Mathematical Methods for Quantitative Finance

Cheat sheet for MITx 15.455x Mathematical Methods for Quantitative Finance.

Week 2: Stochastic Processes

Time series models

Time series models

- a ts process is **stationary** if the join distribution of all of its values is invariant under time translation.
- a ts is **weakly stationary** if the first and second moments are invariant.
- MA(1): $r_t = \mu + \sigma z_t + \phi z_{t-1}$
- AR(p): $R_t = c_0 + c_1 R_{t-1} + \cdots + c_p R_{t-p} + \sigma z_t, z_t \sim IID(0,1)$
- ARMA(p,q):
  $R_t = c_0 + c_1 R_{t-1} + \cdots + c_p R_{t-p} + \sigma z_t + \phi_1 z_{t-1} + \cdots + \phi_q z_{t-q}$
- AR(1) used for **mean reversion**: $R_t = c_0 + c_1 R_{t-1} + \sigma z_t$, $E[R_t] = \frac{c_0}{1-c_1}$, for convenience: $\mu = \frac{c_0}{1-c_1}$, $\lambda = -c_1$. Then: $R_t - \mu = -\lambda(R_t - \mu) + \sigma z_t$, $|\lambda| < 1$. $Var[R_t] = \gamma_0 = \frac{\sigma^2}{1-\lambda^2}$.

  Lag-k autocovariance coefficient: $\gamma_k = (-\lambda)^k \gamma_0 = \frac{(-\lambda)^k}{1-\lambda^2}\sigma^2$

Week 5: Itô Calculus

Black-Scholes equation

Summary of some key formulas

- Itô process: $\mathrm{d}X = a\mathrm{d}t + b\mathrm{d}B$
- Itô formula:

$$\begin{aligned}\mathrm{d}F &= \frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial^2 F}{\partial X^2}\mathrm{d}X + \frac{b^2}{2}\frac{\partial F}{\partial X}\mathrm{d}t \\ &= \left(\frac{\partial F}{\partial t} + a\frac{\partial F}{\partial X} + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}\right)\mathrm{d}t + b\frac{\partial F}{\partial X}\mathrm{d}B\end{aligned}$$

- Stock price: $\mathrm{d}S = \mu S\mathrm{d}t + \sigma\mathrm{d}B \implies \mathrm{d}(\log S) = \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma\mathrm{d}B$
- Black-Scholes: $\Delta = \partial V/\partial S$, $\mathrm{d}\pi = r\pi\mathrm{d}t$,

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Recitation 5

Expectations from Brownian integrals

- $dB \sim N(0, dt)$
- $\int_0^t dB = B_t - B_0 \sim N(0,t)$
- $E[f(B_t - B_0)] = E[f(\sqrt{t}z)] = \frac{1}{\sqrt{2\pi}}\int e^{-z^2/2} f(\sqrt{t}z)dz$
- Example $E[f(B_t - B_0)] = E[(B_t - B_0)^4] = E[(\sqrt{t}z)^4] = t^2 E[z^4] = 3t^2$
  We can pull out $\sqrt{t}$,which is nonstochastic to get $t^2 E[z^4]$, $E[z^4]$ is a well-known Gaussian integal that we use in the kurtosis=3.
- Useful formula: $E[e^{\alpha z + \beta}] = e^{\alpha^2/2+\beta}$

Week 6: Continuous-Time Finance

Itô processes in higher dimensions

Itô's lemma: multiple stochastic variables

- $\mathrm{d}X_i = a_i(t, X_1, X_2, \ldots)\mathrm{d}t + b_i(t, X_1, X_2, \ldots)\mathrm{d}B_i$
  $\mathrm{d}F = \frac{\partial F}{\partial t}\mathrm{d}t + \sum \frac{\partial F}{\partial X_i}\mathrm{d}X_i + \frac{1}{2}\sum \rho_{ij} b_i b_j \frac{\partial^2 F}{\partial X_i X_j}\mathrm{d}t$
- Heuristics "rule of thum" for correlated Brownian motions :
  $(\mathrm{d}B_i)^2 \to \mathrm{d}t$, $(\mathrm{d}B_i)(\mathrm{d}B_j) \to \rho_{ij}\mathrm{d}t$,
  $(\mathrm{d}X_i)^2 \to b_i^2\mathrm{d}t$, $(\mathrm{d}X_i)(\mathrm{d}X_j) \to \rho_{ij} b_i b_j \mathrm{d}t$
- two stochastic variables case: $\mathrm{d}X_1 = a_1\mathrm{d}t + b_1\mathrm{d}B_1, \mathrm{d}X_2 = a_2\mathrm{d}t + b_2\mathrm{d}B_2$
  $\mathrm{d}F =$
  $\frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial X_1}\mathrm{d}X_1 + \frac{\partial F}{\partial X_2}\mathrm{d}X_2 + \left(\frac{b_1^2}{2}\frac{\partial^2 F}{\partial X_1^2} + \frac{b_2^2}{2}\frac{\partial^2 F}{\partial X_2^2} + b_1 b_2 \rho \frac{\partial^2 F}{\partial X_1 \partial X_2}\right)\mathrm{d}t$
- example: $F = X1X2 \implies \mathrm{d}F = X1\mathrm{d}X_2 + X_2\mathrm{d}X_1 + \rho b_1 b_2 \mathrm{d}t$
  since $(\mathrm{d}X_i)(\mathrm{d}X_j) \to \rho_{ij} b_i b_j \mathrm{d}t \implies \mathrm{d}F = X1\mathrm{d}X_2 + X_2\mathrm{d}X_1 + \mathrm{d}X_1\mathrm{d}X_2$

Useful formulas

$\mathrm{Cov}(aX + bY, Z) = a\,\mathrm{Cov}(X,Z) + b\,\mathrm{Cov}(Y,Z)$
$\mathrm{Var}(aX + bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X,Y)$
$\mathrm{Var}(aX - bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) - 2ab\,\mathrm{Cov}(X,Y)$

Recommended Resources

- MITx 15.455x MITx 15.455x Mathematical Methods for Quantitative Finance [Lecture Slides] (`https://learning.edx.org/course/course-v1:MITx+15.455x+3T2020/home`)
- Tsay, Analysis of Financial Time Series (3e), Wiley. (Tsay)
- Capinski and Zastawniak, Mathematics for Finance, Springer. (CZ)
- Olver, Introduction to Partial Differential Equations (2016), Springer. (Olver)
- Campbell, Lo, and MacKinlay, Econometrics of Financial Markets (1997), Princeton. (CLM)
- Lang, Introduction to Linear Algebra (2e), Springer (Lang)
- Axler, Linear Algebra Done Right (3e), Springer (Axler)
- LaTeX File (`github.com/j053g/cheatsheets/15.455x`)

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