

# Foundations of Modern Finance I

Cheat sheet for MITx: 15.415.1x Foundations of Modern Finance I.

## Week 2: Market Prices and Present Value

### State-space model for time and risk

#### State-space model

- Assets can be traded at time  $t = 0$  with payoffs at time  $t = 1$ .
- The price of an asset is  $P$  at  $t = 0$  with payoff  $X = (X_1, \dots, X_N)$  at  $t = 1$ .
- $X$  is a random variable.
- A random payoff is given by the value of its payoff in each state and the corresponding probability:  $[(X_1, \dots, X_N); (p_1, \dots, p_N)]$

#### State Prices

- Consider primitive state-contingent claims (Arrow-Debreu securities) that pay \$1 in a single state and nothing otherwise.
- Denote the price of the A-D claim on state  $j$  by  $\phi_j$ , the state price for state  $j$ .
- No arbitrage requires that all state prices must be positive:  $\phi_j > 0$  for all  $j$ .
- The market is called complete if one can effectively trade A-D securities on each state.

### Arbitrage pricing

#### Arbitrage pricing

- With the prices of A-D securities, we can price other assets/securities.
- Law of One Price: Two assets with the same payoff must have the same market price.
- Suppose the firm is considering a project yielding time-1 cash flow:  $X = (X_1, X_2, \dots, X_N)$
- Using prices of A-D securities, we can attach value to this cash flow as:  $P = \phi_1 X_1 + \dots + \phi_N X_N = PV$
- Example for three assets: riskless bonds pays \$100 in each state currently traded at  $B_{10}$ , stock 1 pays off  $[S_{11}, S_{12}, S_{13}]$  and currently traded at  $S_{10}$ , stock 2 pays off  $[S_{21}, S_{22}, S_{23}]$  and currently traded at  $S_{20}$ :

$$\begin{pmatrix} 100 & 100 & 100 \\ S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \end{pmatrix} \cdot \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} B_{10} \\ S_{10} \\ S_{20} \end{pmatrix}$$

solve system to find state prices  $\phi_1, \phi_2, \phi_3$ .

### Present value and future value

#### Present value and discount rate

- $PV = \frac{E[CF]}{1+r}$

#### Future Value

- $FV$  in  $T$  years:  $FV = (1+r)^T$

### Nominal vs. real cash flows and returns

#### Nominal vs real CFs

- Nominal cash flows  $\implies$  expressed in actual-dollar cash flows.
- Real cash flows  $\implies$  expressed in constant purchasing power.
- At an annual inflation rate of  $i$ , we have:  $(RealCF)_t = \frac{(NominalCF)_t}{(1+i)^t}$

#### Nominal vs real rates

- Nominal rates of return  $\implies$  prevailing market rates.
- Real rates of return  $\implies$  inflation adjusted rates
- Real rate of return:  $r_{real} = \frac{1+r_{nominal}}{1+i} - 1 \approx r_{nominal} - i$

## Week 3: Discounting and Compounding

### Special Cash Flows

#### Special Cash Flows

- Annuity:

$$PV = A \cdot \frac{1}{r} \left[ 1 - \frac{1}{(1+r)^T} \right]$$

$$FV = PV \cdot (1+r)^T$$

- Annuity with constant growth:

$$PV = \begin{cases} A \cdot \frac{1}{r-g} \left[ 1 - \left( \frac{1+g}{1+r} \right)^T \right] & \text{if } r \neq g \\ A \cdot \frac{T}{1+r} & \text{if } r = g \end{cases}$$

- Perpetuity:

$$PV = \frac{A}{r}$$

- Perpetuity with constant growth:

$$PV = \frac{A}{r-g}, \quad r > g$$

### Compounding

#### APR / EAR

- EAR: Effective Annual Rate

$$r_{EAR} = \left( 1 + \frac{r_{APR}}{k} \right)^k - 1$$

- APR: Annual Percentage Rate

$$r_{APR} = k \left[ (1 + r_{EAR})^{\frac{1}{k}} - 1 \right]$$

## Week 4: Fixed Income Securities


### Relative Bond Valuation

#### Arbitrage

- Arbitrage:** The key is to construct the payoffs so that we get \$100 in year 0 and we get \$0 payoffs in all of the subsequent years, i.e. the cashflows of the bonds should offset each other at all times except year 0 when we realize the arbitrage of \$100. For example, suppose there are three bonds with prices  $P_1$ ,  $P_2$  and  $P_3$  with maturities 1, 2 and 3 years and coupons  $C_1$ ,  $C_2$ ,  $C_3$ , respectively, and all with face value  $F$ . There is a fourth bond with price  $P_4$  and coupon  $C_4$  with maturity 3 years (adjust cashflows if maturity is not 3 years) undervalued or overvalued. In order to find the arbitrage strategy we need to solve the following system of equations:

$$\begin{pmatrix} -P_1 & -P_2 & -P_3 & -P_4 \\ C_1 + F & C_2 & C_3 & C_4 \\ 0 & C_2 + F & C_3 & C_4 \\ 0 & 0 & C_3 + F & C_4 + F \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In matrix notation:  $A \cdot X = B \implies X = A^{-1} \cdot B$

 X=MMULT(MINVERSE(A),B)

## Interest Risk, Bond Duration and Bond Convexity

#### Bond Duration

- Modified Duration (MD) for discount bond  $B_t = \frac{1}{(1+y)^t}$

$$MD(B_t) = -\frac{1}{B_t} \frac{dB_t}{dy} = \frac{t}{1+y}$$

- Macaulay Duration is the weighted average term to maturity

$$D = \sum_{t=1}^T \left( \frac{PV(CF_t)}{B} t \right) = \frac{1}{B} \sum_{t=1}^T \left( \frac{CF_t}{(1+y)^t} t \right)$$

- Modified Duration measures bond's interest rate risk by its relative price change with respect to a unit change in yield (with a negative sign):

$$MD = -\frac{1}{B} \frac{dB}{dy} = \frac{D}{1+y}$$

- Convexity (CX) measure the curvature of the bond price as function of the yield:

$$CX = \frac{1}{2} \frac{1}{B} \frac{d^2 B}{dy^2}$$

- Taylor series approximation of bond price changes

$$\Delta B \approx B \left( -MD \cdot \Delta y + CX \cdot (\Delta y)^2 \right)$$

## Week 5: Stocks

### Growth Opportunities and Stock Valuation

#### P/E and PVGO: price/earning and present value of growth opportunities

- P/E and PVGO:

$$P_0 = \frac{EPS_1}{r} + PVGO$$

- if  $PVGO = 0$ :

$$P/E = \frac{1}{r}$$

- if  $PVGO > 0$ :

$$P/E = \frac{1}{r} + \frac{PVGO}{EPS_2} > \frac{1}{r}$$

### Investment and Growth

#### plow-back ratio $b_t$

- Investments:

$$I_t = EPS_t \cdot b_t$$

- Next year earnings  $EPS_{t+1}$ :

$$EPS_{t+1} = EPS_t + ROI_t \cdot I_t$$

- Next year book value  $BVPS_{t+1}$ :

$$BVPS_{t+1} = BVPS_t + I_{t+1}$$

- Dividends  $D_t$ :

$$D_t = EPS_t(1 - b_t)$$

Week 6: Risk

Portfolio mean and variance, two assets

- Expected portfolio return:  $\bar{r}_p = w_1 \bar{r}_1 + w_2 \bar{r}_2$
- Unexpected portfolio return:  $\tilde{r}_p - \bar{r}_p = w_1 (\tilde{r}_1 - \bar{r}_1) + w_2 (\tilde{r}_2 - \bar{r}_2)$
- The variance of the portfolio return:  $\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \sigma_{12}$   
 $\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \rho_{12} \sigma_1 \sigma_2$
- Correlation and covariance:  $\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$

Week 7: Arbitrage Pricing Theory

Factor Models

A single-factor model

- Asset returns:  $\tilde{r}_i = \underbrace{\bar{r}_i}_{\text{expected return}} + \underbrace{b_i \tilde{f} + \tilde{\epsilon}_i}_{\text{risk}}$
- Return variance:  $\sigma_i^2 = \underbrace{b_i^2 \sigma_f^2}_{\text{systematic risk}} + \underbrace{Var(\tilde{\epsilon}_i)}_{\text{idiosyncratic risk}}$
- Return covariance:  $Cov(\tilde{r}_i, \tilde{r}_j) = Cov(b_i \tilde{f} + \tilde{\epsilon}_i, b_j \tilde{f} + \tilde{\epsilon}_j) = b_i b_j \sigma_f^2$   
because of the assumptions:  $Cov(\tilde{f}, \tilde{\epsilon}_i) = 0$   $Cov(\tilde{\epsilon}_i, \tilde{\epsilon}_j) = 0$

Multifactor models

- Asset returns:  $\tilde{r}_i = \underbrace{\bar{r}_i}_{\text{expected return}} + \underbrace{b_{i,1} \tilde{f}_1 + b_{i,2} \tilde{f}_2 + \dots + b_{i,K} \tilde{f}_K}_{\text{systematic component}} + \tilde{\epsilon}_i$
- Assumptions:  $Cov(\tilde{\epsilon}_i, \tilde{\epsilon}_j) = 0, \forall i \neq j$   
 $E[\tilde{f}_k] = 0, k = 1, 2, \dots, K$

Portfolio return

- Portfolio return:

$$\tilde{r}_p = \bar{r}_p + b_{p,1} \tilde{f}_1 + b_{p,2} \tilde{f}_2 + \dots + b_{p,K} \tilde{f}_K + \tilde{\epsilon}_p$$

where,

$$\bar{r}_p = \sum_{i=1}^N w_i \bar{r}_i \quad b_{p,k} = \sum_{i=1}^N w_i b_{i,k} \quad \tilde{\epsilon}_p = \sum_{i=1}^N w_i \tilde{\epsilon}_i$$

- Non-systematic variance:

$$Var(\tilde{\epsilon}_p) = Var\left(\sum_{i=1}^N w_i \tilde{\epsilon}_i\right) = \sum_{i=1}^N w_i^2 Var(\tilde{\epsilon}_i)$$

Expected Returns on Diversified Portfolios

- APT pricing relation:

$$\underbrace{\tilde{r}_p - \bar{r}_f}_{\text{Risk premium}} = \underbrace{\lambda}_{\text{Price of risk}} \cdot \underbrace{b_p}_{\text{Quantity of risk}}$$

$\lambda$  tells us how much compensation one earns in the market for a unit of factor risk exposure.  $\lambda$  is called the market price of risk of the factor, or the factor risk premium.

- APT relation for multi-factor models:

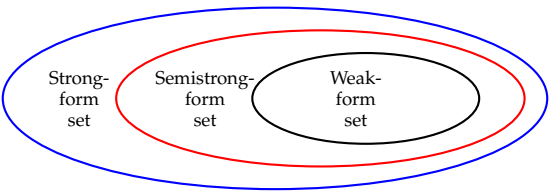
$$\tilde{r}_p - \bar{r}_f = \lambda_1 b_{p,1} + \lambda_2 b_{p,2} + \dots + \lambda_K b_{p,K}$$

Week 8: Market Efficiency

Three forms of market efficiency hypothesis MEH

- Weak-form efficiency: security prices reflect all information contained in past prices  $\implies$  Technical analysis does not provide excess returns.
- Semi-strong-form efficiency : security prices reflect all publicly available information  $\implies$  Fundamental analysis does not provide excess returns.
- Strong-form efficiency : security prices reflect all information, whether publicly available or not  $\implies$  Inside information does not provide excess returns.

Strong-form EMH  $\implies$  Semistrong-form EMH  $\implies$  Weak-form EMH



Week 9: Introduction to Corporate Finance

What is corporate finance?

- Capital budgeting: What projects (real investments) to invest in? Expansions, new products, new businesses, acquisitions, ...
- Financing: How to finance a project? Selling financial assets/securities/claims (bank loans, public debt, stocks, convertibles, ...)
- Payout : What to pay back to shareholders? Paying dividends, buyback shares, ...
- Risk management: What risk to take/to avoid and how?

Task of financial manager

Asset side (LHS): Real investments.  
Liability side (RHS): Financing, payout and risk management.

Week 10: Capital Budgeting 1

NPV Rule

Investment Criteria for NPV and cash flow calculations

- For a single project: Take it if and only if its NPV is positive.
- For many independent projects: Take all those with positive NPV.
- For mutually exclusive projects: Take the one with positive and highest NPV.
- NPV:

$$NPV = CF_0 + \frac{CF_1}{1 + r_1} + \frac{CF_2}{(1 + r_2)^2} + \dots + \frac{CF_T}{(1 + r_T)^T}$$

- Operating Profit:

$$OperatingProfits = OperatingRevenues - OperatingExpensesWithoutDepreciation$$

- Cash Flow:

$$CF = (1 - \tau)(OperatingProfits) - CapEx + \tau \cdot Depreciation - \Delta WC$$

$\tau$  : tax rate

$CapEx$  : Capital Expenditure

$\Delta WC$  : Change in working capital

- Working Capital:

$$WC = Inventory + A/R - A/P$$

$A/R$  : Accounts Receivable

$A/P$  : Accounts Payable

Payback period

Payback period is the minimum length of time  $s$  such that the sum of net cash flows from a project becomes positive:

$$CF_1 + CF_2 + \dots + CF_s \geq -CF_0 = I_0$$

Decision Criterion Using Payback Period

- For independent projects: Accept if  $s$  is less than or equal to some fixed threshold  $t^* : s \leq t^*$ .
- For mutually exclusive projects: Among all the projects having  $s \leq t^*$ , accept the one that has the minimum payback period.

Internal Rate of Return IRR

A project's internal rate of return (IRR) is the number that satisfies:

$$0 = CF_0 + \frac{CF_1}{1 + IRR} + \frac{CF_2}{(1 + IRR)^2} + \dots + \frac{CF_t}{(1 + IRR)^t}$$

Decision Criterion Using IRR

- For independent projects: Accept a project if its IRR is greater than some fixed  $IRR^*$ , the threshold rate/hurdle rate.
- For mutually exclusive projects: Among the projects having IRR's greater than  $IRR^*$ , accept one with the highest IRR.

Profitability index (PI)

Profitability index (PI) is the ratio of the present value of future cash flows and the initial cost of a project:

$$PI = \frac{PV}{-CF_0} = \frac{PV}{I_0}$$

Decision Criterion Using PI

- For independent projects: Accept all projects with PI greater than one (this is identical to the NPV rule).
- For mutually exclusive projects: Among the projects with PI greater than one, accept the one with the highest PI.

Recommended Resources

- Brealey, Myers, and Allen, Principles of Corporate Finance (13e), Irwin/McGraw Hill. (BMA)
- Bodie, Kane, and Marcus, Investments (11e), Irwin/McGraw Hill. (BKM)
- MITx 15.415.1x Foundations of Modern Finance I [Lecture Slides] (<https://courses.edx.org/courses/course-v1:MITx+15.415.1x+1T2020/course/>)
- LaTeX File ([github.com/j053g/cheatsheets/15.415.1x](https://github.com/j053g/cheatsheets/15.415.1x))

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