Research statement

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1 Research synopsis

Broadly speaking my research is concerned with the applied analysis of nonlinear dispersive equations which arise as universal models in a variety of physical settings. My research has two distinct but related goals. The first goal is to uncover the dynamical properties of these equations and their solutions, especially in certain asymptotic limits, and to investigate their surprisingly rich mathematical structure. This direction of research requires tools from both pure and applied mathematics. The second goal is related to the application of these equations to concrete physical systems such as water waves and nonlinear optics, aiming at producing results of practical usefulness. This direction of research makes use of techniques such as approximation theory, exact methods, and numerical simulations. Specific current research topics include the study of soliton and breather gases in integrable dispersive hydrodynamics, long-time asymptotic analysis of integrable systems and their non-integrable generalizations, modulations of periodic wave trains, and methods for the analysis of non-self-adjoint operators.

My research can be split as follows:

Program–Nonlinear dynamics. This direction of research concerns the study of nonlinear phenomena, particularly in the case where the governing equation is *integrable*, i.e., admits a Lax pair [29]. Nonlinear wave phenomena comprise an important class of problems in mathematical physics and nonlinear partial differential equations (PDEs). Remarkably, many of the governing equations, which arise as universal models in a variety of physical settings, are completely integrable infinite-dimensional Hamiltonian systems. Moreover, such equations often admit a delicate balance between dispersion and nonlinearity leading to complex and mathematically interesting coherent structures such as solitons in optics, or large-scale eddies in fluid flows. Accordingly, such equations have a rich mathematical structure that can be uncovered by exploiting their integrability. Thus, the study of such equations offers a unique combination of interesting mathematics and concrete applications. Moreover, the tools of integrability such as the inverse scattering transform (IST) [3, 4, 13, 19, 25] and nonlinear steepest descent method for oscillatory Riemann-Hilbert problems (RHPs) put forth in a seminal work by Deift and Zhou [11] allows for mathematically precise descriptions of such phenomena. Recently, my research in this direction has been concerned with the study of soliton gases (random ensembles of interacting solitons) as a type of integrable turbulence–the general theoretical framework for the description of a broad spectrum of stochastic wave phenomena in physical systems modeled by integrable equations [16, 18, 22, 41]. Another direction of my research in this area concerns the proximity between wave dynamics of integrable systems and their non-integrable generalizations [12, 23].

Program–Wave modulation theory. This direction of research concerns the study of the stability of slow modulations of periodic wave trains using a combination of techniques such as multiple-scale perturbation theory, functional analysis, and spectral perturbation theory. A basic strategy in the mathematical study of many real-world phenomena is to identify coherent structures which act as organizing centers for the long-time dynamics. To this end there are many applications including fluid flow, nonlinear optics, atmospheric science, and plasma physics where such long-time dynamics are dominated by spatially periodic structures. Nevertheless, the ability of periodic waves to carry modulation signals makes their dynamics under perturbations rich in multi-scale phenomena. For example, localized perturbations of periodic waves may effect each of the infinitely many fundamental period cells differently, leading to dynamics dominated by slow modulations of the underlying periodic structure and thus making the problem infinite dimensional in nature. A formal approach to the study of slow modulational dynamics is *Whitham modulation theory* put forth by Gerald Whitham in the 1960's [39]. Since then Whitham modulation theory has become the foundational cornerstone for the study of modulated periodic waves and their applications (including dispersive shock wave theory, small dispersion limits, soliton gases, etc.) [16, 17]. A major goal of my work in this direction concerns the application and development of mathematically rigorous techniques which justify the formal predictions of Whitham modulation theory, both at the level of (spectral) modulational instability and at the level of the dynamics of the wave train. Modulational instability (MI) is a fundamentral process in nonlinear physics where a continuous wave (or background) becomes unstable under perturbation. It is a universal phenomenon that explains the formation of coherent structures such as solitons and rogue waves in various systems, including water waves and optics. Recently, my research in this direction has been concerned with the application of these techniques to lattice models as well as to models in two spatial dimensions.

Program–Applied functional analysis. This direction of research concerns the development of the direct and inverse spectral theory of differential and difference operators arising in the study of integrable systems as well as in the study of modulations of nonlinear periodic wave trains. For example, in the study of integrable systems the fundamental concept is that the evolution equation may be viewed as the compatibility condition between two linear problems referred to as a Lax pair [29]. The spatial half of the Lax pair is a linear spectral problem which defines the direct scattering transform while the inverse scattering transform is defined in terms of a Riemann-Hilbert problem (RHP). In practice, the direct and inverse scattering transforms play the key role in the analysis of integrable systems. For example, discrete eigenvalues of the direct scattering transform parameterize multi-soliton solutions of integrable equations. Moreover, the mathematically rigorous study of the modulational stability, or instability, of nonlinear periodic wave trains requires the use of functional-analytic tools such as spectral perturbation theory of linear operators. My research in this direction has been concerned with the spectral theory of non-self-adjoint operators. In particular, non-self-adjoint Schrödinger, Dirac, and Jacobi operators with applications to special algebro-geometric solutions of integrable equations. Another direction of research in this area concerns the spectral perturbation theory of operator pencils.

Importantly, my research program offers several different opportunities for talented undergraduate students to partake in research projects under my guidance. These include projects related to multiple-scale perturbation theory and Whitham modulation theory, the unified transfrom method for linear and nonlinear initial–boundary value problems, numerical analysis of PDEs using spectral techniques, and empirical data fitting in social and biological sciences.

2 Past research

For those in my field the following offers a detailed description of my past research achievements together with some interesting open questions, and future directions.

1 Nonlinear dynamics and integrable systems

A major goal of my work in this direction concerns the study of certain emergent phenomena in completely integrable nonlinear dispersive equations. Emergence is an important concept that plays a fundamental role in many areas of mathematics and physics. Recently, my theoretical work in this direction has resulted in the first experimental observation of breather gases in optics. My work in this direction is presented in two distinct, but related sections.

Semiclassical limits, spectral theory, and algebro-geometric solutions:

A canonical example of a completely integrable nonlinear dispersive equation is the (semiclassical) focusing nonlinear Schrödinger (NLS) equation [40]

$$i\epsilon q_t + \epsilon^2 q_{xx} + 2|q|^2 q = 0, \tag{1.1}$$

where $q : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is the slowly-varying complex envelope of a quasi-monochromatic, weakly dispersive nonlinear wave packet, and $\epsilon > 0$ quantifies the relative strength of dispersion compared to nonlinearity. Solutions to (1.1) in the *semiclassical limit*, i.e., as $\epsilon \to 0^+$ have a remarkable structure which is of interest to applied analysts [27, 37]. (In the quantum-mechanical setting, ϵ is proportional to Planck's constant $\hbar$.) Such phenomena occur when dispersive effects are weak compared to nonlinear ones. These situations can produce a wide variety of physical effects such as supercontinuum generation, dispersive shocks, wave turbulence, and soliton gases to name a few [14, 17, 38].

Within the IST framework the direct and inverse scattering transforms play the key role in the solution of an integrable system. Our main interest here lies in the semiclassical limit of the direct scattering transform for the focusing NLS equation (1.1) which is the singularly perturbed non-self-adjoint Dirac-type operator

$$\mathcal{L}\phi := i\left[\epsilon\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\partial_x - \begin{pmatrix}0 & q\\ \overline{q} & 0\end{pmatrix}\right]\phi = z\phi, \quad \phi(x; z, \epsilon) = (\phi_1, \phi_2)^T, \tag{1.2}$$

$z \in \mathbb{C}$ is the eigenvalue parameter, overbar denotes the complex conjugate, and $q \equiv q(x, 0)$ is a given potential.

Note, according to Floquet theory for (quasi) periodic data the $L^2(\mathbb{R})$-spectrum of (1.2) is given by

$$\sigma(q) = \{z \in \mathbb{C} : -2 \leq \Delta(z, \epsilon; q) \leq 2\},$$

where $\Delta(z, \epsilon; q)$ is the *Floquet discriminant* (trace of the monodromy matrix) [15] which is isospectral under the flow of the focusing NLS equation. For generic periodic q the spectrum is comprised of an at most countable collection of analytic arcs, i.e., *spectral bands* in the complex plane

[21, 34]. Importantly, the case of finitely many spectral bands corresponds to a special class of (quasi) periodic solutions described using algebro-geometric function theory.

The spectral data provides information regarding the dynamical behavior of solutions to integrable equations. Thus the study of (1.2) has become an active area of research and a natural starting point for the study of solutions to the focusing NLS equation [8, 5, 24, 28, 30, 31, 36].

Assumption: Let q be the $2L$-periodic extension of a continuously differentiable function satisfying (i) non-negative, (ii) even, (iii) increasing $(-L, 0)$ and decreasing on $(0, L)$, and (iv) $q''(0) < 0$. We call such a function *single-lobe periodic*.

Let q be single-lobe periodic, and assume $z \in \mathbb{R} \cup i\mathbb{R}$. Then using a WKB asymptotic expansion together with connection formulae we obtain the following leading order description of the $L^2(\mathbb{R})$-spectrum of (1.2) in the limit $\epsilon \to 0^+$:

(1) To leading order the $L^2(\mathbb{R})$-spectrum is comprised of the cross

$$\mathbb{R} \cup (-iq_{\min}, iq_{\min})$$

plus a sequence of spectral bands and gaps in the intervals $(-iq_{\max}, -iq_{\min}) \cup (iq_{\min}, iq_{\max})$.

(2) In the intervals $(-iq_{\max}, -iq_{\min}) \cup (iq_{\min}, iq_{\max})$ the number of bands $N_\epsilon = O(1/\epsilon)$, the band widths $w_n^\epsilon = O(\epsilon e^{-S_1(z_n)/\epsilon})$ (exponential decay), and the gap widths $g_n^\epsilon = O(\epsilon)$ (algebraic decay) (see Figure 1).

(3) These leading-order estimates demonstrate that the resulting dynamics are characterized by the generation of a large number of nonlinear excitations referred to as *effective solitons* (the band width collapses to a point while the band width-to-gap width ratio goes to zero) having zero velocity, i.e., a bound-state ensemble of effective solitons.

(4) A law is obtained describing how the number of effective solitons scales with the semiclassical parameter. In particular

Importantly, this work applies to the spectral theory of soliton and breather gases for the focusing NLS equation [18]. In particular, it lays the groundwork to establish rigorously that semiclassical limits of single-lobe periodic potentials is a mechanism for realizations of soliton and breather gases in focusing nonlinear media.

In a subsequent work a general study of the $L^2(\mathbb{R})$-spectrum of (1.2) is considered. Among other results several estimates on the spectrum were obtained using energy techniques. Together these estimates prove that the $L^2(\mathbb{R})$-spectrum of (1.2) localizes to the real and imaginary axes of the spectral plane as $\epsilon \to 0^+$. This justifies rigorously our above WKB approach to the semiclassical limit of the direct scattering problem for focusing NLS.

Lemma 1.1. *Let $q \in L^\infty(\mathbb{R})$ be periodic. If $z \in \sigma(q)$, then $|\operatorname{Im} z| \leq \|q\|_{L^\infty(\mathbb{R})}$. Moreover, let $q' \in L^\infty(\mathbb{R})$. If $z \in \sigma(q)$, then $|\operatorname{Re} z||\operatorname{Im} z| \leq \frac{\epsilon}{2}\|q'\|_{L^\infty(\mathbb{R})}$.*

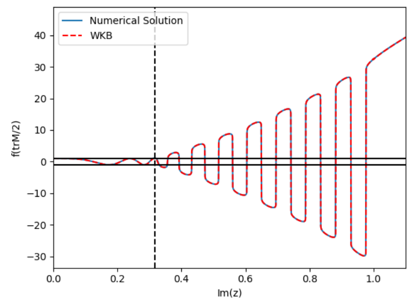

Figure 1: WKB approximation of the spectrum along the imaginary z-axis.

Lemma 1.2. *Let q be real, positive, and periodic. Suppose $q' \in L^\infty(\mathbb{R})$. If $z \in \sigma(q)$ with $\operatorname{Re} z > 0$, then $|\operatorname{Im} z| \leq \frac{\epsilon}{2}\|(\ln q)'\|_{L^\infty(\mathbb{R})}$.*

Theorem 1.3. *Let q be L-periodic, and assume $q' \in L^\infty(\mathbb{R})$. Define $\Sigma_\infty := \mathbb{R} \cup i[-\|q\|_\infty, \|q\|_\infty]$. Moreover, let $N_\delta(\Sigma_\infty)$ be a δ-neighborhood of Σ_∞. Then for any $\delta > 0$,*

$$\sigma(q) \cap (\mathbb{C} \setminus N_\delta(\Sigma_\infty)) = \emptyset \tag{1.3}$$

for all sufficiently small values of ϵ. That is, for any $\delta > 0$ there exists an $\epsilon_o > 0$ such that (1.3) holds for all $0 < \epsilon \leq \epsilon_o$.

A particularly important member of the class of single-lobe periodic potentials is the Jacobi elliptic function

$$\mathrm{dn}(x; m), \quad 0 < m < 1, \tag{1.4}$$

which is $2K$-periodic, where $K = K(m)$ is the complete elliptic integral of the first kind. In a recent paper the following important result in the study of integrable systems was obtained.

Theorem 1.4. *Consider (1.2) ($\epsilon = 1$) with $2K$-periodic potential $q = A\mathrm{dn}(x; m)$, $m \in (0, 1)$, and $A \in \mathbb{R}$. Then q is a finite-band potential if and only if $A \in \mathbb{Z}$. Moreover, if $A \in \mathbb{Z} \setminus \{0\}$ then:*

$$\sigma(q) = \mathbb{R} \cup \left(\cup_{j=1}^{A} \pm i[z_{2j-1}, z_{2j}]\right), \tag{1.5}$$

where $z_1 > 0$, $z_{2j-1} \neq z_{2j} \neq z_{2j+1}$, and $|z_{2A}| < A$. (i.e., q is $2|A|$ band, or genus $2|A| - 1$.)

Thus, initial data $q(x, 0) = A\mathrm{dn}(x; m)$ gives a finite-gap (algebro-geometric) solution to focusing NLS of (arithmetic) genus $2|A| - 1$. Such solutions form a special class of exactly solvable multi-phase solutions of integrable nonlinear dispersive equations (see Figure 2). Further, since

$$\mathrm{dn}(x; m) \to \mathrm{sech}(x) \quad \text{as} \quad m \to 1^+,$$

Theorem 1.4 should be compared to the classic result of Satsuma and Yajima that initial data $q(x, 0) = A\mathrm{sech}(x)$ with $A \in \mathbb{N}$ corresponds to the pure N-soliton solution of focusing NLS [33].

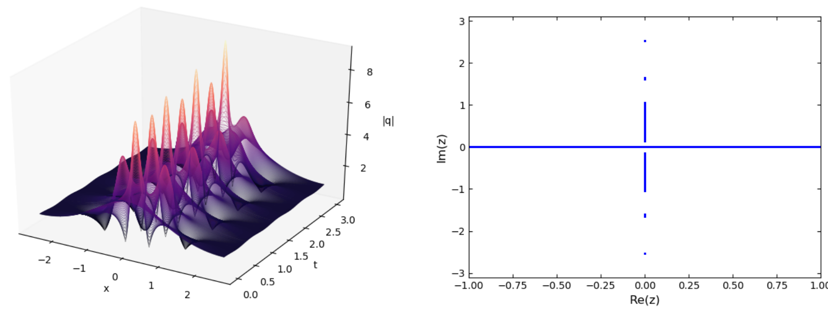

Figure 2: Left: A genus 5 solution of focusing NLS – $q(x,0) \equiv 3\mathrm{dn}(x;0.9)$. Right: The corresponding spectrum of the direct scattering transform.

More recently, I have wrote an independent paper that studies the semiclassical limit of the Dirac-type operator

$$\mathfrak{D}\phi := i\left[\epsilon\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\partial_x + \begin{pmatrix}0 & -q\\ p & 0\end{pmatrix}\right]\phi = z\phi, \quad \phi(x;z,\epsilon) = (\phi_1,\phi_2)^T, \tag{1.6}$$

where p, q are periodic complex-valued potentials. Importantly, (1.6) is the direct scattering transform for the AKNS hierarchy of integrable nonlinear equations [3]. Again, using energy techniques spectral enclosure estimates were obtained which depend on the semiclassical parameter.

Lemma 1.5. *Fix $\epsilon > 0$ and assume p, $q \in L^\infty(\mathbb{R})$ are periodic. If $z \in \sigma(p,q)$, then*

$$|\operatorname{Im} z| \leq \left(\|p\|_{L^\infty(\mathbb{R})}\|q\|_{L^\infty(\mathbb{R})}\right)^{1/2}.$$

Lemma 1.6. *Fix $\epsilon > 0$ and assume p, $q \in AC_{\mathrm{loc}}(\mathbb{R})$ are periodic with p', $q' \in L^\infty(\mathbb{R})$. If $z \in \sigma(p,q)$, then*

$$|\operatorname{Re} z||\operatorname{Im} z| \leq \frac{\epsilon}{2}\left(\|p'\|_{L^\infty(\mathbb{R})} + \|q'\|_{L^\infty(\mathbb{R})}\right) + \frac{1}{4}\|\overline{pq} - pq\|_{L^\infty(\mathbb{R})}.$$

Thus, Lemmas 1.5-1.6 together provide a sufficient condition for the $L^2(\mathbb{R})$-spectrum of $\mathfrak{D}$ to localize to the real and imaginary axes of the spectral plane in the limit $\epsilon \to 0^+$, namely, $\overline{pq} = pq$ almost everywhere. Moreover, by considering the case where p and q are constant we find that the above condition may also be necessary.

Publications:

Spectral enclosure estimates for non-self-adjoint Dirac operators,
J. Oregero, (submitted for publication), arXiv:2504.02236

Elliptic finite-band potentials of a non-self-adjoint Dirac operator,
G. Biondini, X.-D. Luo, J. Oregero and A. Tovbis, Adv. in Math. **429**, 109188 (2023),
DOI 10.1016/j.aim.2023.109188

On the spectrum of the periodic focusing Zakharov-Shabat operator,
G. Biondini, J. Oregero, A. Tovbis, J. Spectr. Theory **12** (3): 939–992 (2022),
DOI 10.4171/JST/432

Semiclassical dynamics and coherent soliton condensates in self-focusing nonlinear media with periodic initial conditions,
G. Biondini and J. Oregero, Stud. Appl. Math. **145** (3): 325–356 (2020),
DOI 10.1111/sapm.12321

Integrable turbulence and soliton gases:

While the classical theory and applications of the focusing NLS equation are mostly concerned with the description of regular, deterministic wave structures, the inherent statistical nature of some physical wave phenomena in focusing media (e.g. the spontaneous noise-induced modulational instability and rogue wave emergence) calls for the study of stochastic focusing NLS solutions, characterized in terms of the probability density function, correlation function, etc.

The study of random nonlinear dispersive waves has been an area of active theoretical and applied research for more than five decades, most notably in the context of water wave dynamics (see [16, 41] and references therein). A recent theme in this direction is the study of *soliton gases*–infinite statistical ensembles of interacting solitons–in integrable systems. It is well known that solitons can form ordered macroscopic coherent structures, such as dispersive shock waves [17]. Moreover, solitons can form unordered statistical ensembles that can be interpreted as soliton gases [16]. The nonlinear wave field in such gases represents a particular case of *integrable turbulence*.

This direction of research concerns showing that the semiclassical limit of focusing NLS with periodic data is a mechanism for the realization of soliton gases in focusing nonlinear media. In particular an analytically tractable model of *breather gas* (a soliton gas on the non-trivial genus zero background) generation via fission, based on the semiclassical limit of the focusing NLS equation (1.1) with initial data in the form of a periodic elliptic "dn" potential with elliptic parameter $m \in (0, 1)$ is presented. Further, it is demonstrated that, upon augmenting the data

$$\mathrm{dn}(x; m) + \eta(x), \quad 0 < m < 1, \tag{1.7}$$

where $\eta(x)$ is a small $\mathrm{iid}(0, 10^{-2})$ Gaussian white noise (which is inevitably present in real physical systems), the solution of the focusing NLS equation evolves into a fully randomized, spatially homogeneous breather gas, a phenomenon we call *breather gas fission*.

Key findings:

- We analytically compute the key quantities in the spectral theory of focusing NLS breather gases. This includes the density of states, logarithmic scaled band width, and nonlinear dispersion relations.

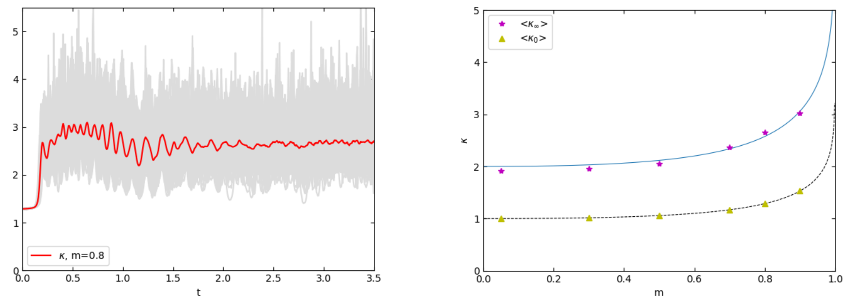

Figure 3: Left: Numerically computed $\kappa(t;m)$ for $m = 0.8$ (red). Right: Theoretical κ_∞ curve for $0 < m < 1$ (blue) together with data points from the numerically computed $\kappa_{\infty,m}$ for different values of m (magenta stars).

- We show $\kappa(t;m) \to 2\kappa(0;m) =: \kappa_{\infty,m}$ as $t \to \infty$, where

$$\kappa(0;m) = \frac{\langle \mathrm{dn}(x;m)^4 \rangle}{\langle \mathrm{dn}(x;m)^2 \rangle^2},$$

  is the normalized fourth moment (a measure of kurtosis used in optics) of the deterministic initial data $\mathrm{dn}(x;m)$ with $0 < m < 1$. Note $\langle \cdot \rangle$ is a spatial average.

- We prove any periodic breather gas of the focusing NLS equation generated by a deterministic real and even single-lobe potential $q(x,0)$ has $\kappa(t) \geq 2$. Moreover, $\kappa(t) = 2$ if and only if $q(x,0) = constant$.

Importantly, $\kappa(t) > 2$ implies heavy tailed non-Gaussian statistics, which is an indication of the possible presence of rogues waves in the dynamics.

To confirm the above analytic results we run ~ 1000 numerical simulations of focusing NLS with initial data (1.7). (Randomness is added to facilitate long-time *thermalization*–relaxation to a spatially homogeneous and statistically stationary state.) We numerically compute $\kappa(t;m)$ and compare numerical experiments to the theory (see Figure 3).

Recently, in collaboration with an optics lab at the University of Lille, Lille, France the fist experimental observation of breather gases in optics was achieved following the above theoretical results. The experiments were done using a recirculating fiber loop (see Figure 4). In the experiments different sinusoidal perturbations of the nonzero background were considered.

Publications:

Experimental observation of the spatio-temporal dynamics of breather gases in a recirculating fiber loop,
F. Copie, G. Biondini, J. Oregero, G. A. El, P. Suret, S. Randoux, (to appear–Optics Letters), arXiv:2507.04787

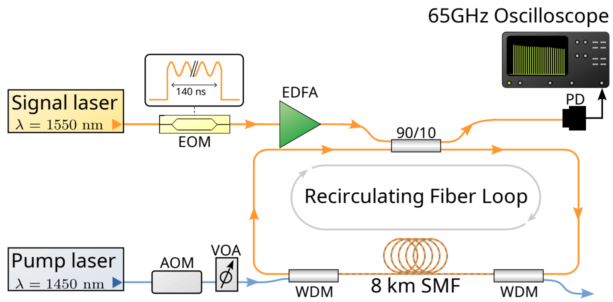

Figure 4: Schematic of the experimental setup.

Breather gas fission from elliptic potentials in self-focusing media,
G. Biondini, G. A. El, X.-D. Luo, J. Oregero, A. Tovbis, Phys. Rev. E **111**, 014214 (2025),
DOI 10.1103/PhysRevE.111.014204

Future directions:

- The work in this section is part of a larger research program concerned with understanding the dynamics of soliton gases and breather gases in integrable systems.
- An interesting open question is whether the special finite-gap solutions of focusing NLS with initial data $q(x,0) = A\mathrm{dn}(x;m)$ with $A \in \mathbb{N}$ are time-periodic.

2 Modulations of one-dimensional periodic wave trains

A major goal of my work in this direction is to provide a mathematically rigorous justification of Whitham's theory of modulations with regards to the modulational stability, or instability, of the underlying periodic wave train. Recall that modulational instability (MI) is a fundamental process in nonlinear physics where a continuous wave (or background) becomes unstable under perturbation. It is a universal phenomenon that explains the formation of coherent structures such as solitons, patterns, and rogue waves in various systems, including water waves and optics.

In the 1960's, Gerald Whitham introduced a formal methodology for studying the slow modulations of periodic wave trains [39]. Since then, the so-called *Whitham modulation theory* has become a cornerstone upon which researchers have studied modulated periodic waves and their applications (including dispersive shock waves, small dispersion limits, soliton gases, etc.). While Whitham modulation theory is a powerful tool the technique is based on formal asymptotic methods and hence its predictions are difficult to justify in a mathematically rigorous, functional-analytic, nonlinear PDE setting.

In an attempt to make the predictions from Whitham's theory of modulations mathematically rigorous I have been engaged in extending methodologies first developed by Jared Bronski and Mat Johnson [6]. In particular, using functional-analytic techniques together with spectral perturbation theory [26] we are able to make a precise and mathematically rigorous connection between the predictions of Whitham modulation theory and the (spectral) modulational stability of the underlying periodic wave train.

Importantly, these techniques due not rely on integrability of the governing equation and thus can be applied to general nonlinear dispesive PDEs.

Recently these techniques were applied and extended to the Camassa-Holm equation (CH)

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{2.1}$$

which is a model for the propagation of unidirectional surface water waves over a flat bottom [9]. The main result of this work is the following theorem.

Theorem 2.1. *Suppose that ϕ_0 is a $T_0 = 1/k_0$-periodic traveling wave solution of (2.1) with wave speed $c_0 > 0$, and that the set of nearby periodic traveling wave profiles ϕ with speed close to ϕ_0 is a 3-dimensional smooth manifold parameterized by $(k, M(\phi), P(\phi))$, where $1/k$ denotes the fundamental period of the wave and M and P denotes the mass and momentum of the wave, respectively. Additionally, assume that the hypothesis of of Theorem 4.4 below holds. If the Whitham system associated with (2.1) is strictly hyperbolic at $(k_0, M(\phi_0), P(\phi_0))$ then the wave ϕ_0 is spectrally modulationally stable, i.e., the $L^2(\mathbb{R})$-spectrum of the linearization of (2.1) about ϕ_0 is purely imaginary in a sufficiently small neighborhood of the origin in the spectral plane. Furthermore, a sufficient condition for ϕ_0 to be spectrally unstable to perturbations in $L^2(\mathbb{R})$ is that the Whitham modulation system be elliptic at $(k_0, M(\phi_0), P(\phi_0))$.*

Importantly, Theorem 2.1 provides a rigorous verification of the hyperbolicity of the Whitham system, a result first obtained by Abenda and Grava [1].

Further, a significant development in these techniques was recently achieved for the Ostrovsky equation

$$(u_t + uu_x + \beta u_{xxx})_x = \gamma u \tag{2.2}$$

which models unidirectional internal waves in a rotating fluid [32]. Here $\gamma > 0$ and $\beta \neq 0$. Interestingly, the case $\beta = 0$ is known as the "reduced" Ostrovsky equation and is integrable.

In this work the Whitham modulation system for the Ostrovsky equation was derived for the first time. Interestingly, the Whitham modulation system for the Ostrovsky equation is a 2×2 first-order quasilinear system of PDEs due to the mass zero constraint present in the equation. Importantly, in this work, in order to handle the mass zero constraint the previous techniques needed to be extended to the case of linear operator pencils. The main result of this work is the following theorem:

Theorem 2.2. *Suppose that ϕ_0 is a $T_0 = 1/k_0$ periodic traveling wave solution of (2.2) with wave speed $c_0 > 0$, and suppose that the set of nearby periodic traveling wave profiles ϕ with speed close to ϕ_0 is a two-dimensional smooth manifold parameterized by $(k, P(\phi))$, where $1/k$ denotes the fundamental period of the wave and P denotes the momentum of the wave. If the Whitham system is strictly hyperbolic at $(k_0, P(\phi_0))$, then the wave ϕ_0 is spectrally modulationally stable,* *i.e., the $L^2(\mathbb{R})$-spectrum associated with the linearization of* (2.2) *about ϕ_0 is purely imaginary in a sufficiently small neighborhood of the origin in the spectral plane. Furthermore, a sufficient condition for ϕ_0 to be spectrally unstable to perturbations in $L^2(\mathbb{R})$ is that the Whitham modulation system is elliptic at $(k_0, P(\phi_0))$.*

Most recently, we've produced numerical stability and instability diagrams associated with the Ostrovsky equation (2.2) for different values $\gamma > 0$ and $\beta \neq 0$. For example, as $\beta \to 0$ we have verified that the Whitham system is hyperbolic when $\beta = 0$. These numerical results serve as a nice demonstration of our mathematically rigorous theory. The numerical stability and instability diagrams are complete and will be part of a forthcoming paper.

Publications:

On the modulation of wave trains in the Ostrovsky equation,
M. A. Johnson, J. Oregero, W. P. Perkins, (submitted for publication),
arXiv:2505.21466

Modulational stability of wave trains in the Camassa-Holm equation,
M. A. Johnson and J. Oregero, J. Diff. Eqs. **446**, 113627 (2025),
DOI 10.1016/j.jde.2025.113627

Computation of Whitham characteristics and the stability of wave trains in the Ostrovsky equation
M. J. Johnson, J. Oregero, P. Sprenger (in preparation)

Future directions:

- An interesting direction of research in this area is to extend the rigorous functional-analytic techniques to the study of modulations in lattice models [7]. This is an ongoing project where the Toda lattice is the model under consideration.
- Another direction of interest and fundamental importance is to extend the techniques to $(2+1)$-dimensions. Recently there have been several advances in the study of Whitham modulation theory in higher spatial dimensions which makes this research direction promising [2].

3 Wave dynamics in integrable systems and their non-integrable generalizations

This direction of research concerns the careful analysis of the wave dynamics for integrable equations and their non-integrable generalizations. The question of whether phenomena that are characteristic to integrable systems persist in the transition to non-integrable settings is a central one in the field of nonlinear dispersive equations. (See for example [10, 12] as well as the so-called *soliton resolution conjecture* by Terence Tao [35]).

Using the tools of integrability such as the IST [20, 4] and nonlinear steepest descent method for oscillatory Riemann-Hilbert problems [11] mathematically precise statement regarding the dynamics of integrable nonlinear equations can be obtained. The goal of this work is to lift these results to the non-integrable setting by using a combination of techniques from integrable systems and nonlinear PDEs.

In a recent work following a nonlinear PDE and energy estimate approach the following proximity results between the wave dynamics of the Korteweg–de Vries (KdV) equation and it's non-integrable generalizations was obtained.

Consider the completely integrable KdV equation [20]

$$u_t + u_{xxx} + uu_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.1}$$

with $u = u(x,t)$ real. Moreover, its family of generalized KdV equations (gKdV) is given by

$$\tilde{u}_t + \tilde{u}_{xxx} + f(\tilde{u})\tilde{u}_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.2}$$

with $\tilde{u}(x,t)$ real and we take $f : \mathbb{R} \to \mathbb{R}$ to be a degree k polynomial satisfying $f(0) = 0$, i.e., $f(x) = \sum_{j=1}^{k} a_j x^j$.

In order to prove the main result a size estimate on the solution must be obtained. In doing this a "minimal lifespan" for which the solution satisfies the size estimate is also obtained. Then using a density argument the size estimate is extended to the space $H_x^s(\mathbb{R})$ for $s > 3/2$. The main result is the following:

Theorem 3.1. *Given $k \in \mathbb{N}$ and $\epsilon > 0$, consider the Cauchy problems for the gKdV and KdV equations (3.2) and (3.1) with initial data $\tilde{u}_0(x)$ and $u_0(x)$, respectively. For $s \in \mathbb{N} \cup \{0\}$, let*

$$T_s = c_{s,k} \min \left\{ \left(\|\tilde{u}_0\|_{H_x^s(\mathbb{R})}^k + A_k \sum_{j=1}^{k-1} \|\tilde{u}_0\|_{H_x^s(\mathbb{R})}^j \right)^{-1}, \|u_0\|_{H_x^{s+1}(\mathbb{R})}^{-1} \right\}, \tag{3.3}$$

where $c_{s,k} = c(s,k) > 0$ and $A_k := \max\{j|a_j| : 1 \le j \le k-1\}$.

<u>L^2 distance:</u> If the initial data satisfy

$$\|u_0\|_{H_x^2(\mathbb{R})} \le c_0 \epsilon, \quad \|\tilde{u}_0\|_{H_x^2(\mathbb{R})} \le C_0 \epsilon \tag{3.4}$$

$$\|u_0 - \tilde{u}_0\|_{L_x^2(\mathbb{R})} \le C \max\{\epsilon^2, \epsilon^{k+1}\} \tag{3.5}$$

for some constants c_0, C_0, and C, then for each fixed $t \in (0, T_1]$ there exists a constant $\tilde{C} = \tilde{C}(k,T)$ such that

$$\sup_{t \in [0,T]} \|u(t) - \tilde{u}(t)\|_{L_x^2(\mathbb{R})} \le \tilde{C} \max\{\epsilon^2, \epsilon^{k+1}\}. \tag{3.6}$$

<u>H^s and L^∞ distance for any $s \in \mathbb{N}$:</u> If the initial data satisfy

$$\|u_0\|_{H_x^{s+1}(\mathbb{R})} \le c_{0,s} \epsilon, \quad \|\tilde{u}_0\|_{H_x^{s+1}(\mathbb{R})} \le C_{0,s} \epsilon \tag{3.7}$$

$$\|u_0 - \tilde{u}_0\|_{H^s_x(\mathbb{R})} \leq C_s \max\{\epsilon^2, \epsilon^{k+1}\} \tag{3.8}$$

for some constants $c_{0,s}$, $C_{0,s}$, and C_s, then for each fixed $t \in (0, T_s]$ there exists a constant $\tilde{C}_s = \tilde{C}_s(s, k, T)$ such that

$$\sup_{t\in[0,T]} \|u(t) - \tilde{u}(t)\|_{H^s_x(\mathbb{R})} \leq \tilde{C}_s \max\{\epsilon^2, \epsilon^{k+1}\}. \tag{3.9}$$

Consequently by the Sobolev embedding theorem, there exists constant $\tilde{C}'_s = \tilde{C}'_s(s, k, T)$ such that

$$\sup_{t\in[0,T]} \|u(t) - \tilde{u}(t)\|_{L^\infty_x(\mathbb{R})} \leq \tilde{C}'_s \max\{\epsilon^2, \epsilon^{k+1}\}. \tag{3.10}$$

By the triangle inequality it is trivial to generalize this result to the proximity of solutions between any two members of the gKdV family with polynomial nonlinearities, say, $f_1 \neq f_2$.

It is also important to point out here that in this work through careful numerical simulations it is demonstrated that KdV and mKdV ($f(\tilde{u}) = \tilde{u}^2$) two-soliton solutions are stable when evolved according to sufficiently small non-integrable perturbations even for large times. (See also [12, 23])

Publications:

On the proximal dynamics between integrable and non-integrable members of a generalized Korteweg–de Vries family of equation,
N. Karachalios, D. Mantzavinos, and J. Oregero (in preparation)

Future directions:

- An interesting direction of research in this area is to obtain distance estimates similar to the ones above for other integrable equations and their non-integrable generalizations.
- Another interesting direction of research in this area is to consider the above problem in the case where $t \to \infty$ for sufficiently small non-integrable perturbations [12].
- A third interesting direction of research is to extend the Deift-Zhou nonlinear steepest descent method to 3×3 oscillatory matrix Riemann-Hilbert problems [11] in order to study the long-time asymptotic behavior of the Manakov system with non-zero boundary conditions. This direction of research is currently under consideration with collaborators.

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