A Mechanistic Model of Under-Representation in Industry

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Abstract

In this work, the population dynamics framework is utilized to model the populations of women and men in industries. A simple model of gendered effects is presented, and its behavior analyzed for a sample industry. A more complex model attempting to capture the effects of toxic workplaces is developed and analyzed. The simple model was most useful in providing a timeline to reach an equilibrium state with regards to the proportional representation of women within industry, and what the final state is. The complex model proved that a 'toxic' environment can significantly slow the time to reach an equilibrium state of representation. Results, problems, and potential improvements of these models are discussed.

1 Introduction

In recent years, the public sphere has given much attention to the role of systemic biases in the demographics of industries, leadership positions, and academic positions. These effects are believed to contribute to the long-standing under-representation of various groups in societal institutions. We propose modeling these collective effects and analyzing the situation using mechanistic modeling techniques inspired by population dynamics.

By doing so, we can ask many interesting and valuable questions regarding the effects of small biases on the demographics of various institutions: given a perfectly representative incoming pipeline of new potential employees but an unrepresentative initial state, how long would we expect to pass before representation is achieved? Are there steady-states for the sub-populations and how do these change as hiring practices and decisions to leave industry change? How do various biases affect representation?

In principle, the model we propose can generalize to a variety of interesting demographic considerations, such as racial identities, economic backgrounds, and so on. Also, our model could be used for analysis at the level of a company, university, or an entire industry. For concreteness and simplicity, we focus our attention on the representation of gender identities at the level of an industry. We restrict ourselves to binary gender identities (women and men), recognizing the non-inclusive nature of this choice, motivated by simplicity.

The qualitative description is fairly straightforward. We imagine an industry with two sub-populations: women and men. Members of the sub-populations sometimes leave the industry, either dropping out of the workforce, moving to a different industry, or retiring. New members sometimes join, coming from any number of prior backgrounds. We suppose that the hire/quit rates may differ for men and women due to a variety of societal and systemic effects. Additional complicating factors can be introduced. With this framework, we can ask a variety of interesting questions. By examining different initial conditions, hire rates, and quit rates, a great number of real-life situations can be modeled and examined.

2 Mathematical Framework

Here we introduce the mathematical framework from which we approach this problem. There are multiple approaches we could take. Though a discrete time-step approach is possible, the problem works well in a continuous time domain. Though a stochastic model might be most appropriate, since we choose to study entire industries, we are working with large numbers of people, and thus individual behaviors should be smoothed out, and average behaviors can be used. As such, a continuous deterministic model via differential equations is both simple yet appropriate. Our decisions were inspired by the population dynamics framework.

2.1 Population Equations

It is easiest to first think about this problem as such: a industry has a total employee base of population N, composed of M men and W women, such that $W + M = N$. The interesting part of the equations are the functions known in population dynamics as the birth rates and death rates, which here we call hire rates and quit rates. We denote the quit rate for women and men as: $Q_W(W, M)$ and $Q_M(W, M)$, and the hire rates as: $H_W(W, M)$ and $H_M(W, M)$. The generalized equations are:

$$\frac{\mathrm{d}W(t)}{\mathrm{d}t} = \Big(H_W(W, M) - Q_W(W, M)\Big) \qquad \frac{\mathrm{d}M(t)}{\mathrm{d}t} = \Big(H_M(W, M) - Q_M(W, M)\Big) \tag{1}$$

The exact form of the hire/quit rates are the interesting part. There are many factors that can contribute to the total number of people being hired into an industry, and the total number of people leaving the industry. As a simple example of what these functions could look like, consider:

$$H_W(t) = h_w N \qquad H_M(t) = h_m N \qquad Q_W(t) = q_w W \qquad Q_M(t) = q_m M$$

Here we have constant per capita hire rates h_w and h_m, which scale with the total population. The per capita quit rates q_w and q_m, on the other hand, must be applied to the sub-population number in order to make sense. Already this model can be useful in describing some real life scenarios. We consider these to be 'baseline' hire/quit functions. We will build on this baseline and discuss these expressions further later.

2.2 Fractional Population Equations

Equation 1 is useful for understanding the problem. However, we are generally interested not in the overall populations numbers, but the fractional populations. It is therefore useful for our analysis to write the corresponding fractional population equations. We introduce $w \equiv W/N$ and $m \equiv M/N$. Since $m + w = 1$, we ignore m and only deal with w and N now. In terms of the previous equations, we could say:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{W}{W+M}\right) = \frac{\frac{\mathrm{d}W}{\mathrm{d}t}M - W\frac{\mathrm{d}M}{\mathrm{d}t}}{N^2} \tag{2}$$

We continue by translating this into a proper differential equation of just w, N, t. We can evaluate eq. 2 using the earlier definitions and equations:

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}t} &= \frac{(H_w - Q_w)}{N}m - \frac{(H_m - Q_m)}{N}w \\ &= \frac{(H_w - Q_w)}{N}(1-w) - \frac{(H_m - Q_m)}{N}w \\ &= \frac{(H_w - Q_w)}{N} - \frac{(H_w + H_m) - (Q_w + Q_m)}{N}w\end{aligned}$$

The change in the total population follows easily from eq. 1 as well, and by using this and the equation above, we arrive at our general form:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{(H_w - Q_w)}{N} - \frac{w}{N}\frac{\mathrm{d}N}{\mathrm{d}t} \tag{3}$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = (H_w + H_m) - (Q_w + Q_m) \tag{4}$$

Thus, the fractional populations are non-linear, first order coupled differential equations. Intuitively, the total population equation make sense, it changes as per the total hiring and quitting of the sub-populations. The fractional equation for women has two terms, one that depends on the hire and quit rates for women only, and other that is coupled to the total change in population in a more complex fashion. We are interested in the the behavior of 3.

3 A First Model: Simple Gender Effects

We now treat a simple scenario of interest. We consider an industry where the new workers entering the industry are an equal mix of men and women. We might say that any early "pipeline" problems in the educational part of the system have been solved in this scenario. The quit rates, however, are a different story. We assume the base quit rate between men and women is slightly biased to lead women to quit at a higher rate. Factors contributing to this are an imbalance between maternal and paternal leave, increased stress or anxiety from the unpaid work of running a household, and a number of other factors. To back-up these claims, we have found multiple examples of this:

> Working mothers have always worked a 'double shift'—a full day of work, followed by hours spent caring for children and doing household labor. [1]

Even more applicable to this current year:

> One in three mothers have considered leaving the workforce or downshifting their careers because of COVID-19. [1].

The numbers tell the story well:

> Of the 1.1 million people ages 20 and over who left the work force (neither working nor looking for work) between August and September, over 800,000 were women according to an analysis by the National Women's Law Center... For comparison, 216,000 men left the job market in the same time period. [2]

These effects only impact the quit function for our population and don't reflect any discrimination based on hiring. We represent this mathematically saying that the female quit rate is a slightly higher than the male rate by some small ϵ. As a simplifying assumption, we also assume an industry whose total population is fixed, and not changing. We can summarize all of this via the hire/quit functions:

$$
\begin{aligned}
H_w &= aN & H_m &= aN \\
Q_w &= (2a+\epsilon)wN & Q_m &= 2a(1-w)N - \epsilon wN
\end{aligned} \tag{5}
$$

When the overall population size is not changing, we do not need to define the male hire/quit functions, but we do so here for clarity in the model. The functions are chosen such that $\mathrm{d}N/\mathrm{d}t = 0$. The $-\epsilon wN$ term on the Q_m looks ad-hoc, but is needed to satisfy $\mathrm{d}N/\mathrm{d}t = 0$ and is in agreement with the scenario setup as intended.

Inserting into the equations and simplifying, we get:

$$
\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{aN - (2a+\epsilon)wN}{N} - \frac{w}{N}\frac{\mathrm{d}N}{\mathrm{d}t} = a - (2a+\epsilon)w \tag{6}
$$

3.1 Solving the model

Without the presence of ϵ, the equilibrium is of course $w^* = 0.5$ since otherwise the two populations have equal hire and quit rates. Due to the ϵ bias the equilibrium becomes:

$$
\begin{aligned}
a - (2a+\epsilon)w &= 0 \\
-(2a+\epsilon)w &= -a \\
w^* &= \frac{a}{2a+\epsilon}
\end{aligned} \tag{7}
$$

And logically,

$$
m^* = 1 - w^* = \frac{a+\epsilon}{2a+\epsilon} \tag{8}
$$

These results satisfy $m^* + w^* = 1$, as they, must.

To analyze the stability of the equilibrium, w^*, we can make a phase-plot to see if the point is stable or not.

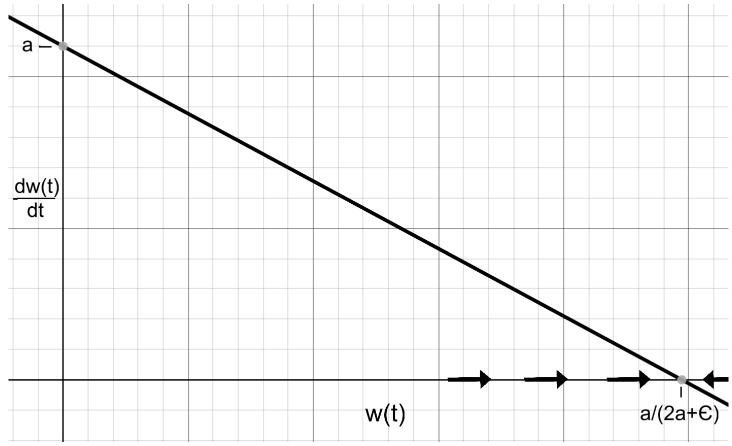

Figure 1: Phase-plot of first scenario

We can see that $w^* = \frac{a}{2a+\epsilon}$ is a stable solution because $\frac{\mathrm{d}w}{\mathrm{d}t} > 0$ when $w < w^*$ and $\frac{\mathrm{d}w}{\mathrm{d}t} < 0$ when $w > w^*$.

Because this is a simple ordinary differential equation we are able to find an exact solution. Below is the solution to this differential equation where w_0 is the initial proportion of women in an industry/company.

$$w(t) = \frac{(-a + (2a+\epsilon)w_0)e^{-t(2a+\epsilon)} + a}{2a+\epsilon} \qquad m(t) = 1 - w(t) \tag{9}$$

3.2 Results and Discussion

Today 25% of computer jobs are held by women [3]. If the prerequisite colleges degrees and other prior 'pipeline' issues were magically resolved today, how long would it be before women makeup a sizable fraction of the industry? What would be the final fraction? When is the steady state (approximately) reached?

For an industry wide analysis such as this, we use the fiducial value of $a = 1/20\text{yr}$, which we could interpret as an average career length in this industry of 20 years. If women, on average, spend four years less this industry due to the effects discussed before (a common number for at-home childcare alone), then we would set $\epsilon = 0.2a$ for this value of a. This sets the equilibrium fraction as $w \approx 0.45$. This result gives us a sense for how systemic pressures differing for material and paternal responsibilities ultimately alter the percent of women in industry in the steady-state. Looking at Figure 2, we see that in 12 years 40% of the computer workforce becomes women, and in 30 years equilibrium at 45% is achieved. Even with this "magic-wand" type fixing of pipeline issues, changes to an industry's workforce take time.

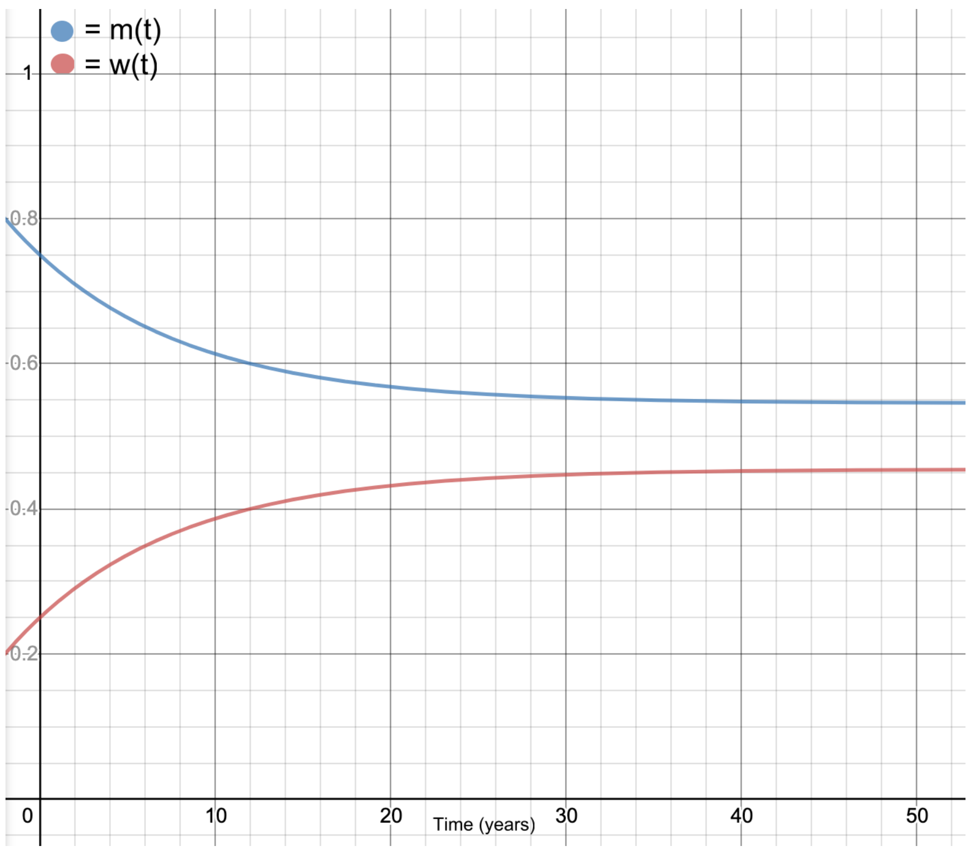

Figure 2: Solution to the first model (eq. 9) using initial condition $w_0 = 0.25$. These initial conditions are the current state of the all jobs in computing [3].

To analyze our parameters, we simply linked ϵ to be a fraction of a: $\epsilon = 0.2a$. If we generalize this to be $\epsilon = pa$ then we can analyze $p \in [0, 1]$ to examine how w^* will change with altering these gendered effects. Attached is a graph of w^* versus the parameter p where $w^* = \frac{a}{2a+pa} = \frac{1}{2+p}$.

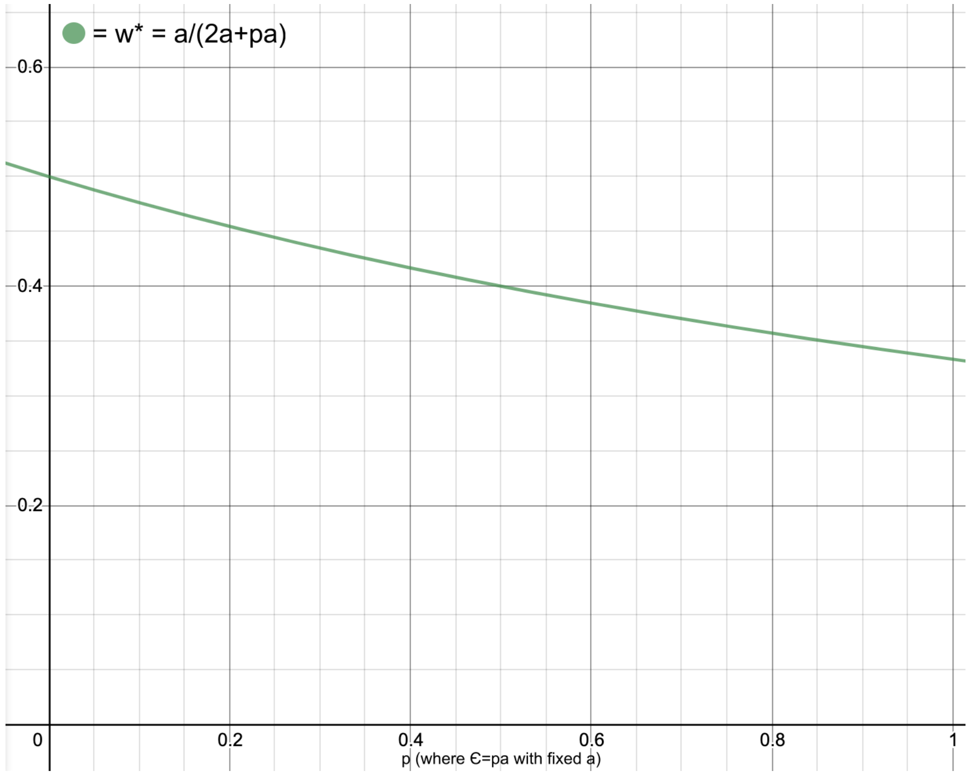

Figure 3: Here we let $a \neq 0$ and let $p \in [0, 1]$ vary and graphed w^*

We can see that a larger $pa = \epsilon$ will result in a smaller w^* which logically makes sense: if more women, on average, leave industry for any reason the proportion of them within that industry will be lower in the long run.

4 Toxic Environments

In the first model, the difference between men and women was a fixed bias in the quit rate resultant from a variety of potential social effects in the real world. Here we attempt to add an element of 'toxic work environments' to our model. By 'toxic work environments', we intend to cover a spectrum of situations, ranging from mild subconscious biases to blatant cultures of misogyny. We postulate that toxic environments are associated with the *degree* of under-representation that the affected group has in the industry. That is: the lower the fractional population of the sub-population, the worse the environment becomes, and thus the affected group is pushed out of the industry at at higher rates. We would expect misogynistic biases to effect the quit rates at all levels to some degree, but as representation decreases, we believe this effect should become increasingly severe as has been reported in many male-dominated industries.

To model this mathematically, we introduce a new term to the fractional quit function that scales $\sim w^{-1}$. This is because the effect we are modeling needs to get much worse as w approaches 0, but be small when $w \gtrapprox 0.5$. We call this parameter β, the toxicity parameter. We consider it's numerical value an arbitrary measure of toxicity. For our model here, we continue with the same other assumptions as the the previous scenario. As such, this model is an extension of the previous model with this new toxicity effect. We change the parameters slightly: we have h, the usual per capita hire rate, and q, the quit rate, which can be set to slightly higher than h to recover the effects discussed before. We continue to study fixed size industries for convenience. We write the hire/quit expressions as:

$$H_w = hN \qquad Q_w = [2qw + (\beta/w)]N \qquad \frac{\mathrm{d}N}{\mathrm{d}t} = 0 \tag{10}$$

The fractional equation becomes:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = h - [2qw + (\beta/w)] \tag{11}$$

4.1 Solving the model

With the $\frac{\mathrm{d}N}{\mathrm{d}t} = 0$ condition, we again only have one equation to work with. The equation is now non-linear. We choose to proceed with analysis without a direct solution. We first find the equilibria:

$$
\begin{aligned}
h - [2qw^* + (\beta/w^*)] &= 0 \\
2qw^* - h + (\beta/w^*) &= 0 \\
w^{*2} - \frac{h}{2q}w^* + \frac{\beta}{2q} &= 0 \\
w^* = \frac{h}{4q} \pm \frac{1}{2}\sqrt{\frac{h^2}{4q^2} - \frac{2\beta}{q}} &
\end{aligned} \tag{12}
$$

There are two equilibria now, instead of one. We proceed to analyze their stability. As an exercise, we can try using the formal framework first. With $F = h - [2qw + (\beta/w)]$:

$$
\frac{\partial F}{\partial w} = \frac{\beta}{w^2} - 2q \tag{13}
$$

The Jacobian matrix is $J = \begin{pmatrix} \frac{\beta}{w^{*2}} - 2q & 0 \\ 0 & 0 \end{pmatrix}$. To do any useful analysis, we need the trace and determinant of J:

$$
T_r\{J\} = \frac{\beta}{w^{*2}} - 2q \qquad\qquad det\{J\} = 0 \tag{14}
$$

One condition for stability is that $T_r\{J\} < 0$, which gives a condition $\frac{\beta}{w^{*2}} < 2q$. However, we need other methods for analyzing stability because the determinant of J is 0. We cannot conclude whether any w^* is stable from this method.

Instead, we turn to graphical means. In Figure 4, we create a phase plot: w vs. $\mathrm{d}w/\mathrm{d}t$ for a few parameter choices. We fix h, since we are not interested in this parameter, and choose various values of for the quit rate q as a multiple of h. We also show various values for the toxicity parameter, β, noting that the numerical value is arbitrary. It is clear from the plot that the previous equilibrium corresponds to the right-side equilibrium, which is the plus case in equation 12. This is a steady-state. The second equilibrium that appears in this model is unstable.

We can see that as the combination of effects represented by q and β become too strong, the steady-state disappears. The red lines on both plots display this situation. This is a new property of the toxicity model that was not present without a non-zero toxicity parameter.

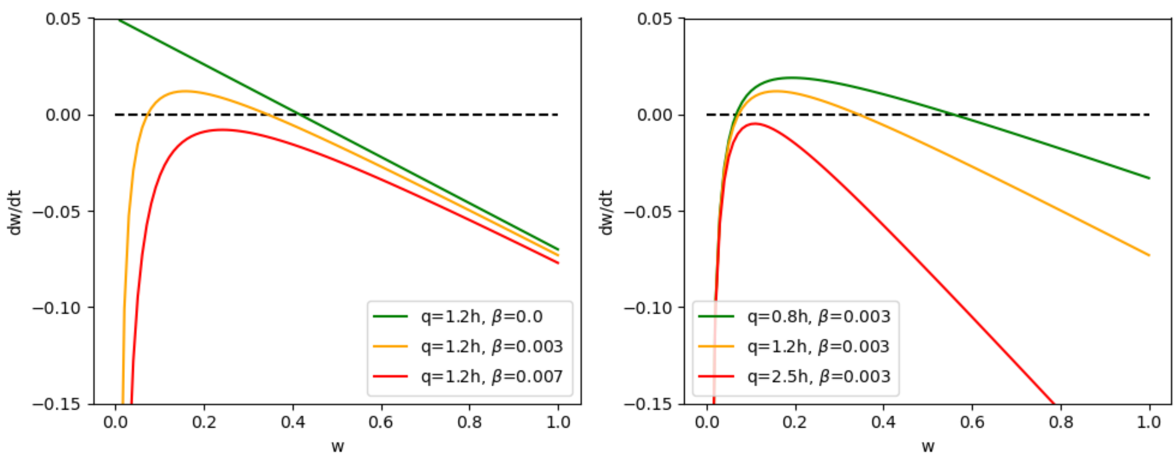

Figure 4: Stability analysis for the toxic environment model. The parameter h is held constant.

4.2 Results and Discussion

Examining figure 4 provides a wealth of information. In our estimation, the green line on the left panel is a good example of an industry with no toxic effects (it is also equivalent to our first simple model), and the orange line is a good example of an industry with some potential for misogyny. While the steady-state for the orange line is of course lower, even more interesting is the large effect that happens around values such as $w \approx 0.2$, which is the current state of a number of industries. In this model, the time it takes for an equal hiring pipeline to increase the number of women in the industry up to the equilibrium around $w = 0.4$ is significantly longer than in the non-toxic model. This can be seen in this figure because the derivative is much lower for the orange line than the green line. In our estimation, this situation may be one that is faced by many industries.

To further enlighten our discussion of industries with toxic environments, let's examine two bifurcation diagrams for the parameters q and β, figures 5 and 6. On the first of these two plots,

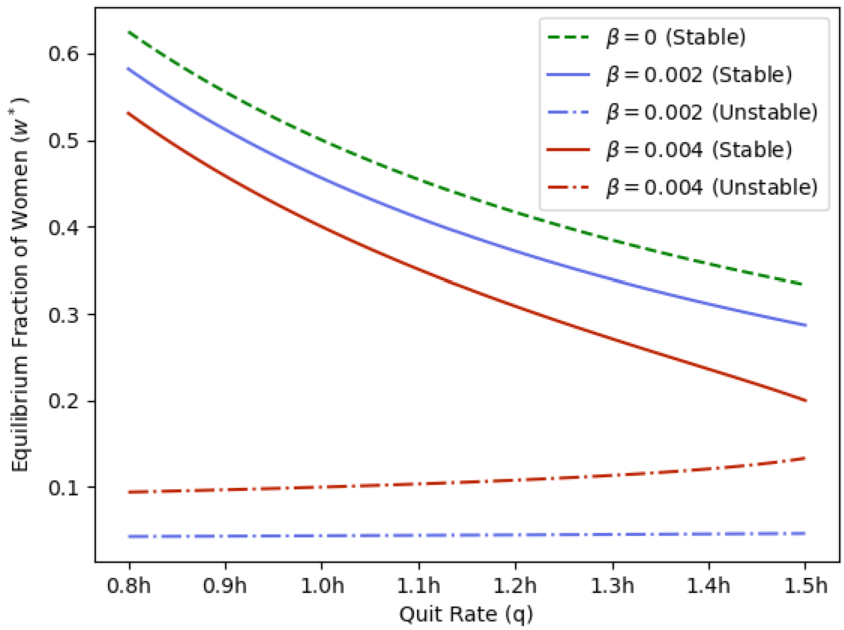

Figure 5: Toxic environment study: Analysis of the fractional population equilibria as the fixed quit rate parameter changes. The green line has $\beta = 0$ and thus is the equivalent of the Simple Gender Effects study (section 3).

we have a dashed green line where there is no toxic environment effects at all. This is again equivalent of the original scenario. The blue and red series add in a mild and more moderate strength toxicity factor, shifting the steady state down. At higher q, corresponding to higher ϵ in the language of the previous model, the equilibrium state gets lower lightly faster in the presence of toxicity, but not to a great degree.

In the figure 6, we can how the equilibria change as the toxicity factor is increased. As revealed earlier in figure 4, if the fractional population ever touches the lower equilibria, it will plummet to 0. Interpretation of this situation is not too difficult - despite the constant hiring of women, the state of toxicity is so bad that they are pushed away out faster than they can be replaced, and men fill their jobs. While this situation makes some sense, is it not very realistic. In reality, w, should not reach 0. This is a pathology in our model which a future improvement could remedy.

Also, when the toxicity factor is too high, the the steady state equilibria disappear entirely.

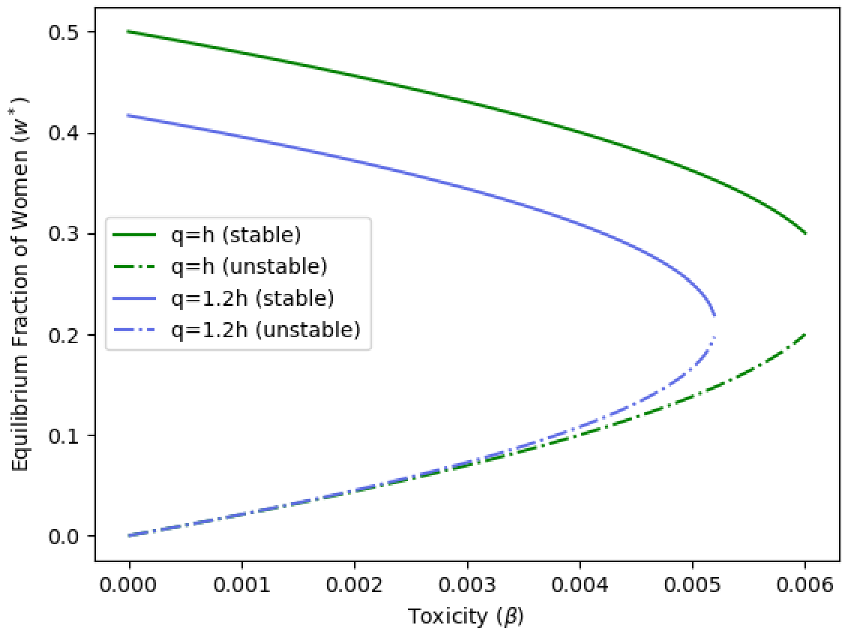

Figure 6: Toxic environment study: Analysis of the fractional population equilibria as the toxic environment parameter changes.

No matter what the initial conditions are, w will go to 0 eventually. These models are unrealistic and the parameter values too high.

5 Improvements

We would like to use our framework to consider similar situations for growing industries, where N is not fixed. Doing so is a natural extension of the progress we have made thus far. While it seems that the overall behaviors of the system should not be affected too strongly in ordinary circumstances, the possibility for new behavior exists, especially with more interesting hire/quit functions.

As discussed before, our modeling of toxic environments has some issues that we do not find realistic, and improving the modeling of this mechanism is an obvious improvement to make. While the principle of an inverse relationship with w makes sense as a measure toxic potential of an environment, there probably should be to be some cap to the strength of this effect.

A problem inherent with our entire approach is that all members of the sub-populations are affected equally. In reality, the coarse graining of having just two populations, men and women, limits our ability to model. For instance, in the tech industry many have noted that there were, in some sense, two groups of women who have worked there (historically): those two "played by the rules of men in charge" and those who were not did not. Considering these as two different groups would allow our model to capture what many say actually happened in the tech industry: the environment became so toxic that one group was completely pushed out (as our model said was possible) while the other subgroup stayed in the industry as a small sub-population.

In our analysis we did not consider very interesting hire functions, having chosen to focus how the industry could behave when the pipeline between men and women is even. In reality, the candidate pool for jobs in an industry is not even, and is always changing. Creating further models with an uneven candidate pool, especially candidate pools that change over time, is a possible way of extending the model, and could be achieved simply with some $H_w = h(t)$.

Additionally, in most companies, hiring and firing practices are influenced by a small group of influential people within the company. These groups are normally made out of experienced employees or the "higher-ups" at a company. The biases of these individuals are very important in the hiring process, but we cannot even create a hire function to express this since this "higher-ups" group is not present in our model. Adding this more realistic functionality, we can obtain a more reasonable model of hiring. This idea can be carried further to include even more stages of the career "pipeline." Indeed, many data sets of under-represented groups in some industry are reported as numbers at various stages of a pipeline: college grads, applicants, employees, managers, executives. Treating all or some subset of these as further sub-populations with "promotion" and "demotion" rates is the path to a much more robust model. This could be done in two separate ways: either simulating the pipeline analogy by introducing more differential equations that represent stages withing the pipeline (say “employee”, “manager”, and “executive”) all with the man/woman sub populations or we could continue with two differential equations but utilize a delay differential equation to represent a ’hiring committee’ that has the gender makeup of a previous time. The latter method would help simulate the time it takes to be promoted to a position of power and more accurately estimate the time needed for a company to reach a more equal balance of men and women.

To use this model to talk about individual companies instead of entire industries, we think a stochastic model would be beneficial. In this view, we imagine the events of employees being hired and quitting as Poisson processes. In the large N limit the predictions of this model should agree with the non-stochastic version, but for smaller institutions the fluctuations can produce interesting and unexpected results that would be present in real life. In this version of the model, running Monte Carlo simulations for a cohort of similar companies might be enlightening.

6 Conclusions

Regarding our initial model I think it is fair to assume that, while a relatively simple model, it can accurately portray the gender makeup of a company with net zero growth. For industries that don’t have diversity issues in hiring that follow our initial conditions, our model predicted it would take ≈ 30 years to reach a value very close to equilibrium. This model is useful because it has an elementary solution that can be graphed and analyzed visually, something that is not as simple in our ‘toxic’ environment model. We think that our simplification of the bias ϵ to be a multiplier of the hiring rate, a, is a good assumption and rids us of the complex task of figuring out a fixed value. While this may reduce the flexibility of our model, it seems negligible. Our first model may not directly point to any certain policy prescription (our research indicates that motherhood plays a significant role in gender imbalances) but it can give us a reasonable timeline to reach a more diverse institution.

Our toxic environment model has demonstrated that even with magically resolved 'pipeline problems,' it would take many years to bring up the numbers of women in industry like engineering (14% women) or computer jobs (25%) up because the toxicity of the environment is particularly bad in this regime, forcing many women out of the industry early. This model proved tricky to build and took several attempts. For moderate parameter values, indicative of realistic scenarios, we think this model performs fairly well, despite some unrealistic issues for certain parameter choices and values of w. We thought it was interesting to see how the phase diagram looked with different parameter choices. The model breaks at very low w values, which was expected given the w^{-1} term. This mild issue does not stop the model from being useful.

After completing the analysis of both of these models we gained knowledge in interpreting phase-plots and bifurcation diagrams of functions that have more parameters and are more complex than ones we have previously done. We learned that there are always trade-offs in adding complexity to a model. For example, we were able to find a full solution to the first model but not the second. The trade-off between complexity and ease of use has now been fully realized by completing this project. We think that our models can be generalized for other systemic biases (with two sub-populations).

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