SOLUTION
Homework 01

EIGEN ANALYSIS OF FLUTTER

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Consider an airfoil in wind tunnel testing with airspeed U (Figure 1). The airfoil is supported by springs at P. The center of mass is at C. The distance from P to C is the static offset x_{CP}. The aerodynamic lift L acts at Q. The distance from P to Q is the aerodynamic offset x_{QP}. The airfoil undergoes oscillatory motion with plunge displacement $h(t)$ and pitch displacement $\theta(t)$. The positive plunge direction is downward. The positive pitch direction is "nose up" (clockwise).

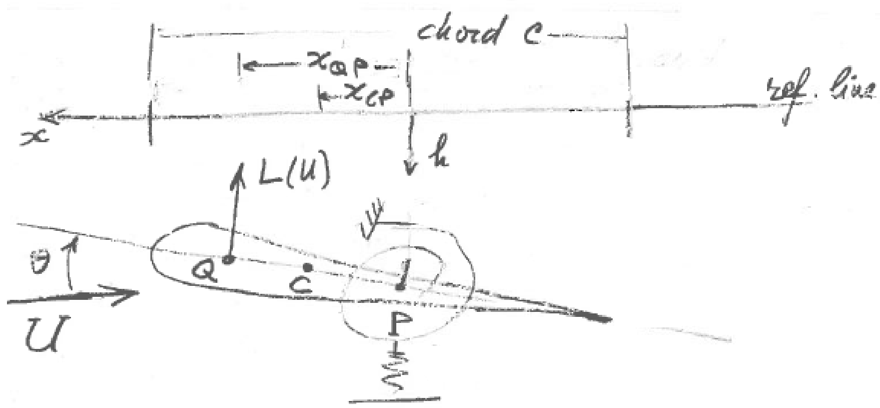


Figure 1: Airfoil in wind tunnel testing

<u>Input data</u>
Air density: ρ=1.225 kg/m^3
Chord: c=0.4 m
Mass, m=2.8 kg
Mass moment of inertia, I_0=0.05 kg.m^2
Plunge frequency, f_h=1.6 Hz
Pitch frequency, f_θ=4.8 Hz
Static offset, $x_{CP=}$ –10% of c
Aerodynamic offset, x_{QP}= 30% of c

NOTES:

- Always display input data!
- Show work and relevant comments
- Partial credit might be given based on work done and MATLAB codes if they are attached; without them, no partial credit can be given.
- Display numerical results to three significant digits (four if the first digit is 1).
- Problems solved beyond the minimum requirements will be given bonus points.
- Challenge problems are optional. They may be solved for extra credit

Notations:

- EOM = equation of motion
- CL = closed loop
- FBD = free body diagram
- FB = feedback
- GVT = ground vibration test
- LHS = left hand side
- LT = Laplace transform
- MIMO = multi-input multi-output
- ODE = ordinary differential equation
- PDE = partial differential equation
- RHS = right hand side
- SISO = single-input single-output
- SS = state space
- TF = transfer function

A. AEROELASTIC MODEL

For the airfoil shown in Figure 1, do the following:

(a) Deduce equations of motion. Clearly state your assumptions.

(b) Make linearity assumption on lift L and express $L(U,\theta)=L_0(U)\theta$, where U is the airspeed

(c) Express equations of motion in terms of the plunge and pitch frequencies ω_h, ω_θ

(d) Assume $h=\hat{h}e^{st}$, $\theta=\hat{\theta}e^{st}$ and cast equations of motion in matrix form as a polynomial eigenvalue problem

$$\left(s^2\mathbf{M}+\mathbf{K}(U)\right)\mathbf{x}=0$$

where

$$x=\begin{bmatrix}\hat{h}\\ \hat{\theta}\end{bmatrix},\qquad \mathbf{M}=\mathbf{M}_S+\mathbf{M}_A,\qquad \mathbf{K}(U)=\mathbf{K}_S+\mathbf{K}_A(U)$$

Answer:

Students should write their own derivation using class notes as a guide

B. GROUND VIBRATION TEST (GVT) ANALYSIS

Consider the airfoil shown in Figure 1 but with zero airspeed U=0. This condition corresponds to ground vibration testing (GVT). Do the following:

(a) Display input data
(b) Calculate the spring stiffnesses K_h and K_θ corresponding to the uncoupled frequencies f_h and f_θ
(c) Let x_{CP}= –10% of c and calculate the following:
  (i) x_{CP} and the moment of inertia I_P about elastic center P
  (ii) structural mass matrix $\mathbf{M}_S$ and structural stiffness matrix $\mathbf{K}_S$
  (iii) eigenvalues s_1, s_2, s_3, s_4 and eigenvectors $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, $\mathbf{V}_4$ (sort eigenvalues in ascending amplitude order, rearrange eigenvectors according to the eigenvalue sorting, and normalize each eigenvector wrt its largest element)
  (iv) coupled vibration frequencies f_I and f_{II}, uncoupled frequencies f_h, f_θ and percentage difference
  (v) modeshapes $\mathbf{V}_I$, $\mathbf{V}_{II}$ corresponding to the frequencies f_I, f_{II}.
(d) Lat x_{CP} = –20%, –10%, –1%, 0, 1%, 10% of c, 20% and calculate in table format the following:
  (i) x_{CP} and the moment of inertia I_P about elastic center P
  (ii) eigenvalues s_1, s_2, s_3, s_4 and eigenvectors $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, $\mathbf{V}_4$ (sort eigenvalues in ascending amplitude order, rearrange eigenvectors according to the eigenvalue sorting, and normalize each eigenvector wrt its largest element)
  (iii) coupled vibration frequencies f_I and f_{II} and percentage difference from the uncoupled frequencies f_h, f_θ.
  (iv) modeshapes $\mathbf{V}_I$, $\mathbf{V}_{II}$ corresponding to the frequencies f_I, f_{II}.
(e) Comment on your results.

<u>Answer</u>:

```
HW01
Section B. Ground Vibration Test (GVT) Analysis
 (a) input data
  c=0.4m, m=2.8 kg/m, I0=0.0500 kg*m^2/m
  uncoupled frequencies fh=1.6 Hz, ft=4.8 Hz

 (b) calculate spring stiffnesses
  plunge stiffness Kh=283.0 N/m
  pitch stiffness  Kt=45.48 N*m/rad
```

```
(c) static offset xCP=-10.0%, -0.0400 m
 moment of inertia Ip= 0.0545 m
 structural mass matrix MS =
   1.0000    0.0400
   2.2400    1.0896
 structural stiffness matrix KS =
 101.0647         0
        0  909.5827
 sorted eigenvalues s =
  0.0000 + 9.9973i   0.0000 - 9.9973i   0.0000 +30.3275i   0.0000 -30.3275i
 sorted and normalized eigenvectors V =
   1.0000    1.0000   -0.0449   -0.0449
   0.2796    0.2796    1.0000    1.0000
 coupled frequencies     fI=1.5911 Hz,   fII=4.8268 Hz
 uncoupled frequencies   fh=1.6000 Hz,   ft=4.8000 Hz
 frequency diff          dfI=-0.5546%,   dfII=0.5577%
 modeshapes, V_I, V_II =
   1.0000   -0.0449
   0.2796    1.0000

(d) other xCP values
CPratio % =
  -20   -10    -1     0     1    10    20
xCP, m =
  -0.0800   -0.0400   -0.0040         0    0.0040    0.0400    0.0800
 Ip, kg*m^2 =
   0.0679    0.0545    0.0500    0.0500    0.0500    0.0545    0.0679
 coupled frequencies, Hz =
   1.5655    1.5911    1.5999    1.6000    1.5999    1.5911    1.5655
   4.9058    4.8268    4.8003    4.8000    4.8003    4.8268    4.9058
 df% =
  -2.1563   -0.5546   -0.0056   -0.0000   -0.0056   -0.5546   -2.1563
   2.2038    0.5577    0.0056         0    0.0056    0.5577    2.2038
 modeshape V_I =
   1.0000    1.0000    1.0000    1.0000    1.0000    1.0000    1.0000
   0.5570    0.2796    0.0280         0   -0.0280   -0.2796   -0.5570
 modeshape V_II =
  -0.0895   -0.0449   -0.0045         0    0.0045    0.0449    0.0895
   1.0000    1.0000    1.0000    1.0000    1.0000    1.0000    1.0000

(e) Every student should insert here discussion of results
```

C. FLUTTER ANALYSIS

C.1 EIGEN ANALYSIS OF FLUTTER

Recall the aeroelastic equations of motion in matrix form expressed as a polynomial eigenvalue problem:

$\left(s^2\mathbf{M}+\mathbf{K}(U)\right)\mathbf{x}=\mathbf{0}$ where $\mathbf{x}=\begin{bmatrix}\hat{h}\\ \hat{\theta}\end{bmatrix}$, $\mathbf{M}=\mathbf{M}_S+\mathbf{M}_A$, $\mathbf{K}(U)=\mathbf{K}_S+\mathbf{K}_A(U)$

Give the expressions for all the matrices

Answer:

Students should write their own derivation using the class notes as a guide

C.2 FLUTTER DIAGRAM

Let x_{CP}= –10% of c, x_{QP}= 30% of c. Let U=0,…,12 m/s with 1001 steps.

(a) Display input data

(b) Plot frequencies and damping vs. airspeed. Use these plots to find the flutter speed U_F and insert datatips appropriately.

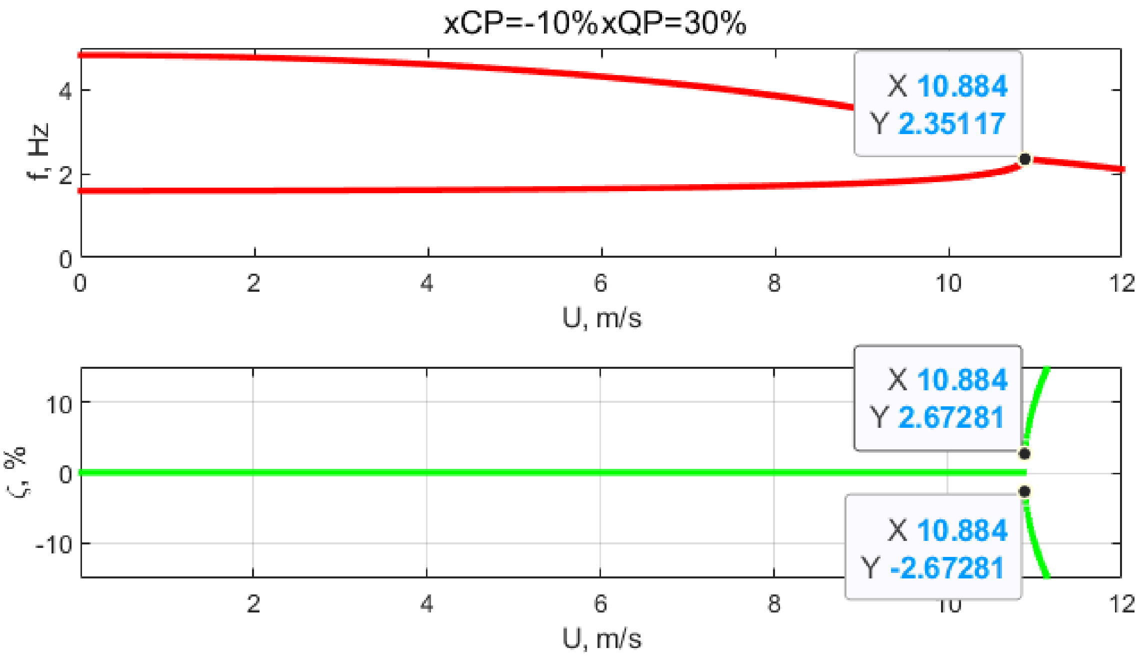
(c) State the value of U_F in m/s and knots.

<u>Answer</u>:

(a)

```
HW01
Section C. Flutter Eigen Analysis
 input data
  rho=1.225 kg/m^3 (air density)
  c=0.4m, m=2.8 kg/m, I0=0.0500 kg*m^2/m
  uncoupled frequencies fh=1.6 Hz, ft=4.8 Hz
  static offset xCP=-10.0%, -0.0400 m
  aerodynamic offset xQP=30.0%, 0.1200 m
  Ustart=0 m/s, Uend=14 m/s, NU=1001
```

(b)



(c)

U_F ≈10.884 m/s = 21.157 knots

C.3 FREQUENCIES AND MODESHAPES NEAR FLUTTER SPEED

Let U= 0, 6m/s, $(1-\varepsilon)U_F$, U_F, $(1+\varepsilon)U_F$, with ε=1%

(a) find frequencies and modeshapes at each of these speeds
(b) discuss your results

<u>Answer</u>:

(a)

```
enter flutter speed read on the flutter diagram UF=10.884
frequencies and modeshapes at various airspeeds
U=0 m/s
  frequency, Hz =
    1.5911    1.5911    4.8268    4.8268
  modeshapes =
    1.0000    1.0000   -0.0449   -0.0449
    0.2796    0.2796    1.0000    1.0000
U=6 m/s
  frequency, Hz =
    1.6439    1.6439    4.3168    4.3168
  modeshapes =
    1.0000    1.0000   -0.0152   -0.0152
    0.3619    0.3619    1.0000    1.0000
U=10.7752 m/s
  frequency, Hz =
    2.1619    2.1619    2.5823    2.5823
  modeshapes =
    0.6764    0.6764    0.3286    0.3286
    1.0000    1.0000    1.0000    1.0000
U=10.884 m/s
  frequency, Hz =
    2.3512    2.3512    2.3512    2.3512
  modeshapes =
   0.4769 + 0.0516i   0.4769 - 0.0516i   0.4769 - 0.0516i   0.4769 + 0.0516i
   1.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
U=10.9928 m/s
  frequency, Hz =
    2.3298    2.3298    2.3298    2.3298
  modeshapes =
   0.4510 + 0.1859i   0.4510 - 0.1859i   0.4510 - 0.1859i   0.4510 + 0.1859i
   1.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
```

(b) **Students should insert here discussion of results**

C.4 STATIC OFFSET EFFECT ON FLUTTER SPEED

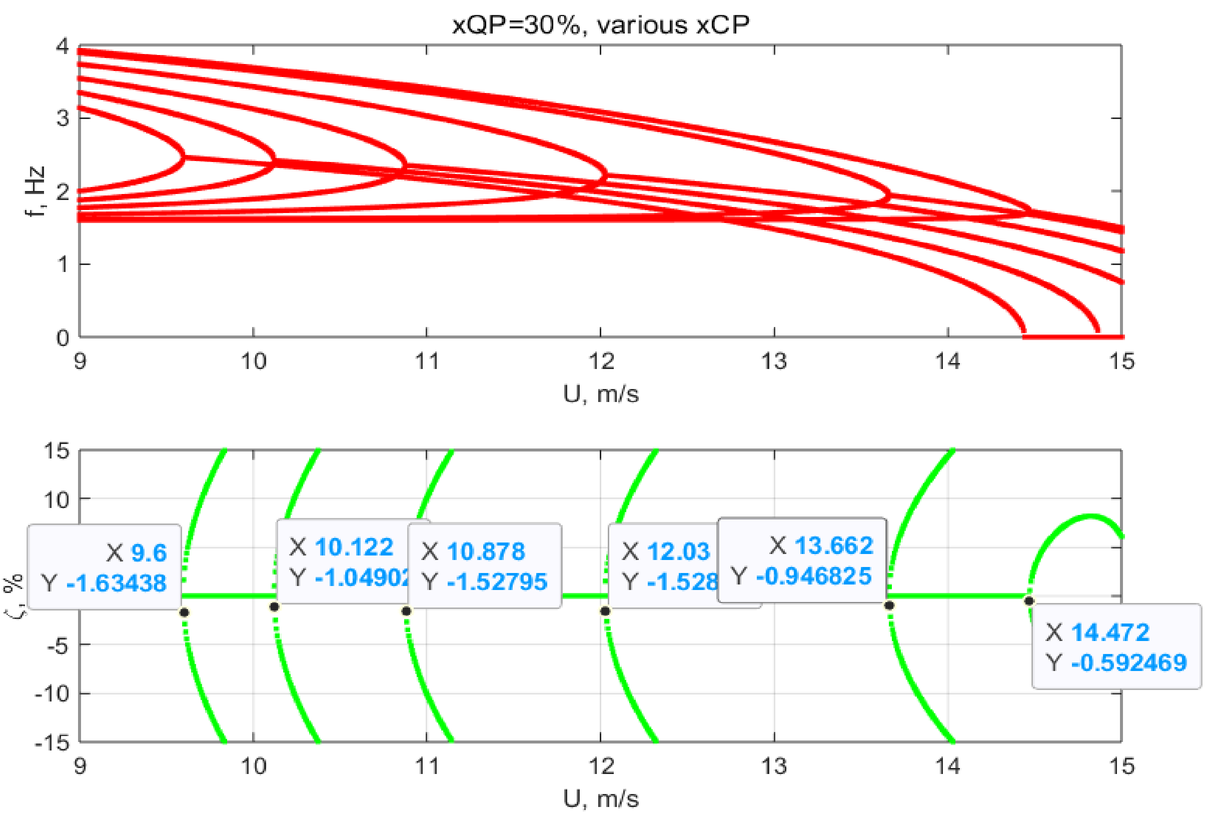
Let U=9,…,15 m/s with 1001 steps. Keep x_{QP}= 30% of c.
Let x_{CP} = –20%, –15%, –10%, –5%, –1%, –0.1% of c. Do the following:

(a) Display input data
(b) Plot overlapped (hold on) frequencies and damping vs. airspeed for each value of x_{CP}. Insert datatips to locate the flutter speed U_F. Present the U_F results in table format.
(c) Plot the variation of flutter speed U_F with x_{CP}%. Make sure the plot covers the whole U range.
(d) Discuss how x_{CP} influences the flutter speed.
(e) Repeat for x_{CP} = 0, 0.1%, 0.5%, 10%, 15%, 20% of c.

<u>Answer</u>:

(a)

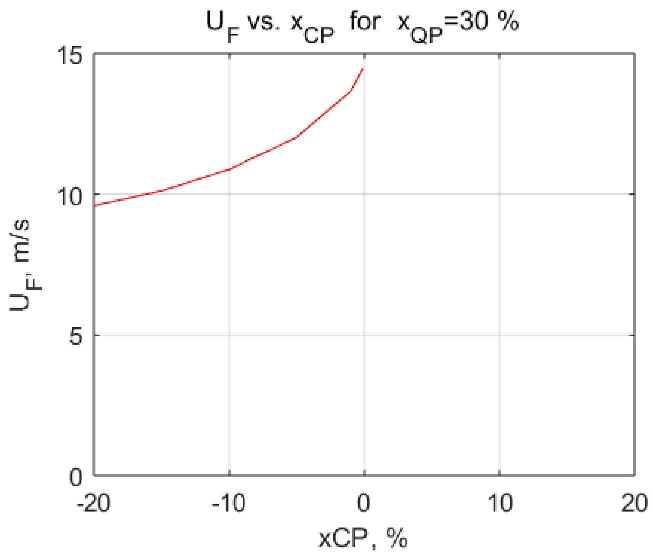
```
Ustart=9 m/s, Uend=15 m/s, NU=1001
QPratio=30%
CPratioRange % =
-20.0000  -15.0000  -10.0000   -5.0000   -1.0000   -0.1000
```



(b)

```
xCP % =
 -20.0000  -15.0000  -10.0000   -5.0000   -1.0000   -0.1000
UF, m/s =
   9.6000   10.1220   10.8780   12.0300   13.6620   14.4720
```

(c)



(d) **Every student should insert here discussion of results**

(e) There is no flutter for $x_{CP} \geq 0$

C.5 Aerodynamic offset effect on flutter speed

Let U=10,…,13 m/s with 1001 steps. Keep x_{CP}= –10% of c.
Let x_{QP} = 20%, 25%, 30% of c. Do the following:

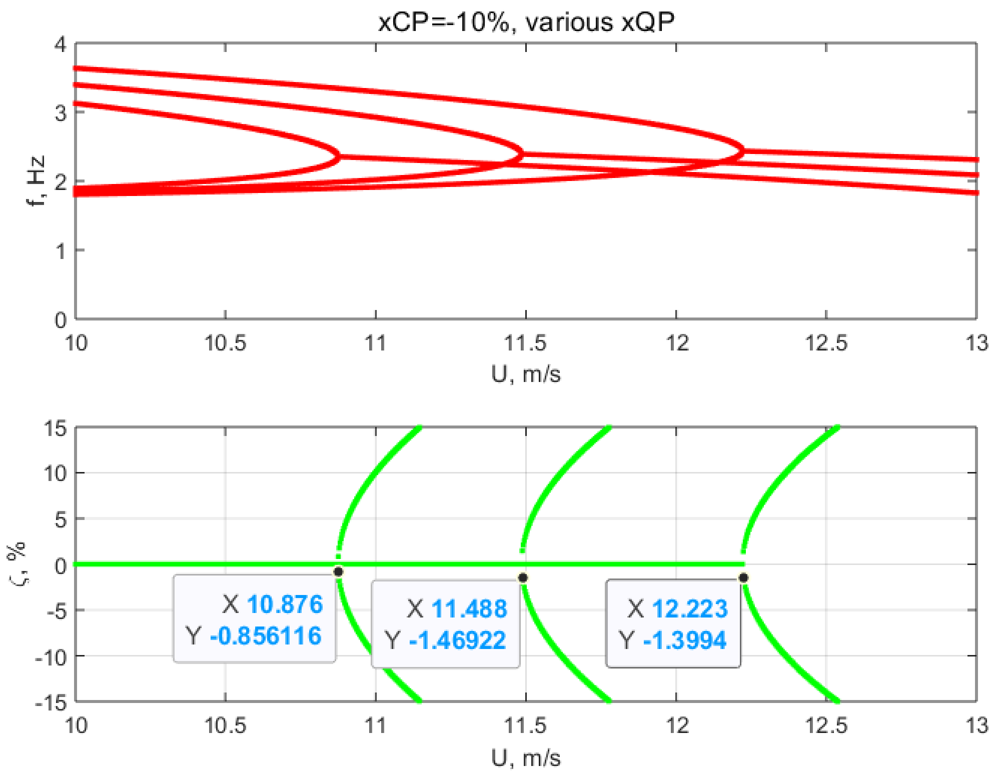
(a) Display input data
(b) Plot overlapped (hold on) frequencies and damping vs. airspeed for each value of x_{CP}. Insert datatips to locate the flutter speed U_F. Present the U_F results in table form.
(c) Plot the variation of flutter speed U_F with x_{QP}%. Make sure the plot covers the whole U range.
(d) Discuss how x_{QP} influences the flutter speed.

Answer:

(a)

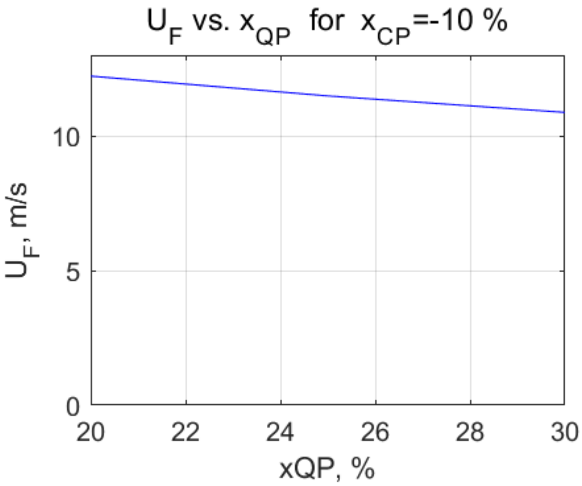
```
Ustart=10 m/s, Uend=13 m/s, NU=1001
CPratio=-10%
QPratioRange % = 20  25  30
```

(b)




```
xQP % =
    20    25    30
UF, m/s =
   12.2230   11.4880   10.8760
```

(c)



(d) **Every student should insert here discussion of results**

C.6 DIVERGENCE SPEED

(a) Let x_{CP}= –10% of c, x_{QP}= 30% of c. Let U=0,…, 20 m/s with 1001 steps.
  (i) Display input data
  (ii) Plot extended flutter diagram
  (iii)Identify with a datatip the location of U_D on the extended flutter diagram

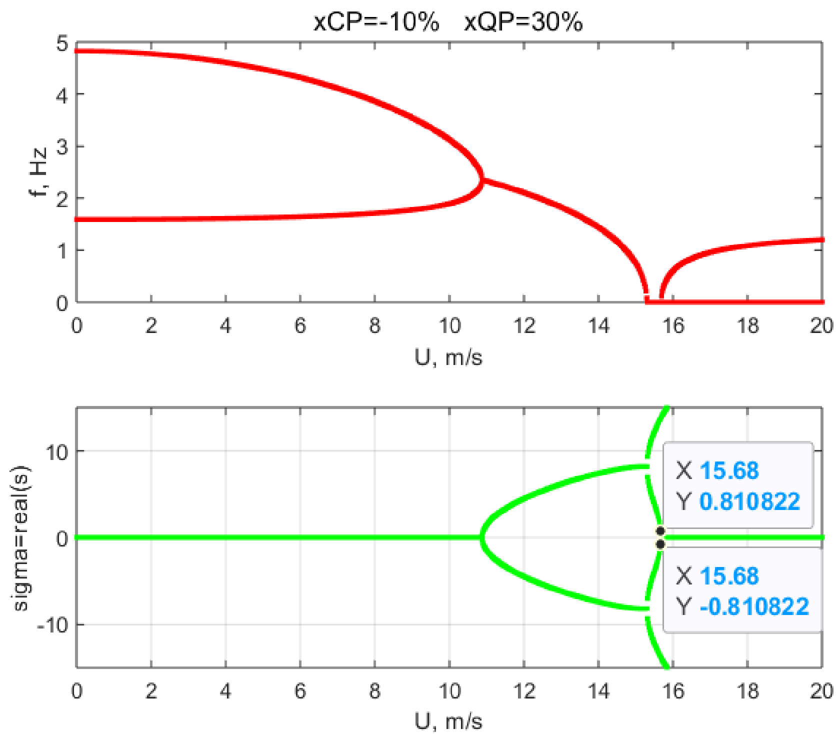
(b) Let x_{QP} = 20%, 25%, 30% of c. Calculate and plot divergence speed U_D vs. x_{QP}%. Make sure the plot covers the whole range of U_D.

(c) Discuss the comparison between the results of items (a) and (b)

Answer:

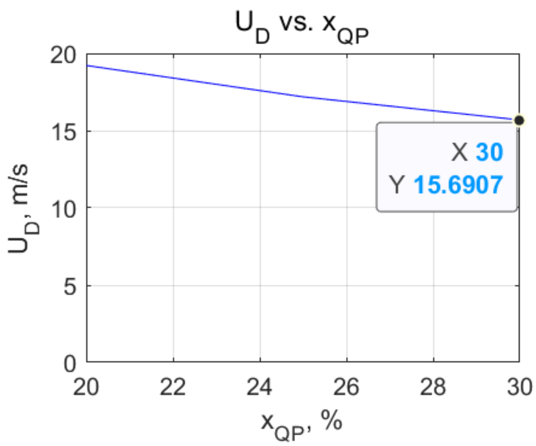
```
static offset xCP=-10.0%, -0.0400 m
aerodynamic offset xQP=30.0%, 0.1200 m
Ustart=0 m/s, Uend=20 m/s, NU=1001
```

(a)



(b)

```
QPratio, % =
    20    25    30
divergence speed UD, m/s =
   19.2171   17.1883   15.6907
```



(c) **Every student should insert here discussion of results**

D. DAMPED GVT AND FLUTTER

D.1 ANALYSIS OF DAMPED GVT AND FLUTTER

(a) Recall from class the damped flutter equations of motion in terms of the plunge and pitch frequencies f_h, f_θ, damping ζ_h, ζ_θ, inertias m, I_0, I_P, and lift function $L_0(U)$. Clearly state your assumptions.

(b) Assume $h = \hat{h}e^{st}$, $\theta = \hat{\theta}e^{st}$ and cast the equations of motion in matrix form as an polynomial eigenvalue problem

$\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}(U)\right)\mathbf{x} = \mathbf{0}$ where $\mathbf{x} = \begin{bmatrix} \hat{h} \\ \hat{\theta} \end{bmatrix}$, $\mathbf{M}=\mathbf{M}_S+\mathbf{M}_A$, $\mathbf{C}=\mathbf{C}_S+\mathbf{C}_A$, $\mathbf{K}(U)=\mathbf{K}_S+\mathbf{K}_A(U)$

Give the expressions for all the matrices.

<u>Answer</u>:

Students should write their own derivation using the class notes as a guide

D.2 SETUP FOR DAMPED GVT AND FLUTTER CALCULATIONS

Augment input data with the following modal damping values:

- Plunge damping ratio: ζ_h=2%
- Pitch damping ratio: ζ_θ=3%

Display updated input data

<u>Answer</u>

```
HW01
Section D. Damped GVT and Flutter
input data
  rho=1.225 kg/m^3 (air density)
  c=0.4m, m=2.8 kg/m, I0=0.0500 kg*m^2/m
  uncoupled frequencies fh=1.6 Hz, ft=4.8 Hz
  damping ratios zh= 2%, zt= 3%,
  static offset xCP=-10.0%, -0.0400 m
  aerodynamic offset xQP=30.0%, 0.1200 m
```

D.3 Damped GVT analysis

Let U=0 and study the effect of damping on GVT:

(a) Calculate damped frequencies and compare them with the undamped frequencies
(b) Calculate modal damping and compare with the uncoupled modal damping
(c) Calculate damped modeshapes and compare them with the undamped modeshapes

Answer:

```
Damped GVT
 (a) damped frequency f, Hz =
    1.5908    1.5908    4.8245    4.8245
     undamped frequency f, Hz =
    1.5911    1.5911    4.8268    4.8268

 (b) coupled modal damping, z, % =
    1.9766    1.9766    3.0542    3.0542
     uncoupled modal damping, zt,zh, % =
     2     3

 (c) damped modeshapes V =
   1.0000 + 0.0000i   1.0000 + 0.0000i  -0.0449 - 0.0003i  -0.0449 + 0.0003i
   0.2796 + 0.0062i   0.2796 - 0.0062i   1.0000 + 0.0000i   1.0000 + 0.0000i
     undamped modeshapes V0 =
    1.0000    1.0000   -0.0449   -0.0449
    0.2796    0.2796    1.0000    1.0000
```

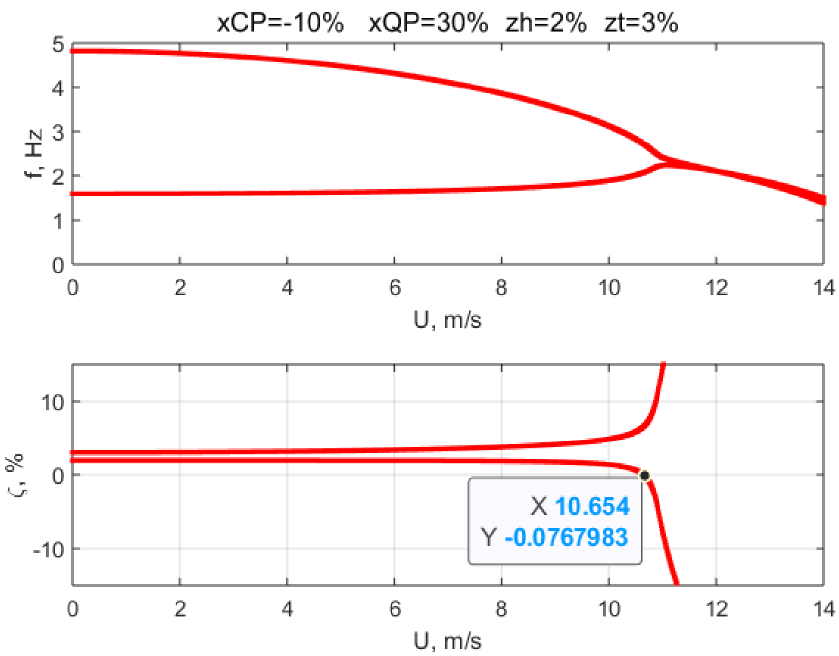
Every student should insert here discussion of results

D.4 DAMPED FLUTTER ANALYSIS

Let U=0…14 m/s with 1001 steps and study the effect of damping on flutter speed U_F

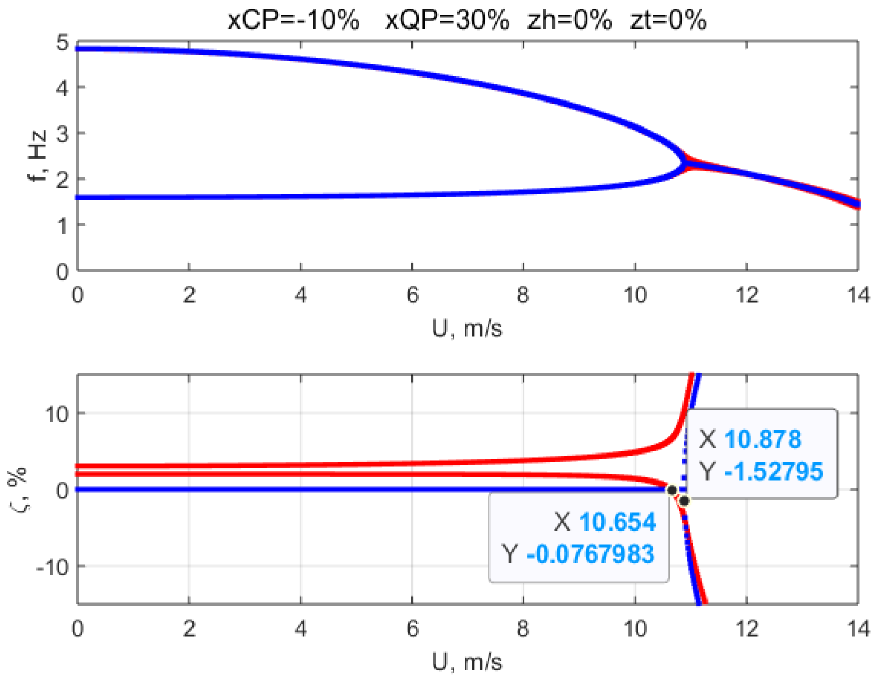
(a) Plot frequencies and damping vs. airspeed; find the damped flutter speed U_F on these plots.
(b) State the value of U_F in m/s and knots
(c) Plot the undamped diagram on top of the damped diagram and identify the undamped flutter speed $U_F^{undamped}$ and the damped flutter speed U_F.
(d) Discuss the difference between the damped and undamped flutter speeds

Answer:
(a)



(b) U_F =10.654 m/s = 20.71 knots

(c)



U_F=10.654 m/s

$U_F^{undamped}$ =10.878 m/s

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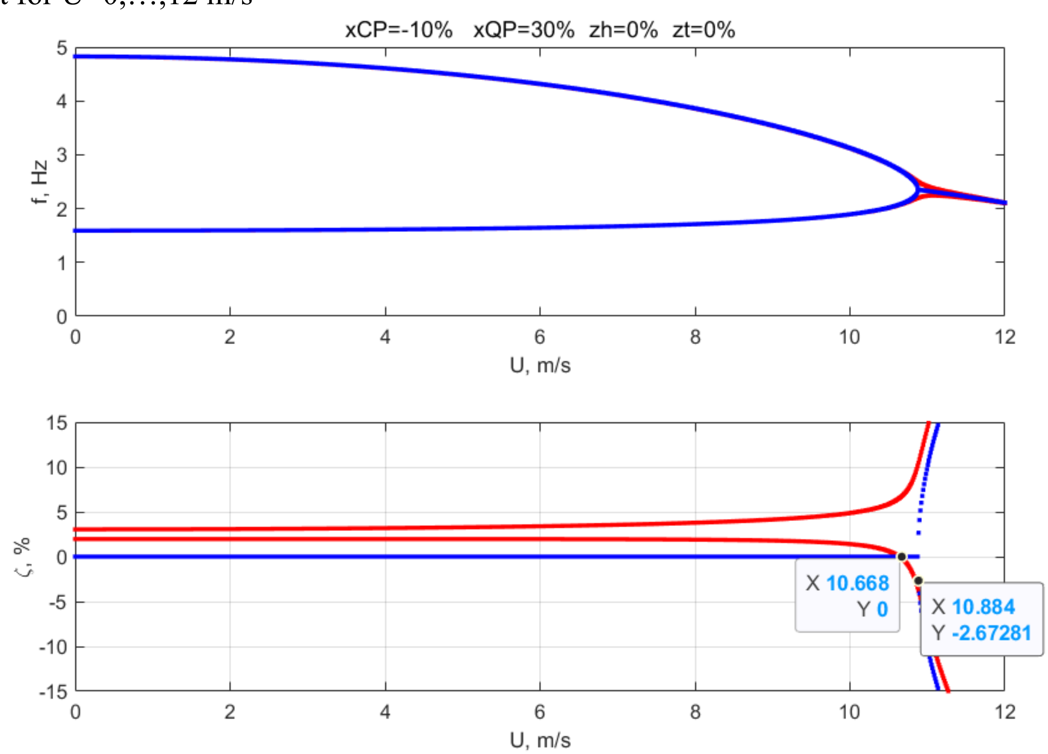
variant for U=0,…,12 m/s



(d) **Every student should insert here discussion of results**

E. EXTRA CREDIT CHALLENGE PROBLEMS

Challenge problems are optional.

E.1 GVT MODELING

Consider the airfoil shown in Figure 1 but with zero airspeed U=0. This condition corresponds to ground vibration testing (GVT). Do the following:

(a) Deduce equations of motion. Clearly state your assumptions.

(b) Assume $h = \hat{h}e^{st}$, $\theta = \hat{\theta}e^{st}$ and cast equations of motion in matrix form as a polynomial eigenvalue problem $\left(s^2\mathbf{M}_S + \mathbf{K}_S\right)\mathbf{x} = \mathbf{0}$ where $\mathbf{x} = \begin{bmatrix} \hat{h} \\ \hat{\theta} \end{bmatrix}$

Answer:
Students should write their own derivation using the class notes as a guide

E.2 DIVERGENCE SPEED MODELING

Derive the expression of the divergence speed U_D

Answer:
Students should write their own derivation using the class notes as a guide