Modal Logic, Winter 2019
Homework 5
due Tuesday, February 12

1 Exercise 3.1

1. $1 \Vdash p \wedge *q$
   $1 \Vdash p \wedge *q$ if and only if $1 \Vdash p$ and $1 \Vdash *q$. We have $1 \Vdash p$ since in room 1, p is true. Notice that $2 \Vdash q$ since in room 2, q is true; then $1 \Vdash *q$. Therefore $1 \Vdash p \wedge *q$.

2. $2 \Vdash *(p \wedge *q)$
   $2 \Vdash *(p \wedge *q)$ if and only if $1 \Vdash p \wedge *q$, which is true by part 1.

3. $1 \Vdash *(*p \wedge q)$
   $1 \Vdash *(*p \wedge q)$ if and only if $2 \Vdash *p \wedge q$ if and only if $2 \Vdash *p$ and $2 \Vdash q$.
   $2 \Vdash *p$ since in room 1, p is true. $2 \Vdash q$ since in room 2, q is true. Hence $2 \Vdash *p \wedge q$; therefore $1 \Vdash *(*p \wedge q)$.

4. $2 \Vdash * * * * q$

$$\begin{aligned} 2 \Vdash * * * * q &\iff 1 \Vdash * * * q \\ &\iff 2 \Vdash * * q \\ &\iff 1 \Vdash * q \\ &\iff 2 \Vdash q \end{aligned}$$

   We know $2 \Vdash q$ since in room 2, q is true. Hence $2 \Vdash * * * * q$.

2 Exercise 3.3

Show the following in the natural deduction proof system:

1. $\vdash *\neg * \varphi \to * * \neg\varphi$

1.	$*\neg * \varphi$	Assume
2.	$\neg * \varphi$	$*_e$
3.	$\neg * \varphi \to *\neg\varphi$	Determinacy Axiom
4.	$*\neg\varphi$	$\to_e, 2, 3$
5.	$* * \neg\varphi$	$*_i$
6.	$*\neg * \varphi \to * * \neg\varphi$	$\to_i, 1-5$

2. $\vdash \mathit{maze} \to * * * * \mathit{maze}$

1.	maze	Assume
2.	$\mathit{maze} \to * * \mathit{maze}$	Involution Lemma
3.	$* * \mathit{maze}$	$\to_e, 1, 2$
4.	$* * \mathit{maze} \to * * * * \mathit{maze}$	Involution Lemma
5.	$* * * * \mathit{maze}$	$\to_e, 1, 2$
6.	$\mathit{maze} \to * * * * \mathit{maze}$	$\to_i, 1-5$

3. $\vdash \neg * \bot$

1.	$\boxed{*\bot}$	Assume
2.	$\neg * \bot$	$\neg_i, 1$

4. Involution Lemma: $\vdash \varphi \to **\varphi$

1.	φ	assume
2.	$\neg**\varphi$	Assume
3.	$\neg**\varphi \to *\neg*\varphi$	Determinacy Axiom
4.	$*\neg*\varphi$	$\to_e, 2, 3$
5.	$\neg*\varphi$	$*_e$
6.	$\neg*\varphi \to *\neg\varphi$	Determinacy Axiom
7.	$*\neg\varphi$	$\to_e, 5, 6$
8.	$**\neg\varphi$	$*_i$
9.	$**\neg\varphi \to \neg\varphi$	Involution Axiom
10.	$\neg\varphi$	$\to_e, 8, 9$
11.	F	$\mathsf{F}_i, 1, 10$
12.	$\neg\neg**\varphi$	$\neg_i, 2-11$
13.	$**\varphi$	$\neg\neg_e, 12$
14.	$\varphi \to **\varphi$	$\to_i, 1-n$

3 Exercise 3.7

Show that for all models M, $1 \Vdash *(p \to q) \to (*p \to *q)$.

Fix a model M. We want to show that $1 \Vdash *(p \to q) \to (*p \to *q)$.
If $1 \nVdash *(p \to q) \to (*p \to *q)$, we're done. So we can assume that $1 \Vdash *(p \to q)$. We want to show that $1 \Vdash (*p \to *q)$. As before, if $1 \nVdash *p$, we're done. So we can assume that $1 \Vdash *p$. It is our job to show that $1 \Vdash *q$. This means that $2 \Vdash q$, so we show that $2 \Vdash q$. Since $1 \Vdash *(p \to q)$, we know $2 \Vdash p \to q$. Similarly, since $1 \Vdash *p$, we have $2 \Vdash p$. Thus we have $2 \Vdash q$, and hence $1 \Vdash *q$. Therefore $1 \Vdash *p \to *q$ and we can conclude that $1 \Vdash *(p \to q) \to (*p \to *q)$.

4 Exercise 3.8

1. $p_1 \to *p_1$

   This is satisfiable, here is such a model that the sentence is true in:

   But it is not valid, here is a model where the sentence is not true

2. $p_1 \to \neg * p_1$

   This is satisfiable, here is a model that it is true in:

   But it is not valid, here is a model where it is false:

3. $(*p_1) \wedge (*\neg p_1)$

   This is not satisfiable, because it is equivalent to $\varphi \wedge \neg\varphi$. Since there is no model where this is true, it is definitely not valid either.

4. $(p_1 \wedge *p_2) \vee \neg(p_1 \wedge *p_2)$

   This is both, because it is equivalent to $\varphi \vee \neg\varphi$.

5. $* * * * *p_2 \to *p_2$

   This is satisfiable, here is such a model it is true in.

To show it is valid we provide a formal proof:

1.	$*****p_2$	Assume
2.	$*****p_2 \to ***p_2$	Involution Axiom
3.	$***p_2$	$\to_e$
4.	$*p_2$	$\to_e$
5.	$*****p_2 \to *p_2$	$\to_i$

By soundness, we know this is a tautology, so it is valid.

5 Exercise 3.10

One of these is true, one is false. Which is which?

1. $\mathit{treadmill} \models_L *\mathit{treadmill}$

   The above statement is false, because you would have to walk into the other room to know whether or not there is a treadmill there.

2. $\mathit{treadmill} \models_G *\mathit{treadmill}$

   This statement is true because φ has to be true in both rooms for global modeling, and if φ is true in both rooms, then $*\varphi$ is true in both as well.

6 Exercise 3.11

Prove that the following are equivalent:

1. $\varphi_1, \ldots, \varphi_n \models_1 \psi$
2. $\varphi_1, \ldots, \varphi_n \models_2 \psi$

Denote $\Gamma := \{\varphi_1, \ldots, \varphi_n\}$. Suppose $\Gamma \models_1 \psi$. Prove $\Gamma \models_2 \psi$. Fix a model M and assume every sentence $\varphi_i \in \Gamma$ is true in room 2 of M. We show $2 \Vdash \psi$ in M. Consider $M*$, the mirror image of M. By our assumption, every sentence $\varphi_i \in \Gamma$ that is true in room 2, is true in room 1 of $M*$, by Lemma 3.11. Then, since $\Gamma \models_1 \psi$, $1 \Vdash \psi$ in $M*$. By Lemma 3.11 again, we must have $2 \Vdash \psi$ in M.

For the other direction, assume $\Gamma \models_2 \psi$, and we must show $\Gamma \models_1 \psi$. Fix a model M and assume every sentence $\varphi_i \in \Gamma$ is true in room 1 of M. We show $1 \Vdash \psi$ in M. Consider $M*$, the mirror image of M. By our assumption, every sentence $\varphi_i \in \Gamma$ that is true in room 1, is true in room 2 of $M*$, by Lemma 3.11. Then, since $\Gamma \models_2 \psi$, $2 \Vdash \psi$ in $M*$. By Lemma 3.11 again, we must have $1 \Vdash \psi$ in M. This completes the proof.