Exercise 3.2. Show by example that a linear combination of entangled states is not necessarily entangled.

Exercise 3.3. Show that the state

$$|W_n\rangle = \frac{1}{\sqrt{n}}(|0\dots001\rangle + |0\dots010\rangle + |0\dots100\rangle + \cdots + |1\dots000\rangle)$$

is entangled, with respect to the decomposition into the n qubits, for every $n > 1$.

Exercise 3.4. Show that the state

$$|GHZ_n\rangle = \frac{1}{\sqrt{2}}(|00\dots0\rangle + |11\dots1\rangle)$$

is entangled, with respect to the decomposition into the n qubits, for every $n > 1$.

Exercise 3.5. Is the state $\frac{1}{\sqrt{2}}(|0\rangle|+\rangle + |1\rangle|-\rangle)$ entangled?

Exercise 3.6. If someone asks you whether the state $|+\rangle$ is entangled, what will you say?

Exercise 3.7. Write the following states in terms of the Bell basis.

a. $|00\rangle$

b. $|+\rangle|-\rangle$

c. $\frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$

Exercise 3.8.

a. Show that $\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$ and $\frac{1}{\sqrt{2}}(|+\rangle|+\rangle + |-\rangle|-\rangle)$ refer to the same quantum state.

b. Show that $\frac{1}{\sqrt{2}}(|0\rangle|0\rangle - |1\rangle|1\rangle)$ refers to the same state as $\frac{1}{\sqrt{2}}(|\mathbf{i}\rangle|\mathbf{i}\rangle + |-\mathbf{i}\rangle|-\mathbf{i}\rangle)$.

Exercise 3.9.

a. Show that any n-qubit quantum state can be represented by a vector of the form

$$a_0|0\dots00\rangle + a_1|0\dots01\rangle + \cdots + a_{2^n-1}|1\dots11\rangle$$

where the first non-zero a_i is real and non-negative.

b. Show that this representation is unique in the sense that any two different vectors of this form represent different quantum states.

Exercise 3.10. Show that for any orthonormal basis $B = \{|\beta_1\rangle, |\beta_2\rangle, \dots, |\beta_n\rangle\}$ and vectors $|v\rangle = a_1|\beta_1\rangle + a_2|\beta_2\rangle + \cdots + a_n|\beta_n\rangle$ and $|w\rangle = c_1|\beta_1\rangle + c_2|\beta_2\rangle + \cdots + c_n|\beta_n\rangle$

a. the inner product $(w|v)$ of $|v\rangle$ and $|w\rangle$ is $\bar{c_1}a_1 + \bar{c_2}a_2 + \cdots + \bar{c_n}a_n$, and

b. the length squared of $|v\rangle$ is $\left||v\rangle\right|^2 = \langle v|v\rangle = |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2$.

Write all steps in Dirac's bra/ket notation.

Exercise 3.11. Let $|\psi\rangle$ be an n-qubit state. Show that the sum of the distances from $|\psi\rangle$ to the standard basis vectors $|j\rangle$ is bounded below by a positive constant that depends only on n,

$$\sum_j \big| |\psi\rangle - |j\rangle \big| \geq C,$$

where $|\vec{v}|$ indicates the length of the enclosed vector. Specify such a constant C in terms of n.

Exercise 3.12. Give an example of a two-qubit state that is a superposition with respect to the standard basis but that is not entangled.

Exercise 3.13.

a. Show that the four-qubit state $|\psi\rangle = \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle)$ of example 3.2.3 is entangled with respect to the decomposition into two two-qubit subsystems consisting of the first and second qubits and the third and fourth qubits.

b. For the four decompositions into two subsystems consisting of one and three qubits, say whether $|\psi\rangle$ is entangled or unentangled with respect to each of these decompositions.

Exercise 3.14.

a. For the standard basis, the Hadamard basis, and the basis $B = \{\frac{1}{\sqrt{2}}(|0\rangle + \mathbf{i}|1\rangle, \frac{1}{\sqrt{2}}|0\rangle - \mathbf{i}|1\rangle\}$, determine the probability of each outcome when the second qubit of a two-qubit system in the state $|00\rangle$ is measured in each of the bases.

b. Determine the probability of each outcome when the second qubit of the state $|00\rangle$ is first measured in the Hadamard basis and then in the basis B of part a).

c. Determine the probability of each outcome when the second qubit of the state $|00\rangle$ is first measured in the Hadamard basis and then in the standard basis.

Exercise 3.15. This exercise analyzes the effectiveness of some simple attacks an eavesdropper Eve could make on Ekert's entangled state based QKD protocol.

a. Say Eve can measure Bob's half of each of the EPR pairs before it reaches him. Say she always measures in the standard basis. Describe a method by which Alice and Bob can determine that there is only a 2^{-s} chance that this sort of interference by Eve has gone undetected. What happens if Eve instead measures each qubit randomly in either the standard basis of the Hadamard basis? What happens if she uniformly at random chooses a basis from all possible bases?

b. Say Eve can pose as the entity sending the purported EPR pairs. Say instead of sending EPR pairs she sends a random mixture of qubit pairs in the states $|00\rangle$, $|11\rangle$, $|+\rangle|+\rangle$, and $|-\rangle|-\rangle$. After Alice and Bob perform the protocol of section 3.4, on how many bits on average do their purported shared secret keys agree? On average, how many of these bits does Eve know?

description, accessible to nonphysicists, of Bell's theorem and the EPR paradox, experimental techniques for generating entangled photon pairs, and Aspect's experiments testing for quantum violation of Bell's inequalities. Detailed results of the experiments by Aspect et al. are published in [25, 26, 24].

Stronger statements than the ones we presented can be made about the sorts of theories that Bell's inequality rules out. The issues here can be relatively subtle. Mermin's article [208] gives a readable account of some of these issues. Peres's book [226] delves into these issues in detail. For a discussion of the various interpretations of quantum mechanics and their perceived strengths and weaknesses, see Sudbery's book [267] and Bub's book [71].

4.6 Exercises

Exercise 4.1. Give the matrix, in the standard basis, for the following operators

a. $|0\rangle\langle 0|$.

b. $|+\rangle\langle 0| - \mathbf{i}|-\rangle\langle 1|$.

c. $|00\rangle\langle 00| + |01\rangle\langle 01|$.

d. $|00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 01| + |10\rangle\langle 11|$.

e. $|\Psi^+\rangle\langle\Psi^+|$ where $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Exercise 4.2. Write the following operators in bra/ket notation

a. The Hadamard operator $H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

b. $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

c. $Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

d. $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

e. $\begin{pmatrix} 23 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}$.

f. $X \otimes X$.

g. $X \otimes Z$.

h. $H \otimes H$.

i. The projection operators $P_1 : V \to S_1$ and $P_2 : V \to S_2$, where S_1 is spanned by $\{|+\rangle|+\rangle, |-\rangle|-\rangle\}$ and S_2 is spanned by $\{|+\rangle|-\rangle, |-\rangle|+\rangle\}$.