Modal Logic, Winter 2019
Homework 5
due Tuesday, February 12

Note In this homework, and in all work in this class: unless otherwise stated, you may use the soundness or completeness of a proof system. But whenever you do, you need to say which one you are using.

1 de Morgan's Laws

1. Give a derivation that shows $\vdash \neg(\varphi \wedge \psi) \rightarrow (\neg\varphi) \vee (\neg\psi)$.

1.	$\neg(\varphi \wedge \psi)$	Assume
2.	φ	Assume
3.	ψ	Assume
4.	$\varphi \wedge \psi$	$\wedge_i, 2-4$
5.	F	F_i,1,4
6.	$\neg\psi$	$\neg_i$, 3-5
7.	$\neg\varphi \vee \neg\psi$	$\vee_i$, 6
8.	$\varphi \rightarrow \neg\varphi \vee \neg\psi$	$\rightarrow_i$,2-7
9.	$\varphi \vee \neg\varphi$	pem
10.	φ	Assume
11.	$\neg\varphi \vee \neg\psi$	$\rightarrow_e$,8,10
12.	$\neg\varphi$	Assume
13.	$\neg\varphi \vee \neg\psi$	or_i,12
14.	$\neg\varphi \vee \neg\psi$	$\vee_e$, 9, 10-11, 12-13
15.	$\neg(\varphi \wedge \psi) \rightarrow \neg\varphi \vee \neg\psi$	$\rightarrow_i$, 1-7

2. Give a derivation that shows $\vdash (\neg\varphi) \vee (\neg\psi) \rightarrow \neg(\varphi \wedge \psi)$.

1.	$(\neg\varphi) \vee (\neg\psi)$	Assume
2.	$\neg\varphi$	Assume
3.	$\varphi \wedge \psi$	Assume
4.	φ	$\wedge_e$,3
5.	$\neg\varphi$	R,2
6.	F	F_i,4,5
7.	$\neg(\varphi \wedge \psi)$	$\neg_i$,3-6
8.	$\neg\psi$	Assume
9.	$\varphi \wedge \psi$	Assume
10.	ψ	$\wedge_e$,9
11.	$\neg\psi$	R,8
12.	F	F_i,10,11
13.	$\neg(\varphi \wedge \psi)$	$\neg_i$,9-12
14.	$(\neg\varphi) \vee (\neg\psi) \rightarrow \neg(\varphi \wedge \psi)$	$\rightarrow_i$,1-13

In one of the two parts, you will need to use the Law of the Excluded Middle.

2 Satisfiability in propositional logic

Recall that a propositional logic sentence φ is *satisfiable* if there is some valuation v such that $[\![\varphi]\!]_v = \mathsf{T}$.

For each of the following sentences, tell whether true or false. For the true ones, give a short proof. For the false ones, give a counterexample.

1. Every sentence φ or its negation $\neg\varphi$ is satisfiable.
   Fix φ. We know that $[\![\varphi]\!]_v = \mathsf{T}$ or $[\![\varphi]\!]_v = \mathsf{F}$. If $[\![\varphi]\!]_v = \mathsf{T}$ then we're done. Otherwise, if $[\![\varphi]\!]_v = \mathsf{F}$, then $[\![\neg\varphi]\!]_v = \neg[\![\varphi]\!] = \neg\mathsf{F} = \mathsf{T}$. Hence $\neg\varphi$ is also satisfiable.

2. Both φ and $\neg\varphi$ are satisfiable. Let $\varphi = p \wedge \neg p$. This sentence and it's negation are not satisfiable.

3. If $\varphi \wedge \psi$ is satisfiable, then both φ and ψ are satisfiable.

   Fix φ and ψ.

   Assume $[\![\varphi \wedge \psi]\!] = \mathsf{T}$. Then $[\![\varphi]\!] \wedge [\![\psi]\!] = \mathsf{T}$, where the $\wedge$ is based on truth-tables. Hence $[\![\varphi]\!] = \mathsf{T}$ and $[\![\psi]\!] = \mathsf{T}$. Therefore both φ and ψ are satisfiable.

4. If both φ and ψ are satisfiable, then $\varphi \wedge \psi$ is satisfiable.

   Fix φ and ψ. Assume $[\![\varphi]\!] = \mathsf{T}$ and $[\![\psi]\!] = \mathsf{T}$. Then

   $$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!] = \mathsf{T} \wedge \mathsf{T} = \mathsf{T}$$

   Hence $\varphi \wedge \psi$ is satisfiable.

5. If $\varphi \vee \psi$ is satisfiable, then either φ or ψ (or both) are satisfiable.

   Fix φ and ψ. Assume $\varphi \vee \psi$ is satisfiable. This means $[\![\varphi \vee \psi]\!] = \mathsf{T}$.
   Then $[\![\varphi]\!] \vee [\![\psi]\!] = \mathsf{T}$, where the $\vee$ is based on truth-tables.
   In the case that $[\![\varphi]\!] = \mathsf{T}$, we see that φ is satisfiable. So either φ or ψ are satisfiable.
   Now consider if $[\![\varphi]\!] = \mathsf{F}$, then $[\![\varphi]\!] \vee [\![\psi]\!] = \mathsf{F} \vee [\![\psi]\!] = \mathsf{T}$. So $[\![\psi]\!] = \mathsf{T}$, and either φ or ψ are satisfiable. The case where both are satisfiable is trivial, since we need not consider $[\![\varphi]\!] = \mathsf{F}$ and one of the cases for $[\![\varphi]\!] \vee [\![\psi]\!] = \mathsf{T}$ must have $[\![\psi]\!] = \mathsf{T}$ again, so both are satisfiable.

6. If either φ or ψ is satisfiable, then $\varphi \vee \psi$ is satisfiable.

   Fix φ and ψ. Assume either φ or ψ is satisfiable. First, consider φ is satisfiable. Then $[\![\varphi]\!] = \mathsf{T}$. And

   $$[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \vee [\![\psi]\!] = \mathsf{T} \vee [\![\psi]\!] = \mathsf{T}$$

   so $\varphi \vee \psi$ is satisfiable.
   Next, consider φ is not satisfiable but ψ is satisfiable. So $[\![\varphi]\!] = \mathsf{F}$ and $[\![\psi]\!] = \mathsf{T}$.

   $$[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \vee [\![\psi]\!] = \mathsf{F} \vee \mathsf{T} = \mathsf{T}$$

   So $\varphi \vee \psi$ is satisfiable.

7. If φ and $\varphi \to \psi$ are satisfiable, then ψ is satisfiable. Fix φ and ψ. Assume φ is

   satisfiable and $\varphi \to \psi$ is satisfiable. We show ψ is satisfiable. Our assumptions mean that $[\![\varphi]\!] = \mathsf{T}$ and $[\![\varphi \to \psi]\!] = \mathsf{T}$. Unfolding the valuation,

   $$\begin{aligned} [\![\varphi \to \psi]\!] &= \mathsf{T} \\ [\![\varphi]\!] \to [\![\psi]\!] &= \mathsf{T} \\ \mathsf{T} \to [\![\psi]\!] &= \mathsf{T} \end{aligned}$$

   Since $\to$ here is based on truth-tables, we must have that $[\![\psi]\!] = \mathsf{T}$. Hence ψ is satisfiable.

8. Every sentence φ or its negation $\neg\varphi$ is a tautology.

   This is not true because we can take φ to be an atomic sentence p, and it is not true that for any valuation v, $[\![p]\!]_v = \mathsf{T}$ or $[\![\neg p]\!] = \mathsf{T}$.

9. If $\varphi \wedge \psi$ is a tautology, then both φ and ψ are tautologies.

   Fix φ and ψ. Assume for any valuation v, $[\![\varphi \wedge \psi]\!]_v = \mathsf{T}$. We prove for all valuations w_1, w_2 that $[\![\varphi]\!]_{w_1} = \mathsf{T}$ and $[\![\psi]\!]_{w_2} = \mathsf{T}$. So $[\![\varphi \wedge \psi]\!]_v = \mathsf{T} = [\![\varphi]\!]_v \wedge [\![\psi]\!]_v$ where $\wedge$ is based on truth tables. So both are true and both are tautologies since we chose a random valuation v.

10. If $\varphi \vee \psi$ is a tautology, then either φ or ψ (or both) are tautologies.

    This is not true. We can take φ to be p and ψ to be $\neg p$, where $\varphi \vee \psi$ is a tautology, but neither p or $\neg p$ is a tautology.

11. If φ and $\varphi \to \psi$ are tautologies, then ψ is a tautology. Fix φ and ψ. Assume for

    any valuation v, $[\![\varphi]\!]_v = \mathsf{T}$ and for any valuation v', $[\![\varphi \to \psi]\!]'_v = \mathsf{T}$. It is our job to prove that for any valuation w, $[\![\psi]\!]_w = \mathsf{T}$.

3 Avoiding confusion

It is easy to confuse the following two assertions:

(i) $\nvdash \varphi \to \psi$

(ii) $\vdash \varphi \to \neg\psi$.

(i) says that there is *no* derivation in our system that has no premises and ends with $\varphi \to \psi$. (ii) says that there *is* a derivation in our system that has no premises and ends with $\varphi \to \neg\psi$.

1. Give an example of two sentences φ and ψ in propositional logic with the property that $\nvdash \varphi \to \psi$ but $\nvdash \varphi \to \neg\psi$. This shows that (i) does not in general imply (ii).

   Fix two atomic sentences p and q such that $\varphi = p$ and $\psi = q$. Then we see that both $\nvdash p \to q$ and $\nvdash p \to \neg q$.

2. Give an example of two sentences φ and ψ in propositional logic with the property that $\vdash \varphi \to \psi$ and also $\vdash \varphi \to \neg\psi$. This shows that (ii) does not in general imply (i).

   Fix an atomic sentence p. Let $\varphi = \mathsf{F}$ and $\psi = p \vee \neg p$. Then we see that $\vdash \mathsf{F} \to (p \vee \neg p)$ and $\vdash \mathsf{F} \to \neg(p \vee \neg p)$

4 A step in the lemma on state descriptions

Recall from class the main lemma on state descriptions. It says: For all sentences φ, and all state descriptions α, if $occ(\varphi) \subseteq occ(\alpha)$, then either $\vdash \alpha \to \varphi$, or else $\vdash \alpha \to \neg\varphi$.

The proof was by induction on φ. In the lecture slides, you can find the induction steps for $\wedge$ and for $\neg$. Your task: prove the induction step for $\to$. Be sure to use the relation between the sets $occ(\varphi \to \psi)$, $occ(\varphi)$, and $occ(\psi)$.

Fix φ and ψ. Assume if $occ(\varphi) \subseteq occ(\alpha)$ then either $\vdash \alpha \to \varphi$ or $\vdash \alpha \to \neg\varphi$.
Also assume that if $occ(\psi) \subseteq occ(\alpha)$ then either $\vdash \alpha \to \psi$ or $\vdash \alpha \to \neg\psi$.
We prove if $occ(\varphi \to \psi) \subseteq occ(\alpha)$ then either $\vdash \alpha \to (\varphi \to \psi)$ or $\vdash \alpha \to \neg(\varphi \to \psi)$.
Suppose $occ(\varphi \to \psi) \subseteq occ(\alpha)$. We show that $\vdash \alpha \to (\varphi \to \psi)$ or $\vdash \alpha \to \neg(\varphi \to \psi)$.

Note that $occ(\varphi \to \psi) = occ(\varphi) \cup occ(\psi)$. So $occ(\varphi) \subseteq occ(\varphi \to \psi)$ and $occ(\psi) \subseteq occ(\varphi \to \psi)$.

Then by our assumption $\vdash \alpha \to \varphi$ or $\vdash \alpha \to \neg\varphi$. By our other assumption $\vdash \alpha \to \psi$ or $\vdash \alpha \to \neg\psi$

We must prove 4 different cases:

$$\alpha \to \varphi, \alpha \to \psi \vdash \alpha \to (\varphi \to \psi)$$

1.	$\alpha \to \varphi$	Premise
2.	$\alpha \to \psi$	Premise
3.	α	Assume
4.	φ	Assume
5.	ψ	$\to_e$, 2, 3
6.	$\varphi \to \psi$	$\to_i$, 4, 5
7.	$\alpha \to (\varphi \to \psi)$	$\to_i$, 3-6

$$\alpha \to \neg\varphi, \alpha \to \psi \vdash \alpha \to (\varphi \to \psi)$$

1.	$\alpha \to \neg\varphi$	Premise
2.	$\alpha \to \psi$	Premise
3.	α	Assume
4.	φ	Assume
5.	ψ	$\to_e$, 2, 3
6.	$\varphi \to \psi$	$\to_i$, 4, 5
7.	$\alpha \to (\varphi \to \psi)$	$\to_i$, 3-6

$$\alpha \to \varphi, \alpha \to \neg\psi \vdash \alpha \to \neg(\varphi \to \psi)$$

1.	$\alpha \to \varphi$	Premise
2.	$\alpha \to \neg\psi$	Premise
3.	α	Assume
4.	$\varphi \to \psi$	Assume
5.	φ	$\to_e$, 1, 3
6.	ψ	$\to_e$, 3, 4
7.	$\neg\psi$	$\to_e$, 2, 3
8.	F	F_i, 5,6
9.	$\neg(\varphi \to \psi)$	$\to_i$, 4, 5
10.	$\alpha \to \neg(\varphi \to \psi)$	$\to_i$, 3-6

$$\alpha \to \neg\varphi, \alpha \to \neg\psi \vdash \alpha \to (\varphi \to \psi)$$

1.	$\alpha \to \neg\varphi$	Premise
2.	$\alpha \to \neg\psi$	Premise
3.	α	Assume
4.	$\neg\psi$	Assume
5.	$\neg\varphi$	$\to_e$, 1, 3
6.	$\neg\psi \to \neg\varphi$	$\to_i$, 4-5
7.	$\varphi \to \psi$	Contrapositive
8.	$\alpha \to (\varphi \to \psi)$	$\to_i$, 3-7

By showing all the possible cases we conclude that $\vdash \alpha \to (\varphi \to \psi)$ or $\vdash \alpha \to \neg(\varphi \to \psi)$. This concludes the induction step for $\to$.

Below is the necessary proof of contrapositive used in the previous result.

$$\vdash (\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$$

1.	$\neg\psi \to \neg\varphi$	Assume
2.	φ	Assume
3.	$\neg\psi$	Assume
4.	$\neg\varphi$	$\to_e$, 1, 3
5.	F	F_i, 2, 4
6.	$\neg\neg\psi$	$\neg_i$, 3-5
7.	ψ	$\neg\neg_e$
8.	$\varphi \to \psi$	$\to_i$, 2-7

5 Consistent and satisfiable

A sentence φ in propositional logic is *consistent* if $\nvdash \neg\varphi$. That is, $\neg\varphi$ is *not* provable.

1. Prove that if φ is consistent, then φ is satisfiable.

   Fix φ. We prove the contrapositive. Assume φ is not satisfiable and we show φ is not consistent. φ not satisfiable means for all valuations v, $[\![\varphi]\!]_v = \mathsf{F}$. Then $[\![\neg\varphi]\!]_v = \mathsf{T}$, so $\models \neg\varphi$. By completeness, $\vdash \neg\varphi$ and therefore φ is inconsistent.

2. Prove that if φ is satisfiable, then φ is consistent.

   Fix φ. We prove the contrapositive. Assume φ is inconsistent. We have to show φ is not satisfiable. This means $\vdash \neg\varphi$. Then by soundness, $\models \neg\varphi$. So there are no valuations such that $[\![\varphi]\!] = \mathsf{T}$, therefore $\not\models \varphi$ and φ is not satisfiable.

You will need to use either the soundness or completeness of our logic, or both. Please be sure to write down exactly where you used these results.