Mathematical Methods for Quantitative Finance

Cheat sheet for MITx 15.455x Mathematical Methods for Quantitative Finance.

Week 1: Probability

Random variables, distributions, and moments

Moments of a distribution

- The **moments** of a distribution are the expectation of powers of the r.v. $\mu_l = E[X^l] = \begin{cases} \sum_k x_k^l p(x_k) \\ \int x^l p(x) dx \end{cases}$
- **Variance** : $\sigma^2 = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$
- **Skewness** - asymmetry parameter : $s = \frac{\mathbb{E}[(X-\mu)^3]}{\sigma^3} = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$
- **Kurtosis** - measure of tail "weights" : $\kappa = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] - 3$

Covariance and correlation

- **covariance**: $\mathrm{Cov}(X,Y) \equiv \mathbb{E}[(X-\mu_x)(Y-mu_y)] = \mathbb{E}[XY] - \mu_x\mu_y$
- the **correlation** is proportional to the covariance: $\rho(X,Y) = \mathrm{Corr}(X,Y) \equiv \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \mathbb{E}\left[\left(\frac{X-\mu_x}{\sigma_x}\right)\left(\frac{Y-\mu_y}{\sigma_y}\right)\right]$

Common distributions

- **Uniform distribution**: $p(x) = \begin{cases} 1, x \in [0,1] \\ 0, \text{otherwise} \end{cases}$, $\mathrm{Prob}(a < X < b) = b - a$
  $\mu = \int_{\infty}^{\infty} xp(x)\mathrm{d}x = \int_0^1 x\mathrm{d}x = \frac{1}{2}, \sigma^2 = \int_{\infty}^{\infty}(x-\frac{1}{2})^2\mathrm{d}x = \frac{1}{12}$
  $u_l = \int_0^1 x_l dx = \frac{1}{l}$
- **Binomial distribution**: $f(x;n,p) = \binom{n}{k} p^k q^{n-k} = \frac{n!}{k!(n-k)!} p^k q^{n-k}$, $\mu = np, \sigma^2 = npq$
- **Gaussian distribution**: $p(x) - \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$,
  $\Phi(x) = Prob(Z < x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz$

Moments:

Order	Non-central moment	Central moment
1	μ	0
2	$\mu^2 + \sigma^2$	σ^2
3	$\mu^3 + 3\mu\sigma^2$	0
4	$\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$	$3\sigma^4$

- **Lognormal distribution**: $X = \log Y$, $x = \log y$, then $g(y) = \frac{p(x)}{|\mathrm{d}y/\mathrm{d}x|} = \frac{1}{y\sqrt{2\pi\sigma^2}} e^{-(\log y-\mu)^2/(2\sigma^2)}$
- **Poisson distribution**: $p(k;\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$. $\mu = \sigma^2 = \lambda$.
  Probability of k "arrivals" during interval t: $p(k;\lambda) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$

Week 2: Stochastic Processes

The Random Walk

Random walk model

- $S_T = z_1 + z_2 + \cdots + z_T$. Each increment is a random IID variable. $\mathbb{E}[z_t] = 0, \mathbb{E}[z_t^2] = 1, \mathbb{E}[z_t z_{t'}] = 0$ if $t \neq t'$. $\mathrm{Var}(S_T) = T$
- **Generalized random walk model**: $r_t = \sigma z_t + \mu$. $\mathbb{E}[r_t] = \mu$, $\mathbb{E}[(r_t-\mu)^2] = \sigma^2$. $X_T \equiv \sum_{t=1}^T r_t, \mathbb{E}[X_T] = T\mu$, $\mathrm{VaR}(X_T) = T\sigma^2$.

Time series models

Time series models

- a ts process is **stationary** if the join distribution of all of its values is invariant under time translation.
- a ts is **weakly stationary** if the first and second moments are invariant.
- MA(1): $r_t = \mu + \sigma z_t + \phi z_{t-1}$
- AR(p): $R_t = c_0 + c_1 R_{t-1} + \cdots + c_p R_{t-p} + \sigma z_t$, $z_t \sim IID(0,1)$
- ARMA(p,q): $R_t = c_0 + c_1 R_{t-1} + \cdots + c_p R_{t-p} + \sigma z_t + \phi_1 z_{t-1} + \cdots + \phi_q z_{t-q}$
- AR(1) used for **mean reversion**: $R_t = c_0 + c_1 R_{t-1} + \sigma z_t$, $E[R_t] = \frac{c_0}{1-c_1}$, for convenience: $\mu = \frac{c_0}{1-c_1}$, $\lambda = -c_1$. Then: $R_t - \mu = -\lambda(R_t - \mu) + \sigma z_t$, $|\lambda| < 1$. $Var[R_t] = \gamma_0 = \frac{\sigma^2}{1-\lambda^2}$.
  Lag-k autocovariance coefficient: $\gamma_k = (-\lambda)^k \gamma_0 = \frac{(-\lambda)^k}{1-\lambda^2}\sigma^2$

Week 3: Time Series Models

Gambler's Ruin

- Repeated set of gambles with probability of success p and of failure $q = 1 - p$
- initial capital is $x > 0$, total capital a. Stop when winning a or **ruin** as $x = 0$.
- Q_x is **probability of ruin** starting from capital x: $Q_x = pQ_{x+1} + pQ_{x-1}$.
- $Q_x = \frac{(q/p)^a - (q/p)^x}{(q/p)^a - 1}$, if $p = q = 1/2$, then: $Q_x = 1 - \frac{x}{a}$

Week 4: Continuous-Time Finance

Itô processes and Itô's lemma

Itô's lemma

- Itô process: $\mathrm{d}X = a\mathrm{d}t + b\mathrm{d}B$
- $\mathrm{d}F = \left(\frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial X}\mathrm{d}X + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(\mathrm{d}X)^2 + \cdots\right) = \left(\frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial X}\mathrm{d}X + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}[b^2\mathrm{d}t]\right) = \left[\frac{\partial F}{\partial t}\mathrm{d}t + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}\right]\mathrm{d}t + \left[\frac{\partial F}{\partial X}\right][a\mathrm{d}t + b\mathrm{d}B]$
  $\mathrm{d}F = \left[\frac{\partial F}{\partial t} + a\frac{\partial F}{\partial X} + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}\right]\mathrm{d}t + b\frac{\partial F}{\partial X}\mathrm{d}B$
- Heuristics: expand and replace $(\mathrm{d}B_t)^2 \to \mathrm{d}t$, $(\mathrm{d}X_t)^2 \to b^2\mathrm{d}t$

Itô processes

- **Brownian motion with drift**: $\mathrm{d}S_t = \mu\mathrm{d}t + b\mathrm{d}B_t$
  $S_T = S_0 + \mu T + \sigma(B_T - B_0)$
- **GBM with drift**: $\mathrm{d}S_t = \mu S_t\mathrm{d}t + bS_t\mathrm{d}B_t$
  $\mathrm{d}(logS_t) = (\mu - \frac{\sigma^2}{2})\mathrm{d}t + \sigma\mathrm{d}B_t$, $S_T = S_0 e^{(\mu-\frac{\sigma^2}{2})T + \sigma(B_T - B_0)}$
- **Ornstein–Uhlenbeck process**: $\mathrm{d}S_t = \lambda(\bar{S} - S_t)\mathrm{d}t + \sigma\mathrm{d}B_t$
- **Cox–Ingersoll–Ross process**: $\mathrm{d}\rho_t = \lambda(\bar{\rho} - \rho_t)\mathrm{d}t + \sigma\sqrt{\rho_t}\mathrm{d}B_t$
  Let $F = \sqrt{\rho}$, $\frac{\partial F}{\partial \rho} = \frac{1}{2\sqrt{\rho}}$, $\frac{\partial^2 F}{\partial \rho^2} = -\frac{1}{4}\rho^{-\frac{3}{2}}$
  $\mathrm{d}F = \left(\frac{4\lambda\bar{\rho} - \sigma^2}{8F} - \frac{1}{2}\lambda F\right)\mathrm{d}t + \frac{1}{2}\sigma\mathrm{d}B_t$

From SDE to PDE: The Black-Scholes equation

Black-Scholes equation

- Itô process: $\mathrm{d}X = a\mathrm{d}t + b\mathrm{d}B$
- $\mathrm{d}F = \left(\frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial X}\mathrm{d}X + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(\mathrm{d}X)^2 + \cdots\right) = \left(\frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial X}\mathrm{d}X + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}[b^2\mathrm{d}t]\right) = \left[\frac{\partial F}{\partial t}\mathrm{d}t + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}\right]\mathrm{d}t + \left[\frac{\partial F}{\partial X}\right][a\mathrm{d}t + b\mathrm{d}B]$
  $\mathrm{d}F = \left[\frac{\partial F}{\partial t} + a\frac{\partial F}{\partial X} + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}\right]\mathrm{d}t + b\frac{\partial F}{\partial X}\mathrm{d}B$
- Heuristics: expand and replace $(\mathrm{d}B_t)^2 \to \mathrm{d}t$, $(\mathrm{d}X_t)^2 \to b^2\mathrm{d}t$

Week 5: Itô Calculus

Black-Scholes equation

Summary of some key formulas

- Itô process: $\mathrm{d}X = a\mathrm{d}t + b\mathrm{d}B$
- Itô formula:

$$\begin{aligned}\mathrm{d}F &= \frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial^2 F}{\partial X^2}\mathrm{d}X + \frac{b^2}{2}\frac{\partial F}{\partial X}\mathrm{d}t \\ &= \left(\frac{\partial F}{\partial t} + a\frac{\partial F}{\partial X} + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}\right)\mathrm{d}t + b\frac{\partial F}{\partial X}\mathrm{d}B\end{aligned}$$

- Stock price: $\mathrm{d}S = \mu S\mathrm{d}t + \sigma\mathrm{d}B \implies \mathrm{d}(\log S) = \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma\mathrm{d}B$
- Black-Scholes: $\Delta = \partial V/\partial S$, $\mathrm{d}\pi = r\pi\mathrm{d}t$,

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Recitation 5

Expectations from Brownian integrals

- $dB \sim N(0, dt)$
- $\int_0^t dB = B_t - B_0 \sim N(0,t)$
- $E[f(B_t - B_0)] = E[f(\sqrt{t}z)] = \frac{1}{\sqrt{2\pi}}\int e^{-z^2/2} f(\sqrt{t}z)dz$
- Example $E[f(B_t - B_0)] = E[(B_t - B_0)^4] = E[(\sqrt{t}z)^4] = t^2E[z^4] = 3t^2$
  We can pull out $\sqrt{t}$,which is nonstochastic to get $t^2E[z^4]$, $E[z^4]$ is a well-known Gaussian integral that we use in the kurtosis=3.
- Useful formula: $E[e^{\alpha z + \beta}] = e^{\alpha^2/2 + \beta}$

Solutions to the diffusion equation

- **diffusion equation**: $\frac{\partial p}{\partial t} - \frac{1}{2}\frac{\partial^2 p}{\partial z^2} = 0$
- special solution $p(z,0) = f(z)$ then the general solution is:
  $p(z,t) = \int p_0(z-w,t)f(w)dw$, where $p_0 = \frac{1}{\sqrt{2\pi t}}e^{-\frac{z^2}{2t}}$
- let $u = \frac{w-z}{\sqrt{t}} \implies du = \frac{dw}{\sqrt{t}} \implies w = u\sqrt{t} + z$. Now general solution can be computed as expectation of standard Gaussian: $p(z,t) = \mathbb{E}[f(\sqrt{t}u + z)]$, where $u \sim N(0,1)$
- **example**: $p(z,0) = z^2, f(z) = z^2$, find $p(z,t)$
  $p(z,t) = \int p_0(z-w,t)f(w)dw = \int \frac{1}{\sqrt{2\pi t}}e^{-\frac{(z-w)^2}{2t}} w^2 dw =$
  $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}(u\sqrt{t} + z)^2 du = \mathbb{E}[f(u\sqrt{t} + z)] =$
  $\mathbb{E}[u^2 t + 2u\sqrt{t}z + z^2] = z^2 + t$

Week 6: Continuous-Time Finance

Itô processes in higher dimensions

Itô's lemma: multiple stochastic variables

- $\mathrm{d}X_i = a_i(t, X_1, X_2, \ldots)\mathrm{d}t + b_i(t, X_1, X_2, \ldots)\mathrm{d}B_i$
  $\mathrm{d}F = \frac{\partial F}{\partial t}\mathrm{d}t + \sum \frac{\partial F}{\partial X_i}\mathrm{d}X_i + \frac{1}{2}\sum \rho_{ij} b_i b_j \frac{\partial^2 F}{\partial X_i X_j}\mathrm{d}t$
- Heuristics "rule of thum" for correlated Brownian motions :
  $(\mathrm{d}B_i)^2 \to \mathrm{d}t$, $(\mathrm{d}B_i)(\mathrm{d}B_j) \to \rho_{ij}\mathrm{d}t$,
  $(\mathrm{d}X_i)^2 \to b_i^2\mathrm{d}t$, $(\mathrm{d}X_i)(\mathrm{d}X_j) \to \rho_{ij} b_i b_j \mathrm{d}t$

- two stochastic variables case: $\mathrm{d}X_1 = a_1\mathrm{d}t + b_1\mathrm{d}B_1, \mathrm{d}X_2 = a_2\mathrm{d}t + b_2\mathrm{d}B_2$ $\mathrm{d}F =$ $\frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial X_1}\mathrm{d}X_1 + \frac{\partial F}{\partial X_2}\mathrm{d}X_2 + \left(\frac{b_1^2}{2}\frac{\partial^2 F}{\partial X_1^2} + \frac{b_2^2}{2}\frac{\partial^2 F}{\partial X_2^2} + b_1 b_2 \rho \frac{\partial^2 F}{\partial X_1 \partial X_2}\right)\mathrm{d}t$
- With two random variables, Ito's formula for $F(t,X,Y)$ is: $dF =$ $\frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}dX + \frac{\partial F}{\partial Y}dY + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(dX)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial Y^2}(dY)^2 + \frac{\partial^2 F}{\partial X \partial Y}(dX)(dY)$
- example: $F = X1X2 \implies \mathrm{d}F = X1\mathrm{d}X_2 + X_2\mathrm{d}X_1 + \rho b_1 b_2 \mathrm{d}t$ since $(\mathrm{d}X_i)(\mathrm{d}X_j) \to \rho_{ij} b_i b_j \mathrm{d}t \implies \mathrm{d}F = X1\mathrm{d}X_2 + X_2\mathrm{d}X_1 + \mathrm{d}X_1\mathrm{d}X_2$

Measures, martingales, and Monte Carlo

Risk-neutral pricing

- Under measure Q, expected return on risky assets equals risk-free rate: $\mathbb{E}_t^Q = \left[\frac{\mathrm{d}S_t}{S_t}\right] = r\mathrm{d}t$
- Heuristic: replace drift with risk-free rate to get risk-neutral process: $\mu \to r$
- All (no-arbitrage) traded assets have discounted price process that are martingales: $e^{-rt}X_t = \mathbb{E}_t^Q[e^{-rT}X_T]$. Example, call option: $C_t = e^{r(T-t)}\mathbb{E}_t^Q[\max(S_T - K, 0)]$
- Monte Carlo implementation: generate ensemble of equiprobable price paths using **risk-neutral** drift and volatility parameters, compute terminal payoffs, and take average of their discounted present value.
- **example**: Let S be the value of a stock that evolves according to GBM. A "cubic contract" has a payoff at expiration of $V_T = (S_T)^3$. What is the value of the contract at V_0 at $t = 0: V_t = e^{-r(T-t)}\mathbb{E}^Q[V_T]$ setting $t = 0$. Then:

$$V_T = S_T^3 = \left[S_0 e^{(r-\sigma^2/2)T + \sigma\sqrt{T}Z}\right]^3$$
$$V_T = \left[S_0^3 e^{3(r-\sigma^2/2)T}\right] e^{3\sigma\sqrt{T}Z}, \quad Z \sim \mathcal{N}(0,1)$$
$$V_t = e^{-r(T-t)}\mathbb{E}^Q[V_T] = \left[S_0^3 e^{-rT} e^{3(r-\sigma^2/2)T}\right]\mathbb{E}^Q\left[e^{3\sigma\sqrt{T}Z}\right]$$
$$V_t = \left[S_0^3 e^{2rT} e^{-3\sigma^2 T/2}\right] e^{9\sigma^2 T/2}\; S_0^3 e^{2rT + 3\sigma^2 T}.$$

Bond pricing

- **one-factor Vasicek model**: The short rate evolves according to the Ornstein-Uhlenbeck process $\mathrm{d}y_t = \alpha(\bar{y} - y_t)\mathrm{d}t + \sigma\mathrm{d}B$,. The bond-pricing equation takes the form $\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial y^2} - yV + \alpha(\bar{y} - y)\frac{\partial V}{\partial y} = 0$.. At the boundary condition $V = 1$ and $t = T$ since the zero-coupon bonds returns \$1 at maturity. Find $V(t,y)$ as a funciton of t, y, and the maturity date T, and the constant parameters α, σ and $\bar{y}$. Ansatz $V = e^{f(t) - yg(t)}$
- Then rearranging : $\left(\alpha g - 1 - \frac{\mathrm{d}g}{\mathrm{d}t}\right)\cdot y + \left(\frac{\mathrm{d}f}{\mathrm{d}t} + \frac{1}{2}\cdot\sigma^2 g^2 - \alpha\bar{y}g\right)$.
- $0 = \alpha g - 1 - \frac{\mathrm{d}g}{\mathrm{d}t}$ and $0 = \frac{\mathrm{d}f}{\mathrm{d}t} + \frac{1}{2}\cdot\sigma^2 g^2 - \alpha\bar{y}g$ which is equivalent to: $\frac{\mathrm{d}g}{\mathrm{d}t} = \alpha g - 1$ and $\frac{\mathrm{d}f}{\mathrm{d}t} = \alpha\bar{y}g - \frac{1}{2}\cdot\sigma^2 g^2$.
- We impose the boundary conditions $f(T) = g(T) = 0$: $f(t) = \bar{y}\left[-(T-t) + \frac{1}{\alpha}\left(1 - e^{-\alpha(T-t)}\right)\right] +$ $\frac{\sigma^2}{4\alpha^3}\left[2\alpha(T-t) - 4\left(1 - e^{-\alpha(T-t)}\right) + \left(1 - e^{-2\alpha(T-t)}\right)\right]$

Week 7: Linear algebra of asset pricing

Linear algebra of asset pricing

- **The payoff matrix** A is a matrix $s \times n$ with s states of the world and n different securities.
- A **portfolio** $\vec{\mathbf{x}}$ is represented as a vector of quantities held for each security.
- The **portfolio payoff** is determined by action on the portfolio vector with the payoff matrix: $A\vec{\mathbf{x}}$
- A **complete market** is one in which every payoff can be generated by some portfolio, $\mathrm{rank}(A) = s$. If $n > s$, select a complete basis and drop $n - s$ redundant securities
- An **incomplete market** s one in which some payoffs cannot be generated by any portfolio. $\mathrm{rank}()A) < s$.
- Present **prices** can be represented by a vector $\vec{\mathbf{S}}$ dimension n.
- Market value of a portfolio is sum of (price) x (quantity):

$$MV = \sum_{i=1}^n S_i x_i = (S_1 S_2 \cdots)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \vec{\mathbf{S}}^\top \vec{\mathbf{x}}$$

- **Type I arbitrage**: Pay nothing now (i.e., negative or zero cost), $\vec{\mathbf{S}}^\top\vec{\mathbf{x}} \le 0$. Receive only non-negative payoffs later: $A\vec{\mathbf{x}} \geq= 0$ and at least one payoff > 0.
- **Type II arbitrage** : There are redundant assets and non-trivial solutions to $A\vec{\mathbf{x}} = 0$. There is arbitrage if the portfolio has a non-zero price, $\vec{\mathbf{S}}^\top\vec{\mathbf{x}} \neq 0$
- Portfolios with the special property that they have **unit** payoff in a single state are called Arrow-Debreu (AD) securities.
- In terms of a general payoff matrix, an elementary AD security can be replicated if there is a portfolio satisfying $A\vec{\mathbf{x}} = \vec{\mathbf{e_j}} = \begin{pmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \end{pmatrix}^\top$. $\vec{\mathbf{x}} = A^{-1}\vec{\mathbf{e_j}}$ is a portfolio with unit payoff. The payoff matrix of a set of AD securities in the complete market is the identity matrix I.
- The prices of AD securities are called **state prices**: $\psi = (\psi_1 \cdots \psi_s)$
- The price of any security with payoff $\vec{\mathbf{b}}$ is $\vec{\psi}^\top\vec{\mathbf{b}}$. Intuition: we can think of $\vec{\mathbf{b}}$ as a portfolio of AD securities.
- **Arbitrage theorem** Consider a market with n securities, s states of the world, payoff matrix $A \in \mathbb{R}^{s\times n}$, $A : \mathbb{R}^n \to \mathbb{R}^s$. There is no arbitrage if and only if there exists a **strictly positive** state-price vector $\vec{\psi}$ **consistent with** the security-price vector, $\vec{\psi} \in \mathbb{R}^s$, $\vec{\mathbf{S}} \in \mathbb{R}^n$, $\vec{\mathbf{S}} = A^\top\vec{\psi}$. For **incomplete markets**, there can be (infinitely) multiple solutions $\vec{\psi}$. If there is **at least one solution** where $\vec{\psi} > 0$, then no arbitrage. If **every solution** has $\vec{\psi} \le 0$, then there is arbitrage.
- If A is not invertible ($s < n$), $A^\top$ has **pseudo-inverse** M: $M \equiv (AA^\top)^{-1}A$, check that $MA^\top = I, I \in \mathbb{R}^s$
- **Arbitrage pricing theorem** Given payoffs A, prices $\vec{\mathbf{S}}$, target asset with payoff $\vec{\mathbf{b}}$: Find all $\vec{\psi} > 0$ such that $A^\top\vec{\psi} = \vec{\mathbf{S}}$. No solution $\implies$ arbitrage. 1 solution $\implies$ complete market. Multiple solution $\implies$ incomplete market.

Week 8: Optimization

Portfolio Optimization

Portfolio risk : $\sigma_p^2 = \vec{\mathbf{w}}^\top C\vec{\mathbf{w}} = \sum w_i^2\sigma_i^2 + 2\sum_{i<j} w_i w_j \sigma_i \sigma_j \rho_{ij}$
$\vec{\mathbf{w}} = (w_1 \quad w_2 \quad \ldots \quad w_n)^\top$

Portfolio optimization with budget

- $\mathcal{L}(\vec{\mathbf{w}}, \ell) = \frac{1}{2}\vec{\mathbf{w}}^\top C\vec{\mathbf{w}} + \ell(1 - \vec{\iota}^\top\vec{\mathbf{w}}^\top)$
- Vary the weights: $\frac{\partial\mathcal{L}}{\partial w_i} = \left(\sum_{j\in[n]} C_{ij}w_j\right) - \ell\iota_i = 0$
- Solve for the weights by inverting matrix: $C\vec{\mathbf{w}} - \ell\vec{\iota} = 0 \implies \vec{\mathbf{w}} = \ell C^{-1}\vec{\iota}$
- Solve Lagrande multiplier : $\vec{\iota}^\top\vec{\mathbf{w}} = 1 = \ell(\vec{\iota}^\top C^{-1}\vec{\iota}) \implies \ell = \frac{1}{\vec{\iota}^\top C^{-1}\vec{\iota}}$
- Solution: $\vec{\mathbf{w}}_{min} = \ell C^{-1}\vec{\iota} = \frac{C^{-1}\vec{\iota}}{\vec{\iota}^\top C^{-1}\vec{\iota}}$, $\quad \sigma_{min}^2 = \ell = \frac{1}{\vec{\iota}^\top C^{-1}\vec{\iota}}$

Portfolio optimization with budget and return constraint generalized to $\sum_i w_i = w_p$

- $\mathcal{L}(\vec{\mathbf{w}}, \ell, m) = \frac{1}{2}\vec{\mathbf{w}}^\top C\vec{\mathbf{w}} + \ell(w_p - \vec{\iota}^\top\vec{\mathbf{w}}) + m(\mu_p - \vec{\mu}^\top\vec{\mathbf{w}})$
- Vary the weights: $\frac{\partial\mathcal{L}}{\partial w_i} = \left(\sum_{j\in[n]} C_{ij}w_j\right) - \ell\iota_i - m\mu_i$
- Solve for the weights: $C\vec{\mathbf{w}} - \ell\vec{\iota} - m\vec{\mu} = \mathbf{0} \implies \vec{\mathbf{w}} = C^{-1}(l\vec{\iota} + m\vec{\mu})$
- Solve for Langrange multipliers with constraints: $\vec{\iota}^\top\vec{\mathbf{w}} = w_p$ and $\vec{\mu}^\top\vec{\mathbf{w}} = \mu_p$ $w_p = \vec{\iota}^\top\vec{\mathbf{w}} = \ell(\vec{\iota}^\top C^{-1}\vec{\iota}) + m(\vec{\mu}^\top C^{-1}\vec{\iota})$ $\mu_p = \vec{\mu}^\top\vec{\mathbf{w}} = \ell(\vec{\mu}^\top C^{-1}\vec{\iota}) + m(\vec{\mu}^\top C^{-1}\vec{\mu})$ as as matrix equation: $\begin{pmatrix} w_p \\ \mu_p \end{pmatrix} = M\begin{pmatrix} \ell \\ m \end{pmatrix}$, where : $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ $a \equiv \vec{\iota}^\top C^{-1}\vec{\iota}, b \equiv \vec{\mu}^\top C^{-1}\vec{\iota}, c \equiv \vec{\mu}^\top C^{-1}\vec{\mu}$.
- Solve for Langrange multiplier by inverting M: $\begin{pmatrix} \ell \\ m \end{pmatrix} = M^{-1}\begin{pmatrix} w_p \\ \mu_p \end{pmatrix}$ $M^{-1} = \frac{1}{ac-b^2}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$, $\ell = \frac{cw_p - b\mu_p}{ac-b^2}$ and $m = \frac{-bw_p + a\mu_p}{ac-b^2}$.
- Solution : $\sigma_p^2 = \begin{pmatrix} \ell & m \end{pmatrix} M \begin{pmatrix} \ell \\ m \end{pmatrix} = \begin{pmatrix} \frac{cw_p - b\mu_p}{ac-b^2} & \frac{-bw_p + a\mu_p}{ac-b^2} \end{pmatrix}\begin{pmatrix} w_p \\ \mu_p \end{pmatrix}$ $= \frac{1}{ac-b^2}\cdot(a\mu_p^2 - 2bw_p\mu_p + cw_p^2)$.

Useful formulas

$\mathrm{Cov}(aX + bY, Z) = a\,\mathrm{Cov}(X,Z) + b\,\mathrm{Cov}(Y,Z)$
$\mathrm{Var}(aX + bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) + 2ab\;\mathrm{Cov}(X,Y)$
$\mathrm{Var}(aX - bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) - 2ab\;\mathrm{Cov}(X,Y)$
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Recommended Resources

- MITx 15.455x MITx 15.455x Mathematical Methods for Quantitative Finance [Lecture Slides] (`https://learning.edx.org/course/course-v1:MITx+15.455x+3T2020/home`)
- Tsay, Analysis of Financial Time Series (3e), Wiley. (Tsay)
- Capinski and Zastawniak, Mathematics for Finance, Springer. (CZ)
- Olver, Introduction to Partial Differential Equations (2016), Springer. (Olver)
- Campbell, Lo, and MacKinlay, Econometrics of Financial Markets (1997), Princeton. (CLM)
- Lang, Introduction to Linear Algebra (2e), Springer (Lang)
- Axler, Linear Algebra Done Right (3e), Springer (Axler)
- LaTeX File (`github.com/j053g/cheatsheets/15.455x`)

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