Derivatives Markets: Advanced Modeling and Strategies

Cheat sheet for MITx 15.435x Derivatives Markets: Advanced Modeling and Strategies.

Week 1: Forward Contracts

Forward contract basics

Forward Contract

- A **forward contract** is an agreement between two counterparties to trade a pre-specified amount of goods or securities at a pre-specified future date, T, for a pre- specified price, F_0.
- The **Profit/Loss (P/L)** at the contract maturity T for each counterparty is: $P/L_{long} = N(S_T - F_0)$, $P/L_{short} = N(F_0 - S_T)$
- Price of a zero coupong bond with face value Z: $P = e^{-r_T T} Z$
  $f(0, T_1, T_2)$ denotes the **forward rate** between time T_1 and T_2, as of time 0 : $f(0, T_1, T_2) = \frac{T_2 r_{T_2} - T_1 r_{T_1}}{T_2 - T_1}$
- Long forward positions are equivalent to borrowing and going long in the underlying asset
- Forward short positions are equivalent to lending and going short the underlying

Pricing formulas

Pricing formulas

- An **arbitrage opportunity** is a trading strategy that either (1) Yields a positive profit today, and zero cash flows in the future; or (2) Costs nothing today and yields a positive profit in the future
- **The Law of One Price**: Securities with identical payoffs must have the same price
- **Stock** with known dividend D at time $t < T$: $F_0 = (P_{S,0} - De^{-rt})e^{rT}$
  Stock with known dividend yield q: $F_0 = P_{S,0} e^{(r-q)T}$
- **Bond** with coupon C at time $t < T$: $F_0 = (P_{B,0} - Ce^{-rt})e^{rT}$
- **Currencies**. $r_\$$ ($r_€$) the USD (EUR) risk-free rate. S_t is the exchange rate (USD per EUR) at time t: $F_0 = S_0 e^{(r_\$ - r_€)T}$

Forward prices for commodities

- Forward price with lump-sum storage cost U: $F_{0,T} = (S_0 + PV(U))e^{rT}$
- Forward price with proportional storage cost u: $F_{0,T} = S_0 e^{(r+u)T}$
- Forward price with convenience yield y: $F_{0,T} = S_0 e^{(r-y)T}$
- Forward price with proportional storage cost u and convenience yield y: $F_{0,T} = S_0 e^{(r+u-y)T}$
- **Contango** is a pattern of forward prices that increases with contract maturity
- **Backwardation** is a pattern of forward prices over time that decreases with contract maturity

Key concepts for hedging and speculating

Valuing a forward contract over time

- Suppose that $K = F_0$ the original delivery price, initial value of contract $f_0 = 0$.
- Value of a **long** forward contract at time t : $f_{long,t,T} = (F_t - K)e^{-r(T-t)}$
- Value of a **short** forward contract at time t : $f_{short,t,T} = (K - F_t)e^{-r(T-t)}$
- **Basis** is the difference between the spot and forward price of a security or commodity.
- **Cross-hedging** involves using a contract type to hedge which differs from the security or commodity being hedged.
- The **hedge ratio** is the relative number of forward contracts to units of the asset being hedged that maximizes the effectiveness of the hedge: $N_S \mathbb{E}[\mathrm{d}S] = N_F \mathbb{E}[\mathrm{d}F]$ then: $\frac{N_S}{N_F} = \frac{\mathbb{E}[\mathrm{d}F]}{\mathbb{E}[\mathrm{d}S]}$. If long in spot then short in forwards, and vice versa.

Week 2: Futures and Swaps Contracts

Futures

Forward Contract

- Daily settlement of gains and losses and have to maintain a minimum balance in a margin account.
- If margin balance falls below the maintenance margin, so the investors has to deposit into the account to restore the initial margin requirement.

Swaps

Swap Pricing

- **Interest Rate Swap:** Given spot yield curve $s_1, s_2, \ldots, s_N$ the coupon rate of the swap solves $F = \frac{cF}{(1+s_1)} + \frac{cF}{(1+s_2)^2} + \cdots + \frac{(1+c)F}{(1+s_N)^N}$, solving for c: $c = \frac{1-B_N}{\sum_{i=1}^N B_i}$, where $B_i = \frac{1}{(1+s_i)^i}$ is the discount factor for period i.
- **Currency Swap**: The currency swap rate equals the current exchange rate multiplied by the ratio of the relative risk-free borrowing costs in the two currencies. Example: US firm pays bank $1M€$ on $T = 0.5, 1, \ldots, 2.5$. US Bank firm $1M \cdot K\$$ then: $K = S_0 \frac{e^{-0.5r_€} + e^{-1r_€} + \cdots + e^{-2.5r_€}}{e^{-0.5r_\$} + e^{-1r_\$} + \cdots + e^{-2.5r_\$}}$

Week 3 – Duration and convexity-based strategies for risk management

Duration and Convexity

Duration and Convexity

- **Modified Duration (MD) for discount bond** $P_t = \frac{1}{(1+y)^t}$, then $MD(P_t) = -\frac{1}{P_t}\frac{dP_t}{dy} = \frac{t}{1+y}$
- **Macaulay Duration** is the weighted average term to maturity $D = \sum_{t=1}^T \left(\frac{PV(CF_T)}{P} t\right) = \frac{1}{P}\sum_{t=1}^T \left(\frac{CF_t}{(1+y)^t} t\right)$
- **Modified Duration** measures bond's interest rate risk by its relative price change with respect to a unit change in yield (with a negative sign): $D_M = -\frac{1}{P}\frac{dP}{dy} = \frac{D}{1+y}$
- **Convexity (CX)** measure the curvature of the bond price as function of the yield: $CX = \frac{1}{2}\frac{1}{P}\frac{d^2 B}{dy^2}$

$$CX = \frac{1}{2}\frac{1}{P}\frac{1}{(1+y)^2}\sum_{t=1}^T \frac{t(t+1)CF_t}{(1+t)^t} = \frac{1}{2}\frac{1}{P}\frac{1}{(1+y)^2}\sum_{t=1}^T PV(CF_t)t(t+1)$$

- **Taylor series approximation of bond price changes** $\Delta P \approx P\left(-D_M \cdot \Delta y + CX \cdot (\Delta y)^2\right)$
- **Dollar Duration:** Dollar duration is the modified duration multiplied by the price: $D_d = D_M \cdot P$. It's useful for hedging strategies and for understanding risk of zero NPV portolios.

Hedging

Delta/Gamma Hedging

- A **delta neutral** portfolio equates the hedge ratio of assets and liabilities : $P_A D_{M,A} = P_L D_{M,L}$
- A **Gamma neutral** portfolio is delta neutral and equates gammas of assets and liabilities. Example hedge Liability L with $D_{M,L}$ and C_L, with two assets A_1, A_2 with $D_{M,1} C_1$ and $D_{M,2}, C_2$ then :
  equate delta: $LD_{M,L} = A_1 D_{M,1} + A_2 D_{M,2}$
  equate gamma: $LC_L = A_1 C_1 + A_2 C_2$, i.e. solve system to find A_1, A_2: $\begin{pmatrix} D_{M,1} & D_{M,2} \\ C_1 & C_2 \end{pmatrix} \cdot \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} LD_{M,L} \\ LC_L \end{pmatrix}$
- **Swap Dollar Duration** for fix receiver: $D_{dollar,rec} = P_{fix} D_{M,fix} - P_{flt} D_{eff,flt}$, The effective duration of a (pure) floating rate bond is the time until the next reset, divided by $1 + \frac{Y_{APR}}{k}$, k is the number of compounding periods in a year, i.e.: $D_{eff,flt} = \frac{t_{nextreset}}{1+\frac{Y_{APR}}{k}}$, for a new swap $P_{flt} = Pfix = 1$, then: $D_{dolar,rec} = D_{M,fix} - D_{eff,flt}$

Week 4: Options Strategies and Pricing Basics

Option Basics

- Put call parity: $Put - Call = e^{-rT}(K - F_{0,T})$
- For a non-dividend paying stock: $Put = Call + e^{-rT}K - S_0$
- *Important: This formula only holds for European options!*

Option Strategies

- Protective put: Long put, Long stock. Payoff at T: $S_T + \max(K - S_T, 0)$
- Covered call: Long stock, Short call. Payoff at T: $S_T - \max(S_T - K, 0)$
- Bear spread: Short OTM put (strike K_1) and long ITM put ($K_2 > K_1$)
- Bull spreads: Long ITM call (strike K_1) and short OTM call ($K_2 > K_1$)
- Buttefly spread: Long 1 call with strike K_0, short 2 calls with strike K_1 and long 1 call with strike K_2, with $K_0 < K_1 < K_2$ and $K_1 = \frac{K_0 + K_2}{2}$
- Straddle: Bet on high volatility. Long a call and a put with the same strike.
- Strangle: Bet on high movements. Long put with K_0 and call with $K_1 > K_0$

Binomial trees

- One step: $S_0 = \frac{E[S_1]}{1+R} = \frac{qS_{1,u} + (1-q)S_{1,d}}{1+R}$
- Expected (gross) Return: $\mathbb{E}\left[\frac{S_1}{S_0}\right] = q\frac{S_{1,u}}{S_0} + (1-q)\frac{S_{1,d}}{S_0}$
- Variance: $\mathbb{E}\left[\left(\frac{S_1}{S_0} - \mathbb{E}\left[\frac{S_1}{S_0}\right]\right)^2\right] = q\left(\frac{S_{1,u}}{S_0} - \mathbb{E}\left[\frac{S_1}{S_0}\right]\right)^2 + (1-q)\left(\frac{S_{1,d}}{S_0} - \mathbb{E}\left[\frac{S_1}{S_0}\right]\right)^2$
- **replicating portfolio** :
  $\Delta \cdot S_{1,u} + B_0 e^{rT} = V_{1,u}$
  $\Delta \cdot S_{1,d} + B_0 e^{rT} = V_{1,d}$
  Solution: $\Delta = \frac{V_{1,u} - V_{1,d}}{S_{1,u} - S_{1,d}}$, then we solve for $B_0 = e^{-rT}(V_{1,u} - \Delta \cdot S_{1,u})$
  no arbitrage $\implies V_0 = \Delta \cdot S_0 + B_0$
- **risk neutral pricing**: we choose q^* so that all risky assets earn the risk-free rate: $q^* S_{1,u} e^{-rT} + (1 - q*)S_{1,d} e^{-rT} = S_0 \implies q^* = \frac{S_0 e^{rT} - S_{1,d}}{S_{1,u} - S_{1,d}}$
  $S_0 = \mathbb{E}^*[e^{-rT} S_1]$. In general: **Price of derivative** $= \mathbb{E}^*[e^{-rT}$**payoff**$]$
- American options. Compare the value of immediate exercise with the value of the option. Exercise if an only if (for put): $K - S >$ **Discounted value of future distribution of payoffs if wait.**
- **multi-step trees**: (i, j) time: $i = 0, 1, 2, \ldots, n$; node: $j = 1, 2, \ldots n$
  with European derivative: $V_{i,j}^E = e^{-rh}\mathbb{E}^*[V_{i+1}^E | (i,j)]$, where $h = \frac{T}{n}$
  with American derivative: $V_{i,j}^A = \max\left(g_{i,j}, e^{-rh}\mathbb{E}^*[V_{i+1}^A | (i,j)]\right)$, where $h = \frac{T}{n}$ where $g_{i,j}$ is the payoff from the American derivative

Week 5 – Black-Scholes-Merton and the Greeks

Multi-step Binomial Trees

- Chop interval $[0, T]$ into n little intervals of time $h = \frac{T}{n}$.
- $u = e^{\sigma h}$; $d = \frac{1}{u}$; and $q^* = \frac{e^{rh} - d}{u - d}$.
- BSM: $C(S, K, T-t, r, \sigma) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$
  $d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$
  $P(S, K, T-t, r, \sigma) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1)$
- BSM with **knwon** dividend. Define $S* = S - PV(D)$, $PV(D) =$ present value of Dividends before Expiration. Use BSM Formula with S^* instead of S.
- BSM with **knwon** dividend yield δ :
  $C = Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$
  $P = Ke^{-r(T-t)}N(-d_2) - Se^{-\delta(T-t)}N(-d_1)$
  $d_1 = \frac{\ln(\frac{S}{K}) + (r - \delta + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$
- **interpretation** of BSM with n calls or puts
  $nc = \underbrace{nS_0N(d_1)}_{\text{value of stock}} - \underbrace{nKe^{-rT}N(d_2)}_{\text{value of bonds}}, \Delta_c = N(d_1)$
  $np = \underbrace{nKe^{-rT}N(-d_2)}_{\text{value of bonds}} - \underbrace{nS_0N(-d_1)}_{\text{value of stock}}, \Delta_p = -N(-d_1)$
  Number of shares $= \frac{\text{value of stock}}{S_0}$

Week 9 – Credit risk

The Merton Model

The Merton Model

- The payoff to equity holders is then the one of a call option
- If we denote E_0 the value of equity today

Recommended Resources

- MITx 15.435x Derivatives Markets: Advanced Modeling and Strategies Lecture Slides
- John Hull's, Options Futures and Other Derivatives, 10th edition
- Bruce Tuckman and Angel Serrat, Fixed Income Securities; Tools for Today's Markets, 3rd Edition (BTAS)
- LaTeX File (`github.com/j053g/cheatsheets/15.435x`)

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