Research Statement

Junrong Yan

My research lies in the interface of geometry, topology, PDE, and mathematical physics, with the goal of understanding mirror symmetry and QFT from the index theoretic points of view. My recent research focuses on the geometry and topology of the Landau-Ginzburg (LG) models, and the renormalization of QFT and its application. In my work, the analysis of elliptic and parabolic PDE on noncompact manifolds plays a crucial role, leading to understanding the asymptotic growth of eigenvalues and the decay of eigenfunctions near the infinity as well as the expansion and the estimate of heat kernel for Schördinger type operators on noncompact manifolds. Moreover, these analysis, for example, paves the way for connecting the L^2 cohomology of the Witten deformation on noncompact manifolds (Quantum vacuum space of LG models) to other cohomologies and makes it possible to define important geometric/topological invariants such as the Ray-Singer analytic torsion for Witten deformation on noncompact manifolds. More specifically, my current research can be divided into the following three topics:

1. Analysis of Witten Laplacian on noncompact manifolds.
2. BCOV torsion, holomorphic anomaly formula, and Weil-Peterson geometry for LG B-models.
3. Renormalization theory and its application to BCOV theory.

In the remainder of this statement, I briefly review the background and results of my research in Section 1, and interested readers who want to know more detail could refer to Section 2, 3, and 4.

1 Motivations and results

1.1 Analysis of Witten deformation on noncompact manifolds

In this topic, we explore the analytic torsion for Witten deformation on noncompact manifolds.
What is Witten deformation? Witten deformation, introduced in the extremely influential paper [29], is a deformation of the de Rham complex. It simply deforms the exterior derivative by

$$d_{Tf} := e^{-Tf} \circ d \circ e^{Tf}$$

where f is a smooth function and T is the deformation parameter. Witten observed that when T is large enough, the eigenfunctions (with bounded eigenvalues) of the Hodge-Laplacian for d_{Tf}, the so-called Witten Laplacian, concentrate at the critical points of f. This beautiful idea has produced a whole range of excellent applications, from Demailly's holomorphic Morse inequalities [13], to the new proof of generalized Ray-Singer conjecture by Bismut-Zhang [4], to the instigation of the development of Floer homology theory (an infinite-dimensional version of Thom-Smale-Witten complex). In all these development, the compactness of the manifolds is a crucial assumption.
Why do we explore Witten deformation on noncompact manifolds? In an ongoing project with Xianzhe Dai, we develop the theory of Witten deformation on noncompact manifolds. This is not only a natural question but also motivated by the recent advances in mirror symmetry, the Calabi-Yau/Landau-Ginzburg (CY/LG) correspondence.

A typical Calabi-Yau manifold can be defined as a (quasi-)homogeneous polynomial f (of suitable degree) on $\mathbb{C}^n$. String theory predicts that the sigma model on the Calabi-Yau manifold is closely related to the so-called Landau-Ginzburg model $(\mathbb{C}^n, g_0, f)$, i.e., the study of the Witten deformation on the noncompact manifold $(\mathbb{C}^n, g_0)$. For example, for the famous quintic $f = x_0^5 + ... + x_4^5$, the quantum information of $X_f = \{p \in \mathbb{C}P^4 : f(p) = 0\}$ can be read from the Landau-Ginzburg model $\big(\mathbb{C}^5, g_0, f\big)$.

We consider the more general case (M, g, f) where (M, g) is a complete noncompact Riemannian manifold with bounded geometry, and f satisfies certain growth conditions called tameness condition. In a paper published in [11], we establish the isomorphism of the L^2-cohomology of the Witten deformation and the cohomology of the Thom-Smale complex of f (Theorem 2.2). In particular, we show that the Morse inequalities hold in this case as well. It is important to note that the tameness condition is natural. In fact, without the tameness conditions, the Thom-Smale complex may not be a complex at all, i.e., the square of its boundary map may not be zero.

What is analytic torsion? The notion of analytic torsion is the basic building block for the so-called BCOV-type torsion for LG B-models, which should be related to counting higher genus curves. Naively, it is simply the determinant $\det(\Delta)$ of some Laplacian Δ (In particular, in our case, we consider Witten Laplacian). However, since Δ is an operator on infinite-dimensional vector space, to define the Ray-Singer analytic torsion for the Witten deformation, we study the heat kernel, heat trace, and local index theory for the Witten Laplacian (Theorem 2.3 and Theorem 2.4).

The difficulties and how we overcome them. Since when dealing with noncompact manifolds, the issue of integrability comes out naturally, in our proof of Theorem 2.2, the Agmon estimate (Theorem 2.1) for the eigenfunctions of the Witten Laplacian, which controls the speed of the decay of eigenfunctions near infinity, plays a crucial role.

Still, for heat kernel expansion, we have issues of integrability. To resolve them, we find two crucial ingredients: one is the Li-Yau's parabolic distance which motivates Perelman's reduced volume in Ricci flow, the other is the coupling $tT^2 = 1$ (here t is the time parameter for the heat kernel). It turns out that our proof of local index theorem for Witten deformation has a taste of 't Hooft's limits in physics: We keep tT^2 to be constant and let $t \to 0$. The interested reader may refer to the last paragraph in section 2.1 and the paragraph above Theorem 2.4 for more details.

1.1.1 Ongoing and future projects in this direction

In a closely related direction, we establish a non-semiclassical Weyl's law for Schördinger operators on noncompact manifolds in [10], generalizing the classical results for Euclidean spaces. Finally, in [8], making use of our work in [11, 12], we define the Ray-Singer analytic torsion for Witten deformation on noncompact manifolds and prove a Cheeger-Müller/Bismut-Zhang Theorem in this setting. Moreover, when the dimension of the manifold is odd, the analytic torsion (or, more precisely, the Ray-Singer-Quillen metric)is independent of the choice of the geometric data as long as tameness conditions hold.

We find an interesting relation between the analytic torsion of the Witten deformation on a noncompact manifold and the usual analytic torsion of its compact core. More explicitly, attaching a cylindrical end $\partial M \times [0, \infty)$ to a compact manifold M with boundary and taking $f = u^2$ on the end (u is the coordinate for the end), the eigenvalues, eigenfunctions and analytic torsion for Witten Laplacian Δ_{Tf} (or Δ_{-Tf}) converges, as $T \to \infty$, to the eigenvalues, eigenfunctions and analytic torsion for Hodge Laplacian on M with absolute (or relative) boundary conditions. This, by our expectation, should lead to another approach for purely analytic proof of gluing formula for analytic torsion (c.f. [5, 23]). Moreover, this approach extends to analytic torsion form naturally (c.f. [22]).

1.2 BCOV torsion, holomorphic anomaly formula, and Weil-Peterson geometry for LG B-models

In various parts joint with Xianzhe Dai and Xinxing Tang, the second part of my research concerns the so-called BCOV torsion. From the index theoretic viewpoint, we generalize the previous discussion in two directions: the family situation and the complex/holomorphic setting.

Here we review the story in CY's side first: The natural setting in mirror symmetry is to consider families of Calabi-Yau manifolds. Indeed, it is well known that the CY B-model concerns the deformation of complex structures. The genus 0 theory is equivalent to the variation of Hodge structures. The study of higher genus theory is much more challenging and interesting. In this direction, Bershadshy-Cecotti-Ooguri-Vafa (BCOV) showed that the genus one term F_1 for N=2 supersymmetric field theories admits a holomorphic anomaly equation as follows (see Section 3 for further explanation of each term)

$$\partial_i \bar{\partial}_j F_1 = \frac{1}{2} \operatorname{tr} C_i \bar{C}_{\bar{j}} - \frac{G_{i\bar{j}}}{24} \operatorname{Tr}(-1)^F. \tag{1}$$

Geometrically, on CY's side, the holomorphic anomaly equation above is nothing but a Bismut-Gillet-Soulé (BGS) type curvature formula for some determinant line bundle on the complex moduli of Calabi-Yau manifolds (Interested reader may refer to section 3.1 for more details).

Then in the spirit of the LG/CY correspondence, we should have similar stories for the LG B-model, which concerns the deformation of singularities. Indeed, its genus 0 theory is given by Saito's theory of primitive forms and higher residue pairing(c.f. [24, 25]), originated in Saito's study of period integrals over vanishing cycles associated to an isolated singularity. Comparing with the CY B-model, it is reasonable to conjecture that the genus one term could be expressed as a torsion type invariant. In [14] and [26], Fan-Fang and Shen-Xu-Yu defined a counterpart of the BCOV torsion for LG models. Moreover, in [27], X. Tang derived a holomorphic anomaly formula for this torsion. There are several subtle issues for LG models: for example, the space is noncompact and also, the usual bi-grading is not compatible with Witten deformed Dolbeault operator $\bar{\partial} + \partial f \wedge$.

1.2.1 Ongoing and future projects in this direction

Whether the LG/CY correspondence holds for the genus one term is an interesting and challenging question. To answer this question, in [28], X. Tang and I plan to investigate the CY/LG correspondence for the real structures in tt^* geometry on CY and LG B-models. Here the tt^* geometry comes from the study of isolated hypersurface singularities and moduli spaces of N = 2 supersymmetric QFT, which, geometrically, can be realized as a generalization of a polarized mixed Hodge structure. The interested reader may refer to section 3 for more details.

As mentioned, in [8], we develop the theory of analytic torsion for Witten deformation on noncompact manifolds. In [9], we extend the definition to analytic torsion forms on families of LG models on noncompact Kähler manifolds which admit nice $U(1)$ actions (instead of $\mathbb{C}^n$). We also obtain a BGS type curvature formula (c.f. [3]). Then we want to explore the generalized LG/CY correspondence, which, by our expectation, relates the BCOV type invariants for LG models on some general noncompact Kähler manifold and BCOV invariants for CY pairs (c. f. [30]).

1.3 Renormalization theory and its application to BCOV theory

This topic is independent of the two topics above. Roughly speaking, the Mirror symmetry conjecture states that one can count the numbers of rational curves on a CY 3-fold X (genus 0 A-model) by looking at the variations of the Hodge structure of a mirror CY 3-folds $\hat{X}$ (genus 0 B-model). It was rigorously proved by Givental [17] and Lian-Liu-Yau [21] for several cases.

Here we are concerned with the theory of the higher genus B-model, which is still mysterious. In the physics literature, since the genus 0 part of the B-model on a Calabi-Yau X concerns the deformations of the complex structure of X, BCOV proposed a string field theory whose classical equations of motion is precisely the Kodaira-Spencer equation. In [7], as a nice application of effective Batalin-Vilkovisky (BV) quantization, Costello and S. Li initiated the generalized BCOV theory and obtained a renormalization theory of quantum BCOV theory via heat kernel.

1.3.1 Ongoing and future projects in this direction

S. Li and I found another approach to renormalize QFTs using heat kernel (Proposition 4.1). We expect that by using Getzler's rescaling technique [16] in index theorem, some geometric quantities will be more computable in this formulation. Moreover, this method avoids introducing non-canonical counterterms in Costello's renormalization theory. We attempt to say something about BCOV theory for CY and LG B-models with this renormalization theory. Furthermore, we also plan to compare it with the regularized integral introduced in [20].

2 Analysis of Witten deformation on noncompact manifolds

2.1 Basic setting and definition

We study Witten deformation on noncompact space in this project. To state our main results, we introduce the following notations:

Tameness conditions. Let (M, g) be a complete Riemannian manifold with bounded geometry. We introduced several tameness conditions to control the behavior of f near infinity, for example :

The triple (M, g, f) is said to be strongly (well) tame, if

$$\limsup_{p\to\infty} \frac{\left|\nabla^2 f\right|(p)}{|\nabla f|^2(p)} = 0(<\infty) \text{ and } \lim_{p\to\infty} |\nabla f| \to \infty(>0),$$

where $\nabla f, \nabla^2 f$ are the gradient and Hessian of f respectively.

Agmon distance. S. Agmon discovered the Agmon estimate in his study of N-body Schördinger operators in the Euclidean setting (c. f. [1]). Here we found that Agmon metric and Agmon distance play significant roles in estimating eigenforms.

So what is Agmon distance? Let $\tilde{g}_T := b^2T^2|\nabla f|^2 g$, $\tilde{g}_T$ is called the Agmon metric on M with respect to f (It a metric with discrete conical singularities). Fix a compact subset $K \subset M$, let $\rho_T(p)$ be the distance between p and K induced by $\tilde{g}_T$, then ρ_T is called an Agmon distance function.

Li-Yau's parabolic distance. In the estimate of the remainder of the asymptotic expansion of heat kernel for Witten Laplacian, the Li-Yau's parabolic distance play an essential role:

Firstly, since we are dealing with Schördinger-type operator, which is almost like $\Delta + T^2|\nabla f|^2$, to handle the potential term $|\nabla f|^2$, we should approach heat kernel by $\frac{1}{(4\pi t)^{\frac{n}{2}}} \exp(-\frac{d^2(x,y)}{4t} - th_T(x, y))$, where $h_T \sim T^2|\nabla f|^2$ would be introduced in next subsection. Secondly, to get the remainder estimate of heat kernel, in the case of $f = 0$, the following triangular inequality play an essential roles: $\frac{d^2(x,z)}{t-s} + \frac{d^2(z,y)}{s} \geq \frac{d^2(x,y)}{t}$, where $0 \leq s \leq t$. However, when $f \neq 0$, the exponential power term $\frac{d^2(x,y)}{4t} + th_T(x, y)$ doesn't satisfy similar triangular ineuqality. Here the Li-Yau parabolic distance $\tilde{d}_T(t, x, y)$ (See the definition below) enter the stage: We have $\frac{d^2(x,y)}{4t} + th_T(x, y) \geq \tilde{d}_T(t, x, y)$ and $\tilde{d}_T(t, x, y)$ satisfy similar triangular inequality. So what is Li-Yau's parabolic distance?

For any piecewise smooth curve $c : [0, t] \mapsto M$, s.t. $c(0) = x, c(t) = y$, we define

$$S_{t,x,y}(c) = \int_0^t \left(\frac{|c'(s)|^2}{4} + T^2|\nabla f|^2(c(s)) \right) ds.$$

Let $C_{t,x,y} := \{c : [0, t] \mapsto M \text{ is piecewise smooth}, c(0) = x, c(t) = y\}$. Li-Yau's parabolic distance is defined as

$$\tilde{d}_T(t, x, y) := \inf_{c\in C_{t,x,y}} S_{t,x,y}(c).$$

2.2 Main results

We first show that an eigenform of Δ_f with small eigenvalues has exponential decay, expressed in the Agmon distance defined above.

Theorem 2.1 (X. Dai and Y.,[11]) *Let (M, g, f) be well tame, and ω be an eigenform of Δ_{Tf} whose eigenvalue is uniformly bounded in T. Then for any $a \in (0, 1)$,*

$$|\omega(p)| \leq CT^{(n+2)/2} \exp\left(-a\rho_T(p)\right) \|\omega\|_{L^2}.$$

We then make essential use of this Agmon estimate to carry out the quasi-isomorphism among the Witten instanton complex, de Rham relative complex, and the Thom-Smale complex:

Theorem 2.2 (X. Dai and Y.,[11]) *If (M, g, f) is well tame, and f is a Morse function, then Witten instanton complex (for large enough T), Thom-Smale-Witten complex, and the relative de Rham complex $(\Omega^*(M, U_c), d)$ are quasi-isomorphic. In particular, Morse inequality holds. Here $U_c = \{p \in M : f(p) < -c\}$ for sufficiently big $c > 0$.*

To define analytic torsion, we study the heat kernel for Witten Laplacian. Let (M, g, f) satisfies suitable tameness conditions, and $K_{Tf}(t, x, y)$ denote the heat kernel of the Witten Laplacian Δ_{Tf}. Denote by $h_T(x, y)$ the average of $T^2|\nabla f|^2$ on the geodesic segment from x to y. With the help of Li-Yau's parabolic distance, one has

Theorem 2.3 (X. Dai and Y.,[12]) *The heat kernel K_{Tf} has the following complete pointwise asymptotic expansion near the diagonal,*

$$K_{Tf}(t, x, y) \sim \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-d^2(x, y)/4t\right) \exp\left(-th_T(x, y)\right) \sum_{j=0}^{\infty} t^j \Theta_{T,j}(x, y)$$

as $t \to 0$. Moreover, we have the following remainder estimate

$$|K_{Tf}(t, x, y) - \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp(-d^2(x, y)/4t) \exp(-th_T(x, y)) \sum_{j=0}^{k} t^j \Theta_{T,j}(x, y)| \leq Ct^{\beta(k)} \exp(-a\tilde{d}_T(t, x, y))$$

for $t \in (0, 1]$, $T \in \left(0, t^{-\frac{1}{2}}\right]$, and $\beta(k) > 0$ when k is big enough.

Lastly, with a pointwise expansion of heat kernel, it is still difficult to derive heat trace expansion, even for the very simple case $M = \mathbb{R}, f = x^2 + x^3$. This is because we don't have a nice asymptotic expansion of $\int_M \exp(-tT^2|\nabla f|^2)\mathrm{dvol}_M$ for general f. To resolve this issue, we make the coupling $tT^2 = 1$, then $\int_M \exp(-tT^2|\nabla f|^2)\mathrm{dvol}_M|_{tT^2=1}$ is independent of t. In particular, we derive a local index theorem:

Theorem 2.4 (X. Dai and Y.,[12])

$$\mathrm{ind}\,(d_f) = \frac{(-1)^{\left[\frac{n+1}{2}\right]}}{\pi^{\frac{n}{2}}} \int_M \exp\left(-|\nabla f|^2\right) \int^B \exp\left(-\frac{\tilde{R}}{2} - \tilde{\nabla}^2 f\right).$$

Here $\tilde{R}, \tilde{\nabla}^2 f \in \Omega^*(M)\hat{\otimes}\Omega^*(M)$ are defined as

$$\tilde{R} = -\sum_{i<j,k<l} R_{ijkl} e^i e^j \hat{e}^k \hat{e}^l, \qquad \tilde{\nabla}^2 f = \nabla^2_{e_i, e_j} f e^i \hat{e}^j$$

for some orthonormal frame $\{e^i\}$ in T^*M, and $\{\hat{e}_i\}$ denotes the same frame in the second copy of T^*M. $\int^B$ denotes the Berezin integral.

With heat kernel expansion and the coupling $tT^2 = 1$ explained above, we are able to define analytic torsion. In [8], we defined Ray-Singer metric $\log \|\cdot\|^{RS}_{H^*(M, d_f^F)}$ for the Witten deformation $d_f^F := e^{-f} \circ \nabla^F \circ e^f$ for some flat vector bundle $F \mapsto M$ with flat connection ∇^F. Let $R(M, g, h, F, f) = \log \|\cdot\|^{RS}_{H^*(M,F,d_f)} + \log \|\cdot\|^{RS}_{H^*(M,F,d_{-f})}$ be our modified Ray-Singer metric on (M, g, f).

Theorem 2.5 (X. Dai and Y.,[8]) *Suppose (M, g_l, h_l, f) is a family of metric satisfies suitable tameness conditions, then when n is odd, $\frac{\partial}{\partial l} R(M, g_l, h_l, F, f) = 0$, i.e. R is independent of metric; when n is even, we obtain an explicit formula (anomaly formula) for $\frac{\partial}{\partial l} R(M, g_l, h_l, F, f)$ as in [4].*

Moreover, we show that the famous Ray-Singer conjecture, which claims the combinatorial (Reidemeister) and the analytic (Ray-Singer) torsion are equal, is true, i. e.

Theorem 2.6 (X. Dai and Y,[8]) *Cheeger-Muller/Bismut-Zhang holds in this setting.*

Moreover, we find an interesting relation between the analytic torsion of the Witten deformation on a noncompact manifold and the usual analytic torsion of its compact core. More explicitly, attaching a cylindrical end $\partial M \times [0, \infty)$ to a compact manifold M with boundary and taking $f = u^2$ on the end (u is the coordinate for the end). Let $\|\cdot\|^{RS}_{abs}$ and $\|\cdot\|^{RS}_{rel}$ denote the Ray-Singer metric on $\det(H^*(M, F))$ with absolute and relative boundary conditions respectively, we showed that

Theorem 2.7 (X. Dai and Y,[8])

$$\lim_{T\to\infty} \|\cdot\|^{RS}_{H^*(M, d^F_{Tf})} = \|\cdot\|^{RS}_{abs} \text{ and } \lim_{T\to\infty} \|\cdot\|^{RS}_{H^*(M, d^F_{-Tf})} = \|\cdot\|^{RS}_{rel}$$

.

This will lead to another approach for purely analytic proof of gluing formula for analytic torsion (c.f. [5, 23]). Moreover, this approach extends to analytic torsion form naturally (c.f. [22]).

3 BCOV torsion, holomorphic anomaly formula, and Weil-Peterson geometry for LG B-models

3.1 Basic setting and definition

To explain the holomorphic anomaly formula and study CY/LG correspondence for genus one term, we introduced tt^* geometry which captures the 2D vacuum geometry in string theory:

Roughly, a tt^* geometry structure $(K \to M, \kappa, (\cdot, \cdot), D, C, \bar{C})$, which generalizes the variation of Hodge structures, consists of the following data(c.f. [18, 15, 27])
(1) $K \to M$ is a smooth vector bundle;
(2) a complex conjugate $\kappa : K \to K$.
(3) a bilinear symmetric pairing $(\cdot, \cdot) : \Gamma(K) \times \Gamma(K) \mapsto \mathbb{C}$, such that $g(u, v) := (u, \kappa v)$ is a Hermitian metric;
(4) a 1-parameter family of flat connections $\nabla^z = D + \frac{1}{z} C + z\bar{C}$, where D is the Chern connection of $g, C, \bar{C}$ are the $C^\infty(M)$-linear map

$$C : C^\infty(K) \to C^\infty(K) \otimes \mathcal{A}^{(1,0)}_M, \quad \bar{C} : C^\infty(K) \to C^\infty(K) \otimes \mathcal{A}^{(0,1)}_M$$

that compatible with κ and $(\cdot, \cdot)$ in some sense.

In [2], BCOV show that the genus one term F_1 satsifies

$$\partial\bar{\partial} F_1 = \frac{1}{2} \operatorname{tr}(C\bar{C}) + \text{correction term}. \tag{2}$$

Here the correction term is related to the Weil-Peterson type metric on the moduli of $N = 2$ supersymmetry field theories.

It is known that both LG and CY B-model enjoy tt^* geometry structure (c.f. [27, 15, 19]), denoted by $(K^{LG} \to M, \kappa^{LG}, (\cdot, \cdot)^{LG}, D^{LG}, C^{LG}, \bar{C}^{LG})$ and $(K^{CY} \to M, \kappa^{CY}, D^{CY}, C^{CY}, \bar{C}^{CY})$ respectively.

In the CY's case, BCOV showed that

$$\partial\bar{\partial} \log \tau^{CY}_{BCOV} = \frac{1}{2} \operatorname{tr}(C^{CY} \bar{C}^{CY}) - \frac{1}{24} \omega^{CY}_{WP} \chi(Z) \tag{3}$$

for a family of Calabi-Yau $\pi : X \to M$ with typical fiber Z, where τ^{CY}_{BCOV} is the BCOV torsion for CY manifolds, ω^{CY}_{WP} is the Weil-Peterson metric and $\chi(Z)$ is the Euler number of Z. More explicitly, (3) is a BGS type formula: condsider the determinant line bundle $\lambda = \bigwedge_{0\le p,q\le n} (\det R^q \pi_* \Omega^p(X/M))^{(-1)^{p+q}p}$, where $\Omega^p(X/M)$ is the sheaf of relative p-forms. Then on line bundle $\lambda \to M$, there are two natural metrics, the usual L^2 metric $\|\cdot\|_{L^2}$ is induced by the harmonic forms, and the Quillen metric $\|\cdot\|_Q$ is given by

$$\|\cdot\|_Q = \|\cdot\|_{L^2}\tau_{BCOV},$$

where τ_{BCOV} is the BCOV torsion. Then in the equation (3), the first term on the right-hand side is precisely the curvature of λ for $\|\cdot\|_{L^2}$. The second term is the curvature of λ for $\|\cdot\|_Q$, which turns out to be related to the Weil-Peterson metric on M. Thus, comparing with (2), the F_1 term should be $\log\tau_{BCOV}$ (BCOV torsion), a certain combination of analytic torsions, up to some holomorphic and anti-holomorphic corrections.

In the LG's case, X. Tang (c.f. [27]) showed similar formula.

To see whether τ^{CY}_{BCOV} and τ^{LG}_{BCOV} and its holomorphic anomaly formula are related via CY/LG correspondence, we need to explore the CY/LG correspondence for tt^* geometry.

In [15], Fan-Lan-Yang partially prove that two tt^* structure are isomorphic via CY/LG correspondence: There is a bundle isomorphism $\Phi : K^{LG} \to K^{CY}$ such that the following holds:
(1)$\Phi^*(\cdot,\cdot)^{CY} = (\cdot,\cdot)^{LG}$.
(2) $\Phi^*C^{CY} = C^{LG}$,$\Phi^*\bar{C}^{CY} = \bar{C}^{LG}$.

To show that two tt^* structures are isomorphic, one still needs to show that $\Phi^*\kappa^{CY} = \kappa^{LG}$. Recently, X. Tang and I started a project in this direction. In [28], we figure out that there is a holomorphic line bundle $L \to M^{LG}$ on the moduli M^{LG} of LG models with a canonical metric (which is corresponding to $R\pi_*\Omega^n(X/M)$ for a family $\pi : X \to M$ in CY's side via CY/LG correspondence), and show that moduli of LG B-model carry a Weil-Peterson type metric:

Theorem 3.1 (X.Tang and Y.,[28]) *Let ϕ_0 be a (local) holomorphic section of L, then $\partial\bar{\partial}\log(|\phi|^2)$ defines a positve definite metric on M^{LG}.*

We hope that this theorem will tell us something about the CY/LG correspondence for real structures.

Another project in this direction (with X. Dai) is that, in [9], we extend our discussion on Section 2 to analytic torsion forms on families of LG models on noncompact Kähler manifolds which admit nice $U(1)$ actions (instead of $\mathbb{C}^n$). Also, A BGS type curvature formula (c.f. [3]) is obtained. Then we expect a generalized LG/CY correspondence, which relates the BCOV type invariants for LG models on some general noncompact Kähler manifold and BCOV invariants for CY pairs (c. f. [30]).

4 Renormalization theory and its application to BCOV theory

4.1 Basic settings

To quantize theories with gauge symmetries, physicists invented BV quantization. To explain our project, I briefly illustrate how Costello and Li's effective BV quantization works here:
Effective BV quantization. Typically, a classical field theory in the BV formalism consists of $(\mathcal{E}, Q, \omega, S_0)$, where
(1) fields $\mathcal{E} = \Gamma(X, E)$, which is the space of (L^2 or smooth) sections of a graded bundle E on a manifold X. In BCOV's original theory, $\mathcal{E}$ is the so-called poly-vector field.
(2) $Q : \mathcal{E} \to \mathcal{E}$ is a differential operator on E that makes $(\mathcal{E}, Q)$ into an elliptic complex.
(3) ω : local symplectic pairing

$$\omega(\alpha,\beta) = \int_X \langle\alpha,\beta\rangle, \quad \forall\alpha,\beta \in \mathcal{E}$$

where $\langle -,-\rangle$ is skew-symmetric pairing on $\mathcal{E}$ valued in the density line bundle on X.
(4) A classical action $S_0 = \omega(Q(-),-) + I_0$ satisfying the classical master equation, where $I_0 \in \mathcal{O}(\mathcal{E}) := \prod_{n\geq 0}\mathrm{Sym}^n(\mathcal{E}^*)$.

In this paragraph, to enlighten the main ideas, we sacrifice the mathematical rigor for a moment. Naively, the Poisson kernel $K_0 := \omega^{-1} \in \mathrm{Sym}^2(\mathcal{E})$. Let Δ_{K_0} denote the second-order operator

$$\Delta_{K_0} : \mathrm{Sym}^n(\mathcal{E}^*) \to \mathrm{Sym}^{n-2}(\mathcal{E}^*)$$

by contracting with the kernel $K_0 \in \mathrm{Sym}^2(\mathcal{E})$. Quantizing the classical theory I_0 amounts to solving Quantum master equation (QME)

$$(Q + \hbar\Delta_{K_0})e^{I_{K_0}/\hbar} = 0$$

for I_{K_0}, where $I_{K_0} = I_0 + I_1\hbar + ... \in \mathcal{O}(\mathcal{E})[[\hbar]]$ with the leading order term I_0, and then studing the $\hbar$ series expansion of $e^{I_{K_0}/\hbar}$. For example, in BCOV's original theory (for Calabi-Yau threefolds), I_0 was taken to be an action whose Euler-Lagrangian equation gives Kodaira-Spencer equation, and the genus g term F_g was computed by the $\hbar$ expansion of $e^{I_K/\hbar}$.

However, since $\Delta_{K_0} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$ is NOT WELL-DEFINED, Costello's lemma and heat kernel enter the stage. Let Q^* be the conjugate of Q, K_t be the heat kernel w. r. t. Hodge type Laplacian $[Q, Q^*]$, Δ_{K_t} be the operator of contracting with K_t. Now Δ_{K_t} is well-defined.

What's the relation between solution for QME w.r.t K_0 and solution for QME w.r.t. K_t? The answer is: set $P^t = Q^* \int_0^t K_s ds$, one formally has

$$[Q, P^t] = \Delta_{K_0} - \Delta_{K_t}. \tag{4}$$

Hence, by Costello's lemma: if (4) holds, we can solve QME w. r. t. Δ_{K_t} first and recover the original one by Feynman amplitude $W_\Gamma(P^t, I_{K_t})$ for all graph Γ, where $W_\Gamma(P^t, I_{K_t})$ is computed by the graph Γ with P^t being the propagator and I_{K_t} being the vertex.

However, even replace K_0 with K_t, usually $W_\Gamma(P^t, I_{K_t})$ is still ill-defined. To resolve this issue, in [6], Costello considers $P_\epsilon^t = Q^* \int_\epsilon^t K_s ds$, and choose a nice counter term I_ϵ^{CT}, such that the limit $\lim_{\epsilon\to 0} W(P_\epsilon^t, I_{K_t} + I_\epsilon^{CT})$ exists.

In the discussion above, the choice of I_ϵ^{CT} is not canonical. S. Li and I figured out another way to make sense of $W_\Gamma(P^t, I_{K_t})$, which avoids introducing the non-canonical counterterms. We introduced one more complex parameter s, such that when $s >> 0$, $W_{\Gamma,s}(P^t, I_{K_t})$ is well defined. And Formally, $\lim_{s\to 0} W_{\Gamma,s}(P^t, I_{K_t}) = W_\Gamma(P^t, I_{K_t})$. Moreover, one has

Proposition 4.1 *$W_{\Gamma,s}(P^t, I_{K_t})$ depends holomorphically on s when $s >> 0$, and could be extend to a meromophic function near $s = 0$.*

As a result, we renormalize $W_\Gamma(P^t, I_{K_t})$ as the regular value of $W_{\Gamma,s}(P^t, I_{K_t})$ at $s = 0$.

We predict that some geometric quantities will be more computable in our formulation by using some index theoretic tools, for example, Getzler's rescaling technique [16]. We also plan to say something about BCOV theory for CY and LG B-models with this renormalization theory. Furthermore, compare this renormalization formulation with the regularized integral introduced in [20] is also very interesting.

References

[1] S. Agmon. *Lectures on exponential decay of solutions of second-order elliptic equations: Bounds on eigenfunctions of N-body Schrödinger operations.(MN-29).* Princeton University Press, 2014.

[2] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa. Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes. *Communications in Mathematical Physics*, 165:311–427, 1994.

[3] J. Bismut, H. Gillet, and C. Soulé. Analytic torsion and holomorphic determinant bundles I, II, and III. *Communications in Mathematical Physics*, 115:79–126, 1988.

[4] J.-M. Bismut and W. Zhang. An extension of a theorem by Cheeger and Müller. *Astérisque*, 205, 1992.

[5] J. Brüning and X. Ma. On the gluing formula for the analytic torsion. *Mathematische Zeitschrift*, 273:1085–1117, 2013.

[6] K. Costello. *Renormalization and effective field theory.* Number 170. American Mathematical Soc., 2011.

[7] K. Costello and S. Li. Quantum BCOV theory on Calabi-Yau manifolds and the higher genus B-model. *arXiv:1201.4501.*

[8] X. Dai and J. Yan. Analytic torsion for Witten deformation for noncompact manifolds. *In preparation.*

[9] X. Dai and J. Yan. BCOV torsion for LG models. *In preparation.*

[10] X. Dai and J. Yan. The non-semiclassical Weyl law for Schrödinger operators on non-compact manifolds. *In preparation.*

[11] X. Dai and J. Yan. Witten deformation for noncompact manifolds with bounded geometry. *Journal of the Institute of Mathematics of Jussieu*, pages 1–38, 2021.

[12] X. Dai and J. Yan. Witten deformation on non-compact manifolds-heat kernel expansion and local index theorem. *arXiv:2011.05468*, 31 pages.

[13] J. P. Demailly. Champs magnétiques et inégalités de Morse pour la d-cohomologie. *C. R. Acad. Sci. and Ann. Inst. Fourier*, 301, 35:119–122, 185–229, 1985.

[14] H. Fan and H. Fang. Torsion type invariants of singularities. *arXiv:1603.0653.*

[15] H. Fan, T. Lan, and Z. Yang. LG/CY correspondence between tt^* geometries. *arXiv:2011.14658.*

[16] E. Getzler. A short proof of the local Atiyah-Singer index theorem. *Topology*, 25:111–117, 1986.

[17] A. Givental. A mirror theorem for toric complete intersections. In *Topological field theory, primitive forms and related topics*, pages 141–175. Springer, 1998.

[18] C. Hertling. tt* geometry, Frobenius manifolds, their connections, and the construction for singularities. *Crelle's Journal*, 2003:77–161, 2002.

[19] S. Li. *Calabi-Yau geometry and higher genus mirror symmetry.* PhD thesis, Harvard University, 2011.

[20] S. Li and J. Zhou. Regularized integrals on riemann surfaces and modular forms. *arXiv:2008.07503.*

[21] B. Lian, K. Liu, and S. Yau. Mirror principle i. *Asian J. Math.*, 1:729–763, 1997.

[22] M. Puchol, Y. Zhang, and J. Zhu. Adiabatic limit, Witten deformation and analytic torsion forms. *arXiv:2009.13925.*

[23] M. Puchol, Y. Zhang, and J. Zhu. Scattering matrix and analytic torsion. *Comptes Rendus Mathematique*, 355(10):1089–1093, 2017.

[24] K. Saito. Primitive forms for a universal unfolding of a function with an isolated critical point. *Journal of the Faculty of Science, the University of Tokyo. Sect. 1 A, Mathematics*, 28:775–792, 1982.

[25] K. Saito. The higher residue pairings $\mathcal{K}_F$ for a family of hypersurface singular points. In *Proceedings of Symposia in pure mathematics*, volume 40, pages 441–463. Amer Mathematical SOC 201 Charles ST, Providence, RI 02940-2213, 1983.

[26] S. Shen, G. Xu, and J.Yu. Analytic torsion in Laudau-Ginzburg B-model. *In preparation.*

[27] X. Tang. tt^* geometry, singularity torsion and anomaly formulas. *arXiv:1710.03915.*

[28] X. Tang and J. Yan. Weil-Peterson metric for LG models. *In preparation.*

[29] E. Witten. Supersymmetry and Morse theory. *J. Diff. Geom*, 17(4):661–692, 1982.

[30] Y. Zhang. BCOV invariant for Calabi-Yau pairs. *arXiv: 1902.08062.*