Research Statement

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My research focuses on the intersection of geometry, topology, and PDEs, particularly in studying mirror symmetry and the Landau-Ginzburg (LG) model through index theory.

Research in the LG B-model has been hindered by the lack of tools for non-compact spaces. Thus, my doctoral thesis primarily involved analysis on non-compact manifolds with potential functions [11, 12]. The analysis results in [11, 12] not only allowed for defining geometric/topological invariants, like the BCOV-type invariant for the LG B-model, but also introduced new techniques. For example, using Witten deformation for non-Morse functions, I present a concise and novel proof for the gluing formula of the analytic torsion form [48, 47], a crucial step in establishing the higher Cheeger-Müller/Bismut-Zhang theorem. Moreover, compared to well-established methods such as b-calculus [27, 18, 19], adiabatic limits [36, 38], and Vishik's moving boundary conditions theory [42, 24], the use of Witten deformation are more straightforward and intuitive when dealing with gluing formulas of global spectral invariants. Also, by applying the heat kernel expansion for non-compact manifolds derived in [12], we provided a simple proof of the Weyl's law for Schrödinger operators on non-compact manifolds [10]. This approach may extend to studying Weyl's law in singular spaces, such as Ricci limit spaces and RCD spaces, with relevance to the study of singular sets in Ricci limit spaces [9].

The proposed research is to harness the novel tools developed in LG B-model research to tackle problems within index theory and related topics. More specifically, we will explore:

(a) Higher Cheeger-Müller/Bismut-Zhang theorem, see §2.1;

(b) The application of Witten deformation for non-Morse functions, including the study of the gluing formula for eta forms and spectral geometry of minimal hypersurfaces, see §3 and §4.

1 Preliminary and a summary of my results

1.1 Witten deformation

Given that the term "Witten deformation" will be frequently mentioned in this research statement, I would like to provide some breif background information on Witten deformation.

In his influential paper [44], Edward Witten introduced the concept of Witten deformation. Classical Witten deformation is a deformation of the de Rham complex on a manifold M. It simply deforms the exterior derivative by

$$d_{Tf} := e^{-Tf} \circ d \circ e^{Tf} = d + Tdf\wedge$$

where f is a smooth function and T is the deformation parameter.

As the parameter T varies from 0 to ∞, Witten deformation establishes a connection between geometric/topological invariants on the manifold M and a small neighborhood of critical point set of f.

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1.2 A synopsis of my results

To lay the foundation for studying the LG B-model, my doctoral thesis primarily involved analysis on non-compact manifolds with potential functions. We gave estimates for the asymptotic behavior of eigenfunctions of Schrödinger-type operators on noncompact manifolds at infinity [11]. Using these estimates, I extended the Thom-Smale-Witten cohomology theory to Morse functions on non-compact manifolds. We also study the asymptotic expansion of heat kernels for Schrödinger-type operators on non-compact manifold [12]. As an application of this expansion, we extend local index theorem to non-compact manifolds with potentials. These findings laid the groundwork for defining the genus-1 term for the LG B-model, which was conjectured to be some analytic torsion.

In collaboration with Xinxing Tang [41], we study the CY/LG correspondence for the genus-0 theory of B-model. Our findings revealed that, apart from the Jacobian ring, the space of harmonic forms for a Witten deformed Laplacian could serve as the state space for the LG B-model. Moreover, as suggested by [14, 40], this space carries a natural tt^* structure (a generalized variation of Hodge structures). Furthermore, we show that this tt^* structure is compatible with the tt^* structure on CY's side. This discovery not only showed the CY/LG correspondence for the tt^* structure, but also paved the way for subsequent research into the CY/LG correspondence for the genus 1 theory.

Apart from studying the genus 1 theory of LG model, I also apply the techniques developed during my study of the LG model to explore problems related to the index theorem and other topics. For example, using Witten deformation for non-Morse functions, I present a concise and novel proof for the gluing formula of the analytic torsion form [47, 48]. In addition, applying the heat kernel expansion mentioned earlier, we provide a simple heat kernel proof of the semiclassical/non-semiclassical Weyl's law for Schrödinger operators on non-compact manifolds [10]. Finally, my interest extends to comparison geometry of submanifolds. In conjunction with Fagui Li [25], we have deduced a lower-bound estimate for the first eigenvalue of hypersurfaces. Subsequently, in our ongoing collaborative efforts (see §4), we will employ Witten deformation as a tool to study the spectral geometry of submanifolds.

2 Witten deformations for non-Morse functions

Previous research on Witten deformation has mostly focused on compact manifolds with Morse functions, while we discovered that Witten deformation for a family of non-Morse functions f_T parametrized by $T \in \mathbb{R}^+$ ([48, 47]) offers a new approach to studying the gluing formulas of global spectral invariants (such as eta invariant, analytic torsion e.t.c.).

Let me briefly explain the philosophy of Witten deformation for non-Morse functions.

Let $Y \subset M$ be a hypersurface cutting M into two pieces M_1 and M_2 (see Figure 1).

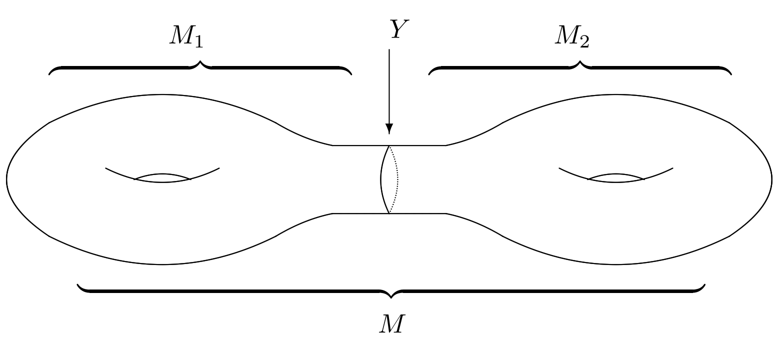

Figure 1:

We then construct a family of non-Morse functions f_T such that as $T \to \infty$, the "critical sets" of f_T consist of M_1 and M_2, and we could roughly think "Morse index" of M_1 and M_2 as 0 and 1 respectively.

Let $d_T := d + df_T\wedge$ be the Witten deformation w.r.t. f_T, and Δ_T be the Hodge Laplacian for d_T.

Then we observe that when $T = 0$, Δ_T corresponds to the original Laplacian on M. As $T \to \infty$, the eigenvalues of Δ_T converge to the eigenvalues of the Laplacians on M_1 and M_2 under appropriate boundary conditions. Based upon this, I establish the gluing formula for analytic torsion and analytic torsion forms [48, 47].

Next we will explore further applications of Witten deformation for non-Morse functions.

2.1 Exploring higher Cheeger-Müller/Bismut-Zhang Theorem

In [39], Ray and Singer introduced the analytic torsion for a unitary flat vector bundle over a closed Riemannian manifold M, and conjectured that this analytic torsion coincides with the classical Reidemeister torsion (R-torsion), a topological invariant that distinguishes homotopy equivalent but non-homeomorphic CW-spaces and manifolds (cf. [31]). This conjecture was later proven in by Cheeger [8] and Müller [33] independently. Bismut and Zhang extend the Cheeger-Müller theorem to general flat vector bundles using Witten deformation [5].

It was conjectured that R-torsion and Ray-Singer torsion can be extended to invariants of a C^∞ fibration $\pi : M \to S$ of a closed fiber Z, associated with a flat complex vector bundle $F \to M$ [43]. Bismut and Lott [4] then construct analytic torsion forms (BL-torsion), which are even forms on S. Igusa, motivated by the work of Bismut and Lott, developed the Igusa-Klein (IK) torsion, a higher topological torsion [20]. As an application of IK-torsion, Goette, Igusa, and Williams [17, 16] uncover fiber bundles' exotic smooth structure. Then it becomes a natural and significant question to ask

Problem 2.1. *What's the relationship between these higher torsion invariants, i.e., do we have higher Cheeger-Müller/Bismut-Zhang theorem?*

Two approaches exist for attacking Problem 2.1. First, under the assumption that there exists a fiberwise Morse function [3, 15], Bismut and Goette established a higher version of the Cheeger-Müller/Bismut-Zhang theorem.

The second method is the so-called axiomatization method: higher torsion invariants were axiomatized by Igusa [21], and Igusa showed that IK-torsion complies with his axioms. His axiomatization contains two axioms: the axiom of additivity and the axiom of transfer. And any higher torsion invariant that satisfies Igusa's axioms is simply a linear combination of IK-torsion and the higher Miller-Morita-Mumford class [32, 35, 30]. BL-torsion is proven to satisfy the transfer axiom thanks to the work of Ma [26]. The axiom of additivity was recently established by Puchol-Zhang-Zhu in [36]. I also provide a simple proof using the techniques described in §2. Using [26, 36] and Igusa's axiom of higher torsion invariants, Puchol-Zhang-Zhu were able to prove the higher Cheeger-Müller theorem for trivial bundles in [37].

It is important to note that, even if a fiberwise Morse function does not exist, a fiberwise framed function $p : M \to \mathbb{R}$ can still exist for a smooth fibration $\pi : M \to S$ with a closed fiber Z (c.f. [22, Theorem 2.1]). Here we say $f : M \to \mathbb{R}$ is a framed function (c.f. [22, Definition 2.2]), if f is a smooth function with only non-degenerate and birth-death critical points. Near a birth-death, f is given by

$$x_1^3 - \sum_{j=2}^{i} x_j^2 + \sum_{k=i+1}^{n} x_k^2 + C.$$

Recently, combining the two methods and the techniques described in §2, Yeping Zhang, Martin Puchol and I initiated a program to study the higher Cheeger-Müller/Bismut-Zhang theorem for general flat bundles. This program involves studying a two-parameter Witten deformation, which can be roughly expressed as $\bar{f}_{T_1} + T_2 p$. Here, $\bar{f}_T$ refers to a family of non-Morse smooth functions that are closely related to f_T described above, p is a fiberwise framed function.

3 On gluing formula of eta invariants and eta forms

Certain spectral invariants have a nice behavior when it comes to operations like cutting and pasting. For instance, the index of a Dirac operator exhibits additive behavior upon the gluing of manifolds, a property that aligns with the index's inherent locality. However, for some global spectral invariants such as analytic torsion and the η-invariant, the surprising properties related to cutting and pasting pose nontrivial challenges in their proofs.

The eta invariant can be understood as the boundary component of the index theorem for manifolds with boundaries [1].

Now consider a hypersurface $Y \subset M$ that divides the manifold M into two parts: M_1 and M_2. Let $(E \to M, h^E)$ represent a Clifford bundle with a Hermitian metric h^E, and let D^E be the corresponding Dirac operator and D_i^E its restriction on M_i. Let $\eta(M, E)$ be the assocaited eta invariant.

When defining eta invariants for M_1 and M_2, boundary conditions must be imposed on Y. The most natural choice is the Atiyah-Patodi-Singer (APS) boundary conditions. Additionally, there exist generalized APS boundary conditions, such as spectral sections [28, 29] and the self-adjoint Fredholm Grassmannian [23]. Roughly, the self-adjoint Fredholm Grassmannian consists of a set of unitary projections $P : L^2(Y, E|_Y) \to L^2(Y, E|_Y)$, possessing certain nice properties. These properties allow the Dirac operator D_i^E with the domain:

$$\left\{\phi \in L^2(M_i, E) \mid \phi \in W^{1,2}(M_i, E) \text{ and } P\left(\phi_{|Y}\right) = 0\right\} \subset L^2(M_i, E)$$

to become a self-adjoint Fredholm operator. When one has such a projection P_i, it becomes possible to define the eta invariant, denoted as $\eta(M_i, E, P_i)$, on the manifold M_i.

The gluing formula for eta invariants can be expressed as follows:

$$\begin{gathered}\eta(M, E) - \eta(M_1, E, P) - \eta(M_2, E, 1 - P) \equiv 0 \quad \mathrm{mod}\ (\mathbb{Z}) \\ \text{or } \eta(M, E) - \eta(M_1, E, P) - \eta(M_2, E, 1 - P) = \mathrm{SF}(D_t(P, Y)),\end{gathered} \tag{1}$$

where $\mathrm{SF}(D_t(P, Y))$ denotes the spectral flow of a family of Dirac operators $D_t(P, Y)$, which is parametrized by $t \in [0, 1]$ and depends on P and Y.

There are several proofs for (1) (c.f.[6, 7, 45, 46, 34]). In my ongoing research [49], given a nice unitary projection P on the boundary and the non-Morse functions f_T described in §2, I construct a family of Dirac operators D_T^P on M. Through this construction, I establish a similar limit as seen in §2, thus offering a novel proof of the gluing formula for eta invariants.

In the family case, the Bismut-Cheeger eta form is well-defined for a fibration of manifolds, when either the fiberwise Dirac operator's kernel forms a vector bundle [2] or, more generally, when the fiberwise Dirac operator admits a spectral section [13].

A natural question is,

Problem 3.1. *Assuming the kernel of fiberwise Dirac operator forms a vector bundle, do we have a gluing formula in the family case, replacing the spectral flow in* (1) *with higher spectral flow [13]?*

Problem 3.1 is still open in index theorem. Extending (1) to the family case encounters several essential challenges, including ensuring the well-definedness of the Bismut-Cheeger eta form for a family of manifolds with boundaries, selecting appropriate boundary conditions e.t.c. It is reasonable to anticipate that the methods outlined in §2 can be applied to address Problem 3.1.

4 Witten deformation for non-Morse function and Yau's conjecture

Consider a hypersurface $Y \subset M$ that divides the manifold M into two parts: M_1 and M_2. One can also construct a family of non-Morse function $\tilde{f}_T$ as in [36], such that as $T \to \infty$, the critical sets of $\tilde{f}_T$ consists of M_1, M_2 and Y with "Morse index" $0, 0$ and 1 respectively.

Let Δ_T be the Witten deformed Beltrami-Laplacian (i.e., Laplacian operator acting on functions instead of differential forms) associated with the non-Morse functions $\tilde{f}_T$. Let Δ_i be the Beltrami-Laplacian on M_i with Dirichlet boundary conditions, Δ_Y be the Beltrami-Laplacian on $Y \subset M$ with induced metric.

Let λ_k be the k-th eigenvalues of $\Delta_1 \oplus \Delta_2 \oplus \Delta_Y$, $\lambda_k(T)$ be the k-th eigenvalue of Δ_T, then one still have limit $\lim_{T\to\infty} \lambda_k(T) = \lambda_k$ as described in §2 if the metric is of **product-type** near Y. However, Fagui Li and I observed that if Y is a **minimal hypersurface**, we similarly has $\lim_{T\to\infty} \lambda_k(T) = \lambda_k$. Moreover, if the Ricci curvature Ric_M of M has lower bound $n-1$, where $n = \dim(M)$, then $\lambda_1(T) \to \lambda_1(\Delta_Y)$ as $T \to \infty$, where $\lambda_1(\Delta_Y)$ is the first non-zero eigenvalue of Y. By estimating $\partial_T \lambda_1(T)$, one can obtain a lower bound of $\lambda_1(Y)$.

We hope that this observation will offer some new insight to attack the famous Yau's Conjecture on the first eigenvalue of minimal hypersurfaces in the unit sphere:

Problem 4.1 (Yau's conjecture [50]). *If Σ^n is a closed embedded minimal hypersurface of the unit sphere $\mathbb{S}^{n+1}$, then the first nonzero eigenvalue of the Laplacian on Σ is equal to n.*

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