

Mathematical Methods for Quantitative Finance

Cheat sheet for MITx 15.455x Mathematical Methods for Quantitative Finance.

Week 1: Probability

Random variables, distributions, and moments

Moments of a distribution

- The **moments** of a distribution are the expectation of powers of the r.v.

$$\mu_l = E[X^l] = \begin{cases} \sum_k x_k^l p(x_k) \\ \int x^l p(x) dx \end{cases}$$

- Variance** : $\sigma^2 = \mathbb{E}[(X - \mu)^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

- Skewness** - asymmetry parameter : $s = \frac{\mathbb{E}[(X-\mu)^3]}{\sigma^3} = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$

- Kurtosis** - measure of tail "weights" : $\kappa = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] - 3$

Covariance and correlation

- covariance**: $\text{Cov}(X, Y) \equiv \mathbb{E}[(X - \mu_x)(Y - \mu_y)] = \mathbb{E}[XY] - \mu_x \mu_y$

- the **correlation** is proportional to the covariance:

$$\rho(X, Y) = \text{Corr}(X, Y) \equiv \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} = \mathbb{E}\left[\left(\frac{X-\mu_x}{\sigma_x}\right)\left(\frac{Y-\mu_y}{\sigma_y}\right)\right]$$

Common distributions

- Uniform distribution**: $p(x) = \begin{cases} 1, x \in [0, 1] \\ 0, \text{otherwise} \end{cases}$, $\text{Prob}(a < X < b) = b - a$

$$\mu = \int_0^\infty xp(x)dx = \int_0^1 xdx = \frac{1}{2}, \sigma^2 = \int_0^\infty (x - \frac{1}{2})^2 dx = \frac{1}{12}$$
$$u_l = \int_0^1 x_l dx = \frac{1}{l}$$

- Binomial distribution**: $f(x; n, p) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{k!(n-k)!} p^k q^{n-k}$,
 $\mu = np, \sigma^2 = npq$

- Gaussian distribution**: $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$,

$$\Phi(x) = \text{Prob}(Z < x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-z^2/2} dz$$

	Order	Non-central moment	Central moment
Moments:	1	μ	0
	2	$\mu^2 + \sigma^2$	σ^2
	3	$\mu^3 + 3\mu\sigma^2$	0
	4	$\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$	$3\sigma^4$

- Lognormal distribution**: $X = \log Y$, $x = \log y$, then

$$g(y) = \frac{p(x)}{|dy/dx|} = \frac{1}{y\sqrt{2\pi\sigma^2}} e^{-(\log y - \mu)^2/(2\sigma^2)}$$

- Poisson distribution**: $p(k; \lambda) = \frac{e^{-\lambda} \lambda^k}{k!}$. $\mu = \sigma^2 = \lambda$.

$$\text{Probability of } k \text{ "arrivals" during interval } t: p(k; \lambda) = \frac{e^{-\lambda t} (\lambda t)^k}{k!}$$

Week 2: Stochastic Processes

The Random Walk

Random walk model

- $S_T = z_1 + z_2 + \dots + z_T$. Each increment is a random IID variable.
 $\mathbb{E}[z_t] = 0, \mathbb{E}[z_t^2] = 1, \mathbb{E}[z_t z_{t'}] = 0$ if $t \neq t'$. $\text{Var}(S_T) = T$

- Generalized random walk model**: $r_t = \sigma z_t + \mu$. $\mathbb{E}[r_t] = \mu$,
 $\mathbb{E}[(r_t - \mu)^2] = \sigma^2$. $X_T \equiv \sum_{t=1}^T r_t, \mathbb{E}[X_T] = T\mu, \text{Var}(X_T) = T\sigma^2$.

Time series models

Time series models

- a ts process is **stationary** if the joint distribution of all of its values is invariant under time translation.
- a ts is **weakly stationary** if the first and second moments are invariant.
- MA(1): $r_t = \mu + \sigma z_t + \phi z_{t-1}$
- AR(p): $R_t = c_0 + c_1 R_{t-1} + \dots + c_p R_{t-p} + \sigma z_t, z_t \sim IID(0, 1)$
- ARMA(p,q):
 $R_t = c_0 + c_1 R_{t-1} + \dots + c_p R_{t-p} + \sigma z_t + \phi_1 z_{t-1} + \dots + \phi_q z_{t-q}$
- AR(1) used for **mean reversion**: $R_t = c_0 + c_1 R_{t-1} + \sigma z_t$, $E[R_t] = \frac{c_0}{1-c_1}$,
for convenience: $\mu = \frac{c_0}{1-c_1}, \lambda = -c_1$. Then: $R_t - \mu = -\lambda(R_t - \mu) + \sigma z_t$,
 $|\lambda| < 1$. $\text{Var}[R_t] = \gamma_0 = \frac{\sigma^2}{1-\lambda^2}$.
Lag- k autocovariance coefficient: $\gamma_k = (-\lambda)^k \gamma_0 = \frac{(-\lambda)^k}{1-\lambda^2} \sigma^2$

Week 3: Time Series Models

Gambler's Ruin

- Repeated set of gambles with probability of success p and of failure $q = 1 - p$
- initial capital is $x > 0$, total capital a . Stop when winning a or **ruin** as $x = 0$.
- Q_x is **probability of ruin** starting from capital x : $Q_x = pQ_{x+1} + qQ_{x-1}$.
- $Q_x = \frac{(q/p)^a - (q/p)^x}{(q/p)^a - 1}$, if $p = q = 1/2$, then: $Q_x = 1 - \frac{x}{a}$

Week 4: Continuous-Time Finance

Itô processes and Itô's lemma

Itô's lemma

- Itô process: $dX = a dt + b dB$
- $dF = \left(\frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial X} dX + \frac{1}{2} \frac{\partial^2 F}{\partial X^2} (dX)^2 + \dots \right) = \left(\frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial X} dX + \frac{1}{2} \frac{\partial^2 F}{\partial X^2} [b^2 dt] \right) = \left[\frac{\partial F}{\partial t} dt + \frac{b^2}{2} \frac{\partial^2 F}{\partial X^2} \right] dt + \left[\frac{\partial F}{\partial X} \right] [adt + b dB]$
 $dF = \left[\frac{\partial F}{\partial t} + a \frac{\partial F}{\partial X} + \frac{b^2}{2} \frac{\partial^2 F}{\partial X^2} \right] dt + b \frac{\partial F}{\partial X} dB$
- Heuristics: expand and replace $(dB_t)^2 \rightarrow dt, (dX_t)^2 \rightarrow b^2 dt$

Itô processes

- Brownian motion with drift**: $dS_t = \mu dt + b dB_t$
 $S_T = S_0 + \mu T + \sigma(B_T - B_0)$
- GBM with drift**: $dS_t = \mu S_t dt + b S_t dB_t$
 $d(\log S_t) = (\mu - \frac{\sigma^2}{2}) dt + \sigma dB_t$, $S_T = S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma(B_T - B_0)}$
- Ornstein-Uhlenbeck process**: $dS_t = \lambda(\bar{S} - S_t) dt + \sigma dB_t$
- Cox-Ingersoll-Ross process**: $d\rho_t = \lambda(\bar{\rho} - \rho_t) dt + \sigma \sqrt{\rho_t} dB_t$
Let $F = \sqrt{\rho}$, $\frac{\partial F}{\partial \rho} = \frac{1}{2\sqrt{\rho}}$, $\frac{\partial^2 F}{\partial \rho^2} = -\frac{1}{4} \rho^{-\frac{3}{2}}$
 $dF = \left(\frac{4\lambda\bar{\rho} - \sigma^2}{8F} - \frac{1}{2} \lambda F \right) dt + \frac{1}{2} \sigma dB_t$

From SDE to PDE: The Black-Scholes equation

Black-Scholes equation

- Itô process: $dX = a dt + b dB$
- $dF = \left(\frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial X} dX + \frac{1}{2} \frac{\partial^2 F}{\partial X^2} (dX)^2 + \dots \right) = \left(\frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial X} dX + \frac{1}{2} \frac{\partial^2 F}{\partial X^2} [b^2 dt] \right) = \left[\frac{\partial F}{\partial t} dt + \frac{b^2}{2} \frac{\partial^2 F}{\partial X^2} \right] dt + \left[\frac{\partial F}{\partial X} \right] [adt + b dB]$
 $dF = \left[\frac{\partial F}{\partial t} + a \frac{\partial F}{\partial X} + \frac{b^2}{2} \frac{\partial^2 F}{\partial X^2} \right] dt + b \frac{\partial F}{\partial X} dB$
- Heuristics: expand and replace $(dB_t)^2 \rightarrow dt, (dX_t)^2 \rightarrow b^2 dt$

Week 5: Itô Calculus

Black-Scholes equation

Summary of some key formulas

- Itô process: $dX = a dt + b dB$
- Itô formula:

$$dF = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial X} dX + \frac{b^2}{2} \frac{\partial^2 F}{\partial X^2} dt$$
$$= \left(\frac{\partial F}{\partial t} + a \frac{\partial F}{\partial X} + \frac{b^2}{2} \frac{\partial^2 F}{\partial X^2} \right) dt + b \frac{\partial F}{\partial X} dB$$

- Stock price: $dS = \mu S dt + \sigma dB \implies d(\log S) = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dB$
- Black-Scholes: $\Delta = \partial V / \partial S$, $d\pi = r\pi dt$,

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

Recitation 5

Expectations from Brownian integrals

- $dB \sim N(0, dt)$
- $\int_0^t dB = B_t - B_0 \sim N(0, t)$
- $E[f(B_t - B_0)] = E[f(\sqrt{t}z)] = \frac{1}{\sqrt{2\pi}} \int e^{-z^2/2} f(\sqrt{t}z) dz$

- Example $E[f(B_t - B_0)] = E[(B_t - B_0)^4] = E[(\sqrt{t}z)^4] = t^2 E[z^4] = 3t^2$
We can pull out \sqrt{t} , which is nonstochastic to get $t^2 E[z^4]$, $E[z^4]$ is a well-known Gaussian integral that we use in the kurtosis=3.

- Useful formula: $E[e^{\alpha z + \beta}] = e^{\alpha^2/2 + \beta}$

Solutions to the diffusion equation

- diffusion equation**: $\frac{\partial p}{\partial t} - \frac{1}{2} \frac{\partial^2 p}{\partial z^2} = 0$
- special solution $p(z, 0) = f(z)$ then the general solution is:
 $p(z, t) = \int p_0(z - w, t) f(w) dw$, where $p_0 = \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}}$
- let $u = \frac{w-z}{\sqrt{t}} \implies du = \frac{dw}{\sqrt{t}} \implies w = u\sqrt{t} + z$. Now general solution can be computed as expectation of standard Gaussian: $p(z, t) = \mathbb{E}[f(u\sqrt{t} + z)]$, where $u \sim N(0, 1)$
- example**: $p(z, 0) = z^2, f(z) = z^2$, find $p(z, t)$
 $p(z, t) = \int p_0(z - w, t) f(w) dw = \int \frac{1}{\sqrt{2\pi t}} e^{-\frac{(z-w)^2}{2t}} w^2 dw = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} (u\sqrt{t} + z)^2 du = \mathbb{E}[f(u\sqrt{t} + z)] = \mathbb{E}[u^2 t + 2u\sqrt{t}z + z^2] = z^2 + t$

Week 6: Continuous-Time Finance

Itô processes in higher dimensions

Itô's lemma: multiple stochastic variables

- $dX_i = a_i(t, X_1, X_2, \dots) dt + b_i(t, X_1, X_2, \dots) dB_i$
 $dF = \frac{\partial F}{\partial t} dt + \sum \frac{\partial F}{\partial X_i} dX_i + \frac{1}{2} \sum \rho_{ij} b_i b_j \frac{\partial^2 F}{\partial X_i \partial X_j} dt$
- Heuristics "rule of thumb" for correlated Brownian motions :
 $(dB_i)^2 \rightarrow dt, (dB_i)(dB_j) \rightarrow \rho_{ij} dt$,
 $(dX_i)^2 \rightarrow b_i^2 dt, (dX_i)(dX_j) \rightarrow \rho_{ij} b_i b_j dt$

- two stochastic variables case: $dX_1 = a_1 dt + b_1 dB_1$, $dX_2 = a_2 dt + b_2 dB_2$
 $dF = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial X_1} dX_1 + \frac{\partial F}{\partial X_2} dX_2 + \left(\frac{b_1^2}{2} \frac{\partial^2 F}{\partial X_1^2} + \frac{b_2^2}{2} \frac{\partial^2 F}{\partial X_2^2} + b_1 b_2 \rho \frac{\partial^2 F}{\partial X_1 \partial X_2} \right) dt$
- With two random variables, Ito's formula for $F(t, X, Y)$ is: $dF = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial X} dX + \frac{\partial F}{\partial Y} dY + \frac{1}{2} \frac{\partial^2 F}{\partial X^2} (dX)^2 + \frac{1}{2} \frac{\partial^2 F}{\partial Y^2} (dY)^2 + \frac{\partial^2 F}{\partial X \partial Y} (dX)(dY)$
- example: $F = X_1 X_2 \implies dF = X_1 dX_2 + X_2 dX_1 + \rho b_1 b_2 dt$
since $(dX_i)(dX_j) \rightarrow \rho_{ij} b_i b_j dt \implies dF = X_1 dX_2 + X_2 dX_1 + dX_1 dX_2$

Week 7: Linear algebra of asset pricing

Linear algebra of asset pricing

- **The payoff matrix** A is a matrix $s \times n$ with s states of the world and n different securities.
- A **portfolio** \vec{x} is represented as a vector of quantities held for each security.
- The **portfolio payoff** is determined by action on the portfolio vector with the payoff matrix: $A\vec{x}$
- A **complete market** is one in which every payoff can be generated by some portfolio, $\text{rank}(A) = s$. If $n > s$, select a complete basis and drop $n - s$ redundant securities
- An **incomplete market** s one in which some payoffs cannot be generated by any portfolio. $\text{rank}(A) < s$.
- Present **prices** can be represented by a vector \vec{S} dimension n .
- Market value of a portfolio is sum of (price) \times (quantity):

$$MV = \sum_{i=1}^n S_i x_i = (S_1 S_2 \dots) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \vec{S}^\top \vec{x}$$

- **Type I arbitrage**: Pay nothing now (i.e., negative or zero cost), $\vec{S}^\top \vec{x} \leq 0$. Receive only non-negative payoffs later: $A\vec{x} \geq 0$ and at least one payoff > 0 .
- **Type II arbitrage**: There are redundant assets and non-trivial solutions to $A\vec{x} = 0$. There is arbitrage if the portfolio has a non-zero price, $\vec{S}^\top \vec{x} \neq 0$
- Portfolios with the special property that they have **unit** payoff in a single state are called Arrow-Debreu (AD) securities.
- In terms of a general payoff matrix, an elementary AD security can be replicated if there is a portfolio satisfying $A\vec{x} = \mathbf{e}_j = (0 \quad 0 \quad \dots \quad 1 \quad \dots \quad 0)^\top$. $\vec{x} = A^{-1} \mathbf{e}_j$ is a portfolio with unit payoff. The payoff matrix of a set of AD securities in the complete market is the identity matrix I .
- The prices of AD securities are called **state prices**: $\psi = (\psi_1 \dots \psi_s)$
- The price of any security with payoff \vec{b} is $\vec{\psi}^\top \vec{b}$. Intuition: we can think of \vec{b} as a portfolio of AD securities.
- **Arbitrage theorem** Consider a market with n securities, s states of the world, payoff matrix $A \in \mathbb{R}^{s \times n}$, $A : \mathbb{R}^n \rightarrow \mathbb{R}^s$. There is no arbitrage if and only if there exists a **strictly positive** state-price vector $\vec{\psi}$ **consistent with** the security-price vector, $\vec{\psi} \in \mathbb{R}^s$, $\vec{S} \in \mathbb{R}^n$, $\vec{S} = A^\top \vec{\psi}$. For **incomplete markets**, there can be (infinitely) multiple solutions $\vec{\psi}$. If there is **at least one solution** where $\vec{\psi} > 0$, then no arbitrage. If **every solution** has $\vec{\psi} \leq 0$, then there is arbitrage.
- If A is not invertible ($s < n$), A^\top has **pseudo-inverse** M : $M \equiv (AA^\top)^{-1} A$, check that $MA^\top = I$, $I \in \mathbb{R}^s$

Week 8: Optimization

Portfolio Optimization

Portfolio risk: $\sigma_p^2 = \vec{w}^\top C \vec{w} = \sum w_i^2 \sigma_i^2 + 2 \sum_{i < j} w_i w_j \sigma_i \sigma_j \rho_{ij}$

$$\vec{w} = (w_1 \quad w_2 \quad \dots \quad w_n)^\top$$

Portfolio optimization with budget

- $\mathcal{L}(\vec{w}, \ell) = \frac{1}{2} \vec{w}^\top C \vec{w} + \ell(1 - \vec{t}^\top \vec{w})$
- Vary the weights: $\frac{\partial \mathcal{L}}{\partial w_i} = \left(\sum_{j \in [n]} C_{ij} w_j \right) - \ell t_i = 0$
- Solve for the weights by inverting matrix: $C \vec{w} - \ell \vec{t} = 0 \implies \vec{w} = \ell C^{-1} \vec{t}$
- Solve Lagrange multiplier: $\vec{t}^\top \vec{w} = 1 = \ell(\vec{t}^\top C^{-1} \vec{t}) \implies \ell = \frac{1}{\vec{t}^\top C^{-1} \vec{t}}$
- Solution: $\vec{w}_{min} = \ell C^{-1} \vec{t} = \frac{C^{-1} \vec{t}}{\vec{t}^\top C^{-1} \vec{t}}, \quad \sigma_{min}^2 = \ell = \frac{1}{\vec{t}^\top C^{-1} \vec{t}}$

Portfolio optimization with budget and return constraint

generalized to $\sum_i w_i = w_p$

- $\mathcal{L}(\vec{w}, \ell, m) = \frac{1}{2} \vec{w}^\top C \vec{w} + \ell(w_p - \vec{t}^\top \vec{w}) + m(\mu_p - \vec{\mu}^\top \vec{w})$
- Vary the weights: $\frac{\partial \mathcal{L}}{\partial w_i} = \left(\sum_{j \in [n]} C_{ij} w_j \right) - \ell t_i - m \mu_i$
- Solve for the weights: $C \vec{w} - \ell \vec{t} - m \vec{\mu} = \mathbf{0} \implies \vec{w} = C^{-1}(\ell \vec{t} + m \vec{\mu})$
- Solve for Lagrange multipliers with constraints: $\vec{t}^\top \vec{w} = w_p$ and $\vec{\mu}^\top \vec{w} = \mu_p$
 $w_p = \vec{t}^\top \vec{w} = \ell(\vec{t}^\top C^{-1} \vec{t}) + m(\vec{\mu}^\top C^{-1} \vec{t})$
 $\mu_p = \vec{\mu}^\top \vec{w} = \ell(\vec{\mu}^\top C^{-1} \vec{t}) + m(\vec{\mu}^\top C^{-1} \vec{\mu})$
as as matrix equation: $\begin{pmatrix} w_p \\ \mu_p \end{pmatrix} = M \begin{pmatrix} \ell \\ m \end{pmatrix}$, where: $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$
 $a \equiv \vec{t}^\top C^{-1} \vec{t}$, $b \equiv \vec{\mu}^\top C^{-1} \vec{t}$, $c \equiv \vec{\mu}^\top C^{-1} \vec{\mu}$.
- Solve for Lagrange multiplier by inverting M : $\begin{pmatrix} \ell \\ m \end{pmatrix} = M^{-1} \begin{pmatrix} w_p \\ \mu_p \end{pmatrix}$
 $M^{-1} = \frac{1}{ac-b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}, \ell = \frac{cw_p - b\mu_p}{ac-b^2} \quad \text{and} \quad m = \frac{-bw_p + a\mu_p}{ac-b^2}.$
- Solution: $\sigma_p^2 = (\ell \quad m) M \begin{pmatrix} \ell \\ m \end{pmatrix} = \begin{pmatrix} cw_p - b\mu_p & -bw_p + a\mu_p \\ ac-b^2 & ac-b^2 \end{pmatrix} \begin{pmatrix} w_p \\ \mu_p \end{pmatrix}$
 $= \frac{1}{ac-b^2} \cdot (a\mu_p^2 - 2bw_p\mu_p + cw_p^2).$

Useful formulas

$$\begin{aligned} \text{Cov}(aX + bY, Z) &= a \text{Cov}(X, Z) + b \text{Cov}(Y, Z) \\ \text{Var}(aX + bY) &= a^2 \text{Var}(X) + b^2 \text{Var}(Y) + 2ab \text{Cov}(X, Y) \\ \text{Var}(aX - bY) &= a^2 \text{Var}(X) + b^2 \text{Var}(Y) - 2ab \text{Cov}(X, Y) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} &= \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \end{aligned}$$

Recommended Resources

- MITx 15.455x MITx 15.455x Mathematical Methods for Quantitative Finance [Lecture Slides]
(<https://learning.edx.org/course/course-v1:MITx+15.455x+3T2020/home>)
- Tsay, Analysis of Financial Time Series (3e), Wiley. (Tsay)
- Capinski and Zastawniak, Mathematics for Finance, Springer. (CZ)
- Olver, Introduction to Partial Differential Equations (2016), Springer. (Olver)
- Campbell, Lo, and MacKinlay, Econometrics of Financial Markets (1997), Princeton. (CLM)
- Lang, Introduction to Linear Algebra (2e), Springer (Lang)
- Axler, Linear Algebra Done Right (3e), Springer (Axler)
- LaTeX File (github.com/j053g/cheatsheets/15.455x)

Last Updated September 27, 2021