Derivatives Markets: Advanced Modeling and Strategies

Cheat sheet for MITx 15.435x Derivatives Markets: Advanced Modeling and Strategies.

Week 1: Forward Contracts

Forward contract basics

Forward Contract

- A **forward contract** is an agreement between two counterparties to trade a pre-specified amount of goods or securities at a pre-specified future date, T, for a pre- specified price, F_0.
- The **Profit/Loss (P/L)** at the contract maturity T for each counterparty is: $P/L_{long} = N(S_T - F_0)$, $P/L_{short} = N(F_0 - S_T)$
- Price of a zero coupong bond with face value Z: $P = e^{-r_T T} Z$
  $f(0, T_1, T_2)$ denotes the **forward rate** between time T_1 and T_2, as of time 0 : $f(0, T_1, T_2) = \frac{T_2 r_{T_2} - T_1 r_{T_1}}{T_2 - T_1}$
- Long forward positions are equivalent to borrowing and going long in the underlying asset
- Forward short positions are equivalent to lending and going short the underlying

Pricing formulas

Pricing formulas

- An **arbitrage opportunity** is a trading strategy that either (1) Yields a positive profit today, and zero cash flows in the future; or (2) Costs nothing today and yields a positive profit in the future
- **The Law of One Price**: Securities with identical payoffs must have the same price
- **Stock** with known dividend D at time $t < T$: $F_0 = (P_{S,0} - De^{-rt})e^{rT}$
  Stock with known dividend yield q: $F_0 = P_{S,0}e^{(r-q)T}$
- **Bond** with coupon C at time $t < T$: $F_0 = (P_{B,0} - Ce^{-rt})e^{rT}$
- **Currencies**. $r_{\$}$ ($r_{€}$) the USD (EUR) risk-free rate. S_t is the exchange rate (USD per EUR) at time t: $F_0 = S_0 e^{(r_{\$} - r_{€})T}$

Forward prices for commodities

- Forward price with lump-sum storage cost U: $F_{0,T} = (S_0 + PV(U))e^{rT}$
- Forward price with proportional storage cost u: $F_{0,T} = S_0 e^{(r+u)T}$
- Forward price with convenience yield y: $F_{0,T} = S_0 e^{(r-y)T}$
- Forward price with proportional storage cost and convenience yield: $F_{0,T} = S_0 e^{(r+u-y)T}$
- **Contango** is a pattern of forward prices that increases with contract maturity
- **Backwardation** is a pattern of forward prices over time that decreases with contract maturity

Key concepts for hedging and speculating

Valuing a forward contract over time

- Suppose that $K = F_0$ the original delivery price, initial value of contract $f_0 = 0$.
- Value of a **long** forward contract at time t : $f_{long,t,T} = (F_t - K)e^{-r(T-t)}$
- Value of a **short** forward contract at time t : $f_{short,t,T} = (K - F_t)e^{-r(T-t)}$
- **Basis** is the difference between the spot and forward price of a security or commodity.
- **Cross-hedging** involves using a contract type to hedge which differs from the security or commodity being hedged.
- The **hedge ratio** is the relative number of forward contracts to units of the asset being hedged that maximizes the effectiveness of the hedge: $N_S \mathbb{E}[\mathrm{d}S] = N_F \mathbb{E}[\mathrm{d}F]$ then: $\frac{N_S}{N_F} = \frac{\mathbb{E}[\mathrm{d}F]}{\mathbb{E}[\mathrm{d}S]}$. If long in spot then short in forwards, and vice versa.

Week 4: Options Strategies and Pricing Basics

Option Basics

- Put call parity: $Put - Call = e^{-rT}(K - F_{0,T})$
- For a non-dividend paying stock: $Put = Call + e^{-rT}K - S_0$
- *Important: This formula only holds for European options!*

Option Strategies

- Protective put: Long put, Long stock. Payoff at T: $S_T + \max(K - S_T, 0)$
- Covered call: Long stock, Short call. Payoff at T: $S_T - \max(S_T - K, 0)$
- Bear spread: Short OTM put (strike K_1) and long ITM put ($K_2 > K_1$)
- Bull spreads: Long ITM call (strike K_1) and short OTM call ($K_2 > K_1$)
- Buttefly spread: Long 1 call with strike K_0, short 2 calls with strike K_1 and long 1 call with strike K_2, with $K_0 < K_1 < K_2$ and $K_1 = \frac{K_0 + K_2}{2}$
- Straddle: Bet on high volatility. Long a call and a put with the same strike.
- Strangle: Bet on high movements. Long put with K_0 and call with $K_1 > K_0$

Binomial trees

- One step: $S_0 = \frac{E[S_1]}{1+R} = \frac{qS_{1,u} + (1-q)S_{1,d}}{1+R}$
- Expected (gross) Return: $\mathbb{E}\left[\frac{S_1}{S_0}\right] = q\frac{S_{1,u}}{S_0} + (1-q)\frac{S_{1,d}}{S_0}$
- Variance:
  $\mathbb{E}\Big[\Big(\frac{S_1}{S_0} - \mathbb{E}\big[\frac{S_1}{S_0}\big]\Big)^2\Big] = q\Big(\frac{S_{1,u}}{S_0} - \mathbb{E}\big[\frac{S_1}{S_0}\big]\Big)^2 + (1-q)\Big(\frac{S_{1,d}}{S_0} - \mathbb{E}\big[\frac{S_1}{S_0}\big]\Big)^2$
- **replicating portfolio** :
  $\Delta \cdot S_{1,u} + B_0 e^r = V_{1,u}$
  $\Delta \cdot S_{1,d} + B_0 e^r = V_{1,d}$
  Solution: $\Delta = \frac{V_{1,u} - V_{1,d}}{S_{1,u} - S_{1,d}}$, then we solve for $B_0 = e^{-r}(V_{1,u} - \Delta \cdot S_{1,u})$
  no arbitrage $\implies V_0 = \Delta \cdot S_0 + B_0$
- **risk neutral pricing**: we choose q^* so that all risky assets earn the risk-free rate: $q^* S_{1,u} e^{-rT} + (1 - q*)S_{1,d}e^{-rT} = S_0 \implies q^* = \frac{S_0 e^{rT} - S_{1,d}}{S_{1,u} - S_{1,d}}$
  $S_0 = \mathbb{E}^*[e^{-rT}S_1]$. In general: **Price of derivative** $= \mathbb{E}^*[e^{-rT}$**payoff**$]$
- American options. Compare the value of immediate exercise with the value of the option. Exercise if an only if (for put): $K - S >$**Discounted value of future distribution of payoffs if wait.**
- **multi-step trees**: (i, j) time: $i = 0, 1, 2, \ldots, n$; node: $j = 1, 2, \ldots n$
  with European derivative: $V^E_{i,j} = e^{-rh}\mathbb{E}^*[V^E_{i+1}|(i,j)]$, where $h = \frac{T}{n}$
  with American derivative: $V^A_{i,j} = \max\Big(g_{i,j}, e^{-rh}\mathbb{E}^*[V^A_{i+1}|(i,j)]\Big)$, where $h = \frac{T}{n}$ where $g_{i,j}$ is the payoff from the American derivative

Recommended Resources

- MITx 15.435x Derivatives Markets: Advanced Modeling and Strategies Lecture Slides
- John Hull's, Options Futures and Other Derivatives, 10th edition
- Bruce Tuckman and Angel Serrat, Fixed Income Securities; Tools for Today's Markets, 3rd Edition (BTAS)
- LaTeX File (`github.com/j053g/cheatsheets/15.435x`)

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