M482 Homework 3

Joshua Larkin

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1 Satisfiability in modal logic

A modal sentence ϕ is *satisfiable* if there is some model M and some $x \in M$ such that $x \models \phi$. Show that each of the following sentences is satisfiable.

1. $(\Diamond\Box p) \wedge (\Box\Diamond p)$

   Let $M = (W, R, \text{val})$ be a model where...
   $W = \{x, y\}$, $R = \{(x, y), (y, y)\}$, and val $= \{(x, \{\}), (y, \{p\})\}$
   We know $(\Diamond\Box p)$ is true because $x \to y$ and $y \to y$, and $y \models p$.
   By the same argument, $(\Box\Diamond p)$ is true – (because both x and y only arrow y).

2. $(\Diamond\Box p) \wedge (\neg\Box\Diamond p)$

   Let $M = (W, R, \text{val})$ be a model where...
   $W = \{x, y, z\}$, $R = \{(x, y), (x, z), (y, y)\}$, and val $= \{(x, \{\}), (y, \{p\}), (z, \{\})\}$.
   We know $(\Diamond\Box p)$ is true because $x \to y$ and $y \to y$ (note that this is y's only arrow), and $y \models p$.
   We see that $(\neg\Box\Diamond p)$ because $x \to z$ and $z \not\models p$

3. $(\neg\Diamond\Box p) \wedge (\Box\Diamond p)$

   Let $M = (W, R, \text{val})$ be a model where...
   $W = \{x, y, z, w\}$, $R = \{(x, y), (y, w), (y, z)\}$,
   and val $= \{(x, \{\}), (y, \{\}), (z, \{p\}), (w, \{\})\}$.

   (a) Show $x \vDash (\neg\Diamond\Box p)$.
   $x \vDash \neg\Diamond\Box \longleftrightarrow x \nvDash \Diamond\Box p$.
   The only node that x can go to is y, so we must show $y \nvDash \Box p$.
   This follows from $(y, w) \in R$ and $(w, \{\}) \in$ val.
   Hence $y \nvDash \Box p$ and $x \nvDash \Diamond\Box p$.
   Therefore, $x \vDash \neg\Diamond\Box p$.

   (b) Show $x \vDash \Box\Diamond p$
   $x \vDash \Box\Diamond p \longleftrightarrow \forall x'.\ x \to x'.\ x' \vDash \Diamond p$
   The only such x' is y. We must show $y \vDash \Diamond p$.
   $y \vDash \Diamond p \longleftrightarrow \exists y'.\ y' \vDash p$. We have (y, w) and (y, z) in R.
   Since val$(z) = p$, we know $z \vDash p$; then $y \vDash \Diamond p$
   Hence, $x \vDash \Box\Diamond p$.

4. $(\neg\Diamond\Box p) \wedge (\neg\Box\Diamond p)$

   Let $M = (W, R, \mathrm{val})$ be a model where...
   $W = \{x, y, z, w\}$, $R = \{(x, y), (x, w), (w, w), (y, y), (y, z)\}$,
   and $\mathrm{val} = \{(x, \{\}), (y, \{p\}), (z, \{\}), (w, \{\})\}$.

   (a) $x \vDash \neg\Diamond\Box p$
   $x \vDash \neg\Diamond\Box p \longleftrightarrow x \nvDash \Diamond\Box p$.
   Since (x, y) and (x, w) in R, we must check that $y \nvDash \Box p$ and $w \nvDash \Box p$.
   In the case of $w \nvDash \Box p$, we only have (w, w) in R, and
   since $\mathrm{val}(w) = \varnothing$, we know $w \nvDash \Box p$.
   In the case of $y \nvDash \Box p$, we see that $z \nvDash p$ because $\mathrm{val}(z) = \varnothing$, hence
   $y \nvDash \Box p$.
   Therefore $x \nvDash \Diamond\Box p$.

   (b) $x \vDash \neg\Box\Diamond p$ $x \vDash \neg\Box\Diamond p \longleftrightarrow x \nvDash \Box\Diamond p$.
   Since (x, y) and (x, w) in R, we must check that $y \nvDash \Diamond p$ or $w \nvDash \Diamond p$.
   In the case of $w \nvDash \Box p$, we only have (w, w) in R, and
   since $\mathrm{val}(w) = \varnothing$, we know $w \nvDash \Diamond p$.
   Therefore $x \nvDash \Diamond\Box p$.

2 Satisfiability on a symmetric model

A modal sentence ϕ is *satisfiable on a symmetric model* if there is some *symmetric modelM* and some $x \in M$ such that $x \models \phi$. Which of the following sentences is satisfiable on a symmetric model?

1. $(\Diamond\Box p) \wedge (\Box\Diamond p)$

   Let $M = (W, R, \mathrm{val})$ be a model where...
   $W = \{x, y\}$, $R = \{(x, y), (y, x)\}$,
   and $\mathrm{val} = \{(x, \{p\}), (y, \{\})\}$.

   Show $x \vDash (\Diamond\Box p) \wedge (\Box\Diamond p)$

   (a) $x \vDash \Diamond\Box p$
   This if and only if $\exists x'.\ x \to x'.\ x' \vDash \Box p$.
   The only arrow from x goes to y. Since the only arrow from y goes
   to x, and $\mathrm{val}(x) = \{p\}$, we know $y \vDash \Box p$.
   Hence $x \vDash \Diamond\Box p$.

(b) $x \vDash \Box\Diamond p$
This if and only if $\forall x'.\ x \to x'.\ x' \vDash \Box p$.
The only arrow from x goes to y. Since the only arrow from y goes to x, and $\mathrm{val}(x) = \{p\}$, we know $y \vDash \Box p$.
Hence $x \vDash \Box\Diamond p$.

2. $(\Diamond\Box p) \wedge (\neg\Box\Diamond p)$
This is not satisfiable on any symmetric model.
Suppose for some world x, $x \vDash \Diamond\Box p$. Then in trying to show $x \nvDash \neg\Box\Diamond p$, we arrive at a contradiction. The contradiction lies in our graph being symmetric. If there were a world y such that $x \to y$ then there is always an arrow back to x, so p would have to be both true and not true at x, a contradiction.

3. $(\neg\Diamond\Box p) \wedge (\Box\Diamond p)$
Let $M = (W, R, \mathrm{val})$ be a model where...
$W = \{x, y, z\}$, $R = \{(x, y), (y, x), (y, z), (z, y)\}$,
and $\mathrm{val} = \{(x, \{p\}), (y, \{\}), (z, \{\})\}$.
Show $x \vDash (\neg\Diamond\Box p) \wedge (\Box\Diamond p)$.

(a) $x \vDash \neg\Diamond\Box p$
This if and only if $\exists x'.\ x \to x'.\ x' \nvDash \Box p$.
The only arrow from x goes to y so we must check $y \nvDash \Box p$.
There is an arrow from y to z and $\mathrm{val}(z) = \{\}$, hence $y \nvDash \Box p$.
Therefore $x \vDash \neg\Diamond\Box p$.

(b) $x \vDash \Box\Diamond p$
This if and only if $\forall x'.\ x \to x'.\ x' \vDash \Diamond p$.
The only arrow from x goes to y so we must check $y \vDash \Diamond p$.
There is an arrow from y to x, since we are working with a symmetric model, and $\mathrm{val}(x) = \{p\}$, hence $y \vDash \Diamond p$. Therefore $x \vDash \Box\Diamond p$

4. $(\neg\Diamond\Box p) \wedge (\neg\Box\Diamond p)$
Let $M = (W, R, \mathrm{val})$ be a model where...
$W = \{x, y, z\}$, $R = \{(x, y), (y, x)\}$,
and $\mathrm{val} = \{(x, \{\}), (y, \{p\})\}$.
Show $x \vDash (\neg\Diamond\Box p) \wedge (\Box\Diamond p)$.

(a) $x \vDash \neg\Diamond\Box p$
This if and only if there does not exist a world x' such that $x' \vDash \Box p$. Our only choice of x' is y, so we must show that $y \nvDash \Box p$. Since the model is symmetric, y arrows to x and the valuation of x gives an empty-set, hence $y \nvDash \Box p$. Therefore we conclude $x \vDash \neg\Diamond\Box p$.

(b) $x \vDash \neg\Box\Diamond p$ For this we need to show that y does not satisfy $\Diamond p$. Since y arrows only to x and x has a valuation of empty-set, we know this is satisfiable. Hence $x \vDash \neg\Box\Diamond p$.

3 Partitions on a fixed set X, and an order on them

1. Find five partitions of $\{1, 2, 3, 4\}$. There are fifteen in total, but you only need to find five of them. Call your partitions $\Pi_1, \ldots, \Pi_5$.

   (a) $\Pi_1 = \{\{1, 4\}, \{2, 3\}\}$
   (b) $\Pi_2 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$
   (c) $\Pi_3 = \{\{2, 4\}, \{1, 3\}\}$
   (d) $\Pi_4 = \{\{2\}, \{1, 3, 4\}\}$
   (e) $\Pi_5 = \{\{1\}, \{4\}, \{2, 3\}\}$

2. Let X be a set. Let Π and Π' be partitions on X. We say that Π' *refines* Π if every cell of Π' is a subset of a cell of Π. In other words, if $s \equiv_{\Pi'} t$, then $s \equiv_{\Pi} t$. We write $\Pi \leq \Pi'$. We also say that Π is *coarser* than Π', or that Π' is *finer* than Π.

   For example, let $X = \{1, 2, 3, 4, 5\}$. Then

   $$\{\{1, 2\}, \{3, 4\}\{5\}\} \leq \{\{1\}\{2\}\{3, 4\}\{5\}\}.$$

   Continuing with X as in part (1), consider

   $$(\{\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5\}, \leq)$$

   Is $\leq$ a preorder? Is it a partial order? Is it an equivalence relation?

   We give the mathematics of the $\leq$ relation on our set X of five partitions. We denote the relation as R.

   $$R = \{(\Pi_1, \Pi_1), (\Pi_2, \Pi_2), (\Pi_3, \Pi_3), (\Pi_4, \Pi_4), (\Pi_5, \Pi_5),$$
   $$(\Pi_1, \Pi_5), (\Pi_1, \Pi_2), (\Pi_4, \Pi_2), (\Pi_3, \Pi_2), (\Pi_5, \Pi_2)\}$$

   (a) Preorder
   True, every partition is a refinement of itself because cells are subsets of themselves, therefore the graph is reflexive. To show transitivity we need not consider the cases where we only see reflexive nodes.
   Similarly, in the case of Π_3 and Π_4, they only go to Π_2 which is reflexive, so transitivity holds here as well.
   Lastly, in the case of Π_1, we see (Π_1, Π_5) and (Π_5, Π_2) in R, then (Π_1, Π_2) being in R finishes showing that the graph is transitive.
   Since the graph is reflexive and transitive, $\leq$ is a preorder relation on X.

(b) partial order
To show the graph is a partial order, we need to know that $\leq$ has an additional property: anti-symmetry.
Anti-symmetry follows from the graph being reflexive and there being no such distinct node Π', i.e. $\Pi_i \neq \Pi'$, where $(\Pi_i, \Pi') \to (\Pi', \Pi_i)$

(c) equivalence relation
No, $\leq$ is not symmetric.

3. Prove that no matter what set X we start with, (Partitions(X), $\leq$) is always a partial order. Here, Partitions(X) is the set of all partitions of X.

Proof. Fix a set X. We prove $\leq$ is a partial order on the Partitions of X.

(a) reflexive
Take a partition in the set of all partitions of X, say Π. To say $\Pi \leq \Pi$ means every cell of Π is a subset of a cell of Π. Since any set is a subset of itself, we know that $\Pi \leq \Pi$ holds.

(b) transitive
Take three partitions, Π_1, Π_2, Π_3.
Assume $\Pi_1 \leq \Pi_2$ and $\Pi_2 \leq \Pi_3$. Show $\Pi_1 \leq \Pi_3$.
Take a cell of Π_3, say π_3. We must show there exists a $\pi_1 \in \Pi_1$ such that $\pi_3 \subset \pi_1$. Since $\Pi_2 \leq \Pi_3$, we know there is a $\pi_2 \in \Pi_2$ so that $\pi_3 \subset \pi_2$, and because $\Pi_1 \leq \Pi_2$ there is a $\pi_1 \in \Pi_1$ so that $\pi_2 \subset \pi_1$. Therefore we know that this π_1 must contain the π_3 we have (because subset is transitive), and we know $\pi_3 \subset \pi_1$. Hence $\Pi_1 \leq \Pi_3$.

(c) anti-symmetric
Take two partitions, Π_1 and Π_2.
Assume $\Pi_1 \leq \Pi_2$ and $\Pi_2 \leq \Pi_1$. Show $\Pi_1 = \Pi_2$.
Take a cell $s \in Pi_1$ and $t \in Pi_2$. Assume $s \subset t$.
Take another cell $s' \in Pi_1$ and assume $t \subset s'$. We must show that $s = s'$. By our assumption, we have $s \subset s'$ and since cells are disjoint in a partition, we must have that $s = s'$. Hence $\Pi_1 = \Pi_2$.

By showing that $\leq$ is reflexive, transitive and anti-symmetric on the partitions of an arbitrarily chosen set X, we conclude that $\leq$ is a partial order on the Partitons of X. $\square$

4 Partition refinement on graphs

Draw any graph on the set $\{1, 2, 3, 4\}$. Take the five partitions which you found in problem 3, part 1, and call them $\Pi_1, \ldots, \Pi_5$. Then find refine(G,Π_1), ..., refine refine(G,Π_5).
We reproduce our five partitions first:

1. $\Pi_1 = \{\{1, 4\}, \{2, 3\}\}$
2. $\Pi_2 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$
3. $\Pi_3 = \{\{2, 4\}, \{1, 3\}\}$
4. $\Pi_4 = \{\{2\}, \{1, 3, 4\}\}$
5. $\Pi_5 = \{\{1\}, \{4\}, \{2, 3\}\}$

Next we give a graph on the set $\{1, 2, 3, 4\}$
$G = (\{1, 2, 3, 4\}, \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (3, 2), (3, 4), (4, 4)\}$
And now to compute the refinements.

1. $\text{refine}(G, \Pi_1) = \{1, 3\}, \{2\}, \{4\}$
2. $\text{refine}(G, \Pi_2) = \{1\}, \{2\}, \{3\}, \{4\}$
3. $\text{refine}(G, \Pi_3) = \{1, 2\}, \{3, 4\}$
4. $\text{refine}(G, \Pi_4) = \{1, 2, 3\}, \{4\}$
5. $\text{refine}(G, \Pi_5) = \{1, 3\}, \{2\}, \{4\}$

5 Partition refinement on graphs

Let G be a graph. Consider the function

$$f : \text{partitions}(G) \to \text{partitions}(G)$$

given by $f(\Pi) = \text{refine}(G, \Pi)$.

1. Show that f is *monotone*: if $\Pi \leq \Pi'$, then $f(\Pi) \leq f(\Pi')$. You need to prove this for all graphs G and all partitions of the set of nodes of G.

   Proof. Fix a graph G and two partitions of G, say Π_1 and Π_2.
   Assume $\Pi_1 \leq \Pi_2$. Show refine(G,Π_1) $\leq$ refine(G,Π_2).
   Let $\Pi_1 = \{X_1, \ldots, X_n\}$ and $\Pi_2 = \{Y_1, \ldots, Y_m\}$
   Let refine(G,Π_1) $= \{C_1, \ldots, C_r\}$ and refine(G,Π_2) $= \{D_1, \ldots, D_t\}$.
   Take an element from refine(G,Π_1), say C.
   We must show there is a $D \in$ refine(G,Π_2) and $C \subset D$.
   Take all the Y_i from Π_2 such that the points of C arrow into Y_i. Since $\Pi_1 \leq \Pi_2$, we know there is a collection of $X_j \in \Pi_1$ so that $Y_i \subset X_j$. Then the points in C arrow to these X_j and no others because any other X would not be a superset of one of our Y's – which would be a contradiction. Therefore this set D is a cell of refine(G,Π_2). To see $C \subset D$ we assume the intersection of C and D is non-empty.
   Assume towards a contradiction that there is a $y \in C$ that is not in D. Take an element x from the intersection, and notice that both $x, y \in C$. Since $y \notin D$ whlie x is, there exists a cell of Π_1 such either both x and y are in it or neither, but this is a contradiction. Hence no such y exists and $C \subset D$. □

2. Give an example of a graph G and a partition Π of its nodes such that $\Pi \not\leq f(\Pi)$.
   We keep the same graph from part 4. Let $\Pi = \{\{1,4\},\{2,3\}\}$. Then $f(\Pi) = \{\{1,3\},\{2\},\{4\}\}$. Since $\{1,3\}$ is not a subset of any cell of Π we know $\Pi \not\leq f(\Pi)$

3. Give an example of a graph G and a partition Π of its nodes such that $f(\Pi) \not\leq \Pi$.
   We keep the same graph from part 4. Let $\Pi = \{\{1\},\{4\},\{2,3\}\}$. Then $f(\Pi) = \{\{1,3\},\{2\},\{4\}\}$. Since $\{2,3\}$ is not a subset of any element of $f(\Pi)$ we know $f(\Pi) \not\leq \Pi$.

4. Let G be the graph shown in problem 4. Find a partition Π such that $f(\Pi) = \Pi$.
   Such a Π is $\{\{1\},\{2\},\{3\},\{4\}\}$ which cannot be refined any further.