# Modal Logic, Winter 2019 Homework 8 due Tuesday, March 19

I know this is after Spring Break. I'll try to make sure that everyone gets a good start this week.

I also am almost certainly not around on Monday, March 18, until the afternoon sometime. But I will try hard to be in my office in the late afternoon or early evening.

## 1 Sentences from the past

We shall use the following notation at several points later on.

$$\alpha_{1} = \Box F \wedge p \qquad \alpha_{5} = \Box \neg p \wedge \neg \Diamond p \wedge p 
\alpha_{2} = \Box F \wedge \neg p \qquad \alpha_{6} = \Box \neg p \wedge \neg \Diamond p \wedge \neg p 
\alpha_{3} = \Box p \wedge \Diamond p \wedge p \qquad \alpha_{7} = \Diamond p \wedge \Diamond \neg p \wedge p 
\alpha_{4} = \Box p \wedge \Diamond p \wedge \neg p \qquad \alpha_{8} = \Diamond p \wedge \Diamond \neg p \wedge \neg p$$
(1)

Check that  $\alpha_1, \ldots, \alpha_8$  are satisfiable and hence consistent in K.

[Hint: it's not hard to do this directly by drawing pictures. But if you wish, you might look back at Homework 2, problem 5.]

Recall the  $\alpha_i$  sentences from Exercise 1 above. Fill in the following chart, putting a  $\sqrt{}$  in the box if the given sentence is consistent in the given logic, and an  $\times$  if it is not. A few of the entries are given for you.

	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\alpha_4$	$\alpha_5$	$\alpha_6$	$\alpha_7$	$\alpha_8$
K	<b>V</b>							
KT	×	×		×	×			
KD	×	×					×	
KB								
<i>K</i> 4				×	×			
S4	×	X		X	×			
<i>S</i> 5	×	X		X	X			

[Hint: to show that a sentence is consistent in a given logic, it is enough (by soundness) to show that it is satisfiable on a model in the corresponding semantic class. For example to show that a sentence  $\varphi$  is consistent in KT, it is enough to find a reflexive model with a point x such that  $x \models \varphi$ . But to show that  $\varphi$  is *inconsistent* in a logic, we should use the logic to get a derivation of  $\vdash \neg \varphi$ . That is, in this problem it is ok to use the soundness of our logics, but not the completeness. That is because this exercise is a step in the completeness proofs.]

### 2 An unprovable sentence

Show that the B axiom  $p \to \Box \Diamond p$  is not provable in KD45. [The way to start is to find a graph which is serial, transitive, and euclidean, and at the same time is not symmetric. Then one needs to find a point and put a valuation on the graph so that the point does not satisfy the given B-axiom.

Consider the following model  $M = (\{x, y\}, \{(x, y), (y, y)\}, Val)$  where  $Val(x) = \{p\}$  and  $Val(y) = \emptyset$ . This model is serial, transitive, and euclidean while also not symmetric. At point x, p is true but  $\Box \Diamond p$  is not. By this counter-example, we know the B axiom is not provable in KD45

## 3 The logics KB and KB'

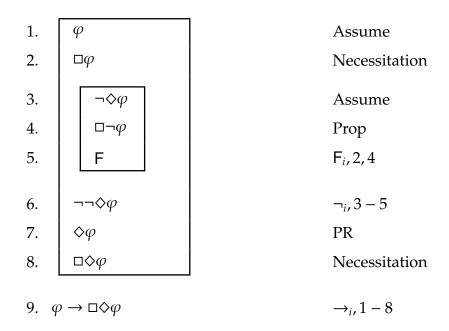
Recall that the B axioms are the sentences of the form  $\varphi \to \Box \Diamond \varphi$ . Let's call the B' axioms the sentences of the form  $(\varphi \land \Diamond \psi) \to \Diamond (\psi \land \Diamond \varphi)$ . So we get a logical system KB' by adding the B' axioms to K.

Prove that  $KB' \leq KB$ .

1.	$\varphi \wedge \diamond \psi$	Assume		
2.	$\neg \diamondsuit (\psi \land \diamondsuit \varphi)$	Assume		
3.	$\neg\neg\Box\neg(\psi\land\Diamond\varphi)$	◊ ≡ ¬□¬		
4.	$\Box \neg (\psi \land \Diamond \varphi)$	PR		
5.	$\mid \; \mid \; \varphi \; \mid \; $	$\wedge_e$ , 1		
6.	$\Box \Diamond \varphi$	B axiom, 5		
7.	F	$F_{i}, 4, 6$		
8.	$\neg\neg \diamond (\psi \land \diamond \varphi)$	$\neg_i, 2-7$		
9.	$\Diamond(\psi \land \Diamond\varphi)$	PR		
10. (	$\rightarrow_i$ , 1 – 9			

#### 4 The Converse

Continuing with the last problems, show that  $KB \leq KB'$ .



It follows that the two logical systems KB and KB' can prove each other's axioms, and hence prove all the same sentences. That is, if  $\vdash \varphi$  in KB, then  $\vdash \varphi$  in KB'; and vice-versa.

#### **5 A** result on *KB*

1. Assume that  $\psi \land \Diamond \varphi$  is inconsistent in KB'. (This means that in KB',  $\vdash \neg(\psi \land \Diamond \varphi)$ .) Show that in KB',  $\varphi \land \Diamond \psi$  is also inconsistent. [Hint: you will need to use the following axiom of KB':

$$(\varphi \land \Diamond \psi) \to \neg \Box \neg (\psi \land \Diamond \varphi).$$

(I have "unabbreviated the ♦ on the right.) ]

- 2. Using the last part, prove the following: if  $\varphi \land \Diamond \psi$  is consistent in KB', then  $\psi \land \Diamond \varphi$  is also consistent in KB'.
- 3. Show that KB and KB' have the same consistent sentences.
- 4. Putting things together, show that if  $\varphi \land \Diamond \psi$  is consistent in *KB*, then  $\psi \land \Diamond \varphi$  is also consistent in *KB*.