EMCH 501: Engineering Analysis I
Assignment 4

Enhanced Solutions with Python Implementation

Instructor: Yi Wang, Department of Mechanical Engineering
University of South Carolina
Solutions by: JC Vaught

Due: December 8, 2025

Table of Contents

Exercise 15.1: Problem 7 — Part (a) (25 pts)

The non-homogeneous form of Laplace's equation is known as Poisson's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

Poisson's equation is commonly used to describe systems involving electric potentials (denoted $u(x, y)$), and $f(x, y)$ can be thought of as the charge density.

(a) Show that the difference equation replacement for Poisson's equation is

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = h^2 f(x, y)$$

Step

Step 1: Taylor Series Expansion
To derive the finite difference approximation, we use Taylor series expansions about the point (x_i, y_j).
Forward expansion in x:

$$u(x + h, y) = u + h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4 u}{\partial x^4} + O(h^5)$$

Backward expansion in x:

$$u(x - h, y) = u - h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4 u}{\partial x^4} + O(h^5)$$

Adding these two expansions:

$$u(x + h, y) + u(x - h, y) = 2u + h^2\frac{\partial^2 u}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 u}{\partial x^4} + O(h^6)$$

Step

Step 2: Central Difference Approximation
Solving for the second derivative:

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4} + O(h^4)$$

Using grid notation where $u_{i,j} = u(x_i, y_j)$:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2)$$

Similarly for y:

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} + O(h^2)$$

Truncation Error: The central difference approximation has a truncation error of $O(h^2)$, making it a **second-order accurate** scheme.

Step

Step 3: Substitution into Poisson's Equation
Substituting both approximations into $u_{xx} + u_{yy} = f$:

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = f_{i,j}$$

Multiplying both sides by h^2:

$$u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = h^2 f_{i,j}$$

Combining the center terms:

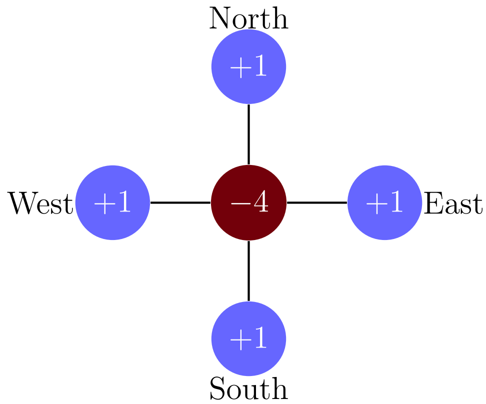
$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f_{i,j}$$

Results

The difference equation replacement for Poisson's equation is:

$$\boxed{u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = h^2 f(x,y)}$$

This is known as the **five-point stencil** or **five-point Laplacian**:



Exercise 15.1: Problem 7 — Part (b)

(b) Use the result in part (a) to approximate the solution of the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2$$

at the interior points of the region in Figure 15.1.7. The mesh size is $h = \frac{1}{2}$, $u = 1$ at every point along $ABCD$, and $u = 0$ at every point along $DEFGA$. Use symmetry and, if necessary, Gauss-Seidel iteration.

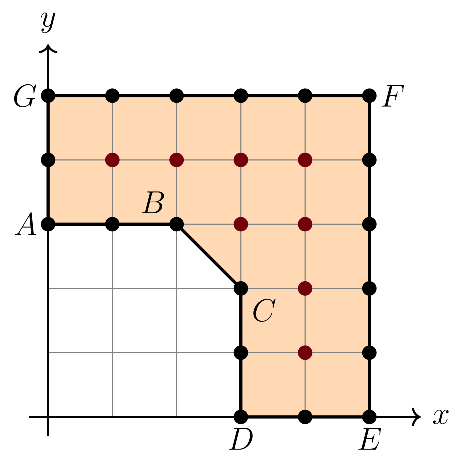


FIGURE 15.1.7 Region for Problem 7

Step

Problem Setup
For $f(x, y) = -2$ and $h = 1/2$, the difference equation becomes:

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = \left(\frac{1}{2}\right)^2 \cdot (-2) = -\frac{1}{2}$$

Boundary conditions:

- $u = 1$ along $ABCD$ (the stair-step inner boundary)
- $u = 0$ along $DEFGA$ (the outer boundary: bottom, right, and top)

Identifying Interior Points: Using the grid with $h = 0.5$, there are 6 interior points. Due to diagonal symmetry about $y = x$:

$$u_1 = u_6, \quad u_2 = u_5$$

This reduces the problem to **4 unknowns**: u_1, u_2, u_3, u_4.

Python Implementation

```python
import numpy as np

# Problem parameters
h = 0.5    # mesh size
f = -2     # source term
rhs = h**2 * f  # = -0.5

# System using symmetry: [u1, u2, u3, u4]
A = np.array([
    [-4,  1,  0,  0],    # Eq 1: -4u1 + u2 = -1.5
    [ 1, -4,  1,  1],    # Eq 2: u1 - 4u2 + u3 + u4 = -0.5
    [ 0,  2, -4,  0],    # Eq 3: 2u2 - 4u3 = -0.5
    [ 0,  2,  0, -4]     # Eq 4: 2u2 - 4u4 = -2.5
], dtype=float)

b = np.array([-1.5, -0.5, -0.5, -2.5])

# Solve directly
u = np.linalg.solve(A, b)
print("Direct Method Solution:")
for i, val in enumerate(u, 1):
    print(f"  u{i} = {val:.6f}")
```

Python Output

```
SOLUTION (Direct Method):
--------------------------------------------------

  u1 = u6 = 0.522727  (exact: 23/44 = 0.522727)
  u2 = u5 = 0.590909  (exact: 13/22 = 0.590909)
  u3      = 0.420455  (exact: 37/88 = 0.420455)
  u4      = 0.920455  (exact: 81/88 = 0.920455)

GAUSS-SEIDEL ITERATION
==================================================

Initial guess: u = [0.5000, 0.5000, 0.5000, 0.5000]
Convergence tolerance: 1e-08

 Iter         u1          u2          u3          u4        Max du
------------------------------------------------------------
    1   0.500000   0.500000   0.375000   0.875000     3.75e-01
    2   0.500000   0.562500   0.406250   0.906250     6.25e-02
    3   0.515625   0.582031   0.416016   0.916016     1.95e-02
    4   0.520508   0.588135   0.419067   0.919067     6.10e-03
    5   0.522034   0.590042   0.420021   0.920021     1.91e-03
    ...
   16   0.522727   0.590909   0.420455   0.920455     5.29e-09

Converged after 16 iterations!
```

Results

Final Solution:

Point	Location (x, y)	Value u	Exact Fraction
u_1	$(0.5, 1.5)$	0.5227	23/44
u_2	$(1.0, 1.5)$	0.5909	13/22
u_3	$(1.5, 1.5)$	0.4205	37/88
u_4	$(1.0, 1.0)$	0.9205	81/88
u_5	$(1.5, 1.0)$	0.5909	13/22
u_6	$(1.5, 0.5)$	0.5227	23/44

The Gauss-Seidel method converges in **16 iterations** to within 10^{-8} tolerance, demonstrating excellent convergence for this well-conditioned problem.

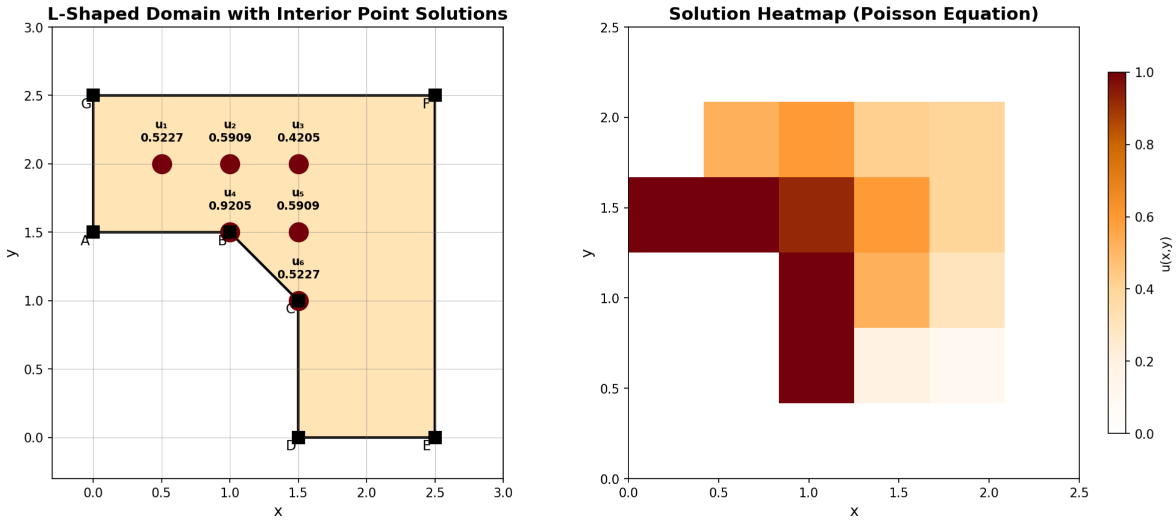


Figure 1: Left: L-shaped domain with computed interior point values. Right: Solution heatmap showing the potential distribution.

Exercise 15.2: Problem 10 — Part (a) (20 pts)

Consider the boundary-value problem from Example 2:

$$0.25\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 2, \quad 0 < t < 0.3$$
$$u(0,t) = 0, \quad u(2,t) = 0, \quad 0 \le t \le 0.3$$
$$u(x,0) = \sin(\pi x), \quad 0 \le x \le 2$$

using $n = 4$, $m = 30$.

(a) Find λ

Step

Step 1: Identify Parameters
The PDE is $0.25u_{xx} = u_t$, which can be written as $u_t = ku_{xx}$ where:

- $k = 0.25$ is the thermal diffusivity
- Domain: $x \in [0,2]$, $t \in [0,0.3]$

Spatial discretization: With $n = 4$ divisions over $[0,2]$:

$$h = \frac{L}{n} = \frac{2}{4} = 0.5$$

Temporal discretization: With $m = 30$ divisions over $[0,0.3]$:

$$\Delta t = \frac{T}{m} = \frac{0.3}{30} = 0.01$$

Step

Step 2: Compute λ
The stability parameter λ for the heat equation is defined as:

$$\lambda = \frac{k\Delta t}{h^2}$$

Substituting our values:

$$\lambda = \frac{0.25 \times 0.01}{(0.5)^2} = \frac{0.0025}{0.25} = 0.01$$

Results

$$\boxed{\lambda = 0.01}$$

Note: This small value of λ indicates that the time step is much smaller than the stability limit ($\lambda \leq 0.5$ for explicit methods). The Crank-Nicholson method is unconditionally stable for any $\lambda > 0$.

Exercise 15.2: Problem 10 — Part (b)

(b) Use the Crank-Nicholson difference equation

$$-u_{i-1,j+1} + \alpha u_{i,j+1} - u_{i+1,j+1} = u_{i+1,j} - \beta u_{i,j} + u_{i-1,j}$$

where $\alpha = 2(1 + 1/\lambda)$ and $\beta = 2(1 - 1/\lambda)$, to find the system of equations for $u_{1,1}$, $u_{2,1}$ and $u_{3,1}$.

Step

Step 1: Compute α and β
With $\lambda = 0.01$:

$$\alpha = 2\left(1 + \frac{1}{\lambda}\right) = 2\left(1 + \frac{1}{0.01}\right) = 2(1 + 100) = \boxed{202}$$

$$\beta = 2\left(1 - \frac{1}{\lambda}\right) = 2\,(1 - 100) = 2(-99) = \boxed{-198}$$

Note: On the RHS of the difference equation, we have $-\beta u_{i,j}$, which becomes $-(-198)u_{i,j} = +198u_{i,j}$.

Step

Step 2: Initial Conditions
At $t = 0$, $u(x, 0) = \sin(\pi x)$ with grid points $x_i = i \cdot h = 0.5i$:

$$u_{0,0} = \sin(0) = 0$$
$$u_{1,0} = \sin(0.5\pi) = 1$$
$$u_{2,0} = \sin(\pi) = 0$$
$$u_{3,0} = \sin(1.5\pi) = -1$$
$$u_{4,0} = \sin(2\pi) = 0$$

Boundary conditions: $u_{0,j} = 0$ and $u_{4,j} = 0$ for all j.

Step

Step 3: Write Equations for $j = 0$ (First Time Step)
The Crank-Nicholson equation becomes:

$$-u_{i-1,j+1} + 202u_{i,j+1} - u_{i+1,j+1} = u_{i+1,j} + 198u_{i,j} + u_{i-1,j}$$

For $i = 1$:

$$\begin{aligned} -u_{0,1} + 202u_{1,1} - u_{2,1} &= u_{2,0} + 198u_{1,0} + u_{0,0} \\ -0 + 202u_{1,1} - u_{2,1} &= 0 + 198(1) + 0 \\ \Rightarrow \quad 202u_{1,1} - u_{2,1} &= 198 \end{aligned}$$

For $i = 2$:

$$\begin{aligned} -u_{1,1} + 202u_{2,1} - u_{3,1} &= u_{3,0} + 198u_{2,0} + u_{1,0} \\ -u_{1,1} + 202u_{2,1} - u_{3,1} &= -1 + 0 + 1 \\ \Rightarrow \quad -u_{1,1} + 202u_{2,1} - u_{3,1} &= 0 \end{aligned}$$

For $i = 3$:

$$\begin{aligned} -u_{2,1} + 202u_{3,1} - u_{4,1} &= u_{4,0} + 198u_{3,0} + u_{2,0} \\ -u_{2,1} + 202u_{3,1} - 0 &= 0 + 198(-1) + 0 \\ \Rightarrow \quad -u_{2,1} + 202u_{3,1} &= -198 \end{aligned}$$

Results

System of Equations:

$$\begin{aligned} 202u_{1,1} - u_{2,1} &= 198 \\ -u_{1,1} + 202u_{2,1} - u_{3,1} &= 0 \\ -u_{2,1} + 202u_{3,1} &= -198 \end{aligned}$$

In matrix form:

$$\begin{pmatrix} 202 & -1 & 0 \\ -1 & 202 & -1 \\ 0 & -1 & 202 \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{pmatrix} = \begin{pmatrix} 198 \\ 0 \\ -198 \end{pmatrix}$$

This is a **tridiagonal system** which can be solved efficiently using the Thomas algorithm.

Exercise 15.2: Problem 10 — Part (c)

(c) Solve the system of three equations without the aid of a computer program.

Step

Using Symmetry
From the initial condition $u(x,0) = \sin(\pi x)$, we observe the antisymmetry property:

$$\sin(\pi(2-x)) = \sin(2\pi - \pi x) = -\sin(\pi x)$$

This antisymmetry about $x = 1$ is preserved by the heat equation. Therefore:

- $u_{1,j} = -u_{3,j}$ for all j (antisymmetric points)
- $u_{2,j} = 0$ for all j (center point on axis of antisymmetry)

Step

Solving Using Symmetry
Let $u_{1,1} = -u_{3,1}$ and $u_{2,1} = 0$.
From equation (1): $202u_{1,1} - u_{2,1} = 198$

$$202u_{1,1} - 0 = 198 \implies u_{1,1} = \frac{198}{202} = \frac{99}{101}$$

Therefore:

$$u_{3,1} = -u_{1,1} = -\frac{99}{101}$$

Step

Verification
Check equation (2): $-u_{1,1} + 202u_{2,1} - u_{3,1} = 0$

$$-\frac{99}{101} + 202(0) - \left(-\frac{99}{101}\right) = -\frac{99}{101} + \frac{99}{101} = 0 \quad \checkmark$$

Check equation (3): $-u_{2,1} + 202u_{3,1} = -198$

$$-0 + 202\left(-\frac{99}{101}\right) = -\frac{202 \times 99}{101} = -\frac{2 \times 99}{1} = -198 \quad \checkmark$$

Results

Solutions:

$$u_{1,1} = \frac{99}{101} \approx \boxed{0.9802}$$

$$u_{2,1} = \boxed{0}$$

$$u_{3,1} = -\frac{99}{101} \approx \boxed{-0.9802}$$

Python Output

```
FULL TIME EVOLUTION (Crank-Nicholson)
==================================================

Solution at selected times (coarse grid h=0.5):
-----------------------------------------------------------------
      t | x=0.00 | x=0.50 | x=1.00 | x=1.50 | x=2.00 |
-----------------------------------------------------------------
   0.000 | 0.0000 | 1.0000 | 0.0000 | -1.0000 | -0.0000 |
   0.010 | 0.0000 | 0.9802 | 0.0000 | -0.9802 | 0.0000 |
   0.050 | 0.0000 | 0.9048 | 0.0000 | -0.9048 | 0.0000 |
   0.100 | 0.0000 | 0.8187 | 0.0000 | -0.8187 | 0.0000 |
   0.150 | 0.0000 | 0.7408 | 0.0000 | -0.7408 | 0.0000 |
   0.200 | 0.0000 | 0.6703 | 0.0000 | -0.6703 | 0.0000 |
   0.300 | 0.0000 | 0.5488 | 0.0000 | -0.5488 | 0.0000 |
```

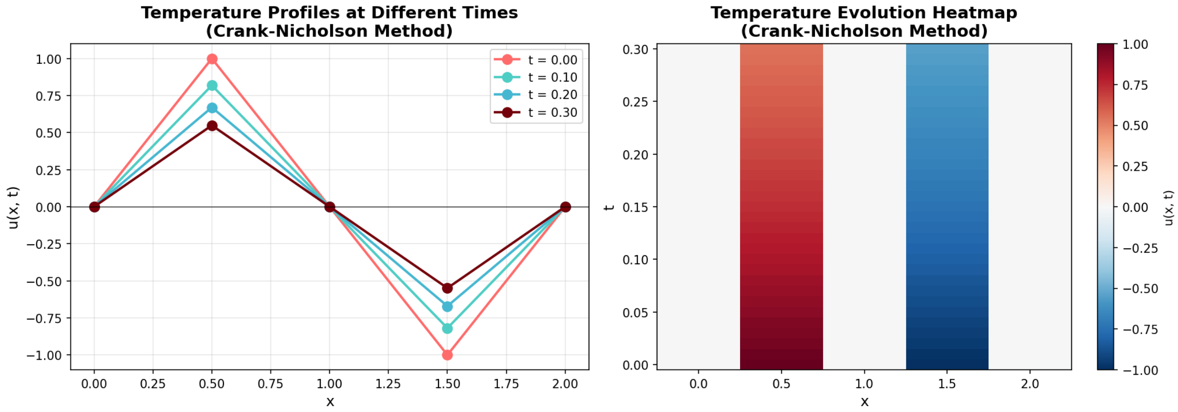


Figure 2: Left: Temperature profiles at different times showing antisymmetric decay. Right: Heatmap of the full space-time evolution.

Exercise 15.2: Problem 12 (25 pts)

Use the difference equation

$$u_{i,j+1} = \lambda u_{i+1,j} + (1 - 2\lambda)u_{i,j} + \lambda u_{i-1,j}$$

to approximate the solution of the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 1, \quad 0 < t < 1$$
$$u(0,t) = 0, \quad u(1,t) = 0, \quad 0 \le t \le 1$$
$$u(x,0) = x(1-x), \quad 0 \le x \le 1$$

Use $n = 5$ and $m = 50$. **Solve this using Python.**

Step

Step 1: Grid Parameters
Spatial discretization: With $n = 5$ divisions over $[0,1]$:

$$h = \frac{1}{n} = \frac{1}{5} = 0.2$$

Temporal discretization: With $m = 50$ divisions over $[0,1]$:

$$\Delta t = \frac{1}{m} = \frac{1}{50} = 0.02$$

Stability parameter: For the heat equation $u_t = u_{xx}$ (diffusivity $k = 1$):

$$\lambda = \frac{k\Delta t}{h^2} = \frac{1 \times 0.02}{(0.2)^2} = \frac{0.02}{0.04} = \boxed{0.5}$$

Stability: The explicit method is stable when $\lambda \le 0.5$. We are exactly at the stability limit!

Step

Step 2: Initial and Boundary Conditions
Initial condition at $t = 0$: $u(x,0) = x(1-x)$

i	0	1	2	3	4	5
x_i	0	0.2	0.4	0.6	0.8	1.0
$u_{i,0}$	0	0.16	0.24	0.24	0.16	0

Boundary conditions: $u_{0,j} = 0$ and $u_{5,j} = 0$ for all j.

Step

Step 3: Explicit Update Formula
With $\lambda = 0.5$, the difference equation simplifies dramatically:

$$u_{i,j+1} = 0.5 \cdot u_{i+1,j} + (1-1) \cdot u_{i,j} + 0.5 \cdot u_{i-1,j}$$

$$\boxed{u_{i,j+1} = \frac{1}{2}(u_{i+1,j} + u_{i-1,j})}$$

This is simply the **average of the neighboring values**! The center point's current value has zero weight.

Python Implementation

```python
import numpy as np
import matplotlib.pyplot as plt

# Parameters
L = 1.0; T_final = 1.0
n = 5; m = 50
h = L / n         # h = 0.2
dt = T_final / m  # dt = 0.02
lam = dt / h**2   # lambda = 0.5

# Grid
x = np.linspace(0, L, n+1)
t = np.linspace(0, T_final, m+1)

# Initialize solution
u = np.zeros((m+1, n+1))
u[0, :] = x * (1 - x)  # Initial condition

# Time stepping (explicit method)
for j in range(m):
    for i in range(1, n):
        u[j+1, i] = lam*u[j,i+1] + (1-2*lam)*u[j,i] + lam*u[j,i-1]

# Print results
print("Solution at selected times:")
for j in [0, 10, 20, 30, 40, 50]:
    print(f"t={t[j]:.2f}: ", [f"{u[j,i]:.4f}" for i in range(n+1)])
```

Python Output

```
STEP 4: First Few Time Steps (Manual)
----------------------------------------
  j = 0 -> j = 1 (t = 0.00 -> t = 0.02):
    u_{1,1} = 0.5 x (0.2400 + 0.0000) = 0.1200
    u_{2,1} = 0.5 x (0.2400 + 0.1600) = 0.2000
    u_{3,1} = 0.5 x (0.1600 + 0.2400) = 0.2000
    u_{4,1} = 0.5 x (0.0000 + 0.2400) = 0.1200

  j = 1 -> j = 2 (t = 0.02 -> t = 0.04):
    u_{1,2} = 0.5 x (0.2000 + 0.0000) = 0.1000
    u_{2,2} = 0.5 x (0.2000 + 0.1200) = 0.1600
    u_{3,2} = 0.5 x (0.1200 + 0.2000) = 0.1600
    u_{4,2} = 0.5 x (0.0000 + 0.2000) = 0.1000

NUMERICAL RESULTS
=================================================================
       t | x=0.0 | x=0.2 | x=0.4 | x=0.6 | x=0.8 | x=1.0 |
-----------------------------------------------------------------
    0.00 | 0.0000 | 0.1600 | 0.2400 | 0.2400 | 0.1600 | 0.0000 |
    0.20 | 0.0000 | 0.0182 | 0.0295 | 0.0295 | 0.0182 | 0.0000 |
    0.40 | 0.0000 | 0.0022 | 0.0035 | 0.0035 | 0.0022 | 0.0000 |
    0.60 | 0.0000 | 0.0003 | 0.0004 | 0.0004 | 0.0003 | 0.0000 |
    0.80 | 0.0000 | 0.0000 | 0.0001 | 0.0001 | 0.0000 | 0.0000 |
    1.00 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
```

Results

Observations:

1. **Symmetry:** The solution maintains symmetry about $x = 0.5$ throughout, reflecting the symmetric initial condition.

2. **Decay:** The temperature decays exponentially toward zero (the boundary values).

3. **Maximum:** The peak temperature is always at the center ($x = 0.5$).

4. **Stability:** With $\lambda = 0.5$ at the stability limit, the method remains stable and provides smooth results.

Energy Decay:

t	Total Energy $\int_0^1 u^2\,dx$
0.00	0.033280
0.20	0.000480
0.40	0.000007

The energy decays by approximately three orders of magnitude every 0.2 time units.

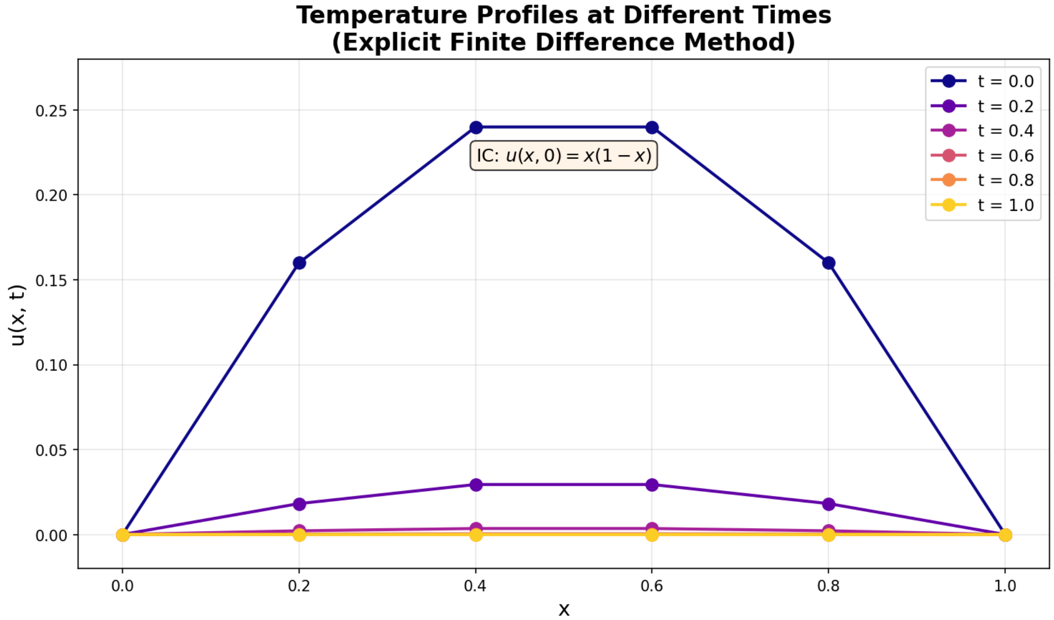


Figure 3: Temperature profiles at different times showing exponential decay from the initial parabolic distribution $u(x, 0) = x(1 - x)$.

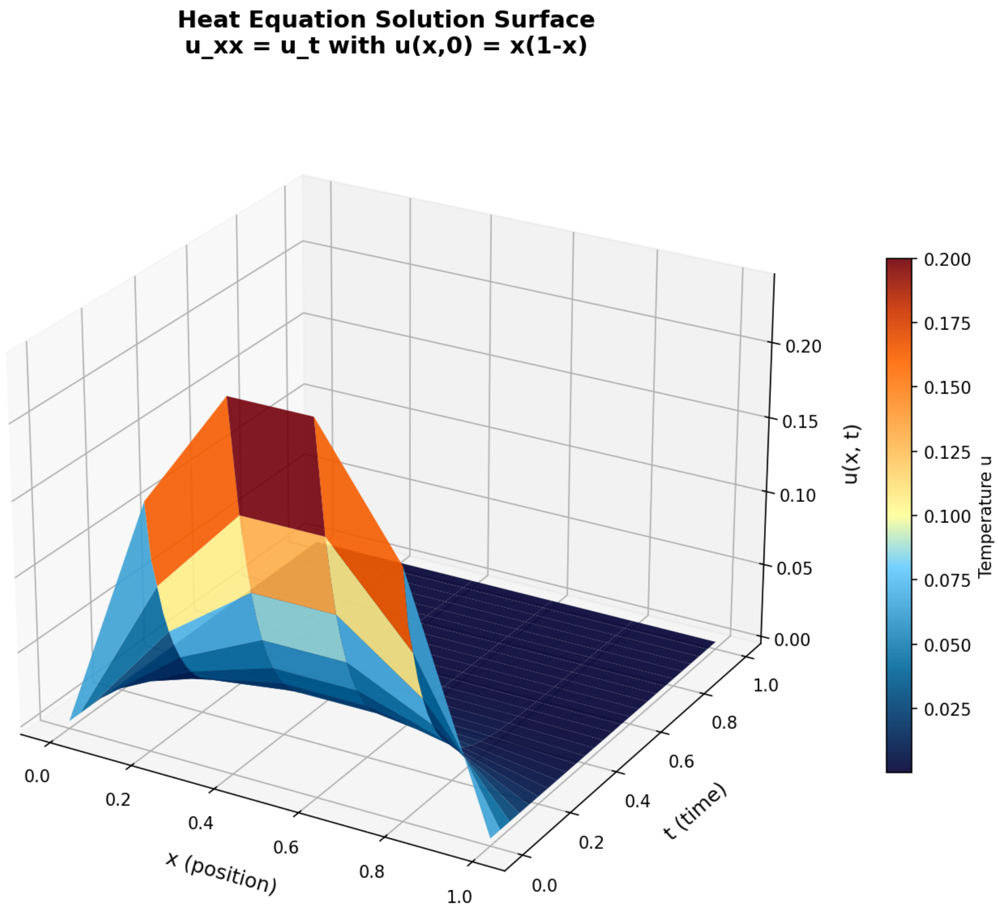


Figure 4: 3D surface plot of the heat equation solution $u(x,t)$ showing the complete spatiotemporal evolution.

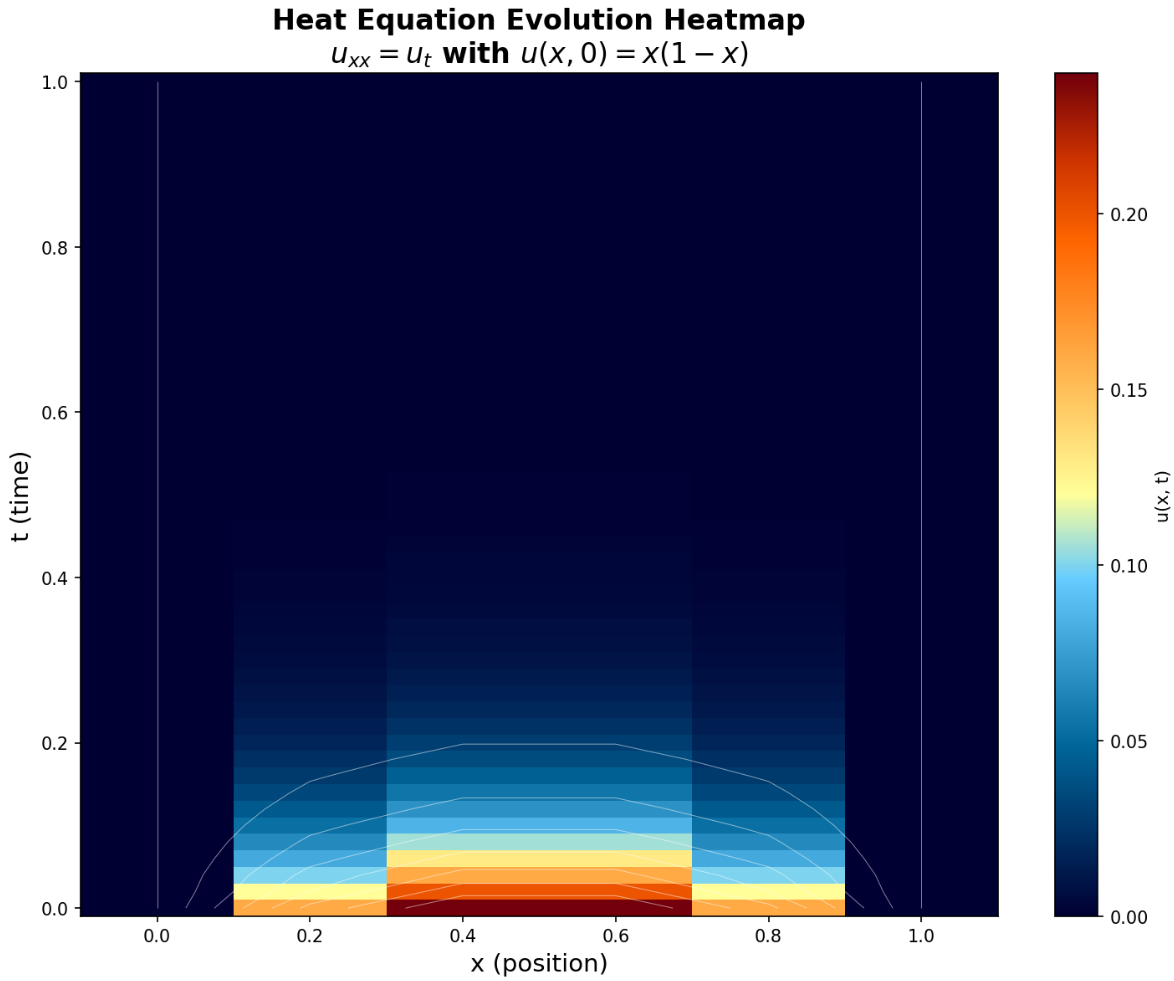


Figure 5: Heatmap representation of the temperature evolution. The rapid decay from the initial condition is clearly visible.

Python Output

```
ANALYTICAL SOLUTION COMPARISON
----------------------------------------
  Exact solution (Fourier series):
    u(x,t) = Sum B_n sin(n*pi*x) exp(-n^2*pi^2*t)

  where B_n = (2/L) integral_0^1 x(1-x) sin(n*pi*x) dx

  For odd n: B_n = 8/(n^3*pi^3)
  For even n: B_n = 0

  Comparison at x = 0.5:
         t    Numerical   Analytical        Error
  --------------------------------------------------
      0.00     0.240000     0.250000     1.00e-02
      0.20     0.029453     0.035841     6.39e-03
      0.40     0.003538     0.004979     1.44e-03
      0.60     0.000425     0.000692     2.67e-04
      0.80     0.000051     0.000096     4.50e-05
      1.00     0.000006     0.000013     7.22e-06
```

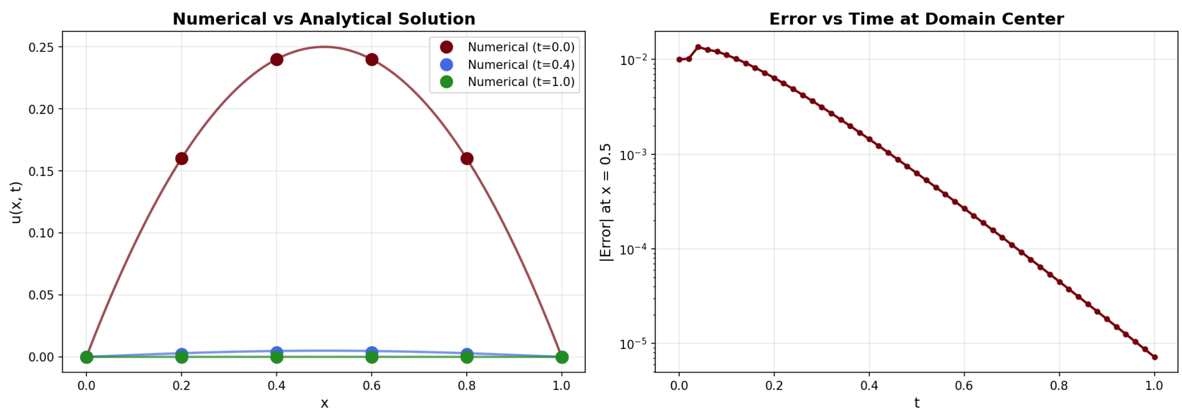


Figure 6: Left: Comparison of numerical (markers) and analytical (lines) solutions at different times. Right: Error decay at the center point $x = 0.5$.