Lecture of March 31st 2020

March 31, 2020

1 Basics

We express the action of some of the basic gates in both the computational basis and in the hadamard basis.

$$\begin{aligned}
X|0\rangle &= |1\rangle \\
X|1\rangle &= |0\rangle
\end{aligned}$$

$$\begin{aligned}
X|+\rangle &= X(|0\rangle + |1\rangle) = |1\rangle + |0\rangle \\
&= |+\rangle \\
X|-\rangle &= X(|0\rangle - |1\rangle) = |1\rangle - |0\rangle = -(|0\rangle - |1\rangle) \\
&= -|-\rangle
\end{aligned}$$

$$\begin{aligned}
H|0\rangle &= |+\rangle \\
H|1\rangle &= |-\rangle
\end{aligned}$$

$$\begin{aligned}
H|+\rangle &= H(|0\rangle + |1\rangle) = H|0\rangle + H|1\rangle = |0\rangle + |1\rangle + |0\rangle - |1\rangle \\
&= |0\rangle \\
H|-\rangle &= H(|0\rangle - |1\rangle) = H|0\rangle - H|1\rangle = |0\rangle + |1\rangle - |0\rangle + |1\rangle \\
&= |1\rangle
\end{aligned}$$

$$\begin{aligned}
CX|0x\rangle &= |0x\rangle \\
CX|1x\rangle &= |1\overline{x}\rangle \qquad \text{where } \overline{\cdot} \text{ is negation by applying } X
\end{aligned}$$

$$\begin{aligned}
CX|+x\rangle &= CX(|0x\rangle + |1x\rangle) = CX|0x\rangle + CX|1x\rangle = |0x\rangle + |1\overline{x}\rangle \\
CX|-x\rangle &= CX(|0x\rangle - |1x\rangle) = CX|0x\rangle - CX|1x\rangle = |0x\rangle - |1\overline{x}\rangle
\end{aligned}$$

$$\begin{aligned}
CX|++\rangle &= |0+\rangle + |1+\rangle = (|0\rangle + |1\rangle)|+\rangle \\
&= |++\rangle \\
CX|+-\rangle &= |0-\rangle - |1-\rangle = (|0\rangle - |1\rangle)|-\rangle \\
&= |--\rangle \\
CX|-+\rangle &= |0+\rangle - |1+\rangle = (|0\rangle - |1\rangle)|+\rangle \\
&= |-+\rangle \\
CX|--\rangle &= |0-\rangle + |1-\rangle = (|0\rangle + |1\rangle)|-\rangle \\
&= |+-\rangle
\end{aligned}$$

2 Two Optimizations

The figure below shows two optimizations that we want to verify:

(a) An implementation of *cnot*(q_1, q_4)   (b) *SWAP*(q_0, q_4)

Figure 3: An equivalent circuit of cnot(q_1, q_4), where *SWAP*(q_0, q_4) is implemented by (b).

In order to avoid confusion we will first rewrite the figures in textual form:

$$\begin{array}{rcl} \mathit{Circuitb}|q_0q_4\rangle & = & |q_4q_0\rangle \leftarrow CX|q_4q_0\rangle \\ & & |q_4q_0\rangle \leftarrow (H \otimes H)|q_4q_0\rangle \\ & & |q_4q_0\rangle \leftarrow CX|q_4q_0\rangle \\ & & |q_4q_0\rangle \leftarrow (H \otimes H)|q_4q_0\rangle \\ & & |q_4q_0\rangle \leftarrow CX|q_4q_0\rangle \\ & & \mathit{return}|q_0q_4\rangle \end{array}$$

The convention is that the circuit is a sequence of gate applications, one on each line. Each gate's input and output are named to avoid confusion. To confirm that the circuit performs a swap, let's symbolically execute it on each possible value starting with $|00\rangle$. The initial state of the execution can be written as where only the inputs to the first gate are known:

$$\begin{array}{rcl} \mathit{Circuitb}|00\rangle & = & |q_4q_0\rangle \leftarrow CX|00\rangle \\ & & |q_4q_0\rangle \leftarrow (H \otimes H)|q_4q_0\rangle \\ & & |q_4q_0\rangle \leftarrow CX|q_4q_0\rangle \\ & & |q_4q_0\rangle \leftarrow (H \otimes H)|q_4q_0\rangle \\ & & |q_4q_0\rangle \leftarrow CX|q_4q_0\rangle \\ & & \mathit{return}|q_0q_4\rangle \end{array}$$

After the first gate is applied, we know its result is $|00\rangle$ which becomes the input to the next gate (not that we match names not positions):

$$\begin{array}{rcl} \mathit{Circuitb}|00\rangle & = & |q_4q_0\rangle \leftarrow CX|00\rangle \\ & & |q_4q_0\rangle \leftarrow (H \otimes H)|00\rangle \\ & & |q_4q_0\rangle \leftarrow CX|q_4q_0\rangle \\ & & |q_4q_0\rangle \leftarrow (H \otimes H)|q_4q_0\rangle \\ & & |q_4q_0\rangle \leftarrow CX|q_4q_0\rangle \\ & & \mathit{return}|q_0q_4\rangle \end{array}$$

Execution then continues to produce:

$$\begin{array}{rcl} \mathit{Circuitb}|00\rangle & = & |q_4q_0\rangle \leftarrow CX|00\rangle \\ & & |q_4q_0\rangle \leftarrow (H \otimes H)|00\rangle \\ & & |q_4q_0\rangle \leftarrow CX|++\rangle \\ & & |q_4q_0\rangle \leftarrow (H \otimes H)|++\rangle \\ & & |q_4q_0\rangle \leftarrow CX|00\rangle \\ & & \mathit{return}|00\rangle \end{array}$$

Now we repeat for the next input $|01\rangle$:

$$
\begin{array}{rcl}
Circuitb|01\rangle & = & |q_4q_0\rangle \leftarrow CX|10\rangle \\
 & & |q_4q_0\rangle \leftarrow (H \otimes H)|11\rangle \\
 & & |q_4q_0\rangle \leftarrow CX|--\rangle \\
 & & |q_4q_0\rangle \leftarrow (H \otimes H)|+-\rangle \\
 & & |q_4q_0\rangle \leftarrow CX|01\rangle \\
 & & return|10\rangle
\end{array}
$$