Homework #5

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Math 381 - Discrete Modeling
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February 12th, 2021

Dice Game

In this write-up I will explain a game that can be played with one die and analyze its probabilistic mechanics.
This game is relatively simple: starting with a score of zero, you roll a die. That first roll is then your score.
After you have a score from 1-6, you will roll again. The next roll will be added to your score.
However, if the sum of your roll and current score is ever a prime number, you will subtract the roll from your score.
This rule only applies to rolls when your score is not 0. If you return to zero the next roll will be your score (even if it is prime).
The goal of the game is to reach the score of 26. If you reach that score, the game is over.

Mathematical Model

To analyze how this game will play out probabilistically, we will use Markov chains.
There will be 27 states where each state's number is your score: 0 as the starting state, 1-25 as intermediary states, and 26 as the final, absorbing state. States 0-25 are transient as there is a non-zero possibility that we will never be in that state again. State 26 is a recurrent state because there is a zero probability we will leave that state.
For the initial state, where your score is 0, there is a relatively simple probability distribution: each state 1-6 has a $\frac{1}{6}$ likelihood of occurring. After the first state, it gets a little more complicated.
As my code will show, we will create the Markov chain's transition matrix that we will call A.
The entry A_{ij} of the transition matrix will be the probability of having a score of j (and thus transitioning to the j^{th} state) after rolling with a score of i.
This matrix is 27 by 27 which makes it incredibly tedious to do all the calculations for each state one at a time.
I have included my Sage code below on how to create the transition matrix:

```
# Returns true if n is a prime number. Returns false otherwise.
def isPrime(n) :
    if (n <= 1) :
        return False
    if (n <= 3) :
        return True

    if (n % 2 == 0 or n % 3 == 0) :
        return False

    i = 5
    while(i * i <= n) :
        if (n % i == 0 or n % (i + 2) == 0) :
            return False
        i = i + 6

    return True

# Initializes a transition matrix full of zeroes
A=zero_matrix(QQ,27);
# The first roll doesn't follow the same rules as the other rolls,
   each state (1-6) has an equal probability of being rolled.
for i in range(6):
    A[0,i+1]=1/6

# This makes a score of 26 an absorbing state
A[26,26]=1;

for i in range(1,26):
    for j in range(1,7):
        # Add 1/6 probability if the next roll + current score has
           reached absorption
        if i+j >= 26:
            A[i, 26] += 1/6
        # If next roll + current score is not prime, add 1/6 to that
            state
        elif not isPrime(i+j):
            A[i, i+j] += 1/6
        # Making sure that if the score is negative we make it 0
           instead.
        elif i-j >= 0:
            A[i, i-j] += 1/6
        else:
            A[i, 0] += 1/6
```

This Sage code will produce the desired transition matrix as shown on the next page.

$$A = \begin{pmatrix}
0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{3} & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{5}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

A question we might have is: what is the average amount of steps it will take to reach the absorption state?

This question can be answered by, first, changing A into its canonical form. Any absorbing Markov chain can be put into canonical form.

This form is this, where J is an identity matrix, O is a matrix of all zeros, and Q and R consist of the rest of the transition probabilities:

$$P = \left(\begin{array}{c|c} Q & R \\ \hline O & J \end{array}\right)$$

An important matrix for gleaning information from the Markov chain is $N = (I - Q)^{-1}$.

To answer our question of mean rolls to reach the absorption state, we need the following theorem:

Theorem 1. Let P be the transition matrix for an absorbing chain in canonical form. Let $N = (I - Q)^{-1}$. Then the sum of the i-th row of N gives the mean number of steps until absorption when the chain is started in state i.

This theorem gives us the answer to our question. All we need to do is calculate the matrix N and then sum the entries of the first row (because the game starts at score 0).

```
# The matrix Q is all but the last row and column of A.
Q = A[:26,:26]
N = (identity_matrix(QQ, 26) - Q).inverse()
# This gives us a numerical value (instead of fractional) for the
    summation of the first row.
exp = numerical_approx(N.row(0).norm(1))
print(exp)
```

We end up with a mean value of 15.1148382307449 rolls before we will reach the absorption state.
Another question one might ask about this Markov chain is: what is the probability that at least half the expected number of rolls are needed to get to the absorption state?
This is a relatively simple calculation with the following theorem:

Theorem 2. Let A be the transition matrix for a Markov chain. Then A^n_{ij} gives the probability that the chain will be in state j exactly n steps after being in state i.

The initial question is asking $\Pr(x \geq \frac{E[x]}{2})$ where x is a random variable denoting the number of rolls it takes to reach a score of 26 and $E[x]$ is the expected value.
We already calculated the expected value and can now write the problem as so:

$$\Pr(x \geq \frac{15.1148382307449}{2} = 7.55741911537 = 8)$$

These simplifications are valid because we are only dealing with discrete values.
One last fact that can help us is this

$$\Pr(x \leq 7) + \Pr(x \geq 8) = 1$$

$$1 - \Pr(x \leq 7) = \Pr(x \geq 8)$$

We will now solve for the left-hand side using Theorem 2.

```
(A^7)[0][26]
```

This value is $\frac{29839}{279936}$ (or 0.10659222107) which means the probability of needing AT LEAST 8 rolls to reach a score of 26 is 89.340777892%.