Quantum Measurement

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1 Classical Probability Spaces

A *probability space* specifies the necessary conditions for reasoning coherently about collections of uncertain events. The conventional definition of a probability space builds upon the field of real numbers. In more detail, a probability space consists of a *sample space* Ω, a space of *events* $\mathcal{E}$, and a *probability measure* μ mapping events in $\mathcal{E}$ to the real interval $[0,1]$. In these notes, we will only consider *finite* sets of events: we therefore restrict our attention to non-empty finite sets Ω as the sample space. The space of events $\mathcal{E}$ includes every possible subset of Ω: it is the powerset $2^\Omega = \{E \mid E \subseteq \Omega\}$.

Definition 1 (Probability Measure). Given the set of events $\mathcal{E}$, a *probability measure* is a function $\mu : \mathcal{E} \to [0,1]$ such that:

- $\mu(\varnothing) = 0$.
- $\mu(\Omega) = 1$.
- For any event E, the probability of the complement event $\Omega \backslash E$ is $\mu(\Omega \backslash E) = 1 - \mu(E)$
- For each pair of disjoint events $E_1 \cap E_2 = \varnothing$, we have that $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$.

□

Example 1 (Two-coins experiment). Consider an experiment that tosses two coins. We have four possible outcomes that constitute the sample space $\Omega = \{HH, HT, TH, TT\}$. There are 16 total events including for example the event $\{HH, HT\}$ that the first coin lands heads up, the event $\{HT, TH\}$ that the two coins land on opposite sides, and the event $\{HT, TH, TT\}$ that at least one coin lands tails up. Here is a possible probability measure for these events:

$$
\begin{array}{rclrcl}
\mu(\varnothing) & = & 0 & \mu(\{HT,TH\}) & = & 2/3 \\
\mu(\{HH\}) & = & 1/3 & \mu(\{HT,TT\}) & = & 0 \\
\mu(\{HT\}) & = & 0 & \mu(\{TH,TT\}) & = & 2/3 \\
\mu(\{TH\}) & = & 2/3 & \mu(\{HH,HT,TH\}) & = & 1 \\
\mu(\{TT\}) & = & 0 & \mu(\{HH,HT,TT\}) & = & 1/3 \\
\mu(\{HH,HT\}) & = & 1/3 & \mu(\{HH,TH,TT\}) & = & 1 \\
\mu(\{HH,TH\}) & = & 1 & \mu(\{HT,TH,TT\}) & = & 2/3 \\
\mu(\{HH,TT\}) & = & 1/3 & \mu(\{HH,HT,TH,TT\}) & = & 1
\end{array}
$$

The assignment satisfies the two constraints for probability measures: the probability of the entire sample space is 1, and the probability of every collection of disjoint events (e.g., $\{HT\} \cup \{TH\} = \{HT, TH\}$) is the sum of the individual probabilities. The probability of collections of non-disjoint events (e.g., $\{HT, TH\} \cup \{TH, TT\} = \{HT, TH, TT\}$) may add to something different than the probabilities of the individual events. It is useful to think that this probability measure is completely determined by the two coins in question and their characteristics, in the sense that each pair of coins induces a measure, and each measure must correspond to some pair of coins. The measure above is induced by two coins such that the first coin is twice as likely to land tails up than heads up and the second coin is double-headed. □

Although specifying a probability measure for every event looks complex, it can be simply constructed by

$$\mu(E) = \sum_{\omega \in E} m(\omega) \ , \tag{1}$$

where $m : \Omega \to [0,1]$, $\sum_{\omega \in \Omega} m(\omega) = 1$, and m is called the Radon-Nikodym derivative of μ. The converse is also valid and called the Radon-Nikodym theorem.

Theorem 1 (Radon-Nikodym theorem for finite probability space)*.* For every probability measure μ, there exists a unique Radon-Nikodym derivative m such that equation (1) holds. □

For example, the Radon-Nikodym derivative of μ in example 1 is:

$$m(HH) = 1/3, \quad m(HT) = 0, \quad m(TH) = 2/3, \quad m(TT) = 0 \ .$$

2 Quantum Probability Spaces

The mathematical framework above assumes that there exists a predetermined set of events that are generated uniformly from the sample space in a way that is independent of the particular experiment. However, in many practical situations, the structure of the event space is only partially known and the precise dependence of two events on each other cannot, a priori, be determined with certainty. In the quantum framework, this partial knowledge is compounded by the fact that there exist non-commuting events which cannot happen simultaneously. To accommodate these more complex situations, we abandon the sample space Ω and reason directly about events. A quantum probability space therefore consists of just two components: a set of events $\mathcal{E}$ and a probability measure $\mu : \mathcal{E} \to [0,1]$. To properly explain the quantum probability, we will first discuss projection operators as quantum events.

Definition 2 (Projection Operators; Orthogonality)*.* Given a Hilbert space $\mathcal{H}$, an event[1] mathematically is represented as a projection operator $P : \mathcal{H} \to \mathcal{H}$ onto a linear subspace S of $\mathcal{H}$. A *projection operators* P satisfies the condition $P^2 = P$.

The set of all events $\mathcal{E}$ can be defined recursively as follow: [2]

- $\mathbb{0}$ is a projection (the empty projection)
- $\mathbb{1}$ is a projection (the identity projection)
- For any pure state $|\psi\rangle$, $|\psi\rangle\langle\psi|$ is a projection operator.
- Sums of *orthogonal* projections P_0 and P_1 with $P_0 P_1 = P_1 P_0 = \mathbb{0}$ are projections,
- Products of *commuting* projections P_0 and P_1 with $P_0 P_1 = P_1 P_0$ are projections.

□

Because quantum events are projection operators, operations and properties of quantum events can be written in terms of those of operators.

Definition 3 (Ideal Measurement; Complement; Commutativity; disjunction)*.*

- A set of mutually orthogonal projections $\{P_i\}_{i=0}^{N-1}$ is called an *ideal measurement* if it is a partition of the identity, i.e., $\sum_{i=0}^{N-1} P_i = \mathbb{1}$.
- If P is a projection operator, then $\mathbb{1} - P$ is also a projection operator, called the *complement* of P. It is orthogonal to P, and corresponds to the complement event $\Omega \backslash E$ in classical probability. The product of commuting projections corresponds to the classical intersection between sets.

[1] An event is formally called an experimental proposition, a question, or an elementary quantum test.
[2] "Projection" is sometimes called "orthogonal projection" or "self-adjoint projection" to emphasize $P^\dagger = P$.

- The union $\cup$ of any two events always gives an event classically, but the operator addition $+$ of two projections may not be a projection. For two commuting projection operators P_0 and P_1, their *disjunction* $P_0 \vee P_1$ is defined to be $P_0 + P_1 - P_0 P_1$.

$\square$

Definition 4 (Quantum Probability Measure). Given a Hilbert space $\mathcal{H}$ with its set of events $\mathcal{E}$, a *quantum probability measure* is a function $\mu : \mathcal{E} \to [0,1]$ such that:

- $\mu(\mathbb{0}) = 0$.
- $\mu(\mathbb{1}) = 1$.
- For any projection P, $\mu(\mathbb{1} - P) = 1 - \mu(P)$
- For each pair of *orthogonal* projections P_0 and P_1: $\mu(P_0 + P_1) = \mu(P_0) + \mu(P_1)$

$\square$

Example 2 (One-qubit quantum probability space). Consider a one-qubit Hilbert space with states $|\phi\rangle$, which can be expressed as a linear combination of $|0\rangle$ and $|1\rangle$, i.e., $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ such that $|\alpha|^2 + |\beta|^2 = 1$, $\alpha, \beta \in \mathbb{C}$. The set of events associated with this Hilbert space consists of all projection operators. Each event is interpreted as a possible post-measurement state of a quantum system in current state $|\phi\rangle$. For example, the event $|0\rangle\langle 0|$ indicates that the post-measurement state will be $|0\rangle$; the event $|1\rangle\langle 1|$ indicates that the post-measurement state will be $|1\rangle$; the event $|+\rangle\langle +|$ where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ indicates that the post-measurement state will be $|+\rangle$; the event $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$ indicates that the post-measurement state will be a linear combination of $|0\rangle$ and $|1\rangle$; and the empty event $\mathbb{0}$ states that the post-measurement state will be the empty state. As in the classical case, a probability measure is a function that maps events to $[0,1]$. Here is a partial specification of a possible probability measure:

$$\mu(\mathbb{0}) = 0, \quad \mu(\mathbb{1}) = 1, \quad \mu(|0\rangle\langle 0|) = 1, \quad \mu(|1\rangle\langle 1|) = 0, \quad \mu(|+\rangle\langle +|) = 1/2, \quad \ldots$$

Note that, similarly to the classical case, the probability of $\mathbb{1}$ is 1 and the probability of collections of orthogonal events (e.g., $|0\rangle\langle 0| + |1\rangle\langle 1|$) is the sum of the individual probabilities. A collection of non-orthogonal events (e.g., $|0\rangle\langle 0|$ and $|+\rangle\langle +|$) is however not even a valid event. $\square$

In the classical example, we argued that each probability measure is uniquely determined by two actual coins. A similar (but much more subtle) argument is valid also in the quantum case. By the postulates of quantum mechanics, it turns out that for large enough quantum systems, each probability measure is uniquely determined by an actual quantum state. In more detail, recall that a classical probability measure can be constructed from its Radon-Nikodym derivative. Similarly, a quantum probability measure can be constructed by states according to the Born rule. For each pure normalized ($\langle\phi|\phi\rangle = 1$) quantum state $|\phi\rangle$, the Born rule induces a probability measure μ_ϕ^B as follows:

$$\mu_\phi^B(P) = \langle\phi|P\phi\rangle \ .$$

We conclude with two remarkable theorems of quantum information.

Theorem 2 (Gleason's Theorem.). Given a Hilbert space of dimension $d \geq 3$ every probability measure μ is consistent with exactly one state and the correspondence is given by the Born rule.

Theorem 3 (The Kochen-Specker Theorem.). Given a Hilbert space of dimension $d \geq 3$ there is no probability measure μ that maps every event to 0 or 1.