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Foundations of Modern Finance II

11 Forward and Futures

Forward Interest rates

The forward interest rate between time $t-1$ and t satisfies:
$(1+r_t)^t = (1+r_{t-1})^{t-1}(1+f_t)$
$f_t = \frac{B_{t-1}}{B_t} - 1 = \frac{(1+r_t)^t}{(1+r_{t-1})^{t-1}} - 1$

Forward price

$F_T = e^{(r-y)T} S_0$

Swaps

The swap rate is a weighted average of forward rates:
$r_s = \frac{\sum_{t=1}^{T} B_t f_t}{\sum_{u=1}^{T} B_u} = \sum_{t=1}^{T} w_t f_t$, with the weights $w_t = \frac{B_t}{\sum_{u=1}^{T} B_u}$

12 Options, Part 1

Net option payoff

The break-even point is given by S_T at which Net Payoff is zero:
$Netpayoff = max[S_T - K, 0] - C(1+r)^T$

Option strategies

- Protective Put: Buy stock + Buy a put
- Bull Call Spread: Buy a Call K1 + Write a Call K2, $K1 < K2$
- Straddle: Buy a Call at K + Buy a Put at K

Corporate securities as options

- Equity (E): A call option on the firm's assets (A) with the exercise price equal to its bond's redemption value.
- Debt (D): A portfolio combining the firm's assets (A) and a short position in the call with the exercise price equal to its bond's face value (F):

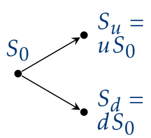
$A = D + E \implies D = A - E$
$E \equiv max(0, A-F)$
$D = A - E = A - max[0, A-F]$

Put-Call parity for European options

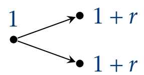
$C + B \cdot K = P + S$

Binomial option pricing model

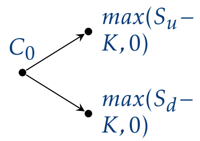
Stock



Bond

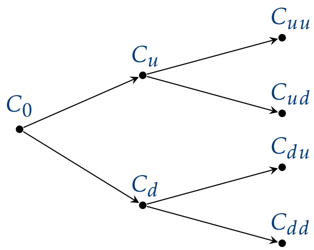


Option at Expiration



Replicating portfolio (call option)
$\begin{pmatrix} S_u & (1+r) \\ S_d & (1+r) \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} max(K-S_u, 0) \\ max(K-S_d, 0) \end{pmatrix}$
solve system to form a portfolio of stock and bond that replicates the call's payoff: a shares of the stock; b dollars in the riskless bond

Binomial option pricing model with multiple periods



Compute the time-0 value working backwards: first C_u and C_d then C_0. In summary:

- Replication strategy gives payoffs identical to those of the call.
- Initial cost of the replication strategy must equal the call price

13 Options, Part 2

Binomial model: risk-neutral pricing

Solve by replication δ shares of the stock, b dollars of riskless bond
$\delta u S_0 + b(1+r) = C_u$
$\delta d S_0 + b(1+r) = C_d$
Solution: $\delta = \frac{C_u - C_d}{(u-d)S_0}$, $b = \frac{1}{1+r}\frac{uC_d - dC)u}{u-d}$
Then: $C_0 = \delta S_0 + b$
$C_0 = \frac{C_u - C_d}{(u-d)}\frac{1}{1+r}\frac{uC_d - dC)u}{u-d}$

Risk neutral probability

$q_u = \frac{(1+r)-d}{u-d}$, $q_d = \frac{u-(1+r)}{u-d}$ Then
$C_0 = \frac{q_u C_u + q_d C_d}{1+r} = \frac{E^Q[C_T]}{1+r}$ where $E^Q[\cdot]$ is the expectation under probability $Q = (1, 1-q)$, which is called the risk-neutral probability

State prices and risk-neutral probabilities

$\Phi_u = \frac{q}{1+r}$, $\Phi_d = \frac{1-q}{1+r}$
$\Phi_{uu} = \frac{q^2}{(1+r)^2}$, $\Phi_{ud} = \Phi_{du} = \frac{q(1-q)}{(1+r)^2}$,
$\Phi_{dd} = \frac{(1-q)^2}{(1+r)^2}$ With state prices, can price any state-contingent payoff as a portfolio of state-contingent claims: mathematically equivalent to the risk-neutral valuation formula.

Implementing binominal model

- As we reduce the length of the time step, holding the maturity fixed, the binomial distribution of log returns converges to Normal distribution.
- Key model parameters u, and d need to be chosen to reflect the distribution of the stock return

Once choice is: $u = exp(\sigma \frac{T}{n})$, $d = \frac{1}{u}$, $p = \frac{1}{2} + \frac{1}{2}\frac{u}{\sigma}\sqrt{\frac{T}{n}}$

Black-Scholes-Merton formula

$C_0 = S_0 N(x) - Ke^{-rT} N(x - \sigma\sqrt{(T)})$
$x = \frac{ln(\frac{S_0}{Ke^{-rT}})}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$

The call is equivalent to a levered long position in the stock; $S_0 N(x)$ is the amount invested in the stock; $Ke^{-rT}N(x - \sigma\sqrt{(T)})$ is the dollar amount borrowed.
Equivalent formulation: $C(S_t, t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)}$
$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$
$d_2 = d_1 - \sigma\sqrt{T-t}$
The price of a corresponding put option based on put-call parity is: $P(S_t, t) = Ke^{-r(T-t)} - S_t + C(S_t, t) = N(-d_2)Ke^{-r(T-t)} - N(-d_1)S_t$

Option Greeks

Delta: $\delta = \frac{\partial C}{\partial S}$ Omega: $\Omega = \frac{\partial C}{\partial S}\frac{S}{C}$ Gamma: $\Gamma = \frac{\partial \delta}{\partial S} = \frac{\partial^2 C}{\partial S^2}$ Theta: $\Theta = \frac{\partial C}{\partial S}$ Vega: $\mathcal{V} = \frac{\partial C}{\partial S}$

14 Portfolio Theory

Preliminaries

Portfolio return and variance, two assets
$\bar{r}_p = w_1\bar{r}_1 + w_2\bar{r}_2$
$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}$
$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2$
$\rho_{12} = \frac{\sigma_{12}}{\sigma_1\sigma_2}$

Quadratic formula is handy when solving for weights in the two asset case: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

In general, for a vector of weights $\boldsymbol{w}$, returns $\boldsymbol{r}$ and covariance matrix Σ: $r_p = \boldsymbol{w} \cdot \boldsymbol{r}$
and $\sigma_p^2 = \boldsymbol{w}' \cdot \Sigma \cdot \boldsymbol{w}$ in Excel σ_p^2=MMULT(MMULT(TRANSPOSE($\boldsymbol{w}$),Σ),$\boldsymbol{w}$)

Sharpe Ratio

$SR = \frac{r - r_f}{\sigma}$, Return-to-Risk Ratio $RRR_{i,p} = \frac{r_i - r_f}{Cov(r_i, r_p)/\sigma_p}$

Portfolio and individual assets

In the presence of a risk-free asset, portfolio's return is: $\bar{r}_p = r_F + \sum_{i=1}^{N} w_i(\bar{r}_i - r_F)$

Individual contribution to expected return $\frac{\partial \bar{r}_p}{\partial w_i} = \bar{r}_i - r_F$

Individual contribution to volatility $\frac{\partial \sigma_p}{\partial w_i} = \frac{Cov(\bar{r}_i, \bar{r}_p)}{\sigma_p}$

Tangency Portfolio

Note $\bar{\boldsymbol{x}}$ is vector of excess returns and $\vec{\mathbf{1}}$ is a vector of ones of size N, number of stocks.
min $\boldsymbol{w}' \cdot \Sigma \cdot \boldsymbol{w}$ s.t. $\boldsymbol{w}' \cdot \bar{\boldsymbol{x}} = m$ solution: $\boldsymbol{w}_T = \lambda\Sigma^{-1}\bar{\boldsymbol{x}}$ where $\lambda = \frac{1}{\bar{\boldsymbol{x}}'\Sigma^{-1}\vec{\mathbf{1}}}$.
In summary, the tangency weights are: $\boldsymbol{w}_T = \frac{1}{\bar{\boldsymbol{x}}'\Sigma^{-1}\vec{\mathbf{1}}}\Sigma^{-1}\bar{\boldsymbol{x}}$. In Excel λ =MMULT(MMULT(TRANSPOSE($\bar{\boldsymbol{x}}$),MINVERSE(Σ)),$\vec{\mathbf{1}}$), lambda is a scalar, then $\boldsymbol{w}_T = \lambda$ MMULT(MINVERSE(Σ),$\bar{\boldsymbol{x}}$).
Once the tangency portfolio is found, RRR is the same for all stocks, meaning that we cannot perturb the weights of individual assets in this portfolio to further increase its risk return trade off.

R14Q4

$r_i = b_1F_1 + b_{2,i}F_2 + \epsilon_i$
inputs: $b_1 = 10$, $b_{2,i} = i$, F_1 has $E[F_1] = 0\%$ and $\sigma_{F_1} = 1\%$, F_2 has $E[F_2] = 1\%$ and $\sigma_{F_2} = 1\%$, and ϵ_i has $E[\epsilon_i] = 0\%$ and $\sigma_{\epsilon_i} = 30\%$. F_1, F_2, ϵ_i are indep. of each other, and $r_f = 0.75\%$.
find sharpe ratio and return-to-risk ratio RRR
$E[r_i] = b_1E[F_1] + b_2, iE[F_2] + E[\epsilon_i] = i \cdot 1\%$

$V[r_i] = V[b_1F_1 + b_{2,i}F2 + \epsilon_i] = b_1^2V[F_1] + b_{2,i}^2V[F_2] + V[\epsilon_i] = 10^2 \times 0.01^2 + i^2 \times 0.01^2 + 0.3^2$
$Cov(r_i, r_j) = Cov(b_1F_1 + b_{2,i}F + 2 + \epsilon_i, b_1F_1 + b_{2,j}F_2 + \epsilon_j) = V[b_1F_1] + Cov(b_{2,i}F_2, b_{2,j}F_2) = b_1^2V[F_1] + b_{2,i}b_{2,j}V[F_2] = 10^2 \times 0.01^2 + i \times j \times 0.01^2$.
With the variances (note variances have an extra-term) and covariances now we can form Σ and compute sharpe ratio.
$RRR_{i,p} = \frac{r_i - r_f}{Cov(r_i, r_p)/\sigma_p}$ note $Cov(r_i, r_p) = Cov(r_i, \sum_{j=1}^{I} w_jr_j) = \sum_{j=I}^{I} w_jCov(r_i, r_j)$

15 CAPM

For the market porfolio to be optional the RRR of all risky assets must be the same

$RRR_i = \frac{\bar{r}_i - r_F}{\sigma_{iM}/\sigma_M} = SR_M = \frac{\bar{r}_M - r_F}{\sigma_M}$

then

$\bar{r}_i - r_F = \frac{\sigma_{iM}}{\sigma_M^2}(\bar{r}_M - r_F) = \beta_{iM}(\bar{r}_M - r_F)$

β_{iM} is a measure of asset it's systematic risk: exposure to the market. $\bar{r}_M - r_F$ gives the premium per unit of systematic risk.

Risk and return in CAPM

We can decompose an asset's return into three pieces:

$\tilde{r}_i - r_F = \alpha_i\beta_{iM}(\tilde{r}_M - r_F) + \tilde{\epsilon}_i$

$E[\tilde{\epsilon}_i] = 0$, $Cov[\tilde{r}_M - r_F), \tilde{\epsilon}_i] = 0$

Three characteristics of an asset: Alpha, according to CAPM, alpha should be zero for all assets. Beta: measures an asset's systematic risk. SD[$\tilde{\epsilon}_i$] measures non-systematic risk.

Leverage: equity beta vs asset betas

The assets of the firm serve to pay all investors, and so: $A = E + D$

$\beta_A = \frac{E}{E+D}\beta_E + \frac{D}{E+D}\beta_D$

R15Q1

inputs: $E[r_M] = 14\%$, $E[r_P] = 16\%$, $r_f = 6\%$, $\sigma_M = 25\%$

$E[r_P] = r_F + \beta_P(E[r_M] - r_f) = 16\% = 6\% + \beta_P(14\% - 6\%) = 1.25$

to make a portfolio to be located in the capital line, one must use the risk-free asset with weight w: $E[r_P] = wr_f + (1-w)E[r_M] = 6\%w + 14\%(1-w) \implies w = \frac{E[r_P] - r_M}{r_F - r_M} = -0.25$

$VaR[r_P] = Var[wr_f + (1-w)E[r_M]] = (1-w)^2Var[r_M] \implies \sigma_P = 31.25\%$

to find correlation use: $\beta_P = \frac{Cov(r_P, r_M)}{Var(r_M)} = \frac{\rho_{P,M}\sigma_M\sigma_P}{\sigma_M^2} \implies \rho_{P,M} = \frac{\beta_P\sigma_M}{\sigma_P} = 1$