Research statement

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My research concerns the analysis of integrable nonlinear partial differential equations (PDEs). A primary goal of this work is to uncover the dynamical behavior of such solutions, especially in certain asymptotic limits. Some recurring topics in my research include the direct and inverse spectral theory of Schrödinger and Dirac operators, singular perturbation theory, potential theory, and Riemann-Hilbert problems.

Current research. My current research agenda is related to the analysis of integrable nonlinear PDEs such as the Korteweg-de Vries (KdV), nonlinear Schrödinger (NLS), and Camassa-Holm (CH) equations within the physically relevant contexts of dense soliton gases and modulational instability. In the field of integrable dispersive hydrodynamics a "soliton gas" is an infinite statistical ensemble of interacting solitons and was introduced by Zakharov in [38] who derived the kinetic equation for a rarefied gas of solitons using the KdV equation and inverse scattering-based reasoning. The generalization of Zakharov's kinetic equation to the case of a gas of finite density was obtained by El in [13]. In a dense gas the solitons exhibit significant overlap and, as a result, are continuously involved in a strongly nonlinear interaction with each other. As a result for a dense gas the aggregate soliton (or large scale) dynamics are of primary interest. Two current directions of this work are the following:

- Further development of the mathematical theory of a dense soliton gas in dispersive hydrodynamics. In particular, to demonstrate that semiclassical limits of the direct scattering map for the NLS equation with a periodic input correspond to realizations of soliton gases.
- To study the modulational stability, or instability of periodic wave trains modeled by nonlinear dispersive equations using rigorous functional-analytic techniques.

An important aspect of this research is the study of Fredholm integral equations of the third kind which appear as nonlinear dispersion relations for soliton gases. Using methods of potential theory one seeks to prove existence and uniqueness of solutions as well as to study the properties of such solutions. Indeed, part of this work extends some recent results of Kuijlaars and Tovbis [25]. A long-term goal of this research is the rigorous analysis of the large-scale dynamics corresponding to soliton gases in dispersive hydrodynamics (e.g., see [19]). Importantly, the theory of soliton gas and its generalization breather gas provides a possible framework for uncovering connections between classical integrable systems and the modern integrable probabilistic systems. Recently research in this direction has been very active.

Future work. To grow as a mathematician I am interested in exploring new areas of research as well as in broadening my knowledge of analysis of PDEs, direct and inverse spectral theory, and approximation theory. Topics of interest include:

- Nonlinear steepest descent for oscillatory Riemann-Hilbert problems with applications to integrable nonlinear PDEs and orthogonal polynomials.

- Using soliton gas as a theoretical framework can we establish connections between dynamical integrable systems and probability theory?

1 Background

Nonlinear waves. Nonlinear wave phenomena comprise an important class of problems in mathematical physics. Remarkably, many of the governing equations, which arise as universal models in a variety of physical settings, are completely integrable infinite-dimensional Hamiltonian systems. Accordingly, such equations have a rich mathematical structure that can be uncovered by exploiting their integrability. Thus, the study of such equations offers a unique combination of interesting mathematics and concrete applications.

Solitons. Localized solitary waves are ubiquitous in nonlinear dispersive wave phenomena. If the equation governing the wave propagation is integrable the solitary waves exhibit particle-like properties such as pairwise elastic interactions, i.e., they remain unchanged, except for a phase shift. Such objects are called *solitons*. Due to their fascinating properties solitons have been studied extensively (see [31] and references therein).

Integrable systems. The main tool for the analysis of integrable nonlinear PDEs in (1+1) dimensions is the so-called inverse scattering transform (IST) [2]. The method is based on a reformulation of the nonlinear equation as the compatibility of two linear problems, the Lax pair [26]. Importantly, the spatial half of the Lax pair is a linear spectral problem whose discrete eigenvalues, which parameterize the solitons, are time-independent under the flow of the governing nonlinear equation. We say a nonlinear evolution equation that has a Lax pair is "integrable".

The IST is a major achievement in mathematical physics over the last 50 years. It was first developed for rapidly decaying [1, 17, 26, 39] and (quasi) periodic [7, 10, 21, 27, 28, 32] initial data. Later the method was generalized to include initial boundary-value problems, i.e., the unified transform method of Fokas [16]. Remarkably, one can study rigorously the behavior of solutions to these integrable equations in certain asymptotic limits by formulating the inverse scattering map as an oscillatory Riemann-Hilbert problem and then applying the Deift–Zhou nonlinear steepest descent method [9].

2 Past research

Preliminaries. A particularly well-known example of an integrable nonlinear PDE appearing in my work is the (semiclassical) focusing NLS equation

$$i\epsilon q_t + \epsilon^2 q_{xx} + 2|q|^2 q = 0\,, \tag{2.1}$$

where $q : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, is the slowly-varying complex envelope of a quasi-monochromatic, weakly dispersive nonlinear wave packet, subscripts x and t denote partial derivatives, and $\epsilon > 0$ quantifies the relative strength of dispersion compared to nonlinearity. Solutions to (2.1) in the "semiclassical limit", i.e., as $\epsilon \to 0^+$ have a remarkable structure which is of interest to applied analysts [23, 36]. (In the quantum-mechanical setting, ϵ is also proportional to Planck's constant $\hbar$.) Such phenomena occur when dispersive effects are weak compared to nonlinear ones. These situations can produce a wide variety of physical effects such as supercontinuum generation, dispersive shocks, and wave turbulence, to name a few [11, 14, 37, 40]. Within the IST framework the semiclassical limit of (2.1) corresponds to the singularly perturbed non-self-adjoint Dirac operator:

$$L\phi := i\epsilon \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \phi_x - \begin{pmatrix} 0 & iq \\ i\bar{q} & 0 \end{pmatrix} \phi = z\phi\,, \quad \phi(x; z, \epsilon) = (\phi_1, \phi_2)^T\,, \tag{2.2}$$

where superscript T denotes matrix transpose, overbar denotes complex conjugation, $z \in \mathbb{C}$ is the eigenvalue parameter, and $q \equiv q(x, 0)$ is a given "potential". The system (2.2) defines the scattering and inverse scattering transforms for (2.1) (Lax's "L" operator for focusing NLS) [39].

Note for (quasi) periodic data the inverse spectral theory of (2.2) concerns the Floquet spectrum $\Sigma(q) := \{z \in \mathbb{C} : -2 \le \Delta(z, \epsilon; q) \le 2\}$ where $\Delta(z; \epsilon)$ is the "Floquet discriminant" (trace of monodromy matrix) [12] which is isospectral under the flow of the governing equation. For generic periodic q the Floquet spectrum is comprised of an at most countable collection of analytic arcs, i.e., "bands" in the complex z-plane [18, 34]. The case of finitely many spectral bands is especially important as it corresponds to the so-called "finite-gap" integration theory [10, 21, 32].

Importantly, the spectral data provides information regarding the dynamical behavior of solutions to integrable equations. For rapidly decaying initial data as $x \to \pm\infty$ soliton velocity and amplitude corresponds to the real and imaginary part of the corresponding discrete eigenvalue, respectively [2]. Thus the study of (2.2) has become an active area of research and a natural starting point for the study of solutions to the focusing NLS equation [6, 24, 30, 33, 35].

Semiclassical dynamics in self-focusing nonlinear media. In [3] the singularly perturbed focusing Zakharov-Shabat (ZS) spectral problem (2.2) with a τ-periodic potential q was studied analytically and numerically in the limit $\epsilon \to 0^+$. For simplicity attention was restricted to a class of potentials coined *periodic single-lobe*, i.e., the τ-periodic extension of a continuously differentiable function satisfying: (a) positive; (b) even; (c) increasing on $(-\frac{\tau}{2}, 0)$ and decreasing on $(0, \frac{\tau}{2})$. Some key findings:

- Using a WKB expansion for $\Delta(z; \epsilon)$ as $\epsilon \to 0^+$ asymptotic expressions for the number and location of the spectral bands were obtained as well as corresponding leading-order expressions for the relative band and gap widths. It was shown the interval $\pm(iq_{\min}, iq_{\max})$ deforms into a sequence of bands and gaps with exponentially shrinking band widths as $\epsilon \to 0^+$ (see Fig. 1).

- The estimates show the resulting dynamics are characterized by the generation of a large number of nonlinear excitations referred to as "effective solitons" (the band/gap ratio goes to zero and the band width collapses to a point in the limit $\epsilon \to 0^+$) having zero velocity, i.e., a bound-state soliton ensemble.

- A law is obtained describing how the number of effective solitons scales with the semiclassical parameter.

This work finds application in the spectral theory of soliton and breather gases of the focusing NLS equation [15]. In particular, it lays the groundwork to establish rigorously that semiclassical limits of periodic potentials are realizations of soliton and breather grases, i.e., "periodic gases". Indeed, this is an interesting future direction of this work.

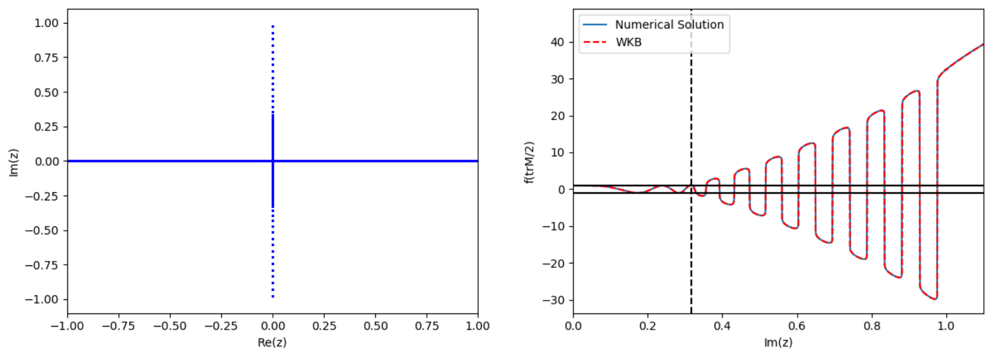

Figure 1: Potential $q \equiv \mathrm{dn}(x; m)$ with $m = 0.9$ and $\epsilon = 1/20$. (Left) Simulation of the Floquet spectrum using Hill's method [8]. (Right) WKB approximation of the Floquet spectrum along the imaginary z-axis.

Non-self-adjoint Dirac operator. In [4] the spectral theory of the non-self-adjoint Dirac operator (2.2) with a periodic potential is studied and several new results obtained. In particular, bounds on the Floquet spectrum are obtained by using classical methods such as Floquet theory and energy techniques. Moreover, these bounds allow one to prove under fairly weak hypotheses that the Floquet spectrum clusters on the real and imaginary axes of the spectral variable in the semiclassical limit. Finally, basic geometric properties of the Floquet spectrum are studied using symmetries of the monodromy matrix. Some key findings:

Lemma 2.1. *Let $q \in L^\infty(\mathbb{R})$ be τ-periodic. If $z \in \Sigma(q)$, then $|\operatorname{Im} z| \leq \|q\|_{L^\infty(\mathbb{R})}$. Moreover, let $q_x \in L^\infty(\mathbb{R})$. If $z \in \Sigma(q)$, then $|\operatorname{Re} z||\operatorname{Im} z| \leq \frac{\epsilon}{2}\|q_x\|_{L^\infty(\mathbb{R})}$.*

Lemma 2.2. *Let q be real, positive, and τ-periodic. Suppose $q_x \in L^\infty(\mathbb{R})$. If $z \in \Sigma(q)$ with $\operatorname{Re} z > 0$, then $|\operatorname{Im} z| \leq \frac{\epsilon}{2}\|(\ln q)_x\|_{L^\infty(\mathbb{R})}$.*

Theorem 2.3. *Let q be τ-periodic, and assume $q_x \in L^\infty(\mathbb{R})$. Define*

$$\Sigma_\infty := \mathbb{R} \cup i[-\|q\|_\infty, \|q\|_\infty]\,. \tag{2.3}$$

Moreover, let $N_\delta(\Sigma_\infty)$ be a δ-neighborhood of Σ_∞. Then for any $\delta > 0$,

$$\Sigma(q) \cap (\mathbb{C} \setminus N_\delta(\Sigma_\infty)) = \emptyset \tag{2.4}$$

for all sufficiently small values of ϵ. That is, for any $\delta > 0$ there exists an $\epsilon_o > 0$ such that (2.4) holds for all $0 < \epsilon \leq \epsilon_o$.

Theorem 2.4. *Let the τ-periodic potential q satisfy at least one of the following conditions: a) it is real; b) it is even; c) it is odd. If the periodic and anti-periodic Floquet spectra are real and purely imaginary only, then the entire Floquet spectrum is contained within the real and imaginary axes, that is, $\Sigma(q) \subset \Sigma_\infty$.*

The study of the non-self-adjoint spectral problem (2.2) is motivated by the role it plays as the direct and inverse scattering transform in the IST for focusing NLS and focusing modified

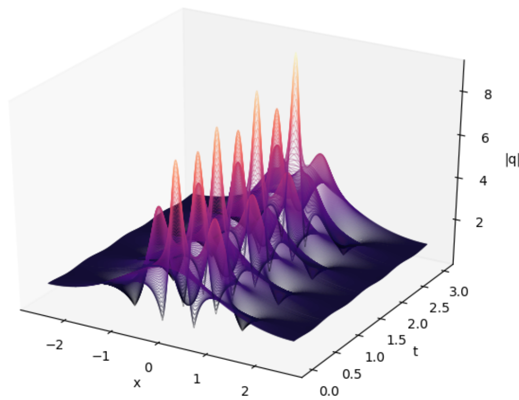

Figure 2: Genus 5 solution of focusing NLS – $q(x, 0) = 3\text{dn}(x; 0.9)$.

Korteweg-de Vries (mKdV) equations. Importantly, Theorem 2.3 gives rigorous justification of the WKB approach employed in [3]. An interesting open question is whether Theorem 2.3 holds for the time-dependent Dirichlet spectrum of (2.2) (the so-called "angle" coordinates).

Jacobi elliptic finite-band potentials. In [5] the spectrum of (2.2) ($\epsilon = 1$) with a two-parameter family of Jacobi elliptic potentials was studied. Recall, $\text{dn}(x; m)$ is a Jacobi elliptic function with period $2K(m)$ where $m \in (0, 1)$ is the elliptic parameter and $K = K(m)$ is the complete elliptic integral of the first kind. In this work the Floquet spectrum is characterized by relating the operator L to a corresponding unbounded tridiagonal operator acting on Fourier coefficients in a weighted Hilbert space. Some key findings:

Theorem 2.5. *Consider (2.2) ($\epsilon = 1$) with $2K$-periodic potential $q \equiv A\text{dn}(x; m)$, $m \in (0, 1)$, $A \in \mathbb{R}$, and K the complete elliptic integral of the first kind. Then q is a finite-band potential if and only if $A \in \mathbb{Z}$. Moreover, if $A \in \mathbb{Z} \setminus \{0\}$ then:*

$$\Sigma(q) = \mathbb{R} \cup \left(\cup_{j=1}^{A} \pm i[z_{2j-1}, z_{2j}]\right), \tag{2.5}$$

where $z_1 > 0$, $z_{2j-1} \neq z_{2j} \neq z_{2j+1}$, and $|z_{2A}| < A$. (i.e., q is $2|A|$ band, or genus $2|A| - 1$.)

Theorem 2.6. *Consider (2.2) ($\epsilon = 1$) with $q \equiv A\text{dn}(x; m)$, $m \in (0, 1)$, and $A \in \mathbb{R}$. If $A \notin \mathbb{Z}$, then there is no coexistence of (anti) periodic eigenfunctions.*

As a consequence of Theorem 2.6 when $A \notin \mathbb{Z}$ there are infinitely many bands intersecting the real z-axis transversally, i.e., "spines" [20, 29].

- This work also analyzed the process of formation of new bands along the imaginary z-axis as $A \to A + 1$. Spines evolve from the real axis to form new bands along the imaginary axis.

Finite-band (algebro-geometric) solutions to the focusing NLS (2.1) ($\epsilon = 1$) are (quasi) periodic solutions that represent nonlinear multi-phase waves (see Fig. 2). Thus periodic potentials having such properties are special and their identification important as they can be integrated using the finite-gap IST. Such solutions for focusing NLS were first constructed by Its and Kotlyarov in [22].

Later Gesztesy and Weikard gave a characterization of the elliptic algebro-geomtric solutions for the AKNS hierarchy in [18]. Importantly, since $\mathrm{dn}(x;m)$ interpolates between a plane wave ($m = 0$) and a sech ($m = 1$) it should be an interesting object in the study of periodic soliton/breather gases (small perturbations lead to chaotic behavior due to integrable turbulence [40]).

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