Quantum Programming HW10

B490 : Spring 2020

Joshua Larkin

1 Calculating Probabilities

We know the state of the quantum system: $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. We show the probabilities of both detectors flashing the same color for all 9 experiments.

Experiment $\rightarrow\rightarrow$. The associated projector is:

$$
\begin{aligned}
P_{\rightarrow\rightarrow} &= (P_{\rightarrow} \otimes P_{\rightarrow}) + (P_{\uparrow} \otimes P_{\uparrow}) \\
&= |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \\
&= |00\rangle\langle 00| + |11\rangle\langle 11|
\end{aligned}
$$

The associated probability is $\langle\phi|P_{\rightarrow\rightarrow}|\phi\rangle\rangle = \frac{1}{\sqrt{2}}\langle\phi|(|00\rangle + |11\rangle) = \frac{1}{2}(\langle 00|00\rangle + \langle 11|11\rangle) = 1$.

Experiment $\nwarrow\nwarrow$. When calculating the associated operator we only focus on the projectors that will actually interact with $|\phi\rangle$:

$$
\begin{aligned}
P_{\nwarrow\nwarrow} &= (P_{\nwarrow} \otimes P_{\nwarrow}) + (P_{\swarrow} \otimes P_{\swarrow}) \\
&= \ldots + \tfrac{1}{16}|00\rangle\langle 00| + \tfrac{3}{16}|00\rangle\langle 11| + \tfrac{3}{16}|11\rangle\langle 00| + \tfrac{9}{16}|11\rangle\langle 11| \\
&\quad + \tfrac{9}{16}|00\rangle\langle 00| + \tfrac{3}{16}|00\rangle\langle 11| + \tfrac{3}{16}|11\rangle\langle 00| + \tfrac{1}{16}|11\rangle\langle 11| \\
&= \ldots + \tfrac{10}{16}|00\rangle\langle 00| + \tfrac{6}{16}|00\rangle\langle 11| + \tfrac{6}{16}|11\rangle\langle 00| + \tfrac{10}{16}|11\rangle\langle 11|
\end{aligned}
$$

The associated probability is:

$$
\frac{1}{2}\langle\phi|(\frac{16}{16}|00\rangle + \frac{16}{16}|11\rangle) = \frac{1}{\sqrt{2}}\langle\phi|(|00\rangle + |11\rangle) = 1
$$

Experiment $\swarrow\swarrow$. We show some work in calculating. First:

$$\begin{aligned}
P_{\swarrow} \otimes P_{\swarrow} &= (\tfrac{1}{4}|0\rangle\langle 0| + \tfrac{\sqrt{3}}{4}|0\rangle\langle 1| + \tfrac{\sqrt{3}}{4}|1\rangle\langle 0| + \tfrac{3}{4}|1\rangle\langle 1|) \otimes (\tfrac{1}{4}|0\rangle\langle 0| + \tfrac{\sqrt{3}}{4}|0\rangle\langle 1| + \tfrac{\sqrt{3}}{4}|1\rangle\langle 0| + \tfrac{3}{4}|1\rangle\langle 1|) \\
&= \cdots + \tfrac{1}{16}|00\rangle\langle 00| + \tfrac{3}{16}|00\rangle\langle 11| + \tfrac{3}{16}|11\rangle\langle 00| + \tfrac{9}{16}|11\rangle\langle 11|
\end{aligned}$$

$$\begin{aligned}
P_{\searrow} \otimes P_{\searrow} &= (\tfrac{3}{4}|0\rangle\langle 0| - \tfrac{\sqrt{3}}{4}|0\rangle\langle 1| - \tfrac{\sqrt{3}}{4}|1\rangle\langle 0| + \tfrac{1}{4}|1\rangle\langle 1|) \otimes (\tfrac{3}{4}|0\rangle\langle 0| - \tfrac{\sqrt{3}}{4}|0\rangle\langle 1| - \tfrac{\sqrt{3}}{4}|1\rangle\langle 0| + \tfrac{1}{4}|1\rangle\langle 1|) \\
&= \cdots + \tfrac{9}{16}|00\rangle\langle 00| + \tfrac{3}{16}|00\rangle\langle 11| + \tfrac{3}{16}|11\rangle\langle 00| + \tfrac{1}{16}|11\rangle\langle 11|
\end{aligned}$$

The projector:

$$\begin{aligned}
P_{\swarrow\swarrow} &= (P_{\swarrow} \otimes P_{\swarrow}) + (P_{\searrow} \otimes P_{\searrow}) \\
&= \cdots + \tfrac{10}{16}|00\rangle\langle 00| + \tfrac{6}{16}|00\rangle\langle 11| + \tfrac{6}{16}|11\rangle\langle 00| + \tfrac{10}{16}|11\rangle\langle 11|
\end{aligned}$$

Now we apply $P_{\swarrow\swarrow}$ to $|\phi\rangle$:

$$\begin{aligned}
P_{\swarrow\swarrow}|\phi\rangle &= (\tfrac{10}{16}|00\rangle\langle 00| + \tfrac{6}{16}|00\rangle\langle 11| + \tfrac{6}{16}|11\rangle\langle 00| + \tfrac{10}{16}|11\rangle\langle 11|)(\tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)) \\
&= \tfrac{1}{\sqrt{2}}(\tfrac{10}{16}|00\rangle\langle 00|00\rangle + \tfrac{10}{16}|00\rangle\langle 00|11\rangle + \tfrac{6}{16}|00\rangle\langle 11|00\rangle + \tfrac{6}{16}|00\rangle\langle 11|11\rangle \\
&\quad + \tfrac{6}{16}|11\rangle\langle 00|00\rangle + \tfrac{6}{16}|11\rangle\langle 00|11\rangle + \tfrac{10}{16}|11\rangle\langle 11|00\rangle + \tfrac{10}{16}|11\rangle\langle 11|11\rangle) \\
&= \tfrac{1}{\sqrt{2}}(\tfrac{10}{16}|00\rangle\langle 00|00\rangle + \tfrac{6}{16}|00\rangle\langle 11|11\rangle + \tfrac{6}{16}|11\rangle\langle 00|00\rangle + \tfrac{10}{16}|11\rangle\langle 11|11\rangle) \\
&= \tfrac{1}{\sqrt{2}}(\tfrac{10}{16}|00\rangle + \tfrac{6}{16}|00\rangle + \tfrac{6}{16}|11\rangle + \tfrac{10}{16}|11\rangle) \\
&= \tfrac{1}{\sqrt{2}}(\tfrac{16}{16}|00\rangle + \tfrac{16}{16}|11\rangle) \\
&= \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)
\end{aligned}$$

Next we calculate $\langle\phi|P_{\swarrow\swarrow}|\phi\rangle\rangle = \left\langle \phi \middle| \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \right\rangle$

$$\begin{aligned}
\left\langle \phi \middle| \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \right\rangle &= \left\langle \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \middle| \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \right\rangle \\
&= \tfrac{1}{2}(\langle 00|00\rangle + \langle 11|11\rangle) \\
&= \tfrac{1}{2}(1 + 1) \\
&= 1
\end{aligned}$$

Therefore the associated probability for $\langle\phi|P_{\swarrow\swarrow}|\phi\rangle\rangle$ is 1.

Experiment $\rightarrow\nwarrow$. The associated projector is:

$$\begin{aligned} P_{\rightarrow\nwarrow} &= (P_{\rightarrow} \otimes P_{\nwarrow}) + (P_{\uparrow} \otimes P_{\swarrow}) \\ &= \tfrac{1}{4}\,|00\rangle\langle 00| + \tfrac{1}{4}\,|11\rangle\langle 11| \end{aligned}$$

The associated probability is:

$$\frac{1}{2}\,\langle\phi|\,(\frac{1}{4}\,|00\rangle + \frac{1}{4}\,|11\rangle) = \frac{1}{2}(\frac{1}{4} + \frac{1}{4}) = \frac{1}{4}$$

Experiment $\rightarrow\swarrow$. We show some work in calculating. First:

$$\begin{aligned} P_{\rightarrow} \otimes P_{\swarrow} &= |0\rangle\langle 0| \otimes (\tfrac{1}{4}\,|0\rangle\langle 0| + \tfrac{\sqrt{3}}{4}\,|0\rangle\langle 1| + \tfrac{\sqrt{3}}{4}\,|1\rangle\langle 0| + \tfrac{3}{4}\,|1\rangle\langle 1|) \\ &= \cdots + \tfrac{1}{4}\,|00\rangle\langle 00| \\ P_{\uparrow} \otimes P_{\searrow} &= |1\rangle\langle 1| \otimes (\tfrac{3}{4}\,|0\rangle\langle 0| - \tfrac{\sqrt{3}}{4}\,|0\rangle\langle 1| - \tfrac{\sqrt{3}}{4}\,|1\rangle\langle 0| + \tfrac{1}{4}\,|1\rangle\langle 1|) \\ &= \cdots + \tfrac{1}{4}\,|11\rangle\langle 11| \end{aligned}$$

The projector:

$$\begin{aligned} P_{\rightarrow\swarrow} &= (P_{\rightarrow} \otimes P_{\swarrow}) + (P_{\uparrow} \otimes P_{\searrow}) \\ &= \tfrac{1}{4}\,|00\rangle\langle 00| + \tfrac{1}{4}\,|11\rangle\langle 11| \end{aligned}$$

Now we apply $P_{\rightarrow\swarrow}$ to $|\phi\rangle$:

$$\begin{aligned} P_{\rightarrow\swarrow}\,|\phi\rangle &= (\tfrac{1}{4}\,|00\rangle\langle 00| + \tfrac{1}{4}\,|11\rangle\langle 11|)(\tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)) \\ &= \tfrac{1}{\sqrt{2}}(\tfrac{1}{4}\,|00\rangle\,\langle 00|00\rangle + \tfrac{1}{4}\,|00\rangle\,\langle 00|11\rangle + \tfrac{1}{4}\,|11\rangle\,\langle 11|00\rangle + \tfrac{1}{4}\,|11\rangle\,\langle 11|11\rangle) \\ &= \tfrac{1}{\sqrt{2}}(\tfrac{1}{4}\,|00\rangle\,\langle 00|00\rangle + \tfrac{1}{4}\,|11\rangle\,\langle 11|11\rangle) \\ &= \tfrac{1}{\sqrt{2}}(\tfrac{1}{4}\,|00\rangle + \tfrac{1}{4}\,|11\rangle) \end{aligned}$$

Next we calculate $\langle\phi|P_{\rightarrow\swarrow}\,|\phi\rangle\rangle = \left\langle\phi\middle|\tfrac{1}{\sqrt{2}}(\tfrac{1}{4}\,|00\rangle + \tfrac{1}{4}\,|11\rangle)\right\rangle$

$$\begin{aligned} \left\langle\phi\middle|\tfrac{1}{\sqrt{2}}(\tfrac{1}{4}\,|00\rangle + \tfrac{1}{4}\,|11\rangle)\right\rangle &= \left\langle\tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\middle|\tfrac{1}{\sqrt{2}}(\tfrac{1}{4}\,|00\rangle + \tfrac{1}{4}\,|11\rangle)\right\rangle \\ &= \tfrac{1}{2}\left\langle|00\rangle + |11\rangle\middle|\tfrac{1}{4}(|00\rangle + |11\rangle)\right\rangle \\ &= \tfrac{1}{2}(\tfrac{1}{4}\,\langle|00\rangle + |11\rangle|00\rangle + \tfrac{1}{4}\,\langle|00\rangle + |11\rangle|11\rangle) \\ &= \tfrac{1}{8}(\langle 00|00\rangle + \langle 11|00\rangle + \langle 00|11\rangle + \langle 11|11\rangle) \\ &= \tfrac{1}{8}(\langle 00|00\rangle + \langle 11|11\rangle) \\ &= \tfrac{1}{4} \end{aligned}$$

Experiment ${\color{blue}\nwarrow}{\color{cyan}\rightarrow}$. We show some work in calculating. First:

$$\begin{aligned}
P_{\color{blue}\nwarrow} \otimes P_{\color{cyan}\rightarrow} &= (\tfrac{1}{4}|0\rangle\langle 0| - \tfrac{\sqrt{3}}{4}|0\rangle\langle 1| - \tfrac{\sqrt{3}}{4}|1\rangle\langle 0| + \tfrac{3}{4}|1\rangle\langle 1|) \otimes |0\rangle\langle 0| \\
&= \tfrac{1}{4}|00\rangle\langle 00| \\
P_{\color{blue}\swarrow} \otimes P_{\color{cyan}\uparrow} &= (\tfrac{3}{4}|0\rangle\langle 0| + \tfrac{\sqrt{3}}{4}|0\rangle\langle 1| + \tfrac{\sqrt{3}}{4}|1\rangle\langle 0| + \tfrac{1}{4}|1\rangle\langle 1|) \otimes |1\rangle\langle 1| \\
&= \tfrac{1}{4}|11\rangle\langle 11|
\end{aligned}$$

The projector:

$$\begin{aligned}
P_{{\color{blue}\nwarrow}{\color{cyan}\rightarrow}} &= (P_{\color{blue}\nwarrow} \otimes P_{\color{cyan}\rightarrow}) + (P_{\color{blue}\swarrow} \otimes P_{\color{cyan}\uparrow}) \\
&= \tfrac{1}{4}|00\rangle\langle 00| + \tfrac{1}{4}|11\rangle\langle 11|
\end{aligned}$$

Now we apply $P_{{\color{blue}\nwarrow}{\color{cyan}\rightarrow}}$ to $|\phi\rangle$:

$$\begin{aligned}
P_{{\color{blue}\nwarrow}{\color{cyan}\rightarrow}}|\phi\rangle &= (\tfrac{1}{4}|00\rangle\langle 00| + \tfrac{1}{4}|11\rangle\langle 11|)(\tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)) \\
&= \tfrac{1}{\sqrt{2}}(\tfrac{1}{4}(|00\rangle + |11\rangle))
\end{aligned}$$

Next we calculate $\langle\phi|P_{{\color{blue}\nwarrow}{\color{cyan}\rightarrow}}|\phi\rangle\rangle = \left\langle\phi\middle|\tfrac{1}{\sqrt{2}}(\tfrac{1}{4}(|00\rangle + |11\rangle))\right\rangle$

$$\begin{aligned}
\left\langle\phi\middle|\tfrac{1}{\sqrt{2}}(\tfrac{1}{4}(|00\rangle + |11\rangle))\right\rangle &= \tfrac{1}{2}\left\langle|00\rangle + |11\rangle\middle|\tfrac{1}{4}(|00\rangle + |11\rangle)\right\rangle \\
&= \tfrac{1}{8}(\langle 00||00\rangle + |11\rangle\rangle + \langle 11||00\rangle + |11\rangle\rangle) \\
&= \tfrac{1}{8}(\langle 00|00\rangle + \langle 00|11\rangle + \langle 11|00\rangle + \langle 11|11\rangle) \\
&= \tfrac{1}{8}(1+1) \\
&= \tfrac{1}{4}
\end{aligned}$$

Experiment ${\color{blue}\nwarrow}{\color{red}\swarrow}$. We show some work in calculating. First:

$$\begin{aligned}
P_{\color{blue}\nwarrow} \otimes P_{\color{red}\swarrow} &= (\tfrac{1}{4}|0\rangle\langle 0| - \tfrac{\sqrt{3}}{4}|0\rangle\langle 1| - \tfrac{\sqrt{3}}{4}|1\rangle\langle 0| + \tfrac{3}{4}|1\rangle\langle 1|) \otimes (\tfrac{1}{4}|0\rangle\langle 0| + \tfrac{\sqrt{3}}{4}|0\rangle\langle 1| + \tfrac{\sqrt{3}}{4}|1\rangle\langle 0| + \tfrac{3}{4}|1\rangle\langle 1|) \\
&= \cdots + \tfrac{1}{16}|00\rangle\langle 00| - \tfrac{3}{16}|00\rangle\langle 11| - \tfrac{3}{16}|11\rangle\langle 00| + \tfrac{9}{16}|11\rangle\langle 11|
\end{aligned}$$

$$\begin{aligned}
P_{\color{blue}\swarrow} \otimes P_{\color{red}\searrow} &= (\tfrac{3}{4}|0\rangle\langle 0| + \tfrac{\sqrt{3}}{4}|0\rangle\langle 1| + \tfrac{\sqrt{3}}{4}|1\rangle\langle 0| + \tfrac{1}{4}|1\rangle\langle 1|) \otimes (\tfrac{3}{4}|0\rangle\langle 0| - \tfrac{\sqrt{3}}{4}|0\rangle\langle 1| - \tfrac{\sqrt{3}}{4}|1\rangle\langle 0| + \tfrac{1}{4}|1\rangle\langle 1|) \\
&= \cdots + \tfrac{9}{16}|00\rangle\langle 00| - \tfrac{3}{16}|00\rangle\langle 11| - \tfrac{3}{16}|11\rangle\langle 00| + \tfrac{1}{16}|11\rangle\langle 11|
\end{aligned}$$

The projector:

$$\begin{aligned}
P_{{\color{blue}\nwarrow}{\color{red}\swarrow}} &= (P_{\color{blue}\nwarrow} \otimes P_{\color{red}\swarrow}) + (P_{\color{blue}\swarrow} \otimes P_{\color{red}\searrow}) \\
&= \tfrac{1}{16}|00\rangle\langle 00| - \tfrac{3}{16}|00\rangle\langle 11| - \tfrac{3}{16}|11\rangle\langle 00| + \tfrac{9}{16}|11\rangle\langle 11| \\
&\quad + \tfrac{9}{16}|00\rangle\langle 00| - \tfrac{3}{16}|00\rangle\langle 11| - \tfrac{3}{16}|11\rangle\langle 00| + \tfrac{1}{16}|11\rangle\langle 11| \\
&= \tfrac{10}{16}|00\rangle\langle 00| - \tfrac{6}{16}|00\rangle\langle 11| - \tfrac{6}{16}|11\rangle\langle 00| + \tfrac{10}{16}|11\rangle\langle 11|
\end{aligned}$$

Now we apply $P_{\nwarrow\swarrow}$ to $|\phi\rangle$:

$$\begin{aligned}
P_{\nwarrow\swarrow}|\phi\rangle &= \left(\tfrac{10}{16}|00\rangle\langle 00| - \tfrac{6}{16}|00\rangle\langle 11| - \tfrac{6}{16}|11\rangle\langle 00| + \tfrac{10}{16}|11\rangle\langle 11|\right)\left(\tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right) \\
&= \tfrac{1}{\sqrt{2}}\left(\tfrac{10}{16}|00\rangle\langle 00|00\rangle + \tfrac{10}{16}|00\rangle\langle 00|11\rangle - \tfrac{6}{16}|00\rangle\langle 11|00\rangle - \tfrac{6}{16}|00\rangle\langle 11|11\rangle\right. \\
&\quad \left. - \tfrac{6}{16}|11\rangle\langle 00|00\rangle - \tfrac{6}{16}|11\rangle\langle 00|11\rangle + \tfrac{10}{16}|11\rangle\langle 11|00\rangle + \tfrac{10}{16}|11\rangle\langle 11|11\rangle\right) \\
&= \tfrac{1}{\sqrt{2}}\left(\tfrac{10}{16}|00\rangle\langle 00|00\rangle - \tfrac{6}{16}|00\rangle\langle 11|11\rangle - \tfrac{6}{16}|11\rangle\langle 00|00\rangle + \tfrac{10}{16}|11\rangle\langle 11|11\rangle\right) \\
&= \tfrac{1}{\sqrt{2}}\left(\tfrac{4}{16}|00\rangle + \tfrac{4}{16}|11\rangle\right) \\
&= \tfrac{1}{\sqrt{2}}\left(\tfrac{1}{4}|00\rangle + \tfrac{1}{4}|11\rangle\right)
\end{aligned}$$

Next we calculate $\langle\phi|P_{\nwarrow\swarrow}|\phi\rangle\rangle = \left\langle \phi \middle| \frac{1}{\sqrt{2}}\left(\frac{1}{4}|00\rangle + \frac{1}{4}|11\rangle\right)\right\rangle$

$$\begin{aligned}
\left\langle \phi \middle| \tfrac{1}{\sqrt{2}}\left(\tfrac{1}{4}|00\rangle + \tfrac{1}{4}|11\rangle\right)\right\rangle &= \tfrac{1}{2}\left(\left\langle |00\rangle + |11\rangle \middle| \tfrac{1}{4}(|00\rangle + |11\rangle)\right\rangle\right) \\
&= \tfrac{1}{8}(\langle 00|00\rangle + \langle 11|11\rangle) \\
&= \tfrac{1}{4}
\end{aligned}$$

Experiment $\swarrow\rightarrow$. We show some work in calculating. First:

$$\begin{aligned}
P_{\swarrow} \otimes P_{\rightarrow} &= \left(\tfrac{1}{4}|0\rangle\langle 0| + \tfrac{\sqrt{3}}{4}|0\rangle\langle 1| + \tfrac{\sqrt{3}}{4}|1\rangle\langle 0| + \tfrac{3}{4}|1\rangle\langle 1|\right) \otimes |0\rangle\langle 0| \\
&= \cdots + \tfrac{1}{4}|00\rangle\langle 00| \\
P_{\searrow} \otimes P_{\uparrow} &= \left(\tfrac{3}{4}|0\rangle\langle 0| - \tfrac{\sqrt{3}}{4}|0\rangle\langle 1| - \tfrac{\sqrt{3}}{4}|1\rangle\langle 0| + \tfrac{1}{4}|1\rangle\langle 1|\right) \otimes |1\rangle\langle 1| \\
&= \cdots + \tfrac{1}{4}|11\rangle\langle 11|
\end{aligned}$$

The projector:

$$\begin{aligned}
P_{\swarrow\rightarrow} &= (P_{\swarrow} \otimes P_{\rightarrow}) + (P_{\searrow} \otimes P_{\uparrow}) \\
&= \tfrac{1}{4}|00\rangle\langle 00| + \tfrac{1}{4}|11\rangle\langle 11|
\end{aligned}$$

Now we apply $P_{\swarrow\rightarrow}$ to $|\phi\rangle$:

$$\begin{aligned}
P_{\swarrow\rightarrow}|\phi\rangle &= \left(\tfrac{1}{4}|00\rangle\langle 00| + \tfrac{1}{4}|11\rangle\langle 11|\right)\left(\tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right) \\
&= \tfrac{1}{\sqrt{2}}\left(\tfrac{1}{4}(|00\rangle\langle 00|00\rangle + |00\rangle\langle 00|11\rangle + |11\rangle\langle 11|00\rangle + |11\rangle\langle 11|11\rangle)\right) \\
&= \tfrac{1}{\sqrt{2}}\left(\tfrac{1}{4}(|00\rangle\langle 00|00\rangle + |11\rangle\langle 11|11\rangle)\right) \\
&= \tfrac{1}{\sqrt{2}}\left(\tfrac{1}{4}|00\rangle + \tfrac{1}{4}|11\rangle\right)
\end{aligned}$$

Next we calculate $\langle\phi|P_{\swarrow\rightarrow}|\phi\rangle\rangle = \left\langle\phi\middle|\frac{1}{\sqrt{2}}(\frac{1}{4}|00\rangle + \frac{1}{4}|11\rangle)\right\rangle$

$$\begin{aligned}
\left\langle\phi\middle|\frac{1}{\sqrt{2}}(\frac{1}{4}|00\rangle + \frac{1}{4}|11\rangle)\right\rangle &= \left\langle\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\middle|\frac{1}{\sqrt{2}}(\frac{1}{4}|00\rangle + \frac{1}{4}|11\rangle)\right\rangle \\
&= \frac{1}{2}\left\langle|00\rangle + |11\rangle\middle|\frac{1}{4}(|00\rangle + |11\rangle)\right\rangle \\
&= \frac{1}{2}(\frac{1}{4}\langle|00\rangle + |11\rangle|00\rangle + \frac{1}{4}\langle|00\rangle + |11\rangle|11\rangle) \\
&= \frac{1}{8}(\langle00|00\rangle + \langle11|00\rangle + \langle00|11\rangle + \langle11|11\rangle) \\
&= \frac{1}{8}(\langle00|00\rangle + \langle11|11\rangle) \\
&= \frac{1}{4}
\end{aligned}$$

Experiment $\swarrow\nwarrow$. We show some work in calculating. First:

$$\begin{aligned}
P_{\swarrow} \otimes P_{\nwarrow} &= (\frac{1}{4}|0\rangle\langle0| + \frac{\sqrt{3}}{4}|0\rangle\langle1| + \frac{\sqrt{3}}{4}|1\rangle\langle0| + \frac{3}{4}|1\rangle\langle1|) \otimes (\frac{1}{4}|0\rangle\langle0| - \frac{\sqrt{3}}{4}|0\rangle\langle1| - \frac{\sqrt{3}}{4}|1\rangle\langle0| + \frac{3}{4}|1\rangle\langle1|) \\
&= \cdots + \frac{1}{16}|00\rangle\langle00| - \frac{3}{16}|00\rangle\langle11| - \frac{3}{16}|11\rangle\langle00| + \frac{9}{16}|11\rangle\langle11|
\end{aligned}$$

$$\begin{aligned}
P_{\searrow} \otimes P_{\swarrow} &= (\frac{3}{4}|0\rangle\langle0| - \frac{\sqrt{3}}{4}|0\rangle\langle1| - \frac{\sqrt{3}}{4}|1\rangle\langle0| + \frac{1}{4}|1\rangle\langle1|) \otimes (\frac{3}{4}|0\rangle\langle0| + \frac{\sqrt{3}}{4}|0\rangle\langle1| + \frac{\sqrt{3}}{4}|1\rangle\langle0| + \frac{1}{4}|1\rangle\langle1|) \\
&= \cdots + \frac{9}{16}|00\rangle\langle00| - \frac{3}{16}|00\rangle\langle11| - \frac{3}{16}|11\rangle\langle00| + \frac{1}{16}|11\rangle\langle11|
\end{aligned}$$

The projector:

$$\begin{aligned}
P_{\swarrow\nwarrow} &= (P_{\swarrow} \otimes P_{\nwarrow}) + (P_{\searrow} \otimes P_{\swarrow}) \\
&= \cdots + \frac{10}{16}|00\rangle\langle00| - \frac{6}{16}|00\rangle\langle11| - \frac{6}{16}|11\rangle\langle00| + \frac{10}{16}|11\rangle\langle11|
\end{aligned}$$

Now we apply $P_{\swarrow\nwarrow}$ to $|\phi\rangle$:

$$\begin{aligned}
P_{\swarrow\nwarrow}|\phi\rangle &= (\frac{10}{16}|00\rangle\langle00| - \frac{6}{16}|00\rangle\langle11| - \frac{6}{16}|11\rangle\langle00| + \frac{10}{16}|11\rangle\langle11|)(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)) \\
&= \frac{1}{\sqrt{2}}(\frac{10}{16}|00\rangle\langle00|00\rangle + \frac{10}{16}|00\rangle\langle00|11\rangle - \frac{6}{16}|00\rangle\langle11|00\rangle - \frac{6}{16}|00\rangle\langle11|11\rangle \\
&\quad -\frac{6}{16}|11\rangle\langle00|00\rangle - \frac{6}{16}|11\rangle\langle00|11\rangle + \frac{10}{16}|11\rangle\langle11|00\rangle + \frac{10}{16}|11\rangle\langle11|11\rangle) \\
&= \frac{1}{\sqrt{2}}(\frac{10}{16}|00\rangle\langle00|00\rangle - \frac{6}{16}|00\rangle\langle11|11\rangle - \frac{6}{16}|11\rangle\langle00|00\rangle + \frac{10}{16}|11\rangle\langle11|11\rangle) \\
&= \frac{1}{\sqrt{2}}(\frac{10}{16}|00\rangle - \frac{6}{16}|00\rangle - \frac{6}{16}|11\rangle + \frac{10}{16}|11\rangle) \\
&= \frac{1}{\sqrt{2}}(\frac{4}{16}|00\rangle + \frac{4}{16}|11\rangle) \\
&= \frac{1}{\sqrt{2}}(\frac{1}{4}|00\rangle + \frac{1}{4}|11\rangle)
\end{aligned}$$

Next we calculate $\langle\phi|P_{\swarrow\nwarrow}\,|\phi\rangle\rangle = \left\langle\phi\middle|\frac{1}{\sqrt{2}}(\frac{1}{4}\,|00\rangle + \frac{1}{4}\,|11\rangle)\right\rangle$

$$\begin{aligned}
\frac{1}{\sqrt{2}}\left\langle\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\middle|(\frac{1}{4}\,|00\rangle + \frac{1}{4}\,|11\rangle)\right\rangle &= \frac{1}{2}(\frac{1}{4}\,\langle 00|00\rangle + \frac{1}{4}\,\langle 11|11\rangle) \\
&= \frac{1}{2}(\frac{1}{2}) \\
&= \frac{1}{4}
\end{aligned}$$

2 Insights

Now that you have seen the precise mathematical description of the experiment, do you have any new insights regarding the *mystery*?

It is hard to say now since the mystery has been explained as a different way of viewing the problem. I think the new insight is that there is not tangible connectedness, but the context of each question is built on the projectors and their orientations, which have this interesting property in the probabilities both say yes/no. My best explanation would be that by setting the devices to the same setting, you are aligning the projection for the partciles, so when you ask yes or no, their answers are aligned. With different projectors, there's 4 possibilities of yes/yes, yes/no, no/yes, no/no – then the alignment cannot help really, so a quarter of the time they line up (yes/yes).

3 Qiskit

Implement the 9 experiments in Qiskit. Experiment-$\rightarrow\rightarrow$ uses the default projector so it is straightforward to realize in Qiskit using the build-in measurement operator. We illustrate how to realize a measurement using another projector below.

First our projector operators are all of the form $|R\,|0\rangle\rangle\langle R\,|0\rangle|$ for some unitary operator R. It is important to recall that for unitary operators R, we have $R^\dagger = R^{-1}$ and $\langle a|b\rangle = \langle Ra|Rb\rangle$. Now our desired probabilities involve computations $\langle\phi|R\,|0\rangle\rangle\,\langle R\,|0\rangle|\phi\rangle$ and these can be simplified as follows:

$$\begin{aligned}
\langle\phi|R\,|0\rangle\rangle\,\langle R\,|0\rangle|\phi\rangle &= \left\langle R^\dagger\phi\middle|R^\dagger R\,|0\rangle\right\rangle\left\langle R^\dagger R\,|0\rangle\middle|R^\dagger\phi\right\rangle \\
&= \left\langle R^\dagger\phi\middle|0\right\rangle\left\langle 0\middle|R^\dagger\phi\right\rangle
\end{aligned}$$

In other words, instead of using projector $|R\,|0\rangle\rangle\langle R\,|0\rangle|$ on state $|\phi\rangle$, it is equivalent to use projector $|0\rangle\langle 0|$ on state $R^\dagger\,|\phi\rangle$.