Coordinating Monetary Contributions in Participatory Budgeting

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Abstract

We formalize a framework for coordinating funding and selecting projects, the costs of which are shared among agents with quasi-linear utility functions and individual budgets. Our model contains the classical discrete participatory budgeting model as a special case, while capturing other well-motivated problems. We propose several important axioms and objectives and study how well they can be simultaneously satisfied. One of our main results is that whereas welfare maximization admits an FPTAS, welfare maximization subject to a well-motivated and very weak participation requirement leads to a strong inapproximability. We show that this result is bypassed if we consider some natural restricted valuations. Our analysis for one of these restrictions leads to the discovery of a new class of tractable instances for the Set Union Knapsack problem, a classical problem in combinatorial optimization. Lastly, we evaluate a greedy heuristic for the problem using both real and synthetic datasets.

1 Introduction

Participatory budgeting (PB) is an exciting grassroots democratic paradigm in which members of a community collectively decide on which relevant public projects should be funded (see, e.g., [7, 38, 42]). The funding decisions take into account the preferences and valuations of the members.

The positive influence of PB is apparent in the many implementations across the globe at the country, city and community level. For example, a PB scheme in the Govanhill area of Glasgow, Scotland empowered local residents to direct funds towards imperative projects like addiction family support groups, a community justice partnership, and refurbishment of locally significant public baths.[1] In the 2014-15 New York City PB process, 51,000 people voted to fund $32 million of

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neighbourhood improvements.[2] As both examples demonstrate, efficient use of public funds and a significant improvement in community member involvement are two of the foremost advantages of PB processes around the world.

In the above examples, and in fact all implementations of PB that we are aware of, the process relies upon a central authority to determine and provide a budget to fund projects. In practice, this requirement excludes groups who lack the institutional structure required to pool resources from initiating a PB process. For example, several neighbouring municipalities may wish to collaborate on funding infrastructure projects which can benefit residents from multiple communities simultaneously. On a smaller scale, a group of friends may need to decide which furnishings and appliances to buy for their shared apartment. Or, a number of organisations co-hosting an event may need to agree upon a list of speakers, each of which charge their own fee. In each of these cases, while PB seems a natural process of arriving at a mutually beneficial outcome, the classical discrete PB model is insufficient because PB's focus is on the selection of projects and not on efficient resource pooling. In this paper, we present a framework which captures both of these components simultaneously.

Furthermore, in the classical PB model, it is typical to assume that agents' utilities depend only on their valuations for selected projects and that agents are indifferent toward the amount of the budget used. While this is appropriate in that setting because the agents do not necessarily believe leftover budget will benefit them, this is not the case in our setting since agents can use their leftover funds directly. For this reason, we model agents with utilities dependent upon their contributions (i.e. *quasi-linear utilities*).

In summary, we consider a flexible and general framework which we term *PB with Resource Pooling* in which (i) agents can have their own budgets and money from their budget can be directed to fund desirable projects and (ii) agents care both about which projects are funded, and how much monetary contribution they make due to quasi-linear utilities. We point out that our framework captures the classical discrete PB model without the above features in the following way: an artificial altruistic agent can be introduced who provides money equivalent to the central budget and is happy to fund as many projects as needed.

Taking cues from what we see as the key successes of PB implementations, we primarily focus on the *efficiency* of the outcome while ensuring each involved agent benefits from the outcome and is thus incentivized to participate in the process. Towards this, we study the possibility of achieving optimal *utilitarian welfare* alongside a very basic participation notion, namely *weak participation*, that guarantees positive utility to all of the agents involved.

Contributions Our first contribution is a meaningful model of collective decision making that connects various problems including participatory budgeting, cost sharing problems, and crowdsourcing. The model can also be viewed as a bridge between voting problems and mechanism design with money.

We lay the groundwork for axiomatic research for the problem by formalizing meaningful axioms that capture efficiency and participation incentives. In particular, to capture a PB program which is both useful and sustainable, we focus our study on mechanisms that satisfy *weak participation*, which requires that every agent benefits from participation.

We show that whereas welfare maximization admits a *fully polynomial-time approximation*

[2] hudexchange.info/programs/participatory-budgeting

scheme (FPTAS)[3], welfare maximization subject to weak participation leads to a strong inapproximability result for as few as two agents (Theorem 2). We then show that the same objective is inapproximable even for the case with identical project costs despite there being an exact, polynomial-time algorithm for welfare maximization.

We give an FPTAS for welfare maximization subject to participation in the single-minded setting with laminar demand sets, drawing upon a result from the *knapsack with conflict graphs* problem (Theorem 4). We use the insight from this argument to reveal a tractable case for the *set-union knapsack problem*, making a novel contribution to a well-studied generalization of the classical knapsack problem (Appendix B). We also present a polynomial-time algorithm for the natural class of valuations called *symmetric valuations*. Finally, we demonstrate through varied simulations that a greedy approach achieves close to optimal welfare on an average basis even when subjected to our participation requirement. Our results are summarized in Table 1.

Table 1: Summary of our computational results

	Welfare Maximization	Welfare Maximization subject to Weak Participation
General case	NP-hard; FPTAS	Inapproximable for any $n \geq 2$
Identical costs	Polynomial-time exact	Inapproximable
Laminar Single-minded valuations	NP-hard	FPTAS
Symmetric valuations	Polynomial-time exact	Polynomial-time exact

* Inapproximability results hold assuming $P \neq NP$.

2 Related Work

Discrete Participatory Budgeting. As mentioned, the model we formalize has some connections with discrete PB models [7, 6, 5, 20, 22, 40, 28] in which agents do not have their own budgets and contributions. Benade et al. [8] and Lu and Boutilier [31] study various aspects of preference elicitation in PB. These models are more general than multi-winner voting [19]. The special case of our model with identical project costs has some connections to multi-winner elections with a variable number of winners [29], with increased generality due to individual budgets and quasi-linear utilities.

Hershkowitz et al. [26] considered a PB setting in which PB is done at a city level but with each district having its own budget and preferences. When districts are viewed as agents, the model is closely related to ours, but we focus on different utility assumptions and axioms. The authors consider an approval-based setting, where the social welfare is the total number of votes selected projects have received irrespective of costs. In our model, the agents/districts care how much of their own funding is used up, which more realistically captures the trade-offs inherent in budgeting decisions made by agents with limited resources.

Funding Public Goods/Projects. The research on funding publicly availed projects is not limited to PB. Brandl et al. [11] considered a simple model without costs and without quasi-linear utilities.

[3] An algorithm which approximates the optimal solution by a factor of at least $1 - \epsilon$ in time polynomial in the instance size and ϵ for any $\epsilon > 0$.

The model is more suitable for making donations to long-term projects. Similarly, Wagner and Meir [43] considered projects without fixed value and cost, where the value obtained is proportional to contribution. Unlike our model, they assume a utility model where the value obtained from a project is scaled by normalized agent preferences. The authors employ the VCG mechanism to elicit agent preferences truthfully and each agent donates a fixed value (VCG payment) that is collectively decided. Aziz and Ganguly [4] considered a model that allows for personal contributions and costs. However, their perspective is that of charitable coordination without the quasi-linear utility assumption. Buterin et al. [13] present a mechanism for the divisible public goods problem which they argue effectively avoids the "free-rider" problem. Although their model also considers quasi-linear utilities, projects the divisible setting differs for ours significantly and agents do not have budgets.

Cost-sharing/Crowdfunding. PB without budget-constrained agents overlaps significantly with the setting of Cost Sharing Mechanisms (CSM) [32, 17]. Specifically, it can be equated to CSM with non-rivalrous projects, i.e. each project has a fixed cost [35, 44]. The focus is on designing truthful and efficient mechanisms for sharing the cost of availing a certain service or project among the members. The agents do not have budget restrictions. Other research considers truthful mechanisms for combinatorial cost sharing across multiple projects [10, 18]. Birmpas et al. [10] focused on symmetric submodular valuations and provided an Iterative Ascending Cost Sharing Mechanism (IACSM) which is H_n-approximate to social cost.[4] The social cost considered is equivalent to the quasi-linear utility that we consider in our setting. Despite the similarities, these works focus on agents without budget restrictions.

Utility-based PB bears resemblance to certain civic crowdfunding models [15, 46, 47]. However, these papers consider neither budget constraints nor the multiple project case. In this line of work, the goal is to analyze what happens at equilibrium when the agents are strategic, as opposed to our own goal of determining a welfare maximizing subset of projects to fund provided budget constraints. We also introduce a natural intuitive criterion - weak participation - to be satisfied.

Single-minded settings. The special case of our model with single-minded valuations is inspired by the single-minded agents in combinatorial auctions, mechanism design, and fair division [1, 14, 16, 3, 12]. Birmpas et al. [9] study socal welfare maximization in cost-sharing settings with single-minded buyers. However, their model interprets valuations as the willingness to pay and thus does not differentiate between valuations and budgets. Although welfare maximization subject to weak participation bears resemblance to the *set-union knapsack problem* [23, 34, 2] in the single-minded case, none of the existing results from this literature imply ours.

3 Preliminaries

In this work, we provide a framework for collective funding decisions in which agents derive quasi-linear utility and lack an exogenously defined shared budget. We assume agents instead have individual budgets, along with valuations, both of which are *public information*. The setting, more formally, has the following components.

[4] $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

3.1 Model

An instance of *PB with Resource Pooling* contains a set of participating agents $N = [n]$ and a set of projects, $M = [m]$. Each project j has associated cost C_j and we denote $\mathbf{C} = (C_j)_{j \in M}$. Each agent i has valuation $v_i : 2^M \to \mathbb{R}_{\geq 0}$ and a budget b_i which is the maximum amount they are willing to contribute. We denote $\mathbf{v} = (v_i)_{i \in N}$ and $\mathbf{b} = (b_i)_{i \in N}$. We assume v_i to be monotonic, that is $v_i(S) \leq v_i(T), S \subseteq T \subseteq M$. The valuation towards a single project j, i.e., $v_i(\{j\})$ is denoted by v_{ij} for ease. Without loss of generality, we assume that the projects included are such that their overall valuation is higher than the cost. Formally,

$$\sum_{i \in N} v_{ij} \geq C_j, \ \forall j \in M \tag{1}$$

We will consider some important restrictions on valuations of agents. Agents are said to have *additive valuations* when for any $S \subseteq M$, $v_i(S) = \sum_{j \in S} v_{ij}$. Section 4 focuses on broad inapproximability results that hold for additive valuations. We also consider restricted settings with *single-minded valuations* and *symmetric valuations*, which will be covered in detail in Sections 5 and 6, respectively.

In summary, a PB with Resource Pooling instance is given by $I = \langle N, M, \mathbf{v}, \mathbf{b}, \mathbf{C} \rangle$. A mechanism $\mathscr{M}$ takes an instance I and computes an outcome $\mathscr{M}(I) = (W, \mathbf{x})$. We close with two illustrative motivating examples, the first of which assumes additive valuations.

	Budget \ Cost	Auditorium	Shelter	Pool
		5	4	2
Town A	2	2	1	2
Town B	3	1	2	2
Town C	1	4	3	1

Table 2: An example with 3 projects and 3 agents.

Example 1. *Three neighboring towns have performed cost-benefit analyses for three proposed projects to be covered by the remainder of their annual budget, namely an auditorium for large events, a homeless shelter, and a community swimming pool. The town budgets, the project costs, and the town valuations for each project are given in Table 2. While Town A and Town B have the funds to pay for the pool on their own, they each only value it at its cost and will receive zero utility from doing so. Furthermore, no town has the required budget to fund either of the other two projects on its own. Now observe that, together, the three towns have just enough funds to build the shelter and the pool, the result of which will be a strictly positive utility for each of the three towns. Lastly, we point out that, while the towns could instead coordinate to fund the auditorium and the sum of their values will strictly exceed the cost, Town B will be required to pay at least 2, but its valuation for the auditorium is only 1 and it will thus receive negative utility.*

Example 1 shows the usefulness of our framework in the traditional setting of PB - the allocation of government budgets. The following example shows that our framework has broader appeal and can serve any group with common interests that lacks the institutional structure required for a common budget.

Example 2. *A group of friends move into a shared house and want to buy various items for communal use. They have drafted a list of items that various members of the household would be interested in sharing, such as a TV for the living room, an espresso machine for the kitchen, and a pool table for the lounge. Because no single friend necessarily values any of the items enough to buy it on their own, they decide to initiate an instance of PB with Resource Pooling. The outcome will be (i) a bundle of items to buy for the house and (ii) a quantity to be paid by each friend to cover the costs of the items. Each friend will receive utility equal to their valuation for the items selected less their required payment.*

3.2 Desirable Axioms

We now formalize properties we would like any mechanism to satisfy. These properties are related to budget balance, participation incentives, and efficiency.

Budget Balanced

We define a weakly budget balanced mechanism as one that always satisfies the basic feasibility criterion that the sum of collected payments is at least the cost of the selected projects.

Definition 1 (Weakly Budget Balanced (WBB))**.** *An outcome $(W, \mathbf{x})$ is Weakly Budget Balanced if $C(W) \leq \sum_{i \in N} x_i$. When $C(W) = \sum_{i \in N} x_i$, we say the outcome is Budget Balanced (BB). A mechanism $\mathscr{M}$ is WBB if for every $I \in \mathscr{I}$, $\mathscr{M}(I)$ is WBB.*

Participation

We now turn our attention to properties that incentivize agents to participate in the mechanism. Weak Participation (WP) ensures that each agent obtains non-negative utility from the mechanism.

Definition 2 (Weak Participation (WP))**.** *An outcome $(W, \mathbf{x})$ is WP if $\forall i \in N$, $u_i(W, \mathbf{x}) \geq 0$, i.e., $v_i(W) \geq x_i$. A mechanism $\mathscr{M}$ is WP if $\forall I \in \mathscr{I}$, $\mathscr{M}(I) = (W, \mathbf{x})$ satisfies WP.*

We note that WP is a minimal requirement and considerably weaker than Individual Rationality (IR), which ensures that the utility of each agent obtained from the mechanism is more than the maximum utility it can obtain solely with its own budget. A mechanism $\mathscr{M}$ is *IR* if $\forall I$, $u_i(\mathscr{M}(I)) \geq \max_{S \subseteq M}(v_i(S) - C(S))$, where $C(S) = \sum_{j \in S} C_j \leq b_i$.

In other words, whereas IR captures maximizing individual profit, WP ensures that no agent is making a loss. Whereas not funding any project gives a WP outcome, computing an IR outcome is NP-hard. Hence, to ensure there is incentive to participate, we will focus on desirable outcomes within the space of WP outcomes. The following lemma points out that the satisfaction of WP and WBB by a set of projects can be captured by a single inequality.

Lemma 1. *A set of projects $W \subseteq M$ can be funded in a WBB and WP manner if and only if*

$$\sum_{j \in W} C_j \leq \sum_{i \in N} \min(b_i, v_i(W))$$

Proof. A set of projects $W \subseteq M$ is WBB if $\sum_{j \in W} C_j \leq \sum_{i \in N} x_i \leq \sum_{i \in N} b_i$. The set of project is WP if $x_i \leq v_i(W), \forall i$, and thus $\sum_{j \in W} C_j \leq \sum_{i \in N} x_i \leq \sum_{i \in N} v_i(W)$. Hence if set of projects W satisfies both WBB and WP then $\sum_{j \in F} C_j \leq \sum_{i \in N} \min(b_i, v_i(F))$.

As for the other direction, suppose each agent contributes $x_i = \min(b_i, v_i(W))$. Then, the WP constraint is satisfied since $v_i(W) \geq x_i = \min(b_i, v_i(W))$. WBB is also satisfied since $\sum_{j \in W} C_j \leq \sum_{i \in N} \min(b_i, v_i(W)) = \sum_{i \in N} x_i$. $\square$

Efficiency

We measure the efficiency of the outcome with respect to the utilitarian welfare, i.e. the sum of utilities. This definition of efficiency is common in the cost sharing literature [33, 21]. Since we model quasi-linear utility for each agent, the utilitarian social welfare (which we refer to simply as social welfare) is given by

$$\mathrm{SW}(W, x) = \sum_{i \in N} u_i(W, x) = \sum_{i \in N} v_i(W) - \sum_{i \in N} x_i \tag{2}$$

The utilitarian welfare optimal (UWO) outcome is the social welfare maximizing outcome which is feasible, i.e. can be afforded by the sum of individual budgets.

Definition 3 (Utilitarian Welfare Optimal (UWO))**.** *An outcome $(W, \mathbf{x})$ is UWO if it maximizes social welfare as given by Equation 2, subject to budget constraints, i.e. $\sum_{j \in W} C_j \leq \sum_{i \in N} b_i$.*

We define UWO-WP to represent the welfare-optimal outcome among those which ensure WP and WBB.

Definition 4 (Welfare Optimal among Weak Participation (UWO-WP))**.** *An outcome $(W, \mathbf{x})$ is UWO-WP if it is welfare optimal among outcomes that ensure WP, i.e. $\forall i \in N, v_i(W) \geq x_i$, and WBB, i.e. $C(W) \leq \sum_{i \in N} x_i$.*

Remark 1. *For any outcome (W, x) which is WBB, there is an outcome which is BB, maintains WP, and achieves weakly greater social welfare.*

To see why Remark 1 is true, consider for any WBB outcome an alternative outcome with the same project selection, but with some set of agents' payments decreased such that the outcome is BB. By Remark 1 and our objectives stated in Definitions 3 and 4, we can restrict our attention to outcomes which are BB ($\sum_{i \in N} x_i = \sum_{j \in W} C_j$) and refer to social welfare from now on as the following:

$$\mathrm{SW}(W) = \sum_{i \in N} v_i(W) - \sum_{j \in W} C_j \tag{3}$$

In the next section, we prove that the problem of computing a utilitarian welfare optimal outcome subject to weak pariticpation (UWO-WP) is inapproximable under standard complexity theoretic assumptions.

4 Inapproximability of UWO-WP

In this section, we first show the hardness of finding the welfare-optimal project bundle subject to budget constraints i.e. UWO. Next, we show that the same objective with the additional constraint of WP, i.e. UWO-WP is not only NP-hard but also inapproximable. Our inapproximability result is striking: the problem of finding UWO admits an FPTAS but after imposing the very simple and weak requirement of WP, the same problem does not admit any polynomial approximation guarantees, even in the setting with only two agents.

The results in this section focus entirely on the setting with additive valuations. We point out that the results that follow also hold in any broader class of valuations which contains additive valuations (e.g. superadditive, subadditive, general setting). We first make some remarks on computation of UWO, that is welfare maximization subject to budget restrictions (Definition 3). Due to Equation 3, the formulation of the UWO problem is given by

$$\max_{W \subseteq M} \quad \sum_{i \in N} v_i(W) - \sum_{j \in W} C_j$$
$$s.t \sum_{j \in W} C_j \leq \sum_{i \in N} b_i.$$

Remark 2. *The UWO problem is NP-hard even for a single agent. However, in the setting with additive valuations, UWO can be approximated to any specified degree, i.e., it has an FPTAS [30].*

We defer the proof of the above remark to the Appendix. Despite the positive approximation results, approximation guarantees to UWO may be possible only with huge costs to certain agents who incur large disutility, thus motivating the search for outcomes which satisfy WP. We now point out that it is not possible to guarantee an outcome which is both UWO and WP.

Proposition 1. *It is impossible to design a mechanism $\mathscr{M}$ that satisfies both WP and UWO.*

Proof. Consider an instance with two agents and a single project with cost $C_1 = 1$. The valuations are $v_{11} = 2$ and $v_{21} = 0$, with budgets being $b_1 = 0$ and $b_2 = 1$. Observe that the UWO outcome is to fund the project. However, this cannot be done in a WP manner since the only agent with nonzero budget does not attribute any value to the project. □

Given the impossibility result stated in Proposition 1, we are interested in finding solutions which are welfare optimal amongst those which satisfy WP, i.e. that are UWO-WP (Definition 4), for additive valuations. The following remark follows from the reduction to prove NP-hardness of UWO in Remark 2.

Remark 3. *Finding a UWO-WP allocation is NP-hard, even for one agent.*

4.1 Gap Introducing Reduction for UWO-WP

As finding UWO-WP is NP-hard, we look for an approximation algorithm. To see why finding such an approximation algorithm will be difficult, consider the following illustrative example.

		Proj 1	Proj 2	Proj 3	Proj 4
	Cost / Budget	1	2	1	1
Agent 1	2	0	20	2-ϵ	2
Agent 2	0	H	0	20	0

Table 3: An instance with $n = 2$ and $m = 4$ where the UWO-WP outcome must find the unique set of projects (Proj 4) which allows the feasible funding of a high-welfare project (Proj 1).

Example 3. *Observe that in the example given in Table 3, agent 1 holds the entire collective budget and maximizes her utility by funding project 2, which can be funded in a WP manner. But, no budget remains to fund other projects if project 2 is selected. Project 3, on the other hand, gives strictly positive utility to both agents and leaves enough leftover budget to fund project 4 in a WP manner (but not project 1).*

However, note that $W = \{1, 4\}$ can be funded in a WP and WBB manner and achieves $SW(W) = H$. Although project 4 seems inferior to projects 2 or 3 on its own, it must be selected in order to fund project 1. Furthermore, since H can be arbitrarily high, any algorithm which hopes to give a good approximation of UWO-WP should be able to find the set of projects that makes project 1 affordable if such a project set exists.

Inspired by Example 3, where excess money must be extracted optimally to fund high value projects, we introduce a related problem.

MAXIMUM EXCESS PAYMENT EXTRACTION (MaxPE):

$$\max_{W \subseteq M} \mathrm{PE}(W) = \sum_{i \in N} \min(b_i, v_i(W)) - \sum_{j \in W} C_j$$

The above quantity measures the amount of excess money we can extract after paying for the costs of the projects provided that the agents are WBB and WP. The decision version of the MaxPE problem asks if $\mathrm{MaxPE} \geq t$, that is whether t dollars can be extracted while maintaining WBB and WP. Note that the decision version of the problem can be used to compute MaxPE simply by using binary search. The following theorem states that any polynomial approximation for UWO-WP can be used to solve the MaxPE decision problem in polynomial time.

Theorem 1. *Let $f(n, m)$ be a polynomial time computable function. Any polynomial time algorithm with $f(n, m)$-approximation guarantee for UWO-WP can be used to decide* MaxPE *in polynomial time.*

Proof. Consider an arbitrary instance $I = \langle N, M, \mathbf{v}, \mathbf{b}, \mathbf{C} \rangle$ of the $\mathrm{MaxPE} \geq t$ problem where there are n agents and m projects. Let $\mathrm{Opt}(I)$ be the social welfare obtained from the UWO-WP solution to instance I. We know that $\mathrm{Opt}(I) \leq \sum_{i \in N} v_i(M)$. We now create a new instance by adding an agent 0 and a project 0 to I such that:

- $v_{00} = (2f(n+1, m+1)) \sum_{i \in N} v_i(M)$, budget $b_0 = 0$
- $v_{0j} = 0$ for all $j \in M$

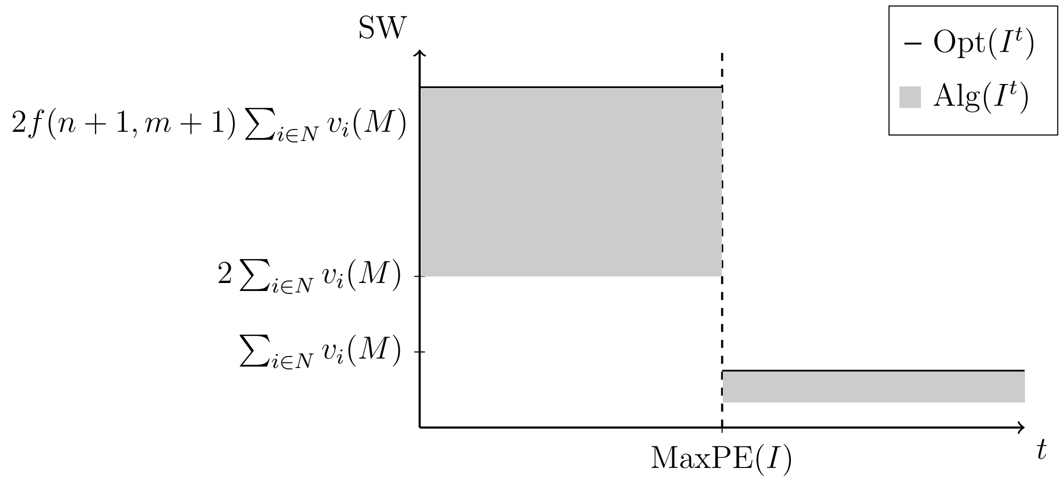

Figure 1: The modified instance I^t creates a large gap in optimal social welfare, depending on t, as shown by the solid lines. This welfare gap will remain for any f-approximation algorithm Alg, as shown by the shaded regions, and illustrates how such an approximation can be used to solve the MaxPE problem.

- $v_{i0} = 0$ for all $i \in N$
- $C_0 = t$

We denote the transformed instance as I^t. It has $n+1$ agents and $m+1$ projects. Note that the transformation is polynomial time as $f(n, m)$ is polynomial-time computable. The modified instance creates a large gap in the optimal social welfare depending on the cost of project 0, as seen in Figure 1. As the figure illustrates, any f-approximation algorithm Alg will maintain such a gap and can thus be used to retrieve information about MaxPE. As we will see, this observation can be used to show approximation hardness.

Consider now a polynomial time algorithm Alg for **UWO-WP** with f-approximation guarantee, that is

$$f(n+1, m+1)\,\mathrm{SW}(\mathrm{Alg}(I^t)) \geq \mathrm{Opt}(I^t). \tag{4}$$

We now claim that $\mathrm{MaxPE}(I) \geq t$ accepts (i.e., t dollars can be extracted from agents in I while satisfying WBB and WP) if and only if $\mathrm{SW}(\mathrm{Alg}(I^t)) \geq 2\sum_{i\in N} v_i(M)$. The forward direction follows since if t dollars can be extracted from agents in I then project 0 can be afforded. Hence, $\mathrm{Opt}(I^t) \geq (2f(n+1, m+1)\sum_{i\in N} v_i(M)$ since there is a solution that satisfies WBB and WP which also funds project zero. Thus combining with inequality (4) we see that $\mathrm{SW}(\mathrm{Alg}(I^t)) \geq 2\sum_{i\in N} v_i(M)$.

As for the backward direction, first observe that if $\mathrm{SW}(\mathrm{Alg}(I^t)) \geq 2\sum_{i\in N} v_i(M)$ then $0 \in \mathrm{Alg}(I^t)$. To see this, suppose on the contrary $0 \notin \mathrm{Alg}(I^t)$. Then, $\mathrm{SW}(\mathrm{Alg}(I^t)) \leq \mathrm{Opt}(I) \leq \sum_{i\in N} v_i(M)$, leading to a contradiction. Thus, we have $0 \in \mathrm{Alg}(I^t)$, which implies that the agents in I must be able to pay for project 0 in a WP and WBB manner. This is due to the fact that agent 0 cannot contribute any money towards funding project 0 since she has budget equal to zero, and agents in I do not value project 0. Hence, t dollars can be extracted from agents in I in a WP and WBB manner. Thus, we may use Alg to decide $\mathrm{MaxPE} \geq t$ in polynomial time. $\square$

In the subsections that follow, we use Theorem 1 to show that unlike UWO, which admits an FPTAS, UWO-WP does not admit *any* bounded polynomial approximation guarantees in the additive setting, even if we limit the number of agents to two or restrict the costs to be identical.

4.2 Two Agents with Additive Valuations

As we will show in Lemma 2, MaxPE is NP-hard even for a single agent with additive valuation. The following theorem uses this result to demonstrate a strong inapproximability of UWO-WP in the additive setting, even in the very restricted case with two agents.

Theorem 2. *Let $f(m)$ be a polynomial time computable function. Even for two agents, there is no polynomial time $f(m)$-approximation algorithm for the UWO-WP problem, assuming $P \neq NP$.*

Proof. Suppose, on the contrary, that a polynomial time algorithm exists with approximation guarantee $f(m)$ for UWO-WP with $n = 2$. By the argument used to prove Theorem 1, we see that such an algorithm can be used to solve MaxPE for one agent in polynomial time. However, as we will show in Lemma 2, MaxPE is NP-hard even for one agent, contradicting $P \neq NP$. □

Lemma 2. *It is NP-hard to compute* MaxPE*, even for a single agent with additive valuation.*

Proof. We give a reduction from the NP-complete $\mathrm{PARTITION}$ problem: given integers $a_1, \cdots, a_m$, the problem asks if there exists a subset S such that $\sum_{i \in S} a_i = \frac{1}{2}\sum_{i\in M} a_i$? For ease of notation, we denote $\gamma = \frac{1}{2}\sum_{i \in M} a_i$. Given any instance of the partition problem, we construct an instance of $\mathrm{MaxPE} \geq t$ with a single agent such that $v_{1i} = a_i$ for each $i \in M$, $b_1 = \gamma$, $C_i = \frac{a_i}{2}$, and $t = \frac{\gamma}{2}$.

The $\mathrm{MaxPE} \geq t$ decision problem for this instance is then

$$\max_{W \subseteq M} \ \min\left(\gamma, \sum_{i \in W} a_i\right) - \sum_{i \in W} \frac{a_i}{2} \geq t$$

If there exists a set S^* such that $\sum_{i \in S^*} a_i = \gamma$, then the corresponding $\mathrm{MaxPE} \geq t$ instance is a `Yes` instance. This holds since

$$\min\left(\gamma, \sum_{i \in S^*} a_i\right) - \sum_{i \in S*} \frac{a_i}{2} \ = \ \gamma - \frac{\gamma}{2} = \frac{\gamma}{2} = t$$

On the other hand, if for any $S \subseteq \{1, \cdots, m\}$ we have $\sum_{i \in S} a_i \neq \gamma$, then the corresponding instance of $\mathrm{MaxPE} \geq t$ is a `No` instance. This can be seen by the following observation:

- If $\sum_{i \in S} a_i < \gamma$, then $\min(\gamma, \sum_{i \in S} a_i) - \sum_{i \in S} \frac{a_i}{2} = \frac{1}{2}\sum_{i \in S} a_i < \frac{\gamma}{2} = t$
- If $\sum_{i \in S} a_i > \gamma$, then $\min(\gamma, \sum_{i \in S} a_i) - \sum_{i \in S} \frac{a_i}{2} = \gamma - \frac{1}{2}\sum_{i \in S} a_i < \frac{\gamma}{2} = t$

Thus, we see that a solution to the $\mathrm{PARTITION}$ problem exists if and only if the corresponding $\mathrm{MaxPE} \geq t$ instance has a valid solution. □

Remark 4. *For arbitrary number of agents $n \geq 2$, Theorem 2 can be restated so that no $f(n, m)$-approximation is possible for any polynomial time computable function $f(n, m)$. This is because* MaxPE *remains NP-hard for any number of agents and m is part of the input to* MaxPE*.*

Even though UWO admits an FPTAS, with the inclusion of the simple and intuitive constraint of WP, we obtain the above strong inapproximability results. Next, we will show that the inapproximability holds even if all projects costs are identical.

4.3 Identical Costs Setting

We consider a well-motivated special case where all projects have identical costs. This special case is meaningful in several scenarios, for example in the case where agents must decide the number and locations of units of some type of infrastructure such as public toilets. In this case, each project may have the same cost but agents won't necessarily value the projects the same due to other considerations such as location. Alternatively, a group of consultants forming a partnership may wish to choose a set of assistants to hire, but have varying available funds and preferences which depend on the alignment of a candidate's background with their own expertise.

More formally, in this setting we have $C_j = C, \forall j \in M$, for some constant C. We assume $C = 1$ for simplicity. We first note that the UWO allocation in the identical costs setting is computable in polynomial time. Specifically, we can compute UWO by sorting the projects according to their social welfare (the sum of agent valuations minus cost) and greedily selecting them until there is no remaining project with non-negative social welfare or until no budget remains.

Given UWO is polynomial time computable in the identical costs setting, and the same problem is NP-hard in the general setting, we may take this as an encouraging sign that a tractable algorithm for UWO-WP may exist in our restricted setting. However, as we will show, UWO-WP does not admit any polynomial approximation guarantees, even in this restricted setting. Our argument follows similarly to that used in the general setting. Intuitively, to be able to afford any single project in our setting, we must be able to identify a subset of projects that extracts a single dollar of excess payment when such a subset exists. Thus we begin by defining the decision version of the MaxPE problem with $t = 1$ ($\text{MaxPE} \geq 1$), which in this setting becomes:

$$\max_{W \subseteq M} \sum_{i \in N} \min(b_i, v_i(W)) - |W| \geq 1$$

Lemma 3. *It is NP-hard to decide* $\text{MaxPE} \geq 1$*, even with identical budgets.*

Proof. We give a reduction from the NP-complete Exact Cover by 3-Sets (X3C) problem: given a set $Z = \{z_1, \ldots, z_{3q}\}$ (with cardinality a multiple of 3) and a family S of 3-element subsets (triples) of Z, the problem asks if there is a subfamily $S' \subseteq S$ such that every element in Z is contained in exactly one triple of S'. Given any instance of X3C, we construct an instance of $\text{MaxPE} \geq 1$ such that we have:

- an agent $i \in N$ for each $z_i \in Z$, each with identical budget of $b_i = \frac{1}{3} + \frac{1}{n}$
- a project $j \in M$ for each triple $s_j \in S$
- agent valuations $v_{ij} = \frac{1}{3} + \frac{1}{n}$ if $z_i \in S_j$ and 0 otherwise.

The $\text{MaxPE} \geq 1$ decision problem becomes

$$\max_{W \subseteq M} \sum_{i \in N} \min\left(\frac{1}{3} + \frac{1}{n}, (\frac{1}{3} + \frac{1}{n}) \cdot |\{j \in W : z_i \in S_j\}|\right) - |W| \geq 1$$

Note that an agent $i \in N$ contributes 0 to the above summation if a bundle does not contain any project j for which $z_i \in S_j$ and they contribute exactly $\frac{1}{3} + \frac{1}{n}$ otherwise. Thus, the maximum this summation can evaluate to is $n \cdot (\frac{1}{3} + \frac{1}{n}) = \frac{n}{3} + 1$.

If there exists a subfamily $S' \subseteq S$ such that every element in Z is contained in exactly one triple of S', then the corresponding $\mathrm{MaxPE} \geq 1$ instance is a `Yes` instance. To see this, consider $W = \{j : s_j \in S'\}$. Note that $|W| = \frac{n}{3}$ since this is the number of triples required to cover a set of size n. By the transformation, each agent attributes non-zero value to exactly one project in W. Thus, the MaxPE expression becomes $n \cdot (\frac{1}{3} + \frac{1}{n}) - \frac{n}{3} = 1$.

In the other direction, if the $\mathrm{MaxPE} \geq 1$ instance is a `Yes` instance, let W be a bundle for which the excess payment extraction is weakly greater than 1. First note that $\sum_{i \in N} b_i = \frac{n}{3} + 1$ so $|W| \leq \frac{n}{3} + 1$ by WBB. However, if $|W| = \frac{n}{3} + 1$, then the MaxPE expression is upper bounded by $\frac{n}{3} + 1 - (\frac{n}{3} + 1) = 0$. If, on the other hand, $|W| < \frac{n}{3}$, because each project is valued by exactly 3 agents, the MaxPE expression is upper bounded by $3 \cdot |W|(\frac{1}{3} + \frac{1}{n}) - |W| = \frac{3 \cdot |W|}{n} < \frac{3(n/3)}{n} = 1$. Thus, $|W| = \frac{n}{3}$.

Consider the subfamily $S' = \{s_j : j \in W\}$. We wish to show S' constitutes an exact cover of Z. If it does not, it either does not cover Z or it contains two subsets which are not disjoint. In the former case, there is an element $z \in Z$ which is not contained in any triple in S' which means there is an agent in the $\mathrm{MaxPE} \geq 1$ instance whose total valuation is 0. This means the MaxPE expression is upper bounded by $(n-1)(\frac{1}{3} + \frac{1}{n}) - \frac{n}{3} = 2/3 - 1/n < 1$, which is a contradiction. In the latter case, there exist $s_j, s_k \in S'$ such that $s_j \cap s_k \neq \emptyset$, which means there exists an agent in the corresponding $\mathrm{MaxPE} \geq 1$ instance with non-zero valuations for both j and k. Because there are exactly 3 agents with non-zero valuations for each of the $n/3$ projects in W, this means there must be some agent who has zero valuation for every project in W and we reach a contradiction. Therefore, S' constitutes an exact cover of Z and the X3C instance is a `Yes` instance. $\square$

With the preceding lemma, the following inapproximability result follows from an analogous argument to that used in the proof of Theorem 2 and the justification of Remark 4.

Theorem 3. *Let $f(n,m)$ be a polynomial time computable function. Unless $P \neq NP$, there is no polynomial time $f(n,m)$-approximation algorithm for UWO-WP problem in the identical costs setting.*

Having found a strong inapproximability in various settings with additive valuations, we now study a different well-motivated setting and provide an approximation algorithm for UWO-WP.

5 An Algorithm for Laminar Single-minded Valuations

In this section, we study a setting in which each agent derives non-zero value from a project bundle if and only if the bundle contains the set of projects desired by that agent, which we refer to as the agent's *demand set*. We refer to this as the setting with *single-minded* valuations, following the precedent of single-minded bidders in the combinatorial auctions literature [1, 14]. Single-minded agents are also well-studied in the mechanism design literature [16, 3] and fair division literature [12]. In contrast to the additive setting, the single-minded setting captures project complementarities in agent valuations. Note that this setting does not encompass additive valuations and thus the results from Section 4 do not hold for single-minded valuations. We now define the single-minded setting formally.

Definition 5 (Single-minded Valuations)**.** *Agent valuations* $\mathbf{v} = (v_i)_{i \in N}$ *are single-minded if, for each* $i \in N$*, there exists* $D_i \subseteq M$ *and* $z_i \in \mathbb{R}_{\geq 0}$ *such that*

$$v_i(W) = \begin{cases} z_i & \textit{if } D_i \subseteq W \\ 0 & \textit{otherwise} \end{cases}$$

We refer to D_i *as agent* i*'s demand set.*

First note that for any set of agents with the same demand set, i.e. $D_i = T$ for all $i \in N' \subseteq N$ and some $T \subseteq M$, the problem is equivalent to the problem where N' is replaced by a single agent i' with $b_{i'} = \sum_{i \in N'} b_i$ and $v_{i'}(W) = \sum_{i \in N'} v_i(W)$ for all $W \subseteq M$. After carrying out this step for all such sets of agents, we are left with an equivalent instance with distinct demand sets, $\mathbf{D}$. Henceforth, we assume without loss of generality that all demand sets are distinct.

We refer to any set which maximizes the PE quantity, as defined in Section 4.1, as a MaxPE *set*. The following observation will be helpful in simplifying the forthcoming results.

Observation 1. *For any set of projects* $W \subseteq M$*,*

$$SW(W) = \sum_{i \in N} v_i(W) - \sum_{j \in W} C_j = \sum_{\{i \in N | D_i \subseteq W\}} z_i - \sum_{j \in W} C_j.$$

Similarly,

$$\mathrm{PE}(W) = \sum_{i \in N} \min(b_i, v_i(W)) - \sum_{j \in W} C_j = \sum_{\{i \in N | D_i \subseteq W\}} \min(b_i, z_i) - \sum_{j \in W} C_j.$$

In this section, we focus our attention on a particular restriction of the single-minded setting, namely that of laminar demand set families. We give an FPTAS for the problem with single-minded laminar valuations, a fortunate byproduct of which is a novel contribution to the set-union knapsack problem (see Appendix B for more detail). We begin by defining the laminar single-minded setting as that with *laminar single-minded valuations*.

Definition 6 (Laminar Single-minded Valuations)**.** *We define agent valuations* $\mathbf{v} = (v_i)_{i \in N}$ *to be laminar single-minded if they are single-minded and the demand set family* $\mathbf{D} = (D_i)_{i \in N}$ *is a laminar set family, i.e. if for every* $D_i, D_j \in \mathbf{D}$*, the intersection of* D_i *and* D_j *is either empty, or equals* D_i*, or equals* D_j*.*

The laminar single-minded case describes instances in which the project set is naturally partitioned into categories and sub-categories. Relating back to Example 2 where members of a houseshare are funding items, we might consider a scenario in which each housemate spends all of their time in shared space in a single room and thus only wants to buy items for that room (say the basement, lounge, and terrace). Further, those that prefer to spend time in the basement might be divided by whether they use the basement to play music or video games. In this case, the agents' valuations would comprise a laminar set family.

We next show that, despite the tractibility of MaxPE, which was NP-hard for a single agent in the additive setting, UWO and UWO-WP remain NP-hard, even in our restricted setting with laminar demand sets.

Remark 5. *Computation of UWO in the single-minded laminar setting is NP-hard as it can be reduced from* KNAPSACK.

The above remark follows from a straightforward reduction from KNAPSACK to an instance of our problem with disjoint (and thus, laminar) demand sets which maps knapsack items to agent demand sets. If we set the budgets of each agent equal to their valuation of their demand set, i.e. $b_i = z_i$ for all $i \in N$, we have that the UWO-WP outcome will also be UWO and thus the same reduction shows NP-hardness of UWO-WP in the single-minded laminar setting.

Given the hardness results, we search for an approximation of UWO-WP in the laminar single-minded setting. What follows is the central theorem of the present section, which provides an FPTAS for UWO-WP in the single-minded setting with laminar demand sets.

Theorem 4. *In the laminar single-minded setting, the UWO-WP problem admits an FPTAS.*

Before we prove this result, we state and prove three key lemmas used in the proof. The first lemma shows somewhat surprisingly that, despite being NP-hard even for a single agent in the additive setting, MaxPE is polynomial time solvable in the single-minded setting.

Lemma 4. *The* MaxPE *problem can be solved in polynomial time in the single-minded setting.*

Proof. Consider the following ILP:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{i \in N} \min(b_i, z_i) \cdot x_i - \sum_{j \in M} C_j \cdot y_j & \\ \text{subject to} & x_i \leq y_j, & \forall i \in N, \forall j \in D_i \\ & x_i \in \{0, 1\}, & \forall i \in N \\ & y_j \in \{0, 1\}, & \forall j \in M \end{array}$$

We claim that the above ILP describes our MaxPE problem. For any feasible solution to MaxPE, W, let $x_i = 1$ iff $D_i \subseteq W$ and $y_j = 1$ iff $j \in W$. It is apparent that the objective function of the ILP is equivalent to that of MaxPE. Furthermore, for each $i \in N$ with $x_i = 1$, we have $y_j = 1$ $\forall j \in D_i \subseteq W$. Thus, the constraints are satisfied and we have a corresponding feasible solution to our ILP.
Because $\min(b_i, z_i)$ is constant for a given agent i, it can be treated as a single variable, and our ILP is thus identical to that given by Birmpas et al. [9]. As shown by Birmpas et al. [9], the LP relaxation of this ILP yields a totally unimodular constraint matrix and thus the MaxPE problem is polynomial time solvable. □

The following lemma, which states that every MaxPE set is contained in at least one UWO-WP outcome, also holds for the single-minded setting in general.

Lemma 5. *In the single-minded setting, for any MaxPE set Q, there exists a UWO-WP outcome $W^\star$ such that $W^\star \supseteq Q$.*

Proof. Let Q be any MaxPE set and let W be any UWO-WP outcome. If $Q \subseteq W$, we are done. Assume $Q \not\subseteq W$. Then, there is a non-empty set $Q' = Q \backslash W$. We will now show that $W' = W \cup Q'$ is UWO-WP, which implies that Q is a subset of a UWO-WP outcome and concludes our proof.

First note that because Q is a MaxPE set, we have that $PE(Q) \geq PE(Q \setminus Q')$, which implies that

$$\sum_{\{i \in N | D_i \subseteq Q\}} \min(b_i, z_i) - \sum_{j \in Q} C_j \geq \sum_{\{i \in N | D_i \subseteq Q \setminus Q'\}} \min(b_i, z_i) - \sum_{j \in Q \setminus Q'} C_j \implies$$

$$\sum_{\{i \in N | D_i \subseteq Q\}} \min(b_i, z_i) - \sum_{\{i \in N | D_i \subseteq Q, D_i \cap Q' = \emptyset\}} \min(b_i, z_i) - \sum_{j \in Q'} C_j \geq 0 \implies$$

$$\sum_{\{i \in N | D_i \subseteq Q, D_i \cap Q' \neq \emptyset\}} \min(b_i, z_i) - \sum_{j \in Q'} C_j \geq 0 \tag{5}$$

We now show that W' is WP and WBB. Because W and Q' are disjoint and $W \supseteq Q \setminus Q'$, observe that

$$\sum_{\{i \in N | D_i \subseteq W'\}} \min(b_i, z_i) - \sum_{j \in W'} C_j$$

$$= \left[\sum_{\{i \in N | D_i \subseteq W', D_i \cap Q' = \emptyset\}} \min(b_i, z_i) - \sum_{j \in W} C_j \right] + \left[\sum_{\{i \in N | D_i \subseteq W', D_i \cap Q' \neq \emptyset\}} \min(b_i, z_i) - \sum_{j \in Q'} C_j \right]$$

$$\geq \left[\sum_{\{i \in N | D_i \subseteq W\}} \min(b_i, z_i) - \sum_{j \in W} C_j \right] + \left[\sum_{\{i \in N | D_i \subseteq Q, D_i \cap Q' \neq \emptyset\}} \min(b_i, z_i) - \sum_{j \in Q'} C_j \right] \geq 0$$

The final inequality follows from Equation 5 in conjunction with W being WP.

All that remains is to show that the addition of Q' to W weakly improves social welfare. By a similar argument as above, it follows that:

$$\mathrm{SW}(W') = \left[\sum_{\{i \in N | D_i \subseteq W\}} z_i - \sum_{j \in W} C_j \right] + \left[\sum_{\{i \in N | D_i \subseteq Q, D_i \cap Q' \neq \emptyset\}} z_i - \sum_{j \in Q'} C_j \right]$$

$$\geq \mathrm{SW}(W) + \left[\sum_{\{i \in N | D_i \subseteq Q, D_i \cap Q' \neq \emptyset\}} \min(b_i, z_i) - \sum_{j \in Q'} C_j \right] \geq \mathrm{SW}(W).$$

□

We construct an FPTAS in the laminar single-minded setting, by providing an approximation-preserving reduction [41] to another generalization of the classical knapsack problem called *knapsack on conflict graphs* (KCG) [36], which we now describe.

An instance of KCG is defined by a knapsack capacity C and an undirected graph $G = (V, E)$ referred to as a *conflict graph*, where each node $i \in V$ has an associated profit p_i and weight w_i. The KCG problem is then expressed by the following optimization problem:

$$\begin{aligned} \max_{S \subseteq V} \quad & \sum_{i \in S} p_i \\ \text{subject to} \quad & \sum_{i \in S} w_i \leq C \\ & |\{(i, j) \in E | i, j \in S\}| = 0 \end{aligned}$$

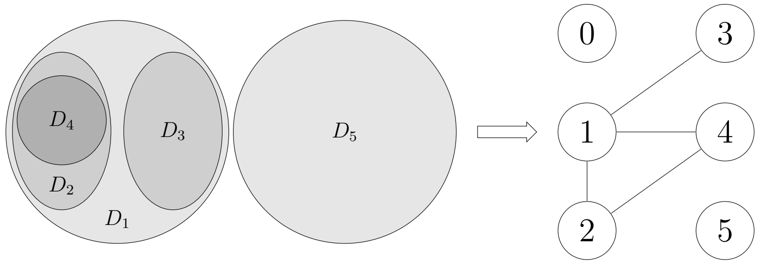

Figure 2: The mapping from an instance of the laminar single-minded UWO-WP problem to an instance of KCG as described in the proof of Theorem 4.

The following lemma states the conflict graph defined by a laminar set family is *chordal*, i.e. all cycles of four or more vertices have a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle.

Lemma 6. *Given a laminar set family* $\mathbf{D} = \{D_1, \cdots, D_n\}$*, the graph* $G = (V, E)$ *where* $V = [n]$ *and* $E = \{(i, j) | D_i \subseteq D_j\}$ *is chordal.*

Proof. Consider such a graph $G = (V, E)$. Let $E^c \subseteq E$ be a cycle of length four or more. Pick any edge $(i, j) \in E^c$ and assume without loss of generality that $D_i \subseteq D_j$. Now pick any edge $(i, k) \in E^c$, which must exist because E^c is a cycle. We have that $D_i \cap D_k \neq \emptyset$ by the construction of the graph and thus $D_k \cap D_j \neq \emptyset$. Since $\mathbf{D}$ is a laminar set family, this means that D_j and D_k form a subset relation, which implies $(j, k) \in E$. (j, k) is a chord since it connects two vertices in E^c and cannot be in E^c since it would form a triangle, contradicting E^c being a cycle of length four or more. $\square$

Proof of Theorem 4. Let $I = \langle N, M, \mathbf{v}, \mathbf{b}, \mathbf{C}\rangle$ be an instance of UWO-WP problem with laminar single-minded demand sets. Let Q be a MaxPE set computed by the polynomial time method given in the proof of Theorem 4 and let N' be the set of agents' whose demand sets are not in Q, i.e. $N' = \{i \in N | D_i \not\subseteq Q\}$. We know that a UWO-WP solution which contains Q is guaranteed to exist by Lemma 5. We proceed by reducing the problem of computing such a UWO-WP solution $W^\star \supseteq Q$ to an instance of KCG as follows:

- $V = \{0\} \cup N$
- $p_i = \begin{cases} \displaystyle\sum_{\{j \in N \setminus N'\}} z_j \quad - \quad \sum_{j \in Q} C_j & \text{if } i = 0 \\ \displaystyle\sum_{\{j \in N' | D_j \subseteq D_i \cup Q\}} z_j \quad - \quad \sum_{j \in D_i \setminus Q} C_j & \text{otherwise} \end{cases}$
- $w_i = \begin{cases} 0 & \text{if } i = 0 \\ \displaystyle\sum_{j \in D_i \setminus Q} C_j \quad - \quad \sum_{\{j \in N' | D_j \subseteq D_i\}} \min(b_j, z_j) & \text{otherwise} \end{cases}$
- $C = PE(Q)$
- $E = \{(i, j) | D_i \subseteq D_j, \forall i, j \in N\}$

See Figure 2 for an example of the mapping between an instance of UWO-WP to an instance of KCG. In words, each node in the conflict graph besides $\{0\}$ corresponds to an agent in the UWO-WP instance I. The knapsack capacity corresponds to the slack in our feasibility constraint brought out by the selection of the MaxPE set. Lastly, node $\{0\}$ corresponds to the agents whose demand sets are in the MaxPE outcome, i.e. $N \setminus N'$. Thus, since we seek a UWO-WP solution containing Q, this node provides profit equal to $\mathrm{SW}(Q)$ for zero cost and is not connected by an edge to any other node.

Let S be a feasible solution to the instance of KCG reduced from I, as given above, and denote $S^0 = S \setminus \{0\}$. We posit that $W = Q \cup \bigcup_{i \in S^0} D_i$ satisfies WP and WBB. Since S is a feasible solution to KCG, we know $\sum_{i \in S} w_i = \sum_{i \in S^0} w_i \leq C$, which we can write as

$$\sum_{i \in S^0} \left[\sum_{j \in D_i \setminus Q} C_j - \sum_{\{j \in N' | D_j \subseteq D_i\}} \min(b_j, z_j) \right] \leq \sum_{\{i \in N | D_i \subseteq Q\}} \min(b_i, z_i) - \sum_{j \in Q} C_j.$$

Since $\mathbf{D}$ is a laminar set family, we know that the sets are either disjoint or form a subset relation. However, since S is feasible, we know that no two members of $\{D_i\}_{i \in S}$ can form a subset relation and thus all members are disjoint. From this, we can also conclude that $\{j \in N' | D_j \subseteq D_{i_1}\} \cap \{j \in N' | D_j \subseteq D_{i_2}\} = \emptyset$ for all $i_1, i_2 \in S$. Thus, we can rewrite the above inequality as

$$\sum_{j \in \bigcup\limits_{i \in S^0} D_i \setminus Q} C_j - \sum_{\{j \in N' | D_j \subseteq \bigcup\limits_{i \in S^0} D_i\}} \min(b_j, z_j) \leq \sum_{\{i \in N | D_i \subseteq Q\}} \min(b_i, z_i) - \sum_{j \in Q} C_j$$

Now see that $\{j \in N' | D_j \subseteq \bigcup\limits_{i \in S^0} D_i\} = \{j \in N | D_j \subseteq \bigcup\limits_{i \in S^0} D_i, D_j \nsubseteq Q\}$ by the definition of N'. Thus, we have

$$\begin{aligned} 0 &\leq \sum_{\{i \in N | D_i \subseteq Q\}} \min(b_i, z_i) + \sum_{\{i \in N | D_i \nsubseteq Q, D_i \subseteq \bigcup\limits_{j \in S^0} D_j\}} \min(b_i, z_i) - \sum_{j \in Q \cup \bigcup\limits_{i \in S^0} D_i} C_j \\ &= \sum_{\{i \in N | D_i \subseteq Q \cup \bigcup\limits_{i \in S^0} D_i\}} \min(b_i, z_i) - \sum_{j \in Q \cup \bigcup\limits_{i \in S^0} D_i} C_j \end{aligned}$$

which means W is WP and WBB and thus feasible.

All that remains to complete our approximation-preserving reduction is to equate the objective values of corresponding solutions of our original and reduced instances. Since node $\{0\}$ has weight 0 and non-negative profit, we can restrict our attention to KCG solutions containing $\{0\}$ without loss of generality. We now show that the objective value of any feasible KCG solution, S, is equal to that of the resulting project bundle, W:

$$
\begin{aligned}
\sum_{i\in S} p_i = & \sum_{i\in S^0}\left[\sum_{\{j\in N'|D_j\subseteq D_i\cup Q\}} z_j - \sum_{j\in D_i\setminus Q} C_j\right] + \sum_{j\in N\setminus N'} z_j - \sum_{j\in Q} C_j \\
= & \sum_{\{i\in N'|D_i\subseteq Q\cup \bigcup_{j\in S^0} D_j\}} z_i + \sum_{\{i\in N|D_i\subseteq Q\}} z_i - \sum_{j\in \bigcup_{i\in S^0} D_i\setminus Q} C_j - \sum_{j\in Q} C_j \\
= & \sum_{\{i\in N|D_i\subseteq W\}} z_i - \sum_{j\in W} C_j \\
= & \ \mathrm{SW}(W)
\end{aligned}
$$

It is apparent that the reduction runs in polynomial time. Pferschy and Schauer [36] provide an FPTAS for KCG on chordal graphs. By Lemma 6, we know that G is a chordal graph. Thus, laminar single-minded UWO-WP admits an FPTAS. □

In the single-minded setting, the problem of welfare maximization resembles the well-studied *set union knapsack problem* (SUKP), which is hard to approximate in the general case [23, 34, 2]. Despite SUKP's resemblance to our problems of interest, approximation results from SUKP or its restricted cases, such as those described by Goldschmidt et al. [23], do not imply any of our results. However, an argument similar to that given to prove Theorem 4 reveals a new restricted setting in which SUKP admits an FPTAS, namely the restriction to laminar item sets. Since this problem is not the topic of the current research, we leave the theorem and proof of this finding, along with a formal description of SUKP, to Appendix B.

6 An Algorithm for Symmetric Valuations

Given that finding UWO-WP for additive valuations is inapproximable, we analyze the simpler valuation domain of symmetric valuations. We say a valuation function v_i is *symmetric* if it only depends on the size of the input, i.e $v_i(S) = v_i(T)$ whenever $|S| = |T|$. Intuitively, this domain corresponds to a setting where multiple units of the same project or item are available and agents' valuations are solely dependent on the quantity of items selected. Note that symmetric valuations are not necessarily additive. For instance, the valuations can be $v_i(S) = \sqrt{|S|}$ or $v_i(S) = e^{|S|} - 1$, which are submodular and supermodular valuations, respectively, and neither of which are additive.

Symmetric valuations are standard restrictions in the mechanism design literature. For example, Hoefer and Kesselheim [27] use symmetric valuations to capture "a spectrum auction of equally sized channels which all offer very similar conditions." Other research has investigated cost sharing mechanisms for symmetric valuations [10].

Theorem 5. *For symmetric valuations, Algorithm 1 finds UWO-WP in polynomial time.*

Proof. As outlined in Algorithm 1, we assume the projects are ordered by increasing cost. We denote the set of projects $\{1, ..., k\}$ by $[k]$ for simplicity of notation. Let W^* be the solution to the UWO-WP problem with $\ell = |W^*|$, and W be the set of projects returned by Algorithm 1. Note that $v_i(W^*) = v_i([\ell])$ for each $i \in N$, since the valuations are symmetric. We see that $[\ell]$ can be

Algorithm 1 Symmetric Valuation

Input: $\langle N, M, \mathbf{v}, \boldsymbol{b}, \mathbf{C}\rangle$ where valuations are symmetric

Order the projects by increasing cost, i.e $C_1 \leq C_2 \leq \cdots \leq C_m$

$\Omega = \emptyset$

for $k = 1$ to m **do**

  if $\sum_{i\in N} \min\{v_i([k]), b_i\} \geq C([k])$ **then**

    $\Omega \leftarrow \Omega \cup k$

$k^* \leftarrow \arg\max_{k\in\Omega} \mathrm{SW}([k])$

$x_i = \begin{cases} \min\{v_i([k^*]), b_i\}, & \text{if } \sum_{j=1}^{i} \min\{v_j([k^*]), b_j\} < C(W) \\ C([k^*]) - \sum_{j=1}^{i-1} \min\{v_j([k^*]), b_j\}, & \text{otherwise} \end{cases}$

return $W = [k^*]$ and $\mathbf{x} = (x_1, \cdots, x_n)$

funded in a WP and WBB manner since W^* can be funded in such a manner and the cost of W^* is weakly greater than that of $[\ell]$. From Lemma 1, it follows that $\sum_{i\in N} \min\{v_i([\ell]), b_i\} \geq C([\ell])$, and hence $\ell \in \Omega$. Thus we have,

$$\begin{aligned}
\mathrm{SW}(W) &\geq \mathrm{SW}([\ell]) \\
&= \sum_{i\in N} v_i([\ell]) - C([\ell]) \\
&= \sum_{i\in N} v_i(W^*) - C([\ell]) \\
&\geq \sum_{i\in N} v_i(W^*) - C(W^*) \\
&= \mathrm{SW}(W^*)
\end{aligned}$$

where the first inequality follows since $\ell \in \Omega$ and $W = [k^*]$ with $k^* = \arg\max_{k\in\Omega} \mathrm{SW}([k])$. Also note that for each $k \in \Omega$, the set of projects $[k]$ can be funded in a WP and WBB manner by Lemma 1, and thus W can also be funded in such a way. Hence, W is indeed a UWO-WP solution. □

Remark 6. *Algorithm 1 can be adapted to compute UWO-WP for the case when project costs are identical, i.e.* $\forall j, C_j = C$, *and agents' valuations follow identical rankings over the projects.*

7 Experiments

We know that UWO-WP is not just hard but inapproximable for general additive valuations. In this section, we use real PB elections and synthetically generated data to construct simulations evaluating how well a simple greedy algorithm approximates UWO-WP welfare.

In the algorithm we henceforth refer to as $greedy$, we sort the projects in descending order of bang per buck, i.e. welfare obtained per cost. We pick a new project in this order, provided it does not violate WP or WBB, and continue in this fashion until no such project remains. For each of our simulation instances, we evaluate $greedy$'s performance by calculating the *welfare ratio* between the greedy solution and the optimal solution, which is computed through brute force search, i.e. $\frac{\mathrm{SW}(greedy)}{\mathrm{SW}(\text{UWO-WP})}$.

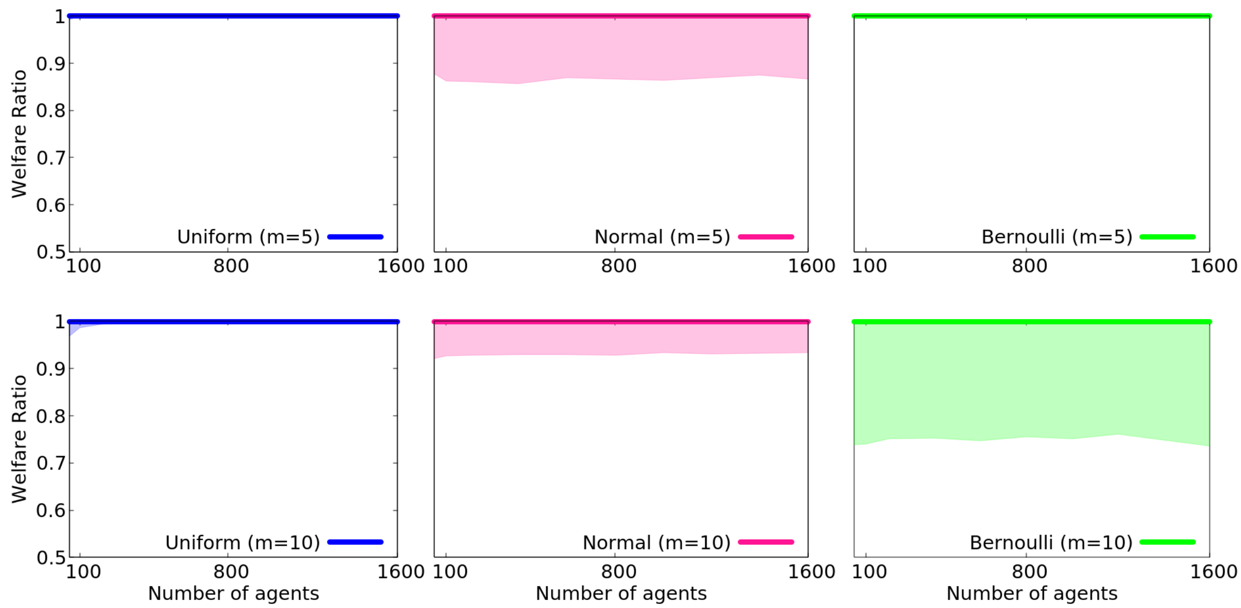

Figure 3: Welfare Ratio $\mathrm{SW}(greedy)/\mathrm{SW}(\textbf{UWO-WP})$ for $m = 5$ (top) and $m = 10$ (bottom). Solid line is the median welfare ratio across 10,000 instances; shaded region indicates the range of welfare ratio for 90% of the instances

Pabulib Experiments

We begin by constructing instances of our problem using real PB election data from Pabulib [39]. We convert all approval elections in the dataset to instances of our problem, restricting only to those elections with 20 projects or fewer due to computational constraints associated with calculating the optimal solution. To construct the instance, we distribute the election's budget evenly among all voters and assume that agents derive non-zero utility from a project if and only if they approve of that project in their ballot. We then scale all agents' utilities up so that the sum of all agents' utilities for the grand bundle is equal to the total cost of the grand bundle. We choose this scaling to avoid creating instances with trivial optima. In the absence of more detailed valuation information, we consider this assumption fair as trivial instances would be unlikely to warrant the initiation of a PB process to begin with.

For the 163 elections meeting the criteria laid out above, we find that the $greedy$ algorithm achieves a welfare ratio of greater than 0.98 in 50% of elections and achieves a welfare ratio of greater than 0.75 in 90% of elections.

Synthetic Experiments

Having found that $greedy$ performs quite well on data from real PB elections, we set out to test $greedy$ on a synthetic dataset with more varied input assumptions and a much larger sample size. The instances we use in our synthetic simulations are of the three following types:

1. For *uniform* instances, we sample agent valuations from a Uniform distribution, $v_{ij} \sim \mathscr{U}[0, 1]$.

2. For *normal* instances, we uniformly sample a mean and variance for each project to define a Normal distribution from which all agents' valuations are drawn for that project. That is, for each $j \in M$, we sample $\mu_j \sim \mathscr{U}[0,1]$ and $\sigma_j \sim \mathscr{U}[0,0.5]$, and draw $v_{ij} \sim \mathscr{N}(\mu_j, \sigma_j)$, shifting to ensure positive valuations.

3. For *bernoulli* instances, for each project j, we first uniformly sample a probability, $p_j \sim \mathscr{U}[0,1]$, and a constant, $\nu_j \sim \mathscr{U}[0,1]$. We then draw agents' valuations from a Bernoulli distribution with success probability p_j and multiply the result by ν_j, i.e. $v_{ij} \sim \nu_j \cdot \text{Bernoulli}(p_j)$.

For all instances, project costs are sampled uniformly, $C_j \sim \mathscr{U}[0.75 \cdot \sum_i v_{ij}, \sum_i v_{ij}]$. By ensuring costs are close to the sum of agents' valuations for each project, we make WP more difficult to satisfy and avoid trivial instances. The total budget is set to half of the total of project costs and individual budgets are fractions of this overall budget, drawn uniformly at random and normalized to sum to 1. The *uniform* instance family captures a setting where agents' valuations are perfectly random and not correlated with one another. Since this is not the case in most realistic settings, we use the *normal* instance type to capture scenarios in which there is a degree of consensus amongst all agents on the value of each project. Lastly, we study the *bernoulli* instance type, which captures approval-based utilities, a modeling assumption well-studied in PB [6, 40].

For each combination of instance family, $m = \{5, 10\}$, and n ranging from 10 to 1,600, we generate 10,000 synthetic instances. The results are reported in Figure 3 for each value of n and m, where the solid line is the median welfare ratio across all 10,000 instances and the shaded region below outlines the 10^{th} percentile welfare ratio across the same instances. Figure 3 shows that $greedy$ obtains optimal welfare (i.e. UWO-WP) in at least 50% of instances for each instance type and number of agents and projects. The shaded region of the charts shows that the 10^{th} percentile performance of $greedy$ drops significantly below the optimum for *normal* instances with $m = 5$ and *bernoulli* instances with $m = 10$. Nevertheless, even for these instances, we point out that $greedy$ achieves at least 70% of optimal welfare in 90% of instances.

Altogether, we observe that the $greedy$ algorithm achieves close-to-optimal performance in the vast majority of a wide variety of instances with reasonable input assumptions. Such performance from an algorithm as unsophisticated as $greedy$ comes as a notable surprise given our strong inapproximability results.

8 Discussion

In this paper, we laid the groundwork for an important and general PB framework which captures resource pooling. We focused on welfare maximization subject to minimal participation axioms that agents would expect the mechanism to satisfy and proved a number of hardness and inapproximability results. We also identified and gave explicit algorithms for two natural classes of instances.

We did not focus on strategic issues. There are existing results, which show that it is impossible to have mechanisms that are strictly budget balanced, efficient, and strategy-proof [24, 37]. Similar results are shown for cost sharing mechanisms [33]. Note that, in our case we already relax the notion of strict budget balanced to WBB and further we also consider efficient among WP and WBB outcomes. Despite the relaxations, one can see that WP and WBB in itself are incompatible with strategy-proofness. Consider an example in which the project 1 will be selected regardless of agent 1's valuation for it. Even if agent 1 has a value of 1, towards project 1, i.e., $v_{11} = 1$, the agent is incentivized to lie. Due to WP, the agent's payment is upper bounded by the agent's valuation for the selected bundle, so agent 1 is incentivized to report $v_{11} = 0$. Thus, strategy-proofness is incompatible with WP and WBB.

Our work leaves interesting open problems, and provides a framework for further exploration of PB with resource pooling. Specifically, while we showed an FPTAS for UWO-WP in the laminar single-minded setting, it remains open whether there exists a bounded approximation algorithm for UWO-WP in the general single-minded case. Furthermore, future work can focus on other restrictions which bypass our inapproximability. Lastly, while our participation axiom captures some notion of fairness, it would be interesting to explore other fairness concepts within our framework.

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A Proof of Remark 2

Remark 7. *The problem is NP-hard even for a single agent with additive valuation as it can be reduced from the NP-complete* KNAPSACK *problem. However, in the setting with additive valuations, UWO can be approximated to any specified degree, i.e., it has an FPTAS [30].*

Proof. To show NP-hardness of UWO with a single agent with additive valuation, we first reduce the problem from the NP-complete KNAPSACK problem: given m items with weights $w_1, \cdots, w_m$ and values $s_1, \cdots, s_m$, capacity B and value V, does there exist a subset $F \subseteq \{1, \cdots, m\}$ such that $\sum_{j \in F} w_j \leq B$ and $\sum_{j \in F} s_j \geq V$? Given an instance of the KNAPSACK problem, we build an instance of the UWO problem as follows:

- The set of projects is the set of items. Each project has cost $C_j = w_j$.
- The budget of the agent is B.
- The value of each project to the agent is $v_j = s_j + w_j$.

We can see that the social welfare obtained by choosing a set of projects F is exactly equal to the value of choosing set of items F in the KNAPSACK problem i.e, $\mathrm{SW}(F) = \sum_{j \in F} v_j - C_j = \sum_{j \in F} s_j$. Also note that the WBB condition is satisfied if and only if the capacity constraint is satisfied. It follows that a solution with value at least V exists in the KNAPSACK problem if and only if there exists a set of projects whose social welfare in the corresponding UWO-WP instance is at least V. The NP-hardness follows immediately.

We prove existence of an FPTAS for UWO via an approximation-preserving reduction to KNAPSACK. Given an instance of UWO, we build an instance of KNAPSACK as follows:

- The set of items is the set of projects. Each item has weight $w_j = C_j$.
- The knapsack capacity is the sum of agent budgets, i.e. $B = \sum_{i \in N} b_i$.
- The value of each item is $s_j = v_j - C_j$.

Again, we note that the social welfare obtained by choosing a set of items F is exactly equal to the value of choosing a set of projects F in the KNAPSACK problem, i.e.

$$
\begin{aligned}
\sum_{j \in F} s_j &= \sum_{j \in F} \left(\sum_{i \in N} v_{ij} - C_j \right) \\
&= \sum_{i \in N} \sum_{j \in F} v_{ij} - \sum_{j \in F} C_j \\
&= \sum_{i \in N} v_i(F) - \sum_{j \in F} C_j \\
&= SW(F)
\end{aligned}
$$

It is clear to see that the knapsack capacity constraint is satisfied if and only if the corresponding set of projects are WBB. We have shown that a set of projects is feasible if and only if the corresponding set of items is feasible and equated the objective functions. Thus, since KNAPSACK admits an FPTAS, we have that UWO admits an FPTAS in the additive valuations setting. $\square$

B Set-Union Knapsack Problem FPTAS

The *set-union knapsack problem* (SUKP) is a well-studied generalization of the classical knapsack problem, wherein items have values and are composed of elements, which may occur in multiple items and are associated with costs [23, 2, 45, 25]. The SUKP solution maximizes the total value of the items in the knapsack while ensuring the weight of the elements in the items' union does not exceed the given knapsack capacity. In this section, we show that under the laminar item set family restriction, the problem admits an FPTAS. To do so, we provide an approximation-preserving reduction to *knapsack on chordal conflict graphs* (defined in Section 5).

SUKP takes as input a set of n items and a universe of elements U where each item i corresponds to an *element set*, L_i, such that $\bigcup_{i=1}^{n} L_i = U$. Each item i has a non-negative value denoted v_i, and each element j has a non-negative weight, ω_j. The knapsack has a capacity of B.

We denote the item set family as $\mathbf{L} = \{L_1, \ldots, L_n\}$, the item values as $\mathbf{v}$, the element weights as $\boldsymbol{\omega}$, and denote an instance of SUKP as $I = \langle \mathbf{L}, \mathbf{v}, \boldsymbol{\omega}, B \rangle$. We henceforth use the notation $[k] = \{1, \ldots, k\}$. The SUKP problem is to find a subset of items such that the total value is maximized and the weight of the elements in the union of the items does not exceed the capacity. Formally, the solution can be written

$$\begin{aligned} \max_{K \subseteq \{1,\cdots,n\}} \quad & \sum_{i \in K} v_i \\ \text{subject to} \quad & \sum_{j \in \bigcup\limits_{i \in K} L_i} \omega_j \leq B \end{aligned}$$

Definition 7 (Laminar SUKP)**.** *We call an instance of SUKP laminar if the items' element sets constitute a laminar set family, i.e. if for every $L_i, L_j \in \mathbf{L}$, the intersection of L_i and L_j is either empty, or equals L_i, or equals L_j.*

Next, we prove two lemmas which will help us to connect Laminar SUKP with KCG.

Lemma 7. *Given an instance of laminar SUKP $I = \langle \mathbf{L}, \mathbf{v}, \boldsymbol{\omega}, B \rangle$, the undirected graph $G = (V, E)$ where $V = [n]$ and $E = \{(i, j) | L_i \subseteq L_j\}$ is chordal.*

Proof. Consider such a graph $G = (V, E)$. Let $E^c \subseteq E$ be a cycle of length four or more. Pick any edge $(i, j) \in E^c$ and assume without loss of generality that $L_i \subseteq L_j$. Now pick any edge $(i, k) \in E^c$, which must exist because E^c is a cycle. We have that $L_i \cap L_k \neq \emptyset$ by the construction of the graph and thus $L_k \cap L_j \neq \emptyset$. Since $\mathbf{L}$ is a laminar set family, this means that L_j and L_k form a subset relation, which implies $(j, k) \in E$. (j, k) is a chord since it connects two vertices in E^c and cannot be in E^c since it would form a triangle, contradicting E^c being a cycle of length four or more. $\square$

Lemma 8. *Given an instance of laminar SUKP $I = \langle \mathbf{L}, \mathbf{v}, \boldsymbol{\omega}, B \rangle$ and any set $K \subseteq [n]$, let $K' = \{i \in [n] | L_i \subseteq \bigcup_{j \in K} L_j\}$. K' is feasible.*

Proof. Let $I = \langle \mathbf{L}, \mathbf{v}, \boldsymbol{\omega}, B \rangle$ be an instance of laminar SUKP and let $K \subseteq [n]$ be a set of items. Consider $i \in [n]$ such that $i \notin K$ and $L_i \subseteq \bigcup_{j \in K} L_j$. It suffices to show that $\{K \cup i\}$ is a feasible solution. This is true because $e \in L_i \implies e \in \bigcup_{j \in K} L_j$ so the weights of i's element set are already included in the knapsack constraint. $\square$

The following theorem makes a novel contribution to the literature around the set-union knapsack problem by identifying a natural restriction which admits an FPTAS.

Theorem 6. *Laminar SUKP admits an FPTAS.*

Proof. Let $I = \langle \mathbf{L}, \mathbf{v}, \boldsymbol{\omega}, B \rangle$ be an instance of laminar SUKP. We proceed by an approximation-preserving reduction to an instance of KCG on chordal graphs as follows:

- $V = [n]$
- $p_i = \displaystyle\sum_{\{j \in [n] | L_j \subseteq L_i\}} v_j$
- $w_i = \displaystyle\sum_{e \in L_i} \omega_e$
- $C = B$
- $(i, j) \in E$ for each distinct $i, j \in [n]$ with $L_i \subseteq L_j$

Let S be a feasible solution to the instance of KCG reduced from I, as given above. We posit that $K' = \{i \in [n] | L_i \subseteq \bigcup_{j \in S} L_j\}$ is a feasible solution to I. Since S is a feasible solution to KCG, we know

$$\sum_{i \in S} w_i \leq C \iff \sum_{i \in S} \sum_{e \in L_i} \omega_e \leq B$$

Since $\mathbf{L}$ is a laminar set family, we know that the sets are either disjoint or form a subset relation. However, since S is feasible, we know that no two members of $\{L_i\}_{i \in S}$ can form a subset relation and thus all members are disjoint. Thus, we can rewrite the above as

$$\sum_{e \in \bigcup\limits_{i \in S} L_i} \omega_e \leq B$$

and we have that S is a feasible solution to I. From Lemma 8, we can thus conclude that K' is a feasible solution to the SUKP instance I.

We now show that the objective value of the KCG solution, S, is equal to that of the resulting SUKP solution K':

$$\begin{aligned} \sum_{i \in S} p_i &= \sum_{i \in S} \sum_{\{j \in [n] | L_j \subseteq L_i\}} v_j \\ &= \sum_{\{j \in [n] | L_j \subseteq \bigcup\limits_{i \in S} L_i\}} v_j \\ &= \sum_{j \in K'} v_j \end{aligned}$$

It is apparent that the reduction given above runs in polynomial time. Furthermore, by Lemma 7 we have that G is a chordal graph and thus the reduced instance admits an FPTAS [36]. Since our reduction equated the objective values of each problem, it is approximation-preserving and we therefore have that Laminar SUKP admits an FPTAS. $\square$