11 Forward and Futures

Forward Interest rates

The forward interest rate between time $t-1$ and t satisfies:
$(1+r_t)^t = (1+r_{t-1})^{t-1}(1+f_t)$
$f_t = \frac{B_{t-1}}{B_t} - 1 = \frac{(1+r_t)^t}{(1+r_{t-1})^{t-1}} - 1$

Forward price

$F_T = e^{(r-y)T} S_0$

Swaps

The swap rate is a weighted average of forward rates:
$r_s = \frac{\sum_{t=1}^T B_t f_t}{\sum_{u=1}^T B_u} = \sum_{t=1}^T w_t f_t$, with the weights $w_t = \frac{B_t}{\sum_{u=1}^T B_u}$

12 Options, Part 1

Net option payoff

The break-even point is given by S_T at which Net Payoff is zero:
$Netpayoff = max[S_T - K, 0] - C(1+r)^T$

Option strategies

- Protective Put: Buy stock + Buy a put
- Bull Call Spread: Buy a Call K1 + Write a Call K2, $K1 < K2$
- Straddle: Buy a Call at K + Buy a Put at K

Corporate securities as options

- Equity (E): A call option on the firm's assets (A) with the exercise price equal to its bond's redemption value.
- Debt (D): A portfolio combining the firm's assets (A) and a short position in the call with the exercise price equal to its bond's face value (F):

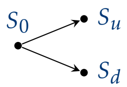
$A = D + E \implies D = A - E$
$E \equiv max(0, A - F)$
$D = A - E = A - max[0, A - F]$

Put-Call parity for European options

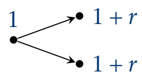
$C + B \cdot K = P + S$

Binominal option pricing model

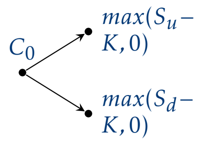
Stock



Bond

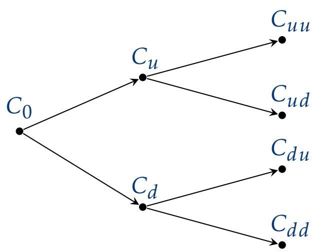


Option at Expiration



Replicating portfolio (call option)
$\begin{pmatrix} S_u & (1+r) \\ S_d & (1+r) \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} max(K - S_u, 0) \\ max(K - S_d, 0) \end{pmatrix}$
solve system to form a portfolio of stock and bond that replicates the call's payoff: a shares of the stock; b dollars in the riskless bond

Binominal option pricing model with multiple periods



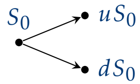
Compute the time-0 value working backwards: first C_u and C_d then C_0. In summary:

- Replication strategy gives payoffs identical to those of the call.
- Initial cost of the replication strategy must equal the call price

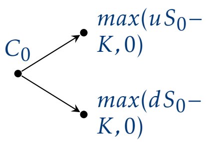
13 Options, Part 2

Binomial model: risk-neutral pricing

Stock



Option at Expiration



Solve by replication δ shares of the stock, b dollars of riskless bond
$\delta u S_0 + b(1+r) = C_u$
$\delta d S_0 + b(1+r) = C_d$
Solution: $\delta = \frac{C_u - C_d}{(u-d)S_0}$, $b = \frac{1}{1+r}\frac{uC_d - dC)u}{u-d}$
Then: $C_0 = \delta S_0 + b$
$C_0 = \frac{C_u - C_d}{(u-d)} \frac{1}{1+r} \frac{uC_d - dC)u}{u-d}$

Risk neutral probability

$q_u = \frac{(1+r)-d}{u-d}$, $q_d = \frac{u-(1+r)}{u-d}$ Then $C_0 = \frac{q_u C_u + q_d C_d}{1+r} = \frac{E^Q[C_T]}{1+r}$ where $E^Q[\cdot]$ is the expectation under probability $Q = (1, 1-q)$, which is called the risk-neutral probability

State prices and risk-neutral probabilities

$\Phi_u = \frac{q}{1+r}$, $\Phi_d = \frac{1-q}{1+r}$

$\Phi_{uu} = \frac{q^2}{(1+r)^2}$, $\Phi_{ud} = \Phi_{du} = \frac{q(1-q)}{(1+r)^2}$, $\Phi_{dd} = \frac{(1-q)^2}{(1+r)^2}$ With state prices, can price any state-contingent payoff as a portfolio of state-contingent claims: mathematically equivalent to the risk-neutral valuation formula.

Implementing binominal model

- As we reduce the length of the time step, holding the maturity fixed, the binomial distribution of log returns converges to Normal distribution.
- Key model parameters u, and d need to be chosen to reflect the distribution of the stock return

Once choice is: $u = exp(\sigma \frac{T}{n})$, $d = \frac{1}{u}$, $p = \frac{1}{2} + \frac{1}{2}\frac{u}{\sigma}\sqrt{\frac{T}{n}}$

Black-Scholes-Merton formula

$C_0 = S_0 N(x) - Ke^{-rT} N(x - \sigma\sqrt(T))$
$x = \frac{ln(\frac{S_0}{Ke^{-rT}})}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$

The call is equivalent to a levered long position in the stock; $S_0 N(x)$ is the amount invested in the stock; $Ke^{-rT} N(x - \sigma\sqrt(T))$ is the dollar amount borrowed;