M482 Homework 2

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1 Practice with definitions.

(G_1) Formally, G_1 is the pairing of the set of states $S = \{a, b, c, d\}$ with the relation $R = \{(a, a), (b, b), (c, c), (d, d), (b, a), (b, c), (c, d)\}$.

reflexive	True, all nodes a, b, c, d bear R to themselves.
symmetric	False, counter-example: bRc but cRb does not exist.
anti-symmetric	False, counter-example: $bRc \wedge cRc \Rightarrow b = c$. $\Rightarrow\Leftarrow$
transitive	False, counter-example: bRc and cRd but bRd does not exist.
euclidean	False, in the case of b when bRa and bRb, aRb does not exist.
equivalence relation	False, only reflexive, i.e. not symmetric and not transitive.
partial function	False, counter-example: bRa and bRc but $a \neq c$.

(G_2) Formally, G_2 is the pairing of the set of states $S = \{a, b, c\}$ with the relation $R = \{(a, b), (a, c), (b, c)\}$.

reflexive	False, there is no node $s \in S$ such that sRs.
symmetric	False, counter-example: aRb but bRa does not exist.
anti-symmetric	False, counter-example: $aRb \wedge bRc \Rightarrow a = c$. $\Rightarrow\Leftarrow$
transitive	True, aRb and bRc and aRc, and $bRc \wedge cRb$ is false so vacuously true, and similar case for c.

euclidean False, counter-example: $aRc \wedge aRb \Rightarrow cRb$. $\Rightarrow\Leftarrow$

equivalence relation False, only transitive – i.e. not reflexive and not symmetric.

partial function False, $aRb \wedge aRc \Rightarrow b = c$. $\Rightarrow\Leftarrow$

(G_3) Formally, G_3 is the pairing of the set of states $S = \{a, b\}$ with the relation $R = \{(a, a), (a, b), (b, a)\}$.

reflexive False, $(b, b) \notin R$

symmetric True, $aRa \Rightarrow aRa\checkmark$ $aRb \Rightarrow bRa\checkmark$ $bRa \Rightarrow aRb\checkmark$

anti-symmetric False, $aRb \wedge bRa \Rightarrow a = b$. $\Rightarrow\Leftarrow$

transitive False, $bRa \wedge aRb \Rightarrow bRb$. $\Rightarrow\Leftarrow$

euclidean False, counter-example: $aRb \wedge aRb \Rightarrow bRb$. $\Rightarrow\Leftarrow$

equivalence relation False, not transitive and not reflexive.

partial function False, $aRa \wedge aRb \Rightarrow a = b$. $\Rightarrow\Leftarrow$

(G_4) Formally, G_4 is the pairing of the set of states $S = \{a\}$ with the relation $R = \{(a, a)\}$.

The single point, reflexive graph is trivially true for all the properties.

(G_5) Formally, G_5 is the pairing of the set of states $S = \{a, b, c\}$ with the relation $R = \{(a, b), (b, a), (b, c), (c, b)\}$.

reflexive False, $(a, a) \notin R$

symmetric True, $aRb \Rightarrow bRa\checkmark$ $bRc \Rightarrow cRb\checkmark$, similar for converse.

anti-symmetric False, $aRb \wedge bRa \Rightarrow a = b$. $\Rightarrow\Leftarrow$

transitive False, $aRb \wedge bRc \Rightarrow aRc$. $\Rightarrow\Leftarrow$

euclidean False, $bRc \wedge bRa \Rightarrow cRa$. $\Rightarrow\Leftarrow$

equivalence relation False, not transitive.

partial function True. $aRb \wedge aRb \Rightarrow b = b\checkmark$ $bRc \wedge bRc \Rightarrow cRc\checkmark$

(G_6) Formally, G_6 is the pairing of the set of states $S = \{a, b, c\}$ with the relation $R = \{(a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b)\}$.

reflexive True.

symmetric True.

anti-symmetric False, $aRb \wedge bRa \Rightarrow a = b$. $\Rightarrow\Leftarrow$

transitive False, $aRb \wedge bRc \Rightarrow aRc$. $\Rightarrow\Leftarrow$

euclidean False, $bRa \wedge bRc \Rightarrow aRc$. $\Rightarrow\Leftarrow$

equivalence relation False, not transitive.

partial function False, $aRa \wedge aRb \Rightarrow a = b$. $\Rightarrow\Leftarrow$

(G_7) Formally, G_7 is the pairing of the set of states $S = \{a, b, c\}$ with the relation $R = \{(a, a), (b, b), (c, c), (b, a), (b, c)\}$.

reflexive True.

symmetric False, $bRc \Rightarrow cRb$. $\Rightarrow\Leftarrow$

anti-symmetric True, $aRa \wedge aRa \Rightarrow a = a\checkmark$
$bRa \wedge aRa \Rightarrow a = a\checkmark$
Similar for c instead of a in both verifications.

transitive True, $bRa \wedge aRa \Rightarrow bRa\checkmark$ Similar for c instead of a.
Reflexive nodes are trivial.

euclidean False, $bRa \wedge bRc \Rightarrow aRc$. $\Rightarrow\Leftarrow$

equivalence relation False, not symmetric.

partial function False, $bRb \wedge bRc \Rightarrow b = c$. $\Rightarrow\Leftarrow$

(G_8) Formally, G_8 is the pairing of the set of states $S = \{a, b, c\}$ with the relation $R = \{(a, a), (a, b), (b, b), (b, a), (c, b), (c, a)\}$.

reflexive	False, $(c, c) \notin R$.
symmetric	False, counter-example: $cRb \Rightarrow bRc$. $\Rightarrow\Leftarrow$
anti-symmetric	False, counter-example: $aRb \wedge bRa \Rightarrow a = b$. $\Rightarrow\Leftarrow$
transitive	True, $aRb \wedge bRb \Rightarrow bRb$✓ $cRa \wedge aRb \Rightarrow cRb$✓ Similar in other cases, where a and b are switched.
euclidean	True, $aRb \wedge aRa \Rightarrow bRa$✓ $bRb \wedge bRa \Rightarrow bRa$✓ $cRa \wedge cRb \Rightarrow aRb$. Similar in other cases, where a and b are switched
equivalence relation	False, only transitive; i.e. not reflexive and not symmetric.
partial function	False, $cRa \wedge cRb \Rightarrow a = b$. $\Rightarrow\Leftarrow$

2 True or false, and why?

The statements below are about relations on a fixed set X. For each statement below, say whether it is true or false. If it is true, give a short proof. If it is false, give a counter-example.

1. Every reflexive relation on X is serial.

   Proof. Take a reflexive relation R on a set X.
   Show $\forall x.\exists x'.(x, x') \in R$.
   Fix x. Show there exists an x'.
   We can choose $x' = x$ since $(x, x) \in R$ since R is reflexive. □

2. Every relation on X which is reflexive and transitive is also symmetric.

   False, counter-example:
   Let $X = \{a, b, c\}$ and $R = \{(a, a), (b, b), (c, c), (a, b), (a, c), (b, c)\}$.
   R is reflexive since $(a, a), (b, b),$ and $(c, c) \in R$.
   When showing transitivity we need not consider the reflexive pairs (otherwise the implication is trivial), so we need only observe that the following holds: $(a, b) \in R \wedge (b, c) \in R \Rightarrow (a, c) \in R$. To show that R is not symmetric we need only see that $(a, b) \in R \Rightarrow (b, a) \in R$ is false. We're done.

3. Every relation on X which is Euclidean, symmetric and transitive is reflexive.

   False, consider the empty relation as a counter-example. Vacuously, it is Euclidean, symmetric and transitive, but it is not reflexive because it is empty.

4. Every relation on X which is Euclidean and reflexive is symmetric.

   Proof. Take such an R.
   Assume xRy. Show yRx.
   R Reflexive $\Rightarrow xRx$.
   R Euclidean $\Rightarrow xRy \wedge xRx \Rightarrow yRx$. Done. □

3 Multiply two graphs

$G_1 = (H_1, R_1)$ where $H_1 = \{a, b, c\}$ and $R_1 = \{(a, b), (a, c), (b, b), (c, b)\}$
$G_2 = (H_2, R_2)$ where $H_2 = \{d, e\}$ and $R_2 = \{(d, d), (d, e), (e, e)\}$
For $G_1 \times G_2$, we first take the Cartesian product of H_1 and H_2.
$H_1 \times H_2 = \{(a, d), (a, e), (b, d), (b, e), (c, d), (c, e)\}$
Then the new relation $R = \{((g, h), (g', h')) : gR_1g' \wedge hR_2h'\}$
Computing R gives:
$\{((a, d), (b, d)), ((a, d), (b, e)), ((a, d), (c, d)), ((a, d), (c, e)),$
$((a, e), (b, e)), ((a, e), (c, e)), ((b, d), (b, d)), ((b, d), (b, e)),$
$((b, e), (b, e)), ((c, d), (b, d)), ((c, d), (b, e)), ((c, e), (b, e))\}$
We provide a picture for clarity:

4 True or false, and why?

For each of the following assertions, tell whether it is true or false. (As always in a problem like this, "true" means *true for all graphs*. So "false" means *false for some graphs*.) Give a short proof of the true ones, and a counter-example for the false ones.

1. The product of two reflexive graphs is reflexive. True.

   Proof. Take two reflexive graphs, G, H.
   Show $G \times H$ is reflexive.
   Let $(g, h) \in G \times H$. Show $((g, h), (g, h)) \in G \times H$.
   This means we must show $(g, g) \in G$ and $(h, h) \in H$.
   This follows from G and H both being reflexive.
   Hence $G \times H$ is reflexive. □

2. The product of two euclidean graphs is reflexive.
   False, we provide a counter-example:

   $G = (G', R_G)$ where $G' = \{a, b, c\}$
   $H = (H', R_H)$ where $H' = \{d, e\}$

   (A piece of scratch paper is attached showing that G and H are Euclidean, and hope that a picture helps illustrates this as well.)

   We show that $G \times H$ is not reflexive by computing the new nodes that start with (a, d). We have $(a, b), (a, c) \in R_G$, and $(d, d), (d, e) \in R_H$.
   Then $((a, d), (b, d)), ((a, d), (b, e)), ((a, d), (c, d)), ((a, d), (c, e)) \in R_{G \times H}$
   Yet $((a, d), (a, d)) \notin R_{G \times H}$, so $G \times H$ is not reflexive.

3. If the product of two graphs is reflexive, then both of the given graphs are reflexive.

   Proof. Assume $G \times H$ is reflexive. Show G is reflxive and H is reflexive.
   Take $(g, h) \in G \times H$.
   Then $G \times H$ is reflexive means $((g, h), (g, h)) \in R_{G \times H}$.
   By the definition of $R_{G \times H}$, we know that $(g, g) \in R_G$ and $(h, h) \in R_H$.
   It follows that G is reflexive and H is reflexive since we chose an arbitrary (g, h) in $G \times H$. □

4. If the product of two graphs is transitive, then both of the given graphs are transitive.

   False, counter-example: consider the cross product of any non-transitive graph G and the empty graph. This product graph is vacuously transitive, since there are no nodes. Hence the product is transitive, but it is not true that BOTH of the given graphs are transitive, since we took G non-transitive.

5. Let G be a one-point graph with no arrows. Then for all $H, G \times H$ looks like H. False, take $H = (\{a, b\}, \{(a, a), (a, b)\})$. Then label the point in G as c. $G \times H = (\{(c, a), (c, b), \{\})$ since there are no arrows from c in G. Obviously this two-point graph with no arrows does not look like the graph H.

5 Modal semantics with boxes and diamonds rather than K-operators

Attached.