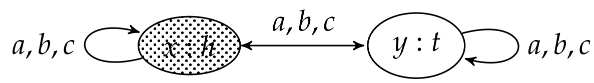
Modal Logic, Winter 2019
Homework 12 (last homework of the course)
Due Tuesday, April 23

1. Suppose that we start with a coin scenario model W with three agents named Amina (A), Bao (B), and Cynthia (C). They enter a room together, just as in the two person scenario. The coin really lies heads up. So the initial model $\mathcal{M}$ is



Suppose that A "cheats" and looks at the coin, while B and C are distracted. Then after this, B also cheats while A and C are distracted.

   (a) We want to draw the model at the end. For this, find two action models that are related to this problem. Call them Σ_1 and Σ_2. Draw those action models. (You get them by looking at the lecture slides where we did the case of two people and one cheater, and then modifying the model to have three people.) Then what we want is
   $$(M \otimes \Sigma_1) \otimes \Sigma_2.$$

   (b) Figure out what $(M \otimes \Sigma_1) \otimes \Sigma_2$ is, using the formal definitions.

2. A the end of this problem, you will find a picture of the three muddy children after their father has announced that at least one child is muddy. The real world is w_7, indicating that all three are muddy.

Suppose that a and b tell each other that they don't know whether they are muddy or not, and they do this in a way which is completely private, with c not suspecting anything. We model this by a private announcement to a and b of φ, where φ is

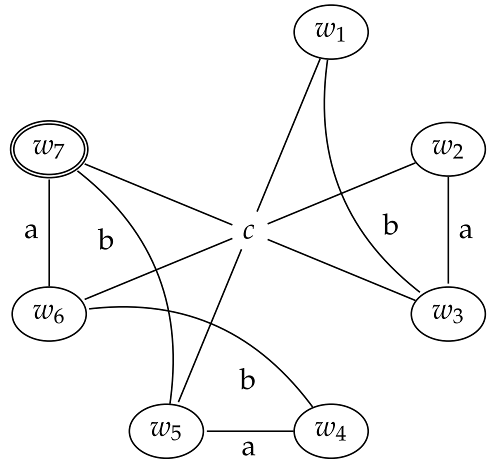
$$\neg K_a \textsc{dirty}_a \wedge \neg K_a \neg \textsc{dirty}_a \wedge \neg K_b \textsc{dirty}_b \wedge \neg K_b \neg \textsc{dirty}_b.$$

   (a) Write the action model corresponding to the action we just described.

   (b) Calculate $\mathcal{M} \otimes \Sigma$. You may draw your answer. You don't have to list all of the worlds, but be sure to list enough to make it clear what the whole thing looks like.

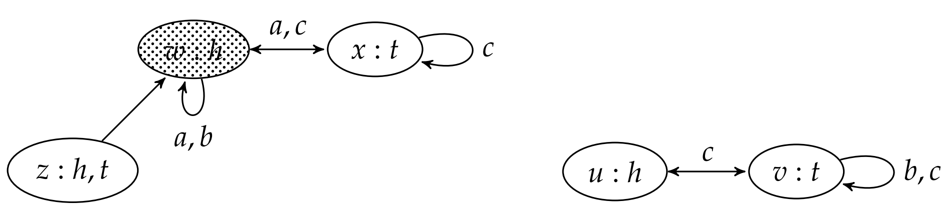
   (c) Is it true or not that *after the action*, a knows that she is muddy? That is, is the sentence $K_a \textsc{dirty}_a$ true in the "real world" of $\mathcal{M} \otimes \Sigma$, or not?

w_1 $\text{DIRTY}_a, \neg\text{DIRTY}_b, \neg\text{DIRTY}_c$
w_2 $\neg\text{DIRTY}_a, \text{DIRTY}_b, \neg\text{DIRTY}_c$
w_3 $\text{DIRTY}_a, \text{DIRTY}_b, \neg\text{DIRTY}_c$
w_4 $\neg\text{DIRTY}_a, \neg\text{DIRTY}_b, \text{DIRTY}_c$
w_5 $\text{DIRTY}_a, \neg\text{DIRTY}_b, \text{DIRTY}_c$
w_6 $\neg\text{DIRTY}_a, \text{DIRTY}_b, \text{DIRTY}_c$
w_7 $\text{DIRTY}_a, \text{DIRTY}_b, \text{DIRTY}_c$

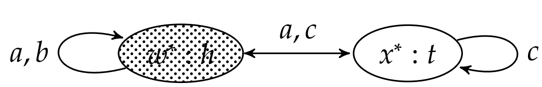


3. Let $\mathcal{M}$ be any model, and let w be one of the worlds in $\mathcal{M}$. Let $M(w)$ be the set of worlds of the model which are reachable in zero or more steps from w following *any* arrows. We make $M(w)$ into a model that I'll call $\mathcal{M}(w)$ by setting $x \to y$ in $\mathcal{M}(w)$ iff $x \to y$ in $\mathcal{M}$; we also define a valuation *val* by *val*(x) in $\mathcal{M}(w)$ is the same set of atomic sentences as *val*(x) in $\mathcal{M}$.

   (For example, let $\mathcal{M}$ be the five-world model shown below, and w is the world that is shaded in.



   Then $\mathcal{M}(w)$ is shown below



   This is just an example, but it shows the general idea.) Note that we can't go from w to z, u, or v, so they are left out of $M(w)$. Note also that I put stars $*$ on the worlds in $M(w)$ just to emphasize that they are different than the corresponding worlds in the original M. Each point z^* is similar to the corresponding z, but technically different.

Your problem: fix a model $\mathcal{M}$, and a world w of it. (You must prove this in general, not just for the example above.) Prove that for all sentence φ, the following holds: For all worlds z which are reachable from w by some sequence of arrows,

$$z^* \models \varphi \text{ in } \mathcal{M}(w) \quad \text{if and only if} \quad z \models \varphi \text{ in } \mathcal{M}$$

Use induction on φ. You can just do the base cases and the induction steps for an operator like K_a.

Proof. Base Case: $\varphi = p$ (atomic). We prove for all worlds z reachable from w,

$$z^* \models \varphi \text{ in } \mathcal{M}(w) \quad \text{if and only if} \quad z \models \varphi \text{ in } \mathcal{M}$$

Because reducing the model to reachable worlds doesn't affect the valuation, p remains true at z^* if it is true at z and vice versa.

Inductive Step: We assume for all worlds x reachable from w,

$$x^* \models \varphi \text{ in } \mathcal{M}(w) \quad \text{if and only if} \quad x \models \varphi \text{ in } \mathcal{M}$$

And we prove for all worlds y reachable from w,

$$y^* \models K_a\varphi \text{ in } \mathcal{M}(w) \quad \text{if and only if} \quad y \models K_a\varphi \text{ in } \mathcal{M}$$

For this, assume $y^* \models K_a\varphi$ in $\mathcal{M}(w)$and suppose, towards a contradiction, that $y \models \neg K_a\varphi$ in $\mathcal{M}$. This means there exists a world v such that $y \xrightarrow{a} v$ and $v \models \neg\varphi$. Our first assumption means that $v^* \models \varphi$, but applying our induciton hypothesis tells us $v^* \models \varphi \iff v \models \varphi$. So we have a contradiciton, and we must have $y \models K_a\varphi$.

Our proof by induction shows that for all worlds z which are reachable from w by some sequence of arrows,

$$z^* \models \varphi \text{ in } \mathcal{M}(w) \quad \text{if and only if} \quad z \models \varphi \text{ in } \mathcal{M}$$

$\square$

4. Let φ and ψ be any sentences. Prove that

$$\models_{all} (CK\varphi) \to (\langle\varphi\rangle\psi) \leftrightarrow \psi).$$

Here, CK means $CK_{\mathcal{A}}$. (That is, common knowledge to the group of all agents.) Intuitively, this says that if a sentence φ is common knowledge, then announcing it has no effect whatsoever: another sentence ψ is true after the announcement if and only if ψ was true before the announcement.

Proof. Fix a model $\mathcal{M}$ and a world m of $\mathcal{M}$. Assume about m that $m \models (CK\varphi)$.
We prove $m \models \langle\varphi\rangle\psi \leftrightarrow \psi$.

- ($\to$) For this we first assume $m \models \langle\varphi\rangle\psi$ in $\mathcal{M}$ and we prove $m \models \psi$ in $\mathcal{M}$.
Since $m \models (CK\varphi)$ we know that $m \models \varphi$, and so m is in $\mathcal{M} \upharpoonright \varphi$. Then our assumption means $m \models \psi$ in $\mathcal{M} \upharpoonright \varphi$, and by problem 3 we must have $m \models \psi$ in $\mathcal{M}$.
- ($\leftarrow$) Next we assume $m \models \psi$ in $\mathcal{M}$ and we show $m \models \langle\varphi\rangle\psi$ in $\mathcal{M}$
Consider the model $\mathcal{M}(m)$ and notice that this is the same model as $(\mathcal{M} \upharpoonright \varphi)(m)$. Then it follows that $m \models \psi$ in $(\mathcal{M} \upharpoonright \varphi)(m) \leftrightarrow w \models \psi$ in $\mathcal{M}(m) \upharpoonright \varphi$. By problem 3, we have $m \models \psi$ in $\mathcal{M} \upharpoonright \varphi$, and thus $w \models \langle\varphi\rangle\psi$ in $\mathcal{M}$.

□

5. Let φ and ψ be any sentences in modal logic, including sentences involving common knowledge or announcements.

   (a) Let $\mathcal{M}$ be a model. Show that $(\mathcal{M} \upharpoonright \varphi) \upharpoonright \psi = \mathcal{M} \upharpoonright \langle\varphi\rangle\psi$. That is, the two models $(\mathcal{M} \upharpoonright \varphi) \upharpoonright \psi$ and $\mathcal{M} \upharpoonright \langle\varphi\rangle\psi$ are the same.
   This boils down to showing that

$$\{w \in M \upharpoonright \varphi : w \Vdash \psi \text{ in } \mathcal{M} \upharpoonright \varphi\} \quad = \quad \{w \in M : w \Vdash \langle\varphi\rangle\psi \text{ in } \mathcal{M}\}$$

   Proof. We prove this by showing that each set is a subset of the other.

   - $\subset$ Fix $w \in \mathcal{M} \upharpoonright \varphi$ such that $w \Vdash \psi$ in $\mathcal{M} \upharpoonright \varphi$. We prove $w \in \mathcal{M}$ such that $w \Vdash \langle\varphi\rangle\psi$ in $\mathcal{M}$. This is true if and only if $w \models \psi$ in $\mathcal{M} \upharpoonright \varphi$, which is true from our assumption.
   - $\supset$ Fix $w \in \mathcal{M}$ such that $w \Vdash \langle\varphi\rangle\psi$ in $\mathcal{M}$. We prove $w \Vdash \psi$ in $\mathcal{M} \upharpoonright \varphi$. This also follows from our assumption about w.

   This finishes the proof of set equality and shows the two models $(\mathcal{M} \upharpoonright \varphi) \upharpoonright \psi$ and $\mathcal{M} \upharpoonright \langle\varphi\rangle\psi$ are the same. □

(b) Use the previous part to prove the *iterated announcement law*: For all φ, ψ, and χ,

$$\models \langle\varphi\rangle\langle\psi\rangle\chi \leftrightarrow \langle\langle\varphi\rangle\psi\rangle\chi$$

Proof. Fix φ, ψ, and χ. Then fix a model $\mathcal{M}$ and a world d of $\mathcal{M}$.
Notice that $d \models \langle\varphi\rangle\langle\psi\rangle\chi \leftrightarrow w \models \langle\psi\rangle\chi$ such that $w \in \mathcal{M} \upharpoonright \varphi$. And this holds if and only if $w \models \chi$ such that $w \in (\mathcal{M} \upharpoonright \varphi) \upharpoonright \psi$.
Similarly, $w \models \langle\langle\varphi\rangle\psi\rangle\chi \leftrightarrow w \models \chi$ such that $w \in \mathcal{M} \upharpoonright \langle\varphi\rangle\psi$.
Now we see that part (a) above applies, and the proof is done. $\square$