Modal Logic, Winter 2019
Homework 11
Due Tuesday, April 16

1. This problem builds on Homework 10, problem 3. Recall that we had sentences φ_{ht}, φ^*, ψ, and χ. It won't matter what those sentences actually were, but it will be important to read again the problem and all parts of what you showed.

   Prove the following: Let φ be a modal sentence.[1] Let $\mathcal{M}$ and $\mathcal{N}$ be any two models.

   (a) Suppose that $x \in M$ and $y \in N$ both satisfy χ_h. Then $x \Vdash \varphi$ in $\mathcal{M}$ if and only if $y \Vdash \varphi$ in $\mathcal{N}$.

   (b) Suppose that $x \in M$ and $y \in N$ both satisfy χ_t. Then $x \Vdash \varphi$ in $\mathcal{M}$ if and only if $y \Vdash \varphi$ in $\mathcal{N}$.

   We prove (a) and (b) together, by induction on φ. Do the base case of h, and the induction step for K_a.

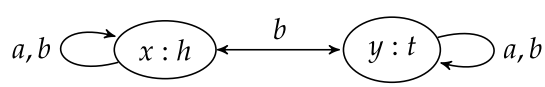
   (In the base case for h, you cannot use the fact that $\models_{all} t \leftrightarrow \neg h$. This is because h and t are just two atomic sentences, so the sentence $t \leftrightarrow \neg h$ is *not* true at all points of all models. Instead, we need to know that $\vdash \chi_h \to (h \oplus t)$, and also $\vdash \chi_t \to (h \oplus t)$.)

   In the induction step for K_a, you will need to use many of the parts of Homework 10, problem 3. So be sure to refer to those parts correctly. If you think that (b) in the induction step is similar to (a), then you can just say this.

2. This problem is a continuation of the previous one, and indeed is an application of that problem. (So you can solve this problem without getting the previous one.) Let φ be any modal sentence. Prove that either $\models_{all} \chi_h \to \varphi$ or else $\models_{all} \chi_h \to \neg\varphi$.

   [This one is almost an immediate consequence of the previous problem, if you look at it correctly. Given φ, you want to determine which of the two alternatives in our problem holds. It might make your life easier to look back at the model shown in Homework 10, problem 3, and to check that $x \Vdash \chi_h$.]

3. This problem concerns the model



   Tell whether the following are true or false, with a short explanation:

   (a) $x \models \langle h \rangle C K_{a,b} h$.

   (b) $x \models \langle t \rangle C K_{a,b} h$.

   (c) $x \models [t] C K_{a,b} h$.

   (d) $x \models \langle K_b\, h \rangle K_a\, K_b\, h$.

[1]For the purposes of Problems 1 and 2, a modal sentence is one built from the atomic sentences t and f, using $\wedge$, $\vee$, $\neg$, $\to$, and the operators K_a and K_b.

The explanations should involve restrictions of the model.

4. Let φ be the sentence $\neg K_a D_a$.

    (a) Show by example that $\not\models_{all} [\varphi] CK_{\mathcal{A}} \varphi$. [Hint: use an example from class.]

    (b) Now go back and read Homework 10, problem 2. What does part (2) tell us about Homework 10, problem 2?

5. This problem has to do with one of the important logical principles about announcements, a valid sentence that relates announcement and negation.

    (a) Prove that for all sentences φ and ψ that

    $$\models_{all} \langle\varphi\rangle\neg\psi \leftrightarrow (\varphi \wedge \neg\langle\varphi\rangle\psi)$$

    (b) The sentence above should remind us of the determinacy axiom from the $*$-logic. Stating this as a biconditional determinacy is $\neg * \varphi \leftrightarrow *\neg\varphi$. However, there is a small difference. The determinacy axiom for announcement logic would seem to be

    $$\langle\varphi\rangle\neg\psi \leftrightarrow \neg\langle\varphi\rangle\psi$$

    Show by an example that the sentence above is *not* valid. (That is, it is not true at all points of all models.) [Hint: do this when φ and ψ are some atomic sentences, say p and q. Consider a model and a world where p is false.]

    (c) Go back and read (a) again. Use it to show that for all sentences φ and ψ,

    $$\models_{all} [\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi)$$

    [Hint: fix φ and ψ, and write out the sentence in (a) with ψ replaced by $\neg\psi$. Then also use the fact that for all α and β, $[\alpha]\beta \leftrightarrow \neg\langle\alpha\rangle\neg\beta$, and also that $\neg(\gamma \rightarrow \delta) \leftrightarrow (\gamma \wedge \neg\delta)$ is a tautology of propositional logic.]