Modal Logic, Winter 2019
Homework 8
due Tuesday, March 19

I know this is after Spring Break. I'll try to make sure that everyone gets a good start this week.

I also am almost certainly not around on Monday, March 18, until the afternoon sometime. But I will try hard to be in my office in the late afternoon or early evening.

1 Sentences from the past

We shall use the following notation at several points later on.

$$\begin{array}{lcllcl} \alpha_1 & = & \Box \mathsf{F} \wedge p & \alpha_5 & = & \Box\neg p \wedge \neg\Diamond p \wedge p \\ \alpha_2 & = & \Box \mathsf{F} \wedge \neg p & \alpha_6 & = & \Box\neg p \wedge \neg\Diamond p \wedge \neg p \\ \alpha_3 & = & \Box p \wedge \Diamond p \wedge p & \alpha_7 & = & \Diamond p \wedge \Diamond\neg p \wedge p \\ \alpha_4 & = & \Box p \wedge \Diamond p \wedge \neg p & \alpha_8 & = & \Diamond p \wedge \Diamond\neg p \wedge \neg p \end{array} \tag{1}$$

Check that $\alpha_1, \ldots, \alpha_8$ are satisfiable and hence consistent in K.

[Hint: it's not hard to do this directly by drawing pictures. But if you wish, you might look back at Homework 2, problem 5.]

Recall the α_i sentences from Exercise 1 above. Fill in the following chart, putting a $\surd$ in the box if the given sentence is consistent in the given logic, and an $\times$ if it is not. A few of the entries are given for you.

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
K	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
KT	$\times$	$\times$	$\surd$	$\times$	$\times$	$\surd$	$\surd$	$\surd$
KD	$\times$	$\times$	$\surd$	$\surd$	$\surd$	$\surd$	$\times$	$\surd$
KB	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$K4$	$\surd$	$\surd$	$\surd$	$\times$	$\times$	$\surd$	$\surd$	$\surd$
$S4$	$\times$	$\times$	$\surd$	$\times$	$\times$	$\surd$	$\surd$	$\surd$
$S5$	$\times$	$\times$	$\surd$	$\times$	$\times$	$\surd$	$\surd$	$\surd$

[Hint: to show that a sentence is consistent in a given logic, it is enough (by soundness) to show that it is satisfiable on a model in the corresponding semantic class. For example to show that a sentence φ is consistent in KT, it is enough to find a reflexive model with a point x such that $x \Vdash \varphi$. But to show that φ is *inconsistent* in a logic, we should use the logic to get a derivation of $\vdash \neg\varphi$. That is, in this problem it is ok to use the soundness of our logics, but not the completeness. That is because this exercise is a step in the completeness proofs.]

2 An unprovable sentence

Show that the B axiom $p \to \Box\Diamond p$ is not provable in $KD45$. [The way to start is to find a graph which is serial, transitive, and euclidean, and at the same time is not symmetric. Then one needs to find a point and put a valuation on the graph so that the point does not satisfy the given B-axiom.

Consider the following model $M = (\{x, y\}, \{(x, y), (y, y)\}, Val)$ where $Val(x) = \{p\}$ and $Val(y) = \varnothing$. This model is serial, transitive, and euclidean while also not symmetric. At point x, p is true but $\Box\Diamond p$ is not. By this counter-example, we know the B axiom is not provable in $KD45$

3 The logics KB and KB'

Recall that the B axioms are the sentences of the form $\varphi \to \Box\Diamond\varphi$. Let's call the B' axioms the sentences of the form $(\varphi \wedge \Diamond\psi) \to \Diamond(\psi \wedge \Diamond\varphi)$. So we get a logical system KB' by adding the B' axioms to K.

Prove that $KB' \leq KB$.

1.	$\varphi \wedge \Diamond\psi$	Assume
2.	$\neg\Diamond(\psi \wedge \Diamond\varphi)$	Assume
3.	$\neg\neg\Box\neg(\psi \wedge \Diamond\varphi)$	$\Diamond \equiv \neg\Box\neg$
4.	$\Box\neg(\psi \wedge \Diamond\varphi)$	PR
5.	φ	$\wedge_e, 1$
6.	$\Box\Diamond\varphi$	B axiom, 5
7.	F	$\mathsf{F}_i, 4, 6$
8.	$\neg\neg\Diamond(\psi \wedge \Diamond\varphi)$	$\neg_i, 2-7$
9.	$\Diamond(\psi \wedge \Diamond\varphi)$	PR
10.	$(\varphi \wedge \Diamond\psi) \to \Diamond(\psi \wedge \Diamond\varphi)$	$\to_i, 1-9$

4 The Converse

Continuing with the last problems, show that $KB \leq KB'$.

1.	φ	Assume
2.	$\Box\varphi$	Necessitation
3.	$\neg\Diamond\varphi$	Assume
4.	$\Box\neg\varphi$	Prop
5.	F	$\mathsf{F}_i, 2, 4$
6.	$\neg\neg\Diamond\varphi$	$\neg_i, 3-5$
7.	$\Diamond\varphi$	PR
8.	$\Box\Diamond\varphi$	Necessitation
9.	$\varphi \to \Box\Diamond\varphi$	$\to_i, 1-8$

It follows that the two logical systems KB and KB' can prove each other's axioms, and hence prove all the same sentences. That is, if $\vdash \varphi$ in KB, then $\vdash \varphi$ in KB'; and vice-versa.

5 A result on *KB*

1. Assume that $\psi \wedge \Diamond\varphi$ is inconsistent in KB'. (This means that in KB', $\vdash \neg(\psi \wedge \Diamond\varphi)$.) Show that in KB', $\varphi \wedge \Diamond\psi$ is also inconsistent. [Hint: you will need to use the following axiom of KB':

   $$(\varphi \wedge \Diamond\psi) \to \neg\Box\neg(\psi \wedge \Diamond\varphi).$$

   (I have "unabbreviated the $\Diamond$ on the right.)]

2. Using the last part, prove the following: if $\varphi \wedge \Diamond\psi$ is consistent in KB', then $\psi \wedge \Diamond\varphi$ is also consistent in KB'.

3. Show that KB and KB' have the same consistent sentences.

4. Putting things together, show that if $\varphi \wedge \Diamond\psi$ is consistent in KB, then $\psi \wedge \Diamond\varphi$ is also consistent in KB.