Research Statement

Junrong Yan

My research lies in the interface of geometry, topology, PDEs, and mathematical physics, with the goal of understanding mirror symmetry and QFT from the index theoretic points of view. My recent research focuses on the geometry and topology of the Landau-Ginzburg (LG) models, and the renormalization of QFTs and their application. In my work, the analysis of elliptic and parabolic PDEs on noncompact manifolds plays a crucial role, leading to understanding the asymptotic growth of eigenvalues and the decay of eigenfunctions near infinity as well as the expansion and the estimate of the heat kernel for Schördinger type operators on noncompact manifolds. Moreover, these analyses, for example, pave the way for connecting the L^2 cohomology of the Witten deformation on noncompact manifolds (Quantum vacuum space of LG models) to other cohomologies and makes it possible to define important geometric/topological invariants such as the Ray-Singer analytic torsion for Witten deformation on noncompact manifolds. More specifically, my current research can be divided into the following three topics:

1. Analysis of the Witten Laplacian on noncompact manifolds.
2. BCOV torsion, the holomorphic anomaly formula, and Weil-Peterson geometry for LG B-models.
3. Renormalization theory and its application to BCOV theory.

In the remainder of this statement, I will briefly review the background and results of my research.

1 Analysis of Witten deformation on noncompact manifolds

In this direction, we explore the analytic torsion for Witten deformation on noncompact manifolds.
What is Witten deformation? Witten deformation, introduced in the extremely influential paper [27], is a deformation of the de Rham complex. It simply deforms the exterior derivative by

$$d_{Tf} := e^{-Tf} \circ d \circ e^{Tf}$$

where f is a smooth function and T is the deformation parameter. Witten observed that when T is large enough, the eigenfunctions (with bounded eigenvalues) of the Hodge-Laplacian for d_{Tf}, the so-called Witten Laplacian, concentrate at the critical points of f. This beautiful idea has produced a whole range of excellent applications, from Demailly's holomorphic Morse inequalities [11], to the new proof of the generalized Ray-Singer conjecture by Bismut-Zhang [2], to the instigation of the development of Floer homology theory (an infinite-dimensional version of Thom-Smale-Witten complex). In all these developments, the compactness of the manifolds is a crucial assumption.
Why do we explore Witten deformation on noncompact manifolds? In an ongoing project with Xianzhe Dai, we develop the theory of Witten deformation on noncompact manifolds. This is not only a natural question to ask, but it is also motivated by a recent advance in mirror symmetry, namely the Calabi-Yau/Landau-Ginzburg (CY/LG) correspondence: for example, for the famous quintic $f = x_0^5 + ... + x_4^5$, the quantum information of $X_f = \{p \in \mathbb{C}P^4 : f(p) = 0\}$ can be read from the LG model $\big(\mathbb{C}^5, g_0, f\big)$. For LG B-models, it amounts to study Witten deformation on $\mathbb{C}^5$ with potential f.

We consider the more general case (M, g, f) where (M, g) is a complete noncompact Riemannian manifold with bounded geometry, and f satisfies certain growth conditions called "tameness conditions" first. In a paper published in [9], we establish an isomorphism of the L^2-cohomology of the Witten deformation and the cohomology of the Thom-Smale complex of f:

Theorem 1.1 (X. Dai and Y.,[9]) *If (M, g, f) satisfies certain weak tameness conditions, and f is a Morse function, then Witten instanton complex (for large enough T), Thom-Smale-Witten complex, and the relative de Rham complex $(\Omega^*(M, U_c), d)$ are quasi-isomorphic. In particular, Morse inequality holds. Here $U_c = \{p \in M : f(p) < -c\}$ for sufficiently big $c > 0$.*

It is important to note that the tameness condition is natural. In fact, without the tameness conditions, the Thom-Smale complex may not be a complex at all.

What is analytic torsion? The notion of analytic torsion is the basic building block for the so-called BCOV-type torsion for LG B-models, which should be related to counting higher genus curves. Naively, it is simply the determinant $\det(\Delta)$ of some Laplacian Δ (In particular, in our case, we consider the Witten Laplacian). However, since Δ is an operator on infinite-dimensional vector space, to define the Ray-Singer analytic torsion for the Witten deformation, we study the heat kernel, heat trace, and local index theory for the Witten Laplacian $\Delta_{Tf} := d_{Tf}d^*_{Tf} + d^*_{Tf}d_{Tf}$ (d_{Tf} is the formal adjoint of d_{Tf})

Theorem 1.2 (X. Dai and Y.,[10]) *The heat kernel K_{Tf} for Δ_{Tf} has the following complete point-wise asymptotic expansion near the diagonal,*

$$K_{Tf}(t, x, y) \sim \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-d^2(x, y)/4t\right) \exp\left(-th_T(x, y)\right) \sum_{j=0}^{\infty} t^j \Theta_{T,j}(x, y)$$

as $t \to 0$ with the following remainder estimate for $t \in (0, 1]$, and some $\beta(k)$ ($\beta(k) > 0$ if k is big)

$$|K_{Tf}(t, x, y) - \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp(-d^2(x, y)/4t) \exp(-th_T(x, y)) \sum_{j=0}^{k} t^j \Theta_{T,j}(x, y)| \leq Ct^{\beta(k)} \exp(-a\tilde{d}_T(t, x, y)).$$

Here $\tilde{d}_T(t, x, y)$ is Li-Yau's parabolic distance defined in Section 1.1, and $h_T(x, y)$ is the average of $T^2|\nabla f|^2$ on the geodesic segment from x to y.

Theorem 1.3 (X. Dai and Y.,[10])

$$\operatorname{ind}\left(d_f + d_f^*\right) = \frac{(-1)^{\left[\frac{n+1}{2}\right]}}{\pi^{\frac{n}{2}}} \int_M AS(f, R).$$

Here $AS(f, R)$ is an Atiyah-Singer local index density depending on f and curvature operator R.

Notice that if (M, g) is compact, $\operatorname{ind}(d_f + d_f^*)$ and $AS(f, R)$ should be independent of f. However, for noncompact case, they actually depends on f. For example, consider the case of $M = \mathbb{R}$, $f = x^2$. Then $\operatorname{ind}(d_f + d_f^*) = -1$, $\operatorname{ind}(d_{-f} + d_{-f}^*) = 1$.

The difficulties and how we overcome them. Since when dealing with noncompact manifolds, the issue of integrability comes out naturally. In our proof of Theorem 1.1, the Agmon estimate (Theorem 1.4) for the eigenfunctions of the Witten Laplacian, which controls the speed of the decay of eigenfunctions near infinity, plays a crucial role:

Theorem 1.4 (X. Dai and Y.,[9]) *Let (M, g, f) be well tame, and ω be an eigenform of Δ_{Tf} whose eigenvalue is uniformly bounded in T. Then for any $a \in (0, 1)$,*

$$|\omega(p)| \leq CT^{(n+2)/2} \exp\left(-a\rho_T(p)\right) \|\omega\|_{L^2}.$$

Here $\rho_T(p)$ is a distance function induced by the metric $T^2|\nabla f|^2 g$.

Still, for heat kernel expansion, we have issues of integrability. To resolve them, we need two crucial ingredients: one is Li-Yau's parabolic distance which motivates Perelman's reduced volume in Ricci flow, the other is the coupling $tT^2 = 1$ (here t is the time parameter for the heat kernel). It turns out that our proof of local index theorem (Theorem 1.3) for Witten deformation has a taste of 't Hooft's limits in physics: We keep tT^2 to be constant and let $t \to 0$. Interested readers may refer to Section 1.1. for more technical details.

1.1 Technical details

Li-Yau's parabolic distance. In the estimate of the remainder of the asymptotic expansion of the heat kernel for the Witten Laplacian, Li-Yau's parabolic distance play an essential role:

Firstly, since we are dealing with Schördinger-type operator (which is almost like $\Delta + T^2|\nabla f|^2$). To handle the potential term $|\nabla f|^2$, we should approach the heat kernel by $\frac{1}{(4\pi t)^{\frac{n}{2}}}\exp(-\frac{d^2(x,y)}{4t} - th_T(x,y))$. Secondly, to get the remainder estimate of the heat kernel, in the case of $f = 0$, the following triangular inequality play an essential roles: $\frac{d^2(x,z)}{t-s} + \frac{d^2(z,y)}{s} \geq \frac{d^2(x,y)}{t}$, where $0 \leq s \leq t$. However, when $f \neq 0$, the exponential power term $\frac{d^2(x,y)}{4t} + th_T(x,y)$ doesn't satisfy a similar triangular ineuqality. Here Li-Yau's parabolic distance $\tilde{d}_T(t,x,y)$ (See the definition below) enters the stage: we have $\frac{d^2(x,y)}{4t} + th_T(x,y) \geq \tilde{d}_T(t,x,y)$ and $\tilde{d}_T(t,x,y)$ satisfy a similar triangular inequality. So, what is Li-Yau's parabolic distance?

Li-Yau's parabolic distance $\tilde{d}_T(t,x,y)$ is given by $d_{Tf}(t,x,y) = \inf \int_0^t |c'(s)|^2 + T^2|\nabla f|^2 \circ c(s)ds$, where the infimum is taking over all piecewise smooth curve $c : [0,t] \to M$, s.t. $c(0) = x, c(t) = y$.

Coupling $\mathbf{tT^2 = 1}$. With a pointwise expansion of the heat kernel (Theorem 1.2), it is still difficult to derive a heat trace expansion, even for the very simple case $M = \mathbb{R}, f = x^2 + x^3$. This is because we don't have a nice asymptotic expansion of $\int_M \exp(-tT^2|\nabla f|^2)\mathrm{dvol}_M$. To resolve this issue, we make the coupling $tT^2 = 1$, then $\int_M \exp(-tT^2|\nabla f|^2)\mathrm{dvol}_M|_{tT^2=1}$ is independent of t.

1.2 Ongoing and future projects in this direction

In a closely related direction, we establish a non-semiclassical Weyl's law for Schördinger operators on noncompact manifolds in [8], generalizing the classical results for Euclidean spaces. Finally, in [6], making use of our work in [9, 10] (heat kernel expansion and the coupling $tT^2 = 1$), we defined the Ray-Singer metric $\log \|\cdot\|^{RS}_{H^*(M,d_f^F)}$ for the Witten deformation $d_f^F := e^{-f} \circ \nabla^F \circ e^f$ associated to some flat vector bundle $F \mapsto M$ with flat connection ∇^F. Let $R(M,g,h,F,f) = \log \|\cdot\|^{RS}_{H^*(M,F,d_f)} + \log \|\cdot\|^{RS}_{H^*(M,F,d_{-f})}$ be our modified Ray-Singer metric on (M,g,f).

Theorem 1.5 (X. Dai and Y.,[6]) *Suppose (M,g_l,h_l,f) is a family of metric satisfies suitable tameness conditions, then when n is odd, $\frac{\partial}{\partial l}R(M,g_l,h_l,F,f) = 0$, i.e. R is independent of metric; when n is even, we obtain an explicit formula (anomaly formula) for $\frac{\partial}{\partial l}R(M,g_l,h_l,F,f)$ as in [2].*

Moreover, we show that the famous Ray-Singer conjecture, which claims the combinatorial (Reidemeister) and the analytic (Ray-Singer) torsion are equal, is true, i.e., Cheeger-Muller/Bismut-Zhang holds in this setting.

Moreover, we find an interesting relation between the analytic torsion of the Witten deformation on a noncompact manifold and the usual analytic torsion of its compact core. More explicitly, attaching a cylindrical end $\partial M \times [0,\infty)$ to a compact manifold M with boundary and taking $f = u^2$ on the end (u is the coordinate for the end), let $\|\cdot\|^{RS}_{abs}$ and $\|\cdot\|^{RS}_{rel}$ denote the Ray-Singer metric on $\det(H^*(M,F))$ with absolute and relative boundary conditions respectively. We showed that

Theorem 1.6 (X. Dai and Y,[6])

$$\lim_{T\to\infty} \|\cdot\|^{RS}_{H^*(M,d^F_{Tf})} = \|\cdot\|^{RS}_{abs} \text{ and } \lim_{T\to\infty} \|\cdot\|^{RS}_{H^*(M,d^F_{-Tf})} = \|\cdot\|^{RS}_{rel}.$$

This will lead to another approach for purely analytic proof of gluing formula for analytic torsion (c.f. [3, 21]). Moreover, this approach could extend to analytic torsion form naturally (c.f. [20]).

2 BCOV torsion and holomorphic anomaly formula

In various parts joint with Xianzhe Dai and Xinxing Tang, the second part of my research concerns the so-called BCOV torsion. From the index theoretic viewpoint, we generalize the previous discussion in two directions: the case of families of LG models and the complex/holomorphic setting.

Here we review the story in the CY side first: mirror symmetry leads us to consider families of Calabi-Yau manifolds. Indeed, it is well known that the CY B-model concerns the deformation of the complex structure. The genus 0 theory is equivalent to the variation of Hodge structure. The study of the higher genus theory is much more challenging and interesting. In this direction, Bershadshy-Cecotti-Ooguri-Vafa (BCOV) showed that the genus one term F_1 for N=2 supersymmetric field theories admits a holomorphic anomaly equation as follows (see Section 2.1 for further explanation of each term)

$$\partial\bar{\partial}F_1 = \frac{1}{2}\operatorname{tr} C\bar{C} - \frac{G_{WP}}{24}\operatorname{Tr}(-1)^F. \tag{1}$$

Geometrically, on CY's side, the holomorphic anomaly equation above is nothing but a Bismut-Gillet-Soulé (BGS) type curvature formula for some determinant line bundle on the complex moduli of Calabi-Yau manifolds (Interested readers may refer to section 2.1 for more technical details).

Then in the spirit of the LG/CY correspondence, we should have a similar story for the LG B-model, which concerns the deformation of singularities. Indeed, its genus 0 theory is given by Saito's theory of primitive forms and higher residue pairing(c.f. [22, 23]), originating in Saito's study of period integrals over vanishing cycles associated to an isolated singularity. Comparing with the CY B-model, it is reasonable to conjecture that the genus one term could be expressed as a torsion type invariant. In [12] and [24], Fan-Fang and Shen-Xu-Yu defined such a counterpart of the BCOV torsion for LG models. Moreover, in [25], X. Tang derived a holomorphic anomaly formula for this torsion for the case of $\mathbb{C}^n$. Lastly, we would like to emphasize that there are several subtle issues for LG models: for example, the space is noncompact and also, the usual bi-grading is not compatible with the Witten deformed Dolbeault operator $\bar{\partial} + \partial f \wedge$.

2.1 Technical details

tt* geometry. To explain the holomorphic anomaly formula and study the CY/LG correspondence for genus one terms, we introduced tt^* geometry which captures the 2D vacuum geometry in string theory:

Roughly, a tt^* geometry structure $(K \to M, \kappa, (\cdot,\cdot), D, C, \bar{C})$, which generalizes the variation of Hodge structures, consists of the following data(c.f. [16, 13, 25])
(1) a smooth vector bundle $K \to M$;
(2) a complex conjugate $\kappa : K \to K$.
(3) a bilinear symmetric pairing $(\cdot,\cdot) : \Gamma(K) \times \Gamma(K) \mapsto \mathbb{C}$, such that $g(u, v) := (u, \kappa v)$ is a Hermitian metric;
(4) a 1-parameter family of flat connections $\nabla^z = D + \frac{1}{z}C + z\bar{C}$, where D is the Chern connection of g, and $C, \bar{C}$ are the $C^\infty(M)$-linear map

$$C : C^\infty(K) \to C^\infty(K) \otimes \mathcal{A}_M^{(1,0)}, \quad \bar{C} : C^\infty(K) \to C^\infty(K) \otimes \mathcal{A}_M^{(0,1)}$$

that compatible with κ and $(\cdot,\cdot)$ in some sense. Notice that operator C and $\bar{C}$ has been appeared in (1).

BGS type formula and holomorphic anomaly. It is known that both the LG and the CY B-model enjoy tt^* geometry structure (c.f. [25, 13, 17]), denoted by $(K^{LG} \to M, \kappa^{LG}, (\cdot,\cdot)^{LG}, D^{LG}, C^{LG}, \bar{C}^{LG})$ and $(K^{CY} \to M, \kappa^{CY}, D^{CY}, C^{CY}, \bar{C}^{CY})$ respectively. Moreover, in the CY's case, BCOV showed that

$$\partial\bar{\partial}\log\tau^{CY}_{BCOV} = \frac{1}{2}\operatorname{tr}(C^{CY}\bar{C}^{CY}) - \frac{1}{24}\omega^{CY}_{WP}\chi(Z) \tag{2}$$

for a family of Calabi-Yau $\pi : X \to M$ with typical fiber Z, where τ^{CY}_{BCOV} is the BCOV torsion for CY manifolds, ω^{CY}_{WP} is the Weil-Peterson metric and $\chi(Z)$ is the Euler number of Z. More explicitly, (2) is a BGS type formula. Condsider the determinant line bundle $\lambda = \bigwedge_{0\le p,q\le n}(\det R^q\pi_*\Omega^p(X/M))^{(-1)^{p+q}p}$, where $\Omega^p(X/M)$ is the sheaf of relative p-forms. Then on line bundle $\lambda \to M$, there are two natural metrics, the usual L^2 metric $\|\cdot\|_{L^2}$ induced by the harmonic forms, and the Quillen metric $\|\cdot\|_Q$ given by $\|\cdot\|_Q = \|\cdot\|_{L^2}\tau_{BCOV}$, where τ_{BCOV} is the BCOV torsion. Then in the equation (2), the first term on the right-hand side is precisely the curvature of λ for $\|\cdot\|_{L^2}$. The second term is the curvature of λ for $\|\cdot\|_Q$, which turns out to be related to the Weil-Peterson metric on M. Thus, comparing with (**??**), the F_1 term should be $\log\tau_{BCOV}$ (BCOV torsion), a certain combination of analytic torsions, up to some holomorphic and anti-holomorphic corrections. In the LG's case, X. Tang (c.f. [25]) showed similar formula for the case of $\mathbb{C}^n$.

To see whether τ^{CY}_{BCOV} and τ^{LG}_{BCOV} and its holomorphic anomaly formulas are related via the CY/LG correspondence, we need to explore the CY/LG correspondence for tt^* geometry.

In [13], Fan-Lan-Yang partially prove that the two tt^* structures are isomorphic via the CY/LG correspondence: There is a bundle isomorphism $\Phi : K^{LG} \to K^{CY}$ such that the following holds:
(1)$\Phi^*(\cdot,\cdot)^{CY} = (\cdot,\cdot)^{LG}$.
(2) $\Phi^*C^{CY} = C^{LG}$,$\Phi^*\bar{C}^{CY} = \bar{C}^{LG}$.

To show that two tt^* structures are isomorphic, one still needs to show that $\Phi^*\kappa^{CY} = \kappa^{LG}$. Recently, X. Tang and I started a project in this direction. In [26], we determine that there is a holomorphic line bundle $L \to M^{LG}$ on the moduli M^{LG} of LG models with a canonical metric (which corresponds to $R\pi_*\Omega^n(X/M)$ for a family $\pi : X \to M$ in the CY side via the CY/LG correspondence), and show that the moduli of LG B-models carry a Weil-Peterson type metric:

Theorem 2.1 (X.Tang and Y.,[26]) *Let ϕ_0 be a (local) holomorphic section of L, then $\partial\bar\partial\log(|\phi|^2)$ defines a positve definite metric on M^{LG}.*

We hope that this theorem will tell us something about the CY/LG correspondence for real structures.

2.2 Ongoing and future projects in this direction

Whether the LG/CY correspondence holds for the genus one term is an interesting and challenging question. To answer this question, in [26], X. Tang and I plan to investigate the CY/LG correspondence for the real structures in tt^* geometry on CY and LG B-models. The interested reader may refer to section 2.1 for more details about tt^* geometry.

Another project in this direction is that, in [7], we extend the definition to analytic torsion forms on families of LG models on noncompact Kähler manifolds which admit nice $U(1)$ actions (instead of $\mathbb{C}^n$). We also obtain a BGS type curvature formula (c.f. [1]). Then we want to explore the generalized LG/CY correspondence, which, by our expectation, relates the BCOV type invariants for LG models on some general noncompact Kähler manifold and BCOV invariants for CY pairs (c. f. [28]).

3 Renormalization theory and its application to BCOV theory

This topic is independent of the two topics above. Roughly speaking, the Mirror symmetry conjecture states that one can count the number of rational curves on a CY 3-fold X (genus 0 A-model) by looking at the variation of Hodge structure of a mirror CY 3-fold $\hat{X}$ (genus 0 B-model). It was rigorously proved by Givental [15] and Lian-Liu-Yau [19] for several cases.

Here we are concerned with the theory of the higher genus B-model, which is still mysterious. In the physics literature, since the genus 0 part of the B-model on a Calabi-Yau X concerns the deformations of the complex structure of X, BCOV proposed a string field theory whose classical equations of motion is precisely the Kodaira-Spencer equation. In [5], as a nice application of effective Batalin-Vilkovisky (BV) quantization, Costello and S. Li initiated the generalized BCOV theory and obtained a renormalization theory of quantum BCOV theory via the heat kernel.

3.1 Technical details

To quantize theories with gauge symmetries, physicists invented BV quantization. To explain our project, I briefly illustrate how Costello and Li's effective BV quantization works here:
Effective BV quantization. Typically, a classical field theory in the BV formalism consists of

$(\mathcal{E}, Q, \omega, S_0)$, where
(1) $\mathcal{E} = \Gamma(X, E)$ is the set of fields, which is the space of (L^2 or smooth) sections of a graded bundle E on a manifold X. In BCOV's original theory, $\mathcal{E}$ is the so-called poly-vector field.
(2) $Q : \mathcal{E} \to \mathcal{E}$ is a differential operator on E that makes $(\mathcal{E}, Q)$ into an elliptic complex.
(3) ω is a local symplectic pairing

$$\omega(\alpha, \beta) = \int_X \langle \alpha, \beta \rangle, \quad \forall \alpha, \beta \in \mathcal{E}$$

where $\langle -, - \rangle$ is skew-symmetric pairing on $\mathcal{E}$ valued in the density line bundle on X.
(4) $S_0 = \omega(Q(-), -) + I_0$ is a classical action satisfying the classical master equation, where $I_0 \in \mathcal{O}(\mathcal{E}) := \prod_{n \geq 0} \mathrm{Sym}^n (\mathcal{E}^*)$.

In this paragraph, to enlighten the main ideas, we sacrifice the mathematical rigor for a moment. Naively, the Poisson kernel is defined as $K_0 := \omega^{-1} \in \mathrm{Sym}^2(\mathcal{E})$. Let Δ_{K_0} denote the second-order operator

$$\Delta_{K_0} : \mathrm{Sym}^n (\mathcal{E}^*) \to \mathrm{Sym}^{n-2} (\mathcal{E}^*)$$

by contracting with the kernel $K_0 \in \mathrm{Sym}^2(\mathcal{E})$. Quantizing the classical theory I_0 amounts to solving Quantum master equation (QME)

$$(Q + \hbar \Delta_{K_0}) e^{I_{K_0}/\hbar} = 0$$

for I_{K_0}, where $I_{K_0} = I_0 + I_1 \hbar + ... \in \mathcal{O}(\mathcal{E})[[\hbar]]$ with the leading order term I_0, and then studing the $\hbar$ series expansion of $e^{I_{K_0}/\hbar}$. For example, in BCOV's original theory (for Calabi-Yau 3-folds), I_0 was taken to be an action whose Euler-Lagrangian equations gives the Kodaira-Spencer equation, and the genus g term F_g was computed by the $\hbar$ expansion of $e^{I_K/\hbar}$.

However, since $\Delta_{K_0} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$ is *is not well-defined*, Costello's lemma and the heat kernel enter the stage. Let Q^* be the conjugate of Q, K_t be the heat kernel w.r.t. the Hodge type Laplacian $[Q, Q^*]$, and Δ_{K_t} be the operator of contracting with K_t. Now Δ_{K_t} is well-defined.

What's the relation between the solution for the QME w.r.t. K_0 and the solution for the QME w.r.t. K_t? The answer is as follows: set $P^t = Q^* \int_0^t K_s ds$, so that one formally has

$$[Q, P^t] = \Delta_{K_0} - \Delta_{K_t}. \tag{3}$$

Hence, by Costello's lemma: if (3) holds, we can solve the QME w.r.t. Δ_{K_t} first and recover the solution w.r.t. Δ_{K_0} by Feynman amplitude $W_\Gamma(P^t, I_{K_t})$ for all graphs Γ, where $W_\Gamma(P^t, I_{K_t})$ is computed by the graph Γ, with P^t being the propagator and I_{K_t} being the vertex.

However, even replacing K_0 with K_t, usually $W_\Gamma(P^t, I_{K_t})$ is still ill-defined. To resolve this issue, in [4], Costello considers $P_\epsilon^t = Q^* \int_\epsilon^t K_s ds$, and chooses a nice counter term I_ϵ^{CT}, such that the limit $\lim_{\epsilon \to 0} W(P_\epsilon^t, I_{K_t} + I_\epsilon^{CT})$ exists.

In the discussion above, the choice of I_ϵ^{CT} is not canonical. S. Li and I determined another way to make sense of $W_\Gamma(P^t, I_{K_t})$, which avoids introducing the non-canonical counterterms. We introduced one more complex parameter s, such that when $s >> 0$, $W_{\Gamma,s}(P^t, I_{K_t})$ is well defined. And Formally, $\lim_{s \to 0} W_{\Gamma,s}(P^t, I_{K_t}) = W_\Gamma(P^t, I_{K_t})$. Moreover, one has

Proposition 3.1 *$W_{\Gamma,s}(P^t, I_{K_t})$ depends holomorphically on s when $s >> 0$, and could be extend to a meromophic function near $s = 0$.*

As a result, we renormalize $W_\Gamma(P^t, I_{K_t})$ as the regular value of $W_{\Gamma,s}(P^t, I_{K_t})$ at $s = 0$.

3.2 Ongoing and future projects in this direction

We expect that by using Getzler's rescaling technique [14] in the index theorem, some geometric quantities will be more computable in our formulation. Moreover, this method avoids introducing the non-canonical counterterms in Costello's renormalization theory. We will attempt to say something about BCOV theory for CY and LG B-models with this renormalization theory. Furthermore, we also plan to compare it with the regularized integral introduced in [18].

References

[1] J. Bismut, H. Gillet, and C. Soulé. Analytic torsion and holomorphic determinant bundles I, II, and III. *Communications in Mathematical Physics*, 115:79–126, 1988.

[2] J.-M. Bismut and W. Zhang. An extension of a theorem by Cheeger and Müller. *Astérisque*, 205, 1992.

[3] J. Brüning and X. Ma. On the gluing formula for the analytic torsion. *Mathematische Zeitschrift*, 273:1085–1117, 2013.

[4] K. Costello. *Renormalization and effective field theory.* Number 170. American Mathematical Soc., 2011.

[5] K. Costello and S. Li. Quantum BCOV theory on Calabi-Yau manifolds and the higher genus B-model. *arXiv:1201.4501.*

[6] X. Dai and J. Yan. Analytic torsion for Witten deformation for noncompact manifolds. *In preparation.*

[7] X. Dai and J. Yan. BCOV torsion for LG models. *In preparation.*

[8] X. Dai and J. Yan. The non-semiclassical Weyl law for Schrödinger operators on non-compact manifolds. *In preparation.*

[9] X. Dai and J. Yan. Witten deformation for noncompact manifolds with bounded geometry. *Journal of the Institute of Mathematics of Jussieu*, pages 1–38, 2021.

[10] X. Dai and J. Yan. Witten deformation on non-compact manifolds-heat kernel expansion and local index theorem. *arXiv:2011.05468*, 31 pages.

[11] J. P. Demailly. Champs magnétiques et inégalités de Morse pour la d-cohomologie. *C. R. Acad. Sci. and Ann. Inst. Fourier*, 301, 35:119–122, 185–229, 1985.

[12] H. Fan and H. Fang. Torsion type invariants of singularities. *arXiv:1603.0653.*

[13] H. Fan, T. Lan, and Z. Yang. LG/CY correspondence between tt^* geometries. *arXiv:2011.14658.*

[14] E. Getzler. A short proof of the local Atiyah-Singer index theorem. *Topology*, 25:111–117, 1986.

[15] A. Givental. A mirror theorem for toric complete intersections. In *Topological field theory, primitive forms and related topics*, pages 141–175. Springer, 1998.

[16] C. Hertling. tt* geometry, Frobenius manifolds, their connections, and the construction for singularities. *Crelle's Journal*, 2003:77–161, 2002.

[17] S. Li. *Calabi-Yau geometry and higher genus mirror symmetry.* PhD thesis, Harvard University, 2011.

[18] S. Li and J. Zhou. Regularized integrals on riemann surfaces and modular forms. *arXiv:2008.07503.*

[19] B. Lian, K. Liu, and S. Yau. Mirror principle i. *Asian J. Math.*, 1:729–763, 1997.

[20] M. Puchol, Y. Zhang, and J. Zhu. Adiabatic limit, Witten deformation and analytic torsion forms. *arXiv:2009.13925.*

[21] M. Puchol, Y. Zhang, and J. Zhu. Scattering matrix and analytic torsion. *Comptes Rendus Mathematique*, 355(10):1089–1093, 2017.

[22] K. Saito. Primitive forms for a universal unfolding of a function with an isolated critical point. *Journal of the Faculty of Science, the University of Tokyo. Sect. 1 A, Mathematics*, 28:775–792, 1982.

[23] K. Saito. The higher residue pairings $\mathcal{K}_F$ for a family of hypersurface singular points. In *Proceedings of Symposia in pure mathematics*, volume 40, pages 441–463. Amer Mathematical SOC 201 Charles ST, Providence, RI 02940-2213, 1983.

[24] S. Shen, G. Xu, and J.Yu. Analytic torsion in Laudau-Ginzburg B-model. *In preparation.*

[25] X. Tang. tt^* geometry, singularity torsion and anomaly formulas. *arXiv:1710.03915.*

[26] X. Tang and J. Yan. Weil-Peterson metric for LG models. *In preparation.*

[27] E. Witten. Supersymmetry and Morse theory. *J. Diff. Geom*, 17(4):661–692, 1982.

[28] Y. Zhang. BCOV invariant for Calabi-Yau pairs. *arXiv: 1902.08062.*