Mermin's Experiment

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1 Summary: Projectors and Measurements

Projectors P are special operators that satisfy: $P^\dagger = P$ and $P^2 = P$. The first condition holds iff $\langle x|Py\rangle = \langle Px|y\rangle$. In that case, we can unambiguously write $\langle x|P|y\rangle$ which could be interpreted as either $\langle x|Py\rangle$ or $\langle Px|y\rangle$ depending on how to choose to associate the operations.

We give the following definition of projectors P:

- $\mathbb{0}$ is a projector that maps every vector to the zero vector
- $\mathbb{1}$ is a projector that maps every vector to itself
- $|a\rangle\langle a|$ is a projector when $|a\rangle$ is a quantum state (a normalized vector)
- $\mathbb{1} - P$ is a projector
- $P_1 + P_2$ are projectors when P_1 and P_2 are orthogonal
- $P_1 P_2$ are projectors when P_1 and P_2 commute

By the Born rule, each state $|\phi\rangle$ induces a probability measure μ_ϕ which maps projectors to a probability. Specifically given a state $|\phi\rangle$ and a projector P, the probability of observing the event P is $\mu_\phi(P) = \langle\phi|P|\phi\rangle$ and the post-measurement state is $\frac{P|\phi\rangle}{\sqrt{\mu_\phi(P)}}$.

2 Mermin's Experiment

We have two particles and two detectors with each detector having three settings which we label $\rightarrow$, $\nwarrow$ and $\swarrow$. Each setting corresponds to a **yes/no** question that the detector can answer. For example, the setting $\rightarrow$ on a detector is asking the detector to decide if a particle passing through it is spinning in the $\rightarrow$-direction. If the answer to the question is **yes**, the detector flashes green, and if the answer to the question is **no** the detector flashes red. Each run of the experiment consists of preparing a quantum state, choosing settings for the two detectors, and asking whether the two detectors flash the same color or not.

Outline. Repeated runs of the experiment use a new quantum state each time and new settings for the detectors each time. This meta-experiment is outside the formalism and is more about the frequency interpretation of probability. Each individual run of the experiment deals with one quantum state and one pair of settings for the detectors and this is what we formalize. Given settings s_1 and s_2 for the two detectors, the trivial part of each run of the experiment is to prepare the quantum state:

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

This is an entangled state of two qubits which could be easily prepared starting from classical bits using a Hadamard gate and a controlled-not gate. All the action is in defining the projector that corresponds to the **yes/no** question: did the two detectors flash the same color?

Let's consider a specific experiment in which both detectors are set to $\rightarrow$. Let's denote by $P_{\rightarrow}$ the projector corresponding to the question: *is the particle spinning in the* $\rightarrow$*-direction.* The complement question: *is the particle not spinning in the* $\rightarrow$*-direction* is then denoted by $\mathbb{1} - P_{\rightarrow}$ which we abbreviate as $P_{\uparrow}$ using the visual cue of a label of the same color and in an orthogonal direction. Using these conventions the relevant projector which we denote using $P_{\rightarrow\rightarrow}$ is therefore:

$$P_{\rightarrow\rightarrow} = (P_{\rightarrow} \otimes P_{\rightarrow}) + (P_{\uparrow} \otimes P_{\uparrow})$$

As discussed before $P_1 \otimes P_2$ is equivalent to $(P_1 \otimes \mathbb{1})(\mathbb{1} \otimes P_2)$ which is a valid projector as it is the product of two commuting projectors. It is also straightforward to check that $P_{\rightarrow\rightarrow}$ is well-formed as it is the sum of two orthogonal projectors.

To summarize, the projector $P_{\rightarrow\rightarrow}$ is the projector associated with an experiment in which both detectors are set to $\rightarrow$ and it corresponds to the question: will the two detectors flash the same color (either because both detectors answer **yes** or both answer **no**). We have in total, 9 possible experiments each with a different setting for the detectors and hence each with a different projector. Using the notation above, the 9 projectors are:

$$\begin{array}{rcccc}
P_{\rightarrow\rightarrow} & = & (P_{\rightarrow} \otimes P_{\rightarrow}) & + & (P_{\uparrow} \otimes P_{\uparrow}) \\
P_{\nwarrow\nwarrow} & = & (P_{\nwarrow} \otimes P_{\nwarrow}) & + & (P_{\textcolor{blue}{\swarrow}} \otimes P_{\textcolor{blue}{\swarrow}}) \\
P_{\textcolor{red}{\swarrow}\textcolor{red}{\swarrow}} & = & (P_{\textcolor{red}{\swarrow}} \otimes P_{\textcolor{red}{\swarrow}}) & + & (P_{\searrow} \otimes P_{\searrow}) \\
P_{\rightarrow\nwarrow} & = & (P_{\rightarrow} \otimes P_{\nwarrow}) & + & (P_{\uparrow} \otimes P_{\textcolor{blue}{\swarrow}}) \\
P_{\rightarrow\textcolor{red}{\swarrow}} & = & (P_{\rightarrow} \otimes P_{\textcolor{red}{\swarrow}}) & + & (P_{\uparrow} \otimes P_{\searrow}) \\
P_{\nwarrow\rightarrow} & = & (P_{\nwarrow} \otimes P_{\rightarrow}) & + & (P_{\textcolor{blue}{\swarrow}} \otimes P_{\uparrow}) \\
P_{\nwarrow\textcolor{red}{\swarrow}} & = & (P_{\nwarrow} \otimes P_{\textcolor{red}{\swarrow}}) & + & (P_{\textcolor{blue}{\swarrow}} \otimes P_{\searrow}) \\
P_{\textcolor{red}{\swarrow}\rightarrow} & = & (P_{\textcolor{red}{\swarrow}} \otimes P_{\rightarrow}) & + & (P_{\searrow} \otimes P_{\uparrow}) \\
P_{\textcolor{red}{\swarrow}\nwarrow} & = & (P_{\textcolor{red}{\swarrow}} \otimes P_{\nwarrow}) & + & (P_{\searrow} \otimes P_{\textcolor{blue}{\swarrow}})
\end{array}$$

Each projector is labeled with the relevant setting of the detectors and each projector corresponds, in the setting of its particular experiment, to the question of whether the two detectors are flashing the same color.

The Actual Projectors. As Mermin suggests, the three projectors $P_{\rightarrow}$, $P_{\nwarrow}$, and $P_{\textcolor{red}{\swarrow}}$ represent three subspaces differing by an orientation of $\frac{2\pi}{3}$. We will identify each direction with a point on the Block sphere. Recall the general state of a qubit on the Bloch sphere:

$$\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

We will fix $\phi = 0$ and vary θ to be 0 which we choose to represent the direction $\rightarrow$, $-\frac{2\pi}{3}$ which we choose to represent the direction $\nwarrow$, and $\frac{2\pi}{3}$ which we choose to represent the direction $\textcolor{red}{\swarrow}$. Remember that varying θ like this involves rotations around the Y-axis and that the general rotation around the Y-axis by angle θ is given by the unitary operator $R_y(\theta)$ defined below:

$$\begin{array}{rcl}
R_y(\theta)|0\rangle & = & \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle \\[2ex]
R_y(\theta)|1\rangle & = & -\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle
\end{array}$$

This rotation operation allows us to generate our 3 distinguished directions by rotating $|0\rangle$ around the Y-axis with the three angles above:

$$\begin{array}{rcccl}
|\psi_{\rightarrow}\rangle & = & R_y(0)|0\rangle & = & \cos 0|0\rangle + \sin 0|1\rangle = |0\rangle \\[2ex]
|\psi_{\nwarrow}\rangle & = & R_y(\frac{-2\pi}{3})|0\rangle & = & \cos\frac{\pi}{3}|0\rangle - \sin\frac{\pi}{3}|1\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle \\[2ex]
|\psi_{\textcolor{red}{\swarrow}}\rangle & = & R_y(\frac{2\pi}{3})|0\rangle & = & \cos\frac{\pi}{3}|0\rangle + \sin\frac{\pi}{3}|1\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle
\end{array}$$

And we finally have our three projectors:

- $P_{\rightarrow} = |\psi_{\rightarrow}\rangle\langle\psi_{\rightarrow}| = |0\rangle\langle 0|$
- $P_{\nwarrow} = |\psi_{\nwarrow}\rangle\langle\psi_{\nwarrow}| = \frac{1}{4}|0\rangle\langle 0| - \frac{\sqrt{3}}{4}|0\rangle\langle 1| - \frac{\sqrt{3}}{4}|1\rangle\langle 0| + \frac{3}{4}|1\rangle\langle 1|$
- $P_{\swarrow} = |\psi_{\swarrow}\rangle\langle\psi_{\swarrow}| = \frac{1}{4}|0\rangle\langle 0| + \frac{\sqrt{3}}{4}|0\rangle\langle 1| + \frac{\sqrt{3}}{4}|1\rangle\langle 0| + \frac{3}{4}|1\rangle\langle 1|$

and their complements:

- $P_{\uparrow} = \mathbb{1} - P_{\rightarrow} = (|0\rangle\langle 0| + |1\rangle\langle 1|) - |0\rangle\langle 0| = |1\rangle\langle 1|$
- $P_{\swarrow} = \mathbb{1} - P_{\nwarrow} == \frac{3}{4}|0\rangle\langle 0| + \frac{\sqrt{3}}{4}|0\rangle\langle 1| + \frac{\sqrt{3}}{4}|1\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$
- $P_{\searrow} = \mathbb{1} - P_{\swarrow} == \frac{3}{4}|0\rangle\langle 0| - \frac{\sqrt{3}}{4}|0\rangle\langle 1| - \frac{\sqrt{3}}{4}|1\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$

Calculating the Probabilities. Now that we know the state of the quantum system $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and we know the 9 projectors P_i for the 9 experiments, it is a matter of using the Born rule $\langle\phi|P_i|\phi\rangle$ to calculate the probabilities of both detectors flashing the same color for each of the possible 9 experiments. We show three of the calculations below.

Experiment $\rightarrow\rightarrow$. The associated projector is:

$$\begin{aligned} P_{\rightarrow\rightarrow} &= (P_{\rightarrow} \otimes P_{\rightarrow}) + (P_{\uparrow} \otimes P_{\uparrow}) \\ &= |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \\ &= |00\rangle\langle 00| + |11\rangle\langle 11| \end{aligned}$$

The associated probability is $\langle\phi|P_{\rightarrow\rightarrow}|\phi\rangle\rangle = \frac{1}{\sqrt{2}}\langle\phi|(|00\rangle + |11\rangle) = \frac{1}{2}(\langle 00|00\rangle + \langle 11|11\rangle) = 1$.

Experiment $\nwarrow\nwarrow$. When calculating the associated operator we only focus on the projectors that will actually interact with $|\phi\rangle$:

$$\begin{aligned} P_{\nwarrow\nwarrow} &= (P_{\nwarrow} \otimes P_{\nwarrow}) + (P_{\swarrow} \otimes P_{\swarrow}) \\ &= \ldots + \tfrac{1}{16}|00\rangle\langle 00| + \tfrac{3}{16}|00\rangle\langle 11| + \tfrac{3}{16}|11\rangle\langle 00| + \tfrac{1}{16}|11\rangle\langle 11| \\ &\quad + \tfrac{9}{16}|00\rangle\langle 00| + \tfrac{3}{16}|00\rangle\langle 11| + \tfrac{3}{16}|11\rangle\langle 00| + \tfrac{9}{16}|11\rangle\langle 11| \\ &= \ldots + \tfrac{10}{16}|00\rangle\langle 00| + \tfrac{6}{16}|00\rangle\langle 11| + \tfrac{6}{16}|11\rangle\langle 00| + \tfrac{10}{16}|11\rangle\langle 11| \end{aligned}$$

The associated probability is:

$$\frac{1}{2}\langle\phi|(\frac{16}{16}|00\rangle + \frac{16}{16}|11\rangle) = \frac{1}{2}\langle\phi|(|00\rangle + |11\rangle) = 1$$

Experiment $\rightarrow\nwarrow$. The associated projector is:

$$\begin{aligned} P_{\rightarrow\nwarrow} &= (P_{\rightarrow} \otimes P_{\nwarrow}) + (P_{\uparrow} \otimes P_{\swarrow}) \\ &= \tfrac{1}{4}|00\rangle\langle 00| + \tfrac{1}{4}|11\rangle\langle 11| \end{aligned}$$

The associated probability is:

$$\frac{1}{2}\langle\phi|(\frac{1}{4}|00\rangle + \frac{1}{4}|11\rangle) = \frac{1}{2}(\frac{1}{4} + \frac{1}{4}) = \frac{1}{4}$$

3 Questions

- Calculate the probability of both detectors flashing the same color for the remaining 6 experiments.
- Now that you have seen the precise mathematical description of the experiment, do you have any new insights regarding the *mystery*?
- Implement the 9 experiments in Qiskit. Experiment-$\rightarrow\rightarrow$ uses the default projector so it is straightforward to realize in Qiskit using the build-in measurement operator. We illustrate how to realize a measurement using another projector below.

  First our projector operators are all of the form $|R|0\rangle\rangle\langle R|0\rangle|$ for some unitary operator R. It is important to recall that for unitary operators R, we have $R^\dagger = R^{-1}$ and $\langle a|b\rangle = \langle Ra|Rb\rangle$. Now our desired probabilities involve computations $\langle\phi|R|0\rangle\rangle\,\langle R|0\rangle|\phi\rangle$ and these can be simplified as follows:

$$
\begin{aligned}
\langle\phi|R|0\rangle\rangle\,\langle R|0\rangle|\phi\rangle &= \left\langle R^\dagger\phi\middle|R^\dagger R|0\rangle\right\rangle\left\langle R^\dagger R|0\rangle\middle|R^\dagger\phi\right\rangle \\
&= \left\langle R^\dagger\phi\middle|0\right\rangle\left\langle 0\middle|R^\dagger\phi\right\rangle
\end{aligned}
$$

  In other words, instead of using projector $|R|0\rangle\rangle\langle R|0\rangle|$ on state $|\phi\rangle$, it is equivalent to use projector $|0\rangle\langle 0|$ on state $R^\dagger|\phi\rangle$.