

Unbounded operators in Hilbert spaces and EPDEs with boundary conditions

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1 Basic Settings

Let $(\mathcal{H}_1, \langle \cdot, \cdot \rangle_{\mathcal{H}_1}), (\mathcal{H}_2, \langle \cdot, \cdot \rangle_{\mathcal{H}_2})$ be Hilbert spaces. We say $(T, \mathcal{D}(T))$, $T : \mathcal{H}_1 \mapsto \mathcal{H}_2$ is unbounded linear operator if restricted in a dense subspace $\mathcal{D}(T) \subset \mathcal{H}_1$, T is linear. Moreover, we say $\mathcal{D}(T)$ is a domain of T .

Example 1. Let $\mathcal{H}_1 = \mathcal{H}_2 = L^2(\mathbb{R})$, $T = \frac{d}{dx} \mathcal{D}(T) = C_c^\infty(\mathbb{R})$. Then $(T, \mathcal{D}(T))$ is an unbounded operator.

Let $(T_1, \mathcal{D}(T_1))$ and $(T_2, \mathcal{D}(T_2))$ ($T_i : \mathcal{H}_1 \mapsto \mathcal{H}_2$) be unbounded operators, if $\mathcal{D}(T_1) \subset \mathcal{D}(T_2)$ and $T_2|_{\mathcal{D}(T_1)} = T_1$, we say $(T_2, \mathcal{D}(T_2))$ is an extension of $(T_1, \mathcal{D}(T_1))$, denoted by $(T_1, \mathcal{D}(T_1)) < (T_2, \mathcal{D}(T_2))$.

Remark 1. If there exists $M > 0$, such that $\forall x \in \mathcal{D}(T)$, $\|Tx\| \leq M\|x\|$, Then T could be extended to a linear operator, with domain \mathcal{H}_1 .

Next, we always assume $(T, \mathcal{D}(T))$ is closable: If $\{x_n\}_{n=1}^\infty \subset \mathcal{D}(T)$, such that $\lim_{n \rightarrow \infty} x_n = 0$, and $\lim_{n \rightarrow \infty} Tx_n$ exists, then we must have $\lim_{n \rightarrow \infty} Tx_n = 0$.

Remark 2. 1. The unbounded operator in Example 1 is closable: let $f_0 \in C_c^\infty(\mathbb{R}) \rightarrow 0$ in $L^2(\mathbb{R})$, and $f_n \rightarrow g$ for some $g \in L^2(\mathbb{R})$. If $g \neq 0 \in L^2(\mathbb{R})$, since $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, there exists $h \in C_c^\infty(\mathbb{R})$, s.t. $(g, h)_{L^2(\mathbb{R})} \neq 0$. But

$$(g, h) = \lim_{n \rightarrow \infty} (f'_n, h) = - \lim_{n \rightarrow \infty} (f_n, h') = 0.$$

As a result, we must have $g = 0$.

2. Let $\mathcal{H}_1 = L^2(\mathbb{R}), \mathcal{H}_2 = \mathbb{R}$, $\mathcal{D} = C_c(\mathbb{R}) \subset \mathcal{H}$. Consider the unbounded operator (T, \mathcal{D}) , $f \rightarrow f(0)$. Then (T, \mathcal{D}) is not closable: Let

$$f_n(x) = \begin{cases} n(x + 1/n), & \text{if } x \in (-1/n, 0) \\ n(1/n - x), & \text{if } x \in (0, 1/n) \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Then $f_n \in C_c(\mathbb{R})$ and $f_n \rightarrow 0$ in $L^2(\mathbb{R})$. Moreover, $f_n(0) = 1$, hence $\lim_{n \rightarrow \infty} Tf_n = 1 \neq 0$, which means T is not closable.

Definition 1. We say $(T, \mathcal{D}(T))$ is a close operator, if for a Cauchy Sequence $\{x_n\}_{n=1}^\infty \subset \mathcal{H}_1$ such that $\{Tx_n\} \subset \mathcal{H}_2$ is also a Cauchy sequence, then $x := \lim_{n \rightarrow \infty} x_n \in \mathcal{D}(T)$, and $Tx = \lim_{n \rightarrow \infty} Tx_n$

Definition 2 (close extension). We say $(T_1, \mathcal{D}(T_1))$ is a close extension of $(T_0, \mathcal{D}(T_0))$, if

1. $(T_1, \mathcal{D}(T_1))$ is closed;
2. $\mathcal{D}(T_0) \subset \mathcal{D}(T_1)$;
3. $T_1|_{\mathcal{D}(T_0)} = T_0$.

Let $(T, \mathcal{D}(T))$ be unbounded operator, for $x, y \in \mathcal{D}(T)$, define the inner product $\langle \cdot, \cdot \rangle_T$:

$$\langle x, y \rangle_T := \langle x, y \rangle_{\mathcal{H}_1} + \langle Tx, Ty \rangle_{\mathcal{H}_2}.$$

It's easy to check that if $(T, \mathcal{D}(T))$ is closed, then $\mathcal{D}(T)$ is complete with respect to the norm $\|\cdot\|_T$.

Let $\mathcal{D}(\bar{T}_{min})$ be the completion of $\mathcal{D}(T)$ under the norm $\|\cdot\|_T$. Since $\|x\|_{\mathcal{H}_1} \leq \|x\|_T, \forall x \in \mathcal{D}(T)$, we can think $\mathcal{D}(\bar{T}_{min})$ as a dense subspace of \mathcal{H}_1 . $\forall x \in \mathcal{D}(\bar{T}_{min})$, since T is closable define $\bar{T}_{min}x = \lim_{n \rightarrow \infty} Tx_n$ where $\lim_{n \rightarrow \infty} \|x_n - x\|_T = 0$. Then one can show that $(\bar{T}_{min}, \mathcal{D}(\bar{T}_{min}))$ is a close extension of $(T, \mathcal{D}(T))$, called **minimal extension** of $(T, \mathcal{D}(T))$. One can show that if $(T_1, \mathcal{D}(T_1))$ is another close extension of $(T, \mathcal{D}(T))$, then $(\bar{T}_{min}, \mathcal{D}(\bar{T}_{min})) < (T_1, \mathcal{D}(T_1))$.

Example 2. 1. Let Ω be a bouned domain in \mathbb{R}^n with smooth boundary. Let $\mathcal{H}_1 = L^2(\Omega)$, $\mathcal{H}_2 = \underbrace{L^2(\Omega) \oplus \dots \oplus L^2(\Omega)}_{n \text{ } L^2(\Omega)}$, $\mathcal{D} = C_c^\infty(\Omega)$. Define $T : \mathcal{H}_1 \mapsto \mathcal{H}_2$:

$$\phi \rightarrow (\frac{\partial}{\partial x_1}\phi, \dots, \frac{\partial}{\partial x_n}\phi), \forall \phi \in C_c^\infty.$$

Then $\mathcal{D}(\bar{T}_{min})$ is the Sobolev space $W_0^{1,2}(\Omega)$, T is the weak derivatives (See page 245 in [1] for more details).

2. Now let

$$\mathcal{D} = \{\phi \in C^\infty(\Omega) : \phi \text{ and } \partial_{x_i}\phi \text{ are } L^2\text{-integable}\}.$$

Then $\mathcal{D}(\bar{T}_{min})$ is the Sobolev space $W^{1,2}(\Omega)$ (See Theorem 2 in page 251 of [1]).

2 Adjoint operator

Definition 3 (Formal adjoint operator). We say $(S, \mathcal{D}(S))$ is a formal adjoint operator of $(T, \mathcal{D}(T))$, if $\forall x \in \mathcal{D}(T), y \in \mathcal{D}(S)$,

$$\langle Tx, y \rangle_{\mathcal{H}_2} = \langle x, Sy \rangle_{\mathcal{H}_1}.$$

If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, and $(S, \mathcal{D}(S)) = (T, \mathcal{D}(T))$, then we say $(T, \mathcal{D}(T))$ is symmetric.

It could be check easily that if $(T, \mathcal{D}(T))$ has a formal adjoint operator $(S, \mathcal{D}(S))$, then $(T, \mathcal{D}(T))$ is closable: let $x_n \in \mathcal{D}(T)$, such that $x_n \rightarrow 0$, $Tx_n \rightarrow g$. If $g \neq 0$, one can find $h \in \mathcal{D}(S)$, such that $(g, h) \neq 0$. But

$$(g, h) = \lim_{n \rightarrow \infty} (Tx_n, h) = \lim_{n \rightarrow \infty} (x_n, Sh) = 0,$$

which is a contradiction.

In fact, if $(T, \mathcal{D}(T))$ is closed, then it has a special formal adjoint operator, called adjoint operator:

Definition 4 (Adjoint Operator). We say $(T^*, \mathcal{D}(T^*))$ is $(T, \mathcal{D}(T))$ the adjoint operator of $(T, \mathcal{D}(T))$, if $(T^*, \mathcal{D}(T^*))$ is a formal adjoint operator of $(T, \mathcal{D}(T))$, and

$$\mathcal{D}(T^*) := \{y \in \mathcal{H}_2 : \text{there exists } M_y > 0 \text{ such that } |\langle T x, y \rangle_{\mathcal{H}_2}| \leq M_y \|x\|_{\mathcal{H}_1}, \forall x \in \mathcal{D}(T)\}.$$

If $\mathcal{H}_1 = \mathcal{H}_2$, $(T^*, \mathcal{D}(T^*)) = (T, \mathcal{D}(T))$, then we say $(T, \mathcal{D}(T))$ is self-adjoint.

It's easy to check that if $(T_1, \mathcal{D}(T_1)) < (T_2, \mathcal{D}(T_2))$, then

$$(T_2^*, \mathcal{D}(T_2^*)) < (T_1^*, \mathcal{D}(T_1^*)).$$

Moreover, it follows from the definition that $(T^*, \mathcal{D}(T^*))$ is closed: let $\{y_n\} \subset \mathcal{D}(T^*)$ be a Cauchy sequence, s.t. $T^*(y_n)$ is a Cauchy sequence in \mathcal{H}_1 . Let $y = \lim_{n \rightarrow \infty} y_n \in \mathcal{H}_2$, $z = \lim_{n \rightarrow \infty} T^*(y_n) \in \mathcal{H}_1$, then for all $x \in \mathcal{D}(T)$,

$$|\langle T x, y \rangle| = \lim_{n \rightarrow \infty} |\langle T x, y_n \rangle| = \lim_{n \rightarrow \infty} |\langle x, T^* y_n \rangle| = |\langle x, z \rangle| \leq \|z\|_{\mathcal{H}_1} \|x\|_{\mathcal{H}_1}.$$

Hence, one can see that $y \in \mathcal{D}(T^*)$, moreover $T^* y = z$.

In fact, one has $(T^{**}, \mathcal{D}(T^{**})) = (\bar{T}_{min}, \mathcal{D}(\bar{T}_{min}))$.

If $(S, \mathcal{D}(S))$ is a formal adjoint operator of $(T, \mathcal{D}(T))$ then $(S^*, \mathcal{D}(S^*))$ is a close extension of $(T, \mathcal{D}(T))$.

Example 3. Let Ω be a bounded domain in \mathbb{R}^n with smooth boundary. Let $\mathcal{H}_1 = L^2(\Omega)$, $\mathcal{H}_2 = L^2(\Omega) \oplus \dots \oplus L^2(\Omega)$, $\mathcal{D} = C_c^\infty(\Omega)$. Set $T : \mathcal{H}_1 \mapsto \mathcal{H}_2$:

$\underbrace{L^2(\Omega) \oplus \dots \oplus L^2(\Omega)}_{n \text{ copies of } L^2(\Omega)}$

$$\phi \rightarrow (\frac{\partial}{\partial x_1} \phi, \dots, \frac{\partial}{\partial x_n} \phi), \forall \phi \in C_c^\infty.$$

Set $\mathcal{D}^n := \underbrace{C_c^\infty(\Omega) \oplus \dots \oplus C_c^\infty(\Omega)}_{n \text{ copies of } C_c^\infty(\Omega)}$, and $S : \mathcal{H}_2 \mapsto \mathcal{H}_1$,

$$(\phi_1, \dots, \phi_n) \rightarrow - \sum_{k=1}^n \frac{\partial}{\partial x_k} \phi_k, \phi_k \in C_c^\infty(\Omega),$$

Then (S, \mathcal{D}^n) is a formal adjoint operator of (T, \mathcal{D}) . Moreover, it follows from the definition of Sobolev space that $\mathcal{D}(S^*) = W^{1,2}(\Omega)$. Here we give another description of Sobolev space $W^{1,2}(\Omega)$.

3 Friedrichs Extension and Essential self-adjoint

Let $(T, \mathcal{D}(T))$ be a nonnegative symmetric operator, that is, for all $\phi \in \mathcal{D}(T)$,

$$\langle T\phi, \phi \rangle_{\mathcal{H}} = \langle \phi, T\phi \rangle_{\mathcal{H}} \geq 0.$$

Then, on $\mathcal{D}(T)$,

$$\langle \phi, \psi \rangle_{T^{1/2}} := \langle \phi, \psi \rangle_{\mathcal{H}} + \langle \phi, T\psi \rangle_{\mathcal{H}}, \phi, \psi \in \mathcal{H}$$

defines an inner product. Let \mathcal{H}_1 be the completion of $\mathcal{D}(T)$ under the norm $\|\cdot\|_{T^{1/2}}$ then \mathcal{H}_1 could be think as a subspace of \mathcal{H} . Set

$$\mathcal{D}^F := \{\phi \in \mathcal{H}_1 : \langle \eta, \phi \rangle_H + \langle T\eta, \phi \rangle_{\mathcal{H}} \leq M_\phi \|\eta\|_{\mathcal{H}} (\forall \eta \in \mathcal{D}(T)) \text{ for some } M_\phi > 0.\}$$

By Riesz representation theorem, there exists $u \in \mathcal{H}$, such that

$$\langle \eta, \phi \rangle_H + \langle T\eta, \phi \rangle_{\mathcal{H}} = \langle \eta, u \rangle_{\mathcal{H}}. \quad (2)$$

Now set $T^F(\phi) = u - \phi$. We called (T^F, \mathcal{D}^F) be Friedrichs extension of $(T, \mathcal{D}(T))$. One can check that (T^F, \mathcal{D}^F) is a closed extension of $(T, \mathcal{D}(T))$, and is self-adjoint.

Proposition 1. *Let Ω be a bounded domain in \mathbb{R}^n with smooth boundary. $\mathcal{H}_1 = \mathcal{H}_2 = L^2(\mathbb{R}^n)$, $\mathcal{D} = C_c^\infty(\mathbb{R}^n)$, then the operator $T = \Delta$, $\phi \rightarrow \Delta\phi := -\sum_i \partial_i^2 \phi$ is symmetric. Then, $u \in \mathcal{D}^F$ iff $u \in W_0^{1,2}(\Omega)$ solve EPDEs below weakly for some $g \in L^2(\mathbb{R}^n)$:*

$$\begin{cases} \Delta u = g, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (3)$$

i.e., for all $v \in W_0^{1,2}(\Omega)$,

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} h v.$$

Furthermore, $T^F u = g$.

Next, let $\mathcal{D}^N = \{u \in C^\infty(\bar{\Omega}) : \partial_\nu u = 0 \text{ on } \Omega\}$ be the domain of $T^N = \Delta$, then $u \in (\mathcal{D}^N)^F$ iff $u \in W^{1,2}(\Omega)$ solves EPDEs below weakly for some $h \in L^2(\mathbb{R}^n)$:

$$\begin{cases} \Delta u = h, & \text{in } \Omega; \\ \partial_\nu u = 0, & \text{on } \partial\Omega, \end{cases} \quad (4)$$

i.e., for all $v \in W^{1,2}(\Omega)$,

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} h v.$$

Furthermore, $(T^F)^* u = g$.

Here ν is the normal direction on $\partial\Omega$.

Proof. If $u \in \mathcal{D}^F$, then there exists $g \in L^2(\Omega)$, s.t. for any $\eta \in C_c^\infty(\Omega)$

$$\langle \Delta\eta, u \rangle_{L^2(\Omega)} = \langle \eta, g - u \rangle_{L^2(\Omega)}.$$

While integration by parts shows that $\langle T\eta, u \rangle_{L^2(\Omega)} = \int_{\Omega} \nabla \eta \cdot \nabla u = \int_{\Omega} \eta(g - u)$. Since $C_c^\infty(\Omega)$ is dense in $W_0^{1,2}(\Omega)$, one can see that u solves

$$\begin{cases} \Delta u = g - u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (5)$$

On the other hand, if $u \in W_0^{1,2}(\Omega)$ solves (3) for some g , integration by parts shows that

$$\langle \Delta\eta, u \rangle_{\Omega} + \langle \eta, u \rangle_{L^2(\Omega)} \leq (\|g\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \|\eta\|_{L^2(\Omega)}$$

for all $u \in C_c^\infty(\Omega)$. Hence $u \in \mathcal{D}^F$

For Neumann's case, the proof is similar. The only somewhat nontrivial part is to show that \mathcal{D}^N is dense in $W^{1,2}(\Omega)$ (w.r.t. to the norm $\|\cdot\|_{W^{1,2}(\Omega)}$): First, since $C^\infty(\bar{\Omega})$ is dense in $W^{1,2}(\Omega)$, for $u \in \mathcal{D}^N$, any $\epsilon > 0$, there exists $v \in C^\infty(\bar{\Omega})$, s.t. $\|u - v\|_{W^{1,2}(\Omega)} < \epsilon/2$. Fix $\eta \in C_c^\infty(\mathbb{R})$, s.t. $\text{supp } \eta \supset (-1, 1)$, $\eta|_{(-1/2, 1/2)} \equiv 1$. Set $M = \int_{\partial\Omega} |\partial_\nu v|^2 + \int_{\partial\Omega} |\nabla^{\partial\Omega} \partial_\nu v|^2$. Let $d(x) := \text{dist}(x, \Omega)$, $w(x) = d(x)\eta(NMd(x))\partial_\nu v$, then when $N > 0$ is big, $\|w\|_{W^{1,2}(\Omega)} \leq \frac{C}{N}$ for some $C > 0$ depending only on Ω . Furthermore, $\partial_\nu w = \partial_\nu v$. Then for N is big enough, $\|u - (v + w)\| \leq \epsilon$, and $\partial_\nu(v + w) = 0$. Hence, \mathcal{D}^N is dense in $W^{1,2}(\Omega)$. \square

Moreover, one has

Theorem 1. *When $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, then T^F (or $(T^N)^F$) has discrete spectrum $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \dots$ (or respectively, $0 \leq \lambda_{1,N} \leq \lambda_{2,N} \leq \dots \leq \lambda_{k,N} \dots$). Moreover, their eigenfunctions $\{e_k\}$ (or $\{e_{k,N}\}$) respectively) forms an orthonormal basis of $L^2(\Omega)$. Furthermore, $\lim_{k \rightarrow \infty} \lambda_k = \infty$ (or $\lim_{k \rightarrow \infty} \lambda_{k,N} = \infty$ respectively).*

4 min-max principle and EPDEs with boundary conditions

In this section, we would like to present another description of eigenvalues of Laplacian operator. For any vector space L , let $\Phi_k(L)$ denote the set of k -dimensional vector spaces.

Let Ω be a bounded domain in \mathbb{R}^n with smooth boundary.

For $u \in W_0^{1,2}(\Omega)$ or $u \in W^{1,2}(\Omega)$, consider the functional

$$\mathcal{F}(u) = \frac{\int_{\Omega} |\nabla u|^2}{|u|^2}.$$

Theorem 2. *Let $l_k = \inf_{V \in \Phi_k(W_0^{1,2}(\Omega))} \sup_{u \in V} \mathcal{F}(u)$, then there exists $0 \neq u_k \in W_0^{1,2}(\Omega)$ solves*

$$\begin{cases} \Delta u_k = l_k u_k, & \text{in } \Omega, \\ u_k = 0, & \text{on } \partial\Omega \end{cases} \quad (6)$$

weakly. That is, for any $w \in W_0^{1,2}(\Omega)$,

$$\int_{\Omega} \nabla u_k \cdot \nabla w = l_k \int_{\Omega} u_k w.$$

Moreover, u_k is orthogonal to $\{u_j\}_{j=1}^{k-1}$.

Proof. For simplicity, we prove the case of $k = 1$ only.

Let $l_1 = \inf_{0 \neq u \in W_0^{1,2}(\Omega)} \mathcal{F}(u)$. Let $w_n \in W_0^{1,2}(\Omega)$ such that $\|w_n\|_{L^2(\Omega)} = 1$ $\mathcal{F}(w_n) \rightarrow \lambda$. Then $\|w_n\|_{W^{1,2}(\Omega)} \leq C$ for some $C > 0$. Hence, since $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ compactly, we may assume that $w_n \rightarrow u_1$ for some $u_1 \in L^2(\Omega)$. Moreover, since $\|\nabla w_n\|_{L^2(\Omega)} \leq C$, we may assume that $\nabla w_n \rightarrow \psi$ in weak $L^2(\Omega)$ -topology.

Then for $\rho \in C_c^\infty(\Omega)$,

$$\int_{\Omega} \psi \rho = \lim_{n \rightarrow \infty} \int_{\Omega} \nabla w_n \rho = - \lim_{n \rightarrow \infty} \int_{\Omega} w_n \nabla \rho = - \int_{\Omega} u_1 \nabla \rho$$

Hence, u_1 has weak derivative ψ . Hence $u_1 \in W_0^{1,2}(\Omega)$.

Next, we would like to show that u_1 satisfies the EPDEs (6) weakly.

Fix $0 \neq \rho \in W_0^{1,2}(\Omega)$, $u_t = u_1 + t\rho$, then we must have

$$\frac{d}{dt} \mathcal{F}(u_t)|_{t=0} = 0.$$

Which, by a straightforward computation, implies that

$$\int_{\Omega} \nabla u \nabla \rho = l_1 \int_{\Omega} u \rho.$$

□

Similarly,

Theorem 3. *Let $l_{k,N} = \inf_{V \in \Phi_k(W^{1,2}(\Omega))} \sup_{u \in V} \mathcal{F}(u)$, then there exists $0 \neq u_{k,N} \in W^{1,2}(\Omega)$ solves*

$$\begin{cases} \Delta u_{k,N} = l_{k,N} u_{k,N}, & \text{in } \Omega, \\ \partial_{\nu} u_{k,N} = 0, & \text{on } \partial\Omega \end{cases} \quad (7)$$

weakly. That is, for any $w \in W^{1,2}(\Omega)$,

$$\int_{\Omega} \nabla u_k \cdot \nabla w = l_k \int_{\Omega} u_k w.$$

Moreover, $u_{k,N}$ is orthogonal to $\{u_{j,N}\}_{j=1}^{k-1}$.

Remark 3. *In fact, $\lambda_k = l_k$ and $\lambda_{k,N} = l_{k,N}$. Moreover, one can take $e_k = \frac{u_k}{\|u_k\|_{L^2(\Omega)}}$ and $e_{k,N} = \frac{u_{k,N}}{\|u_{k,N}\|_{L^2(\Omega)}}$.*

References

- [1] Lawrence C. Evans. *Partial differential equations*. American Mathematical Society, Providence, R.I., 2010.