

Heat Equation on Vector Bundles

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1 Basic Settings

Let (M, g) be a closed Riemannian manifold with Levi-Civita connection ∇^{LC} , $E \rightarrow M$ be a hermitian bundle with hermitian metric h . Let ∇^E be a unitary connection on E , then the connection Laplacian $\Delta^E : \Gamma(E) \rightarrow \Gamma(E)$ is defined as

$$\Delta^E := - \sum_i \nabla_{e_i}^E \nabla_{e_i}^E + \nabla_{\nabla_{e_i}^{LC} e_i}^E,$$

where $\{e_i\}$ is a local orthonormal frame.

Let s be a smooth section of $E \rightarrow M$, and for $\epsilon > 0$, denote $|s|_\epsilon = \sqrt{h(s, s) + \epsilon}$. In particular, denote $|s| := \sqrt{h(s, s)}$. Let Δ^{LC} be the Laplace–Beltrami operator on (M, g) , then

Proposition 1.1.

$$\begin{aligned} \Delta^{LC}|s|_\epsilon &= \frac{\operatorname{Re} h(s, \Delta^E s)}{|s|_\epsilon} - \frac{\sum_i h(\nabla_{e_i}^E s, \nabla_{e_i}^E s) |s|_\epsilon^2 - \sum_i (\operatorname{Re} h(\nabla_{e_i}^E s, s))^2}{|s|_\epsilon^3} \\ &\leq \frac{\operatorname{Re} h(\Delta^E s, s)}{|s|_\epsilon} \end{aligned}$$

where the last inequality follows from Cauchy–Schwartz inequality. Here Re denotes the real part of a complex number.

Theorem 1.2 (Maximal Principle). *Let $\Omega \subset M$ be a connected domain in M with smooth boundary. Assume that $s \in \Gamma(E)$ solves*

$$\Delta^E s = 0 \text{ in } \Omega$$

$$\text{Then } \sup_{p \in \Omega} |s|(p) = \sup_{p \in \partial\Omega} |s|(p).$$

Proof. It follows from Proposition 1.1 that

$$\Delta^{LC}|s|_\epsilon \leq 0.$$

Hence, $\sup_{p \in \Omega} |s|_\epsilon = \sup_{p \in \partial\Omega} |s|_\epsilon$. Letting $\epsilon \rightarrow 0$, the result follows. \square

2 Heat Equation

It follows from Proposition 1.1 again that

Theorem 2.1 (Maximal Principle). *Let $s(t)$ be a time dependent section solves*

$$\begin{cases} (\partial_t + \Delta^E)s(t) = 0 \text{ in } M \times (0, T] \\ s(0) = s_0 \end{cases} \quad (1)$$

for some $s_0 \in \Gamma(E)$, $T > 0$. Then $\sup_{(p,t) \in M \times (0,T]} |s| = \sup_{p \in M} |s_0|$.

Set $p_i : M \times M \rightarrow M$, $(p_1, p_2) \rightarrow p_i$, $i = 1, 2$, and let $E^* \rightarrow M$ be the dual bundle of $E \rightarrow M$, then heat kernel $K(t, x, y)$ with respect to Δ^E is a time-dependent section of $p_1^*E \otimes p_2^*E^*$, such that

$$s(t, x) := \int_M (K(t, x, y), s_0(y)) dy$$

solves (1), where (\cdot, \cdot) is the nature pairing of E^* and E .

Proposition 2.2. 1. $K(t, x, y)$ solves

$$\begin{cases} (\partial_t + \Delta_y^E)K(t, x, y) = 0, \\ \lim_{t \rightarrow 0} K(t, x, y) = \delta_x(y) \sum_i e_i(x) \otimes \tilde{e}_j(y), \end{cases}$$

where $\{e_i\}, \{\tilde{e}_j\}$ are orthonormal basis of E and E^* near x and y respectively.

2. $K(t, x, y) = K(t, y, x)^*$.

3. $K(t + s, x, y) = \int_M (K(s, x, z), K(s, z, y)) dz$. Hence,

$$K(2t, x, x) = \int_M (K(s, x, y), K(s, x, y)^*) dy$$

Theorem 2.3. Let $k(t, x, y)$ be the heat kernel with respect to Δ^{LC} , then $|K(t, x, y)| = \sqrt{h(K(t, x, y), K(t, x, y))} \leq nk(t, x, y)$, where h is the metric on $p_1^*E \otimes p_2^*E^*$ induced by h on $E \rightarrow M$, $n = \text{rank}(E)$.

Proof. Notice that $\partial_t |K|_{\epsilon t} = \frac{2\text{Re}h(\partial_t K, K) + \epsilon}{2|K|_{\epsilon t}} \leq \frac{\text{Re}h(\partial_t K, K)}{|K|_{\epsilon t}} + \frac{\sqrt{\epsilon}}{2\sqrt{t}}$.

By Proposition 1.1,

$$\begin{cases} (\partial_t + \Delta_y^{LC}) \left(e^{-\sqrt{\epsilon}t} |K(t, x, y)|_{\epsilon t} - nk(t, x, y) \right) \leq 0, \\ \lim_{t \rightarrow 0} \left(e^{-\sqrt{\epsilon}t} |K(t, x, y)|_{\epsilon t} - nk(t, x, y) \right) = 0. \end{cases}$$

By classical maximal principle,

$$\sup_{M \times [0, \infty)} e^{-\sqrt{\epsilon}t} |K(t, x, y)|_{\epsilon t} - nk(t, x, y) \leq 0.$$

Letting $\epsilon \rightarrow 0$, the result follows. \square

3 Heat equation for Hodge Laplacian

Let Δ^H be the Hodge Laplacian, Δ^C be the connection laplacian on $\Lambda^*(T^*M) \rightarrow M$, then one has Bochner-Lichnerowicz-Weitzenbock formula

$$\Delta^H = \Delta^C + \sum_{e_i, e_j} c(e_i)c(e_j)R(e_i, e_j),$$

where $\{e_i\}$ is a local orthonormal frame, $\{e^i\}$ is its dual frame, $c(e_i) = e^i \wedge -\iota_{e_i}$, R is the curvature operator.

Theorem 3.1. *Let $K^H(t, x, y)$ be the heat kernel with respect to Δ^H , then there exists $C = C(n, R) > 0$, such that*

$$|K^H(t, x, y)| \leq e^{Ct}k(t, x, y),$$

where $n = \dim(M)$.

Proof. By Proposition 1.1 and Bochner-Lichnerowicz-Weitzenbock formula, there exists $C = C(n, R) > 0$, such that

$$\begin{cases} (\partial_t + \Delta_y^{LC}) \left(e^{-\sqrt{\epsilon}t} e^{-Ct} |K(t, x, y)|_{\epsilon t} - nk(t, x, y) \right) \leq 0, \\ \lim_{t \rightarrow 0} \left(e^{-\sqrt{\epsilon}t} e^{-Ct} |K(t, x, y)|_{\epsilon t} - nk(t, x, y) \right) = 0. \end{cases}$$

By classical maximal principle, and letting $\epsilon \rightarrow 0$, the result follows. \square

Corollary 3.2. *There exists $C(n, R) > 0$, such that for all $t \in (0, 1)$,*

$$\int_M |K(t, x, y)| dy \leq C(n, R).$$

Proof. This is because

$$\int_M k(t, x, y) dy = 1.$$

\square