Modal Logic, Winter 2019
Homework 7
due Tuesday, March 5

# 1 Some derivations in $\mathcal{L}(*)$

1. $\vdash (\neg\varphi) \to (*\neg * \varphi)$.

| | | |
|---|---|---|
| 1. | $\neg\varphi$ | Assume |
| 2. | $* * \neg\varphi$ | Involution |
| 3. | $*\neg\varphi$ | $*_e, 2$ |
| 4. | $\neg * \varphi$ | Determinacy |
| 5. | $*\neg * \varphi$ | $*_i, 4$ |
| 6. | $\neg\varphi \to *\neg * \varphi$ | $\to_i, 1-5$ |

2. $\vdash (*\neg * \varphi) \to (\neg\varphi)$.

| | | |
|---|---|---|
| 1. | $*\neg * \varphi$ | Assume |
| 2. | $\neg * \varphi$ | $*_e, 1$ |
| 3. | $*\neg\varphi$ | Determinacy |
| 4. | $* * \neg\varphi$ | $*_i, 3$ |
| 5. | $\neg\varphi$ | Involution |
| 6. | $*\neg * \varphi \to \neg\varphi$ | $\to_i, 1-5$ |

3. $\vdash \varphi \to (\neg * \neg * \varphi)$.

| | | |
|---|---|---|
| 1. | $\varphi$ | Assume |
| 2. | $\neg\neg * \neg * \varphi$ | Assume |
| 3. | $* \neg * \varphi$ | $\neg\neg_e, 2$ |
| 4. | $\neg\varphi$ | Result of 2 above |
| 5. | $\mathsf{F}$ | $\mathsf{F}_i, 1, 4$ |
| 6. | $\neg\neg\neg * \neg * \varphi$ | $\neg_i, 2-5$ |
| 7. | $\neg * \neg * \varphi$ | $\neg\neg_e, 6$ |
| 8. | $\varphi \to \neg * \neg * \varphi$ | $\to_i, 1-7$ |

## 2 Another inductive step in the lemma on $*$-SDs

Here is one of the main results on $*$-SDs:

Let $\alpha$ be a $*$-state description (of some order or other). Let $occ(\varphi) \subseteq occ(\alpha)$.
Then both of the following hold:

1. Either $\vdash \alpha \to \varphi$, or else $\vdash \alpha \to \neg\varphi$
2. Either $\vdash \alpha \to *\varphi$, or else $\vdash \alpha \to *\neg\varphi$

We prove this by induction on sentences in the language $\mathcal{L}(*)$. Do the inductive step for $\neg$. That is, write clearly what you assume and what you prove for that step; then do the proof. If some of it is the same as what we saw for propositional logic, you can just say so.

You only have to do the one inductive step, the one for $\neg$.

Fix $\alpha$ and $\varphi$ and we assume about them the following:

If $occ(\varphi) \subseteq occ(\alpha)$ then either $\vdash \alpha \to \varphi$ or else $\vdash \alpha \to \neg\varphi$ and either $\vdash \alpha \to *\varphi$ or else $\vdash \alpha \to *\neg\varphi$.

We prove if $occ(\neg\varphi) \subseteq occ(\alpha)$ then either $\vdash \alpha \to \neg\varphi$ or else $\vdash \alpha \to \neg\neg\varphi$ and either $\vdash \alpha \to *\neg\varphi$ or else $\vdash \alpha \to *\neg\neg\varphi$

Assume $occ(\neg\varphi) \subseteq occ(\alpha)$. The definition of $occ$ shows that $occ(\neg\varphi) = occ(\varphi)$, so $occ(\varphi) \subseteq occ(\alpha)$. Now our induction hypothesis kicks in and we have four cases:

1. $\vdash \alpha \to \varphi$ and $\vdash \alpha \to *\varphi$

   We prove $\vdash \alpha \to \neg\neg\varphi$ and $\vdash \alpha \to *\neg\neg\varphi$.

| | | |
|---|---|---|
| 1. | $\alpha \to \varphi$ | Premise |
| 2. | $\alpha \to *\varphi$ | Premise |
| 3. | $\alpha$ | Assume |
| 4. | $\varphi$ | $\to_e, 1, 3$ |
| 5. | $\neg\neg\varphi$ | Propositional |
| 6. | $\alpha \to \neg\neg\varphi$ | $\to_i, 3-5$ |
| 7. | $\alpha$ | Assume |
| 8. | $*\varphi$ | $\to_e, 2, 7$ |
| 9. | $\varphi$ | $*_e, 8$ |
| 10. | $\neg\neg\varphi$ | Propositional |
| 11. | $*\neg\neg\varphi$ | $*_i, 10$ |
| 12. | $\alpha \to *\neg\neg\varphi$ | $\to_i, 7-12$ |
| 13. | $\alpha \to \neg\neg\varphi \wedge \alpha \to *\neg\neg\varphi$ | $\wedge_i, 6, 12$ |

2. $\vdash \alpha \to \varphi$ and $\vdash \alpha \to *\neg\varphi$

   We prove $\vdash \alpha \to \neg\neg\varphi$ and $\vdash \alpha \to *\neg\varphi$

| | | |
|---|---|---|
| 1. | $\alpha \to \varphi$ | Premise |
| 2. | $\alpha \to *\neg\varphi$ | Premise |
| 3. | $\alpha$ | Assume |
| 4. | $\varphi$ | $\to_e, 1, 3$ |
| 5. | $\neg\neg\varphi$ | Propositional |
| 6. | $\alpha \to \neg\neg\varphi$ | $\to_i, 3-5$ |
| 7. | $\alpha \to \neg\neg\varphi\alpha \to *\neg\varphi$ | $\wedge_i, 7, 2$ |

3. $\vdash \alpha \to \neg\varphi$ and $\vdash \alpha \to *\varphi$

   We prove $\vdash \alpha \to \neg\varphi$ and $\vdash \alpha \to *\neg\neg\varphi$

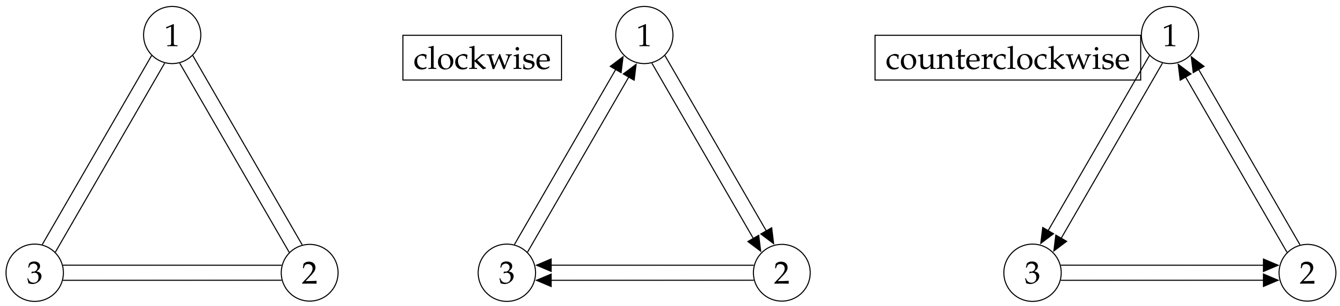
| | | |
|---|---|---|
| 1. | $\alpha \to \neg\varphi$ | Premise |
| 2. | $\alpha \to *\varphi$ | Premise |
| 3. | $\alpha$ | Assume |
| 4. | $*\varphi$ | $\to_e, 2$ |
| 5. | $\varphi$ | $*_e, 4$ |
| 6. | $\neg\neg\varphi$ | Propositional |
| 7. | $*\neg\neg\varphi$ | $*_i, 5-6$ |
| 8. | $\alpha \to *\neg\neg\varphi$ | $\to_i, 3-7$ |
| 9. | $\alpha \to \neg\varphi \wedge \alpha \to *\neg\neg\varphi$ | $_i, 1, 8$ |

4. $\vdash \alpha \to \neg\varphi$ and $\vdash \alpha \to *\neg\varphi$

   This case is trivial because the things we want to prove are given to us by the hypothesis.

# 3 Triangular rat houses

Suppose that we move Ronnie the Rat to a triangular house that looks like the model on the left below:




These houses have three rooms. He can travel *clockwise* (cw) or *counterclockwise* (ccw), as shown.

We'd like to reason in this new setting. So we make a logical language with operators $*_{cw}$ and $*_{ccw}$. We call this language $\mathcal{L}(*_{cw}, *_{ccw})$.

1. Fill in the chart below, telling what the semantics should be for this logic:

| | | | | | |
|---|---|---|---|---|---|
| $1 \Vdash *_{cw}\ \varphi$ | iff | $2 \Vdash \varphi$ | $1 \Vdash *_{ccw}\ \varphi$ | iff | $3 \Vdash \varphi$ |
| $2 \Vdash *_{cw}\ \varphi$ | iff | $3 \Vdash \varphi$ | $2 \Vdash *_{ccw}\ \varphi$ | iff | $1 \Vdash \varphi$ |
| $3 \Vdash *_{cw}\ \varphi$ | iff | $1 \Vdash \varphi$ | $3 \Vdash *_{ccw}\ \varphi$ | iff | $2 \Vdash \varphi$ |

2. In the $*$-logic, we had sound logical principles such as involution ($**\varphi \leftrightarrow \varphi$) and determinacy ($\neg * \varphi \leftrightarrow *\neg\varphi$). What are the corresponding sound principles for $\mathcal{L}(*_{cw}, *_{ccw})$?

   The sound principles are involution for both,

   $*_{cw}\ *_{cw}\ *_{cw}\ \varphi \leftrightarrow \varphi$ and $*_{ccw}\ *_{ccw}\ *_{ccw}\ \varphi \leftrightarrow \varphi$,

   determinacy for both,

   $*_{cw}\ \neg\varphi \leftrightarrow \neg *_{cw}\ \varphi$ and $*_{ccw}\ \neg\varphi \leftrightarrow \neg *_{ccw}\ \varphi$

   and two new principles which we call inversion:

   $*_{cw}\ *_{ccw}\ \varphi \leftrightarrow \varphi$ and $*_{ccw}\ *_{cw}\ \varphi \leftrightarrow \varphi$

3. Suppose we want a proof system for validity, as we have done with our other logics. How should the proof system work regarding subproofs?

   The subproofs will work just like $*$-subproofs – that is, we can eliminate a $*_{cw}$ or $*_{ccw}$ for a subproof, then stick it back on when we close the subproof – but we must make a clarification. If we ever have two sentences in a proof like $*_{cw}\ \varphi$ and $*_{ccw}\ \psi$, if we open a $*_{cw}$-subproof, we get $\varphi$ in the subproof by $*_{cw}$ elimination but we do not get $\psi$. Similar principle holds if we did a $*_{ccw}$-subproof instead.

We take 6 axioms:

1. $*cw \ *cw \ *cw\ \varphi \rightarrow \varphi$
2. $*ccw \ *ccw \ *ccw\ \varphi \rightarrow \varphi$
3. $\neg *cw \ \varphi \rightarrow *cw\ \neg\varphi$
4. $\neg *ccw \ \varphi \rightarrow *ccw\ \neg\varphi$
5. $*cw \ *ccw \ \varphi \rightarrow \varphi$
6. $*ccw \ *cw \ \varphi \rightarrow \varphi$

4. Prove what you said in part (2) in the system which you constructed in part (3).

   We prove the $*cw$ version of the 6 axioms (there are 3 such) and we abstract over these proof methods to conclude the same for the $*ccw$ version. This is because the proofs are be essentially the same, with the exception being that every $*cw$ is replaced by a $*ccw$. To demonstrate, we will do the $\varphi \rightarrow *ccw \ *ccw \ *ccw\ \varphi$ proof and assume the rest follow in similar fashion of proof rewriting.

1. $\vdash \varphi \rightarrow {*cw}\ {*cw}\ {*cw}\ \varphi$

| | | |
|---|---|---|
| 1. | $\varphi$ | Assume |
| 2. | $\neg {*cw}\ {*cw}\ {*cw}\ \varphi$ | Assume |
| 3. | ${*cw} \neg {*cw}\ {*cw}\ \varphi$ | ${*cw}$ Determinacy |
| 4. | $\neg {*cw}\ {*cw}\ \varphi$ | ${*cw}\ _e, 3$ |
| 5. | ${*cw} \neg {*cw}\ \varphi$ | ${*cw}$ Determinacy |
| 6. | $\neg {*cw}\ \varphi$ | ${*cw}\ _e, 5$ |
| 7. | ${*cw} \neg\varphi$ | ${*cw}$ Determinacy |
| 8. | ${*cw}\ {*cw}\ \neg\varphi$ | ${*cw}\ _i, 7$ |
| 9. | ${*cw}\ {*cw}\ {*cw} \neg\varphi$ | ${*cw}\ _i, 8$ |
| 10. | $\neg\varphi$ | ${*cw}$ Involution |
| 11. | $\mathsf{F}$ | $\mathsf{F}_i, 1, 10$ |
| 12. | $\neg\neg {*cw}\ {*cw}\ {*cw}\ \varphi$ | $\neg_i, 2-11$ |
| 13. | ${*cw}\ {*cw}\ {*cw}\ \varphi$ | Propositional |
| 14. | $\varphi \rightarrow {*cw}\ {*cw}\ {*cw}\ \varphi$ | $\rightarrow_i, 1-13$ |

2. $\vdash \varphi \to *ccw\ *ccw\ *ccw\ \varphi$

| | | |
|---|---|---|
| 1. | $\varphi$ | Assume |
| 2. | $\neg *ccw\ *ccw\ *ccw\ \varphi$ | Assume |
| 3. | $*ccw\ \neg *ccw\ *ccw\ \varphi$ | $*ccw$ Determinacy |
| 4. | $\neg *ccw\ *ccw\ \varphi$ | $*ccw_{e}, 3$ |
| 5. | $*ccw\ \neg *ccw\ \varphi$ | $*ccw$ Determinacy |
| 6. | $\neg *ccw\ \varphi$ | $*ccw_{e}, 5$ |
| 7. | $*ccw\ \neg\varphi$ | $*ccw$ Determinacy |
| 8. | $*ccw\ *ccw\ \neg\varphi$ | $*ccw_{i}, 7$ |
| 9. | $*ccw\ *ccw\ *ccw\ \neg\varphi$ | $*ccw_{i}, 8$ |
| 10. | $\neg\varphi$ | $*ccw$ Involution |
| 11. | $\mathsf{F}$ | $\mathsf{F}_i, 1, 10$ |
| 12. | $\neg\neg *ccw\ *ccw\ *ccw\ \varphi$ | $\neg_i, 2-11$ |
| 13. | $*ccw\ *ccw\ *ccw\ \varphi$ | Propositional |
| 14. | $\varphi \to *ccw\ *ccw\ *ccw\ \varphi$ | $\to_i, 1-13$ |

3. $\vdash *cw\ \neg\varphi \rightarrow \neg *cw\ \varphi$

| | | |
|---|---|---|
| 1. | $*cw\ \neg\varphi$ | Assume |
| 2. | $\neg\neg *cw\ \varphi$ | Assume |
| 3. | $*cw\ \varphi$ | Prop |
| 4. | $\varphi$ | $*cw\ _e, 2$ |
| 5. | $\neg\varphi$ | $*cw\ _e, 1$ |
| 6. | $\mathsf{F}$ | $\mathsf{F}_i$ |
| 7. | $\neg\neg\neg *cw\ \varphi$ | $\neg_i$ |
| 8. | $\neg *cw\ \varphi$ | Prop |
| 9. | $*cw\ \neg\varphi \rightarrow \neg *cw\ \varphi$ | $\rightarrow_i, 1-8$ |

4. $\vdash \varphi \rightarrow *cw\ \ *ccw\ \ \varphi$

| | | |
|---|---|---|
| 1. | $\varphi$ | Assume |
| 2. | $\neg *cw\ \ *ccw\ \varphi$ | Assume |
| 3. | $*cw\ \neg *ccw\ \ \varphi$ | $*cw$ Determinacy |
| 4. | $\neg *ccw\ \ \varphi$ | $*cw\ _e, 3$ |
| 5. | $*ccw\ \neg\varphi$ | $*ccw$ Determinacy |
| 6. | $*cw\ \ *ccw\ \ \neg\varphi$ | $*cw\ _i, 5$ |
| 7. | $\neg\varphi$ | Inversion |
| 8. | $\mathsf{F}$ | $\mathsf{F}_i, 1, 7$ |
| 9. | $\neg\neg *cw\ \ *ccw\ \varphi$ | $\neg_i, 2-8$ |
| 10. | $*cw\ \ *ccw\ \ \varphi$ | Prop |
| 11. | $\varphi \rightarrow *cw\ \ *ccw\ \ \varphi$ | $\rightarrow_i, 1-10$ |

Now we have

1. $*cw\ *cw\ *cw\ \varphi \leftrightarrow \varphi$
2. $*ccw\ *ccw\ *ccw\ \varphi \leftrightarrow \varphi$
3. $\neg *cw\ \varphi \leftrightarrow *cw\ \neg\varphi$
4. $\neg *ccw\ \varphi \leftrightarrow *ccw\ \neg\varphi$
5. $*cw\ *ccw\ \varphi \leftrightarrow \varphi$
6. $*ccw\ *cw\ \varphi \leftrightarrow \varphi$

We will use these axioms/principles to prove an interesting theorem: $*cw\ \varphi \leftrightarrow *ccw\ *ccw\ \varphi$

0. But first we would like to use the following lemma:
   $\vdash \neg *ccw\ *ccw\ *ccw\ \varphi \rightarrow \neg\varphi$

| | | |
|---|---|---|
| 1. | $\neg *ccw\ *ccw\ *ccw\ \varphi$ | Assume |
| 2. | $*ccw\ \neg *ccw\ *ccw\ \varphi$ | Determinacy |
| 3. | $\neg *ccw\ *ccw\ \varphi$ | $*ccw_e$ |
| 4. | $*ccw\ \neg *ccw\ \varphi$ | Determinacy |
| 5. | $\neg *ccw\ \varphi$ | $*ccw_e$ |
| 6. | $*ccw\ \neg\varphi$ | Determinacy |
| 7. | $*ccw\ *ccw\ \neg\varphi$ | $*ccw_i$ |
| 8. | $*ccw\ *ccw\ *ccw\ \neg\varphi$ | $*ccw_i$ |
| 9. | $\neg\varphi$ | $*ccw$ Involution |
| 10. | $\neg *ccw\ *ccw\ *ccw\ \varphi \rightarrow \neg\varphi$ | $\rightarrow_i$ |

1. $\vdash *cw\ \varphi \rightarrow *ccw\ \ *ccw\ \ \varphi$

| | | |
|---|---|---|
| 1. | $*cw\ \varphi$ | Assume |
| 2. | $\neg *ccw\ \ *ccw\ \varphi$ | Assume |
| 3. | $*cw\ \ *ccw\ \ \neg *ccw\ \ *ccw\ \varphi$ | Inversion |
| 4. | $*ccw\ \neg *ccw\ \ *ccw\ \varphi$ | $*cw\ _e$ |
| 5. | $\neg *ccw\ \ *ccw\ \ *ccw\ \ \varphi$ | Determinacy |
| 6. | $\neg\varphi$ | Lemma above |
| 7. | $*cw\ \neg\varphi$ | $*cw\ _i$ |
| 8. | $\mathsf{F}$ | $\mathsf{F}_i 1, 7$ |
| 9. | $\neg\neg *ccw\ \ *ccw\ \varphi$ | $\neg_i, 2-8$ |
| 10. | $*ccw\ \ *ccw\ \ \varphi$ | Prop |
| 11. | $*cw\ \varphi \rightarrow *ccw\ \ *ccw\ \ \varphi$ | $\rightarrow_i, 1-10$ |

Note that this proof gives way to an equivalent theorem: $\vdash *ccw\ \varphi \rightarrow *cw\ \ *cw\ \ \varphi$, the only essential difference is the Inversion rule used.

2. $\vdash *ccw\ \ *ccw\ \ \varphi \rightarrow *cw\ \varphi$

| | | |
|---|---|---|
| 1. | $*ccw\ \ *ccw\ \ \varphi$ | Assume |
| 2. | $*cw\ \ *cw\ \ *ccw\ \varphi$ | Previous result |
| 3. | $*cw\ \ *ccw\ \ \varphi$ | $*cw\ _e$ |
| 4. | $\varphi$ | Inversion |
| 5. | $*cw\ \varphi$ | $*cw\ _i$ |
| 6. | $*ccw\ \ *ccw\ \ \varphi \rightarrow *cw\ \varphi$ | $\rightarrow_i$ |

# 4 Eating

Let's go back to $\mathcal{L}(*)$. Suppose that we wish to add a new action, $\Diamond_{eat}$. We want this action to correspond in a given room to there bing food in that room, and then having Ronnie eating all of it up.

So now we have a new syntax, giving us a language $\mathcal{L}(*, \Diamond_{eat})$. We get sentences like

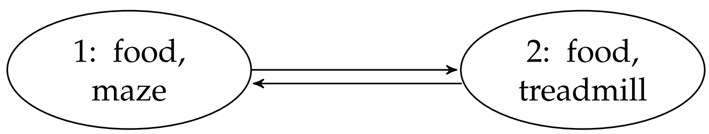
$$\Diamond_{eat} (*\text{water} \wedge \Diamond_{eat} \text{ food}).$$

Now we turn to the semantics. We need to say what it means to say that

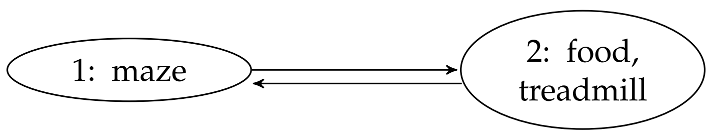
$$i \Vdash \varphi \text{ in } M$$

Where $M$ is a model (a two-room house), and $i$ is one of its two rooms, and $\varphi$ is a sentence in our new language $\mathcal{L}(*, \Diamond_{eat})$. For a model $M$, let $M_{-food,i}$ be the same as $M$, except that we change the valuation so that room $i$ contains no food.

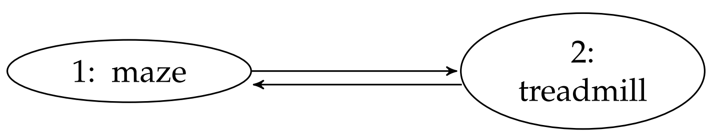
For example, if $M$ is




Then $M_{-food,1}$ is




and $(M_{-food,1})_{-food,2}$ is




Now we define:

$$i \Vdash \Diamond_{eat} \varphi \text{ in } M \qquad \text{iff} \qquad \text{food} \in Val_M(i), \text{ and also } i \Vdash \varphi \text{ in } M_{-food,i}$$

Note that if we have a model $M$ where there is no food in room $i$, then $i \Vdash \Diamond_{eat} \varphi$ is false in $M$.

With $M$ as above, tell whether the following are true or false, with reasons:

1. $1 \Vdash \Diamond_{eat}$ food in $M$.

   False. $1 \Vdash \Diamond_{eat}$ food if and only if food $\in Val_M(1)$ and $1 \Vdash$ food in $M_{-food,1}$. Checking the model we see food is in room 1, but after eating (i.e. in $M_{-food,1}$), there is no food in room 1, so $1 \Vdash$ food is false, meaning the sentence in total is false.

2. $1 \Vdash \Diamond_{eat} * \Diamond_{eat} \top$ in $M$.

   True. We know food $\in Val_M(1)$ so we must check if $1 \Vdash * \Diamond_{eat} \top$ in $M_{-food,1}$. Since $1 \Vdash *\varphi \iff 2 \Vdash \varphi$, we check $2 \Vdash \Diamond_{eat} \top$. The valuation shows that food $\in Val_{M'}(2)$, so we must check if $2 \Vdash \top$ in $M_{-food,1,2}$. Well, $\top$ is true everywhere, so $2 \Vdash \top$ is true, making the sentence true.

And which of the following are valid (true at all rooms of all models)?

3. $\Diamond_{eat}$ ¬food

   Not valid, consider the following valuation: $val_M(1)$ = treadmill and $val_M(2)$ = water. Here, neither 1 nor 2 satisfies $\Diamond_{eat}$ ¬food because food $\notin Val_M(1)$ and food $\notin Val_M(2)$. Since the sentence is not true in either room of this model, we know it is not valid.

4. $\Diamond_{eat}$ food

   Not valid, consider the valuation: $val_M(1)$ = food and $val_M(2)$ = water. $2 \nVdash \Diamond_{eat}$ food because food $\notin val_M(2)$. Since the sentence is not true at room 2 of this model, we know it is not valid.

5. ($\Diamond_{eat}$ water) ↔ (food ∧ water).

   This is valid, we give a proof to demonstrate. Fix a model $M$ and a world $i \in M$.

   → Assume $i \Vdash \Diamond_{eat}$ water. We prove $i \Vdash$ food ∧ water. Our assumption means food $\in Val_M(i)$ and $i \Vdash$ water in $M_{-food,i}$. So we know $i \Vdash$ water in $M$ because the only change was about food. Since food $\in Val_M(i)$, we know $i \Vdash$ food in $M$. Then we can conclude $i \Vdash$ food ∧ water.

   ← Assume $i \Vdash$ food ∧ water. Show $i \Vdash \Diamond_{eat}$ water. Our assumption means $i \Vdash$ food and $i \Vdash$ water, so {food, water} $\subseteq Val_M(i)$. To show $i \Vdash \Diamond_{eat}$ water we need food $\in Val_M(i)$ and $i \Vdash$ water in $M_{-food,i}$. We know food $\in Val_M(i)$ by our assumption, and similarly we know $i \Vdash$ water in $M_{-food,i}$ because removing food does not affect water's presence. Hence we know $i \Vdash \Diamond_{eat}$ water.

   Done!

# 5 Eliminating $\Diamond_{eat}$

We say that two sentences, $\varphi_1$ and $\varphi_2$, in modal logic are *semantically equivalent* if for every model $M$ and every world $w$ in it, $w \Vdash \varphi_1$ if and only if $w \Vdash \varphi_2$.

So for the logic $\mathcal{L}(*, \Diamond_{eat})$, two sentences are semantically equivalent if for every rat house, each room $i$ satisfies one of our sentences if and only if it satisfies the other.

Prove that for every sentence $\varphi$ of $\mathcal{L}(*, \Diamond_{eat})$ there is a sentence $\widehat{\varphi}$ of $\mathcal{L}(*)$ such that $\varphi$ and $\widehat{\varphi}$ are logically equivalent.

You should give a recursive definition of $\widehat{\varphi}$ as a function of $\varphi$. In other words, in defining $\widehat{\varphi}$, you can use $\widehat{\psi}$ for various subsentences of $\varphi$.

You don't have to prove your answer correct by induction (but that would be nice); you should just say what the main steps of the proof would be.

**Clarification** This problem will be really hard to do in full, unless you know some rather advanced topics. What you should try to do is to write down *recursion equations for the elimination of* $\Diamond_{eat}$. That is, you want to come up with facts like the following:

$$\begin{array}{rcl} \Diamond_{eat} \text{ water} & \leftrightarrow & \text{food} \wedge \text{water} \\ \Diamond_{eat} \Diamond_{eat} \varphi & \leftrightarrow & \mathsf{F} \end{array}$$

The point is that these two sentences are valid (true in every room of every house). And so part of the definition of $\widehat{\varphi}$ will be

$$\begin{array}{rcl} (\Diamond_{eat} \text{ water})\hat{} & = & \text{food} \wedge \text{water} \\ (\Diamond_{eat} \Diamond_{eat} \varphi)\hat{} & = & \mathsf{F} \end{array}$$

(I have written $\widehat{\varphi}$ as $\varphi\hat{}$ for typographic reasons.)

And you also will have equations like

$$(\varphi \wedge \psi)\hat{} \;=\; \varphi\hat{} \wedge \psi\hat{}$$

It would be really nice if you could have enough recursion equations to "cover all the cases". That is, you want to have enough equations that *every sentence in the new language* $\mathcal{L}(*, \Diamond_{eat})$ *is covered by one of your equations*.

We define two functions, $f$ and $g$ both have domain $\mathcal{L}(*, \Diamond_{eat})$ and codomain $\mathcal{L}(*)$. Interpretation: $f$ is the $\varphi\hat{}$ function mentioned in the problem; $g$ is a function that returns $\mathsf{F}$ if it ever is called on food, otherwise it runs $f$ on the input.

$$\begin{aligned} f(\mathsf{T}) &= \mathsf{T} \\ f(\mathsf{F}) &= \mathsf{F} \\ f(\neg\varphi) &= \neg f(\varphi) \\ f(\varphi \wedge \psi) &= f(\varphi) \wedge f(\psi) \\ f(\varphi \vee \psi) &= f(\varphi) \vee f(\psi) \\ f(\varphi \to \psi) &= f(\varphi) \to f(\psi) \\ f(\varphi \leftrightarrow \psi) &= (f(\varphi) \to f(\psi)) \wedge (f(\psi) \to f(\varphi)) \\ f(*\varphi) &= *f(\varphi) \\ f(\Diamond_{eat} \varphi) &= \text{food} \wedge g(\varphi) \end{aligned}$$

$$\begin{aligned} g(\text{food}) &= \mathsf{F} \\ g(**\varphi) &= g(\varphi) \\ g(\varphi) &= f(\varphi) \end{aligned}$$