Modal Logic, Winter 2019
Homework 4
due Tuesday, February 5

1 Review of modal semantics on graphs

Recall the following graph from Homework 3.

$$\begin{tikzcd} 1 \arrow[loop left] \arrow[d] & & 2 & & 5 \\ & 6 \arrow[ul] \arrow[ur] & & & \\ 3 \arrow[loop left] & & 4 \arrow[u] \arrow[loop right] & & \end{tikzcd}$$

We saw on that homework that the following partition Π had the property that $(\mathcal{G}, \Pi) = \Pi$: $\{\{1,3\},\{4\},\{6\},\{2,5\}\}$.

1. Find a modal logic sentence φ_1 such that $1 \models \varphi_1$, $4 \not\models \varphi_1$, $6 \not\models \varphi_1$, and $2 \not\models \varphi_1$.

   $\varphi_1 = \Box\Box\Diamond\mathsf{T} \wedge \Box\Diamond\mathsf{T}$

2. Does $3 \models \varphi_1$ as well?

   Indeed!

3. Find a modal logic sentence φ_4 such that $1 \not\models \varphi_4$, $4 \models \varphi_4$, $6 \not\models \varphi_4$, and $2 \not\models \varphi_4$.

   $\varphi_4 = \neg\Box\Diamond\mathsf{T} \wedge \Diamond\neg\Box\Diamond\mathsf{T}$

4. Find a modal logic sentence φ_6 such that $1 \not\models \varphi_6$, $4 \not\models \varphi_6$, $6 \models \varphi_6$, and $2 \not\models \varphi_6$.

   $\varphi_6 = \Diamond\Box\mathsf{F} \wedge \neg\Box\Diamond\mathsf{T} \wedge \neg\Diamond\neg\Diamond\mathsf{T}$

5. Find a modal logic sentence φ_2 such that $1 \not\models \varphi_2$, $4 \not\models \varphi_2$, $6 \not\models \varphi_2$, and $2 \models \varphi_2$.

   $\varphi_2 = \Box\mathsf{F}$

6. Does $5 \models \varphi_2$ as well?

   Indeed!

You can use the inductive method from Tuesday's class, or you could just guess the answer. You don't have to justify your assertions, but I trust that you *could* write out the full justifications if you needed to.

2 Facts about modal semantics on graphs

1. Let $\mathcal{G} = (G, \to)$ be a Euclidean graph. Show that for all $g \in G$ $g \models \Diamond p \to \Box\Diamond p$.

   Fix $g \in G$. Assume about g that $g \models \Diamond p$. We must prove $g \models \Box\Diamond p$.
   Take $k \in G$ such that $g \to k$. We must show $k \models \Diamond p$.
   Since $g \models \Diamond p$, fix $h \in G$ such that $g \to h$ and $h \models p$.
   Since $g \to k$ and $g \to h$, we must have $k \to h$, because the graph is Euclidean.
   We know $h \models p$, so we know $k \models \Diamond p$.
   Therefore $g \models \Box\Diamond p$.

2. Is the following true: for every modal sentence φ, the sentence $\Diamond\varphi \to \Box\Diamond\varphi$ is true at every point in every Euclidean graph?

   Yes, we consider p the atomic sentences in standard semantics on graphs, and using modal sentences with modal semantics would not change the proof we have just provided since the sentences are essentially the same.

3. Let $\mathcal{G} = (G, \to)$ be a graph which is a partial function.
   Show that for all $g \in G$ $g \models \Diamond p \to \Box p$.

   Fix $g \in G$. Assume about g that $g \models \Diamond p$.
   We prove $g \models \Box p$.
   Take $k \in G$ so that $g \to k$. We must show $k \models p$.
   Since $g \models \Diamond p$, take $h \in G$ such that $g \to h$, and $h \models p$.
   Because the graph is a partial function, having $g \to k$ and $g \to h$ means $h = k$.
   And since $h \models p$, we know $k \models p$. Hence $g \models \Box p$.

4. What would the analog of part (2) be for partial function graphs?

   For every modal sentence φ, the sentence $\Diamond\varphi \to \Box\varphi$ is true at every point in every graph which is a partial function graph.

3 A fact about fixed points of monotone functions on posets

Recall that a *partially-ordered set* (also called a *poset*) is a preorder $(P, \leq)$ which is anti-symmetric.

Let $(P, \leq)$ be a poset, and suppose that P has a least element, $\perp$. (This is read "bottom.") Let $f : P \to P$ monotone, and let p^* be a fixed point of f. Show that for all n, $f^n(\perp) \leq p^*$.

Here is what this means in more detail. We assume that f has the following property: if $p \leq q$, then $f(p) \leq f(q)$. We define elements $f^n(\perp)$ of P by the following recursion: $f^0(\perp) = \perp$, and $f^{n+1}(\perp) = f(f^n(\perp)$. Finally, we fix an element p^* such that $f(p^*) = p^*$. We want to prove that for all n $f^n(\perp) \leq p^*$.

We prove that for all n, $f^n(\bot) \leq p*$. We proceed by induction on n.

1. Base Case, n = 0.
   $f^0(\bot) = \bot \leq p*$ trivially, since $\bot$ is the least element of P.

2. Inductive Step, n = k + 1
   Assume that $f^k(\bot) \leq p*$. We must show that $f^{k+1}(\bot) \leq p*$.
   Since f is monotone we know $f(f^k(\bot)) \leq f(p*)$.
   And we know $p* = f(p*)$ since $p*$ is a fixed point, so $f(f^k(\bot)) \leq p*$.

4 Another fact about fixed points of monotone functions on posets, this time finite posets

As in problem 3, let $(P, \leq)$ be a poset, and suppose that P has a least element, $\bot$. Let $f : P \rightarrow P$ monotone,

1. If P is *finite*, then show that there is some n such that $f^n(\bot) = f^{n+1}(\bot)$.

   [Use the following fact: If A is any finite set, and if $a_0, a_1, a_2, \ldots, a_n, \ldots$ is an infinite sequence from A, then there must be $i < j$ such that $a_i = a_j$.]

   We consider the infinite sequence $f^0(\bot), f^1(\bot), \ldots, f^n(\bot), \ldots$. Since P is a finite set, and for every i, $f^i(\bot) \in P$, because $f : P \rightarrow P$. By the provided fact, there must be $i < j$ such that $f^i(\bot) = f^j(\bot)$. By induction on n, we know either $j = i + i$, in which case $f^i(\bot) = f^{i+1}(\bot)$. Or $j = i + k$ and we assume $f^i(\bot) = f^j(\bot)$ and $f^i(\bot) = f^k(\bot)$, so we know $f^k(\bot) = f^j(\bot)$.

2. Given an example to show that we need P to be finite in order to have the result in part (1).

   Consider the function on the natural numbers that adds two to any given number. Because this is an infinite set and a monotone function, it is not guaranteed that in an infinite sequence of numbers produced by this function that two numbers $n_i = n_j$ where $i < j$.

3. Use part (1) and results from Homework 3 to show that for every *finite* graph $\mathcal{G}$, there is a partition Π of the nodes of the graph such that $\Pi = (\mathcal{G}, \Pi)$. [Hint: Recall from Homework 3, problem 3 that the set P of partitions of G is a poset, where $\leq$ is the refinement order. What is the $\bot$ of this order? And what monotone function on P should we think about?]

We prove that for every finite graph $\mathcal{G}$, there is a partition Π of the nodes such that $\Pi = \texttt{refine}(\mathcal{G}, \Pi)$.
Fix a graph $\mathcal{G} = (G, \rightarrow)$. Consider the set of partitions on the nodes of the graph. We know from homework 3 that $\leq$ is a partial order on the set of partitions of any graph.
We also know that the function $f : P \rightarrow P$ where $f(\Pi) = \texttt{refine}(G, \Pi)$ is monotone. The least element, or $\perp$, of the partitions is the set of nodes G.
Let $\Pi = f^n(\perp)$ for some n. Since $f^n(\perp) = f^{n+1}(\perp)$, by part 1,
we know $\Pi = f^n(\perp) = f^{n+1}(\perp) = f(f^n(\perp)) = f(\Pi)$. Therefore $\Pi = \texttt{refine}(G, \Pi)$

5 Modal sentences and partitions

Let $\mathcal{G}$ be a graph. Recall that we interpret the modal language with T, F, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\Box$, and $\Diamond$ on $\mathcal{G}$. Let Π be a partition of the nodes of $\mathcal{G}$ such that $\Pi = (\mathcal{G}, \Pi)$. In this problem, we are going to prove that for all modal sentences φ the following holds:

$\underline{\text{for all } g, h \in G,}$
$\underline{\text{if } g \text{ and } h \text{ belong to the same cell of } \Pi, \text{ then } g \models \varphi \text{ if and only if } h \models \varphi.}$

1. Write out the underlined statement with φ being T, and say why the statement you wrote is true. (The same holds for F, but we'll skip this.)

   For all $g, h \in G$,
   if g and h belong to the same cell of Π, then $g \models \mathsf{T}$ if and only if $h \models \mathsf{T}$.
   The statement is true because every node satisfies T.

2. This next part is the induction step for $\wedge$. Take sentences φ and ψ, and assume the underlined statement for φ and also the underlined statement for ψ. Write out these assumptions. Then write your goal: the underlined statement for the sentence $\varphi \wedge \psi$. Prove this statement. The argument should be easy.

   We are going to skip the induction steps for $\neg$, $\vee$, $\rightarrow$, and $\leftrightarrow$, since they are basically the same as the one for $\wedge$.

   Fix φ and ψ. Take $g, h \in G$ in the same cell of Π and assume that $g \models \varphi$ if and only if $h \models \varphi$, and $g \models \psi$ if and only if $h \models \psi$. We want to show that $g \models \varphi \wedge \psi$ if and only if $h \models \varphi \wedge \psi$.
   g $\models \varphi \wedge \psi \leftrightarrow g \models \varphi$ and $g \models \psi$
   $\leftrightarrow h \models \varphi$ and $h \models \psi$ (by the induction hypothesis)
   $\leftrightarrow h \models \varphi \wedge \psi$. We're done.

3. This next part is the induction step for $\Diamond$. Take a sentence φ, assume the underlined statement for φ, and prove it for $\Diamond\varphi$. You will definitely need to use the assumption that $\Pi = (\mathcal{G}, \Pi)$ in this part, so be sure to indicate where exactly this assumption was used.

   Fix $g, h \in G$ such that they are in the same cell C of Π.
   Take a sentence φ and assume about φ that $g \models \varphi$ if and only if $h \models \varphi$. We must prove $g \models \Diamond\varphi$ if and only if $h \models \Diamond\varphi$. $g \models \Diamond\varphi \leftrightarrow \exists k.\ g \to k$ and $k \models \varphi$. Given that $g, h \in C$ and $\Pi = (\mathcal{G}, \Pi)$, we know everything g arrows is in the same cell as everything h arrows. So we have k and k' in the same cell of Π such that $g \to k$ and $h \to k'$, and by the inductive hypothesis, $k \models \varphi$ if and only if $k' \models \varphi$, so $g \models \Diamond\varphi$ if and only if $h \models \Diamond\varphi$.

4. This next part is the induction step for $\Box$. Take a sentence φ, assume the underlined statement for φ, and prove it for $\Box\varphi$. You will definitely again need to use the assumption that $\Pi = (\mathcal{G}, \Pi)$ in this part, so be sure to indicate where exactly this assumption was used. [The step for $\Box$ is quite close to the step for $\Diamond$, but you should start with the one for $\Diamond$ because it is a little easier.]

   Fix $g, h \in G$ such that $g, h \in C \in \Pi$. Fix a sentence φ and assume about φ that $g \models \varphi$ if and only if $h \models \varphi$. We must prove $g \models \Box\varphi$ if and only if $h \models \Box\varphi$. Assume towards a contradiction that $g \models \Box\varphi$ but $h \not\models \Box\varphi$. Take $x, y \in G$ so that $g \to x$ and $h \to y$. Since $\Pi = (\mathcal{G}, \Pi)$, we know x, y in the same cell of Π and $x \models \varphi$ if and only if $y \models \varphi$ (by our induction hypothesis). So $h \not\models \Box\varphi$ is a contradiction. Hence $y \models \Box\varphi$, therefore $g \models \Box\varphi$ if and only if $h \models \Box\varphi$.

6 Putting it all together

Let $\mathcal{G}$ be a finite graph.

1. Let $\equiv$ be the following relation on the nodes of G:

$$g \equiv h \quad \text{iff} \quad g \text{ and } h \text{ satisfy the same modal sentences}$$

   Check that $\equiv$ is an equivalence relation. [This should be easy.]

   (a) $\equiv$ reflexive
   $g \equiv g \quad$ iff $\quad g$ and g satisfy the same modal sentences
   This is trivially true because $g = g$.

(b) $\equiv$ symmetric
Assume $g \equiv h$. Show $h \equiv g$.
$g \equiv h$ iff g and h satisfy the same modal sentences
Then we know h and g satisfy the same modal sentences.
Hence $h \equiv g$.

(c) $\equiv$ transitive
Assume $g \equiv h$ and $h \equiv k$. Show $g \equiv k$.
$g \equiv h$ iff g and h satisfy the same modal sentences
and $h \equiv k$ iff h and k satisfy the same modal sentences
then we can rewrite using our assumptions,
and we see $g \equiv k$ iff g and k satisfy the same modal sentences

2. Let Π_1 be the partition associated to $\equiv$. In the special example of the graph $\mathcal{G}$ in Exercise 1, what exactly are the cells of Π_1, and why?

   $\Pi_1 = \{\{1,3\},\{4\},\{6\},\{2,5\}\}$ because 2 and 5 are end points they satisfy the same sentences, and similarly for the rest of the cells (each cells' members all satisfy the same sentences as their fellow members).

3. Recall from problem 4 that for every finite graph $\mathcal{G}$, there is some number n so that $f^n(\perp)$ which is a fixed point of refinement: $(f^n(\perp)) = f^n(\perp)$. (That is, for some particular n, this holds.) Let Π_2 be such a partition. So, for some number N, $\Pi_2 = f^N(\perp)$. Prove that for all finite graphs $\mathcal{G}$, $\Pi_1 = \Pi_2$. [The point here is to quote other results: so look at this homework and the previous one, and also what we did in class.]

   Fix a finite graph $\mathcal{G}$. To prove $\Pi_1 = \Pi_2$, it suffices to show that $\Pi_1 \leq \Pi_2$ and $\Pi_2 \leq \Pi_1$, because $\leq$ is a partial order. To show $\Pi_1 \leq \Pi_2$ we take an arbitrary cell C of Π_2 and show that there is a cell $D \in \Pi_1$ such that $C \subset D$. Fix $x \in C$ (since cells of a partition are nonempty). Then there must be some cell D of Π_1 so that $x \in D$. Next we take an arbitrary point $y \in C$ and we must show $y \in D$. Since $x, y \in C$ and $\Pi_2 = (\mathcal{G}, \Pi_2)$, x and y must satisfy the same modal sentences, by problem 5. Then we must have $x, y \in D$, by definition of Π_1. Hence $C \subset D$ and $\Pi_1 \leq \Pi_2$. We have proven $\Pi_2 \leq \Pi_1$ in class, so we conclude that $\Pi_1 = \Pi_2$.