Quantum Measurement (II)

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1 Notation

An operator O on a Hilbert space is a map from vectors to vectors. We write $O^\dagger$ for the adjoint of O mapping dual vectors to dual vectors. Recall that in the bra-ket notation, we write $\langle x|$ for $|x\rangle^\dagger$. It follows that if O maps $|x\rangle$ to $|y\rangle$, then $O^\dagger$ maps $\langle x|$ to $\langle y|$.

2 Definition of Projectors

Projectors P are special operators that satisfy: $P^\dagger = P$ (i.e., they are Hermitian operators) and $PP = P$. The first condition holds iff $\langle x|Py\rangle = \langle Px|y\rangle$. In that case, we can unambiguously write $\langle x|P|y\rangle$ which could be interpreted as either $\langle x|Py\rangle$ or $\langle Px|y\rangle$ depending on how to choose to associate the operations.

Example 0. $P|x\rangle = \bullet$ (the zero vector)
Trivially $\langle x|Py\rangle = 0 = \langle Px|y\rangle$
Trivially $PP = P$
This is called the *empty* projector $\mathbb{0}$.

Example I. $P|x\rangle = |x\rangle$
Trivially $\langle x|Py\rangle = \langle x|y\rangle = \langle Px|y\rangle$
Trivially $PP = P$
This is called the *identity* projector $\mathbb{1}$.

Example II. $P = |a\rangle\langle a|$.
$|Py\rangle = |a\rangle\langle a|y\rangle$ and $\langle Px| = \langle x|a\rangle\langle a|$.
Then $\langle x|Py\rangle = \langle x|a\rangle\langle a|y\rangle$ and $\langle Px|y\rangle = \langle x|a\rangle\langle a|y\rangle$
$PP = (|a\rangle\langle a|)(|a\rangle\langle a|) = |a\rangle\langle a|a\rangle\langle a| = |a\rangle\langle a| = P$.

Example III. $P = |a\rangle\langle a| + |b\rangle\langle b|$ with $a \neq b$

$$\begin{aligned}
\langle x|Py\rangle &= \langle x|a\rangle\langle a|y\rangle + \langle x|b\rangle\langle a|y\rangle \\
\langle Px|y\rangle &= \langle x|a\rangle\langle a|y\rangle + \langle x|b\rangle\langle a|y\rangle
\end{aligned}$$

$$\begin{aligned}
PP &= (|a\rangle\langle a| + |b\rangle\langle b|)(|a\rangle\langle a| + |b\rangle\langle b|) \\
&= (|a\rangle\langle a|)(|a\rangle\langle a|) + (|a\rangle\langle a|)(|b\rangle\langle b|) + (|b\rangle\langle b|)(|a\rangle\langle a|) + (|b\rangle\langle b|)(|b\rangle\langle b|) \\
&= |a\rangle\langle a| + |b\rangle\langle b|
\end{aligned}$$

It is crucial that $\langle a|b\rangle = 0$, i.e., that $|a\rangle$ and $|b\rangle$ are orthogonal.

Example IV. $P = P_1 + P_2$ with $P_1 P_2 = P_2 P_1 = \mathbb{0}$

$$\begin{aligned}
\langle x|(P_1+P_2)y\rangle &= \langle x|P_1 y\rangle + \langle x|P_2 y\rangle \\
&= \langle P_1 x|y\rangle + \langle P_2 x|y\rangle \qquad \text{induction} \\
\langle (P_1+P_2)x|y\rangle &= \langle P_1 x|y\rangle + \langle P_2 x|y\rangle
\end{aligned}$$

$$\begin{aligned}
PP &= (P_1+P_2)(P_1+P_2) \\
&= P_1P_1 + P_1P_2 + P_2P_1 + P_2P_2 \\
&= P_1P_1 + P_2P_2 \\
&= P_1 + P_2 \\
&= P
\end{aligned}$$

Sums of *orthogonal* projectors are also projectors.

Example V. $P = P_1 P_2 = P_2 P_1$

$$\begin{aligned}
\langle x|P_1P_2 y\rangle &= \langle P_1 x|P_2 y\rangle \qquad \text{induction} \\
&= \langle P_2 P_1 x|y\rangle \qquad \text{induction} \\
&= \langle P_1 P_2 x|y\rangle \qquad \text{assumption}
\end{aligned}$$

$$\begin{aligned}
PP &= (P_1P_2)(P_1P_2) \\
&= P_1(P_2P_1)P_2 \\
&= P_1(P_1P_2)P_2 \\
&= (P_1P_1)(P_2P_2) \\
&= P_1P_2 \\
&= P
\end{aligned}$$

Products of *commuting* projectors are also projectors.

Example VI. Let P be a projector. Then $\mathbb{1} - P$ is a projector.
$\langle x|(\mathbb{1}-P)y\rangle = \langle x|y\rangle - \langle x|Py\rangle = \langle x|y\rangle - \langle Px|y\rangle = \langle(\mathbb{1}-P)x|y\rangle$
$(\mathbb{1}-P)(\mathbb{1}-P) = (\mathbb{1}-2P+PP) = (\mathbb{1}-2P+P) = (\mathbb{1}-P)$

Example VII. $P = |00\rangle\langle 00| + |01\rangle\langle 01|$ (valid because $\langle 00|01\rangle = 0$.)
Note: $|00\rangle\langle 00| + |01\rangle\langle 01| = |0\rangle\langle 0| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) = |0\rangle\langle 0| \otimes \mathbb{1}$

Example VIII. $P = P_1 P_2$ where $P_1 = |00\rangle\langle 00| + |01\rangle\langle 01|$ and $P_2 = |01\rangle\langle 01| + |11\rangle\langle 11|$.
Commutative because $P_1 = |0\rangle\langle 0| \otimes \mathbb{1}$ and $P_2 = \mathbb{1} \otimes |1\rangle\langle 1|$ and hence $P_1P_2 = P_2P_1 = |0\rangle\langle 0| \otimes |1\rangle\langle 1|$.

3 Projectors = Subspace

Projectors are in 1-1 correspondence with the subspaces of the Hilbert space, i.e, each projector identifies a subset of the vectors that is closed under linear combinations.

- $P = \mathbb{0}$ identifies the empty subspace
- $P = \mathbb{1}$ identifies the entire Hilbert space

- $P = |a\rangle\langle a|$ identifies the subspaces of vectors of the form $\alpha |a\rangle$ (where $\alpha \neq 0$)
- $P = P_1 + P_2$ for orthogonal P_1 and P_2 identifies the subspace of linear combinations of states in the orthogonal (!) subspaces corresponding to P_1 and P_2
- $P = P_1 P_2$ for commuting P_1 and P_2 identifies the subspace corresponding to the intersection of the subspaces for P_1 and P_2 (intersection commutative!)
- $P = \mathbb{1} - P$ identifies the subspace of vectors orthogonal to the vectors in the subpace corresponding to P

4 Quantum Events

Each projector P corresponds to the following **yes/no** question about a state $|\phi\rangle$.

Does the state ϕ have a non-zero component that lies in the subspace corresponding to P ?

Example. $P = |0\rangle\langle 0|$. For state $|\phi\rangle$, the event is whether $|\phi\rangle$ has a non-zero component $\alpha |0\rangle$?

$P|0\rangle = |0\rangle \langle 0|0\rangle = |0\rangle$ **yes**

$P|1\rangle = |0\rangle \langle 0|1\rangle = \bullet$ **no**

$P|+\rangle = |0\rangle\langle 0| 1/\sqrt{2}(|0\rangle + |1\rangle) = 1/\sqrt{2}(|0\rangle \langle 0|0\rangle + |0\rangle \langle 0|1\rangle) = 1/\sqrt{2}\,|0\rangle$ **yes**

Example. $P = |0\rangle\langle 0| + |1\rangle\langle 1|$ (valid because $\langle 0|1\rangle = 0$.) For state $|\phi\rangle$, the event is whether $|\phi\rangle$ has a non-zero component that is a linear combinations of $\alpha |0\rangle$ and $\beta |1\rangle$. **yes** because $P = \mathbb{1}$ because $\{|0\rangle, |1\rangle\}$ is an orthonormal basis for the entire space.

Example. $P = |00\rangle\langle 00| + |01\rangle\langle 01|$ (valid because $\langle 00|01\rangle = 0$.) For state $|\phi\rangle$, the event is whether $|\phi\rangle$ has a non-zero component that is a linear combinations of $\alpha |00\rangle$ and $\beta |01\rangle$.
Note: $|00\rangle\langle 00| + |01\rangle\langle 01| = |0\rangle\langle 0| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) = |0\rangle\langle 0| \otimes \mathbb{1}$

$P(1/2(|00\rangle + |01\rangle + |10\rangle + |11\rangle)) = 1/2(|00\rangle\langle 00| + |01\rangle\langle 01|)(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = 1/2(|00\rangle + |01\rangle)$ **yes**

$P(1/\sqrt{2}(|00\rangle + |11\rangle) = 1/\sqrt{2}(|00\rangle\langle 00| + |01\rangle\langle 01|)(|00\rangle + |11\rangle) = 1/\sqrt{2}\,|00\rangle$ **yes**

Example. $P = P_1 P_2$ where $P_1 = |00\rangle\langle 00| + |01\rangle\langle 01|$ and $P_2 = |01\rangle\langle 01| + |11\rangle\langle 11|$. Check commutativity. $P_1 = |0\rangle\langle 0| \otimes \mathbb{1}$ and $P_2 = \mathbb{1} \otimes |1\rangle\langle 1|$. Hence $P_1 P_2 = P_2 P_1 = |0\rangle\langle 0| \otimes |1\rangle\langle 1|$.
Let L_1 be the space of linear combinations of $|00\rangle$ and $|01\rangle$
Let L_2 be the space of linear combinations of $|01\rangle$ and $|11\rangle$
Their intersection is the set of scaled vectors $\alpha |01\rangle$
For state $|\phi\rangle$, the event is whether $|\phi\rangle$ has a non-zero component $\alpha |01\rangle$.

$P|00\rangle = P_1(P_2 |00\rangle) = P_1 \bullet = \bullet$ **no**

$P|01\rangle = P_1(P_2 |01\rangle) = P_1 |01\rangle = |01\rangle$ **yes**

5 Measurements

So far we defined projectors. Applying a projector loses information and is stable (further projections have no effect.) Each projector identifies a subspace. A given state $|\phi\rangle$ may or may not have a component that lies in the subspace identified by P. The idea is that we can build an apparatus that can detect whether this is the case.

The following is a *postulate*. Each state $|\phi\rangle$ induces a probability measure $\mu_\phi : \mathit{Events} \to [0,1]$ defined as follows:

$$\mu_\phi(P) = \langle\phi|P|\phi\rangle$$

This is the *Born rule*. The rule assigns a probability to the event "does ϕ have a component that lies in the space corresponding to P."

Example . Let our projector $P = |00\rangle\langle 00|$. This projector identifies the following **yes/no** event: given a state $|phi\rangle$ does it have a component that lies in the space generated by $|00\rangle$. Given a system in the current state $|\phi\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle$, we reason as follows:

- Apply the projector P to $|\phi\rangle$, we get $1/\sqrt{2}(|00\rangle\langle 00|00\rangle + |00\rangle\langle 00|11\rangle) = 1/\sqrt{2}\,|00\rangle$
- As we have previously concluded the answer to our event is **yes**
- The Born rule further refines this answer by assigning it the probability $\mu_\phi(P) = \langle\phi|P|\phi\rangle = \left\langle\phi\middle|1/\sqrt{2}\,|00\rangle\right\rangle = 1/2(\langle 00|00\rangle + \langle 11|00\rangle) = 1/2$
- This probability is not only useful to reason about the possible observations but it also enables us to normalize the result of the projection to get a valid state. Formally, the state after the projection is given by $\frac{P|\phi\rangle}{\sqrt{\langle\phi|P|\phi\rangle}}$, which in our case reduces to $|00\rangle$.

We claimed above that the Born rule can be used to define a probability measure. If this is true, we expect certain properties which we check below.

- Projecting on the empty subspace is an impossible event: $\mu_\phi(\mathbb{0}) = 0$

  $\mu_\phi(\mathbb{0}) = \langle\phi|\mathbb{0}|\phi\rangle = \langle\phi|\bullet\rangle = 0$
- Projecting on the entire space is a certain event: $\mu_\phi(\mathbb{1}) = 1$

  $\mu_\phi(\mathbb{1}) = \langle\phi|\mathbb{1}|\phi\rangle = \langle\phi|\phi\rangle = 1$
- Projecting on a primitive subspace has a real (non-imaginary) probability $\mu_\phi(|a\rangle\langle a|)$ is a real (not imaginary) number.

  $\mu_\phi(|a\rangle\langle a|) = \langle\phi|\,|a\rangle\langle a|\,|\phi\rangle = \langle\phi|a\rangle\,\langle a|\phi\rangle = \langle a|\phi\rangle^*\,\langle a|\phi\rangle = |\,\langle a|\phi\rangle\,|^2$ which is a real number
- The probability of projecting on the sum of two subspaces is the sum of the individual probabilities $\mu_\phi(P_1 + P_2) = \mu_\phi(P_1) + \mu_\phi(P_2)$

  $\mu_\phi(P_1 + P_2) = \langle\phi|P_1 + P_2|\phi\rangle = \langle\phi|P_1\phi + P_2\phi\rangle = \langle\phi|P_1\phi\rangle + \langle\phi|P_2\phi\rangle = \mu_\phi(P_1) + \mu_\phi(P_2)$
- The probability of projecting on the complement of a subspace is one minus the probability of projecting on the subspace itself $\mu_\phi(\mathbb{1} - P) = 1 - \mu_\phi(P)$

  $\mu_\phi(\mathbb{1} - P) = \langle\phi|(\mathbb{1} - P)\phi\rangle = \langle\phi|\phi - P\phi\rangle = \langle\phi|\phi\rangle - \langle\phi|P\phi\rangle = 1 - \mu_\phi(P)$.
- The probability of projecting on the product of two subspaces is the conditional probability of the second event given the first event $\mu_\phi(P_1P_2) = \mu_\phi(P_2)\mu_{\phi_2}(P_1)$ where $|\phi_2\rangle = \frac{P_2|\phi\rangle}{\sqrt{\mu_\phi(P_2)}}$. The intuition is that we first project using P_2: with some probability $\mu_\phi(P_2)$ we get the state $\frac{P_2(|\phi\rangle)}{\sqrt{\mu_\phi(P_2)}}$. The probability of the entire event is now the probability of this latter state satisfying the event P_1.

$$
\begin{aligned}
RHS &= \mu_\phi(P_2)\mu_{\frac{P_2|\phi\rangle}{\sqrt{\mu_\phi(P_2)}}}(P_1) \\
&= \frac{\mu_\phi(P_2)\,\langle P_2\phi|P_1|P_2\phi\rangle}{\mu_\phi(P_2)} \\
&= \langle P_2\phi|P_1|P_2\phi\rangle \\
&= \langle \phi|P_2P_1|P_2\phi\rangle \\
&= \langle \phi|P_2P_1P_2|\phi\rangle \\
&= \langle \phi|P_1P_2P_2|\phi\rangle \qquad \text{commutative} \\
&= \langle \phi|P_1P_2|\phi\rangle \\
&= LHS
\end{aligned}
$$

6 Theorems

Gleason. Every quantum probability measure is induced by exactly one state according to the Born rule.
Kochen-Specker. There is no quantum probability measure that maps every event to 0 or 1

7 Mermin's Experiment

Main insight: Express the **yes/no** questions of the experiment as projectors!