Lecture 6: The Quantum Fourier Transform

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Introduction

Last lecture, we:

- showed how to analyse some simple quantum logic circuits

In this lecture we will:

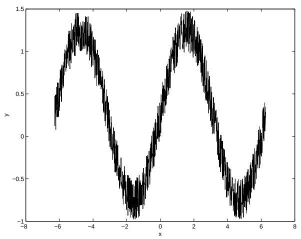
- introduce the discrete Fourier transform (DFT)
- show how the quantum Fourier transform (QFT) emerges from the DFT
- construct a quantum circuit that performs the QFT

The Fourier Transform

The Fourier Transform (FT) is one of the most useful mathematical tools in modern science and engineering. It is used, amongst many other things, to:

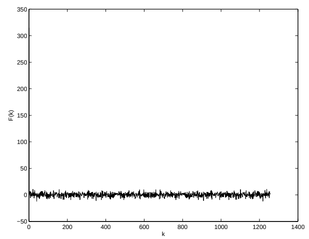
- remove noise from data
- examine the properties of crystals
- produce holograms

The FT is especially useful when we have something with underlying periodicity: imagine a sine wave with a bit of high-frequency noise

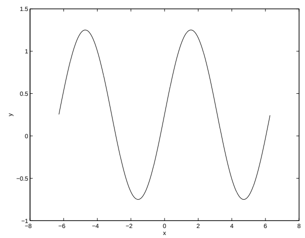


The Fourier Transform

We can Fourier Transform the data to get a frequency spectrum:



We then remove the high-frequency components (the noise) from the spectrum, and inverse-FT to give a clean set of data:



The Fourier Transform

- FT allows us to extract the underlying periodic behaviour of a function
- Period finding is the basis for Shor's factoring algorithm, and wewill use the QFT in this important application of quantum computing
- We must begin by defining the discrete version of the Fourier Transform, which will form the basis for the quantum algorithm

Discrete Fourier Transform

- The DFT is a version of the FT which works on discrete data sets
- Mathematically, the DFT is written as

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \mathrm{e}^{2\pi \mathrm{i} jk/N}$$

- Looks formidable, but isn't too hard to calculate
- x_j are complex numbers, with $j = 0 \ldots N-1$
- y_k are complex numbers, with $k = 0 \ldots N-1$
- $\mathrm{i} = \sqrt{-1}$; j and k are indices
- An example will show us how to calculate this in practice

An Example Calculation

- Given $x_j = \{1, 2\}$, calculate y_k
- $x_0 = 1$, $x_1 = 2$, and $N = 2$
- So $y_k = \frac{1}{\sqrt{2}} \sum_{j=0}^{1} x_j \mathrm{e}^{2\pi \mathrm{i} jk/2}$
- For $k = 0$, $y_0 = \frac{1}{\sqrt{2}} \sum_{j=0}^{1} x_j = \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{3}{\sqrt{2}}$
- For $k = 1$, $y_1 = \frac{1}{\sqrt{2}} \sum_{j=0}^{1} x_j \mathrm{e}^{2\pi \mathrm{i} j/2} = \frac{1}{\sqrt{2}} \left(1 + 2\mathrm{e}^{\pi \mathrm{i}}\right) = -\frac{1}{\sqrt{2}}$
- So it's really not that hard to calculate
- The Fast Fourier Transform (FFT) algorithm of Cooley and Tukey allows us to compute the DFT very rapidly
- The FFT is often use in sound and image processing for the removal of noise
- In general, the FT is useful when there is underlying periodicity
- We will later see that the FT allows us to manipulate quantum state vectors to allow us to measure the result of quantum computations

The Quantum Fourier Transform

- Since our state vectors for qubits are just vectors of complex numbers, we should not be surprised to learn that the DFT can be applied to them
- Given a state vector $|\psi\rangle = \sum_{j=0}^{N-1} a_j |j\rangle = \begin{pmatrix} a_0 \\ \vdots \\ a_{N-1} \end{pmatrix}$
- We can compute the DFT of this state as

$$\begin{aligned} F|\psi\rangle &= \sum_{k=0}^{N-1} b_k |k\rangle \\ \text{where} \quad b_k &= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} a_j \mathrm{e}^{2\pi \mathrm{i} jk/N} \end{aligned}$$

- It can be shown that this is unitary, and so can be implemented

Example of a QFT

- Consider the 2-qubit state $|\psi\rangle = a_{00}\,|00\rangle + a_{01}\,|00\rangle + a_{10}\,|10\rangle + a_{11}\,|11\rangle$, which has $N = 4$.
- Then $b_k = \frac{1}{2}\sum_{j=0}^{3} a_j \mathrm{e}^{2\pi \mathrm{i} jk/4}$, and we have

$$
\begin{aligned}
b_0 &= \frac{1}{2}\sum_{j=0}^{3} a_j = \frac{1}{2}\left(a_{00} + a_{01} + a_{10} + a_{11}\right) \\
b_1 &= \frac{1}{2}\sum_{j=0}^{3} a_j \mathrm{e}^{2\pi \mathrm{i} j/4} = \frac{1}{2}\left(a_{00} + a_{01}\mathrm{e}^{\mathrm{i}\pi/2} + a_{10}\mathrm{e}^{\mathrm{i}\pi} + a_{11}\mathrm{e}^{3\mathrm{i}\pi/2}\right) \\
b_2 &= \frac{1}{2}\sum_{j=0}^{3} a_j \mathrm{e}^{4\pi \mathrm{i} j/4} = \frac{1}{2}\left(a_{00} + a_{01}\mathrm{e}^{\mathrm{i}\pi} + a_{10}\mathrm{e}^{2\mathrm{i}\pi} + a_{11}\mathrm{e}^{3\mathrm{i}\pi}\right) \\
b_3 &= \frac{1}{2}\sum_{j=0}^{3} a_j \mathrm{e}^{6\pi \mathrm{i} j/4} = \frac{1}{2}\left(a_{00} + a_{01}\mathrm{e}^{3\mathrm{i}\pi/2} + a_{10}\mathrm{e}^{3\mathrm{i}\pi} + a_{11}\mathrm{e}^{9\mathrm{i}\pi/2}\right)
\end{aligned}
$$

Example of a QFT

- Writing $\omega = \mathrm{e}^{\pi \mathrm{i}/2}$, and noting that $\omega^4 = \mathrm{e}^{2\pi\mathrm{i}} = 1$ and so, e.g. $\mathrm{e}^{9\mathrm{i}\pi/2} = \mathrm{e}^{\mathrm{i}\pi/2} = \mathrm{i}$, we can write the 2-qubit QFT in matrix form:

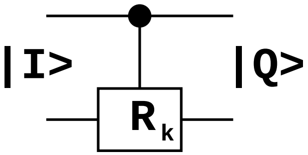
$$
F = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \end{pmatrix}
$$

- This can easily be shown to be a unitary operator
- We can build a fairly simple quantum circuit that performs this transformation

QFT Circuit

- The circuit that implements the QFT is quite simple, but has some new elements in it.
- We need to introduce some new gates:
- The Controlled-R_k gate:

$$
R_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{2\pi\mathrm{i}/2^k} \end{pmatrix}
$$

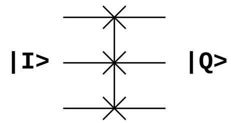


QFT Circuit

- We also need a gate which swaps the order of the qubits
- On three qubits, this gate does the following:

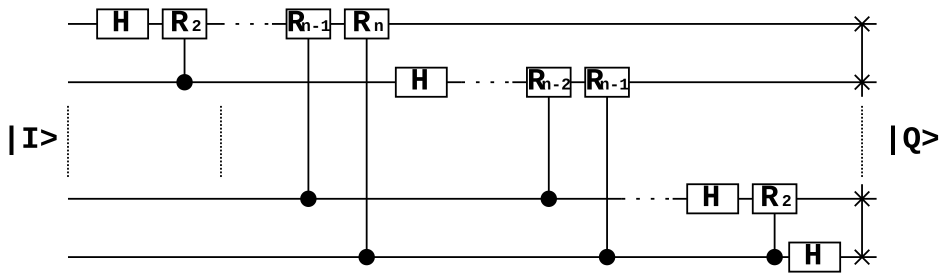
$$
\begin{aligned}
|000\rangle &\mapsto |000\rangle \\
|001\rangle &\mapsto |100\rangle \\
|010\rangle &\mapsto |010\rangle \\
|011\rangle &\mapsto |110\rangle \\
|100\rangle &\mapsto |001\rangle \\
|101\rangle &\mapsto |101\rangle \\
|110\rangle &\mapsto |011\rangle \\
|111\rangle &\mapsto |111\rangle
\end{aligned}
\quad \mapsto S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

- In the circuit, we will use the symbol:



QFT Circuit

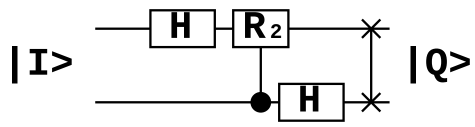
The circuit which performs the QFT is drawn as follows:



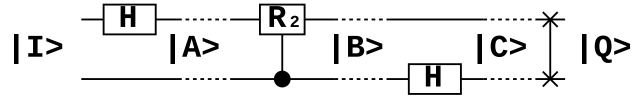
Let's look in detail at an example of this

Two Qubit QFT

The circuit which implements the two-qubit QFT is:



- What is the corresponding transformation matrix?
- We split things up as we have done previously:



- With $|A\rangle = U_1\,|I\rangle$, $|B\rangle = U_2\,|A\rangle$, $|C\rangle = U_3\,|B\rangle$ and $|Q\rangle = U_4\,|C\rangle$
- We already know that $U_2 = R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{i} \end{pmatrix}$

Two Qubit QFT

- It is easy to show that

$$U_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \quad U_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad U_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Then we have that $F = U_4 U_3 U_2 U_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \end{pmatrix}$
- This is the same as we obtained by doing the calculation directly, and we have verified that this circuit does indeed perform the QFT

Conclusions

In this lecture we have:

- Introduced the Discrete Fourier Transform
- Shown how to apply it to quantum states (the Quantum Fourier Transform
- Shown how to implement the QFT using quantum logic gates

Next lecture we will:

- Introduce the first of our applied quantum algorithms, Grover's Algorithm, which can perform searches in time $O(\sqrt{N})$