Amplitude damping channel

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August 5, 2023

1 One qubit

The amplitude damping channel (ADC) is defined by the map [1]

$$|00\rangle \to |00\rangle$$
$$|10\rangle \to \sqrt{1-p}|10\rangle + \sqrt{p}|01\rangle$$

applied on two qubits, the second one being the environment. If we we consider a state of one qubit given by

$$|\psi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle \tag{1}$$

then to apply the ADC we consider a state $|\psi 0\rangle$, where the second qubit belongs to the environment. The final state of the whole system would be

$$|\phi\rangle = \cos\theta|00\rangle + \sin\theta(\sqrt{1-p}|10\rangle + \sqrt{p}|01\rangle)$$

In general a circuit like 1 produces the ADC for any initial state of the qubit 0.

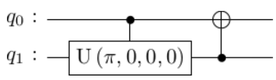

Figure 1: ADC for one qubit. [2]

Here the U gate is defined [3] by

$$U(\theta,\phi,\lambda) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -e^{i\lambda}\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) & e^{i(\phi+\lambda)}\cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

By using only the first parameter[1] of the U gate we are left with a rotation matrix. Given that the qubit 0 may be initialized as equation 1, applying the circuit 1 we are left with

$$|\phi\rangle = \cos\theta|00\rangle + \sin\theta[\cos(\theta/2)|10\rangle + \sin(\theta/2)|01\rangle]$$

where we must solve $\sin(\frac{\theta}{2}) = \sqrt{p}$ to apply a given value of p.

[1] The U gate on figure 1 shows a value of π but, in general, it may take any value. For all circuits in this papers this holds.

2 Two qubit

We may then consider a pure initial state of 2 qubits

$$|\psi_i\rangle = \cos(\theta)|01\rangle + \sin(\theta)|10\rangle$$

where the first qubit on the bracket corresponds to qubit 0 and the second to qubit 1. None belong to the environment yet. Then we consider the evolution of two qubits under a ADC, each with its environment qubit, $|\psi 00\rangle$. We get

$$|\phi_f\rangle = \cos\theta[\sqrt{1-p}|0100\rangle + \sqrt{p}|0001\rangle] + \sin\theta[\sqrt{1-p}|1000\rangle + \sqrt{p}|0010\rangle]$$

To build a quantum circuit that outputs this state, we first, using qubit 0 and 1, build the state $|\psi_i\rangle$, and then using two more qubits apply a ADC to qubit 0 and 1. This way the environment of qubit 0 is qubit 2 and of qubit 1 is qubit 3. The first U gate prepares the initial state and the two remaining U gates, along with the CNOT gates, perform the ADC. Of the U gates only the first parameter is necessary.

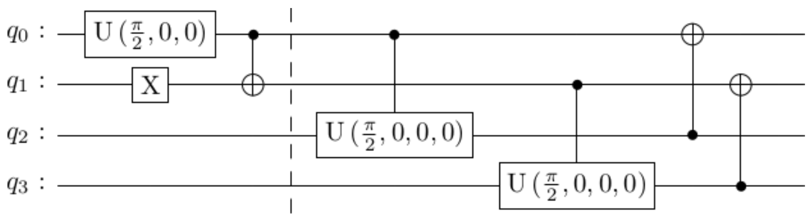

Figure 2: ADC for two qubit using two ancillas.

Using this circuit we seek to show how a change in p evolves a pure state to a mixed one. Using Qiskit quantum computer simulator we ran the circuit 2 with $\theta = \pi/4$, performed a tomography on qubits 0 and 1, and obtained the following density matrices for different values of p.

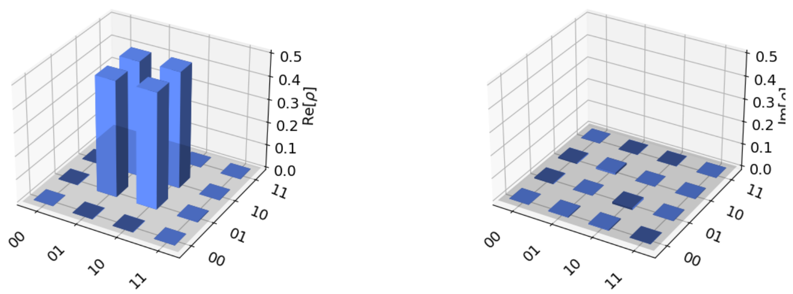

Figure 3: Final state for $p = 0$.

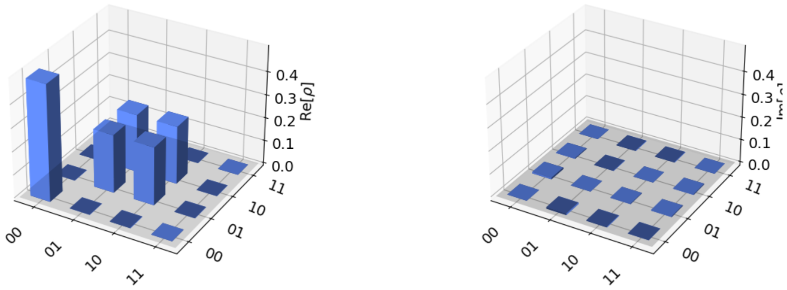

Figure 4: Final state for $p = 0.5$.

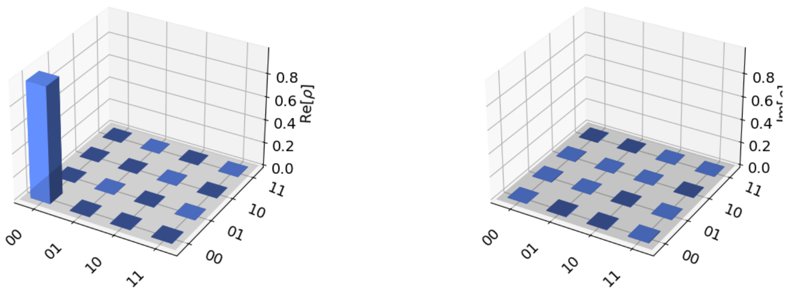

Figure 5: Final state for $p = 1$.

We may also show that the relationship [1]

$$\begin{aligned} C_{01}^2 + (P_0 \pm p)^2 &= (1-p)^2 \\ C_{01}^2 + (P_1 \pm p)^2 &= (1-p)^2 \end{aligned} \tag{2}$$

also holds.

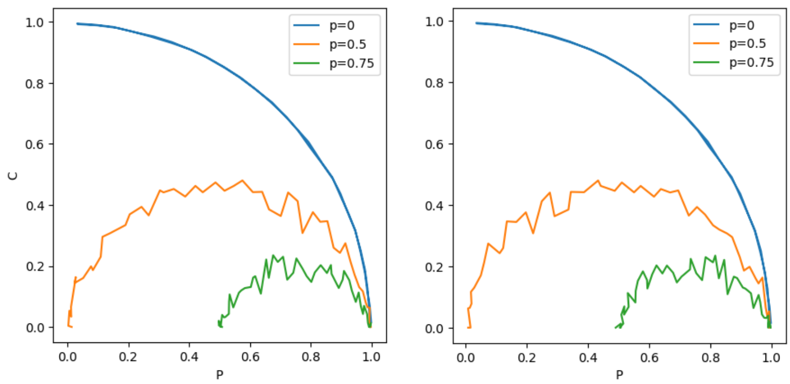

Figure 6: Concurrence and polarization measured for 50 values of θ ranging from 0 to π, and 3 values of p shown on the graphs. The first graph shows the relationship for P_0 and the second one for P_1.

3 Two qubits ADC purification

We may reduce the quantum circuit to just 3 qubits if we consider the purification of the final state. First, we must consider that initial state

$$|\psi_i\rangle = \cos(\theta)|01\rangle + \sin(\theta)|10\rangle$$

will evolve to

$$\rho_{01} = \mathcal{E}\left(\rho_{01}^{(0)}\right) = \sum_{\mu=0}^{1}\sum_{v=0}^{1} M_\mu^0 \otimes M_v^1 \left(\rho_{01}^{(0)}\right) M_\mu^{0^\dagger} \otimes M_v^{1^\dagger},$$

where $\rho_{01}^{(0)} = |\psi_i\rangle\langle\psi_i|$ and

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$

We are left with a final state

$$\rho_{01} = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & (1-p)\cos^2\theta & (1-p)\cos\theta\sin\theta & 0 \\ 0 & (1-p)\cos\theta\sin\theta & (1-p)\sin^2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now to purify this state [2] we must first express our state as

$$\rho^A = \sum_i p_i \left|i^A\right\rangle \left\langle i^A\right|$$

where each $|i^A\rangle$ forms an orthonormal basis. Then our pure state will be

$$|AR\rangle \equiv \sum_i \sqrt{p_i}\left|i^A\right\rangle \left|i^R\right\rangle$$

where $|i^R\rangle$ also forms an orthonormal basis belonging to system R.
Therefore we may express our state of interest as

$$\rho = p|00\rangle\langle 00| + (1-p)|\psi_i\rangle\langle\psi_i|$$

and its pure state will be

$$|\phi\rangle = \sqrt{p}|000\rangle + \sqrt{1-p}(\cos\theta|011\rangle + \sin\theta|101\rangle)$$

where the last qubit belongs to the environment. If we consider that $\cos\varphi = \sqrt{p}$ and $\sin\varphi = \sqrt{1-p}$ we find a circuit that produces the state $|\phi\rangle$.

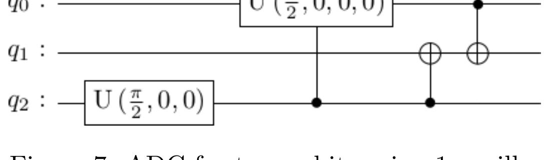

Figure 7: ADC for two qubits using 1 ancilla.

Likewise we have used only the first parameter of U gate to set the values of φ and θ. Running this circuit on Qiskit quantum computer simulator we observe the same effect on the final states for different values of p.

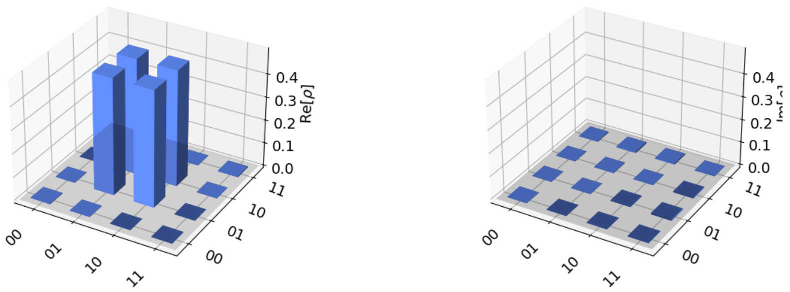

Figure 8: Final state for $p = 0$.

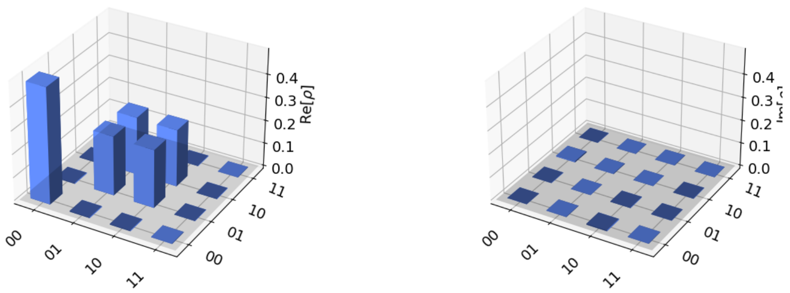

Figure 9: Final state for $p = 0.5$.

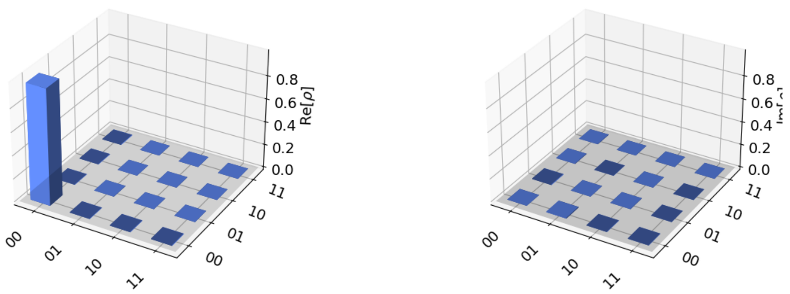

Figure 10: Final state for $p = 1$.

We also see that the equations on 2 hold.

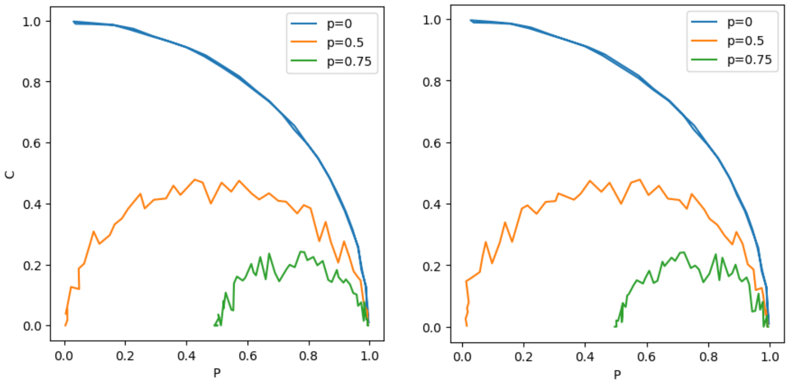

Figure 11: Concurrence and polarization measured for 50 values of θ ranging from 0 to π, and 3 values of p shown on the graphs. The first graph shows the relationship for P_0 and the second one for P_1.

References

[1] Y. Yugra, C. Montenegro, and F. De Zela, "Constraints between concurrence and polarization for mixed states subjected to open system dynamics," *Physical Review A*, vol. 105, p. 063710, June 2022.

[2] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information*. Cambridge ; New York: Cambridge University Press, 10th anniversary ed ed., 2010.

[3] Qiskit contributors, "Qiskit: An open-source framework for quantum computing," 2023.