Research Statement

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My research lies in the interface of geometry, topology, PDE, and mathematical physics, with the goal of understanding mirror symmetry and QFT from the index theoretic points of view. My recent research focuses on the geometry and topology of the Landau-Ginzburg (LG) models, and the renormalization of QFT and its application. In my work, the analysis of elliptic and parabolic PDE on noncompact manifolds plays a crucial role, leading to understanding the asymptotic growth of eigenvalues and the decay of eigenfunctions near the infinity as well as the expansion and the estimate of heat kernel for Schördinger type operators on noncompact manifolds. Moreover, these analysis, in turn, paves the way for connecting the L^2 cohomology of the Witten deformation on noncompact manifolds (Quantum vacuum space of LG models) to other cohomologies and makes it possible to define important geometric/topological invariants such as the Ray-Singer analytic torsion for Witten deformation on noncompact manifolds. More specifically, my current research can be divided into the following three topics:

1. Analysis of Witten Laplacian on noncompact manifolds.
2. BCOV torsion, holomorphic anomaly formula, and Weil-Peterson geometry for LG B-models.
3. Renormalization theory and its application to BCOV theory.

In the remainder of this statement, I briefly review the background and results of my research, and then give more details of basic setting and notations in Section 2, 3, and 4.

1 Motivations and results

1.1 Analysis of Witten deformation on noncompact manifolds

In this topic, we explore the analytic torsion for Witten deformation on noncompact manifolds.
What is Witten deformation? Witten deformation, introduced in the extremely influential paper [29], is a deformation of the de Rham complex. It simply deforms the exterior derivative by

$$d_{Tf} := e^{-Tf} \circ d \circ e^{Tf}$$

where f is a smooth function and T is the deformation parameter. Witten observed that when T is large enough, the eigenfunctions (with bounded eigenvalues) of the Hodge-Laplacian for d_{Tf}, the so-called Witten Laplacian, concentrate at the critical points of f. As a result, Witten deformation builds a direct bridge between the Betti numbers and the critical point information of f, leading immediately to an analytic proof of the Morse inequalities.

This beautiful idea has produced a whole range of amazing applications, from Demailly's holomorphic Morse inequalities [13], to the new proof of Ray-Singer conjecture and its generalization by Bismut-Zhang [4], to the instigation of the development of Floer homology theory (an infinite-dimensional version of Thom-Smale-Witten complex). In all these development, the compactness of the manifolds is an important assumption.
Why do we explore Witten deformation on noncompact manifolds? In an ongoing project with Xianzhe Dai, we develop the theory of Witten deformation on noncompact manifolds. This is not only a natural question but also motivated by the recent advances in mirror symmetry, the Calabi-Yau/Landau-Ginzburg (CY/LG) correspondence.

A typical Calabi-Yau manifold can be defined as a (quasi-)homogeneous polynomial f (of suitable degree) on $\mathbb{C}^n$. String theory predicts that the sigma model on the Calabi-Yau manifold is closely related to the so-called Landau-Ginzburg model $(\mathbb{C}^n, g_0, f)$, i.e., the study of the Witten deformation on the noncompact manifold $(\mathbb{C}^n, g_0)$. For example, for the famous quintic $f = x_0^5 + ... + x_4^5$, the quantum information of $X_f = \{p \in \mathbb{C}P^4 : f(p) = 0\}$ can be read from the Landau-Ginzburg model $\left(\mathbb{C}^5, g_0, f\right)$.

We consider the more general case (M, g, f) where (M, g) is a complete noncompact Riemannian manifold with bounded geometry, and f satisfies certain growth conditions called tameness condition. In a paper published in [11] we establish the isomorphism of the L^2-cohomology of the Witten deformation and the cohomology of the Thom-Smale complex of f (Theorem 2.2). In particular, we show that the Morse inequalities hold in this case as well. It is important to note that the tameness condition is natural. In fact, without the tameness conditions, the Thom-Smale complex may not be a complex at all, i.e., the square of its boundary map may not be zero. Since when dealing with noncompact manifolds, the issue of integrability comes out naturally, in our proof, the Agmon estimate (Theorem 2.1) for the eigenfunctions of the Witten Laplacian, which controls the speed of the decay of eigenfunctions near infinity, plays a crucial role.

What is analytic torsion? The notion of analytic torsion is the basic building block for the so-called BCOV-type torsion for LG B-models, which should be related to counting higher genus curves. Naively, it is simply the determinant $\det(\Delta)$ of some Laplacian Δ (In particular, in our case, we consider Witten Laplacian). However, since Δ is an operator on infinite-dimensional vector space, to define the Ray-Singer analytic torsion for the Witten deformation, we study the heat kernel, heat trace, and local index theory for the Witten Laplacian (Theorem 2.3 and Theorem 2.4). This is closely related to Li-Yau's estimate for heat kernels of Schrödinger operators derived from their famous Harnack inequality. Still, several issues are coming from doing integration on noncompact manifolds. To resolve them, we find that the Li-Yau's parabolic distance, which motivates Perelman's reduced volume in Ricci flow, and the coupling $tT^2 = 1$ (here t is the time parameter for the heat kernel) become crucial ingredients in our study. It turns out that our proof of local index theorem for witten deformation has a taste of 't Hooft's limits: We keep tT^2 to be constant and let $t \to 0$. The interested reader may refer to the last paragraph in section 2.1 and the paragraph above Theorem 2.4 for more details.

1.1.1 Ongoing and future projects in this direction

In a closely related direction, we establish a non-semiclassical Weyl's law for Schördinger operators on noncompact manifolds in [10], generalizing the classical results for Euclidean spaces. Finally, in [8], making use of our work in [11, 12] we define the Ray-Singer analytic torsion for Witten deformation on noncompact manifolds and prove a Cheeger-Müller/Bismut-Zhang Theorem in this setting. Moreover, when the dimension of the manifold is odd, the analytic torsion (or, more precisely, the Ray-Singer-Quillen metric)is independent of the choice of the geometric data as long as tameness conditions hold; when the dimension of the manifold is even, we obtain an anomaly formula.

We find an interesting relation between the analytic torsion of the Witten deformation on a noncompact manifold and the usual analytic torsion of its compact core (with suitable boundary conditions). More explicitly, attaching a cylindrical end $\partial M \times [0, \infty)$ to a compact manifold M with boundary and taking $f = u^2$ on the end (u is the coordinate for the end), the eigenvalues, eigenfunctions and analytic torsion for Witten Laplacian Δ_{Tf} (or Δ_{-Tf}) converges, as $T \to \infty$, to the eigenvalues, eigenfunctions and analytic torsion for Hodge Laplacian on M with absolute (or relative) boundary conditions. This leads to another approach for purely analytic proof of gluing formula for analytic torsion (c.f. [5, 22]). Moreover, this approach extends to analytic torsion form naturally (c.f. [23]).

1.2 BCOV torsion, holomorphic anomaly formula, and Weil-Peterson geometry for LG B-models

The second part of my research, in various parts joint with Xianzhe Dai and Xinxing Tang, concerns the so-called BCOV torsion. From the index theoretic viewpoint, we generalize the previous discussion in two directions: the family situation and the complex/holomorphic setting.

Here we review the story in CY's side first: The natural setting in mirror symmetry is to consider families of Calabi-Yau manifolds. Indeed, it is well known that the CY B-model concerns the deformation of complex structures. The genus 0 theory is equivalent to the variation of Hodge structures. The study of higher genus theory is much more challenging and interesting. In this direction, Bershadshy-Cecotti-Ooguri-Vafa (BCOV) showed that the genus one term F_1 for N=2 supersymmetric field theories admits a holomorphic anomaly equation as follows (see Section 3 for further explanation of each term)

$$\partial_i \bar{\partial}_j F_1 = \frac{1}{2} \operatorname{tr} C_i \bar{C}_{\bar{j}} - \frac{G_{i\bar{j}}}{24} \operatorname{Tr}(-1)^F. \tag{1}$$

Geometrically, on CY's side, the holomorphic anomaly equation above is nothing but a Bismut-Gillet-Soulé (BGS) type curvature formula for some determinant line bundle on the complex moduli of Calabi-Yau manifolds (Interested reader may refer to section 3.1 for more details).

Then in the spirit of the LG/CY correspondence, we should have similar stories for the LG B-model, which concerns the deformation of singularities. Indeed, its genus 0 theory is given by Saito's theory of primitive forms and higher residue pairing(c.f. [24, 25]), originated in Saito's study of period integrals over vanishing cycles associated to an isolated singularity. Comparing with the CY B-model, it is reasonable to conjecture that the genus one term could be expressed as a torsion type invariant. In [14] and [26], Fan-Fang and Shen-Xu-Yu defined a counterpart of the BCOV torsion for LG models. Moreover, in [27], X. Tang derived a holomorphic anomaly formula for this torsion. Whether the LG/CY correspondence holds for the genus one term is an interesting and challenging question. There are several subtle issues for LG models; for example, the space is noncompact and also, the usual bi-grading is not compatible with Witten deformed Dolbeault operator $\bar{\partial} + \partial f \wedge$.

1.2.1 Ongoing and future projects in this direction

In [28], X. Tang and I investigate the CY/LG correspondence for the real structures in tt^* geometry on CY and LG B-models. Here the tt^* geometry comes from the study of isolated hypersurface singularities, which, geometrically, can be realized as a generalization of a polarized mixed Hodge structure (PMHS). The interested reader may refer to section 3 for more details.

As mentioned, in [8], we develop the theory of analytic torsion for Witten deformation on noncompact manifolds. Next, we establish the Cheeger-Müller/Bismut-Zhang theorem in this setting, as well as several fundamental properties. In [9], we extend the definition to analytic torsion forms on families of LG models on noncompact Kähler manifolds which admit nice $U(1)$ actions (instead of $\mathbb{C}^n$). Then we obtain a BGS type curvature formula (c.f. [3]). Then we explore the generalized LG/CY correspondence, which, by our expectation, relates the BCOV type invariants for LG models on some general noncompact Kähler manifold and BCOV invariants for CY pairs (c. f. [30]).

1.3 Renormalization theory and its application to BCOV theory

This topic is independent of the two topics above.

Roughly speaking, the Mirror symmetry conjecture states that one can count the numbers of rational curves on a CY 3-fold X (genus 0 A-model) by looking at the variations of the Hodge structure of a mirror CY 3-folds $\hat{X}$ (genus 0 B-model). It was rigorously proved by Givental [17] and Lian-Liu-Yau [21] for several cases.

Here we are concerned with the theory of the higher genus B-model, which is still mysterious. In the physics literature, since the genus 0 part of the B-model on a Calabi-Yau X concerns the deformations of the complex structure of X, BCOV proposed a string field theory whose classical equations of motion is precisely the Kodaira-Spencer equation in Tian's form. They called this theory the Kodaira-Spencer theory of gravity (or Kodaira-Spencer gauge theory). In [7], as a nice application of effective Batalin-Vilkovisky (BV) quantization, Costello and S. Li initiated the generalized BCOV theory and obtained a renormalization theory of quantum BCOV theory via heat kernel.

1.3.1 Ongoing and future projects in this direction

S. Li and I found another approach to renormalize QFTs using heat kernel (Proposition 4.1). We hope that some geometric quantities will be more computable in this new formulation (we expect that Getzler's rescaling technique[16] works here). Moreover, this new method avoids introducing non-canonical counterterms in Costello's renormalization theory. We hope to say something about BCOV theory for CY and LG B-models with this new renormalization theory. Furthermore, we also plan to compare it with the regularized integral introduced in [20].

2 Analysis of Witten deformation on noncompact manifolds

2.1 Basic setting and definition

We study Witten deformation on noncompact space in this project. To state our main results, we introduce the following notations:

Tameness conditions. Let (M,g) be a noncompact connected complete Riemannian manifold with bounded geometry. We introduced several tameness conditions to control the behavior of f near infinity, for example :

The triple (M,g,f) is said to be strongly (well) tame, if (M,g) has bounded geometry and

$$\limsup_{p\to\infty}\frac{\left|\nabla^2 f\right|(p)}{|\nabla f|^2(p)}=0(<\infty)$$

and

$$\lim_{p\to\infty}|\nabla f|\to\infty(>0),$$

where $\nabla f,\nabla^2 f$ are the gradient and Hessian of f respectively.

Agmon distance. S. Agmon discovered the Agmon estimate in his study of N-body Schördinger operators in the Euclidean setting (c. f. [1]) and has found many important applications. Here we introduced the notation of Agmon metric and Agmon distance, which play significant roles in estimating eigenforms.

Let $\tilde{g}_T:=b^2T^2|\nabla f|^2 g$, $\tilde{g}_T$ is called the Agmon metric on M with respect to f (It a metric with discrete conical singularities). Fix a compact subset $K\subset M$, let $\rho_T(p)$ be the distance between p and K induced by $\tilde{g}_T$, then ρ_T is called an Agmon distance function.

Li-Yau's parabolic distance. In the estimate of the remainder of the asymptotic expansion of heat kernel for Witten Laplacian, the Li-Yau's parabolic distance play an essential role.

Firstly, since we are dealing with Schördinger-type operator, which is almost like $\Delta+T^2|\nabla f|^2$, to handle the potential term $|\nabla f|^2$, we should approach heat kernel by $\frac{1}{(4\pi t)^{\frac{n}{2}}}\exp(-\frac{d^2(x,y)}{4t}-th_T(x,y))$, where $h_T(x,y)$ is the average of the potential function $T^2|\nabla f|^2$ on the geodesic segment from x to y. Secondly, to get the remainder estimate of heat kernel, in the case of $f=0$, the following triangular inequality play an essential roles: $\frac{d^2(x,z)}{t-s}+\frac{d^2(z,y)}{s}\geq\frac{d^2(x,y)}{t}$, where $0\leq s\leq t$. However, when $f\neq 0$, the exponential power term $\frac{d^2(x,y)}{4t}+th_T(x,y)$ doesn't satisfy similar triangular ineuqality. Here the Li-Yau parabolic distance $\tilde{d}_T(t,x,y)$ (See the definition below) enter the stage: We have $\frac{d^2(x,y)}{4t}+th_T(x,y)\geq\tilde{d}_T(t,x,y)$ and $\tilde{d}_T(t,x,y)$ satisfy similar triangular inequality. So what is Li-Yau's parabolic distance? For any piecewise smooth curve $c:[0,t]\mapsto M$, s.t. $c(0)=x,c(t)=y$, we define

$$S_{t,x,y}(c)=\int_0^t\left(\frac{|c'(s)|^2}{4}+T^2|\nabla f|^2(c(s))\right)ds.$$

Let

$$C_{t,x,y}:=\{c:[0,t]\mapsto M\text{ is piecewise smooth, }c(0)=x,c(t)=y\}.$$

Li-Yau's parabolic (meta-)distance is defined as

$$\tilde{d}_T(t,x,y) := \inf_{c\in C_{t,x,y}} S_{t,x,y}(c).$$

2.2 Main results

We first show that an eigenform of Δ_f with small eigenvalues has exponential decay, expressed in terms of the Agmon distance defined above.

Theorem 2.1 (X. Dai and Y.,[11]) *Let (M,g,f) be well tame, and ω be an eigenform of Δ_{Tf} whose eigenvalue is uniformly bounded in T. Then*

$$|\omega(p)| \le CT^{(n+2)/2}\exp\left(-a\rho_T(p)\right)\|\omega\|_{L^2}$$

for any $a\in(0,1)$, provided T is sufficiently large and C is a constant depending on the dimension n, the function f, the curvature bound, and a.

We then make essential use of this Agmon estimate to carry out the quasi-isomorphism among the Witten instanton complex, de Rham relative complex, and the Thom-Smale complex:

Let $c>0$ be big enough, $U_c=\{p\in M: f(p)<-c\}$ and $(\Omega^*(M,U_c), d\ )$ be the relative de Rham complex.

Theorem 2.2 (X. Dai and Y.,[11]) *If (M,g,f) is well tame, and f is a Morse function, then Witten instanton complex (for large enough T), Thom-Smale-Witten complex, and the relative de Rham complex above are quasi-isomorphic.*

To define analytic torsion, we study the heat kernel for Witten Laplacian. Let (M,g,f) satisfies suitable tameness conditions, and $K_{Tf}(t,x,y)$ denote the heat kernel of the Witten Laplacian Δ_{Tf}. Denote by $h_T(x,y)$ the average of the potential function $T^2|\nabla f|^2$ on the geodesic segment from x to y. With the help of Li-Yau's parabolic distance, one has

Theorem 2.3 (X. Dai and Y.,[12]) *The heat kernel K_{Tf} has the following complete pointwise asymptotic expansion near the diagonal,*

$$K_{Tf}(t,x,y)\sim\frac{1}{(4\pi t)^{\frac{n}{2}}}\exp\left(-d^2(x,y)/4t\right)\exp\left(-th_T(x,y)\right)\sum_{j=0}^{\infty}t^j\Theta_{T,j}(x,y)$$

as $t\to 0$. Moreover, we have the following remainder estimate

$$\begin{aligned}&|K_{Tf}(t,x,y)-\tfrac{1}{(4\pi t)^{\frac{n}{2}}}\exp\left(-d^2(x,y)/4t\right)\exp\left(-th_T(x,y)\right)\textstyle\sum_{j=0}^{k}t^j\Theta_{T,j}(x,y)\ |\\ &\le Ct^{\beta(k)}\exp\left(-a\tilde{d}_T(t,x,y)\right)\end{aligned}$$

for $t\in(0,1]$, $T\in\left(0,t^{-\frac{1}{2}}\right]$, and $\beta(k)>0$ when k is big enough.

Lastly, with a pointwise asymptotic expansion of heat kernel, it is still nontrivial to derive asymptotic expansion of heat trace, even for the very simple case $M=\mathbb{R}, f=x^2+x^3$. This is because we don't have a nice asymptotic expansion of $\int_M\exp(-tT^2|\nabla f|^2)\mathrm{dvol}_M$ for general f. To resolve this issue, we make the coupling $tT^2=1$, then $\int_M\exp(-tT^2|\nabla f|^2)\mathrm{dvol}_M$ is independent of t. In particular, we derive a local index theorem:

Theorem 2.4 (X. Dai and Y.,[12])

$$\mathrm{ind}\,(d_f)=\frac{(-1)^{\left[\frac{n+1}{2}\right]}}{\pi^{\frac{n}{2}}}\int_M\exp\left(-|\nabla f|^2\right)\int^B\exp\left(-\frac{\tilde{R}}{2}-\tilde{\nabla}^2 f\right).$$

Remark 2.5 *Here $\tilde{R}, \tilde{\nabla}^2 f \in \Omega^*(M)\hat{\otimes}\Omega^*(M)$ are defined as*

$$\tilde{R} = -\sum_{i<j,k<l} R_{ijkl} e^i e^j \hat{e}^k \hat{e}^l, \qquad \tilde{\nabla}^2 f = \nabla^2_{e_i,e_j} f e^i \hat{e}^j$$

for some orthonormal frame $\{e_i\}$ in TM and its dual frame $\{e^i\}$ in T^*M. We have used $\{\hat{e}_i\}$ to denote the same orthonormal frame in the second copy of T^*M. For any $\omega \in \Omega^*(M)\hat{\otimes}\Omega^*(TM)$, $I = \{i_1, ..., i_k\} \subset 1, 2, ..., n$, we write ω as

$$\omega = \sum_I w_I \hat{e}^I,$$

where $\hat{e}^I = \hat{e}^{i_1} \wedge ... \wedge \hat{e}^{i_k}$. Then the Berezin integral $\int^B$ is defined as

$$\int^B : \Omega^*(M)\hat{\otimes}\Omega^*(M) \mapsto \Omega^*(M), \qquad \int^B \omega = \omega_{1,2,...,n}.$$

Now we are in a position to define analytic torsion. In [8], we defined Ray-Singer metric $\log \|\cdot\|^{RS}_{H^*(M,d^F_f)}$ for the Witten deformation $d^F_f := e^{-f} \circ \nabla^F \circ e^f$ for some flat vector bundle $F \mapsto M$ with flat connection ∇^F on noncompact manifolds M. Let $R(M, g, h, F, f) = \log \|\cdot\|^{RS}_{H^*(M,F,d_f)} + \log \|\cdot\|^{RS}_{H^*(M,F,d_{-f})}$, we showed that for a family of metric g_l and h_l on TM and F respectively (depending on l), one has

Theorem 2.6 (X. Dai and Y.,[8]) *Suppose (M, g_l, h_l, f) satisfies suitable tameness conditions, then When n is odd, $\frac{\partial}{\partial l} R(M, g_l, h_l, F, f) = 0$. when n is even, we have a similar anomaly formula in [4].*

Moreover,

Theorem 2.7 (X. Dai and Y,[8]) *Cheeger-Muller/Bismut-Zhang holds in this setting.*

In particular, attach a geometric cylindrical end $\partial M \times [0, \infty)$ to a compact manifold M with boundaries and take $f = u^2$ on the end (here u is the coordinate for the end), let $\|\cdot\|^{RS}_{abs}$ and $\|\cdot\|^{RS}_{rel}$ denote the Ray-Singer metric on $\det(F) \to M$ with absolute and relative boundary conditions respectively, we showed that

Theorem 2.8 (X. Dai and Y,[8])

$$\lim_{T\to\infty} \|\cdot\|^{RS}_{H^*(M,d^F_{Tf})} = \|\cdot\|^{RS}_{abs}$$

and

$$\lim_{T\to\infty} \|\cdot\|^{RS}_{H^*(M,d^F_{-Tf})} = \|\cdot\|^{RS}_{rel}$$

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3 BCOV torsion, holomorphic anomaly formula, and Weil-Peterson geometry for LG B-models

3.1 Basic setting and definition

To explain the holomorphic anomaly formula and study CY/LG correspondence for genus one term, we introduced tt^* geometry which captures the 2 dimensional vacuum geometry in string theory:

Roughly, a tt^* geometry structure $(K \to M, \kappa, (\cdot,\cdot), D, C, \bar{C})$ consists of the following data(c.f. [18, 15, 27])

- $K \to M$ is a smooth vector bundle;

- a complex conjugate $\kappa : K \to K$.
- a bilinear symmetric pairing $(\cdot,\cdot) : \Gamma(K) \times \Gamma(K) \mapsto \mathbb{C}$, such that $g(u,v) := (u, \kappa v)$ is a Hermitian metric;
- a 1-parameter family of flat connections $\nabla^z = D + \frac{1}{z}C + z\bar{C}$, where D is the Chern connection of $g, C, \bar{C}$ are the $C^\infty(M)$-linear map

$$C : C^\infty(K) \to C^\infty(K) \otimes \mathcal{A}_M^{(1,0)}, \quad \bar{C} : C^\infty(K) \to C^\infty(K) \otimes \mathcal{A}_M^{(0,1)}$$

that compatible with κ and $(\cdot,\cdot)$ in some sense.

To see why tt^* structure generalizes the variation of (polarized) (mixed) Hodge structures, recall that:
Variation of polarized Hodge structures. A variation of polarized Hodge structures of weight w consists of $(K \to M, \kappa, (\cdot,\cdot), \nabla = D + C + \bar{C})$ satisfying all conditions above with $z = 1$ along with: an exhaustive decreasing filtration $F^\bullet$ by holomorphic subbundles $F^p \subset K$ such that

$$K_t = F_t^p \oplus \overline{F_t^{w+1-p}} \quad \text{for any } t \in M$$

and (Griffiths transversality)

$$\nabla : \mathcal{O}\left(F^p\right) \to \Omega_M^1 \otimes \mathcal{O}\left(F^{p-1}\right).$$

In fact, the decomposition $\nabla = D + C + \bar{C}$ is given by Griffiths transversality.

In [2], BCOV show that the genus one term F_1 satsifies

$$\partial\bar{\partial}F_1 = \frac{1}{2}\operatorname{tr}(C\bar{C}) + \text{correction term}. \tag{2}$$

Here the correction term is related to the Weil-Peterson type metric on the moduli of $N = 2$ supersymmetry field theories.

It is known that both LG and CY B-model enjoy tt^* geometry structure (c.f. [27, 15, 19]), denoted by $(K^{LG} \to M, \kappa^{LG}, (\cdot,\cdot)^{LG}, D^{LG}, C^{LG}, \bar{C}^{LG})$ and $(K^{CY} \to M, \kappa^{CY}, D^{CY}, C^{CY}, \bar{C}^{CY})$ respectively.

In the CY's case, BCOV showed that (2) becomes

$$\partial\bar{\partial}\log\tau_{BCOV}^{CY} = \frac{1}{2}\operatorname{tr}(C^{CY}\bar{C}^{CY}) - \frac{1}{24}\omega_{WP}^{CY}\chi(Z)$$

for a family of Calabi-Yau $\pi : X \to M$ with typical fiber Z, where τ_{BCOV}^{CY} is the BCOV torsion for CY manifolds, ω_{WP}^{CY} is the Weil-Peterson metric and $\chi(Z)$ is the Euler number of Z. More explicitly, condsider the determinant line bundle $\lambda = \bigwedge_{0\le p,q\le n}\left(\det R^q\pi_*\Omega^p(X/M)\right)^{(-1)^{p+q}p}$, where $\Omega^p(X/M)$ is the sheaf of relative p-forms. Then on line bundle $\lambda \to M$, there are two natural metrics, the usual L^2 metric $\|\cdot\|_{L^2}$ is induced by the harmonic forms, and the Quillen metric $\|\cdot\|_Q$ is given by

$$\|\cdot\|_Q = \|\cdot\|_{L^2}\tau_{BCOV},$$

where τ_{BCOV} is the BCOV torsion. Then in the equation (2), the first term on the right-hand side is exactly the curvature of λ for $\|\cdot\|_{L^2}$. The second term is the curvature of λ for $\|\cdot\|_Q$, which turns out to be related to the Weil-Peterson metric on M. Thus, the F_1 term should be $\log\tau_{BCOV}$ (BCOV torsion), a certain combination of analytic torsions, up to some holomorphic and anti-holomorphic correction terms.

In the LG's case, X. Tang (c.f. [27]) showed similar formula.

To see whether τ_{BCOV}^{CY} and τ_{BCOV}^{LG} and its holomorphic anomaly formula are related via CY/LG correspondence, we need to explore the CY/LG correspondence for tt^* geometry.

In [15], Fan-Lan-Yang partially prove that two tt^* structure are isomorphic via CY/LG correspondence, i.e.

Theorem 3.1 (Fan-Lan-Yang) *There is a bundle isomorphism* $\Phi : K^{LG} \to K^{CY}$ *such that the following holds:*
(1) $\Phi^*(\cdot,\cdot)^{CY} = (\cdot,\cdot)^{LG}$*.*
(2) $\Phi^* C^{CY} = C^{LG}, \Phi^* \bar{C}^{CY} = \bar{C}^{LG}$*.*

By Theorem 3.1, to show that two tt^* structures are isomorphic, it suffices to show that $\Phi^*\kappa^{CY} = \kappa^{LG}$. Recently, X. Tang and I started a project in this direction.

In [28], we figured out that there is a holomorphic line bundle $L \to M^{LG}$ on the moduli M^{LG} of LG models with a canonical metric (which is corresponding to $R\pi_*\Omega^n(X/M)$ for a family $\pi : X \to M$ in CY's side via CY/LG correspondence), and show that moduli of LG B-model carry a Weil-Peterson type metric:

Theorem 3.2 (X.Tang and Y.,[28]) *Let* ϕ_0 *be a (local) holomorphic section of* L*, then* $\partial\bar{\partial}\log(|\phi|^2)$ *defines a positve definite metric on* M*.*

We hope this theorem could tell us some information about the CY/LG correspondence for real structures.

4 Renormalization theory and its application to BCOV theory

4.1 Basic settings and definition

To quantize theories with gauge symmetries, physicists invented BV quantization. In [7], Costello and S. Li initiated a generalized BCOV theory for any dimension via BV formalism. We briefly illustrate how BV quantization works here, and the interested reader may refer to the next subsection about a toy model for more details:

Effective BV quantization. Typically, a classical field theory in the BV formalism consists of $(\mathcal{E}, Q, \omega, S_0)$, where
(1) fields $\mathcal{E} = \Gamma(X, E)$, which is the space of (L^2 or smooth) sections of a graded bundle E on a manifold X. In BCOV's original theory, $\mathcal{E}$ is the so-called poly-vector field.
(2) $Q : \mathcal{E} \to \mathcal{E}$ is a differential operator on E that makes $(\mathcal{E}, Q)$ into an elliptic complex.
(3) ω : local symplectic pairing of degree (-1)

$$\omega(\alpha,\beta) = \int_X \langle\alpha,\beta\rangle, \quad \forall\alpha,\beta \in \mathcal{E}$$

where $\langle -,-\rangle$ is a degree -1 skew-symmetric pairing on $\mathcal{E}$ valued in the density line bundle on X.
(4) A classical action $S_0 = \omega(Q(-),-) + I_0$ satisfying the classical master equation, where $I_0 \in \mathcal{O}(\mathcal{E}) := \prod_{n\geq 0} \mathrm{Sym}^n(\mathcal{E}^*)$.

In this paragraph, to enlighten the main ideas, we sacrifice the mathematical rigor for a moment. Naively, the Poisson kernel $K_0 := \omega^{-1} \in \mathrm{Sym}^2(\mathcal{E})$. Let Δ_{K_0} denote the second-order operator

$$\Delta_{K_0} : \mathrm{Sym}^n(\mathcal{E}^*) \to \mathrm{Sym}^{n-2}(\mathcal{E}^*)$$

by contracting with the kernel $K_0 \in \mathrm{Sym}^2(\mathcal{E})$. Quantizing the classical theory I_0 amounts to solving Quantum master equation (QME)

$$(Q + \hbar\Delta_{K_0})e^{I_{K_0}/\hbar} = 0$$

for I_{K_0}, where $I_{K_0} = I_0 + I_1\hbar + ... \in \mathcal{O}(\mathcal{E})[[\hbar]]$ with the leading order term I_0, and then studing the $\hbar$ series expansion of $e^{I_{K_0}/\hbar}$. For example, in BCOV's original theory (for 3-dimensional Calabi-Yau manifolds), I_0 was taken to be an action whose Euler-Lagrangian equation is Kodaira-Spencer equation, and the genus g term F_g was computed by the $\hbar$ expansion of $e^{I_K/\hbar}$.

However, since $K_0 = \omega^{-1}$ is given by δ-function,

$$\Delta_{K_0} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$$

is NOT WELL-DEFINED. Then Costello's lemma and heat kernel enter the stage. Let Q^* be the conjugate of Q, K_t be the heat kernel w. r. t. Hodge type Laplacian $[Q, Q^*]$, Δ_{K_t} be the operator of contracting with K_t. Now Δ_{K_t} is well-defined. What's the relation between solution for QME w.r.t K_0 and solution for QME w.r.t. K_t? The answer is: set $P^t = Q^* \int_0^t K_s ds$, one formally has

$$[Q, P^t] = \Delta_{K_0} - \Delta_{K_t}.$$

Hence, by Costello's Lemma (see Lemma 4.2 below) we can solve QME w. r. t. Δ_{K_t} first and recover the original one by Feynman amplitude $W_\Gamma(P^t, I_{K_t})$ for all graph Γ, where $W_\Gamma(P^t, I_{K_t})$ is computed by the graph Γ with P being the propagator and I_{K_t} being the vertex.

However, even replace K_0 with K_t, usually $W_\Gamma(P^t, I_{K_t})$ is still ill-defined. To resolve this issue, in [6], Costello considers $P_\epsilon^t = Q^* \int_\epsilon^t K_s ds$, and choose a nice counter term I_ϵ^{CT}, such that the limit $\lim_{\epsilon\to 0} W(P_\epsilon^t, I_{K_t} + I_\epsilon^{CT})$ exists.

In the discussion above, the choice of I_ϵ^{CT} is not canonical. S. Li and I figured out another way to make sense of $W_\Gamma(P^t, I_{K_t})$, which avoids introducing the non-canonical counterterms. We introduced one more complex parameter s, such that when $s >> 0$, $W_{\Gamma,s}(P^t, I_{K_t})$ is well defined. And Formally, $\lim_{s\to 0} W_{\Gamma,s}(P^t, I_{K_t}) = W_\Gamma(P^t, I_{K_t})$. Moreover, one has

Proposition 4.1 *$W_{\Gamma,s}(P^t, I_{K_t})$ depends holomorphically on s when $s >> 0$, and could be extend to a meromophic function near $s = 0$.*

As a result, we renormalize $W_\Gamma(P^t, I_{K_t})$ as the regular value of $W_{\Gamma,s}(P^t, I_{K_t})$ at $s = 0$.

4.2 Toy model

Let (V, Q, ω) be a finite-dimensional differential graded symplectic vector space. Here the differential $Q : V \to V$ induces a differential on various tensors of V, V^*, still denoted by Q. Moreover, ω is a Q-closed symplectic form of degree -1. Let $K = \omega^{-1} \in \mathrm{Sym}^2(V)$ be the Poisson kernel of degree 1 under the identification via ω

$$\wedge^2 V^* \simeq \mathrm{Sym}^2(V).$$

Let Δ_K denote the second-order operator

$$\Delta_K : \mathrm{Sym}^n(V^*) \to \mathrm{Sym}^{n-2}(V^*)$$

by contracting with the kernel $K \in \mathrm{Sym}^2(V)$. It is straight-forward to see that $(\mathcal{O}(V), Q, \Delta_K)$ defines a so-called differential BV algebra.

Suppose we have a classical action I_0 satisfying the classical master equation (CME), and an element $I_K = I_0 + I_1\hbar + \cdots \in \mathcal{O}(\mathcal{V})[[\hbar]]$ of degree 0 satisfying quantum master equation (QME) with leading order term I_0:

$$(Q + \hbar\Delta_K)e^{I_K/\hbar} = 0.$$

(In fact, CME is nothing but the leading term of QME.)

We also introduce an important lemma which is essential to Costello's homotopic renormalization theory [6]:

Let P be a degree 0 element of $\mathrm{Sym}^2(V)$, and consider the new BV kernel

$$K_P := K + Q(P)$$

We can form a new differential BV algebra $(\mathcal{O}(V), Q, \Delta_{K_P})$.

Roughly speaking, Costello's lemma says that one can recover the solution I_{K_P} (with leading order term I_0) of QME w. r. t. $(\mathcal{O}(V), Q, \Delta_{K_P})$ from the solution I_K (with leading order term I_0) of QME w. r. t. $(\mathcal{O}(V), Q, \Delta_K)$ via Feynman diagram formally. More precisely, one has

Lemma 4.2 (Costello's lemma [6]) *Let*

$$\mathcal{O}^{+}(V) := \mathrm{Sym}^{2}(V^{*}) \oplus \hbar\mathcal{O}(V)[[\hbar]]$$

denote the subspace of $\mathcal{O}(V)[[\hbar]]$ consisting of functions that are at least cubic modulo $\hbar$. Here $\mathcal{O}(V)[[\hbar]]$ denote the formal series of $\hbar$ with coefficient in $\mathcal{O}(V)$.

Let $I_K \in \mathcal{O}^{+}(V)$ be a solution of quantum master equation in $(\mathcal{O}(V)[[\hbar]], Q, \Delta_K)$, ∂_P denote the second order operator on $\mathcal{O}(V)$ by contracting with P. Let I_{K_P} be formally defined by the equation

$$e^{I_{K_P}/\hbar} = e^{\hbar\partial_P} e^{I_K/\hbar}.$$

Then I_{K_P} is a well-defined element of $\mathcal{O}^{+}(V)$ that solves the quantum master equation in $(\mathcal{O}(V)[[\hbar]], Q, \Delta_{K_P})$. Formally,

$$I_{K_P} = \sum_{\Gamma:connected} \frac{\hbar^{g(\Gamma)}}{|\operatorname{Aut}(\Gamma)|} W_{\Gamma}(P, I_K)$$

where the summation is over all connected Feynman graphs with P being the propagator and I_K being the vertex.

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