Foundations of Modern Finance I

Cheat sheet for MITx: 15.415.1x Foundations of Modern Finance I.

Week 1: Introduction

A Framework for Financial Analysis

Income Statement

- Source of funds = Use of funds
  $NI + \Delta D + \Delta E = I + C + Div + T$
- NI: net income, ΔD : funds raised from new debt issue, ΔE : funds raised from new equity issue, I: investment, C: coupon payment, Div: dividend payment, T : tax payment,

Week 2: Market Prices and Present Value

State-space model for time and risk

State-space model

- Assets can be traded at time $t = 0$ with payoffs at time $t = 1$.
- The price of an asset is P at $t = 0$ with payoff $X = (X_1, \ldots, X_N)$ at $t = 1$.
- X is a random variable.
- A random payoff is given by the value of its payoff in each state and the corresponding probability: $[(X_1, \ldots, X_N); (p_1, \ldots, p_N)]$
- **expected** value : $\sum_{i=1}^{N} p_i X_i$

State Prices

- Consider primitive state-contingent claims (Arrow-Debreu securities) that pay $1 in a single state and nothing otherwise.
- Denote the price of the A-D claim on state j by ϕ_j , the state price for state j.
- No arbitrage requires that all state prices must be positive: $\phi_j > 0$ for all j.
- The market is called complete if one can effectively trade A-D securities on each state.

Arbitrage pricing

Arbitrage pricing

- With the prices of A-D securities, we can price other assets/securities.
- Law of One Price: Two assets with the same payoff must have the same market price.
- Suppose the firm is considering a project yielding time-1 cash flow: $X = (X_1, X_2, \ldots, X_N)$
- Using prices of A-D securities, we can attach **market value** to this cash flow as: $P = \phi_1 X_1 + \cdots + \phi_N X_N = PV$
- Example: Suppose there are two states next year. The payoff of a share of stock and the probabilities of the states are $[(X_1, X_2); (p_1, p_2)]$ the state prices for the two states are (ϕ_1, ϕ_2)
  Stock price today: $P = \phi_1 X_1 + \phi_2 X_2$
  Expected rate of return: $\bar{r} = \frac{E[X]-P}{P} = \frac{p_1 X_1 + p_2 X_2}{P} - 1$
- Example for three assets: riskless bonds pays $100 in each state currently traded at $B1_0$, stock 1 pays off $[S1_1, S1_2, S1_3]$ and currently traded at $S1_0$, stock 2 pays off $[S2_1, S2_2, S2_3]$ and currently traded at $S2_0$:

$$\begin{pmatrix} 100 & 100 & 100 \\ S1_1 & S1_2 & S1_3 \\ S2_1 & S2_2 & S3_3 \end{pmatrix} \cdot \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} B1_0 \\ S1_0 \\ S2_0 \end{pmatrix}$$

solve system to find state prices ϕ_1, ϕ_2, ϕ_3.

Present value and future value

Present value and discount rate

- $PV = \frac{E[CF]}{1+\bar{r}}$

Future Value

- FV in T years: $FV = (1+r)^T$

Nominal vs. real cash flows and returns

Nominal vs real CFs

- Nominal cash flows $\implies$ expressed in actual-dollar cash flows.
- Real cash flows $\implies$ expressed in constant purchasing power.
- At an annual inflation rate of i, we have: $(RealCF)_t = \frac{(NominalCF)_t}{(1+i)^t}$

Nominal vs real rates

- Nominal rates of return $\implies$ prevailing market rates.
- Real rates of return $\implies$ inflation adjusted rates
- Real rate of return: $r_{real} = \frac{1+r_{nominal}}{1+i} - 1 \approx r_{nominal} - i$

Week 3: Discounting and Compounding

Special Cash Flows

Special Cash Flows

- **Annuity:** $PV = A \cdot \frac{1}{r}\left[1 - \frac{1}{(1+r)^T}\right]$ $FV = PV \cdot (1+r)^T$
- **Annuity with constant growth:**

$$PV = \begin{cases} A \cdot \frac{1}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^T\right] & \text{if } r \neq g \\ A \cdot \frac{T}{1+r} \quad \text{if } r = g \end{cases}$$

- **Perpetuity:** $PV = \frac{A}{r}$
- **Perpetuity with constant growth:** $PV = \frac{A}{r-g}, \quad r > g$

Compounding

APR / EAR

- **EAR:Effective Annual Rate** $r_{EAR} = \left(1 + \frac{r_{APR}}{k}\right)^k - 1$
- **APR:Annual Percentage Rate** $r_{APR} = k\left[(1 + r_{EAR})^{\frac{1}{k}} - 1\right]$

Week 4: Fixed Income Securities

Bond prices and interest rates

- Prices of discount bonds provide information about spot interest rates:
  $DF_t = \frac{1}{(1+s_t)^t} \implies s_t = (\frac{1}{DF_t})^{(\frac{1}{t})} - 1$
- Price of one price: $B = FV \cdot B_T + \sum_{t=1}^{T} C_t B_t$

Relative Bond Valuation

Arbitrage

obtaining zero coupon bond prices B_1, B_2 and B_3: from a set of three bonds with prices P_1, P_2 and P_3:

$$\begin{pmatrix} C_1 + F & 0 & 0 \\ C_2 & C_2 + F & 0 \\ C_3 & C_3 & C_3 + F \end{pmatrix} \cdot \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

- **Arbitrage:** The key is to construct the payoffs so that we get $100 in year 0 and we get $0 payoffs in all of the subsequent years, i.e. the cashflows of the bonds should offset each other at all times except year 0 when we realize the arbitrage of $100. For example, suppose there are three bonds with prices P_1, P_2 and P_2 with maturities 1, 2 and 3 years and coupons C_1, C_2, C_3, respectively, and all with face value F. There is a fourth bond with price P_4 and coupon C_4 with maturity 3 years (adjust cashflows if maturity is not 3 years) undervalued or overvalued. In order to find the arbitrage strategy we need to solve the following system of equations:

$$\begin{pmatrix} -P_1 & -P_2 & -P_3 & -P_4 \\ C_1 + F & C_2 & C_3 & C_4 \\ 0 & C_2 + F & C_3 & C_4 \\ 0 & 0 & C_3 + F & C_4 + F \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In matrix notation: $A \cdot X = B \implies X = A^{-1} \cdot B$

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X=MMULT(MINVERSE(A),B)
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Interest Risk, Bond Duration and Bond Convexity

Bond Duration

- **Modified Duration (MD) for discount bond** $B_t = \frac{1}{(1+y)^t}$

$$MD(B_t) = -\frac{1}{B_t}\frac{dB_t}{dy} = \frac{t}{1+y}$$

- **Macaulay Duration** is the weighted average term to maturity

$$D = \sum_{t=1}^{T}\left(\frac{PV(CF_T)}{B}t\right) = \frac{1}{B}\sum_{t=1}^{T}\left(\frac{CF_t}{(1+y)^t}t\right)$$

- **Modified Duration** measures bond's interest rate risk by its relative price change with respect to a unit change in yield (with a negative sign):

$$MD = -\frac{1}{B}\frac{dB}{dy} = \frac{D}{1+y}$$

- **Convexity (CX)** measure the curvature of the bond price as function of the yield:

$$CX = \frac{1}{2}\frac{1}{B}\frac{d^2B}{dy^2}$$

$$CX = \frac{1}{2}\frac{1}{P}\frac{1}{(1+y)^2}\sum_{t=1}^{T}\frac{t(t+1)CF_t}{(1+t)^t} = \frac{1}{2}\frac{1}{P}\frac{1}{(1+y)^2}\sum_{t=1}^{T}PV(CF_t)t(t+1)$$

- **Taylor series approximation of bond price changes**

$$\Delta B \approx B\left(-MD \cdot \Delta y + CX \cdot (\Delta y)^2\right)$$

Week 5: Stocks

Growth Opportunities and Stock Valuation

- **P/E and PVGO:** $P_0 = \frac{EPS_1}{r} + PVGO$
- **if** $PVGO = 0$: $P/E = \frac{1}{r}$
- **if** $PVGO > 0$: $P/E = \frac{1}{r} + \frac{PVGO}{EPS_2} > \frac{1}{r}$

Investment and Growth

- plow-back ratio $b_t = 1 - payout = 1 - \frac{DIV}{EPS}$
- **Investments:** $I_t = EPS_t \cdot b_t$
- **Next year earnings** $EPS_{t+1} = EPS_t + ROI_t \cdot I_t$
- **Next year book value:** $BVPS_{t+1} = BVPS_t + I_{t+1}$
- **Dividends:** $D_t = EPS_t(1 - b_t)$
- **Growth rate:** $g = b \cdot ROI$
- $V_{0,noinvt>0} = E_1/r$, $NPV_1 = -E_1 + \frac{E_2 - E_1}{r}$,
  $V_0 = V_{0,noinvt>0} + \frac{NPV_1}{1+r}$

Week 6: Risk

Portfolio mean and variance, two assets

- **Expected portfolio return:** $\bar{r}_p = w_1\bar{r}_1 + w_2\bar{r}_2$
- **Unexpected portfolio return:** $\tilde{r}_p - \bar{r}_p = w_1(\tilde{r}_1 - \bar{r}_1) + w_2(\tilde{r}_2 - \bar{r}_2)$
- **The variance of the portfolio return:** $\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}$
  $\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2$
- **Correlation and covariance:** $\rho_{12} = \frac{\sigma_{12}}{\sigma_1\sigma_2}$

Week 7: Arbitrage Pricing Theory

Factor Models

A single-factor model

- **Asset returns:** $\tilde{r}_i = \underbrace{\bar{r}_i}_{\text{expected return}} + \underbrace{b_i\tilde{f} + \tilde{\epsilon}_i}_{\text{risk}}$
- **Return variance:** $\sigma_i^2 = \underbrace{b_i^2\sigma_f^2}_{\text{systematic risk}} + \underbrace{Var(\tilde{\epsilon}_i)}_{\text{idiosyncratic risk}}$
- **Return covariance:** $Cov(\tilde{r}_i, \tilde{r}_j) = Cov(b_i\tilde{f} + \tilde{\epsilon}_i, b_j\tilde{f} + \tilde{\epsilon}_j) = b_ib_j\sigma_f^2$
  because of the assumptions: $Cov(\tilde{f}, \tilde{\epsilon}_i) = 0$ $Cov(\tilde{\epsilon}_i, \tilde{\epsilon}_j) = 0$

Two-factor model

- **Asset returns:** $E[\tilde{r}_i] - r_f = b_{i1}\lambda_1 + b_{i2}\lambda_2$
- **Return std dev:** $\sigma_i = \sqrt{b_{i,1}^2 Var[f_1] + b_{i,2}^2 Var[f_2] + Var[\tilde{u}_i]}$
- **Return covariance:** $Cov_{1,2} = b_{1,1}b_{2,1}Var[f_1] + b_{1,2}b_{2,2}Var[f_2]$
- **Return correlation:** $Corr_{1,2} = \frac{Cov_{1,2}}{\sigma_1\sigma_2}$

Multifactor models

- **Asset returns:** $\tilde{r}_i = \underbrace{\bar{r}_i}_{\text{expected return}} + \underbrace{b_{i,1}\tilde{f}_1 + b_{i,2}\tilde{f}_1 + \cdots + b_{i,K}\tilde{f}_K}_{\text{systematic component}} + \tilde{\epsilon}_i$
- **Assumptions:** $Cov(\tilde{\epsilon}_i, \tilde{\epsilon}_j) = 0, \forall i \neq j$
  $E[\tilde{f}_k] = 0, k = 1, 2, \ldots, K$

Portfolio return

- **Portfolio return:**

$$\tilde{r}_p = \bar{r}_p + b_{p,1}\tilde{f}_1 + b_{p,2}\tilde{f}_1 + \cdots + b_{p,K}\tilde{f}_K + \tilde{\epsilon}_p$$

  where,

$$\bar{r}_p = \sum_{i=1}^{N} w_i\bar{r}_i \quad b_{p,k} = \sum_{i=1}^{N} w_ib_{i,k} \quad \tilde{\epsilon}_p = \sum_{i=1}^{N} w_i\tilde{\epsilon}_i$$

- **Non-systematic variance:**

$$Var(\tilde{\epsilon}_p) = Var\left(\sum_{i=1}^{N} w_i\tilde{\epsilon}_i\right) = \sum_{i=1}^{N} w_i^2 Var(\tilde{\epsilon}_i)$$

Expected Returns on Diversified Portfolios

- **APT pricing relation:**

$$\underbrace{\bar{r}_p - \bar{r}_f}_{\text{Risk premium}} = \underbrace{\lambda}_{\text{Price of risk}} \cdot \underbrace{b_p}_{\text{Quantity of risk}}$$

  λ tells us how much compensation one earns in the market for a unit of factor risk exposure. λ is called the market price of risk of the factor, or the factor risk premium.

- **APT relation for multi-factor models:**

$$\bar{r}_p - \bar{r}_f = \lambda_1 b_{p,1} + \lambda_2 b_{p,2} + \cdots + \lambda_K b_{p,K}$$

Week 8: Market Efficiency

Three forms of market efficiency hypothesis MEH

- **Weak-form efficiency**: security prices reflect all information contained in past prices $\Longrightarrow$ Technical analysis does not provide excess returns.
- **Semi-strong-form efficiency** : security prices reflect all publicly available information $\Longrightarrow$ Fundamental analysis does not provide excess returns.
- **Strong-form efficiency** : security prices reflect all information, whether publicly available or not $\Longrightarrow$ Inside information does not provide excess returns.

Strong-form EMH $\Longrightarrow$ Semistrong-form EMH $\Longrightarrow$ Weak-form EMH

Week 9: Introduction to Corporate Finance

What is corporate finance?

- **Capital budgeting**: What projects (real investments) to invest in? Expansions, new products, new businesses, acquisitions, ...
- **Financing**: How to finance a project? Selling financial assets/securities/claims (bank loans, public debt, stocks, convertibles, ...)
- **Payout** : What to pay back to shareholders? Paying dividends, buyback shares, ...
- **Risk management**: What risk to take/to avoid and how?

Task of financial manager

Asset side (LHS): Real investments.
Liability side (RHS): Financing, payout and risk management.

Week 10: Capital Budgeting 1

NPV Rule

Investment Criteria for NPV and cash flow calculations

- **For a single project:** Take it if and only if its NPV is positive.
- **For many independent projects:** Take all those with positive NPV.
- **For mutually exclusive projects:** Take the one with positive and highest NPV.
- $NPV = CF_0 + \frac{CF_1}{1+r_1} + \frac{CF_2}{(1+r_2)^2} + \cdots + \frac{CF_T}{(1+r_T)^T}$
- **Operating Profit:**

$$\begin{aligned} OperatingProfits = {} & OperatingRevenues \\ & - OperatingExpensesWithoutDepreciation \end{aligned}$$

- **Cash Flow:**

$$\begin{aligned} CF = {} & (1-\tau)(OperatingProfits) - CapEx + \tau \cdot Depreciation \\ & - \Delta WC \end{aligned}$$

  τ : tax rate
  $CapEx$: Capital Expenditure
  ΔWC : Change in working capital

- **Working Capital:** $WC = Inventory + A/R - A/P$,
  A/R : Accounts Receivable, A/P : Accounts Payable

Payback period

Payback period is the minimum length of time s such that the sum of net cash flows from a project becomes positive:

$$CF_1 + CF_2 + \cdots + CF_S \geq -CF_0 = I_0$$

Decision Criterion Using Payback Period

- **For independent projects:** Accept if s is less than or equal to some fixed threshold $t^* : s \leq t^*$.
- **For mutually exclusive projects:** Among all the projects having $s \leq t^*$, accept the one that has the minimum payback period.

Internal Rate of Return IRR

A project's internal rate of return (IRR) is the number that satisfies:

$$0 = CF_0 + \frac{CF_1}{1+IRR} + \frac{CF_2}{(1+IRR)^2} + \cdots + \frac{CF_t}{(1+IRR)^t}$$

Decision Criterion Using IRR

- **For independent projects:** Accept a project if its IRR is greater than some fixed IRR^*, the threshold rate/hurdle rate.
- **For mutually exclusive projects:** Among the projects having IRR's greater than IRR^* , accept one with the highest IRR.

Profitability index (PI)

Profitability index (PI) is the ratio of the present value of future cash flows and the initial cost of a project:

$$PI = \frac{PV}{-CF_0} = \frac{PV}{I_0}$$

Decision Criterion Using PI

- **For independent projects:** Accept all projects with PI greater than one (this is identical to the NPV rule).
- **For mutually exclusive projects:** Among the projects with PI greater than one, accept the one with the highest PI.

Recommended Resources

- Brealey, Myers, and Allen, Principles of Corporate Finance (13e), Irwin/McGraw Hill. (BMA)
- Bodie, Kane, and Marcus, Investments (11e), Irwin/McGraw Hill. (BKM)
- MITx 15.415.1x Foundations of Modern Finance I [Lecture Slides] (`https://courses.edx.org/courses/course-v1:MITx+15.415.1x+1T2020/course/`)
- LaTeX File (`github.com/j053g/cheatsheets/15.415.1x`)

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