Computationally feasible identification-robust inference on discrete choice demand

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Abstract

I propose an approach to inference on BLP-style demand models when instruments are potentially weak. I show how in practice one can adapt the two-step identification-robust procedure proposed by Andrews (2018) to such models. Direct application of this approach introduces substantial computational complexity, since it requires grid search over a potentially large parameter space. I provide conditions under which the time complexity of the procedure is reduced from the total number of parameters to the number of so-called "nonlinear parameters." Monte Carlo simulations reveal that the two-step confidence set, equipped with my dimension reduction technique, achieves the correct coverage probability. It is also shown that, although the technique is developed under the assumption of homoscedasticity, the resulting confidence set still performs well when the true structural errors are heteroscedastic.

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1 Introduction

This paper proposes an approach to inference in BLP-style demand estimation (Berry, Levinsohn, and Pakes, 1995) when instruments may be weak. Instruments for prices and quantities are essential to identification of such models, and various types of instruments have been considered in the literature. On the other hand, there are growing concerns about instruments that are inherently weak and their implications for inference.

To the best of my knowledge, this paper is the first exploration of applying an econometric method that is robust to weak identification in the context of demand estimation. To do so, I adapt the two-step confidence set proposed by Andrews (2018). One challenge of such identification-robust approach is its computational cost when the parameters are of high dimension. My approach softens this cost by providing conditions under which the dimensionality of expensive grid search can be significantly reduced. In this regard, I provide practical guidance for adapting a recent development in econometrics to an important class of applications in empirical industrial organization.

Monte Carlo simulations show that the coverage probability of the conventional non-robust confidence set is compromised under weak identification, whereas the proposed robust confidence set equipped with the dimension reduction technique does attain the correct coverage probability when the conditions are met. Furthermore, the conditions I impose for computational ease have minimal impact on the performance of the robust confidence set.

Since Berry (1994) provided a way to reformulate demand models into generalized method of moments (GMM) problems, the BLP estimator by Berry, Levinsohn, and Pakes (1995) has been a central tool in empirical industrial organization. Several types of instruments have been proposed in response to different data availability.[1] Those instrumental variables are to provide sufficient exogenous variation to shift the market shares and prices independent of the demand shocks associated with each product–market pair. An important question, of course, is whether the instruments are strong enough, i.e., whether they provide sufficient variation in the endogenous variables. This concern is manifested in the emphasis on optimal instruments by Berry, Levinsohn, and Pakes (1995), Reynaert and Verboven (2014), Gandhi and Houde (2019), and Conlon and Gortmaker (2020).

When instrumental variables are weak, GMM estimators (which include the BLP estimator) are known to not behave according to the conventional asymptotics. In particular, the estimators are not asymptotically normal, and the corresponding confidence sets do not have correct coverage probability under weak identification asymptotics, potentially resulting in misleading policy implications. For example, when determining a relevant market for tradi-

[1] See Berry and Haile (2021) for discussion about various types of instruments.

tional antitrust merger policy, demand estimates play a key role in predicting the changes in the profit due to a marginal increase in the prices of a hypothetical monopolist in a candidate market. If the conventional confidence interval for this profit change suffers under-coverage due to weak identification,[2] then it might fail to capture the true profit changes more often than as intended. Also, a researcher might mistakenly regard the result as more precise than it is, due to the spuriously narrow confidence interval.

The econometrics literature has worked on tests for weak identification and inference that is robust to weak identification, although primarily for simpler empirical settings. This paper implements such robust inference, for the first time in the context of BLP-style models. Specifically, I employ the two-step procedure by Andrews (2018). It has an advantage in that it also provides an "informal" test as to whether the instruments are weak,[3] while guaranteeing a pre-specified level of the resulting confidence set, under both strong and weak identification.

One hurdle in implementing an identification-robust confidence set, including that of Andrews (2018), is the expensive computational cost it incurs. Existing robust confidence sets are constructed by inverting a test statistic, for which grid search is a usual choice due to its simplicity and robustness. However, grid search often requires a large number of points at which the test statistic is to be evaluated. In particular, the number of grid points grows exponentially in the number of parameters, potentially requiring days or weeks to obtain a confidence set. Exacerbating this challenge is the fact that the size and location of the grid to be considered are not clear ex ante. These practical challenges may be increasingly relevant as more product characteristics become available and therefore the dimension of the parameter space increases due to increasing data availability.

To overcome these challenges surrounding grid search, I provide a condition under which the dimension of the grid can be reduced. The condition consists of two assumptions, namely just-identification and homoscedasticity. Under just-identification, the robust statistic used in the two-step method reduces to a nonlinear Anderson-Rubin-type (AR-type) test statistic. Using homoscedasticity, I transform the test statistic into the linear AR test statistic conditional on "nonlinear" parameters, akin to the separation between linear and nonlinear parameters in the nested fixed point algorithm for BLP. Then I apply a method by Dufour and Taamouti (2005) to obtain the analytic representation of the corresponding robust confi-

[2] In general, under-coverage is considered a more serious problem. As for weak identification, there are few results about whether under- or over-coverage prevails.

[3] To the best of my knowledge, in the context of nonlinear instrumental variables models, there is not yet a widely adopted formal test for detecting weak identification, if any, unlike the first-stage F-test by Stock and Yogo (2005). See Berry and Haile (2021) for discussion. One of the few proposals in the BLP context is that of Gandhi and Houde (2019), who propose a test for independence of irrelevant alternatives (IIA) as a test of weak identification. I discuss in Section 2.4 what is meant by the qualifier "informal."

dence set, which then takes a simple geometrical shape known as *quadric*, of which ellipsoids are special cases. The first step of the two-step procedure additionally requires checking whether a robust confidence set is included in the conventional confidence set. Although the confidence sets involved are now quadrics, checking such inclusion is not a trivial task. To this end, I apply the S-lemma from control theory to reduce the problem to an optimization of a single variable concave function, which permits fast numerical solution.

My approach reduces the dimension of the required grid from the total number of parameters to the number of nonlinear parameters only. I detail an algorithm that implements the procedure, with a suggestion of how to form a grid for the remaining nonlinear parameters.

I conduct Monte Carlo simulation exercises to evaluate my proposed approach. The results show that the conventional non-robust confidence set exhibits under-coverage under weak identification scenarios. In contrast, the robust two-step confidence set has coverage probability around the nominal coverage probability under both weak and strong identification. I also investigate how much the homoscedasticity assumption, which is potentially restrictive, distorts the confidence set when the true structural errors are actually heteroscedastic. The results show that the robust confidence set obtained using the dimension reduction technique (assuming homoscedasticity) still performs well. It performs better than a confidence set that is robust against heteroscedasticity but not robust against weak identification. It also approximates well the confidence set that is robust against both weak identification and heteroscedasticity (yet requiring full grid search). This suggests that the method can be useful as a good approximation to the fully robust confidence set, or as guidance for how to form a grid for obtaining the confidence set robust against both weak identification and heteroscedasticity.

The paper is organized as follows. Section 2 introduces the BLP model and describes how to construct the two-step confidence set by Andrews (2018). Section 3 discusses difficulties in applying the two-step procedure, proposes a method to reduce the dimensionality of the grid search under two main assumptions, and provides an algorithm to implement it. Section 4 conducts Monte Carlo simulations to investigate the properties of the two-step confidence set with the low-dimensional grid search, under both strong and weak identification scenarios, with both homoscedastic and heteroscedastic errors.

2 The model and the two-step confidence set

In this section, I lay down the discrete choice demand model of interest (commonly referred to as the BLP model) and describe the construction of the two-step confidence set by Andrews (2018). Along the way, the concept of weak identification and some desirable properties of

the two-step procedure are briefly discussed.

The rest of this paper focuses on inference on the entire vector of the parameters, rather than its sub-vector or a known function of it.[4] Also, I consider the case where a researcher only assumes the demand side model, without restrictions on the supply side, as in Nevo (2000); Conlon and Gortmaker (2020) suggest that with correctly specified supply side and optimal instruments, the BLP estimator tends to be well-behaved in finite sample, even when the instruments are fairly weak.

2.1 The discrete choice model and the BLP estimator

Consider a discrete choice model, where specifically, consumer i's conditional indirect utility from good $j \in \{1, \ldots, J\}$ is[5]

$$u_{ijt} = x'_{jt}\beta - \alpha p_{jt} + \xi_{jt} + \mu_{ijt} + \epsilon_{ijt} \equiv \delta_{jt} + \mu_{ijt} + \epsilon_{ijt},$$

where $t = 1, \ldots, T$ denotes markets. The distribution of μ_{ijt} may be parameterized by γ. Consumer i's utility depends on the characteristics x_{jt}, the price p_{jt}, the unobserved product-level heterogeneity term ξ_{jt}, and the individual-product-level heterogeneity terms μ_{ijt} and ϵ_{ijt}. For example, in a random coefficient model, $\mu_{ijt} = x'_{jt}\nu_i$ with $\nu_i \sim N(0, \Sigma(\gamma))$, where $\Sigma(\gamma)$ is a positive semidefinite matrix that depends on γ. The idiosyncratic error term ϵ_{ijt} usually independently and identically follows the type I extreme value distribution. The outside option, represented by $j = 0$, is assumed to yield the utility $u_{i0t} = \epsilon_{i0t}$.

The market share is then

$$s_{jt} = \int \mathbb{1}(u_{ijt} \geq u_{ikt} \ \forall k) dF(\epsilon, \mu) = \int \frac{\exp(\delta_{jt})}{1 + \sum_{k=1}^{J} \exp(\delta_{kt})} dF(\mu)$$

where F denotes the joint distribution of ϵ and μ. The second equality holds under the assumption that ϵ_{ijt} follows the type I extreme value distribution.

In standard parametric discrete choice models, we can invert the market share to derive

[4] When a researcher is interested only in a subset of parameters, deriving a confidence set for the entire vector and then taking the corresponding projection of the confidence set results in a conservative confidence set; i.e., the coverage probability is higher than the nominal coverage probability. Non-conservative inference on sub-vectors or a known non-stochastic function of the entire parameter vector is an active research area. However, in demand estimation, the objects of interest such as price elasticities or diversion ratios often involve the entire parameter vector. They are indeed functions of parameters, but those functions also depend on the data, for which case I am not aware of a result yet.

[5] I assume the number of products is the same across markets for simplicity.

δ_{jt} from the observable variables, for a given value of γ (Berry, 1994):[6]

$$\delta_{jt}(\gamma) \equiv \delta_{jt}(\gamma; s_t, p_t, x_t).$$

The structural error, given a parameter value $\theta = [\beta', \gamma']'$, is then

$$\xi_{jt}(\theta) = \delta_{jt}(\gamma) - x'_{jt}\beta$$

where (and hereafter) we let x_{jt} and β include p_{jt} and α respectively, and omit the observed variables s_t and x_t in $\delta_{jt}(\gamma; s_t, x_t)$. As γ enters the expression for the error nonlinearly while β does so linearly, the parameters are called "nonlinear" and "linear" parameters respectively.

The identifying assumption is that the structural error $\xi_{jt}(\theta)$ is uncorrelated[7] with instruments z_{jt} at (and only at) the true parameter value θ_0:

$$\mathbb{E}[z_{jt}\xi_{jt}(\theta_0)] = 0.$$

Writing $g_{jt}(\theta) = z_{jt}\xi_{jt}(\theta)$, the BLP estimator $\widehat{\theta}$ (Berry, Levinsohn, and Pakes, 1995) minimizes the generalized methods of moments (GMM) criterion function:[8]

$$\widehat{\theta} = \arg\min_{\theta} \overline{g}_n(\theta)' A_n \overline{g}_n(\theta),$$

where $n = JT$ is the total number of products, $\overline{g}(\theta) = n^{-1}\sum_{jt} g_{jt}(\theta) = n^{-1}\sum_{jt} z_{jt}\xi_{jt}(\theta)$, and A_n is a GMM weighting matrix that can possibly depend on θ as in the continuously updating estimator.

2.2 The standard confidence set

In what follows, $\dim v$ for a vector v denotes the dimension of v, and $\dim f$ for a vector-valued function f is the dimension of the range. $\chi^2(k)$ represents the chi-squared distribution with degree of freedom k, and $\chi^2_{1-\alpha}(k)$ is the $1-\alpha$ quantile of $\chi^2(k)$.

Given the GMM estimator $\widehat{\theta}$, the usual choice of confidence set is obtained by inverting

[6] Berry, Gandhi, and Haile (2013) provide conditions under which such an inversion is possible in a general nonparametric setting.

[7] This is usually an implication of mean-independence assumption: $\mathbb{E}[\xi_{jt}|\widetilde{z}_{jt}] = 0$ almost surely, based on which z is chosen as a function of instruments $\widetilde{z}$.

[8] Since the inversion of the share function to calculate $\delta_{jt}(\gamma)$ typically involves numerical integration via a Monte Carlo simulation, one needs to take into account the error from such simulation when deriving the asymptotic variance (Berry, Linton, and Pakes, 2004). In this paper, however, I assume that the number of simulation draws is sufficiently large or that the numerical integration uses other methods to achieve smaller errors (Conlon and Gortmaker, 2020), and ignore the error from numerical integration.

the Wald statistic $W(\theta)$:

$$W(\theta) = (\widehat{\theta} - \theta)' \widehat{\Sigma}_n^{-1} (\widehat{\theta} - \theta),$$

where $\widehat{\Sigma}_n$ is a consistent estimator for the asymptotic variance of $\sqrt{n}(\widehat{\theta} - \theta)$. The corresponding confidence set with confidence level $1 - \alpha$ is[9]

$$CS_N = \{\theta \in \Theta : W(\theta) \leq \chi^2_{1-\alpha}(\dim \theta)\},$$

where Θ is a suitable parameter space. We shall call CS_N the non-robust confidence set for the reason I explain as follows.

Weak identification and non-robustness of CS_N To discuss what weak identification means, let the expected Jacobian of the moment condition at the true parameter be defined as

$$G \equiv \mathbb{E}\left[\frac{\partial}{\partial \theta'} g_{jt}(\theta_0)\right].$$

Weak identification pertains to cases where G is not full rank or G is "small," whose exact meaning depends on how one models weak identification. A popular choice, for example by Staiger and Stock (1997) or Kleibergen (2005), is to have G drifting to zero at the rate of $\sqrt{n}$ as $n \to \infty$; e.g., G depends on n and $G = C/\sqrt{n}$ where C is a finite matrix.[10] This modeling approach is also adopted by Andrews (2018) in the context of GMM.[11] As this paper does not pursue the derivation of new asymptotic properties of the robust confidence set, I refer readers to Andrews (2018).

In any models of weak identification, it is central to the identification status whether the

[9] We have two α's in this paper depending on the context; one is the mean price coefficient in the indirect utility function, and the other is related to the confidence level as this one.

[10] A more general approach by Andrews and Guggenberger (2017) is to set a parameter space and study the asymptotic size of tests or the asymptotic coverage probability of confidence sets in a uniform sense. An advantage of this approach is that it allows for arbitrary sequences on the parameter space, including the aforementioned $\sqrt{n}$-rate sequences.

[11] However, as mentioned in that paper, the two-step procedure itself is agnostic to the modeling choice regarding weak identification, as long as its high-level assumptions are satisfied.

Jacobian G is small.[12] In the (parametric) BLP model, the Jacobian translates into

$$\left[\mathbb{E}\left[p_{jt}z_{jt}\right], -\mathbb{E}\left[z_{jt}x'_{jt}\right], \mathbb{E}\left[\left(\frac{\partial}{\partial\gamma'}\delta_{jt}(\gamma_0; s_t, p_t, x_t)\right) z_{jt}\right]\right]$$

where we write α and p_{jt} separately from β and x_{jt} (unlike notation as above) for the sake of interpretation.

From the first part, $\mathbb{E}[p_{jt}z_{jt}]$, we see that identification is weak when the instruments z_{jt} are not sufficiently relevant to the endogenous price p_{jt}. The last part is somewhat harder to interpret; we have weak identification when the instruments are not strongly relevant to the variability of the inverse market share function with respect to the nonlinear parameter. In a general sense, as the market share s_t appears in the expression, this shows us that the instruments need to be sufficiently relevant to the market shares, in line with Berry and Haile (2014). Indeed, in a nested logit model, the last part becomes $\mathbb{E}\left[(\log s_{j|g,t})z_{jt}\right]$ where $s_{j|g,t}$ is the within-group market share of j in market t.

It is known that, under such weak identification, the GMM estimator (and hence the BLP estimator) may not be asymptotically normal, and the corresponding standard confidence set may not have correct coverage. This motivates the use of confidence sets that are robust to weak identification, as described in the following subsection.

2.3 Robust confidence sets

To address incorrect coverage probability of the non-robust confidence set under weak identification, I apply the two-step procedure developed by Andrews (2018). The remainder of this section reiterates the method by Andrews (2018) on how to construct confidence sets that are robust against weak identification. Section 3 will discuss how I adapt this method in a computationally feasible way.

The procedure requires some ingredients, including the non-robust confidence set described in the previous subsection, and two robust confidence sets, denoted as CS_P and CS_N. These two sets are constructed by inverting a test statistic, which in turn is a combination of two robust test statistics called S and K, the latter by Kleibergen (2005). In the next section and for the rest of the paper, I assume that the demand model is just-identified. I consider this assumption to be not restrictive, given that a demand model is just-identified

[12] It is worth mentioning that the boundary between strong and weak identification is often not clear-cut in finite sample, or may not be even defined when asymptotics is involved. It is not as straightforward as, for a contrived example, "the identification is weak if the matrix norm of G is smaller than a certain value." Rather, when we model weak identification in terms of drifting sequences of parameters, the identification status is a feature of such *sequences*, rather than a partition of the parameter space given a fixed T. In this paper, however, I use the terms weak identification and strong identification loosely to avoid complication.

when optimal instruments are used, and as those can be conveniently computed using packages like PyBLP (Conlon and Gortmaker, 2020). An implication of just-identification in the current context is that K coincides with S, and hence K does not need to be calculated separately from S. Therefore, for simplicity, I present the robust statistic in its simplified (yet still correct) form that arises under just-identification.[13]

The weak identification literature has developed methods mainly based on the S statistic, which is often called the AR-type test statistic (Anderson and Rubin, 1949; Stock and Wright, 2000). The statistic is a quadratic form of the average moment function $\overline{g}_n(\theta)$:

$$S(\theta) = n\overline{g}_n(\theta)'\Omega_n(\theta)^{-1}\overline{g}_n(\theta)$$

where $\Omega_n(\theta)$ is a consistent estimator for $\text{Var}(g_{jt}(\theta))$. Under the null that $\mathbb{E}[z_{jt}\xi_{jt}(\theta)] = 0$, the test statistic converges in distribution to $\chi^2(\dim g)$, regardless of the Jacobian $\mathbb{E}[\partial g_{jt}(\theta)/\partial\theta']$, i.e., even under weak identification.[14]

To construct the test statistic, a researcher first chooses a number $\zeta > 0$ (e.g., $\zeta = 0.05$) such that they are willing to take $1-\alpha-\zeta$ as the lower bound of the coverage probability under weak identification. It turns out that the larger ζ is, the more likely the two-step procedure will indicate strong identification; we see a trade-off here. If a researcher prefers that the procedure indicates strong identification (by setting a larger ζ), then she takes higher risk of misclassifying the identification situation as strong when the true data generating process in fact is of weak identification. When such misclassification occurs, the coverage probability of the reported non-robust confidence set may deviate from the nominal level $1-\alpha$, more so as ζ becomes larger. Given the choice of ζ, let $a = \chi^2_{1-\alpha}(\dim\theta)/\chi^2_{1-\alpha-\zeta}(\dim\theta) - 1$.

Now define two robust confidence sets:

$$\begin{aligned} CS_P &= \{\theta \in \Theta : S(\theta) < \chi^2_{1-\alpha}(\dim\theta)/(1+a)\} \\ CS_R &= \{\theta \in \Theta : S(\theta) \leq \chi^2_{1-\alpha}(\dim\theta)\}. \end{aligned} \tag{1}$$

We call CS_P the preliminary robust confidence set, as it will be used in the first step of the procedure. As $S(\theta)$ converges in distribution to $\chi^2(\dim g)$ under the null, which is equivalent to $\chi^2(\dim\theta)$ when the model is just-identified, we see that the critical values are designed so that CS_P and CS_R have coverage probability of $1-\alpha-\zeta$ and $1-\alpha$ respectively.

[13] See Andrews (2018) for the construction of the robust statistic under over-identification that uses K.

[14] As can be seen from the construction, the test statistic directly measures how much the empirical moment condition $\overline{g}_n(\theta)$ deviates from zero, instead of being based on the estimator $\widehat{\theta}$ whose asymptotic distribution depends on the Jacobian.

2.4 The two-step confidence set

With the non-robust confidence set CS_N and robust confidence sets CS_P and CS_R, the two-step confidence set is constructed as follows. The first step checks whether $CS_P \subseteq CS_N$. If so, it is interpreted as an indication of strong identification, and weak identification otherwise. The idea behind this is that both robust and non-robust confidence sets CS_N and CS_R behave similarly to each other under strong identification. Therefore a set smaller than CS_R, namely CS_P in this context, will tend to be included in CS_N under strong identification.

In the second step, the non-robust confidence set CS_N is reported if the first step indicates strong identification, and the robust confidence set CS_R is reported for weak identification; i.e., the resulting confidence set is

$$CS_2 = \begin{cases} CS_N & \text{if } CS_P \subseteq CS_N \\ CS_R & \text{if } CS_P \nsubseteq CS_N. \end{cases}$$

The role of ζ becomes clearer in this context. The preliminary set CS_P is constructed by shrinking CS_R. As the value of ζ increases, CS_P becomes smaller, resulting in a reduced coverage probability of CS_P, specifically $1-\alpha-\zeta$. As CS_P becomes smaller, the first step is more likely to indicate strong identification, which would increase the risk of reporting the misbehaving non-robust confidence set CS_N when the true data generating process is actually weakly identified. However, even when such misclassification occurs, the set inclusion relationship between CS_N and CS_P helps bound the extent of the misbehavior; if the first step (wrongly) indicates strong identification, then CS_N contains CS_P by construction. As CS_P has coverage probability of $1-\alpha-\zeta$, the coverage probability of CS_N is at least $1-\alpha-\zeta$.

The two-step method satisfies the following properties: (i) along any strongly identified sequence of parameters, the first step indicates strong identification with probability approaching one, (ii) along any weakly identified sequence of parameters, the two-step confidence set CS_2 has an asymptotic coverage probability of at least $1-\alpha$, and (iii) along any weakly identified sequence of parameters, CS_2 has an asymptotic coverage probability of at least $1-\alpha-\zeta$. For a formal statement and sufficient regularity conditions, see Theorem 1 by Andrews (2018).

The first property suggests that the first step may be considered as an "informal" test of weak identification; it gives us a consistent test for the null of weak identification, in that the rejection probability of the test approaches one when the true data generating process is of strong identification. However, it is not a test in a usual sense, in that it does not control the size of the test; we do not know the probability of falsely rejecting the null when the true

data generating process is indeed of weak identification. The Monte Carlo simulations in Section 4 suggest that the first step does indicate weak identification with high probability (if not with probability one) when the instrumental variables seem weak. Still, the procedure does not pre-specify the type I error.

The third property only guarantees that CS_2 has a coverage probability of at least $1-\alpha-\zeta$, which is due to the coverage probability of CS_N under weak identification as discussed above. However, Monte Carlo simulations in Section 4 suggest that CS_2 tends to achieve a coverage probability around $1-\alpha$ under weak identification. Under weak identification, the first step correctly indicates weak identification with high probability, in which case CS_2 coincides with CS_R, which is designed to have a coverage probability of $1-\alpha$.

3 Dimension reduction in the two-step method

The two-step method introduced in the previous section usually requires grid search, which poses computational challenges. In this section I discuss how the dimensionality of the grid can be significantly reduced, in particular when (i) inverting the robust test statistic to construct CS_P and CS_R, and (ii) checking $CS_P \subseteq CS_N$ in the first step of the procedure. The results hinge on Assumption 1 and 2 which are introduced below. Then, I detail an algorithm that utilizes the dimension reduction technique.

3.1 Inverting the robust test statistic

The confidence sets in the procedure—robust and non-robust ones—are obtained by inverting the corresponding tests; for example, the non-robust confidence set CS_N collects all the parameter values θ (in the parameter space) at which the test statistic $W(\theta)$ does not exceed the critical value $\chi^2_{1-\alpha}(\dim\theta)$.

The non-robust confidence set CS_N is easy to handle due to its simple form. The left hand side of the inequality $(\widehat{\theta}-\theta)'\widehat{\Sigma}_n^{-1}(\widehat{\theta}-\theta) \leq \chi^2_{1-\alpha}(\dim\theta)$ is quadratic in θ. As a result, various properties immediately follow. These include the facts that the confidence set is an ellipsoid centered around $\widehat{\theta}$, and that a closed form solution exists when calculating projection confidence intervals.

On the other hand, inverting a test in general is a non-trivial task in empirical work, since a confidence set can take any shape and its closed form representation can be difficult to derive (if any). The most popular approach in such cases is to conduct a grid search, due to its simple implementation and robustness. A researcher specifies a finite set of points in the parameter space, evaluates the test statistic at each point, and then collects all points

at which the value of the statistic does not exceed the corresponding critical value.

However, grid search has two potential problems: (i) the number of grid points grows exponentially with the number of parameters, and (ii) it is not clear ex ante where to construct the grid in the large parameter space.

As an example, the demand model by Nevo (2000) involves ten linear parameters, let alone nonlinear parameters. If one wants to apply the two-step identification-robust procedure with ten parameters by constructing a coarse grid with ten points for each parameter,[15] it would require 10^{10}, or ten billion grid points. To translate the magnitude into computation time, it takes two minutes to run the two-step procedure for a nested logit with only five parameters with ten points on each direction, in Python on a machine having 2.8 GHz CPU with 20 cores. Increasing the dimension to ten would take 100,000 times that amount of time, i.e., about 140 days. If, in addition, the grid is made slightly denser by increasing the number of points in each direction to 20, then it would take 390 years. While a better implementation, such as running the procedure in C instead, may significantly reduce the computation time, I expect the magnitude to be still large especially with large number of parameters are involved.

Another issue is regarding the location, the size, and the denseness of the grid to be chosen by a researcher. In the mixed logit model by Nevo (2000), the linear parameter estimates show a large variation in magnitude, from 0.03 to 43.04. Obviously significant part of the variability is due to different scales of the characteristics variables, and one can standardize the variables before estimation to reduce the variability. However, even so, we do not know ex ante which characteristics would have larger or smaller coefficients, nor the overall magnitude of the coefficients relative to the idiosyncratic error term ϵ_{ijt}. Robust confidence sets may be even unbounded, as shown by Dufour (1997), aggravating the issue of the width choice of a grid.

Moreover, in combination with the previous point about computational burden, an inappropriate choice of the grid may result in imprecise results; if one chooses too wide a grid but at low resolution due to computational cost, then the researcher might not detect the confidence set. For example, suppose a true confidence interval (given the data) for one of the parameters is $[3.1, 3.8]$, but a researcher forms a coarse grid $\{0, 1, \ldots, 10\}$.[16] Then the grid search would report that none of the grid points belong to the confidence interval, leading the researcher to falsely conclude that the confidence set is an empty set, although

[15] A grid with ten points in each direction is arguably very coarse; for comparison, with two parameters, Andrews (2018) chooses a grid of $201 \times 2,641$ points.

[16] As a reference, in a Stata command `twostepweakiv` by Sun (2018) that implements the two-step method for linear instrumental variables models, the default number of grid points (in each dimension) is set to 100, 25, 11, 7, and 5 as the dimension of the grid increases from one to five.

the confidence interval were to deliver a rather precise information about the parameter.

With these challenges in mind, I consider two assumptions that yield an analytic representation of the robust confidence set, similar to the non-robust Wald confidence set.

Assumption 1 (Just-identified demand model)**.** The demand model is just-identified; i.e., the number of instrumental variables equals the number of parameters.

When estimating the demand side without restricting the supply side (as mentioned at the beginning of Section 2), just-identification is automatically satisfied when optimal instruments are used.[17] When there are only demand-side structural errors ξ_{jt} (rather than having another set of structural errors from the supply side), the number of the optimal instrumental variables coincides with the number of the parameters. Since packages such as PyBLP (Conlon and Gortmaker, 2020) provide a readily available method to approximate the optimal instruments, just-identification is not a restrictive assumption.

The next assumption is about the data generating process.

Assumption 2 (Homoscedasticity)**.** The unobserved product-level heterogeneity ξ_{jt} is homoscedastic across j and t, i.e., $\mathbb{E}[\xi_{jt}|z_{jt}] = \mathbb{E}[\xi_{jt}]$ almost surely.

This assumption is restrictive in general. As the structural error ξ_{jt} represents unobserved product-level heterogeneity, such as product quality and latent taste variation across markets for the product, the condition may not be adequate unless the products are similar in nature, both within and across markets. Many different types of violation may occur; products supplied by a particularly innovative firm may have higher variance of ξ_{jt} conditional on observed characteristics. Markets with consumers that are more sensitive to product quality may have higher variance of ξ_{jt}.

Even if homoscedasticity is considered too restrictive, my method to analytically represent the robust confidence sets as below (under the homoscedasticity assumption) can still provide guidance for constructing a relevant grid, in case one wants to apply the two-step procedure under heteroscedasticity using grid search. I discuss this later with a Monte Carlo simulation.

Before stating the analytic representation result, let us define *partial* confidence sets. For a confidence set CS and a value of nonlinear parameter γ, we call

$$CS(\gamma) = \{\beta : (\beta, \gamma) \in CS\}$$

a partial confidence set (derived from CS at γ).

[17] If one is willing to obtain a conservative confidence set, then this assumption is not needed; see remarks after Proposition 3.1.

With the two assumptions, I derive the following result, which provides a closed form representation of the partial robust confidence sets $CS_R(\gamma)$ and $CS_P(\gamma)$ given nonlinear parameter γ.

Proposition 3.1. *Suppose Assumptions 1 and 2 hold. For a given distortion bound $\zeta > 0$, let*

$$a = \frac{\chi^2_{1-\alpha}(\dim\theta)}{\chi^2_{1-\alpha-\zeta}(\dim\theta)} - 1 > 0.$$

Then the partial robust confidence sets $CS_R(\gamma)$ and $CS_P(\gamma)$ can be written in the following form

$$\{\beta : \beta' A\beta + 2b'\beta + c \leq 0\}$$

where

$$\begin{aligned} A &= X'\left(P_Z - \frac{\mathcal{C}}{n}M_1\right)X \\ b &= -X'\left(P_Z - \frac{\mathcal{C}}{n}M_1\right)\delta \\ c &= \delta'\left(P_Z - \frac{\mathcal{C}}{n}M_1\right)\delta \end{aligned}$$

with $\mathcal{C} = \chi^2_{1-\alpha}(\dim\theta)$ for CS_R and $\mathcal{C} = \chi^2_{1-\alpha}(\dim\theta)/(1+a)$ for CS_P,

$$\begin{aligned} X &= [x'_{1,1}, \ldots, x'_{JT}]' \in \mathbb{R}^{n\times\dim\beta} \\ Z &= [z'_{1,1}, \ldots, z'_{JT}]' \in \mathbb{R}^{n\times\dim\theta} \\ \delta &= [\delta_{1,1}(\gamma), \ldots, \delta_{JT}(\gamma)]' \in \mathbb{R}^n, \end{aligned}$$

and projection matrices $P_Z = Z(Z'Z)^{-1}Z'$ and $M_1 = I_n - \iota_n(\iota'_n\iota_n)^{-1}\iota'_n$ where I_n is the $n \times n$ identity matrix and $\iota_n \in \mathbb{R}^n$ is the vector of ones.

Proof. Under just-identification, the K statistic by Kleibergen (2005) coincides with the S statistic by Stock and Wright (2000). Therefore the robust test statistic by Andrews (2018) can be written as $(1+a)S$, yielding confidence sets as defined in (1). By homoscedasticity, the variance of the moment function is $\mathrm{Var}(g_i(\theta)) = \sigma^2_\xi \mathbb{E} z_{jt} z'_{jt}$, where σ^2_ξ is the (unconditional) variance of $\xi_{jt}(\theta)$. For this we choose an estimator $\widehat{\sigma}^2_\xi(\theta) = n^{-1}\sum_{jt}(\xi_{jt}(\theta) - \overline{\xi}_n(\theta))^2$ where

$\overline{\xi}_n(\theta) = n^{-1}\sum_{jt}\xi_{jt}(\theta)$. Using matrix notation,

$$\widehat{\sigma}^2_\xi(\theta) = \frac{1}{n}\xi(\theta)'M_1\xi(\theta)$$

where $\xi(\theta) = [\xi_{1,1},\ldots,\xi_{J,T}]' \in \mathbb{R}^n$, and also

$$\widehat{\Sigma}_g(\theta) = \frac{\widehat{\sigma}^2_\xi(\theta)}{n}\sum_{jt} z_{jt}z'_{jt} = \frac{\widehat{\sigma}^2_\xi(\theta)}{n}Z'Z$$
$$\overline{g}_n(\theta) = \frac{1}{n}\sum_{jt} z_{jt}\xi_{jt}(\theta) = \frac{1}{n}Z'\xi(\theta).$$

Now the S statistic can be written as

$$\begin{aligned} S(\theta) &= n\overline{g}_n(\theta)'\widehat{\Sigma}_g(\theta)^{-1}\overline{g}_n(\theta) \\ &= \xi(\theta)'Z\left(\widehat{\sigma}^2_\xi(\theta)Z'Z\right)^{-1}Z'\xi(\theta) \\ &= \frac{\xi(\theta)'P_Z\xi(\theta)}{\xi(\theta)'M_1\xi(\theta)/n} \\ &= \frac{(\delta - X\beta)'P_Z(\delta - X\beta)}{(\delta - X\beta)'M_1(\delta - X\beta)/n}. \end{aligned}$$

The inequality for the robust test with a critical value $\mathcal{C}$ is then $S(\theta) \leq \mathcal{C}$, which is equivalent to

$$(\delta - X\beta)'P_Z(\delta - X\beta) \leq \frac{\mathcal{C}}{n}(\delta - X\beta)'M_1(\delta - X\beta),$$
$$(\delta - X\beta)'\left[P_Z - \frac{\mathcal{C}}{n}M_1\right](\delta - X\beta) \leq 0.$$

Observing that the left hand side is quadratic in β given γ (and hence given δ), the result follows from arranging the terms. □

The proof combines ideas by Stock and Wright (2000) and Dufour and Taamouti (2005); it first exploits the structure of the BLP-like models, to transform the nonlinear AR-type test statistic into a linear AR test statistic (conditional on nonlinear parameters). Then it derives a quadratic representation of the confidence sets. The resulting partial confidence sets $CS_R(\gamma)$ and $CS_P(\gamma)$ are *quadrics*, i.e., the shapes defined by quadratic inequalities, whose geometrical properties are known. Dufour and Taamouti (2005) provide such properties, including conditions under which a quadric is bounded as well as the analytic solution of its projections. For later reference, I collect them (from Theorems 4.1 and 5.1–5.3 therein) in

Proposition A.1 under the assumption that A is nonsingular.[18]

By providing analytic representations of $CS_R(\gamma)$ and $CS_P(\gamma)$, Proposition 3.1 reduces the dimensionality of forming a grid and searching over the grid from $\dim\theta$ to $\dim\gamma$. As discussed above, this not only decreases computational burden but also gets rid of having to choose an appropriate grid over β without prior knowledge; as Proposition A.1 shows, robust confidence sets may be unbounded under weak identification. The quadratic representation allows us to immediately check the boundedness of a partial robust confidence set. If it is bounded, then projections of the set can be easily obtained, again by invoking Proposition A.1.

As demonstrated in the proof, the purpose of just-identification assumption is to replace a robust statistic K with S, since they are the same under just-identification. When the model is over-identified, one can still proceed as the procedure prescribes (i.e., by inverting $S(\theta)$ and therefore constructing A, b, and c as in Proposition 3.1). The resulting two-step confidence set, however, will have a higher coverage probability compared to a confidence set that inverts the original statistic $K(\theta) + aS(\theta)$.

Note that we still need grid search over γ.[19] This resembles different roles played by the two sets of parameters in the BLP estimator; in minimization of the BLP-GMM objective function, each trial value of nonlinear parameter γ requires a market share inversion. On the other hand, given γ, the value of β that minimizes the objective function is analytically solved using the linearity of δ with respect to β, thereby reducing the dimensionality of nonlinear optimization from $\dim\theta$ to $\dim\gamma$.

As a non-robust confidence set is always an ellipsoid, its partial version is also an ellipsoid. I conclude this subsection by presenting a quadratic representation of the partial non-robust confidence set. Note that, in the proposition, $[\widehat{\Sigma}^{-1}]_{\beta\beta}$ is *not* the inverse of the top left $(\dim\beta \times \dim\beta)$ block of $\widehat{\Sigma}$, but the top left $(\dim\beta \times \dim\beta)$ block of the inverse of $\widehat{\Sigma}$.

Proposition 3.2. *Let $\widehat{\Sigma}$ be a consistent estimator of the asymptotic variance of $\sqrt{n}(\widehat{\theta} - \theta)$ where the elements of θ are ordered as $\theta = [\beta', \gamma']'$ and analogously for $\widehat{\theta}$. Let the blocks of its inverse be denoted as*

$$\widehat{\Sigma}^{-1} = \begin{bmatrix} [\widehat{\Sigma}^{-1}]_{\beta\beta} & [\widehat{\Sigma}^{-1}]_{\beta\gamma} \\ [\widehat{\Sigma}^{-1}]'_{\beta\gamma} & [\widehat{\Sigma}^{-1}]_{\gamma\gamma} \end{bmatrix}$$

with $[\widehat{\Sigma}^{-1}]_{\beta\beta} \in \mathbb{R}^{\dim\beta\times\dim\beta}$, $[\widehat{\Sigma}^{-1}]_{\beta\gamma} \in \mathbb{R}^{\dim\beta\times\dim\gamma}$, and $[\widehat{\Sigma}^{-1}]_{\gamma\gamma} \in \mathbb{R}^{\dim\gamma\times\dim\gamma}$.

[18] This is also a maintained assumption in Dufour and Taamouti (2005), as A being singular is unlikely. See Dufour and Taamouti (2007) for singular A.

[19] We discuss a rule-of-thumb way to form a grid for nonlinear parameters in Section 3.3.

Then the partial non-robust Wald confidence set $CS_N(\gamma)$ can be written as $\{\beta : \beta' A\beta + 2b'\beta + c \leq 0\}$ where

$$A = [\widehat{\Sigma}^{-1}]_{\beta\beta}$$
$$b = [\widehat{\Sigma}^{-1}]_{\beta\gamma}(\gamma - \widehat{\gamma}) - [\widehat{\Sigma}^{-1}]_{\beta\beta}\widehat{\beta}$$
$$c = \widehat{\beta}'[\widehat{\Sigma}^{-1}]_{\beta\beta}\widehat{\beta} - 2\widehat{\beta}'[\widehat{\Sigma}^{-1}]_{\beta\gamma}(\gamma - \widehat{\gamma}) + (\gamma - \widehat{\gamma})'[\widehat{\Sigma}^{-1}]_{\gamma\gamma}(\gamma - \widehat{\gamma}) - \chi^2_{1-\alpha}(\dim\theta)/n.$$

Proof. The results is a rearrangement of the inequality $(\widehat{\theta} - \theta)'\widehat{\Sigma}_n(\widehat{\theta} - \theta) \leq \chi^2_{1-\alpha}(\dim\theta)$. □

3.2 Checking inclusion of confidence sets

The first step of the two-step procedure requires us to check $CS_P \subseteq CS_N$. The Wald confidence set CS_N is an ellipsoid. On the other hand, the robust (preliminary) confidence set CS_P does not have a known shape in general. Consequently, without any structures, one would resort to a grid search over CS_P to check $CS_P \subseteq CS_N$, which can be computationally expensive, or might not detect a non-inclusion if the grid is too coarse.

Proposition 3.1 provided a condition under which $CS_P(\gamma)$ takes a known form, namely ellipsoid. Then the problem becomes checking whether an ellipsoid $CS_P(\gamma)$ is included in another ellipsoid $CS_N(\gamma)$ for all γ. The following proposition provides a feasible method to check the inclusion, even when the centers of the ellipsoids do not coincide.[20]

Proposition 3.3. *Let $CS_P(\gamma)$ and $CS_N(\gamma)$ be two ellipsoids in $\mathbb{R}^k$ represented by*

$$CS_P(\gamma) = \{x \in \mathbb{R}^k : x'A_P x + 2b_P' x + c_P \leq 0\}$$
$$CS_N(\gamma) = \{x \in \mathbb{R}^k : x'A_N x + 2b_N' x + c_N \leq 0\}.$$

where $A_P, A_N \in \mathbb{R}^{k\times k}$, $b_P, b_N \in \mathbb{R}^k$, $c_P, c_N \in \mathbb{R}$. Define a function $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) = \lambda_{\min}\left(t\begin{bmatrix} A_P & b_P \\ b_P' & c_P \end{bmatrix} - \begin{bmatrix} A_N & b_N \\ b_N' & c_N \end{bmatrix}\right)$$

where $\lambda_{\min}M$ denotes the smallest eigenvalue for a matrix M. Then ϕ is well-defined and concave. Suppose $CS_P(\gamma) \neq \emptyset$. Then $CS_P(\gamma) \subseteq CS_N(\gamma)$ if and only if there exists $t \geq 0$ such that $\phi(t) \geq 0$.

[20] When two ellipsoids share the same center, then it is sufficient to check whether a matrix is positive semidefinite; suppose we have two ellipsoids represented by $E_1 = \{x : (x - x_0)'A_1(x - x_0) \leq 1\}$ and $E_2 = \{x : (x - x_0)'A_2(x - x_0) \leq 1\}$. Then $E_1 \subseteq E_2$ iff $A_1 - A_2$ is positive semidefinite.

Proof. The function ϕ is well-defined since the matrix

$$M(t) = t \begin{bmatrix} A_P & b_P \\ b_P' & c_P \end{bmatrix} - \begin{bmatrix} A_N & b_N \\ b_N' & c_N \end{bmatrix}$$

is symmetric for all $t \in \mathbb{R}$. It is concave because it is the composition of an affine function $t \mapsto M(t)$ and a concave function $M \mapsto \lambda_{\min} M$.

The S-lemma (see e.g., Boyd and Vandenberghe, 2004) shows that $CS_P(\gamma) \subseteq CS_N(\gamma)$ if and only if there exists $t \geq 0$ such that $M(t)$ is positive semidefinite. The result follows since $M(t)$ is positive semidefinite if and only if $\lambda_{\min} M(t) \geq 0$. □

Since ϕ is a single variable concave function, it is easy to check the condition "ϕ attains a non-negative value on $[0, \infty)$"; for example, first check whether $\phi(0) \geq 0$. If that is the case, then the condition holds. If not, and if ϕ is decreasing at 0, then the condition does not hold. Otherwise, apply gradient ascent until ϕ takes a positive value (in which case we conclude that the condition holds) or arrives at the global maximum. If the global maximum is non-negative, then the condition holds. If the global maximum is negative, then the condition does not hold. The only case in which the procedure fails is when $\phi(t)$ increases but does not hit zero as $t \to \infty$.[21] In practice, one can try finding the maximum of ϕ within a large enough bounded interval that the machine can handle, say $[0, 10^{16}]$.

Remarks on dimensionality reduction by Proposition 3.1 pertains here as well; exploiting the structure under the assumptions, checking $CS_P \subseteq CS_N$ no longer requires a grid search on CS_P of dimension $\dim \theta$, but only of $\dim \gamma$.

3.3 The algorithm for two-step confidence set

I summarize the algorithm here using the previous results under Assumptions 1 and 2, and then discuss a few details regarding the algorithm.

1. Obtain the BLP estimate $\widehat{\theta}$ as well as the estimate $\widehat{\Sigma}$ for the asymptotic variance of $\sqrt{n}(\widehat{\theta} - \theta)$ (under homoscedasticity).

2. Choose the bound of distortion ζ, say 0.05, and set $a = \chi^2_{1-\alpha}(\dim \theta)/\chi^2_{1-\alpha-\zeta}(\dim \theta) - 1$.

3. Form a grid $\widetilde{\Gamma}$ in the parameter space for γ.

[21] This may happen in the most general setting; $\lambda_{\min}\left(t \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\right)$ is always negative while approaching zero as $t \to \infty$. However, it is not clear whether such a case may arise in our setting, in which we have restrictions such as A_1 and A_2 must be positive definite.

4. Set a weak instruments indicator variable, say `weakiv`, as false.

5. For each grid point γ do the following:

   (a) Compute $\delta(\gamma) = [\delta_{1,1}(\gamma; s_1, p_1, x_1), \ldots \delta_{J,T}(\gamma; s_T, p_T, x_T)]' \in \mathbb{R}^{J \times T}$.

   (b) Obtain $CS_R(\gamma)$ using Proposition 3.1 and store it.

   (c) If `weakiv` is true, then continue to the next value of γ.

   (d) Obtain $CS_P(\gamma)$ using Proposition 3.1.

   (e) If $CS_P(\gamma)$ is not bounded (using Proposition A.1), then set `weakiv` as true and continue to the next value of γ.

   (f) If $CS_P(\gamma) \not\subseteq CS_N(\gamma)$ (using Proposition 3.3), then set `weakiv` as true.

6. If `weakiv` is true, then report $CS_R = \{(\beta, \gamma) : \beta \in CS_R(\gamma), \gamma \in \widetilde{\Gamma}\}$ as CS_2. Otherwise, report CS_N as CS_2.

Obtaining optimal instruments To ensure just-identification when there are more instrumental variables than parameters, one can apply a convenient method provided by PyBLP to obtain the optimal instruments; before Step 1, obtain the BLP estimate, and then use it to calculate the optimal instruments.[22] Then begin with Step 1 with the newly constructed set of instruments.

Storing confidence sets At Step 5 (b), only the objects A, b, and c (as in Proposition 3.1) for each γ need to be stored in the memory, rather than the set of points that are in $CS_R(\gamma)$, since A, b, and c fully determine $CS_R(\gamma)$.

Grid search over nonlinear parameters The algorithm still requires grid search over nonlinear parameters γ, even though grid search over β is no longer present, which mirrors similar roles taken by nonlinear and linear parameters in BLP estimation procedure, as mentioned earlier. However, unlike the minimization task over γ as in BLP estimation, where several methods have been studied extensively and are readily available in optimization packages, research on effective and accurate grid search over a potentially large parameter space is still growing.[23]

One rule of thumb we may consider is to form a grid *slightly larger* than the set $\{\gamma : CS_N(\gamma) \neq \emptyset\}$, i.e., the projection of CS_N on the space of nonlinear parameters, which can

[22] See `ProblemResult.compute_optimal_instruments` in PyBLP.

[23] Instead of a grid search, one could follow the approach suggested by Chen, Christensen, and Tamer (2018) and use Monte Carlo draws from a quasi-posterior based on the robust test statistic.

be analytically obtained by using Proposition A.1, since (non-partial) CS_N is already an ellipsoid. On that grid we can check whether $CS_P(\gamma) \subseteq CS_N(\gamma)$ for all γ; if it is violated, then say that the first step indicates weak identification (up to the choice of the denseness of the grid), and then obtain CS_R while sequentially enlarging the grid for γ as needed to enclose CS_R. Otherwise, if $CS_P(\gamma) \subseteq CS_N(\gamma)$ on the grid (which was taken slightly large so that it includes values of γ at which $CS_N(\gamma)$ is empty), then say that the first step indicates strong identification, albeit with caution since there might still be a value of γ outside the grid for which $CS_P(\gamma) \neq \emptyset$.

Nested logit model As in the BLP estimator for the nested logit model, the "nesting parameter," which is a nonlinear parameter, can be considered as a linear parameter when it comes to computation. Using the notation by Berry (1994), consider the nested logit model (omitting subscript t)

$$u_{ij} = x_j'\beta + \xi_j + \zeta_{ig} + (1-\sigma)\epsilon_{ij}$$

where $\sigma \in (0, 1]$ is the nesting parameter and g denotes the group j belongs to. The model can be analytically inverted to yield

$$\log s_j - \log s_0 = x_j'\beta + \sigma \log s_{j|g} + \xi_j,$$

where $s_{j|g}$ is the within-group market share of j. Then the algorithm can be applied without a loop over γ after the following renaming: $\delta_{jt} \leftarrow \log s_g - \log s_0$, $\theta \leftarrow [\beta', \sigma]'$, and $x_j \leftarrow [x_j', \log \overline{s}_{j|g}]'$.

Application in transformation models The dimension reduction technique can be also useful when applying the robust two-step inference to transformation models (Horowitz, 1998):

$$T(y_i; \gamma) = x_i'\beta + u_i,$$

where $y_i, u_i \in \mathbb{R}$ and $x_i \in \mathbb{R}^{\dim \beta}$, and the function T is invertible and is parametrized by γ. In this case, the inverse of T with respect to y_i, i.e., $T^{-1}(\gamma; y_i)$, takes the role of $\delta_{jt}(\gamma; s_t, p_t, x_t)$.

4 Monte Carlo simulations

In this section, I conduct Monte Carlo simulations to investigate the performance of the two-step procedure equipped with the dimension reduction technique. I first set the data generating process, and then conduct Monte Carlo simulations under both homoscedasticity and heteroscedasticity. We use the same notation as in Section 2.1, while omitting the market subscript t.

4.1 The data generating process

We vary the number of markets as $T = 100, 200, 500$ in the simulation exercises. In all cases, there are $J = 6$ inside goods in each market. There are 6 firms in each market, each producing one product. The indirect utility is specified as a logit model with a random coefficient on the price, as

$$u_{ij} = 1 - (3 + 0.5\nu_{ij})p_j + 1.5x_{1j} + 1.5x_{2j} + \xi_j + \epsilon_{ij},$$

where x_{1j} and x_{2j} are drawn independently from the standard uniform distribution, ν_{ij} follows the standard normal distribution,[24] and ϵ_{ij} follows the type-I extreme value distribution. Consequently, the true parameters are $\beta = [1, -3, 1.5, 1.5]'$, $\alpha = -3$ (which is part of β), and $\gamma = 0.5$.

The marginal cost c_{ij} is determined by

$$c_j = 2x_{1j} + 2x_{2j} + \rho w_j + \omega_j,$$

where w_j is a cost shifter drawn from the standard uniform distribution. For homoscedastic errors, the unobserved product heterogeneity ξ_j in the previous display and the unobserved cost shifter ω_j are drawn from a bivariate normal distribution such that each has variance one, and the correlation coefficient between ξ_j and ω_j is 0.9.

For heteroscedasticity, I multiply ξ_j with a factor of $\sqrt{2(1 - w_j)}$. As w_j is uniformly distributed on $[0, 1]$ and the original ξ_j has variance one, the new ξ_j has conditional variance between 0 and 2, depending on the cost shifter.

The parameter ρ is set differently across simulations, in order to vary the degree of weak identification; decreasing ρ reduces the correlation between the price and the cost shifter w_j, thereby leading to weak identification. This setting is also used by Conlon and Gortmaker

[24] In a typical empirical setting, the random component of the price coefficient is the exponential of some random variable, to ensure that the price coefficient is negative for each individual. Instead, I take advantage of the fact that $3 + 0.5\nu_{ij}$ is rarely negative; $P(3 + 0.5\nu_{ij} < 0) \approx 9.87 \times 10^{-10}$.

(2020).

The endogenous market shares and prices, namely s_j and p_j, are generated by solving the Bertrand-Nash price-setting game between firms. I use the Python package PyBLP (Conlon and Gortmaker, 2020) to conduct simulation draws and to solve for the endogenous variables.

The excluded instruments I use are the cost shifter w_j and some simple BLP instruments, namely the sums of rival products' characteristics (within the market):

$$z_j = \left[1, x_{1j}, x_{2j}, w_j, \sum_{k \neq j} x_{1k}, \sum_{k \neq j} x_{2k}\right]'.$$

Since we have five parameters and six (included and excluded) instruments, I use PyBLP to compute the approximate optimal instruments and call the resulting vector of instruments z_j, thereby reducing the dimension of z_j to five.

4.2 Confidence sets under homoscedastic errors

Table 1: Coverage of confidence sets under homoscedasticity

	$T = 100$			$T = 200$			$T = 500$		
ρ	1	3	5	1	3	5	1	3	5
$\text{corr}(p_j, w_j)$	0.217	0.558	0.747	0.216	0.548	0.745	0.216	0.556	0.745
Weak IV	1.000	1.000	0.944	1.000	1.000	0.996	1.000	1.000	0.138
Coverage of CS_N	0.820	0.864	0.891	0.844	0.888	0.891	0.865	0.880	0.909
Coverage of CS_R	0.900	0.900	0.894	0.911	0.905	0.890	0.918	0.881	0.910
Coverage of CS_2	0.900	0.900	0.894	0.911	0.905	0.890	0.918	0.881	0.907
Length of $CS_{\alpha,N}$	1.990	1.105	0.669	1.441	0.892	0.606	0.924	0.523	0.355
Length of $CS_{\alpha,R}$	2.840	1.226	0.719	1.665	0.921	0.619	0.973	0.550	0.359
Length of $CS_{\gamma,N}$	0.679	0.291	0.139	0.512	0.251	0.139	0.322	0.149	0.079
Length of $CS_{\gamma,R}$	1.010	0.355	0.153	0.655	0.265	0.143	0.357	0.152	0.080

Notes: $\text{corr}(p_j, w_j)$ is the average correlation between the price and the cost shifter, across simulation draws. Weak IV denotes the sample probability that the first step indicates weak identification. Coverage of CS denotes the sample probability of $\theta \in CS$, i.e., the coverage probability. Length of CS_α and Length of CS_γ are the average lengths of the projection confidence intervals for α and γ, respectively. Each column is obtained using 1,000 simulation draws.

Table 1 tabulates simulated behaviors of confidence sets CS_N, CS_R, and CS_2, when the structural errors are homoscedastic. The nominal coverage rate is set to 0.90, and the coverage distortion bound ζ is set to 0.10. Overall, the robust confidence set CS_R has coverage probabilities around the nominal coverage probability, as expected, for all ρ

considered in the exercise. The robust confidence set behaves well even for a fairly small T. On the other hand, the non-robust confidence set CS_N shows slight under-coverage for low values of ρ, more so as T gets smaller.

The first step tends to indicate strong identification when $T = 500$ and $\rho = 5$, and weak identification otherwise. One thing to note is that the procedure indicates strong identification only when T is large, even at the same level of ρ (and hence with the same correlation between p_j and w_j). This seems to be due to CS_N being less stable with small T, which in turn affects whether the first-step set inclusion holds.

As for the two-step confidence set CS_2, it performs well in all simulation cases, attaining the coverage probability around 0.90. This simulation result is favorable, especially considering the theoretically guaranteed lower bound of 0.80. This happens because the first step correctly forces CS_2 to discard CS_N and use CS_R instead when CS_N performs poorly.

If correct coverage probability is the only concern, then one could have just used CS_R to begin with, as it already attains the correct coverage probability. However, one benefit of using the two-step procedure is that it gives an indicator of weak identification, analogously to the F-test in the linear instrumental variables models. Another potential benefit can be found in the lower part of the table. The lengths of projections of CS_N are smaller than those of CS_R in all cases. Due to this, one might find it desirable to use CS_N when the first step indicates that it is "safe" to use CS_N.

Nonetheless, the results suggest that there is little cost to using CS_R (i.e., a slightly larger confidence set under strong identification), while the benefit is large (i.e., the correct coverage probability as compared to that of CS_N under weak identification). Therefore one might as well consider using CS_R without the two-step procedure, in spirit of Keane and Neal (2023). Even in that case, the dimension reduction technique provided in Proposition 3.1 remains useful.

4.3 Confidence sets under heteroscedastic errors

Table 2 tabulates simulated behaviors of confidence sets CS_N, CS_R, CS_2, and CS_H. Here, CS_H is the Wald confidence set obtained by inverting the Wald test statistic, like CS_N, but using the heteroscedasticity-robust variance estimator; that is, CS_H is robust against heteroscedasticity but not robust against weak identification.[25]

Even though the true data is generated under heteroscedasticity, the weak-identification robust (but not robust against heteroscedasticity) confidence set CS_R does not suffer a

[25] A confidence set that is robust against both heteroscedasticity and weak identification would be an ideal benchmark. However, I do not simulate it due to its extensive computational cost. See Figure 1 for such confidence set plotted based on a single simulation draw.

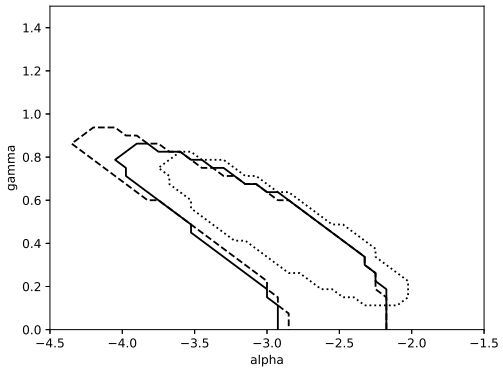
Table 2: Coverage of confidence sets under heteroscedasticity

	$T = 100$			$T = 200$			$T = 500$		
ρ	1	3	5	1	3	5	1	3	5
$\text{corr}(p_j, w_j)$	0.218	0.555	0.745	0.219	0.557	0.745	0.217	0.556	0.745
Weak IV	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.272
Coverage of CS_N	0.832	0.854	0.858	0.813	0.869	0.883	0.833	0.872	0.883
Coverage of CS_R	0.914	0.885	0.870	0.887	0.895	0.885	0.887	0.888	0.885
Coverage of CS_2	0.914	0.885	0.870	0.887	0.895	0.885	0.887	0.888	0.885
Coverage of CS_H	0.748	0.822	0.860	0.756	0.847	0.886	0.815	0.873	0.880
Length of $CS_{\alpha,N}$	2.148	1.285	0.887	1.488	0.878	0.600	0.920	0.533	0.348
Length of $CS_{\alpha,R}$	2.484	1.363	0.924	1.655	0.905	0.614	0.964	0.540	0.353
Length of $CS_{\alpha,2}$	2.484	1.363	0.924	1.655	0.905	0.614	0.964	0.540	0.348
Length of $CS_{\alpha,H}$	2.174	1.315	0.904	1.512	0.896	0.614	0.935	0.546	0.355
Length of $CS_{\gamma,N}$	0.740	0.370	0.208	0.531	0.249	0.138	0.324	0.147	0.077
Length of $CS_{\gamma,R}$	0.862	0.416	0.219	0.646	0.261	0.142	0.354	0.149	0.079
Length of $CS_{\gamma,2}$	0.862	0.416	0.219	0.646	0.261	0.142	0.354	0.149	0.077
Length of $CS_{\gamma,H}$	0.739	0.367	0.204	0.533	0.246	0.136	0.325	0.146	0.076

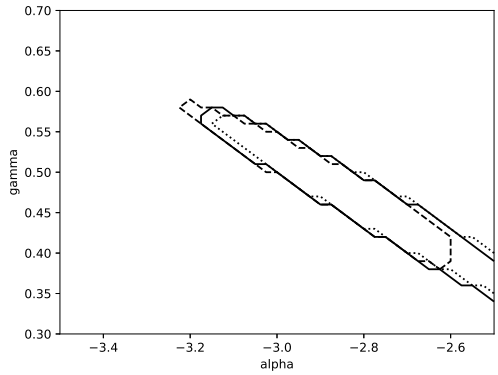
Notes: See Table 1 for a description of terms "corr", "coverage", and "length." "Weak IV" is the sample probability that the first step indicates weak identification, *under the assumption that the errors are homoscedastic*. Likewise, CS_N, CS_R, and CS_2 are obtained based on the assumption that the errors are homoscedastic. CS_H is the Wald confidence set that is robust against heteroscedasticity while not robust against weak identification.

severe distortion in coverage probability, although slight under-coverage is present. CS_N, which is not robust against both heteroscedasticity and weak identification, shows slight under-coverage, more so for smaller T and lower ρ. Heteroscedasticity-robust (but not weak-identification robust) confidence set CS_H exhibits severe under-coverage for smaller T and lower ρ; although CS_H is supposed to correct CS_N to be robust against heteroscedasticity, its performance appears worse than CS_N under weak identification. This might need further research to understand the cause of the particularly poor performance of heteroscedasticity-robust estimator under weak identification. Still, the simulation result hints that addressing weak identification might be more important than addressing heteroscedasticity when there is a concern of potential weak identification.

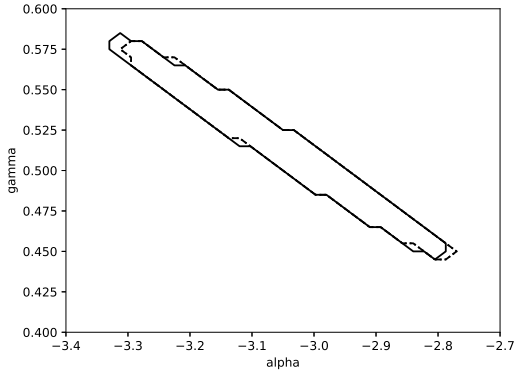
Figure 1 depicts, for each of three scenarios, a simulation draw of each of three confidence sets: CS_2 under the homoscedasticity assumption (solid line), CS_2 without the homoscedasticity assumption (dashed line), and CS_H (dotted line). Let us call the first two confidence sets CS_2^{hom} and CS_2^{het} respectively. Among the three confidence sets, CS_2^{het} is designed to be robust against both heteroscedasticity and weak identification, and therefore serves as a



(a) $T = 100, \rho = 1$



(b) $T = 200, \rho = 3$



(c) $T = 500, \rho = 5$

Figure 1: Confidence sets under heteroscedasticity

Notes: In each panel, a confidence set with a solid line is CS_2 obtained using the dimension reduction technique, under the assumption that errors are homoscedastic. A confidence set with a dashed line is CS_2 obtained by grid search over θ, without assuming that errors are homoscedastic. A confidence set with a dotted line is CS_H, i.e., the confidence set that is robust against heteroscedasticity while not robust against weak identification. All the confidence sets are projected on the α–γ plane. In panel (c), the dotted line and the dashed line overlap. The panels have different scales.

benchmark.

The confidence sets almost overlap under strong identification ($T = 500, \rho = 5$). On the other hand, under weak identification ($T = 100, \rho = 1$), the CS_2^{hom} and CS_H deviate from CS_2^{het}. Between CS_2^{hom} and CS_H, the latter seems particularly worse in capturing the overall shape of CS_2^{het}, since CS_H is bound to be an ellipsoid in every direction.[26] On the other hand, CS_2^{hom}, which takes advantage of dimension reduction, seems to approximate the shape and the location of CS_2^{het}, which requires a grid search over the entire dimension and hence can be computationally prohibitive when the dimensionality is large and the grid is fine. This visual inspection suggests two potential uses for the analytic representation technique: (i) as an approximation for the fully robust confidence set CS_2^{het}, or (ii) as guidance on the location and the breadth of the grid for computing CS_2^{het}.

5 Conclusion

In this paper, in light of concerns about instrumental variables being weak in estimating discrete choice demand models (BLP-style models), I show how one can adapt a recent econometric method proposed by Andrews (2018) that is robust to weak identification. As the procedure involves computationally intensive grid search especially when the number of parameters is large, I propose a computationally feasible method by reducing the dimensionality of the grid search under two assumptions, namely just-identification and homoscedasticity. The dimension reduction is done by deriving an analytic representation of the robust confidence sets in the space of linear parameters and by providing a fast method to check an inclusion relationship between two ellipsoids. As a result, the dimensionality of the required grid is reduced from the total number of parameters to the number of nonlinear parameters.

I conduct Monte Carlo simulations to check the performance of the two-step procedure equipped with my dimension reduction technique. The coverage probability of the two-step confidence set is around the pre-specified level under both strong and weak identification, whereas the non-robust confidence set shows under-coverage when instruments are weak. Another set of Monte Carlo simulations shows that, although the technique is developed under homoscedasticity assumption, the resulting confidence set still performs well when the true structural errors are set to be heteroscedastic.

[26] On the other hand, CS_2^{hom} is an ellipsoid only after fixing γ. In the Figure 1, this shape restriction presents itself in that the projection confidence set for α is an interval for each γ.

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Appendix

A Properties of quadric sets

Proposition A.1 (Dufour and Taamouti, 2005)**.** *Suppose* $\mathcal{C} = \{\beta : \beta' A\beta + 2b'\beta + c \leq 0\}$ *with nonsingular* A*.*

1. *(Boundedness check)* $\mathcal{C}$ *is bounded iff* A *is positive definite.*

2. *(Projection confidence interval) Let* $w \in \mathbb{R}^{\dim \beta} \setminus \{0\}$*, and* $\mathcal{C}_w$ *be the projection of* $\mathcal{C}$*:*

$$\mathcal{C}_w = \{w'\beta : \beta \in \mathcal{C}\}.$$

Let $d = b'A^{-1}b - c$ *and* $\widetilde{\beta} = -A^{-1}b$*.*

(a) Case 1: if A *is positive definite and*

- $d \geq 0$*, then*

$$\mathcal{C}_w = \left[w'\widetilde{\beta} - \sqrt{d(w'A^{-1}w)}, w'\widetilde{\beta} + \sqrt{d(w'A^{-1}w)}\right].$$

- $d < 0$*, then* $\mathcal{C}_w$ *is empty.*

(b) Case 2: if A *has exactly one negative eigenvalue and*

- $w'A^{-1}w < 0$ *and* $d < 0$*, then*

$$\mathcal{C}_w = \left(-\infty, w'\widetilde{\beta} - \sqrt{d(w'A^{-1}w)}\right] \cup \left[w'\widetilde{\beta} + \sqrt{d(w'A^{-1}w)}, \infty\right).$$

- $w'A^{-1}w = 0$ *and* $d < 0$*, then* $\mathcal{C}_w = \mathbb{R} \setminus \{w'\widetilde{\beta}\}$*.*
- *otherwise, then* $\mathcal{C}_w = \mathbb{R}$*.*

(c) Case 3: if A *has more than one negative eigenvalues, then* $\mathcal{C}_w = \mathbb{R}$*.*