AMATH 482/582: HOMEWORK 1

JONATHAN BEAUBIEN

Applied Mathematics Department, University of Washington, Seattle, WA
`beaubj@uw.edu`

Abstract. A submarine in the Puget Sound has been emitting acoustic sound at an unknown frequency. After collecting 3D acoustic data in 30 minute intervals over a period of 24 hours, we were able to find the submarine's emitting frequency by averaging the Discrete Fourier Transforms of this data over time. We used a Gaussian filter on the signature frequency to then remove noise within the original acoustic data and thus locate the submarine's path in 3D.

1. Introduction and Overview

A submarine in the Puget Sound has been emitting acoustic sound at an unknown frequency. The data at our disposal is 3D acoustic data taken at 30 minute intervals over 24 hours. This data is in the form of 3D measurements on a $64 \times 64 \times 64$ uniformly spaced grid. There are 49 occurrences of this 64^3 matrix. We have been tasked with finding the emitting frequency of this submarine and then filtering out the noise that lies in the original data to find the 3D path of the submarine.

To find the signature frequencies, we will use an important fact about Fourier transforms and noise. We can take the Fast Fourier Transform (FFT)–an implementation of the Discrete Fourier Transform (DFT)–of each time snapshot and average them to obtain the signature frequencies. There will be two signature frequencies that we can verify visually and numerically (they are reflected over the origin in frequency space). Then we will apply a Gaussian filter centered on those signature frequencies to filter out noise associated with frequencies other than those from the submarine. The choice of sigma within the Gaussian filter ended up being of minimal importance.

Finally, we can trace the submarine's path by taking the Inverse Fast Fourier Transform (IFFT) of the filtered data in frequency space. The final result produced a relatively smooth path. There is provided a graph of the 3D path as well as the path in 2D (displaying only x, y values) for use by a sub-tracking aircraft.

2. Theoretical Background

Fourier transforms have been invaluable when it comes to signal processing. By mapping a signal from its time or space domain to the frequency domain lets us remove unnecessary frequencies and remove noise. Because we are working in a discrete environment, we must use the DFT:

$$f : [0, 2L] \to \mathbb{R}, c_k := \frac{1}{2L}\int_0^{2L} f(x)\exp(-\frac{i\pi kx}{L})dx$$
$$f(x) \approx \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} c_k \exp(-\frac{ik\pi x}{L}), c_k \in \mathbb{C} \tag{1}$$

Date: February 13, 2022.

This is the equation for a 1D DFT [4]. However, we have 3D acoustic data so we must generalize this to three dimensions [3].

$$f : [0, L] \times [0, L] \times [0, L] \to \mathbb{R}$$

$$f(x, y, z) = \sum_{k,l,m=-\frac{N}{2}}^{\frac{N}{2}-1} c_{klm} \exp\left(i\pi\left(\frac{kx}{L} + \frac{ly}{L} + \frac{mz}{L}\right)\right) \tag{2}$$

Thankfully, we don't need to implement this ourselves. The Fast Fourier Transform (FFT) algorithm applies to n dimensional data and (as the name suggests) is very fast. FFT is faster than most linear solvers because it is $O(N \log N)$ rather than $O(N^2)$ [1]. We also have access to IFFT for finding the inverse when we want to transition back to the spatial domain.

An important insight that is needed to isolate the signature frequency of the submarine is this fact:

It is known that adding mean zero white noise to a signal is equivalent to adding mean zero white noise to its Fourier series coefficients. [**?**]

What does this mean for our problem? Because we can assume our noise is random and mean zero, it will average to zero over many samples in time. Therefore, if we average our measurements in the Fourier domain, we can reduce noise.

While this process will remove noise, we need further methods to remove more. Our next step will be to apply a Gaussian filter centered on the frequency signature of the submarine. This filter will be filtering out unnecessary frequencies so it will live within the frequency domain. Because there are two frequencies, our final filter will be a sum of two of these Gaussian equations centered on (kx_0, ky_0, kz_0):

$$G(kx, ky, kz) = \exp\left(-\frac{(kx - kx_0)^2 + (ky - ky_0)^2 + (kz - kz_0)^2)}{2\sigma^2}\right) \tag{3}$$

3. Algorithm Implementation and Development

The NumPy [2] implementations of FFT and IFFT were used in a Python environment. Plotly was used for creating 2D and 3D plots for visualization. The Gaussian filter was self-implemented.

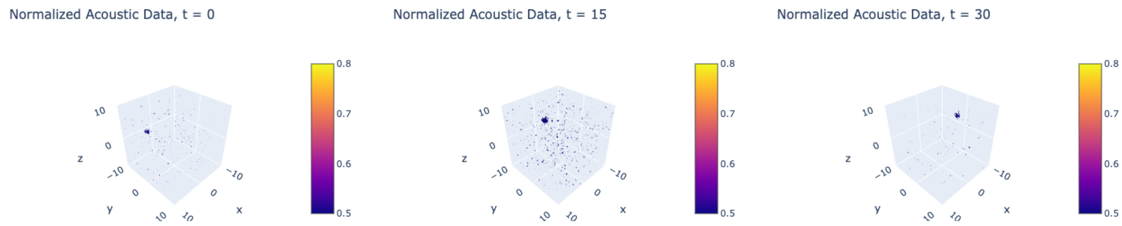
The data that we obtained was in a flattened array with dimensions $262144(64^3) \times 49$. By using the *np.reshape()* command on each column we can remake the $64 \times 64 \times 64$ matrix. Then we can pass that matrix into the *np.fft.fftn()* algorithm for doing the 3D FFT. Averaging these cubes over time gives us magnitudes in frequency space that we can then find the peak of (by using *np.unravel_index()* and *np.argmax()*). This will give use the frequency signature that we can verify visually.

Next, using our two center frequencies we can create a filter that is the sum of two Gaussian filters centered around each frequency. Passing the (kx, ky, kz) grid and each center frequency (kx_0, ky_0, kz_0) and a value for σ into (3) then summing the results.

Lastly, we will apply our filter to each time snapshot in the Fourier domain by multiplying our filter by the FFT (*np.fft.fftn()*) of our spatial data. Once we take the IFFT (*np.fft.ifftn()*) to return to the spatial domain, we can find the location of the peak magnitude of acoustic pressure (again using *np.unravel_index()* and *np.argmax()*) for each time snapshot. We will then plot the 3D and 2D path of the submarine using Plotly.

4. Computational Results

To start off, let us visualize our data at some random times $(t = 0, t = 15, t = 30)$.



(A) (B) (C)

Figure 1. We can see that there is pressure concentrated in certain places (probably where the submarine is) but there is also a lot of noise especially at $t = 15$. Our goal is to remove this noise. Note: time is in units of $\frac{1}{2}$ hours

Now that we know what our data looks like, we can start employing our strategies outlined above to remove this noise. Here is a plot of the time averaged FFTs in frequency space:

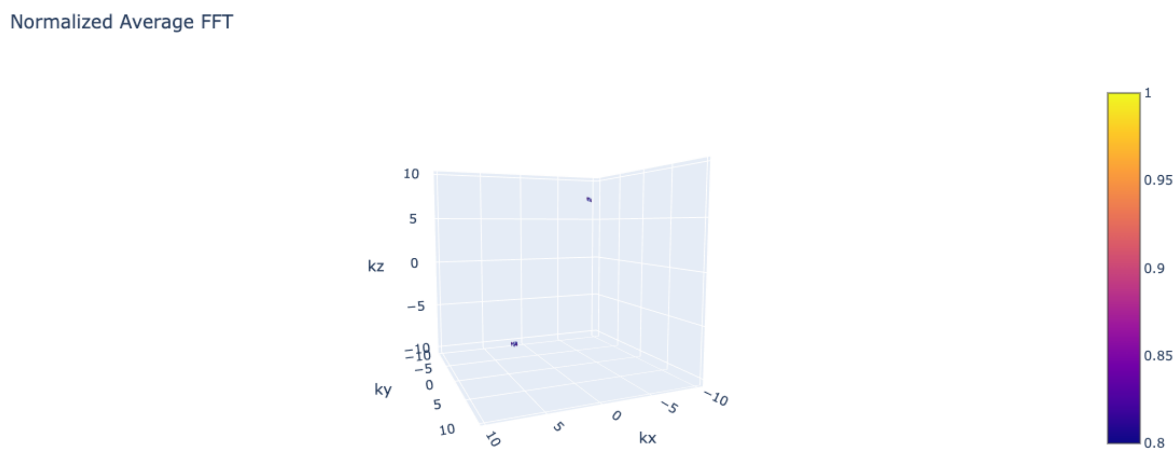
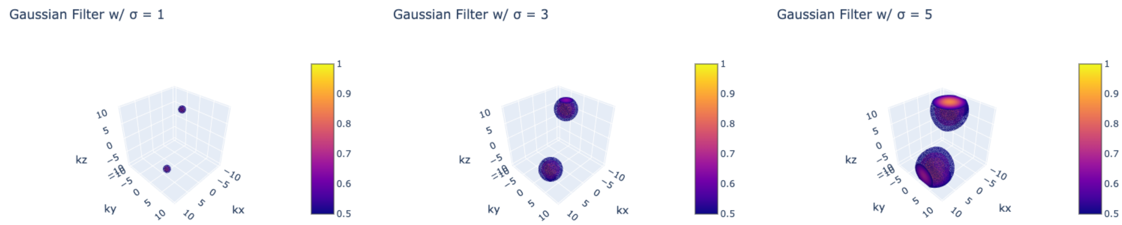


Figure 2. Here we have the signature frequencies of the submarine. As expected, they are reflections of each other over the origin. I have graphed the normalized values only from 0.8 to 1 because we will only be using the point with the maximum value.

It is hard to view the averaged FFTs in 3D so I have provided a table with the numerical values. You can verify their values by comparing them to Figure 2.

kx	ky	kz
5.3407	2.1991	-6.9115
-5.3407	-2.1991	6.9115

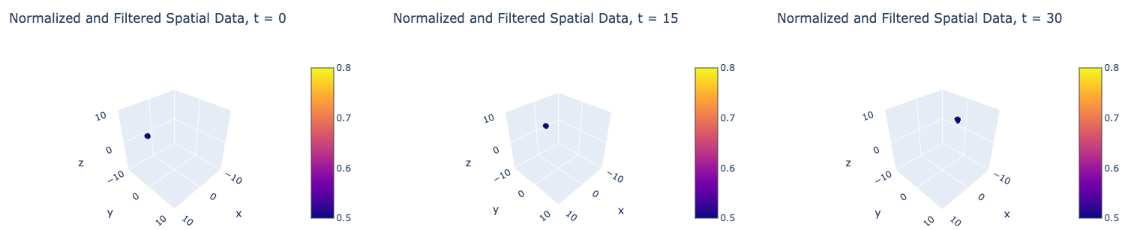
Now we can center our Gaussian filter(s) one these center frequencies. Here are several examples of what the filter looks like with varying σ. The choice of $\sigma = 3$ was decided on because after inspection, the choice did not significantly effect the resulting submarine path.



(A) (B) (C)

Figure 3. Here are 3D Gaussian filters with varying σ according to eq. 3

Now we can look at what the filtered data in the spatial domain looks like to compare. After applying the filter to the Fourier transform with $\sigma = 3$, we can take the IFFT and get back the spatial data. The next figure is the same data as Figure 4 but filtered.



(A) (B) (C)

Figure 4. We can see that there is pressure concentrated in one area but without any noise. This is what we want! Note: time is in units of $\frac{1}{2}$ hours

The next step is to find the actual position of the submarine. As said before, we can take the position of the maximum value of pressure in the spatial domain to be submarine's position. The next two figures are those positions in both 3D and 2D.

5. Summary and Conclusions

We successfully found the submarine's location in 3D and it looks like a spiral. By using the FFT and IFFT algorithms along with an averaging over time and a summation of 3D Gaussian filters we were able to locate the submarine's signature frequency and get rid of noise within our data. This shows that one can use a Gaussian filter on noisy acoustic data within the Puget Sound to locate a submarine with ease.

We can conclude that the submarine's emitting frequency is $\pm(5.3407, 2.1991, -6.9115)$ and the submarine's 2D and 3D path are located in Figures 5 and 6 respectively.

While we didn't try any other filters, further research into relevant and useful filters could be done. One could try to use an Elliptic or Median filter instead of a Gaussian for example.

Acknowledgements

The author is thankful to Katherine Owens for her tips for applying the Gaussian filter on a shifted axis. The author is also thankful to Brian Powers and Ava Mistry for their help regarding plotting and citations respectively.

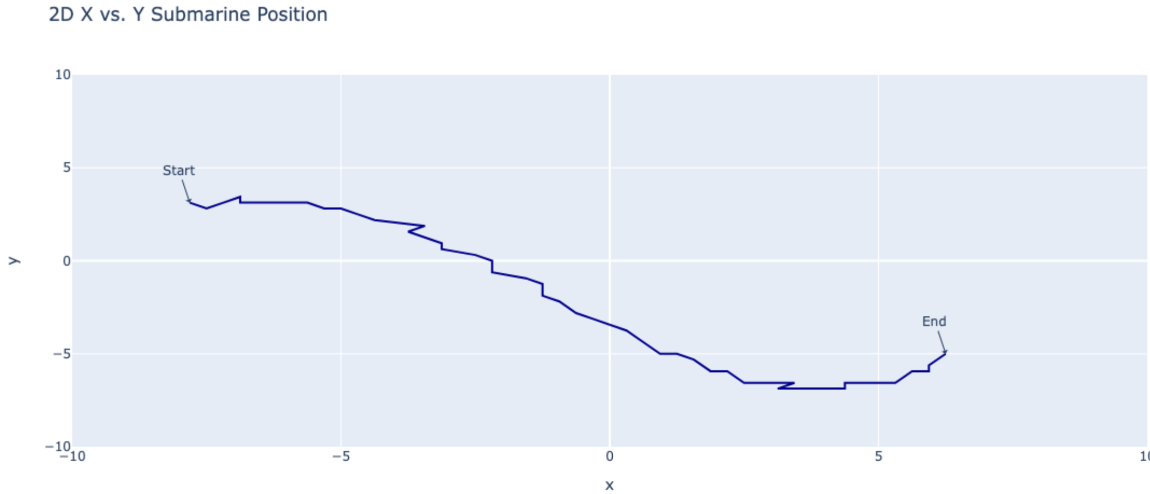


FIGURE 5. The submarine's 2D path. One could deploy a submarine tracking aircraft and follow this path to remain above it.

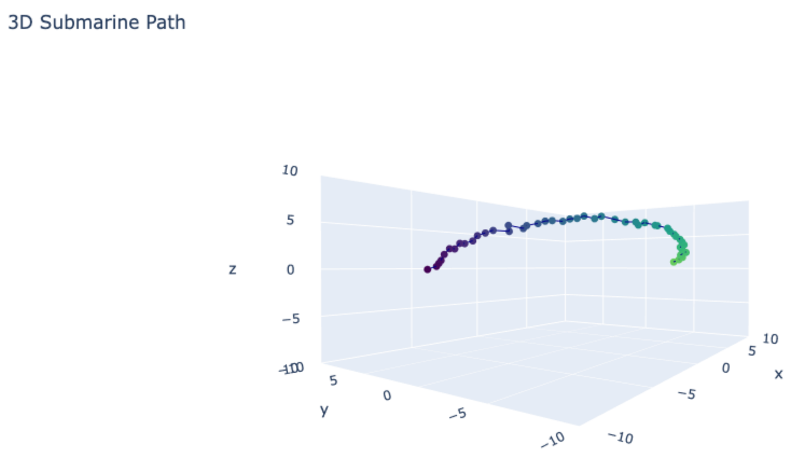


FIGURE 6. The submarine's 3D path. The starting position is purple and transitions to green over time.

REFERENCES

[1] S. Brunton and N. Kutz. *Data Driven Science Engineering*. databook.uw.edu, 2017.

[2] C. R. Harris, K. J. Millman, S. J. van der Walt, R. Gommers, P. Virtanen, D. Cournapeau, E. Wieser, J. Taylor, S. Berg, N. J. Smith, R. Kern, M. Picus, S. Hoyer, M. H. van Kerkwijk, M. Brett, A. Haldane, J. F. del Río, M. Wiebe, P. Peterson, P. Gérard-Marchant, K. Sheppard, T. Reddy, W. Weckesser, H. Abbasi, C. Gohlke, and T. E. Oliphant. Array programming with NumPy. *Nature*, 585(7825):357–362, Sept. 2020.

[3] B. Hosseini. High-dimensional extensions amp; image processing. University of Washington-Seattle (LOW 216), Jan 2022. AMATH 482/582.

[4] B. Hosseini. Signal processing with dft. University of Washington (LOW 216), Jan 2022.