MITx 15.455x Mathematical Methods for Quantitative Finance

Recitation 5

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Module 2: Browninan integrals

Let's look at more expectations involving our stochastic processes. We know that dB is a Gaussian random variable with a mean 0 and variance dt.

$$dB \sim N(0, dt)$$

We also know that if we integrate $\int_0^t dB$, and for convenience, I can write this as:

$$\int_0^t dB = B_t - B_0 \sim N(0, t)$$

Let's agree that $B_0 = 0$.

So since that is the case, we have an easy way that we can write things. We know that the expectation of B is going to be 0. We know that its variance is going to be t.

So let's just replace the random variable $B_t - B_0$ by $\sqrt{t}z$ where $z \sim N(0, 1)$

Where z doesn't have any time dependence. So that's just simplifying things and putting everything in an even more standard form. But it also makes explicit the time dependent. And it's the scaling with time that's going to be important for a lot of financial applications, and for risk management. So, what that means is that any time where we see a B_t, or a B_t minus B_0, we can replace it by that function. So if we want to compute the expectation of some function, we can just write:

$$E[f(B_t - B_0)] = E[f(\sqrt{t}z)]$$

and compute that.

And remember, these are our Gaussian intervals. So we just need to do an integral. That's all it is. So this is equal:

$$E[f(B_t - B_0)] = E[f(\sqrt{t}z)] = \frac{1}{\sqrt{2\pi}} \int e^{-z^2/2} f(\sqrt{t}z) dz$$

Easy.

So for example, suppose we wanted to compute the expectation of the fourth power.

$$f(x) = x^4$$

So we'd like to compute:

$$E[f(B_t - B_0)] = E[(B_t - B_0)^4] = E[(\sqrt{t}z)^4] = t^2 E[z^4] = 3t^2$$

We can pull out $\sqrt{t}$, which is nonstochastic to get $t^2 E[z^4]$, $E[z^4]$ is a well-known Gaussian integal that we use in the kurtosis. It's just equal to 3.

All right.

So this way of standardizing things and writing things in terms of z when we work in the integral form, when we integrate our processes, is very convenient. Let's do an example, and I want to review a very useful trick with you. So let's take a look.

Example

Suppose we wanted to compute
$$E[e^{6X}] \text{ where } dX = \mu dt + \sigma dB$$

where dX, let's just say it's ordinary Brownian motion.

OK, because it's on this form, you can integrate as:

$$\text{integrate:} X_t - X_0 = \mu t + \sigma \sqrt{t} z$$

So if we go to compute the expectation :

$$E[e^{6(\mu t + \sigma \sqrt{t} z)}] = e^{6\mu t} E[e^{6\sigma \sqrt{t} z}]$$

Useful Formula

Suppose that we have $E[e^{\alpha z + \beta}]$. And this shows up a lot in particular in asset pricing formulas. So this is clearly equal to

$$
\begin{aligned}
E[e^{\alpha z + \beta}] &= e^{\beta} E[e^{\alpha z}] \\
&= \frac{e^{\beta}}{\sqrt{2\pi}} \int e^{-z^2/2} e^{\alpha z} dz \\
&= \frac{e^{\beta}}{\sqrt{2\pi}} \int e^{-z^2/2 + \alpha z - \alpha^2/2 + \alpha^2/2} dz \\
&= \frac{e^{\beta}}{\sqrt{2\pi}} \int e^{-(z^2 - 2\alpha z + \alpha^2)/2 + \alpha^2/2} dz \\
&= \frac{e^{\beta}}{\sqrt{2\pi}} \int e^{-(z-\alpha)^2/2 + \alpha^2/2} dz \\
&= e^{\beta + \alpha^2/2} \left[\frac{1}{\sqrt{2\pi}} \int e^{-(z-\alpha)^2/2} dz \right]
\end{aligned}
$$

We do the Gaussian integral, with the trick of completing the square in the exponent. And because the integral goes from minus infinity to infinity, the difference in the exponential between z and $z - \alpha$ makes no difference at all. If you'd like, you can shift the variable of integration. The integral inside the square brackets with $\frac{1}{\sqrt{2\pi}}$ in front is equal to 1. So our final result, our final useful formula, is that:

$$E[e^{\alpha z + \beta}] = e^{\beta + \alpha^2/2}$$

Now, let's apply that to our example.

$$E[e^{6X}] = E[e^{6\mu t + 6\sigma \sqrt{t} z}]$$

with $\alpha = 6\sigma\sqrt{t}$ and $\beta = 6\mu t$, we apply $E[e^{\alpha z + \beta}] = e^{\beta + \alpha^2/2}$

$$
\begin{aligned}
E[e^{6X}] &= e^{6\mu t + (6\sigma\sqrt{t})^2/2} \\
&= e^{6\mu t + 18\sigma^2 t}
\end{aligned}
$$

So there's our answer for finding this expectation. And this works, generally, for a larger class of functions that show up for different kinds of Ito processes.

Module 3: Solutions to the diffusion equation

Let's take a closer look at the diffusion equation. So the general diffusion equation is given right here.

So we say that we have a function $p(z, t)$,

$$p(z, t) = \int p_0(z - w, t) f(w) dw$$

and they need to satisfy this differential equation.

$$\frac{\partial p}{\partial t} - \frac{1}{2} \frac{\partial^2 p}{\partial z^2} = 0$$

And we've seen that there's a very special solution, p_0, that satisfies that equation, which we can find by plugging it in and taking some derivatives. So here's our p_0.

$$p_0 = \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}}$$

Now, I claimed that if we construct this integral on top, that this is the solution to the equation. So what I'd like to do is, let's check that, and then let's apply it. So the first thing to do is to check it. And the way we do it is we take derivatives. And what we find is that if we take that integral expression, and we stick it into the differential equation, what we'll do is, we'll move the differential operators inside. So for example,

$$\frac{\partial}{\partial t} p(z, t) = \int \frac{\partial}{\partial t} \Big[p_0(z - w, t) f(w) dw \Big]$$

But where is the t dependence? The t dependence is only in one place. It's in p_0.

And similarly,

$$\Big(\frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial z^2} \Big) p(z, t) = \int \Big[\frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial z^2} \Big] p_0(z - w, t) f(w) dw$$

Exercise

Suppose that

$$f(z) = z^2$$
$$p(z, 0) = z^2$$
$$\text{find } p(z, t)$$

So what we need to do is, we've got a formula. Let's just plug and chug.

$$p(z, t) = \int p_0(z - w, t) f(w) dw$$
$$= \int \frac{1}{\sqrt{2\pi t}} e^{-\frac{(z-w)^2}{2t}} w^2 dw$$

Remember, it's a Gaussian where the variance is t. So it's not completely standardized. So there is t dependence in this. How do we do this integral? Well, let's change variables. So what we'd like to do is, let's simplify the exponent and pick a new variable.

$$\text{let } u = \frac{w - z}{\sqrt{t}} \implies du = \frac{dw}{\sqrt{t}}$$
$$\implies w = u\sqrt{t} + z$$

So let's make those substitutions.

$$p(z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} (u\sqrt{t} + z)^2 du$$

Because now we almost have things in standardized form for a Gaussian interval, let's just expand that out.

$$p(z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} (u^2 t + 2u\sqrt{t}z + z^2) du$$

$$= t \Big[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} u^2 \Big] + 0 + z^2 \Big[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} u^2 \Big]^1$$

Let's look at each of these terms. The first term, is going to be $u^2 t$. Remember, t is a constant with respect to the variable of integration. The second term, $2u\sqrt{t}z$, is going to vanish. Because it's linear in u, this is an odd function in u. We're going from minus infinity to infinity. And odd functions are going to have varnishing integrals. It's just a plus sign to the minus sign are going to cancel each other out. And then, the last term, z^2, that's just going to be multiplied by a constant a constant equal to 1.

Because $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} u^2$ is just the variance of the standardized Gaussian distribution.

So we do the integrals. And what we're left with is :

$$p(z,t) = z^2 + t$$

And we can check that it satisfies our differential equation.

$$\frac{\partial p}{\partial t} = 1 \;, \; \frac{1}{2} \frac{\partial^2 p}{\partial z} = 1$$

And if we subtract the two of them, we get 0. So we're done. So this is the answer. Let's do one more exercise.

Exercise

Slightly different version of the one that we did in lecture. But now, don't refer back to the lecture. Just take a look at what we've done for the integrals. And do this one yourself. So what I'd like to do is introduce the Gaussian cumulative distribution function, which is going to be useful and show up in a few places. I will call this $\Phi(x)$. It's going to be defined as:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz = Prob(Z < x)$$

So I compute the left side of the integral of the bell curve for the Gaussian distribution up to some point x. So in terms of Gaussian probabilities, this is the same thing as the probability that a Gaussian random variable Z is less than some particular value x. Now, from the fundamental theorem of calculus of course, I know that :

$$\frac{d}{dx} \Phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

So I can differentiate or I can integrate. I can go back and forth. But this basic idea that it's an incomplete integral. So I integrate from minus infinity up to a particular value x that's defined out here. It's going to be useful, and we'll see that it shows up in a bunch of places. Here's a really simple example.

$$f(w) = \begin{cases} 1 & \text{if } w < \kappa \\ 0 & \text{if } w > \kappa \end{cases}$$

$f(w) = p(w,0)$ find $p(z,t)$

So the question is, find $p(z,t)$,

So at time 0, it's a step function. It's either 0 or 1 depending on the value of its argument.

So what we'd like to do is, we want to find $p(z,t)$ for the general case. So take a moment, see if you can work it out from our general definition.

Remember that our definition is, we integrate p_0 against p_0 evaluated at $z-2$ against the function $f(w)$. And we integrate that over w in an explicit expression in terms of z. Go take a moment to do that. And then, we'll take a look at this together.

OK? Let's go. Let's just work from the definition. The reason why this is particularly nice is the integrand is either 1 or 0. So there's an interesting generalization that's important for the Black-Scholes case, where we might– instead of 1 or 0, we might like to let it be 0 in the lower case, but $\kappa - w$ in the upper case. So after you've done this one, I'd suggest trying that one as an extension. But this one's fairly straightforward. So let's do this one together just to make sure we've got the concepts and our definitions of the integral.

So this is going to be:

$$p(z,t) = \int_{-\infty}^{\kappa} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(z-w)^2}{2t}} \, dw$$

$$\text{let } u = \frac{w-z}{\sqrt{t}} \implies du = \frac{dw}{\sqrt{t}}$$

In addition, our upper limit of integration is $w = \kappa$. And that's going to translate into an upper limit of

$$u* = \frac{\kappa - z}{\sqrt{t}}$$

So, now we have:

$$p(z,t) = \int_{-\infty}^{\frac{\kappa - z}{\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \, du$$

$$= \Phi\left(\frac{\kappa - z}{\sqrt{t}}\right)$$

Which is a very well behaved function for positive values of t. And it would be interesting to take a look and plot this as t goes to 0. But right now, we have the result that we saw it. And we can check this, again, by differentiating and putting it into the differential equation, and verifying that it satisfies the equation, and verifying that it satisfies the initial conditions a t equals 0. Which as I said, it requires taking a limit, because we can't immediately set t equals 0 in this case in the way we did previously.