11 Forward and Futures

Forward Interest rates

The forward interest rate between time $t-1$ and t satisfies:
$(1+r_t)^t = (1+r_{t-1})^{t-1}(1+f_t)$
$f_t = \frac{B_{t-1}}{B_t} - 1 = \frac{(1+r_t)^t}{(1+r_{t-1})^{t-1}} - 1$

Forward price

$F_T = e^{(r-y)T} S_0$

Swaps

The swap rate is a weighted average of forward rates: $r_s = \frac{\sum_{t=1}^T B_t f_t}{\sum_{u=1}^T B_u} = \sum_{t=1}^T w_t f_t$, with the weights $w_t = \frac{B_t}{\sum_{u=1}^T B_u}$

12 Options, Part 1

Net option payoff

The break-even point is given by S_T at which Net Payoff is zero:
$Netpayoff = max[S_T - K, 0] - C(1+r)^T$

Option strategies

- Protective Put: Buy stock + Buy a put
- Bull Call Spread: Buy a Call K1 + Write a Call K2, $K1 < K2$
- Straddle: Buy a Call at K + Buy a Put at K

Corporate securities as options

- Equity (E): A call option on the firm's assets (A) with the exercise price equal to its bond's redemption value.
- Debt (D): A portfolio combining the firm's assets (A) and a short position in the call with the exercise price equal to its bond's face value (F):

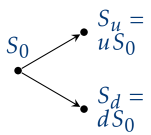
$A = D + E \implies D = A - E$
$E \equiv max(0, A - F)$
$D = A - E = A - max[0, A - F]$

Put-Call parity for European options

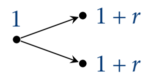
$C + B \cdot K = P + S$

Binominal option pricing model

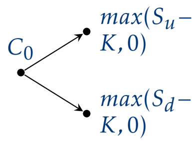
Stock



Bond

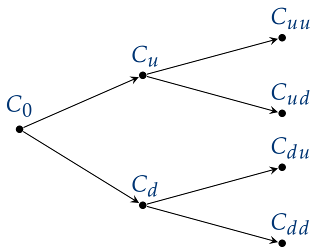


Option at Expiration



Replicating portfolio (call option)
$\begin{pmatrix} S_u & (1+r) \\ S_d & (1+r) \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} max(K - S_u, 0) \\ max(K - S_d, 0) \end{pmatrix}$
solve system to form a portfolio of stock and bond that replicates the call's payoff: a shares of the stock; b dollars in the riskless bond

Binomial option pricing model with multiple periods



Compute the time-0 value working backwards: first C_u and C_d then C_0. In summary:

- Replication strategy gives payoffs identical to those of the call.
- Initial cost of the replication strategy must equal the call price

13 Options, Part 2

Binomial model: risk-neutral pricing

Solve by replication δ shares of the stock, b dollars of riskless bond
$\delta u S_0 + b(1+r) = C_u$
$\delta d S_0 + b(1+r) = C_d$
Solution: $\delta = \frac{C_u - C_d}{(u-d)S_0}$, $b = \frac{1}{1+r}\frac{uC_d - dC)u}{u-d}$
Then: $C_0 = \delta S_0 + b$
$C_0 = \frac{C_u - C_d}{(u-d)} \frac{1}{1+r} \frac{uC_d - dC)u}{u-d}$

Risk neutral probability

$q_u = \frac{(1+r)-d}{u-d}$, $q_d = \frac{u-(1+r)}{u-d}$ Then
$C_0 = \frac{q_u C_u + q_d C_d}{1+r} = \frac{E^Q[C_T]}{1+r}$ where $E^Q[\cdot]$ is the expectation under probability $Q = (1, 1-q)$, which is called the risk-neutral probability

State prices and risk-neutral probabilities

$\Phi_u = \frac{q}{1+r}$, $\Phi_d = \frac{1-q}{1+r}$
$\Phi_{uu} = \frac{q^2}{(1+r)^2}$, $\Phi_{ud} = \Phi_{du} = \frac{q(1-q)}{(1+r)^2}$, $\Phi_{dd} = \frac{(1-q)^2}{(1+r)^2}$ With state prices, can price any state-contingent payoff as a portfolio of state-contingent claims: mathematically equivalent to the risk-neutral valuation formula.

Implementing binominal model

- As we reduce the length of the time step, holding the maturity fixed, the binomial distribution of log returns converges to Normal distribution.
- Key model parameters u, and d need to be chosen to reflect the distribution of the stock return

Once choice is: $u = exp(\sigma\frac{T}{n})$, $d = \frac{1}{u}$, $p = \frac{1}{2} + \frac{1}{2}\frac{u}{\sigma}\sqrt{\frac{T}{n}}$

Black-Scholes-Merton formula

$C_0 = S_0 N(x) - Ke^{-rT} N(x - \sigma\sqrt(T))$
$x = \frac{ln(\frac{S_0}{Ke^{-rT}})}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$

In Excel $N(x)$=`NORM.S.DIST(x,TRUE)`
The call is equivalent to a levered long position in the stock; $S_0 N(x)$ is the amount invested in the stock; $Ke^{-rT}N(x - \sigma\sqrt(T))$ is the dollar amount borrowed.
Equivalent formulation: $C(S_t, t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)}$
$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$
$d_2 = d_1 - \sigma\sqrt{T-t}$
The price of a corresponding put option based on put-call parity is: $P(S_t, t) = Ke^{-r(T-t)} - S_t + C(S_t, t) = N(-d_2)Ke^{-r(T-t)} - N(-d_1)S_t$

Option Greeks

Delta: $\delta = \frac{\partial C}{\partial S}$ Omega: $\Omega = \frac{\partial C}{\partial S}\frac{S}{C}$ Gamma: $\Gamma = \frac{\partial \delta}{\partial S} = \frac{\partial^2 C}{\partial S^2}$ Theta: $\Theta = \frac{\partial C}{\partial S}$ Vega: $\mathcal{V} = \frac{\partial C}{\partial S}$

14 Portfolio Theory

Preliminaries

Portfolio return and variance, two assets
$\bar{r}_p = w_1\bar{r}_1 + w_2\bar{r}_2$
$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}$
$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2$
$\rho_{12} = \frac{\sigma_{12}}{\sigma_1\sigma_2}$

Quadratic formula is handy when solving for weights in the two asset case: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
In general, for a vector of weights $\boldsymbol{w}$, returns $\boldsymbol{r}$ and covariance matrix Σ: $r_p = \boldsymbol{w} \cdot \boldsymbol{r}$ and $\sigma_p^2 = \boldsymbol{w}' \cdot \Sigma \cdot \boldsymbol{w}$ in Excel σ_p^2=MMULT(MMULT(TRANSPOSE($\boldsymbol{w}$),Σ),$\boldsymbol{w}$)

Sharpe Ratio

$SR = \frac{r - r_f}{\sigma}$, Return-to-Risk Ratio $RRR_{i,p} = \frac{r_i - r_f}{Cov(r_i, r_p)/\sigma_p}$

Portfolio and individual assets

In the presence of a risk-free asset, portfolio's return is: $\bar{r}_p = r_F + \sum_{i=1}^N w_i(\bar{r}_i - r_F)$
Individual contribution to expected return: $\frac{\partial \bar{r}_p}{\partial w_i} = \bar{r}_i - r_F$
Individual contribution to volatility: $\frac{\partial \sigma_p}{\partial w_i} = \frac{Cov(\bar{r}_i, \bar{r}_p)}{\sigma_p}$

Tangency Portfolio

Note $\bar{\boldsymbol{x}}$ is vector of excess returns and $\vec{\mathbf{1}}$ is a vector of ones of size N, number of stocks.
min $\boldsymbol{w}' \cdot \Sigma \cdot \boldsymbol{w}$ s.t. $\boldsymbol{w}' \cdot \bar{\boldsymbol{x}} = m$ solution: $w_T = \lambda\Sigma^{-1}\bar{\boldsymbol{x}}$ where $\lambda = \frac{1}{\bar{\boldsymbol{x}}'\Sigma^{-1}\vec{\mathbf{1}}}$.
In summary, the tangency weights are: $w_T = \frac{1}{\bar{\boldsymbol{x}}'\Sigma^{-1}\vec{\mathbf{1}}}\Sigma^{-1}\bar{\boldsymbol{x}}$. In Excel use Σ^{-1}=MINVERSE(Σ), then λ=MMULT(MMULT(TRANSPOSE($\bar{\boldsymbol{x}}$),Σ^{-1})),$\vec{\mathbf{1}}$), lambda is a scalar, then $w_T = \lambda$ MMULT(Σ^{-1}),$\bar{\boldsymbol{x}}$).
Once the tangency portfolio is found, RRR is the same for all stocks, meaning that we cannot perturb the weights of individual assets in this portfolio to further increase its risk return trade off.

R14Q4

$r_i = b_1F_1 + b_{2,i}F_2 + \epsilon_i$
inputs: $b_1 = 10$, $b_{2,i} = i$, F_1 has $E[F_1] = 0\%$ and $\sigma_{F_1} = 1\%$, F_2 has $E[F_2] = 1\%$ and $\sigma_{F_2} = 1\%$, and ϵ_i has $E[\epsilon_i] = 0\%$ and $\sigma_{\epsilon_i} = 30\%$. F_1, F_2, ϵ_i are indep. of each other, and $r_f = 0.75\%$.
find sharpe ratio and return-to-risk ratio RRR
$E[r_i] = b_1E[F_1] + b_2, iE[F_2] + E[\epsilon_i] = i \cdot 1\%$

$V[r_i] = V[b_1F_1 + b_{2,i}F2 + \epsilon_i] = b_1^2V[F_1] + b_{2,i}^2V[F_2] + V[\epsilon_i] = 10^2 \times 0.01^2 + i^2 \times 0.01^2 + 0.3^2$
$Cov(r_i, r_j) = Cov(b_1F_1 + b_{2,i}F + 2 + \epsilon_i, b_1F_1 + b_{2,j}F_2 + \epsilon_j) = V[b_1F_1] + Cov(b_{2,i}F_2, b_{2,j}F_2) = b_1^2V[F_1] + b_{2,i}b_{2,j}V[F_2] = 10^2 \times 0.01^2 + i \times j \times 0.01^2$.
With the variances (note variances have an extra-term) and covariances now we can form Σ and compute sharpe ratio.
$RRR_{i,p} = \frac{r_i - r_f}{Cov(r_i, r_p)/\sigma_p}$ note $Cov(r_i, r_p) = Cov(r_i, \sum_{j=1}^I w_jr_j) = \sum_{j=I}^I w_jCov(r_i, r_j)$

15 CAPM

For the market porfolio to be optional the RRR of all risky assets must be the same
$RRR_i = \frac{\bar{r}_i - r_F}{\sigma_{iM}/\sigma_M} = SR_M = \frac{\bar{r}_M - r_F}{\sigma_M}$
then
$\bar{r}_i - r_F = \frac{\sigma_{iM}}{\sigma_M^2}(\bar{r}_M - r_F) = \beta_{iM}(\bar{r}_M - r_F)$
β_{iM} is a measure of asset it's systematic risk: exposure to the market. $\bar{r}_M - r_F$ gives the premium per unit of systematic risk.

Risk and return in CAPM

We can decompose an asset's return into three pieces:
$\tilde{r}_i - r_F = \alpha_i + \beta_{iM}(\tilde{r}_M - r_F) + \tilde{\epsilon}_i$
$E[\tilde{\epsilon}_i] = 0$, $Cov[\tilde{r}_M - r_F), \tilde{\epsilon}_i] = 0$
Three characteristics of an asset: Alpha, according to CAPM, alpha should be zero for all assets. Beta: measures an asset's systematic risk. SD[$\tilde{\epsilon}_i$] measures non-systematic risk.

Leverage: equity beta vs asset betas

The assets of the firm serve to pay all investors, and so: $A = E + D$
$\beta_A = \frac{E}{E+D}\beta_E + \frac{D}{E+D}\beta_D$

R15Q1

inputs: $E[r_M] = 14\%$, $E[r_P] = 16\%$, $r_f = 6\%$, $\sigma_M = 25\%$
$E[r_P] = r_F + \beta_P(E[r_M] - r_f) = 16\% = 6\% + \beta_P(14\% - 6\%) = 1.25$
to make a portfolio to be located in the capital line, one must use the risk-free asset with weight w: $E[r_P] = wr_f + (1-w)E[r_M] = 6\%w + 14\%(1-w) \implies w = \frac{E[r_P] - r_M}{r_F - r_M} = -0.25$
$VaR[r_P] = Var[wr_f + (1-w)E[r_M]] = (1-w)^2Var[r_M] \implies \sigma_P = 31.25\%$
to find correlation use: $\beta_P = \frac{Cov(r_P, r_M)}{Var(r_M)} = \frac{\rho_{P,M}\sigma_M\sigma_P}{\sigma_M^2} \implies \rho_{P,M} = \frac{\beta_P\sigma_M}{\sigma_P} = 1$

R15Q5

Empirically estimating CAPM. $r_i - r_f = \alpha + \beta_{MKT}^i(r_{MKT} - r_f) + \epsilon_i$ In Excel : LINEST($r_i - r_f$, $r_{MKT} - r_f$,1,0) this yields β_{MKT} and α in that order.

16 Capital Budgeting, Part II

16.1 Real Options

Growth Options An investment includes a growth option if it allows follow-on investments, and the decision whether to undertake the follow-on investments will be made later on the basis of new information, akin to **call** options.
Abandonment options An investment includes an abandonment option if, under certain circumstances, it can be shut down if so chosen, akin to **put** options.
Scale options example: Ability to slow

the rate of mineral extraction from a mine. **Timing options** Flexibility about the timing of an investment can be very valuable, akin to **American call** option.

R16Q1

To find sector 1 an2 β's: β_{s_1} and β_{s_2} for firms A and B:

$\beta_A = w_{A,s_1}\beta_{s_1} + w_{A,s_2}\beta_{s_2}$
$\beta_B = w_{B,s_1}\beta_{s_1} + w_{B,s_2}\beta_{s_2}$

solve system for β_{s_1} and β_{s_2}.

R16Q2

inputs: firm's FCF_1, β_1, β_2, table below:

Portfolio	$E[r_i]$	$\beta_{i,1}$	$\beta_{i,2}$
A	r_A	$\beta_{A,1}$	$\beta_{A,2}$
B	r_B	$\beta_{B,1}$	$\beta_{B,2}$
C	r_C	$\beta_{C,1}$	$\beta_{C,2}$

find firm value.
Recall APT: $E[r_p] = r_f + \lambda_1\beta_{P,1} + \lambda_2\beta_{P,2}$:

$$\begin{pmatrix} 1 & \beta_{A,1} & \beta_{A,2} \\ 1 & \beta_{B,1} & \beta_{B,2} \\ 1 & \beta_{C,1} & \beta_{C,2} \end{pmatrix} \cdot \begin{pmatrix} r_f \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} r_A \\ r_B \\ r_C \end{pmatrix}$$

solve system to find r_f, λ_1, λ_2 then apply APT with firm loadings (β's) to find $E[r] = r_f + \beta_1\lambda_1 + \beta_2\lambda_2$ finally $V_0 = \frac{FCF_1}{E[r]-g}$

17 Financing, Part I

Capital structure theory I

Modigliani-Miller I aka MM-I Theorem: Capital structure is irrelevant (under ïdeal conditions").

$CF_D + CF_D = CF_A \implies PV(CF_D) + PV(CF_D) = PV(CF_A) \implies D + E = A$. A firm's value is determined by the total cash flow on its assets. Capital structure only determines how total cash flow is split between debt and equity holders. Given its assets, capital structure won't affect a firm's total value.

WACC - weighted average cost of capital

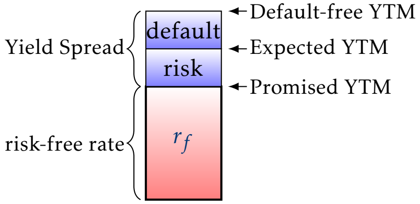
$WACC = \frac{D}{D+E}r_D + \frac{E}{D+E}r_E = w_D r_D + w_E r_E$

Leverage and financial risk

MM-II $r_E = r_A + \frac{D}{E}(r_A - r_D)$

Default premium and risk premium

Promised YTM: the yield if default does not occur. **Expected YTM**: the probability-weighted average of all possible yields. **Default premium**: the difference between promised yield and expected yield. **Risk premium**: the difference between the expected yield on a risky bond and the yield on a risk-free bond of similar maturity and coupon rate.



18 Financing, Part II

MM with taxes

Notation: U: Unlevered, L: Levered, X: Terminal value, τ: taxes.
Firm U: $(1-\tau)X$
Firm L: $(1-\tau)(X - r_D D) + r_D D = (1-\tau)X + \tau r_D D$

$V_U = \frac{(1-\tau)X}{1+r_A}$

$V_L = \frac{(1-\tau)X}{1+r_A} + \frac{\tau r_D D}{1+r_D} = V_U + \frac{\tau r_D D}{1+r_D}$

The value of a levered firm equals the value of the unlevered firm (with the same assets) plus the present value of the **tax shield**

$V_L = V_U + PV(\text{debt tax shield}) = V_U + PVTS$

Costs of financial distress

$V_L = V_U + PV(\text{debt tax shield}) - PV(\text{cost of financial distress}) = APV$.
Where APV: Adjusted present value.

MM with personal tax

Notation: Debt level at D with interest rate r_D. Corporate tax rate τ. Investors pay additional personal taxes:

- Tax rate on equity (dividend and capital gain) π.
- Tax rate on debt (interest) δ.

total after-tax cashflow:

$$\underbrace{(1-\pi)(1-\tau)X}_{\text{all equity firm}} + \underbrace{[(1-\delta)-(1-\pi)(1-\tau)]r_D D}_{\text{tax impact of debt}}$$

$V_L = V_U + [(1-\delta)-(1-\pi)(1-\tau)]PV(r_D D)$

19 Investment and Financing

Leverage and taxes

Notation:
X_t - CF from the firm's assets at time t (independent of leverage),
$V_{U,t}$ - value of firm without leverage at t,
$V_{L,t}$ - value of the firm with leverage at t,
D_t - value of its debt,
E_t - value of its equity,
r_A - required rate of return on the firm's assets of the unlevered firm, r_L - required rate of return on the levered firm, r_D - required rate of return interest on debt,
r_E - required rate of return on equity,
τ - corporate tax rate.

Leverage without tax shield

$V = D + E = \sum_{s=1}^{\infty}\frac{(1-\tau)X_s}{(1+r_A)^s} = V_U = \sum_{s=1}^{\infty}\frac{(1-\tau)X_s}{(1+WACC)^s} = V_L$

MM II: Cost of equity with leverage (D/E) is: $r_E = r_A + \frac{D}{E}(r_a - r_D)$
r_A is independent of D/E (leverage), r_E increases with D/E (assuming riskless debt), r_D may also increase with D/E as debt becomes risky.

Leverage with tax shield - APV

$V_L = E + D = V + PVTS - PDVC = APV$, assuming debt is riskless $V_L = E + D = V + PVTS = APV$

Leverage with tax shield - WACC

Assume **Leverage ratio remains constant over time** $w_D = \frac{D_t}{V_{L,t}}$, $\frac{E_t}{V_{L,t}} = w_E$

first, we have: $r_L = w_D r_D + w_E r_E$
Next, we have: $(1 + r_L - w_D\tau r_D)V_{L,t} = (1-\tau)X_{t+1} + V_{L,t+1}$

define $WACC = w_D(1-\tau)r_D + w_E r_E$, then:

$V_{L,t} = \frac{(1-\tau)X_t + V_{L,t+1}}{1+WACC} = \sum_{s=1}^{\infty}\frac{(1-\tau)X_{t+s}}{(1+WACC)^s}$

WACC with taxes - WACC

$V_L = \sum_{s=1}^{\infty}\frac{(1-\tau)X_s}{(1+WACC)^s}$ where $WACC = w_D(1-\tau)r_D + w_E r_E = \frac{D}{D+E}(1-\tau)r_D + \frac{E}{D+E}r_E$

Implementing APV

1. Find a traded firm with the same business risk: Debt to equity ratio $\frac{D}{E}$, Equity return r_E (by CAPM or APT), Debt return r_D, Tax rate τ. 2. Uncover r_A (the discount rate without leverage). 3. Apply r_A to the after-tax cash flow of the project to get V_U. 4. Compute PV of debt tax shield. 5. Compute APV.

20 Payout & Risk Management

Modigliani-Miller on payout

MM Payout Policy Irrelevance: In a financial market with no imperfections, holding fixed its investment policy (hence its free cash flow), a firm's payout policy is irrelevant and does not affect its initial share price.
Paying dividends is a zero NPV transaction. Firm value before dividend = Firm value dividend + Dividend.

Hedging basics

Let $V_{original}$: Value of the original position (unhedged),
$V_{hedging}$ - Value of the hedging position,
V_{net} - Value of the hedged position. Then,
$V_{net} = V_{original} + (\text{hedge ratio}) \times V_{hedging}$
The hedge is perfect if: 1. $V_{original}$ and $V_{hedging}$ are perfectly correlated, and 2. Hedge ratio is appropriately chosen. Otherwise, the hedging is imperfect.

Managing interest rate risk

Preliminaries

Modified Duration (MD) for discount bond $B_t = \frac{1}{(1+y)^t}$ $MD(B_t) = -\frac{1}{B_t}\frac{dB_t}{dy} = \frac{t}{1+y}$

Macaulay Duration is the weighted average term to maturity $D = \sum_{t=1}^{T}\left(\frac{PV(CF_T)}{B}t\right) = \frac{1}{B}\sum_{t=1}^{T}\left(\frac{CF_t}{(1+y)^t}t\right)$

Modified Duration measures bond's interest rate risk by its relative price change with respect to a unit change in yield (with a negative sign):
$MD = -\frac{1}{B}\frac{dB}{dy} = \frac{D}{1+y}$

Convexity (CX) measure the curvature of the bond price as function of the yield:
$CX = \frac{1}{2}\frac{1}{B}\frac{d^2B}{dy^2}$

$CX = \frac{1}{2}\frac{1}{P}\frac{1}{(1+y)^2}\sum_{t=1}^{T}\frac{t(t+1)CF_t}{(1+t)^t} = \frac{1}{2}\frac{1}{P}\frac{1}{(1+y)^2}\sum_{t=1}^{T}PV(CF_t)t(t+1)$

Taylor series approximation of bond price changes $\Delta B \approx B\left(-MD\cdot\Delta y + CX\cdot(\Delta y)^2\right)$

Bond	Price	Dur	ModDur
A	B_A	D_A	MD_A
B	B_B	D_B	MD_B

$V_P = V_A + V_B = n_A B_A + n_B B_B$
$MD_P = \frac{V_A}{V_A+V_B}MD_A + \frac{V_B}{V_A+V_B}MD_B$

δ is the hedge ratio if bond A is used to hedge bond B $MD_B - \delta MD_A = 0$, then $\delta = \frac{MD_B}{MD_A}$