

Heat Equation on Vector Bundles

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1 Basic Settings

Let (M, g) be a closed Riemannian manifold with Levi-Civita connection ∇^{LC} , $E \rightarrow M$ be a hermitian bundle with hermitian metric h . Let ∇^E a unitary connection on E , then the connection Laplacian $\Delta^E : \Gamma(E) \rightarrow \Gamma(E)$ is defined as

$$\Delta^E := - \sum_i \nabla_{e_i}^E \nabla_{e_i}^E + \nabla_{\nabla_{e_i}^{LC} e_i}^E,$$

where $\{e_i\}$ is a local orthonormal frame.

Let s be a smooth section of $E \rightarrow M$, denote $|s|^2 = h(s, s)$. Let Δ^{LC} be the Laplace-Beltrami operator on (M, g) , then

Theorem 1.1 (Bochner formula).

$$\Delta^{LC}|s|^2 = 2h(\Delta^E s, s) - 2 \sum_i h(\nabla_{e_i}^E s, \nabla_{e_i}^E s)$$

By Bochner Formula, one has

Theorem 1.2 (Maximal Principle). *Let $\Omega \subset M$ be a connected domain in M with smooth boundary. Assume that $s \in \Gamma(E)$ solves*

$$\begin{cases} \Delta^E s = 0 & \text{in } \Omega \\ s = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $\sup_{p \in \Omega} |s|^2(p) = \sup_{p \in \partial\Omega} |s|^2(p)$.

2 Heat Equation

It follows from Bochner Formula again that

Theorem 2.1 (Maximal Principle). *Let $s(t)$ be a time dependent section solves*

$$\begin{cases} (\partial_t + \Delta^E)s(t) = 0 & \text{in } M \times [0, \infty) \\ s(0) = s_0 \\ \lim_{t \rightarrow \infty} s(t) = 0. \end{cases} \tag{1}$$

for some $s_0 \in \Gamma(E)$. Then $\sup_{(p,t) \in M \times [0, \infty)} |s|^2 = \sup_{p \in M} |s_0|^2$.

Set $p_i : M \times M \rightarrow M$, $(p_1, p_2) \rightarrow p_i$, $i = 1, 2$, and let $E^* \rightarrow M$ be the dual bundle of $E \rightarrow M$, then heat kernel $K(t, x, y)$ with respect to Δ^E is a time-dependent section of $p_1^*E \otimes p_2^*E^*$, such that

$$s(t, x) := \int_M (K(t, x, y), s_0(y)) dy$$

solves (1), where (\cdot, \cdot) is the nature pairing of E^* and E .

Proposition 2.2. 1. $K(t, x, y)$ solves

$$\begin{cases} (\partial_t - \Delta_y^E)K(t, x, y) = 0, \\ \lim_{t \rightarrow 0} K(t, x, y) = \delta_x(y) \sum_i e_i(x) \otimes \tilde{e}_j(y), \end{cases}$$

where $\{e_i\}, \{\tilde{e}_j\}$ are orthonormal basis of E and E^* near x and y respectively.

2. $K(t, x, y) = K(t, y, x)^*$.

3. $K(t+s, x, y) = \int_M (K(s, x, z), K(s, z, y)) dz$. Hence, $K(2t, x, x) = \int_M (K(s, x, y), K(s, x, y)^*) dy$.

Theorem 2.3. Let $k(t, x, y)$ be the heat kernel with respect to Δ^{LC} , then $|K(t, x, y)|^2 = h(K(t, x, y), K(t, x, y)) \leq nk(t, x, y)$, where h is the metric on $p_1^*E \otimes p_2^*E^*$ induced by h on $E \rightarrow M$, $n = \text{rank}(E)$.

Proof. By Bochner formula,

$$\begin{cases} (\partial_t - \Delta_y^{LC}) (|K(t, x, y)|^2 - nk(t, x, y)) \leq 0, \\ \lim_{t \rightarrow 0} (|K(t, x, y)|^2 - nk(t, x, y)) = 0. \end{cases}$$

By classical maximal principle, the result follows. \square

3 Heat equation for Hodge Laplacian

Let Δ^H be the Hodge Laplacian, Δ^C be the connection laplacian on $\Lambda^*(T^*M) \rightarrow M$, then one has Bochner-Lichnerowicz-Weitzenbock formula

$$\Delta^H = \Delta^C + \sum_{e_i, e_j} c(e_i)c(e_j)R(e_i, e_j),$$

where $\{e_i\}$ is a local orthonormal frame, $\{e^i\}$ is its dual frame, $c(e_i) = e^i \wedge -\iota_{e_i}$, R is the curvature operator.

Theorem 3.1. Let $K^H(t, x, y)$ be the heat kernel with respect to Δ^H , then there exists $C = C(n, R) > 0$, such that

$$|K^H(t, x, y)|^2 \leq e^{Ct} 2^n k(t, x, y),$$

where $n = \dim(M)$.

Proof. By Bochner formula and Bochner-Lichnerowicz-Weitzenbock formula, there exists $C = C(n, R) > 0$, such that

$$\begin{cases} (\partial_t - \Delta_y^{LC}) (e^{-Ct} |K(t, x, y)|^2 - nk(t, x, y)) \leq 0, \\ \lim_{t \rightarrow 0} (e^{-Ct} |K(t, x, y)|^2 - nk(t, x, y)) = 0. \end{cases}$$

By classical maximal principle, the result follows. \square

Corollary 3.2. *There exists $C(n, R) > 0$, such that for all $t \in (0, 1)$,*

$$\int_M |K(t, x, y)|^2 dy \leq C(n, R).$$

Proof. This is because

$$\int_M k(t, x, y) dy = 1.$$

\square