\frac{\pi^2}{6}$, and it was this solution that made him famous[7] in the first place. We can solve this by using Parseval's identity. To do so, let $f(x) = x$ for $-1 < x <; 1$. Then with a little bit of work we can find the (real) Fourier coefficients:

$$\begin{aligned} a_0 &= 0 \\ a_k &= 0 \text{ for } k \geq 1 \\ b_k &= \frac{2(-1)^k}{\pi k} \end{aligned}$$

By Parseval's identity, it then follows that

$$\frac{1}{2}\int_{-1}^{1} x^2\,dx = \frac{1}{2}\sum_{k=1}^{\infty} \frac{4}{\pi^2 k^2},$$

which simplifies down to

$$\frac{2}{3} = \frac{4}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

In other words, $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

7.4 Approximation by Trigonometric Polynomials

If a function $f(x)$ has a Fourier series and is equal to its Fourier series, i.e.,

$$f(x) = a_0 + \sum_{k=1}^{\infty}[a_k \cos\frac{k\pi}{L}x + b_k \sin\frac{k\pi}{L}x],$$

then the partial sums of the Fourier series should be good approximations of $f(x)$:

$$f(x) \approx a_0 + \sum_{k=1}^{N}[a_k \cos\frac{k\pi}{L}x + b_k \sin\frac{k\pi}{L}x] = \sum_{-N}^{N} c_k e^{i\frac{k\pi}{L}x}.$$

Such a sum is a **trigonometric polynomial of degree** N.

We can also consider approximating f with other trigonometric polynomials of degree N, say

$$F(x) = A_0 + \sum_{k=1}^{N}[A_k \cos\frac{k\pi}{L}x + B_k \sin\frac{k\pi}{L}x].$$

We'd like to know how good the approximation is. To do this, we need to define a measure of error.

[7] Or at least math famous.

Definition 7.4.1 Square Error.

Given a function of period $p = 2L$ $f(x)$ and approximation $F(x)$, we define the **square error** to be

$$E = \int_{-L}^{L} (f(x) - F(x))^2 \, dx,$$

assuming these are real-valued functions.

It turns out that if we are approximating $f(x)$ by trigonometric polynomials $F(x)$, then the square error takes a specific form.

Theorem 7.4.2 Square Error Formula.

Let $f(x)$ be a function of period $p = 2L$ with Fourier coefficients a_k and b_k, and let

$$F(x) = A_0 + \sum_{k=1}^{N} [A_k \cos \frac{k\pi}{L} x + B_k \sin \frac{k\pi}{L} x]$$

be a degree N trigonometric polynomial. Then

$$E \geq \int_{-L}^{L} f^2 \, dx - L \left[2a_0^2 + \sum_{k=1}^{N} (a_k^2 + b_k^2) \right].$$

The error E takes this minimum value if $A_k = a_k, B_k = b_k$.

Example 7.4.3 Error from a Trigonometric Polynomial.

Define $f(x) = x^3$ for $-\pi \leq x < \pi$ as in Example 7.1.10, and recall that the Fourier series is given by

$$\sum_{k=1}^{\infty} (-1)^k \frac{2(6 - \pi^2 k^2)}{k^3} \sin kx.$$

Find the trigonometric polynomial of degree N that best approximates f and give the corresponding error.

Solution. The trigonometric polynomial of degree N that best approximates $f(x)$ is

$$\sum_{k=1}^{N} (-1)^k \frac{2(6 - \pi^2 k^2)}{k^3} \sin kx.$$

The corresponding square error is

$$\int_{-\pi}^{\pi} x^6 \, dx - \pi \sum_{k=1}^{N} \frac{4(6 - \pi^2 k^2)^2}{k^6} = \frac{2\pi^7}{7} - \pi \sum_{k=1}^{N} \frac{4(6 - \pi^2 k^2)^2}{k^6}.$$

Since the square error is a positive value, it follows that

$$2a_0^2 + \sum_{k=1}^{N} (a_k^2 + b_k^2) \leq \frac{1}{L} \int_{-L}^{L} f(x)^2 \, dx.$$

This is known as **Bessel's inequality**. Theorem 7.3.4 states that this inequality becomes equality if we let $N \to \infty$.

Example 7.4.4 Applying Bessel's Inequality.

Let

$$f(x) = \begin{cases} -1 & \text{if } -1 \leq x < 0 \\ 1 & \text{if } 0 < x \leq 1 \end{cases}.$$

Apply Bessel's inequality to this function. What does Parseval's Identity say?

Solution. If we find the Fourier coefficients of f, we get

$$b_k = \frac{2 - (-1)^k}{k\pi}.$$

By Bessel's inequality, we know that

$$\sum_{k=1}^{N} \left(\frac{2 - (-1)^k}{k\pi} \right) \leq 2$$

for any N. As $N \to \infty$, Parseval's gives the identity

$$2\pi^2 = \frac{3^2}{1^2} + \frac{1^2}{2^2} + \frac{3^2}{3^2} + \frac{1^2}{4^2} + \cdots.$$

7.5 The Fourier Transform

If $f(x)$ is a periodic function with period $p = 2L$, then we know how to find its Fourier series, both real and complex. But what do we do if our function $f(x)$ is not periodic? Can we still get a similar representation?

Let $f(x)$ be some piecewise-differentiable function, not necessarily periodic. Then we can't find it's Fourier series. However, we can truncate the graph of $f(x)$, and replace it with a periodic function that is equal to $f(x)$ on some interval $(-L, L)$. Then we can find the Fourier series of *this* function, which by Theorem 7.3.1 is given by $\sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi}{L}x}$ where

$$c_k = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\frac{k\pi}{L}x}\, dx.$$

So we can write

$$f(x) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\frac{k\pi}{L}x}\, dx \right) e^{i\frac{k\pi}{L}x}$$

wherever $f(x)$ is continuous on $-L < x < x$.

The idea now is that the larger that T gets, this expression can be used to represent $f(x)$ for more and more values of x. So we want to see what happens to this as $T \to \infty$. First, we'll clean this up a little bit by writing $w_k = \frac{k\pi}{L}$ and $\Delta w = w_{k+1} - w_k$, so that $\frac{1}{2L} = \frac{\Delta w}{2\pi}$. Then if x is in $(-L, L)$ and f is continuous at x, then we can say

$$f(x) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\frac{k\pi}{L}x}\, dx \right) e^{i\frac{k\pi}{L}x}$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-L}^{L} f(x) e^{-iw_k x}\, dx \right) e^{iw_k x} \Delta w.$$

As *awful* as this looks, we can relate this to a Riemann sum! As $L \to \infty$, $\Delta w \to 0$, we can replace w_k with the new variable ω and this expression becomes

$$f(x) = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx \right) e^{i\omega x}\, d\omega.$$

This leads to the definition of the **Fourier transform**. But first we need another definition.

Definition 7.5.1 Absolutely Integrable Functions.

Let $f(x)$ be a piecewise continuous function. Then $f(x)$ is absolutely integrable if $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$.

Definition 7.5.2 The Fourier Transform.

Let $f(x)$ be an absolutely integrable piecewise continuous function. The Fourier transform of $f(x)$ is the function $\hat{f}(\omega)$ defined by

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx.$$

We often write $\mathcal{F}(f)$ to denote the Fourier transform as well.

Example 7.5.3 Fourier transform of a piecewise exponential.

Let $f(x) = e^{-2x}$ for $x > -5$ and 0 otherwise. Then the Fourier transform of $f(x)$ is

$$\begin{aligned}
\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-5}^{\infty} e^{-2x} e^{-i\omega x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-5}^{\infty} e^{-(2+i\omega)x}\, dx \\
&= -\frac{1}{(2+i\omega)\sqrt{2\pi}} e^{-(2+i\omega)x} \Big]_{-5}^{\infty} \\
&= \frac{e^{(2+i\omega)5}}{(2+i\omega)\sqrt{2\pi}}.
\end{aligned}$$

As with the Laplace transform, the Fourier transform of a function is said to be in the **frequency domain**. In fact, the magnitude of $\hat{f}(\omega)$ represents the **frequency content** of the function $f(x)$ (thought of as a signal) at the frequency ω. It's also a quick jump to get the **inverse Fourier transform**.

Definition 7.5.4 The Inverse Fourier Transform.

The inverse Fourier transform of $\hat{f}(\omega)$ is

$$\mathcal{F}^{-1}(\hat{f}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega.$$

Theorem 7.5.5 Fourier Inversion Theorem.

Let $f(x)$ be an absolutely integrable, piecewise differentiable function. Then $f(x) = \mathcal{F}^{-1}(\hat{f})$ wherever $f(x)$ is continuous.

Example 7.5.6 Inverse Fourier transform of a step function.

Define $\hat{f}(\omega)$ by

$$\hat{f}(\omega) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then we can find the inverse transform using Definition 7.5.4:

$$\begin{aligned} \mathcal{F}^{-1}\{\hat{f}\} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega \\ &= \frac{1}{2\pi} \int_{-1}^{1} e^{i\omega x}\, d\omega \\ &= \frac{1}{2\pi} \frac{e^{i\omega x}}{ix}\bigg]_{-1}^{1} \\ &= \frac{1}{2\pi} \frac{e^{ix} - e^{-ix}}{ix} \\ &= \frac{1}{\pi} \frac{\sin x}{x}. \end{aligned}$$

The Fourier and inverse Fourier transforms are also linear like the Laplace transform: if a, b are constants and f, g are functions, then

$$\mathcal{F}\{af + bg\} = a\hat{f} + b\hat{g}$$

and

$$\mathcal{F}^{-1}(a\hat{f} + b\hat{g}) = af + bg.$$

The Fourier transform also works well with derivatives.

Theorem 7.5.7 Fourier Transforms and Derivatives.

Let $f(x)$ be differentiable with derivative $f'(x)$. Suppose that both $f(x)$ and $f'(x)$ are absolutely integrable. Then

$$\mathcal{F}(f') = i\omega\hat{f}(\omega).$$

Fourier transforms also behave well with another type of convolution.

Theorem 7.5.8 Convolution Theorem.

Suppose that $f(x), g(x)$ *are piecewise continuous, bounded and absolutely integrable. Define* $f * g$ *by*

$$f * g = \int_{-\infty}^{\infty} f(\tau) g(x - \tau) \, d\tau.$$

Then $\mathcal{F}(f * g) = \hat{f}\hat{g}$.

Chapter 8

Partial Differential Equations

Consider a rod that's recently been pulled out of the oven and is now cooling in a room. Let $T(t)$ denote the temperature of the rod at time t, where t represents the amount of time since the rod was pulled out of the oven. Then we can model $T(t)$ by using Newton's Law of Cooling, which states that

$$\frac{dT}{dt} = -k(T - A),$$

where k is a positive constant and A represents the ambient temperature of the room. This is an ordinary differential equation since it involves only one independent variable (t).

This works just fine as a simple model of the temperature of the rod, but in reality the situation is likely to be more complicated. For instance, the temperature of the rod will likely vary over the surface of the rod at any given time t. So it's reasonable to assume that the temperature of the rod should depend on *two* variables, say, x (position on the rod) and t (time). So the temperature of the rod would be more accurately described by a function $u(x, t)$. In fact, as we will see later, the temperature $u(x, t)$ can be described by the **heat equation**.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

This is an example of a **partial differential equation**.

8.1 Basic Concepts

Partial derivatives and PDEs

Given some quantity $u(x)$ that depends solely on the variable x, $\frac{du}{dx} = u'(x)$ represents the rate of change of u with respect to x. More generally, given some quantity $u(x, t)$ that depends on x, t, we can attempt to find the rate of change of u with respect to each of the variables x, t. This idea leads to **partial derivatives**.

Definition 8.1.1 Partial derivatives.

Let $u(x, t)$ denote a function depending on the variables x, t. Then the partial derivative of u with respect to x is found by differentiating $u(x, t)$ while treating t as a constant. The partial derivative of u with respect to x is denoted by

$$\frac{\partial}{\partial x}u \text{ or } \frac{\partial u}{\partial x} \text{ or } u_x.$$

The partial derivative of u with respect to t is found similarly, and is likewise denoted by

$$\frac{\partial}{\partial t}u \text{ or } \frac{\partial u}{\partial t} \text{ or } u_t.$$

From here we can define higher order partial derivatives, such as

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x}$$

or

$$u_{txx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}u\right)\right) = \frac{\partial^3 u}{\partial x^2 \partial t}.$$

The **order** of each of these partial derivatives is 2 and 3, respectively.

Definition 8.1.2 Partial differential equation.

A partial differential equation (PDE) is an equation involving one or more (partial) derivatives of an unknown function u that depends on two or more independent variables, usually thought of as time and position. The highest derivative appearing in a PDE is called the order of the PDE.

Just as ODEs in practice typically appear as initial value problems, PDEs can appear as **boundary value problems**. Boundary value problems involve conditions of the form

$$u(0,t) = u_0, \quad u(L,t) = u_1 \quad \text{and} \quad u(x,0) = f(x).$$

These are examples of **boundary conditions**. In other words, boundary conditions can represent initial data at infinitely many points, as opposed to finitely many points like we had for our IVPs.

Linear homogeneous PDEs and the superposition principle

We will mostly be concerned with **linear PDEs**, which are PDEs where the only thing we're allowed to do to the function u and its derivatives is multiply it by a constant. A linear PDE is **homogeneous** if every term contains the function u or one of its derivatives. A **solution** of a PDE is a function u that satisfies the PDE.

Example 8.1.3 Solution of the heat equation.

Let $u(x,t) = e^{-9t}\sin 3x$. Show that this is a solution of the boundary value problem

$$u_t = u_{xx}, \quad u(0,t) = u(\pi,t) = 0 \quad \text{and} \quad u(x,0) = \sin 3x.$$

Solution. To do so, we need to compute the partial derivatives of $u(x,t)$:

$$u_t(x,t) = \frac{\partial u}{\partial t} = -9e^{-9t}\sin 3x$$

$$u_x(x,t) = \frac{\partial u}{\partial x} = 3e^{-9t}\cos 3x$$
$$u_{xx}(x,t) = \frac{\partial}{\partial x}u_x = 9e^{-9t}\sin 3x.$$

So we see that $u_t = u_{xx}$, which means that $u(x,t) = e^{-9t}\sin 3x$ is a solution of $u_t = u_{xx}$. Now it remains to show that $u(x,t)$ satisfies the boundary conditions, which we can do without too much trouble.

Just as with linear homogeneous ODEs, PDEs that are linear and homogeneous satisfy the **superposition principle**.

Principle 8.1.4 Superposition principle.

Let c_1 and c_2 denote arbitrary constants, and suppose that u_1 and u_2 are both solutions of the same linear homogeneous PDE. Then

$$u = c_1u_1 + c_2u_2$$

is also a solution of the same PDE.

The superposition principle is incredibly useful since it allows us to find general solutions of PDEs, which makes solving linear homogeneous PDEs somewhat tractable. If a PDE fails to be linear or homogeneous, the superposition principle is not guaranteed to hold.

Example 8.1.5 Failure of the superposition principle.

Consider the PDE given by

$$u_t + uu_x = 0.$$

This PDE fails to be linear because the second term involves multiplying u with its derivative u_x. However, it's not too hard to check that $u(x,t) = \frac{x}{t+1}$ is a solution of the PDE, since if we plug this function into the PDE we get

$$\begin{aligned} u_t + uu_x &= -\frac{x}{(t+1)^2} + \frac{x}{t+1}\frac{1}{t+1} \\ &= 0. \end{aligned}$$

However, the closely related function $v(x,t) = 3u(x,t) = \frac{3x}{t+1}$ is *not* a solution of the same PDE, since

$$v_t + vv_x = \frac{6x}{(t+1)^2} \neq 0$$

So the superposition principle does not hold for this PDE.

Important PDEs

As mentioned in the introduction, PDEs are useful for modeling quantities that depend on multiple independent variables. We finish this section by listing several of the simplest and most studied PDEs.

1. $u_t = c^2 u_{xx}$ where $c > 0$. This is called the **heat** or **diffusion equation**. This equation is used for modeling the spread of a quantity, such as how temperature diffuses along a rod.

2. $u_{tt} = c^2 u_{xx}$ where $c > 0$. This is called the **wave equation**, and is used for modeling vibrating motion, such as that along a plucked string.

In both PDEs above, the expression u_{xx} is an example of the **Laplacian** of u. The Laplacian of a function u at a point (x, t) is a measure of how $u(x, t)$ differs from the average value of u at nearby x. In particular, the Laplacian is positive at (x, t) if $u(x, t)$ tends to be less than nearby averages; the Laplacian is negative at (x, t) if $u(x, t)$ tends to be greater than nearby averages; and the Laplacian at (x, t) is 0 if $u(x, t)$ is in equilibrium with its nearby averages.

With this viewpoint, we can assign physical reasoning to the heat and wave equations:

1. The heat equation states that the time rate of change of the temperature is proportional to the difference between the temperature at (x, t) and the average values of nearby temperatures. If the nearby average temperature is greater (i.e., the Laplacian is positive), then the temperature will increase.

2. The wave equation states that the acceleration of the wave height is proportional to the difference between the height of the wave at (x, t) and the average height at nearby points. If the nearby average height is greater (i.e., the Laplacian is positive), then the wave height will accelerate upwards.

Our goal in the next section will be to determine how to solve PDEs such as these.

8.2 The Wave Equation and Separation of Variables

The main difficulty in solving PDEs (even linear ones) as compared with ODEs is that any solution of a PDE typically depends on more than one variable. Adding this extra degree of freedom into the problem greatly complicates matters. However, we can make this problem more reasonable by assuming that our solution $u(x, t)$ depends on each variable *separately*. That is, we'll assume that the function we want to find, $u(x, t)$, satisfies the constraint $u(x, t) = X(x)T(t)$. This technique is known as **separation of variables**.

Consider a one-dimensional string of length L that vibrates in the vertical direction. The vertical displacement of such a string depends on the horizontal position along the string, x, and the time t. So let $u(x, t)$ denote the vertical displacement of the string at position x and at time t. If we assume that the string has constant density and that the force of gravity of the string is negligible, then $u(x, t)$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{8.1}$$

for some constant c.

Suppose that the string is also subject to the **boundary conditions**

$$u(0, t) = 0 \text{ and } u(L, t) = 0. \tag{8.2}$$

In other words, the string is held fixed at both ends. We'll also suppose that we know the initial position of the string and the initial velocity of the string, represented by the **initial conditions**

$$u(x, 0) = f(x) \text{ and } u_t(x, 0) = g(x). \tag{8.3}$$

Our goal will be to find $u(x, t)$ subject to these conditions. To start, assume that $u(x, t) = X(x)T(t)$. If we plug this into (8.1), then we get

$$X(x)T''(t) = c^2 X''(x)T(t).$$

If we assume that $X(x), T(t)$ are both nonzero, then we can rewrite this to get

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}.$$

This may not look that helpful, but it actually places some serious restrictions on X and T. The left hand side of this equation only depends on t whereas the right hand side depends only on x. So the only way for this equation to be true for *all* x, t is if both sides are constant:

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = k$$

for some k. This now gives us two separate *ordinary* differential equations for $T(t)$ and $X(x)$:

$$X''(x) - kX(x) = 0 \tag{8.4}$$
$$T''(t) - kc^2 T(t) = 0. \tag{8.5}$$

We can add a few more restrictions to these ODEs to help us solve them. Note that the boundary conditions (8.2) force either $X(0) = X(L) = 0$ or $T(t) = 0$ for all t, which leads to $u(x, t) = 0$. So to avoid this trivial solution, we'll set $X(0) = X(L) = 0$.

We'll solve (8.4) first since we have extra information to use. So to start, suppose that $k > 0$ and write $k = \mu^2$ for some nonzero μ. Then (8.4) becomes $X'' - \mu^2 X = 0$ and has solution given by

$$X(x) = c_1 \sinh \mu x + c_2 \cosh \mu x.$$

Now, $X(0) = 0$ forces $c_2 = 0$, so we get $X(x) = c_1 \sinh \mu x$. However, since $X(L) = 0$ as well, we get $c_1 \sinh \mu L = 0$. But the only way to solve this is to set $c_1 = 0$ since $\sinh u = 0$ only if $u = 0$. So in other words, if we assume that $k = \mu^2 > 0$, then the only way to solve (8.4) is to set $X(x) = 0$, which also forces $u(x, t) = 0$. Obviously, this isn't very useful. Similarly, if we assume that $k = 0$ then we get the same problem. So let's assume that $k = -\mu^2 < 0$ for some nonzero μ. Then (8.4) becomes $X'' + \mu^2 X = 0$, which has solution

$$X(x) = c_1 \sin \mu x + c_2 \cos \mu x.$$

The condition $X(0) = 0$ forces $c_2 = 0$, and the second boundary condition $X(L) = 0$ forces $c_1 \sin \mu L = 0$. We want to avoid setting c_1 equal to 0 since this would give us $X = 0$ again, so we'll set $\sin \mu L = 0$ instead. *This* tells us that $\mu L = n\pi$ for some integer n, or just $\mu = \frac{n\pi}{L}$. So nontrivial solutions of (8.4) that satisfy the boundary conditions $X(0) = X(L) = 0$ can occur only if $k = -\mu^2$ where $\mu = \frac{n\pi}{L}$ and $n = \pm 1, \pm 2, \ldots$. For each choice of n (ignoring sign), we get the solution $X_n = \sin \frac{n\pi}{L} x$.

Now we move on to solving (8.5), but we still need to keep the condition $k = -(\frac{n\pi}{L})^2$ for $n = \pm 1, \pm 2, \ldots$. If we do so, then (8.5) becomes $T'' + (c\frac{n\pi}{L})^2 T = 0$, which has solutions given by

$$T_n = b_n \cos \lambda_n t + b_n^* \sin \lambda_n t,$$

where $\lambda_n = c\frac{n\pi}{L}$.

So this means that every function of the form

$$u_n(x,t) = X_n T_n = \sin \frac{n\pi}{L} x \left(b_n \cos \lambda_n t + b_n^* \sin \lambda_n t\right)$$

is a solution of (8.1) subject to the boundary conditions (8.2). It also follows from the superposition principle that any (finite) linear combination of these functions will give another solution that satisfies the boundary conditions.

However, this does *not* guarantee that we can solve for the initial conditions in (8.3). To give ourselves as general a solution as possible, we will guess that the solution to the wave equation is actually a linear combination of all possible u_n. That is, we'll say that

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left(b_n \cos \lambda_n t + b_n^* \sin \lambda_n t\right).$$

Now we'll use the initial conditions to actually determine b_n, b_n^*. To start, note that we must have

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x.$$

This is a Fourier series, and in particular it's the Fourier series of the odd extension of $f(x)$ with period $2L$. [8] So it follows that

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx.$$

Similarly, we must have

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} b_n^* \lambda_n \sin \frac{n\pi}{L} x.$$

This is the Fourier series for the odd extension of $g(x)$ with period 2π. Therefore

$$b_n^* \lambda_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx,$$

or just

$$b_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx.$$

We can put all of this together into the following theorem.

Theorem 8.2.1 Solution of the Wave Equation.

The solution of the wave equation (8.1) with boundary conditions (8.2)

[8] See Theorem 7.2.6.

and initial conditions (8.3) is given by

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x (b_n \cos \lambda_n t + b_b^* \sin \lambda_n t),$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L}\, dx \text{ and } b_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x\, dx$$

and $\lambda_n = c\frac{n\pi}{L}$ *for* $n = 1, 2, \ldots$.

Example 8.2.2 A string with fixed ends.

A string at rest has unit length, and is fixed at both ends. Suppose that the string is now stretched into the triangular shape given by the graph of

$$f(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \le x \le 1 \end{cases}.$$

The string is then released at time $t = 0$. Given $c = 4$, find the function $u(x,t)$ that models the vertical displacement of the string at position x at time t.

Solution. We can model $u(x,t)$ as the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 16 \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions $u(0,t) = u(1,t) = 0$ and initial conditions

$$u(x,0) = f(x) \text{ and } u_t(x,0) = 0.$$

We can find $u(x,t)$ from Theorem 8.2.1.
Using the Sage cell below, we get

$$b_n = \frac{8 \sin\left(\frac{1}{2}\pi n\right)}{\pi^2 n^2},$$

and since $g(x) = 0$ this forces $b_n^* = 0$ as well. Hence the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8 \sin\left(\frac{1}{2}\pi n\right)}{\pi^2 n^2} \sin(n\pi x) \cos(4n\pi t).$$

```
# Defines the odd extension of the function f(x).
f1(x) = 2*x
f2(x) = 2*(1-x)
f =
    piecewise([[(-1,-1/2),-f2(-x)],[(-1/2,0),-f1(-x)],[(0,1/2),f1],[(1/2,1),f2]])

# Gets Fourier sine coefficients for odd extension of f(x).
L = 1
var('n')
assume(n,'integer')
```

```
def b(n):
  return
      f.fourier_series_sine_coefficient(n,L).full_simplify()

pretty_print(b(n))
```

8.3 d'Alembert's Solution of the Wave Equation

Although Theorem 8.2.1 solves the wave equation, it's a little complicated to work with. We'll try to express the solution in a simpler way. In particular, we'll start by trying to simplify the solution of the boundary value problem in Example 8.2.2. In that example, we saw that the solution of

$$u_{tt} = 16u_{xx}$$

with boundary conditions $u(0,t) = u(1,t) = 0$ and initial conditions

$$u(x,0) = f(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \le x \le 1 \end{cases}, \quad u_t(x,0) = 0$$

was given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\cos(4n\pi t),$$

where b_n was the n^{th} Fourier coefficient of the odd extension of $f(x)$.

If we look at this, it looks kind of like a Fourier series except that we have a product of sine and cosine. We can make this look more like a Fourier series by using one of the product-to-sum formulas from trigonometry:

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)].$$

Using this formula, we get

$$\sin(n\pi x)\cos(4n\pi t) = \frac{1}{2}[\sin[n\pi(x+4t)] + \sin[n\pi(x-4t)]],$$

which means we can write the solution $u(x,t)$ as

$$u(x,t) = \frac{1}{2}\left[\sum_{n=1}^{\infty} b_n \sin[n\pi(x+4t)] + \sum_{n=1}^{\infty} b_n \sin[n\pi(x-4t)]\right].$$

Here's how this helps us. Since b_n is the n^{th} Fourier coefficient of the odd extension of $f(x)$, each of these sums must be a Fourier series for the odd extension of $f(x)$! In particular, if we denote the odd extension by $f^*(x)$ then we can simply say that

$$u(x,t) = \frac{1}{2}[f^*(x+4t) + f^*(x-4t)].$$

We can extend this to other boundary value problems without an initial velocity component.

Theorem 8.3.1 d'Alembert's Solution without Initial Velocity.

Let $c > 0$, and consider the boundary value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(0,t) &= 0 \\ u(L,t) &= 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= 0. \end{aligned}$$

Assuming that $f(x)$ is piecewise twice differentiable, then the solution of this boundary value problem is given by

$$u(x,t) = \frac{1}{2}[f^*(x-ct) + f^*(x+ct)]$$

where f^* denotes the odd extension of $f(x)$.

Example 8.3.2 Boundary value problem with sinusoidal deflection.

A string of length $L = 1$ has initial deflection, or position, given by $f(x) = \sin \pi x - \frac{1}{2}\sin 2\pi x$ for $0 \le x \le 1$. The string is released at time $t = 0$. Suppose $c = 1$. Find $u(x,t)$.

Solution. We can do so very easily with Theorem 8.3.1, since the initial velocity of the string is 0. So we have

$$u(x,t) = \frac{1}{2}[f^*(x-t) + f^*(x+t)],$$

where f^* is the odd extension of $f(x)$. However, since $f(x)$ is itself an odd function, it follows that the odd extension is simply $\sin \pi x - \frac{1}{2}\sin 2\pi x$. Therefore the solution is

$$u(x,t) = \frac{1}{2}\left[\sin(\pi(x-t)) - \frac{1}{2}\sin(2\pi(x-t)) + \sin(\pi(x+t)) - \frac{1}{2}\sin(2\pi(x+t))\right].$$

d'Alembert's solution is very useful if we want to model a vibrating string with zero initial velocity. But what can we do if the string has an initial velocity? d'Alembert's solution actually suggests a possible approach to take. If we look at Theorem 8.3.1, it essentially states that the solution of the wave equation (assuming zero initial velocity) is the superposition of the rightward traveling wave $f(x-ct)$ with the leftward traveling wave $f(x+ct)$. This suggests that superpositions of waves are fundamental to solutions of the wave equation.

We'll try something similar for the case $u_t(x,0) = g(x)$. We'll assume that adding in this initial velocity also adds in a new rightward traveling wave $G(x-ct)$ and a new leftward traveling wave $H(x+ct)$ into our solution $u(x,t)$, so that we have

$$u(x,t) = \frac{1}{2}[f^*(x-ct) + f^*(x+ct)] + G(x-ct) + H(x+ct).$$

Now we'll try to use the initial conditions to find G and H. Now, since we need $u(x,0) = f(x)$ this forces

$$G(x) + H(x) = 0 \Rightarrow H(x) = -G(x).$$

Therefore our solution becomes

$$u(x,t) = \frac{1}{2}[f^*(x-ct) + f^*(x+ct)] + G(x-ct) - G(x+ct).$$

If we use the second initial condition $u_t(x,0) = g(x)$, then we get

$$g(x) = -cG'(x) - cG'(x) = -2cG'(x) \Rightarrow G'(x) = -\frac{1}{2c}g(x).$$

Now we can integrate both sides to find $G(x)$! So there exists some x_0 such that

$$G(x) = -\frac{1}{2c}\int_{x_0}^{x} g(s)\,ds.$$

Therefore

$$\begin{aligned}
u(x,t) &= \frac{1}{2}[f^*(x-ct) + f^*(x+ct)] + G(x-ct) + H(x+ct)\\
&= \frac{1}{2}[f^*(x-ct) + f^*(x+ct)] + G(x-ct) - G(x+ct)\\
&= \frac{1}{2}[f^*(x-ct) + f^*(x+ct)] - \frac{1}{2c}\int_{x_0}^{x-ct} g(s)\,ds + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,ds\\
&= \frac{1}{2}[f^*(x-ct) + f^*(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.
\end{aligned}$$

This gives the following adjustment to d'Alembert's solution.

Theorem 8.3.3 d'Alembert's Solution with Initial Velocity.

Let $c > 0$, and consider the boundary value problem

$$\begin{aligned}
u_{tt} &= c^2 u_{xx}\\
u(0,t) &= 0\\
u(L,t) &= 0\\
u(x,0) &= f(x)\\
u_t(x,0) &= g(x).
\end{aligned}$$

Assuming that $f(x)$ is piecewise twice differentiable and that $g(x)$ is piecewise continuous, then the solution of this boundary value problem is given by

$$u(x,t) = \frac{1}{2}[f^*(x-ct) + f^*(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds.$$

where f^* denotes the odd extension of $f(x)$ and g^* the odd extension of $g(x)$.

Example 8.3.4 Boundary value problem with sinusoidal deflection and initial velocity.

Consider a string of length $L = 1$, initial deflection $f(x) = \sin \pi x$ and initial velocity $\frac{1}{4}\sin 2\pi x$. Assume that $c = 2$. Find $u(x,t)$, the vertical displacement at (x,t).

Solution. The vertical displacement $u(x,t)$ of the string is given by

$$\begin{aligned} u(x,t) &= \frac{1}{2}[f^*(x-ct)+f^*(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds \\ &= \frac{1}{2}[\sin(\pi(x-2t))+\sin(\pi(x+2t))] + \frac{1}{4}\int_{x-2t}^{x+2t} \frac{1}{4}\sin 2\pi s\,ds \\ &= \frac{1}{2}[\sin(\pi(x-2t))+\sin(\pi(x+2t))] - \frac{1}{32\pi}\cos 2\pi s\Big]_{x-2t}^{x+2t} \\ &= \frac{1}{2}[\sin(\pi(x-2t))+\sin(\pi(x+2t))] - \frac{\cos(2\pi(x+2t))-\cos(2\pi(x+2t))}{32\pi}. \end{aligned}$$

8.4 The Heat Equation

The last equation we will look at is the **heat equation**, which models the temperature distribution of a thin bar of uniform density and constant cross-section placed along the x-axis. We also assume that the bar is perfectly insulated on its surface, so that heat flows along the bar in the x-direction only. With these assumptions, the temperature $u(x,t)$ of the bar at position x and time t satisfies the PDE

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}. \tag{8.6}$$

This is called the **one-dimensional heat equation**.

Bar with ends fixed at 0

We will start by solving the heat equation for the case where the bar has ends which are fixed at temperature 0. If we're given an initial temperature distribution $f(x)$, then $u(x,t)$ is the solution of the boundary value problem

$$u_t = c^2 u_{xx} \tag{8.7}$$
$$u(0,t) = u(L,t) = 0 \tag{8.8}$$
$$u(x,0) = f(x). \tag{8.9}$$

We can solve this boundary value problem using separation of variables, much as we did in Section 8.2. So to start, we assume that $u(x,t) = X(x)T(t)$. If we plug this into the heat equation (8.7), then we get

$$\frac{T'}{c^2T} = \frac{X''}{X} = k. \tag{8.10}$$

Now we have three separate cases to consider for k: $k>0, k=0$ or $k<0$. Just as with the wave equation, the only case that doesn't lead to trivial solutions is $k=-\mu^2<0$. In this case (8.10) leads to the two ODEs given by

$$\begin{aligned} X'' + \mu^2 X &= 0 \\ T' + c^2\mu^2 T &= 0. \end{aligned}$$

The boundary conditions in (8.8) force $X(0) = X(L) = 0$, and the only non-trivial solutions of $X''+\mu^2X=0$ occur when $\mu = \frac{n\pi}{L}$. So we get the solutions $X = X_n = \sin\frac{n\pi}{L}x$, just as with the wave equation.

For the second ODE, we readily solve it to obtain $T = T_n = b_n e^{-\lambda_n^2 t}$ where $\lambda_n = c\frac{n\pi}{L}$ as before. So every function

$$u_n(x,t) = X_n T_n = b_n \sin \frac{n\pi}{L} x e^{-\lambda_n^2 t}$$

is a solution of (8.7) that satisfies the boundary equations (8.8). In order to satisfy the arbitrary initial condition $f(x)$, we take an infinite sum of the functions $u_n(x,t)$ to get

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x e^{-\lambda_n^2 t}.$$

Finally, if we plug in $t = 0$ and use the initial condition $u(x,0) = f(x)$, we get $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$. This is just the Fourier series of the odd extension of $f(x)$, which lets us find b_n. We summarize all of this in the following theorem.

Theorem 8.4.1 Solution of the Heat Equation with Fixed Temperature.

The solution of the heat equation (8.7) satisfying the boundary conditions (8.8) and initial condition (8.9) is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x e^{-\lambda_n^2 t} \tag{8.11}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx \tag{8.12}$$

$$\lambda_n = c\frac{n\pi}{L}. \tag{8.13}$$

Example 8.4.2 Sinusoidal initial temperature.

Consider a thin metal bar of length π placed on the x-axis, with one end at $x = 0$ and the other at $x = \pi$. Assuming that $c = 1$ and that the initial temperature is $f(x) = 30 \sin x$ for $0 \le x \le \pi$, find the temperature distribution using Theorem 8.4.1.

Solution. The temperature is the function $u(x,t)$ given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 t},$$

where b_n is the n^{th} coefficient of the Fourier series of the odd extension of $f(x)$. The odd extension of $f(x)$ is $30 \sin x$. Furthermore, the Fourier series of $30 \sin x$ is clearly just $30 \sin x$.
So in other words,

$$b_n = \begin{cases} 30 & n = 1 \\ 0 & \textit{otherwise}. \end{cases}$$

Hence

$$u(x,t) = 30 e^{-t} \sin x.$$

Appendices

Appendix A

Notation

The following table defines some of the notation used in this book. Page numbers or references refer to the first appearance of each symbol.

Symbol	Description	Page
$\mathbf{x}(t)$	a vector whose elements are functions	59
$W(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})$	the Wronskian of n vectors	61
$J_{\mathbf{f}}(\mathbf{y})$	Jacobian of $\mathbf{y}' = \mathbf{f}(\mathbf{y})$	77
$J_n(x)$	Bessel function of the first kind of order n	104
$\Gamma(x)$	Gamma function	105
$Y_n(x)$	Bessel function of the second kind of order n	105
$F(s) = \mathcal{L}\{f(t)\}$	Laplace transform of $f(t)$	108
$f(t) = \mathcal{L}^{-1}\{F(s)\}$	inverse Laplace transform of $F(s)$	111
$f * g$	convolution of $f(t)$ and $g(t)$	126

Appendix B

Table of Laplace Transforms

Table B.0.1 Table of Common Laplace Transforms

1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n, $n = 0, 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$
$\sin(kt)$	$\frac{k}{s^2 + k^2}$
$\cos(kt)$	$\frac{s}{s^2 + k^2}$
$\sinh(kt)$	$\frac{k}{s^2 - k^2}$
$\cosh(kt)$	$\frac{s}{s^2 - k^2}$
e^{at}	$\frac{1}{s - a}$
$e^{at} f(t)$	$F(s - a)$
$u(t - a)$	$\frac{e^{-as}}{s}$
$u(t - a) f(t - a)$	$e^{-as} F(s)$
$f^{(}n)(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$
$f(t) * g(t)$	$F(s)G(s)$
$\delta(t - a)$	e^{-as}

Appendix C

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