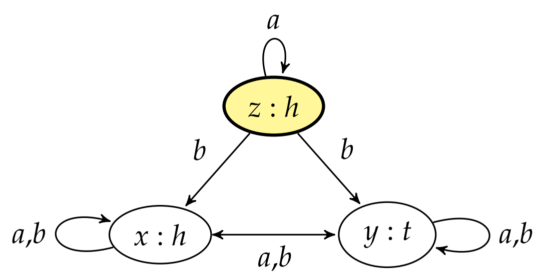
Modal Logic, Winter 2019
Homework 10
Due Tuesday, April 9

1. Here is a model which we've seen before



   (a) True or false: $z \Vdash CK_{a,b}\neg K_b\, K_a\, h$, where h is "heads."

   True. Let $g = \{a, b\}$. We show that for all z' such that $z \xrightarrow{g*} z'$, $z' \Vdash \neg K_b\, K_a h$.

   - (z′ = a): The only world that z is a-connected to is itself (z), so we show $z \Vdash \neg K_b\, K_a h$. For this, we must show there exists a world w such that $z \xrightarrow{b} w$ and $w \Vdash \neg K_a h$. Pick $w = y$ because $z \xrightarrow{b} y$. Then $y \xrightarrow{a} y$ and $y \Vdash t$, so $y \Vdash \neg K_a h$. Hence $z \Vdash \neg K_b\, K_a h$.
   - (z′ = b): There are two possible worlds to go to from z on b. First we consider $z \xrightarrow{b} x$. We must show $x \Vdash \neg K_b\, K_a h$. Consider $x \xrightarrow{b} x$, thus we need $x \Vdash \neg K_a h$, which follows from $x \xrightarrow{a} y$ where $y \Vdash t$. Therefore $x \Vdash \neg K_b\, K_a h$.
     Second we consider $z \xrightarrow{b} y$ and show that $y \Vdash \neg K_b\, K_a h$. Notice that $y \xrightarrow{b} y$ and $y \Vdash \neg K_a h$. Then $y \xrightarrow{a} y$ where $y \Vdash t$, so $y \Vdash \neg K_a h$. Hence $y \Vdash \neg K_b\, K_a h$.
   - ($g* = \emptyset$): In the case that we don't take any arrows, we are still at world z and we have shown $z \Vdash \neg K_b\, K_a h$ in the first case, so we're done.

   (b) True or false: $z \Vdash CK_{a,b}K_b\neg K_a\, h$, where h is "heads."

   True. Let $g = \{a, b\}$. We show that for all z' such that $z \xrightarrow{g*} z'$, $z' \Vdash K_b\, \neg K_a h$.

   - (z′ = a) Show $z \Vdash K_b\, \neg K_a h$. First consider $z \xrightarrow{b} y$; we show $y \Vdash \neg K_a h$. This follows from $y \xrightarrow{a} y$ and $y \Vdash t$, so $y \Vdash \neg K_a h$. Next consider $z \xrightarrow{b} x$; we show $x \Vdash \neg K_a h$. This follows from the same reasoning because $x \xrightarrow{a} y$. Therefore $z \Vdash K_b\, \neg K_a h$.
   - (z′ = b) We must show $y \Vdash K_b\neg K_a h$ as well as $x \Vdash K_b\neg K_a h$. For the first part, we know $y \xrightarrow{b} y$ and we just saw that $y \Vdash \neg K_a h$. We must also consider $y \xrightarrow{b} x$, which we showed in the second step of problem (a) above. Next we show $x \Vdash K_b\neg K_a h$. This means both $x \Vdash \neg K_a h$ and $y \Vdash \neg K_a h$, which we have concluded already.
   - ($g* = \emptyset$) Like before, this is the case where we stay at world z, and we have shown that the sentence holds at z in the first case.

2. Recall that throughout our semester, letters like p denote *atomic sentences*. Show that $\models_{all} [p]CK_{\mathcal{A}}\, p$. Here $\mathcal{A}$ is the set of all agents. Intuitively, $[p]CK_{\mathcal{A}}\, p$ says, "if p is true, then announcing it truthfully to everyone results in p becoming common knowledge."

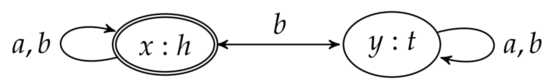
   Fix a model M and a world w in M. We show $w \models [p]CK_{\mathcal{A}}\, p$. By definition, it is equivalent to show that if $w \models p$ in M then in $M \upharpoonright p$, $w \models CK_{\mathcal{A}}\, p$. For this, we assume $w \models p$ and consider $M' = M \upharpoonright p$, and we prove $w \models CK_{\mathcal{A}}\, p$. By the definition of M', p is true at every world of M', therefore every node x such that $w \xrightarrow{\mathcal{A}^*} x$, $x \models p$. Hence $w \models CK_{\mathcal{A}}\, p$, and therefore $\models_{all} [p]CK_{\mathcal{A}}\, p$.

3. Recall the coin scenarios from our worksheet:

Scenario 0 Two people, *Amina (A)* (female) and *Bao (B)* (male), enter a large room. On the table, is a remote-control mechanical coin flipper. One presses a button, and the coin spins through the air, landing in a small box on the table. The box closes. The two people are much too far to see the coin. In reality, the coin shows heads.

Scenario 2 After Scenario 0, A goes up and looks at the coin while B stands at the door. B watches her open the box, but he doesn't see what she sees.

We symbolize the situation after Scenario 2 by



Now consider the following sentences:

$$
\begin{array}{ll}
\varphi_{ht} & h \oplus t = (h \vee t) \wedge \neg(h \wedge t) \\
\varphi^* & ((h \wedge K_a h \wedge P_a h) \vee (t \wedge K_a t \wedge P_a t)) \wedge \ \neg K_b\, h \ \wedge \ \neg K_b\, t \\
\psi & \varphi_{ht} \wedge \varphi^* \wedge CK_{a,b}(\varphi_{ht} \wedge \varphi^*) \\
\chi_h & h \wedge \psi \\
\chi_t & t \wedge \psi
\end{array}
$$

Here, $P\varphi$ is the "$\Diamond$ version of K". That is, $P_a h$ is an abbreviation of $\neg K_a \neg h$. And for all z, $z \models P_a h$ if there is some w such that $z \xrightarrow{a} w$ and $w \models h$.

Let $\mathcal{N}$ be any model of χ_h, *not necessarily the model shown above*. Your job is to show that all of the following hold about satisfaction in $\mathcal{N}$:

Note: ψ is equivalent to just $CK_{a,b}(\varphi_{ht} \wedge \varphi^*)$, so we refer to ψ as such.

(a) If $w \models \chi_h$, then there is some v such that $w \xrightarrow{a} v$ and $v \models \chi_h$.

Assume $w \models \chi_h$. Show there exists such a v. Our assumptions means that $w \models h$ and $w \models \psi$. By the Mix Lemma we have $w \models \varphi_{ht} \wedge \varphi^* \wedge E_{a,b}\ CK_{a,b}(\varphi_{ht} \wedge \varphi^*)$. Then, we must have $w \models h \wedge K_a h \wedge P_a h$ from φ^*. So, by $w \models P_a h$, we know there exists a v such that $w \xrightarrow{a} v$ and $v \models h$. And since $w \models E_{a,b} CK_{a,b}(\varphi_{ht} \wedge \varphi^*)$, $v \models CK_{a,b}(\varphi_{ht} \wedge \varphi^*)$. Therefore $v \models \chi_h$

(b) If $w \models \chi_h$, then every v such that $w \xrightarrow{a} v$ has $v \models \chi_h$.

This follows from the same reasoning above, except that we appeal to $w \models K_a\ h$ instead of $w \models P_a\ h$. This makes v arbitrarily chosen, and shows statement (b).

(c) If $w \models \chi_t$, then there is some v such that $w \xrightarrow{a} v$ and $v \models \chi_t$.

Assume $w \models \chi_t$. Show there exists such a v. Our assumptions means that $w \models t$ and $w \models \psi$. By the Mix Lemma we have $w \models \varphi_{ht} \wedge \varphi^* \wedge E_{a,b}\ CK_{a,b}(\varphi_{ht} \wedge \varphi^*)$. Then, we must have $w \models t \wedge K_a t \wedge P_a t$ from φ^*. So, by $w \models P_a t$, we know there exists a v such that $w \xrightarrow{a} v$ and $v \models t$. And since $w \models E_{a,b} CK_{a,b}(\varphi_{ht} \wedge \varphi^*)$, $v \models CK_{a,b}(\varphi_{ht} \wedge \varphi^*)$. Therefore $v \models \chi_t$

(d) If $w \models \chi_t$, then every v such that $w \xrightarrow{a} v$ has $v \models \chi_t$.

This follows from the same reasoning above, except that we appeal to $w \models K_a\ t$ instead of $w \models P_a\ t$. This makes v arbitrarily chosen, and shows statement (d).

(e) If $w \models \chi_h$, then there is some v such that $w \xrightarrow{b} v$ and $v \models \chi_h$.

Assume $w \models \chi_h$. Show there is some v such that $w \xrightarrow{b} v$ and $v \models \chi_{Heads}$. So $w \models h$ and $w \models \psi$. Then by the Mix Lemma we have $w \models \varphi_{ht} \wedge \varphi^* \wedge E_{a,b}\ CK_{a,b}(\varphi_{ht} \wedge \varphi^*)$. From φ^*, we know $w \models \neg K_b\ h$, while $w \models h$. Since $w \xrightarrow{b} w$ there must be a world v such that $w \xrightarrow{b} v$ and $v \models t$. Otherwise every world v' that $w \xrightarrow{b} v'$ would be such that $v' \models h$, which would be a contradiction. Next, since $w \models E_{a,b}\ \psi$, we have $v \models \psi$. Hence $v \models \chi_h$.

(f) If $w \models \chi_h$, then there is some v such that $w \xrightarrow{b} v$ and $v \models \chi_t$.

This is similar to the reasons of statement (e), except we appeal to $w \models \neg K_b\ t$ instead of $w \models \neg K_b\ h$, and we conclude $v \models h$. every

(g) If $w \models \chi_h$, then every v such that $w \xrightarrow{a} v$ has $v \models \chi_h$ or $v \models \chi_t$.

This is trivially true, by the assumption.

(h) If $w \models \chi_t$, then there is some v such that $w \xrightarrow{b} v$ and $v \models \chi_h$.

This follows from the same reasoning as statement (e), we just replace each occurrence of t with h and h with t.

(i) If $w \models \chi_t$, then there is some v such that $w \xrightarrow{b} v$ and $v \models \chi_t$.

This follows statement (e), we just need to replace all instances of h with t and statement (i) follows.

(j) If $w \models \chi_t$, then every v such that $w \xrightarrow{b} v$ has $v \models \chi_h$ or $v \models \chi_t$.

This is similar to statement (g).

[Hint: All of the parts are fairly short semantic arguments (a few sentences each), and the key steps involve the use of the Mix Axiom. You also can say that some parts are similar to others. For example, (f), (i), and (h) are all similar to (e); you can say this. Similarly, (j) is similar to (g).]

4. (The knowledge flip-flop) Suppose we have four players: Amina, Bao, Chandra, and Dianne. They have a deck with two indistinguishable ♠ cards, one ♢ and one ♣. The cards are dealt, and the players look at their own cards.

We assume that the following are common knowledge: the distribution of cards in the deck, the fact that each player knows which card was dealt to them, and that they do not initially know any other player's card.

The deal is

$$w \quad = \quad (A\spadesuit, B\spadesuit, C\diamondsuit, D\clubsuit).$$

So this is the "real world", and we call this world w.

We take a modal language with atomic sentences

$$A\spadesuit, A\diamondsuit, A\clubsuit, B\spadesuit, B\diamondsuit, B\clubsuit, C\spadesuit, C\diamondsuit, C\clubsuit, D\spadesuit, D\diamondsuit, D\clubsuit,$$

We have agents a, b, c, and d, and so we have knowledge sentences involving those agents. We shall be concerned with the sentence φ, given as

$$\neg K_b\, A\spadesuit \ \wedge\ \neg K_b\, A\diamondsuit \ \wedge\ \neg K_b\, A\clubsuit.$$

It says that Bao doesn't know Amina's that Amina has spaces, he also doesn't know that she has diamonds, and in addition he doesn't know that Amina has clubs.

Reminder: The deal is

$$w \quad = \quad (A\spadesuit, B\spadesuit, C\diamondsuit, D\clubsuit).$$

(a) Show that $w \Vdash K_c\, \varphi$.

Fix y so that $w \xrightarrow{c} y$. There are 3 options of y, because the only accessible worlds are the ones where C has $\diamondsuit$. The possible worlds are

- w
  For this, we first show that $w \Vdash \neg K_b\, A\diamondsuit$. It is enough to find a world x, such that $w \xrightarrow{b} x$ and $x \Vdash \neg A\diamondsuit$. Let $x = (A\diamondsuit, B\spadesuit, C\diamondsuit, D\clubsuit)$. Since $x \Vdash B\spadesuit$ we know x is b-accessible from w. Then we have $x \Vdash A\diamondsuit$, so $x \Vdash \neg A\spadesuit$.
  Thus $w \Vdash \neg K_b\, A\spadesuit$.
  This choice of x also shows that $w \Vdash \neg K_b\, A\clubsuit$ by a similar argument, as well as $w \Vdash \neg K_b\, A\diamondsuit$ by choice of $x = (A\clubsuit, B\spadesuit, C\diamondsuit, D\spadesuit)$.
  Hence $w \Vdash \neg K_b\, A\diamondsuit \wedge \neg K_b\, A\diamondsuit \wedge \neg K_b\, A\diamondsuit$.

- $(A\spadesuit, B\clubsuit, C\diamondsuit, D\spadesuit)$
  Denote $\alpha := (A\spadesuit, B\clubsuit, C\diamondsuit, D\spadesuit)$.
  First, we show $\alpha \Vdash \neg K_b\, A\clubsuit$. For this, consider the world $x = (A\diamondsuit, B\clubsuit, C\spadesuit, D\spadesuit)$. α is b-connected to x because $x \Vdash B\clubsuit$, yet $x \Vdash A\diamondsuit$ so $x \Vdash \neg A\spadesuit$.
  A similar argument can be made for $x \Vdash \neg A\clubsuit$, so $\alpha \Vdash \neg K_b\, A\clubsuit$ also.
  To show $\alpha \Vdash \neg K_b\, A\diamondsuit$ we can pick x to be α by the same reasoning above. Since $\alpha \Vdash A\spadesuit$, we have $\alpha \Vdash \neg A\diamondsuit$.
  Hence $\alpha \Vdash K_b \neg A\diamondsuit \wedge K_b \neg A\diamondsuit \wedge K_b \neg A\diamondsuit$.
- $(A\clubsuit, B, C\diamondsuit, D\spadesuit)$
  Denote $\beta := (A\clubsuit, B\spadesuit, C\diamondsuit, D\spadesuit)$. We show $\beta \Vdash \varphi$.
  First, we show $\beta \Vdash \neg K_b\, A\spadesuit$. Let $x = \beta$, this is trivially b-connected. Because $\beta \Vdash A\clubsuit$, we have $\beta \Vdash A\spadesuit$. This means that $\beta \Vdash \neg K_b\, A\spadesuit$. Notice that this choice of x also does not satisfy $A\diamondsuit$, so $\beta \Vdash \neg K_b\, A\diamondsuit$ as well.
  By similar reasoning we can show $\beta \Vdash \neg K_b\, A\clubsuit$ by choice of $x = (A\spadesuit, B\spadesuit, C\diamondsuit, D\clubsuit)$, better known as w. Since $w \Vdash B\spadesuit$, we know $w \Vdash \neg A\clubsuit$. Thus, $\beta \Vdash \neg K_b\, A\clubsuit$.
  In conclusion, $\beta \Vdash \neg K_b\, A\clubsuit \wedge \neg K_b\, A\spadesuit \wedge \neg K_b\, A\diamondsuit$.

Since at each c-accesible world from w, say w' we have shown that $w' \Vdash \varphi$, we know that $w \Vdash K_c\, \varphi$.

(b) Then Amina announces, "I do not have $\diamondsuit$." Show that now $w \Vdash \neg K_c\, \varphi$.

For this we fix a world w' such that $w \xrightarrow{c} w'$, and we show that $w' \Vdash \neg\varphi$. We do this by showing that $w' \Vdash K_b\, A\spadesuit$. Fix x such that $w' \xrightarrow{b} x$. Now, since A doesn't have $\diamondsuit$, and w' and x are b-connected, we have $x \Vdash \neg A\diamondsuit \wedge B\clubsuit$. This forces $x \Vdash A\spadesuit$. Since we took an arbitrary x that was b-connected to w', we know that $w' \Vdash K_b\, A\spadesuit$. Therefore $w' \Vdash \neg\varphi$ and $w \Vdash \neg K_c\, \varphi$.

(c) After this, Dianne announces, "I do not have $\spadesuit$." Show that now $w \Vdash K_c\, \varphi$.

This means that the only c-accessible world from w is w!
(Recall the options listed in part a, and how only one satisifed $\neg D\spadesuit$.)
We show that $w \Vdash \varphi$, where $w = (A\spadesuit, B\spadesuit, C\diamondsuit, D\clubsuit)$.
First we show that $w \Vdash \neg K_b\, A\spadesuit$.
It is enough to find a world x such that $w \xrightarrow{b} x$ and $x \Vdash \neg A\spadesuit$.
Let $x = (A\clubsuit, B\spadesuit, C\spadesuit, D\diamondsuit)$, we know $w \xrightarrow{b} x$ because the three necessary conditions are true at both worlds: B has $\spadesuit$, A doesn't have $\diamondsuit$ and D doesn't have $\spadesuit$.
Then $x \Vdash A\clubsuit$ and $x \Vdash \neg A\spadesuit$, and we know $w \Vdash \neg K_b\, A\spadesuit$.
This choice of x also gives $x \Vdash \neg A\diamondsuit$, so we know $w \Vdash \neg K_b\, A\diamondsuit$ as well.
To show $w \Vdash \neg K_b\, A\clubsuit$, we choose such an $x = (A\spadesuit, B\spadesuit, C\clubsuit, D\diamondsuit)$. Again, we know x is a valid choice because $x \Vdash \neg A\diamondsuit \wedge B\spadesuit \wedge \neg D\spadesuit$. Then $x \Vdash A\spadesuit$ and $x \Vdash \neg A\clubsuit$, so $w \Vdash \neg K_b\, A\clubsuit$.
Therefore $w \Vdash \neg K_b\, A\spadesuit \wedge \neg K_b\, A\diamondsuit \wedge \neg K_b\, A\clubsuit$, and we conclude $w \Vdash K_c\, \varphi$.