M482 Homework 3

Joshua Larkin

Tuesday, 29 January 2019

1 Satisfiability in modal logic

A modal sentence ϕ is *satisfiable* if there is some model M and some $x \in M$ such that $x \models \phi$. Show that each of the following sentences is satisfiable.

1. $(\Diamond\Box p) \wedge (\Box\Diamond p)$

   Let $M = (W, R, \mathrm{val})$ be a model where...
   $W = \{x, y\}$, $R = \{(x, y), (y, y)\}$, and val $= \{(x, \{\}), (y, \{p\})\}$
   We know $(\Diamond\Box p)$ is true because $x \to y$ and $y \to y$, and $y \models p$.
   By the same argument, $(\Box\Diamond p)$ is true – (because both x and y only arrow y).

2. $(\Diamond\Box p) \wedge (\neg\Box\Diamond p)$

   Let $M = (W, R, \mathrm{val})$ be a model where...
   $W = \{x, y, z\}$, $R = \{(x, y), (x, z), (y, y)\}$, and val $= \{(x, \{\}), (y, \{p\}), (z, \{\})\}$.
   We know $(\Diamond\Box p)$ is true because $x \to y$ and $y \to y$ (note that this is y's only arrow), and $y \models p$.
   We see that $(\neg\Box\Diamond p)$ because $x \to z$ and $z \not\models p$

3. $(\neg\Diamond\Box p) \wedge (\Box\Diamond p)$

   Let $M = (W, R, \mathrm{val})$ be a model where...
   $W = \{x, y, z, w\}$, $R = \{(x, y), (y, w), (y, z)\}$,
   and val $= \{(x, \{\}), (y, \{\}), (z, \{p\}), (w, \{\})\}$.

   (a) Show $x \vDash (\neg\Diamond\Box p)$.
   $x \vDash \neg\Diamond\Box \longleftrightarrow x \nvDash \Diamond\Box p$.
   The only node that x can go to is y, so we must show $y \nvDash \Box p$.
   This follows from $(y, w) \in R$ and $(w, \{\}) \in$ val.
   Hence $y \nvDash \Box p$ and $x \nvDash \Diamond\Box p$.
   Therefore, $x \vDash \neg\Diamond\Box p$.

   (b) Show $x \vDash \Box\Diamond p$
   $x \vDash \Box\Diamond p \longleftrightarrow \forall x'.\ x \to x'.\ x' \vDash \Diamond p$
   The only such x' is y. We must show $y \vDash \Diamond p$.
   $y \vDash \Diamond p \longleftrightarrow \exists y'.\ y' \vDash p$. We have (y, w) and (y, z) in R.
   Since val$(z) = p$, we know $z \vDash p$; then $y \vDash \Diamond p$
   Hence, $x \vDash \Box\Diamond p$.

4. $(\neg\Diamond\Box p) \wedge (\neg\Box\Diamond p)$

   Let $M = (W, R, \text{val})$ be a model where...
   $W = \{x, y, z, w\}$, $R = \{(x, y), (x, w), (w, w), (y, y), (y, z)\}$,
   and $\text{val} = \{(x, \{\}), (y, \{p\}), (z, \{\}), (w, \{\})\}$.

   (a) $x \vDash \neg\Diamond\Box p$
   $x \vDash \neg\Diamond\Box p \longleftrightarrow x \nvDash \Diamond\Box p$.
   Since (x, y) and (x, w) in R, we must check that $y \nvDash \Box p$ and $w \nvDash \Box p$.
   In the case of $w \nvDash \Box p$, we only have (w, w) in R, and
   since $\text{val}(w) = \varnothing$, we know $w \nvDash \Box p$.
   In the case of $y \nvDash \Box p$, we see that $z \nvDash p$ because $\text{val}(z) = \varnothing$, hence
   $y \nvDash \Box p$.
   Therefore $x \nvDash \Diamond\Box p$.

   (b) $x \vDash \neg\Box\Diamond p$ $x \vDash \neg\Box\Diamond p \longleftrightarrow x \nvDash \Box\Diamond p$.
   Since (x, y) and (x, w) in R, we must check that $y \nvDash \Diamond p$ or $w \nvDash \Diamond p$.
   In the case of $w \nvDash \Box p$, we only have (w, w) in R, and
   since $\text{val}(w) = \varnothing$, we know $w \nvDash \Diamond p$.
   Therefore $x \nvDash \Diamond\Box p$.

2 Satisfiability on a symmetric model

A modal sentence ϕ is *satisfiable on a symmetric model* if there is some *symmetric modelM* and some $x \in M$ such that $x \models \phi$. Which of the following sentences is satisfiable on a symmetric model?

1. $(\Diamond\Box p) \wedge (\Box\Diamond p)$
2. $(\Diamond\Box p) \wedge (\neg\Box\Diamond p)$
3. $(\neg\Diamond\Box p) \wedge (\Box\Diamond p)$
4. $(\neg\Diamond\Box p) \wedge (\neg\Box\Diamond p)$

3 Partitions on a fixed set X, and an order on them

1. Find five partitions of $\{1, 2, 3, 4\}$. There are fifteen in total, but you only need to find five of them. Call your partitions $\Pi_1, \ldots, \Pi_5$.

   (a) $\Pi_1 = \{\{1, 4\}, \{2, 3\}\}$
   (b) $\Pi_2 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$
   (c) $\Pi_3 = \{\{2, 4\}, \{1, 3\}\}$
   (d) $\Pi_4 = \{\{2\}, \{1, 3, 4\}\}$
   (e) $\Pi_5 = \{\{1\}, \{4\}, \{2, 3\}\}$

2. Let X be a set. Let Π and Π' be partitions on X. We say that Π' *refines* Π if every cell of Π' is a subset of a cell of Π. In other words, if $s \equiv_{\Pi'} t$, then $s \equiv_{\Pi} t$. We write $\Pi \leq \Pi'$. We also say that Π is *coarser* than Π', or that Π' is *finer* than Π.

   For example, let $X = \{1, 2, 3, 4, 5\}$. Then

   $$\{\{1, 2\}, \{3, 4\}\{5\}\} \leq \{\{1\}\{2\}\{3, 4\}\{5\}\}.$$

   Continuing with X as in part (1), consider

   $$(\{\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5\}, \leq)$$

   Is $\leq$ a preorder? Is it a partial order? Is it an equivalence relation?

   We give the mathematics of the $\leq$ relation on our set X of five partitions. We denote the relation as R.

   $$R = \{(\Pi_1, \Pi_1), (\Pi_2, \Pi_2), (\Pi_3, \Pi_3), (\Pi_4, \Pi_4), (\Pi_5, \Pi_5),$$
   $$(\Pi_1, \Pi_5), (\Pi_1, \Pi_2), (\Pi_4, \Pi_2), (\Pi_3, \Pi_2), (\Pi_5, \Pi_2)\}$$

   (a) Preorder
   True, every partition is a refinement of itself because cells are subsets of themselves, therefore the graph is reflexive. To show transitivity we need not consider the cases where we only see reflexive nodes.
   Similarly, in the case of Π_3 and Π_4, they only go to Π_2 which is reflexive, so transitivity holds here as well.
   Lastly, in the case of Π_1, we see (Π_1, Π_5) and (Π_5, Π_2) in R, then (Π_1, Π_2) being in R finishes showing that the graph is transitive.
   Since the graph is reflexive and transitive, $\leq$ is a preorder relation on X.

(b) partial order
To show the graph is a partial order, we need to know that $\leq$ has an additional property: anti-symmetry.
Anti-symmetry follows from the graph being reflexive and there being no such distinct node Π', i.e. $\Pi_i \neq \Pi'$, where $(\Pi_i, \Pi') \rightarrow (\Pi', \Pi_i)$

(c) equivalence relation
No, $\leq$ is not symmetric.

3. Prove that no matter what set X we start with, (Partitions(X), $\leq$) is always a partial order. Here, Partitions(X) is the set of all partitions of X.

4 Partition refinement on graphs

Draw any graph on the set $\{1, 2, 3, 4\}$. Take the five partitions which you found in problem 3, part 1, and call them $\Pi_1, \ldots, \Pi_5$. Then find refine(G,Π_1), $\ldots$, refine refine(G,Π_5).
We reproduce our five partitions first:

1. $\Pi_1 = \{\{1, 4\}, \{2, 3\}\}$
2. $\Pi_2 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$
3. $\Pi_3 = \{\{2, 4\}, \{1, 3\}\}$
4. $\Pi_4 = \{\{2\}, \{1, 3, 4\}\}$
5. $\Pi_5 = \{\{1\}, \{4\}, \{2, 3\}\}$

Next we give a graph on the set $\{1, 2, 3, 4\}$
$G = (\{1, 2, 3, 4\}, \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (3, 2), (3, 4), (4, 4)\}$
And now to compute the refinements.

1. refine(G, Π_1) = $\{1, 3\}$, $\{2\}$, $\{4\}$
2. refine(G, Π_2) = $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$
3. refine(G, Π_3) = $\{1, 2\}$, $\{3, 4\}$

4. refine$(G, \Pi_4) = \{1, 2, 3\}, \{4\}$
5. refine$(G, \Pi_5) = \{1, 3\}, \{2\}, \{4\}$

5 Partition refinement on graphs

Let G be a graph. Consider the function

$$f : \text{partitions}(G) \to partitions(G)$$

given by $f(\Pi) = \text{refine}(G, \Pi)$.

1. Show that f is *monotone*: if $\Pi \leq \Pi'$, then $f(\Pi) \leq f(\Pi')$. You need to prove this for all graphs G and all partitions of the set of nodes of G.
2. Give an example of a graph G and a partition Π of its nodes such that $\Pi \nleq f(\Pi)$.
3. Give an example of a graph G and a partition Π of its nodes such that $f(\Pi) \nleq \Pi$.
4. Let G be the graph shown in problem 4. Find a partition Π such that $f(\Pi) = \Pi$.