B490 Exam 1 – Spring 2020

Name: ______________________________

Student ID (last 4 digits) or username: ______________________________

Problem	Points/out of
1	/ 26
2	/ 6
3	/ 12
4	/ 16
Total	/ **60**

Problem 1 Whenever you are asked to define a function or parametrized circuit in pidgin Python, write mathematical pseudocode based on Python syntax. Do not take the time to write working Python programs. The following are example lines of pidgin Python: 26 POINTS

```
ind = the indices (from the right) where s and t differ;

c = an empty 3-bit circuit with one register r;

add ccx gate negating r[i] (controlled by remaining bits);
```

Otherwise, whenever you are asked to define a circuit, draw the circuit or write the corresponding QASM code. Specify here the order in which you are mapping register indices to bits (from MSB to LSB or from LSB to MSB):

1. (*4 points*) Define a 3-bit circuit which swaps `000` and `100` and fixes all other strings using only the `ccx` and the `x` gates.

2. (*4 points*) Define a function `flip1` in pidgin Python which takes two 3-bit strings s and t which differ *at only one position* and returns a 3-bit circuit which swaps s and t and fixes all other inputs.

3. (*4 points*) Recall that one transposition may be decomposed into two transpositions as follows:

$$(i\ k) = (i\ j)(j\ k)(i\ j)$$

Define a 3-bit circuit which swaps `000` and `011` and fixes all other strings using only the `ccx` and the `x` gates. **Hint**: `flip1` implements certain transpositions.

4. (*4 points*) Define a function `flip2` in pidgin Python which takes two 3-bit strings s and t which differ *at most at two positions* and returns a circuit which swaps s and t fixes all other inputs. Use `flip1`.

5. (*4 points*) Define a function `flip3` in pidgin Python which takes 3-bit strings s and t returns a circuit which swaps s and t and fixes all other inputs. Use `flip1` and `flip2`.

6. (*6 points*) Assume there exists a parametrized circuit `flip1`(n) which takes two strings s and t which differ at most at one position and returns an $(n+1)$-bit circuit which swaps s and t and fixes all other inputs. (The additional bit is an ancilla bit which can be reused.)

   Define a parametrized circuit `flip`(k, n) in pidgin Python which takes n-bit strings s and t which differ at most at k positions and returns an $(n+1)$-bit circuit which swaps s and t and fixes all other inputs. (Hint: your definition should be recursive in k.)

   Next, use `flip`(k, n) to define `swap`(n) (whether or not you have defined `flip`(k, n)) which takes any two n-bit strings s and t and returns an $(n+1)$-bit circuit which swaps s and t and fixes all other inputs.

   Finally, answer the following question: why does the existence of `swap`(n) allow us to conclude that the gates `ccx` and `x` suffice to implement any n-bit reversible function?

Problem 2 Recall that for a vector v in a complex Hilbert space, $\|v\| := \langle v, v \rangle$. 6 POINTS

1. (*4 points*) Let Q_1 be the complex Hilbert space generated by $|0\rangle$ and $|1\rangle$. Let $s = c_1 |0\rangle + c_2 |1\rangle \in Q_1$ with $\|s\| = 1$. Find $z \in \mathbb{C}$ such that $zs = d_1 |0\rangle + d_2 |1\rangle$ where $\mathrm{Re}(d_1) \geq 0$, $\mathrm{Im}(d_1) = 0$ and $\|zs\| = 1$.

2. (*2 points*) Show that for any complex Hilbert space underlying an n-qubit state space Q with basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ and for any $s \in Q$ with $\|s\| = 1$, there exists an n-qubit state t (with norm 1) equivalent to s whose first coordinate is real and non-negative.

Problem 3 Let $\boxed{\text{12 POINTS}}$

$$
\begin{aligned}
V &:= \left\{ \begin{bmatrix} c_1 & c_2 \end{bmatrix} \mid c_1, c_2 \in \mathbb{C}^2 \right\} \\
\begin{bmatrix} c_1 & c_2 \end{bmatrix} + \begin{bmatrix} d_1 & d_2 \end{bmatrix} &:= \begin{bmatrix} c_1 + d_1 & c_2 + d_2 \end{bmatrix} \\
0 &:= \begin{bmatrix} 0 & 0 \end{bmatrix} \\
z \cdot \begin{bmatrix} c_1 & c_2 \end{bmatrix} &:= \begin{bmatrix} \overline{z} \cdot c_1 & \overline{z} \cdot c_2 \end{bmatrix} \\
\left\langle \begin{bmatrix} c_1 & c_2 \end{bmatrix}, \begin{bmatrix} d_1 & d_2 \end{bmatrix} \right\rangle &:= \overline{\left\langle \begin{bmatrix} c_1 & c_2 \end{bmatrix}^T, \begin{bmatrix} d_1 & d_2 \end{bmatrix}^T \right\rangle}
\end{aligned}
$$

Verify that V with these operations is a Hilbert space (i.e., a vector space over $\mathbb{C}$ with an inner product). (Remember that the inner product for complex Hilbert spaces is linear in the second argument and conjugate linear in the first argument.) You may omit verifying that $+$ is associative, unital, and commutative and the existence of additive inverses.

(Problem 3 continued.)

Problem 4 Which of the following states are superpositions with respect to the given basis? Justify your answers. For each state that is a superposition, give a basis with respect to which it is not a superposition. $\boxed{\text{16 POINTS}}$

1. (*4 points*) $\frac{1}{\sqrt{2}}(i\,|+\rangle - i\,|-\rangle)$ with respect to $\mathcal{S} := \{|\mathbf{i}\rangle\,, |\mathbf{-i}\rangle\}$

2. (*4 points*) $\frac{\sqrt{3}}{2}\,|0\rangle - \frac{1}{2}\,|1\rangle$ with respect to $\mathcal{S} := \{|\mathbf{i}\rangle\,, |\mathbf{-i}\rangle\}$

3. (*4 points*) $\frac{1}{\sqrt{2}}(|\mathbf{i}\rangle + |-\mathbf{i}\rangle)$ with respect to $\mathcal{H} := \{|+\rangle, |-\rangle\}$

4. (*4 points*) $\frac{1}{\sqrt{2}}(i\,|0\rangle + |1\rangle)$ with respect to $\mathcal{S} := \{|\mathbf{i}\rangle, |-\mathbf{i}\rangle\}$