Modal Logic, Winter 2019
Homework 8
due Tuesday, March 19

1 Sentences from the past

We shall use the following notation at several points later on.

$$
\begin{array}{rclrcl}
\alpha_1 & = & \Box\mathsf{F} \wedge p & \alpha_5 & = & \Box\neg p \wedge \neg\Diamond p \wedge p \\
\alpha_2 & = & \Box\mathsf{F} \wedge \neg p & \alpha_6 & = & \Box\neg p \wedge \neg\Diamond p \wedge \neg p \\
\alpha_3 & = & \Box p \wedge \Diamond p \wedge p & \alpha_7 & = & \Diamond p \wedge \Diamond\neg p \wedge p \\
\alpha_4 & = & \Box p \wedge \Diamond p \wedge \neg p & \alpha_8 & = & \Diamond p \wedge \Diamond\neg p \wedge \neg p
\end{array}
\tag{1}
$$

Check that $\alpha_1, \ldots, \alpha_8$ are satisfiable and hence consistent in K.

[Hint: it's not hard to do this directly by drawing pictures. But if you wish, you might look back at Homework 2, problem 5.]

Recall the α_i sentences from Exercise 1 above. Fill in the following chart, putting a $\surd$ in the box if the given sentence is consistent in the given logic, and an $\times$ if it is not. A few of the entries are given for you.

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
K	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
KT	$\times$	$\times$	$\surd$	$\times$	$\times$	$\surd$	$\surd$	$\surd$
KD	$\times$	$\times$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
KB	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$K4$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$S4$	$\times$	$\times$	$\surd$	$\times$	$\times$	$\surd$	$\surd$	$\surd$
$S5$	$\times$	$\times$	$\surd$	$\times$	$\times$	$\surd$	$\surd$	$\surd$

Reasons for above are attached on scratch paper.

2 An unprovable sentence

Show that the B axiom $p \to \Box\Diamond p$ is not provable in $KD45$. [The way to start is to find a graph which is serial, transitive, and euclidean, and at the same time is not symmetric. Then one needs to find a point and put a valuation on the graph so that the point does not satisfy the given B-axiom.

Consider the following model $M = (\{x, y\}, \{(x, y), (y, y)\}, Val)$ where $Val(x) = \{p\}$ and $Val(y) = \varnothing$. This model is serial, transitive, and euclidean while also not symmetric. At point x, p is true but $\Box\Diamond p$ is not. By this counter-example, we know the B axiom is not provable in $KD45$

3 The logics KB and KB'

Recall that the B axioms are the sentences of the form $\varphi \to \Box\Diamond\varphi$. Let's call the B' axioms the sentences of the form $(\varphi \wedge \Diamond\psi) \to \Diamond(\psi \wedge \Diamond\varphi)$. So we get a logical system KB' by adding the B' axioms to K.

Prove that $KB' \leq KB$.

1.	$\varphi \wedge \Diamond\psi$	Assume
2.	$\Diamond\psi$	$\wedge_e, 1$
3.	φ	$\wedge_e, 1$
4.	$\Box\Diamond\varphi$	B axiom, 3
5.	$\Diamond\psi \wedge \Box\Diamond\varphi$	$\wedge_i, 2, 4$
6.	$\Diamond\psi \wedge \Box\Diamond\varphi \to \Diamond(\psi \wedge \Diamond\varphi)$	Powerpoint
7.	$\Diamond(\psi \wedge \Diamond\varphi)$	$\to_e, 5, 6$
8.	$(\varphi \wedge \Diamond\psi) \to \Diamond(\psi \wedge \Diamond\varphi)$	$\to_i, 1 - 9$

4 The Converse

Continuing with the last problems, show that $KB \leq KB'$.

1.	φ	Assume
2.	$\neg\Box\Diamond\varphi$	Assume
3.	$\Diamond\neg\Diamond\varphi$	$\neg\Box \equiv \Diamond\neg$
4.	$\Diamond\Box\neg\varphi$	$\neg\Diamond \equiv \Box\neg$
5.	$\varphi \wedge \Diamond\Box\neg\varphi$	$\wedge_i, 1, 4$
6.	$(\varphi \wedge \Diamond\Box\neg\varphi) \to \Diamond(\Box\neg\varphi \wedge \Diamond\varphi)$	B′ axiom, 5
7.	$\Diamond(\Box\neg\varphi \wedge \Diamond\varphi)$	$\to_e, 5, 6$
8.	$\Box\neg\varphi \wedge \Diamond\varphi$	Assume
9.	$\Box\neg\varphi$	$\wedge_e, 8$
10.	$\Diamond\varphi$	$\wedge_e, 8$
11.	F	$\mathsf{F}_i, 9, 10$
12.	$\neg(\Box\neg\varphi \wedge \Diamond\varphi)$	$\neg_i, 8 - 11$
13.	$\Box\neg(\Box\neg\varphi \wedge \Diamond\varphi)$	$*_i, 8 - 12$
14.	F	$\mathsf{F}_i, 7, 13$
15.	$\neg\neg\Box\Diamond\varphi$	$\neg_i, 2 - 14$
16.	$\Box\Diamond\varphi$	PR
17.	$\varphi \to \Box\Diamond\varphi$	$\to_i, 1 - 16$

It follows that the two logical systems KB and KB' can prove each other's axioms, and hence prove all the same sentences. That is, if $\vdash \varphi$ in KB, then $\vdash \varphi$ in KB'; and vice-versa.

5 A result on *KB*

1. Assume that $\psi \wedge \Diamond\varphi$ is inconsistent in KB'. (This means that in KB', $\vdash \neg(\psi \wedge \Diamond\varphi)$.) Show that in KB', $\varphi \wedge \Diamond\psi$ is also inconsistent. [Hint: you will need to use the following axiom of KB':

$$(\varphi \wedge \Diamond\psi) \to \neg\Box\neg(\psi \wedge \Diamond\varphi).$$

(I have "unabbreviated" the $\Diamond$ on the right.)]

1.	$\varphi \wedge \Diamond\psi$	Assume
2.	$\boxed{\neg(\psi \wedge \Diamond\varphi)}$	Given as assumption
3.	$\Box\neg(\psi \wedge \Diamond\varphi)$	$*_i, 2$
4.	$(\varphi \wedge \Diamond\psi) \to \neg\Box\neg(\psi \wedge \Diamond\varphi)$	B′ axiom
5.	$\neg\Box\neg(\psi \wedge \Diamond\varphi)$	$\to_e, 4, 1$
6.	F	$\mathsf{F}_i, 3, 5$
7.	$\neg(\varphi \wedge \Diamond\psi)$	$\neg_i, 1-6$

2. Using the last part, prove the following: if $\varphi \wedge \Diamond\psi$ is consistent in KB', then $\psi \wedge \Diamond\varphi$ is also consistent in KB'.

   We prove the contrapositive. We assume $\psi \wedge \Diamond\varphi$ is inconsistent in KB' and prove $\varphi \wedge \Diamond\psi$ is inconsistent in KB'. For this, we have just constructed a proof above of $\vdash \neg(\varphi \wedge \Diamond\psi)$ which is the definition of inconsistent. Hence we're done.

3. Show that KB and KB' have the same consistent sentences.

   Fix a sentence φ. Assume about φ that φ is consistent in KB. We prove φ is consistent in KB'. So in KB, $\nvdash \neg\varphi$. Suppose, towards a contradiction, that φ is inconsistent in KB'; this means in KB', $\vdash \neg\varphi$. By the previous result (5.3), $\vdash \neg\varphi$ in KB, a contradiction! Hence φ is consistent in KB'

4. Putting things together, show that if $\varphi \wedge \Diamond\psi$ is consistent in KB, then $\psi \wedge \Diamond\varphi$ is also consistent in KB.

   Assume $\varphi \wedge \Diamond\psi$ is consistent in KB. We prove $\psi \wedge \Diamond\varphi$ is consistent in KB. By part 3, we know that $\varphi \wedge \Diamond\psi$ is also consistent in KB'. By part 2, we know that if $\varphi \wedge \Diamond\psi$ is consistent in KB' then $\psi \wedge \Diamond\varphi$ is consistent in KB'. By part 3 again, we know that $\psi \wedge \Diamond\varphi$ is consistent in KB. Done.