Quantum Measurement (II)

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1 Notation

An operator O on a Hilbert space is a map from vectors to vectors. We write $O^\dagger$ for the adjoint of O mapping dual vectors to dual vectors. Recall that in the bra-ket notation, we write $\langle x|$ for $|x\rangle^\dagger$. It follows that if O maps $|x\rangle$ to $|y\rangle$, then $O^\dagger$ maps $\langle x|$ to $\langle y|$.

2 Definition of Projectors

Projectors P are special operators that satisfy: $P^\dagger = P$ (i.e., they are Hermitian operators) and $PP = P$. The first condition holds iff $\langle x|Py\rangle = \langle Px|y\rangle$. In that case, we can unambiguously write $\langle x|P|y\rangle$ which could be interpreted as either $\langle x|Py\rangle$ or $\langle Px|y\rangle$ depending on how to choose to associate the operations.

Example 0. $P\,|x\rangle = \bullet$ (the zero vector)
Trivially $\langle x|Py\rangle = 0 = \langle Px|y\rangle$
Trivially $PP = P$
This is called the *empty* projector $\mathbb{0}$.

Example I. $P\,|x\rangle = |x\rangle$
Trivially $\langle x|Py\rangle = \langle x|y\rangle = \langle Px|y\rangle$
Trivially $PP = P$
This is called the *identity* projector $\mathbb{1}$.

Example II. $P = |a\rangle\langle a|$.
$|Py\rangle = |a\rangle\,\langle a|y\rangle$ and $\langle Px| = \langle x|a\rangle\,\langle a|$.
Then $\langle x|Py\rangle = \langle x|a\rangle\,\langle a|y\rangle$ and $\langle Px|y\rangle = \langle x|a\rangle\,\langle a|y\rangle$
$PP = (|a\rangle\langle a|)(|a\rangle\langle a|) = |a\rangle\,\langle a|a\rangle\,\langle a| = |a\rangle\langle a| = P$.

Example III. $P = |a\rangle\langle a| + |b\rangle\langle b|$ with $a \neq b$

$$\begin{aligned}
\langle x|Py\rangle &= \langle x|a\rangle\,\langle a|y\rangle + \langle x|b\rangle\,\langle a|y\rangle \\
\langle Px|y\rangle &= \langle x|a\rangle\,\langle a|y\rangle + \langle x|b\rangle\,\langle a|y\rangle
\end{aligned}$$

$$\begin{aligned}
PP &= (|a\rangle\langle a| + |b\rangle\langle b|)(|a\rangle\langle a| + |b\rangle\langle b|) \\
&= (|a\rangle\langle a|)(|a\rangle\langle a|) + (|a\rangle\langle a|)(|b\rangle\langle b|) + (|b\rangle\langle b|)(|a\rangle\langle a|) + (|b\rangle\langle b|)(|b\rangle\langle b|) \\
&= |a\rangle\langle a| + |b\rangle\langle b|
\end{aligned}$$

It is crucial that $\langle a|b\rangle = 0$, i.e., that $|a\rangle$ and $|b\rangle$ are orthogonal.

Example IV. $P = P_1 + P_2$ with $P_1 P_2 = P_2 P_1 = \mathbb{0}$

$$\begin{aligned}
\langle x|(P_1 + P_2)y\rangle &= \langle x|P_1 y\rangle + \langle x|P_2 y\rangle \\
&= \langle P_1 x|y\rangle + \langle P_2 x|y\rangle \qquad \text{induction} \\
\langle (P_1 + P_2)x|y\rangle &= \langle P_1 x|y\rangle + \langle P_2 x|y\rangle
\end{aligned}$$

$$\begin{aligned}
PP &= (P_1 + P_2)(P_1 + P_2) \\
&= P_1P_1 + P_1P_2 + P_2P_1 + P_2P_2 \\
&= P_1P_1 + P_2P_2 \\
&= P_1 + P_2 \\
&= P
\end{aligned}$$

Sums of *orthogonal* projectors are also projectors.

Example V. $P = P_1 P_2 = P_2 P_1$

$$\begin{aligned}
\langle x|P_1P_2 y\rangle &= \langle P_1 x|P_2 y\rangle \qquad \text{induction} \\
&= \langle P_2 P_1 x|y\rangle \qquad \text{induction} \\
&= \langle P_1 P_2 x|y\rangle \qquad \text{assumption}
\end{aligned}$$

$$\begin{aligned}
PP &= (P_1P_2)(P_1P_2) \\
&= P_1(P_2P_1)P_2 \\
&= P_1(P_1P_2)P_2 \\
&= (P_1P_1)(P_2P_2) \\
&= P_1P_2 \\
&= P
\end{aligned}$$

Products of *commuting* projectors are also projectors.

Example VI. Let P be a projector. Then $\mathbb{1} - P$ is a projector.
$\langle x|(\mathbb{1} - P)y\rangle = \langle x|y\rangle - \langle x|Py\rangle = \langle x|y\rangle - \langle Px|y\rangle = \langle (\mathbb{1} - P)x|y\rangle$
$(\mathbb{1} - P)(\mathbb{1} - P) = (\mathbb{1} - 2P + PP) = (\mathbb{1} - 2P + P) = (\mathbb{1} - P)$

Example. $P = |00\rangle\langle 00| + |01\rangle\langle 01|$ (valid because $\langle 00|01\rangle = 0$.)
Note: $|00\rangle\langle 00| + |01\rangle\langle 01| = |0\rangle\langle 0| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) = |0\rangle\langle 0| \otimes \mathbb{1}$

Example. $P = P_1 P_2$ where $P_1 = |00\rangle\langle 00| + |01\rangle\langle 01|$ and $P_2 = |01\rangle\langle 01| + |11\rangle\langle 11|$.
Commutative because $P_1 = |0\rangle\langle 0| \otimes \mathbb{1}$ and $P_2 = \mathbb{1} \otimes |1\rangle\langle 1|$ and hence $P_1P_2 = P_2P_1 = |0\rangle\langle 0| \otimes |1\rangle\langle 1|$.