Notes on Connes fiberation

June 23, 2022

In [1], Connes constructs the so called Connes foliation, which plays an important role in studing flat vector bundles.

1 Connes's construction

Definition 1.1 (Connes fiberation)**.** *Let (M, g) be a smooth manifold. We define Connes fiberation $\pi : \mathcal{M} \mapsto M$ to be a fiber bundle, such that each fiber $\pi^{-1}(p) = \mathrm{GL}^+(T_pM)/\mathrm{SO}(T_pM) \cong \mathrm{GL}_n^+(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$. It's easy to see that a global section of this fiber bundle gives a Riemannian metric on M.*

To better comprehend Connes fiberation, let's examine the homogeneous space

$$\mathcal{H} = \mathrm{GL}_n^+(\mathbb{R})/\mathrm{SO}_n(\mathbb{R}) \cong \text{all positive definite matrix},$$

in depth.

If $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a linear isomorphism that preserves orientation, then it induces a map

$$\begin{aligned} L(f) : \mathcal{H} &\mapsto \mathcal{H}, \\ A &\mapsto f \circ A \circ f^T, \end{aligned}$$

where A is positive definite. Moreover, we can observe that $\mathrm{GL}_n^+(\mathbb{R})$ acts transitively on $\mathcal{H}$.

Proposition 1.1. *$\mathcal{H}$ could carry a metric of non-positive sectional curvature, which is invariant under $\mathrm{GL}_n^+(\mathbb{R})$ in the sense described above.*

Proof. Let I be the identity matrix, then we can deduce that $T_I\mathcal{H} := \{\text{all symmetry matrix}\}$. Let g_I be a metric on $T_I\mathcal{H}$ given by

$$g_I(X, Y) = tr(XY), X, Y \in T_I\mathcal{H}.$$

For any $A \in \mathcal{H}$, we can find $C \in GL_n^+(\mathbb{R})$, s.t. $A = CC^T$, i.e. $A = L(C)I$, therefore, for any $X, Y \in T_A\mathcal{H}$, define

$$g_A(X, Y) = (L(C^{-1})_* g_I)(X, Y) = g_I(L(C^{-1})_* X, L(C^{-1})_* Y).$$

□

Definition 1.2. *Let (M, g) be a Riemannian manifold, F be a sub-bundle of TM, $F^\perp$ be the orthogonal complement of F w.r.t g. We say that a diffeomorphism $f : M \mapsto M$ is almost isometry w.r.t. F if both $f_*|_F$ and $\mathcal{P}^\perp f_*|_{F^\perp}$ are isometric, where $\mathcal{P}^\perp : TM \mapsto F^\perp$ is the orthogonal projection.*

Assume M is orientable. Consider the principle bundle $P = P_{\mathrm{GL}_n^+(\mathbb{R})} \to M$ with respect to $TM \to M$, a connection ∇^{TM} on M gives a lift from TM to $T^H P$, which also naturally gives a lift from TM to $T^H\mathcal{M}$. Each $v \in \mathcal{M}$, by definition, gives a metric on $T_{\pi(v)}M$, and thus also gives a tautological metric g^H on $T^H\mathcal{M}$. By Proposition 1.1, we have a canonical metric g^V on $T^V\mathcal{M}$ that gives a metric of non-positive sectional curvature fiberwisely. Now $g^{\mathcal{M}} := g^H \oplus g^V$ gives a Riemann metric on $\mathcal{M}$.

Any diffeomorphism $f : M \mapsto M$, as discussed above, induces a diffeomorphism $L(f) : \mathcal{M} \mapsto \mathcal{M}$.

Proposition 1.2. *$L(f)$ is almost isometric w.r.t. $T^V\mathcal{M}$.*

Proof. By Proposition 1.1, we can see $L(f)_*|_{T^V\mathcal{M}}$ is isometric. Take any $X \in T^H_v\mathcal{M}, v \in \mathcal{M}$, which is lifted by $\widetilde{X} \in T_pM, p = \pi(v)$. Then $\mathcal{P}^\perp L(f)_*(X)$ is lifted by $f_*(\widetilde{X})$, since $\pi_* L(f)_* = f_*$. Now we have

$$g^H_{L(f)(v)}(\mathcal{P}^\perp L(f)_*(X), \mathcal{P}^\perp L(f)_*(X)) = L(f)(v)(f_*(\widetilde{X}), f_*(\widetilde{X}))$$
$$= v(f_*^{-1} f_*(X), f_*^{-1} f_*(X)) = v(X, X) = g^H_v(X, X).$$

□

2 Zhang's construction (version 1)

In [3] and [2], Zhang generalize the constructions of Connes fiberation. Here we introduce the first version.

Definition 2.1 (Connes fiberation for foliation)**.** *Let (M, F, g^M) be a foliated manifold, and $F^\perp$ be the orthogonal complement of F w.r.t. g. Connes fiberation for foliation $\pi : \mathcal{M} \mapsto M$ is the fiber bundle, whose fiber $\pi^{-1}(p) := \mathrm{GL}^+(F_p^\perp)/\mathrm{SO}(F_p^\perp)$.*

Definition 2.2 (Bott connection)**.** *Let ∇^M be the Levi-Civita Connection w.r.t. g^M, $\mathcal{P}^\perp : TM \mapsto F$ be the orthogonal projection. Then Bott connection ∇^B on $F^\perp$ is defined by*

$$\nabla^B_X Y = \mathcal{P}^\perp \nabla^{TM}_X Y, X, Y \in \Gamma(F^\perp)$$
$$\nabla^B_X Y = \mathcal{P}^\perp [X, Y], X \in \Gamma(F), Y \in \Gamma(F^\perp).$$

It's straightforward to verify that when restricts to each foliation, ∇^B is flat, i.e. $(\nabla^B)^2_{X,Y} = 0, \forall X, Y \in \Gamma(F)$.

Now, we could lift TM to a subbundle $T^H\mathcal{M}$ by Bott connection as what we did in the last section. Moreover, since ∇^B is flat on each leaf, F was lifted to an integral bundle $\mathcal{F}$. Let $g^{\mathcal{F}}$ be the metric on $\mathcal{F}$ given by

$$g^{\mathcal{F}}(X, Y) = g^T M(\pi_* X, \pi_* Y), X, Y \in \Gamma(F).$$

Also, each $v \in \mathcal{M}$ gives a tautological metric $g^{\mathcal{F}^\perp}$ on $\mathcal{F}^\perp := T^H\mathcal{M}/\mathcal{F}$ by

$$g^{\mathcal{F}^\perp}(X,Y) = v(\pi_*(X), \pi_*(Y), X, Y \in \Gamma(\mathcal{F}^\perp).$$

We also have a nature metric g^V on $T^V\mathcal{M}$. Now we have a metric $g^{\mathcal{M}}$ on $T\mathcal{M}$ given by $g^{\mathcal{M}} = g^{\mathcal{F}} \oplus g^{\mathcal{F}^\perp} \oplus g^V$.

Let $\widetilde{\mathcal{F}} := T\mathcal{M}/\mathcal{F} \cong \mathcal{F}^\perp \oplus T^V\mathcal{M}$, $g^{\widetilde{\mathcal{F}}} = g^{\mathcal{F}^\perp} \oplus g^V$.

So far we have a foliated manifold $(\mathcal{M}, \mathcal{F})$, let $\mathcal{G}$ be the holonomy groupoid w.r.t. $\mathcal{F}$ and Bott connection on $\widetilde{\mathcal{F}}$. Each $\tau \in \mathcal{G}$ induces a linear map from $\widetilde{F}_{s(\tau)}$ to $\widetilde{F}_{r(\tau)}$, then we have

Proposition 2.1. *$\tau : \widetilde{\mathcal{F}}_{s(\tau)} \mapsto \widetilde{\mathcal{F}}_{r(\tau)}$ is almost isometric w.r.t. $T^V\mathcal{M}$, i.e. $\tau|_{T^V\mathcal{M}}$ and $\mathcal{P}^{\mathcal{F}^\perp} \circ \tau|_{\mathcal{F}^\perp}$ are isometric w.r.t $g^{\widetilde{F}}$, where $\mathcal{P}^{F^\perp} : \widetilde{\mathcal{F}} \mapsto \mathcal{F}^\perp$ is the orthogonal projection.*

Proof. Let $\tau \in \mathcal{G}$ be generated by a vector field $X \in \mathcal{F}$, extend X to whole $\mathcal{M}$, (still denote the extension as X) s.t. X has compact support on $\mathcal{M}$, and in a neighborhood U of τ, $X|_U \in \Gamma(U, \mathcal{F})$. Let $\widetilde{X} = \pi_* X$, $\widetilde{\phi}^t$ be the flow generated by $\widetilde{X}$, ϕ^t be the flow generated by X.

Let $U \in \widetilde{\mathcal{F}}_{s(\tau)}$, we claim:

1. $L(\widetilde{\phi}^t) = \phi^t$.

2. $$\mathcal{P}^{\widetilde{\mathcal{F}}} \phi^t_*(U) \text{ is parallel along } \tau,$$

   where $\mathcal{P}^{\widetilde{\mathcal{F}}} : T\mathcal{M} \mapsto \widetilde{\mathcal{F}}$ is the orthogonal projection.

By a similar argument as in Proposition 1.1 and Claim 1, $\mathcal{P}^{\widetilde{\mathcal{F}}} \phi^t_*$ is almost isometric. By Claim 2, we can see that τ is almost isometric. So it reduces to prove the claims above.

Let's prove Claim 2 first.

Let $\gamma : (-\epsilon, \epsilon)$ be a smooth curve with $\gamma(0) = s(\tau)$, $\gamma'(0) = U$. Let $F(s,t) = \phi^t(\gamma(s))$, then

$$0 = [F_*(\frac{\partial}{\partial s}), F_*(\frac{\partial}{\partial t})]_{t=0,s=0} = [\phi^t_*(U), X(s(\tau))].$$

Therefore

$$\begin{aligned}
\nabla_X \mathcal{P}^{\widetilde{\mathcal{F}}} \phi^t(U)|_{t=0} &= \mathcal{P}^{\widetilde{\mathcal{F}}}[X, \mathcal{P}^{\widetilde{\mathcal{F}}} \phi^t(U)]_{t=0} \\
&= \mathcal{P}^{\widetilde{\mathcal{F}}}[X, \phi^t(U)] - \mathcal{P}^{\widetilde{\mathcal{F}}}[X, \mathcal{P}^{\mathcal{F}} \phi^t(U)] \\
&= 0 \text{ (Since } \mathcal{F} \text{ is integrable).}
\end{aligned}$$

Now we just need to prove Claim 1 locally. Hence we can assume $\mathcal{M} = M \times \mathrm{GL}_n^+(\mathbb{R})/SO_n(\mathbb{R})$. Let $\widetilde{X} \in F$, then $\widetilde{X}$ is lifted to $(\widetilde{X}, -w(\widetilde{X}))$ when restricts on $M \times I \subset \mathcal{M}$, where $w = \nabla^B - d \in \{\text{Symmetry matrix valued 1-form}\}$. While

$$\frac{\partial L(\widetilde{\phi}^t)}{\partial t}|_{M\times I, t=0} = (\widetilde{X}, \frac{\partial}{\partial t} \widetilde{\phi_*^{-t}} I(\widetilde{\phi^{-t}}_*)^T) = (\widetilde{X}, -w(\widetilde{X})),$$

where the last equality follows from the definition of Bott connection.

$\square$

3 Zhang's construction version 2

Definition 3.1 (Connes fiberation for flat bundle)**.** *Let $p : E \mapsto M$ be a flat vector bundle with flat connection ∇ and fiber F, Connes fiberation $\pi : \mathcal{M} \mapsto M$ is a fiber bundle, s.t. $\pi^{-1}(x) = \mathrm{GL}^+(F_x)/\mathrm{SO}(F_x)$.*

We can lift TM to a integrable subbundle $T^H\mathcal{M}$ of $T\mathcal{M}$. Then we can consider Bott connection on $T^V\mathcal{M}$ for this foliation. Also, we have canonical metric $g^{T^V\mathcal{M}}$ on $T^V\mathcal{M}$.

Since a flat connection is not always preserve the metric, however, we have

Proposition 3.1. *Let $\mathcal{F} := \pi^* p^* F$, we then have a tautological metric $g^{\mathcal{F}}$ on $\mathcal{F}$,*

1. *The Bott connection on $(T^V\mathcal{M}, g^{T^V\mathcal{M}})$ is leafwise Euclidean.*

2. *There exists a canonical Euclidean connection $\nabla^{\mathcal{F}}$ on $(\mathcal{F}, g^{\mathcal{F}})$ such that for any $X, Y \in \Gamma(\mathcal{M}, T^H\mathcal{M})$, one has*

$$(\nabla^{\mathcal{F}})^2_{X,Y} = 0$$

Proof. The proof of first part is similar to what we did in the last section.

For the section part, TM was lifted to a integrable subbundle T^HE of TE, then $p_*E \cong T^VE$ then consider the Connes fiberation $\bar{\pi} : \mathcal{E} \mapsto E$ w.r.t. this foliation. We can see that $\mathcal{E}^0 := T\mathcal{E}/(T^VE \oplus \mathcal{E}^1) \cong \bar{\pi}_* p_* E$, where $\mathcal{E}^1$ was the integrable subbundle lifted by T^HE under Bott connection. By Proposition 2.1, there exists an Euclidean connection on $\mathcal{E}^0$. Since $\mathcal{M} = \bar{\pi}^{-1}(M)$, we can see that there exists an Euclidean connection on $\mathcal{F}$. $\square$

4 Connes fiberation for general lie group

References

[1] A. Connes. Cyclic cohomology and the transverse fundamental class of a foliation. *Geometric methods in operator algebras*, 123:52–144, 1983.

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[3] W. Zhang. Positive scalar curvature on foliations. *Annals of Mathematics*, pages 1035–1068, 2017.