Heat Equation on Vector Bundles

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1 Basic Settings

Let (M, g) be a closed Riemannian manifolds with Levi-Civita connection ∇^{LC}, $E \to M$ be a hermitian bundle with hermitian metric h. Let ∇^E a unitary connection on E, then the connection Laplacian $\Delta^E : \Gamma(E) \to \Gamma(E)$ is defined as

$$\Delta^E := -\sum_i \nabla^E_{e_i}\nabla^E_{e_i} + \nabla^E_{\nabla^{LC}_{e_i}e_i},$$

where $\{e_i\}$ is a local orthonormal frame.

Let s be a smooth section of $E \to M$, and for $\epsilon > 0$, denote $|s|_\epsilon = \sqrt{h(s,s)+\epsilon}$. In particular, denote $|s| := \sqrt{h(s,s)}$. Let Δ^{LC} be the Laplace–Beltrami operator on (M, g), then

Proposition 1.1.

$$\begin{aligned}\Delta^{LC}|s|_\epsilon &= \frac{\mathrm{Re}h(s,\Delta^E s)}{|s|_\epsilon} - \frac{\sum_i h(\nabla^E_{e_i}s,\nabla^E_{e_i}s)|s|^2_\epsilon - \sum_i(\mathrm{Re}h(\nabla^E_{e_i}s,s))^2}{|s|^3_\epsilon}\\ &\le \frac{\mathrm{Re}h(\Delta^E s,s)}{|s|_\epsilon}\end{aligned}$$

where the last inequality follows from Cauchy-Schwartz inequality. Here Re *denotes the real part of a complex number.*

Theorem 1.2 (Maximal Principle)**.** *Let $\Omega \subset M$ be a connected domain in M with smooth boundary. Assume that $s \in \Gamma(E)$ solves*

$$\Delta^E s = 0 \ \textit{in}\ \Omega$$

Then $\sup_{p\in\Omega}|s|(p) = \sup_{p\in\partial\Omega}|s|(p)$.

Proof. It follows from Proposition 1.1 that

$$\Delta^{LC}|s|_\epsilon \le 0.$$

Hence, $\sup_{p\in\Omega}|s|_\epsilon = \sup_{p\in\partial\Omega}|s|_\epsilon$. Letting $\epsilon \to 0$, the result follows. □

2 Heat Equation

It follows from Proposition 1.1 again that

Theorem 2.1 (Maximal Principle)**.** *Let $s(t)$ be a time dependent section solves*

$$\begin{cases} (\partial_t + \Delta^E)s(t) = 0 \ in \ M \times (0,T] \\ s(0) = s_0 \end{cases} \tag{1}$$

for some $s_0 \in \Gamma(E), T > 0$. Then $\sup_{(p,t)\in M\times(0,T]} |s| = \sup_{p\in M} |s_0|$.

Set $p_i : M \times M \to M$, $(p_1, p_2) \to p_i$, $i = 1, 2$, and let $E^* \to M$ be the dual bundle of $E \to M$, then heat kernel $K(t, x, y)$ with respect to Δ^E is a time-dependent section of $p_1^* E \otimes p_2^* E^*$, such that

$$s(t, x) := \int_M (K(t, x, y), s_0(y))\, dy$$

solves (1), where $(\cdot, \cdot)$ is the nature pairing of E^* and E.

Proposition 2.2. *1. $K(t, x, y)$ solves*

$$\begin{cases} (\partial_t + \Delta_y^E)K(t, x, y) = 0, \\ \lim_{t\to 0} K(t, x, y) = \delta_x(y) \sum_i e_i(x) \otimes \widetilde{e}_j(y), \end{cases}$$

where $\{e_i\}, \{\widetilde{e}_j\}$ are orthonormal basis of E and E^* near x and y respectively.

2. $K(t, x, y) = K(t, y, x)^*$.

3. $K(t + s, x, y) = \int_M (K(s, x, z), K(s, z, y))\, dz$. Hence,

$$K(2t, x, x) = \int_M (K(s, x, y), K(s, x, y)^*)\, dy$$

.

Theorem 2.3. *Let $k(t, x, y)$ be the heat kernel with respect to Δ^{LC}, then $|K(t, x, y)| = \sqrt{h(K(t, x, y), K(t, x, y))} \leq nk(t, x, y)$, where h is the metric on $p_1^* E \otimes p_2^* E^*$ induced by h on $E \to M$, $n = \mathrm{rank}(E)$.*

Proof. Notice that $\partial_t |K|_{\epsilon t^2} = \frac{2\mathrm{Re}h(\partial_t K,K)+2\epsilon t}{2|K|_{\epsilon t}} \leq \frac{\mathrm{Re}h(\partial_t K,K)}{|K|_{\epsilon t}} + \sqrt{\epsilon}$.

By Proposition 1.1,

$$\begin{cases} (\partial_t + \Delta_y^{LC}) \left(|K(t, x, y)|_{\epsilon t^2} - \sqrt{\epsilon} t - nk(t, x, y)\right) \leq 0, \\ \lim_{t\to 0} \left(|K(t, x, y)|_{\epsilon t^2} - \sqrt{\epsilon} t - nk(t, x, y)\right) = 0. \end{cases}$$

By classical maximal principle,

$$\sup_{M\times[0,\infty)} |K(t, x, y)|_{\epsilon t^2} - \sqrt{\epsilon} t - nk(t, x, y) \leq 0.$$

Letting $\epsilon \to 0$, the result follows. $\square$

3 Heat equation for Hodge Laplacian

Let Δ^H be the Hodge Laplacian, Δ^C be the connection laplacian on $\Lambda^*(T^*M) \to M$, then one has Bochner-Lichnerowicz-Weitzenbock formula

$$\Delta^H = \Delta^C + \sum_{e_i, e_j} c(e_i)c(e_j)R(e_i, e_j),$$

where $\{e_i\}$ is a local orthonormal frame, $\{e^i\}$ is its dual frame, $c(e_i) = e^i \wedge -\iota_{e_i}$, R is the curvature operator.

Theorem 3.1. *Let $K^H(t,x,y)$ be the heat kernel with respect to Δ^H, then there exists $C = C(n,R) > 0$, such that*

$$|K^H(t,x,y)| \leq e^{Ct} k(t,x,y),$$

where $n = \dim(M)$.

Proof. By Proposition 1.1 and Bochner-Lichnerowicz-Weitzenbock formula, there exists $C = C(n,R) > 0$, such that

$$\begin{cases} (\partial_t + \Delta_y^{LC}) \left(e^{-Ct} |K(t,x,y)|_{\epsilon t^2} - \sqrt{\epsilon} t - nk(t,x,y) \right) \leq 0, \\ \lim_{t \to 0} \left(e^{-Ct} |K(t,x,y)|_{\epsilon t^2} - \sqrt{\epsilon} t - nk(t,x,y) \right) = 0. \end{cases}$$

By classical maximal principle, and letting $\epsilon \to 0$, the result follows. □

Corollary 3.2. *There exists $C(n,R) > 0$, such that for all $t \in (0,1)$,*

$$\int_M |K(t,x,y)| dy \leq C(n,R).$$

Proof. This is because

$$\int_M k(t,x,y) dy = 1.$$

□