Mathematical Methods for Quantitative Finance

Cheat sheet for MITx 15.455x Mathematical Methods for Quantitative Finance.

Week 1: Probability

Random variables, distributions, and moments

Moments of a distribution

- The **moments** of a distribution are the expectation of powers of the r.v.
  $\mu_l = E[X^l] = \begin{cases} \sum_k x_k^l p(x_k) \\ \int x^l p(x)dx \end{cases}$
- **Variance** : $\sigma^2 = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$
- **Skewness** - asymmetry parameter : $s = \frac{\mathbb{E}[(X-\mu)^3]}{\sigma^3} = \mathbb{E}\big[\big(\frac{X-\mu}{\sigma}\big)^3\big]$
- **Kurtosis** - measure of tail "weights" : $\kappa = \mathbb{E}\big[\big(\frac{X-\mu}{\sigma}\big)^4\big] - 3$

Covariance and correlation

- **covariance**: $\text{Cov}(X,Y) \equiv \mathbb{E}[(X-\mu_x)(Y-mu_y)] = \mathbb{E}[XY] - \mu_x\mu_y$
- the **correlation** is proportional to the covariance:
  $\rho(X,Y) = \text{Corr}(X,Y) \equiv \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}} = \mathbb{E}\Big[\Big(\frac{X-\mu_x}{\sigma_x}\Big)\Big(\frac{Y-\mu_y}{\sigma_y}\Big)\Big]$

Common distributions

- **Uniform distribution**: $p(x) = \begin{cases} 1, x \in [0,1] \\ 0, \text{otherwise} \end{cases}$, $\text{Prob}(a < X < b) = b - a$
  $\mu = \int_{\infty}^{\infty} xp(x)\text{d}x = \int_0^1 x\text{d}x = \frac{1}{2}, \sigma^2 = \int_{\infty}^{\infty}(x-\frac{1}{2})^2\text{d}x = \frac{1}{12}$
  $u_l = \int_0^1 x_l dx = \frac{1}{l}$
- **Binomial distribution**: $f(x;n,p) = \binom{n}{k}p^k q^{n-k} = \frac{n!}{k!(n-k)!}p^k q^{n-k}$,
  $\mu = np, \sigma^2 = npq$
- **Gaussian distribution**: $p(x) - \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$,
  $\Phi(x) = Prob(Z < x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2}dz$

Moments:

Order	Non-central moment	Central moment
1	μ	0
2	$\mu^2 + \sigma^2$	σ^2
3	$\mu^3 + 3\mu\sigma^2$	0
4	$\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$	$3\sigma^4$

- **Lognormal distribution**: $X = \log Y$, $x = \log y$, then
  $g(y) = \frac{p(x)}{|\text{d}y/\text{d}x|} = \frac{1}{y\sqrt{2\pi\sigma^2}}e^{-(\log y - \mu)^2/(2\sigma^2)}$
- **Poisson distribution**: $p(k;\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$. $\mu = \sigma^2 = \lambda$.
  Probability of k "arrivals" during interval t: $p(k;\lambda) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$

Week 2: Stochastic Processes

The Random Walk

Random walk model

- $S_T = z_1 + z_2 + \cdots + z_T$. Each increment is a random IID variable.
  $\mathbb{E}[z_t] = 0, \mathbb{E}[z_t^2] = 1, \mathbb{E}[z_t z_{t'}] = 0$ if $t \neq t'$. $\text{Var}(S_T) = T$
- **Generalized random walk model**: $r_t = \sigma z_t + \mu$. $\mathbb{E}[r_t] = \mu$,
  $\mathbb{E}[(r_t - \mu)^2] = \sigma^2$. $X_T \equiv \sum_{t=1}^{T} r_t, \mathbb{E}[X_T] = T\mu$, $\text{VaR}(X_T) = T\sigma^2$.

Time series models

Time series models

- a ts process is **stationary** if the join distribution of all of its values is invariant under time translation.
- a ts is **weakly stationary** if the first and second moments are invariant.
- MA(1): $r_t = \mu + \sigma z_t + \phi z_{t-1}$
- AR(p): $R_t = c_0 + c_1 R_{t-1} + \cdots + c_p R_{t-p} + \sigma z_t$, $z_t \sim IID(0,1)$
- ARMA(p,q):
  $R_t = c_0 + c_1 R_{t-1} + \cdots + c_p R_{t-p} + \sigma z_t + \phi_1 z_{t-1} + \cdots + \phi_q z_{t-q}$
- AR(1) used for **mean reversion**: $R_t = c_0 + c_1 R_{t-1} + \sigma z_t$, $E[R_t] = \frac{c_0}{1-c_1}$,
  for convenience: $\mu = \frac{c_0}{1-c_1}$, $\lambda = -c_1$. Then: $R_t - \mu = -\lambda(R_t - \mu) + \sigma z_t$,
  $|\lambda| < 1$. $Var[R_t] = \gamma_0 = \frac{\sigma^2}{1-\lambda^2}$.
  Lag-k autocovariance coefficient: $\gamma_k = (-\lambda)^k\gamma_0 = \frac{(-\lambda)^k}{1-\lambda^2}\sigma^2$

Week 3: Time Series Models

Gambler's Ruin

- Repeated set of gambles with probability of success p and of failure $q = 1 - p$
- initial capital is $x > 0$, total capital a. Stop when winning a or **ruin** as $x = 0$.
- Q_x is **probability of ruin** starting from capital x: $Q_x = pQ_{x+1} + pQ_{x-1}$.
- $Q_x = \frac{(q/p)^a - (q/p)^x}{(q/p)^a - 1}$, if $p = q = 1/2$, then: $Q_x = 1 - \frac{x}{a}$

Week 4: Continuous-Time Finance

Itô processes and Itô's lemma

Itô's lemma

- Itô process: $\text{d}X = a\text{d}t + b\text{d}B$
- $\text{d}F = \Big(\frac{\partial F}{\partial t}\text{d}t + \frac{\partial F}{\partial X}\text{d}X + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(\text{d}X)^2 + \cdots\Big) = \Big(\frac{\partial F}{\partial t}\text{d}t + \frac{\partial F}{\partial X}\text{d}X + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}[b^2\text{d}t]\Big) = \Big[\frac{\partial F}{\partial t}\text{d}t + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}\Big]\text{d}t + \Big[\frac{\partial F}{\partial X}\Big][a\text{d}t + b\text{d}B]$
  $\text{d}F = \Big[\frac{\partial F}{\partial t} + a\frac{\partial F}{\partial X} + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}\Big]\text{d}t + b\frac{\partial F}{\partial X}\text{d}B$
- Heuristics: expand and replace $(\text{d}B_t)^2 \to \text{d}t$, $(\text{d}X_t)^2 \to b^2\text{d}t$

Itô processes

- **Brownian motion with drift**: $\text{d}S_t = \mu\text{d}t + b\text{d}B_t$
  $S_T = S_0 + \mu T + \sigma(B_T - B_0)$
- **GBM with drift**: $\text{d}S_t = \mu S_t\text{d}t + bS_t\text{d}B_t$
  $\text{d}(logS_t) = (\mu - \frac{\sigma^2}{2})\text{d}t + \sigma\text{d}B_t$, $S_T = S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma(B_T - B_0)}$
- **Ornstein–Uhlenbeck process**: $\text{d}S_t = \lambda(\bar{S} - S_t)\text{d}t + \sigma\text{d}B_t$
- **Cox–Ingersoll–Ross process**: $\text{d}\rho_t = \lambda(\bar{\rho} - \rho_t)\text{d}t + \sigma\sqrt{\rho_t}\text{d}B_t$
  Let $F = \sqrt{\rho}$, $\frac{\partial F}{\partial \rho} = \frac{1}{2\sqrt{\rho}}$, $\frac{\partial^2 F}{\partial \rho^2} = -\frac{1}{4}\rho^{-\frac{3}{2}}$
  $\text{d}F = \Big(\frac{4\lambda\bar{\rho} - \sigma^2}{8F} - \frac{1}{2}\lambda F\Big)\text{d}t + \frac{1}{2}\sigma\text{d}B_t$

From SDE to PDE: The Black-Scholes equation

Black-Scholes equation

- Itô process: $\text{d}X = a\text{d}t + b\text{d}B$
- $\text{d}F = \Big(\frac{\partial F}{\partial t}\text{d}t + \frac{\partial F}{\partial X}\text{d}X + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(\text{d}X)^2 + \cdots\Big) = \Big(\frac{\partial F}{\partial t}\text{d}t + \frac{\partial F}{\partial X}\text{d}X + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}[b^2\text{d}t]\Big) = \Big[\frac{\partial F}{\partial t}\text{d}t + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}\Big]\text{d}t + \Big[\frac{\partial F}{\partial X}\Big][a\text{d}t + b\text{d}B]$
  $\text{d}F = \Big[\frac{\partial F}{\partial t} + a\frac{\partial F}{\partial X} + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}\Big]\text{d}t + b\frac{\partial F}{\partial X}\text{d}B$
- Heuristics: expand and replace $(\text{d}B_t)^2 \to \text{d}t$, $(\text{d}X_t)^2 \to b^2\text{d}t$

Week 5: Itô Calculus

Black-Scholes equation

Summary of some key formulas

- Itô process: $\text{d}X = a\text{d}t + b\text{d}B$
- Itô formula:
  $$\begin{aligned}\text{d}F &= \frac{\partial F}{\partial t}\text{d}t + \frac{\partial^2 F}{\partial X^2}\text{d}X + \frac{b^2}{2}\frac{\partial F}{\partial X}\text{d}t \\ &= \left(\frac{\partial F}{\partial t} + a\frac{\partial F}{\partial X} + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}\right)\text{d}t + b\frac{\partial F}{\partial X}\text{d}B\end{aligned}$$
- Stock price: $\text{d}S = \mu S\text{d}t + \sigma\text{d}B \implies \text{d}(\log S) = \Big(\mu - \frac{\sigma^2}{2}\Big)\text{d}t + \sigma\text{d}B$
- Black-Scholes: $\Delta = \partial V/\partial S$, $\text{d}\pi = r\pi\text{d}t$,
  $$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Recitation 5

Expectations from Brownian integrals

- $dB \sim N(0, dt)$
- $\int_0^t dB = B_t - B_0 \sim N(0,t)$
- $E[f(B_t - B_0)] = E[f(\sqrt{t}z)] = \frac{1}{\sqrt{2\pi}}\int e^{-z^2/2}f(\sqrt{t}z)dz$
- Example $E[f(B_t - B_0)] = E[(B_t - B_0)^4] = E[(\sqrt{t}z)^4] = t^2E[z^4] = 3t^2$
  We can pull out $\sqrt{t}$,which is nonstochastic to get $t^2E[z^4]$, $E[z^4]$ is a well-known Gaussian integral that we use in the kurtosis=3.
- Useful formula: $E[e^{\alpha z + \beta}] = e^{\alpha^2/2 + \beta}$

Solutions to the diffusion equation

- **diffusion equation**: $\frac{\partial p}{\partial t} - \frac{1}{2}\frac{\partial^2 p}{\partial z^2} = 0$
- special solution $p(z,0) = f(z)$ then the general solution is:
  $p(z,t) = \int p_0(z-w,t)f(w)dw$, where $p_0 = \frac{1}{\sqrt{2\pi t}}e^{-\frac{z^2}{2t}}$
- let $u = \frac{w-z}{\sqrt{t}} \implies du = \frac{dw}{\sqrt{t}} \implies w = u\sqrt{t} + z$. Now general solution can be computed as expectation of standard Gaussian: $p(z,t) = \mathbb{E}[f(\sqrt{t}u + z)]$, where $u \sim N(0,1)$
- **example**: $p(z,0) = z^2$, $f(z) = z^2$, find $p(z,t)$
  $p(z,t) = \int p_0(z-w,t)f(w)dw = \int \frac{1}{\sqrt{2\pi t}}e^{-\frac{(z-w)^2}{2t}}w^2dw =$
  $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}(u\sqrt{t} + z)^2du = \mathbb{E}[f(u\sqrt{t} + z)] =$
  $\mathbb{E}[u^2t + 2u\sqrt{t}z + z^2] = z^2 + t$

Week 6: Continuous-Time Finance

Itô processes in higher dimensions

Itô's lemma: multiple stochastic variables

- $\text{d}X_i = a_i(t, X_1, X_2, \ldots)\text{d}t + b_i(t, X_1, X_2, \ldots)\text{d}B_i$
  $\text{d}F = \frac{\partial F}{\partial t}\text{d}t + \sum\frac{\partial F}{\partial X_i}\text{d}X_i + \frac{1}{2}\sum\rho_{ij}b_ib_j\frac{\partial^2 F}{\partial X_i X_j}\text{d}t$
- Heuristics "rule of thum" for correlated Brownian motions :
  $(\text{d}B_i)^2 \to \text{d}t$, $(\text{d}B_i)(\text{d}B_j) \to \rho_{ij}\text{d}t$,
  $(\text{d}X_i)^2 \to b_i^2\text{d}t$, $(\text{d}X_i)(\text{d}X_j) \to \rho_{ij}b_ib_j\text{d}t$

- two stochastic variables case: $\mathrm{d}X_1 = a_1\mathrm{d}t + b_1\mathrm{d}B_1, \mathrm{d}X_2 = a_2\mathrm{d}t + b_2\mathrm{d}B_2$
  $\mathrm{d}F =$
  $\frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial X_1}\mathrm{d}X_1 + \frac{\partial F}{\partial X_2}\mathrm{d}X_2 + \left(\frac{b_1^2}{2}\frac{\partial^2 F}{\partial X_1^2} + \frac{b_2^2}{2}\frac{\partial^2 F}{\partial X_2^2} + b_1 b_2 \rho \frac{\partial^2 F}{\partial X_1 \partial X_2}\right)\mathrm{d}t$
- With two random variables, Ito's formula for $F(t, X, Y)$ is: $dF =$
  $\frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}dX + \frac{\partial F}{\partial Y}dY + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(dX)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial Y^2}(dY)^2 + \frac{\partial^2 F}{\partial X \partial Y}(dX)(dY)$
- example: $F = X1X2 \implies \mathrm{d}F = X1\mathrm{d}X_2 + X_2\mathrm{d}X_1 + \rho b_1 b_2 \mathrm{d}t$
  since $(\mathrm{d}X_i)(\mathrm{d}X_j) \to \rho_{ij} b_i b_j \mathrm{d}t \implies \mathrm{d}F = X1\mathrm{d}X_2 + X_2\mathrm{d}X_1 + \mathrm{d}X_1\mathrm{d}X_2$

Week 8: Optimization

Portfolio Optimization

Portfolio risk : $\sigma_p^2 = \vec{\mathbf{w}}^\top C\vec{\mathbf{w}} = \sum w_i^2\sigma_i^2 + 2\sum_{i<j} w_i w_j \sigma_i \sigma_j \rho_{ij}$
$\vec{\mathbf{w}} = (w_1 \quad w_2 \quad \dots \quad w_n)^\top$

Portfolio optimization with budget

- $\mathcal{L}(\vec{\mathbf{w}}, \ell) = \frac{1}{2}\vec{\mathbf{w}}^\top C\vec{\mathbf{w}} + \ell(1 - \vec{\iota}^\top\vec{\mathbf{w}}^\top)$
- Vary the weights: $\frac{\partial \mathcal{L}}{\partial w_i} = \left(\sum_{j\in[n]} C_{ij} w_j\right) - \ell\iota_i = 0$
- Solve for the weights by inverting matrix: $C\vec{\mathbf{w}} - \ell\vec{\iota} = 0 \implies \vec{\mathbf{w}} = \ell C^{-1}\vec{\iota}$
- Solve Lagrande multiplier : $\vec{\iota}^\top\vec{\mathbf{w}} = 1 = \ell(\vec{\iota}^\top C^{-1}\vec{\iota}) \implies \ell = \frac{1}{\vec{\iota}^\top C^{-1}\vec{\iota}}$
- Solution: $\vec{\mathbf{w}}_{min} = \ell C^{-1}\vec{\iota} = \frac{C^{-1}\vec{\iota}}{\vec{\iota}^\top C^{-1}\vec{\iota}}, \quad \sigma^2_{min} = \ell = \frac{1}{\vec{\iota}^\top C^{-1}\vec{\iota}}$

Portfolio optimization with budget and return constraint generalized to $\sum_i w_i = w_p$

- $\mathcal{L}(\vec{\mathbf{w}}, \ell, m) = \frac{1}{2}\vec{\mathbf{w}}^\top C\vec{\mathbf{w}} + \ell(w_p - \vec{\iota}^\top\vec{\mathbf{w}}) + m(\mu_p - \vec{\mu}^\top\vec{\mathbf{w}})$
- Vary the weights: $\frac{\partial \mathcal{L}}{\partial w_i} = \left(\sum_{j\in[n]} C_{ij} w_j\right) - \ell\iota_i - m\mu_i$
- Solve for the weights: $C\vec{\mathbf{w}} - \ell\vec{\iota} - m\vec{\mu} = \mathbf{0} \implies \vec{\mathbf{w}} = C^{-1}(l\vec{\iota} + m\vec{\mu})$
- Solve for Langrange multipliers with constraints: $\vec{\iota}^\top\vec{\mathbf{w}} = w_p$ and $\vec{\mu}^\top\vec{\mathbf{w}} = \mu_p$
  $w_p = \vec{\iota}^\top\vec{\mathbf{w}} = \ell(\vec{\iota}^\top C^{-1}\vec{\iota}) + m(\vec{\mu}^\top C^{-1}\vec{\iota})$
  $\mu_p = \vec{\mu}^\top\vec{\mathbf{w}} = \ell(\vec{\mu}^\top C^{-1}\vec{\iota}) + m(\vec{\mu}^\top C^{-1}\vec{\mu})$
  as as matrix equation: $\begin{pmatrix} w_p \\ \mu_p \end{pmatrix} = M\begin{pmatrix} \ell \\ m \end{pmatrix}$, where : $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$
  $a \equiv \vec{\iota}^\top C^{-1}\vec{\iota}, b \equiv \vec{\mu}^\top C^{-1}\vec{\iota}, c \equiv \vec{\mu}^\top C^{-1}\vec{\mu}.$
- Solve for Langrange multiplier by inverting M: $\begin{pmatrix} \ell \\ m \end{pmatrix} = M^{-1}\begin{pmatrix} w_p \\ \mu_p \end{pmatrix}$
  $M^{-1} = \frac{1}{ac-b^2}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}, \ell = \frac{cw_p - b\mu_p}{ac-b^2}$ and $m = \frac{-bw_p + a\mu_p}{ac-b^2}.$
- Solution : $\sigma_p^2 = (\ell \quad m)\, M\begin{pmatrix} \ell \\ m \end{pmatrix} = \begin{pmatrix} \frac{cw_p - b\mu_p}{ac-b^2} & \frac{-bw_p + a\mu_p}{ac-b^2} \end{pmatrix}\begin{pmatrix} w_p \\ \mu_p \end{pmatrix}$
  $= \frac{1}{ac-b^2}\cdot(a\mu_p^2 - 2bw_p\mu_p + cw_p^2).$

Useful formulas

$\mathrm{Cov}(aX + bY, Z) = a\,\mathrm{Cov}(X, Z) + b\,\mathrm{Cov}(Y, Z)$
$\mathrm{Var}(aX + bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X, Y)$
$\mathrm{Var}(aX - bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) - 2ab\,\mathrm{Cov}(X, Y)$
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Recommended Resources

- MITx 15.455x MITx 15.455x Mathematical Methods for Quantitative Finance [Lecture Slides] (`https://learning.edx.org/course/course-v1:MITx+15.455x+3T2020/home`)
- Tsay, Analysis of Financial Time Series (3e), Wiley. (Tsay)
- Capinski and Zastawniak, Mathematics for Finance, Springer. (CZ)
- Olver, Introduction to Partial Differential Equations (2016), Springer. (Olver)
- Campbell, Lo, and MacKinlay, Econometrics of Financial Markets (1997), Princeton. (CLM)
- Lang, Introduction to Linear Algebra (2e), Springer (Lang)
- Axler, Linear Algebra Done Right (3e), Springer (Axler)
- LaTeX File (`github.com/j053g/cheatsheets/15.455x`)

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