MATH 102 (SP21, Eggers)

Homework 4

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1. *Inner product on polynomials*

Recall that the standard basis for $\mathbb{P}_3$ is given by

$$\text{Span}\{1, t, t^2, t^3\}$$

Hence it follows that any $f \in \mathbb{P}_3$ can be written as the linear combination

$$f = w + xt + yt^2 + zt^3$$

Likewise, any $g \in \text{Span}\{1, t\}$ may be written

$$g = a + bt$$

We now must find all such f which are orthogonal to the span of 1 and t, that is, for scalars a and b we must find w, x, y, z such that

$$\begin{aligned} 0 = < f, g > &= \int_{-1}^{1} f(t)g(t)\ dt \\ &= \int_{-1}^{1} (a + bt)(w + xt + yt^2 + zt^3)\ dt \end{aligned}$$

Using Mathematica, we obtain

$$\begin{aligned} 0 &= \frac{2}{15}\left(5a(3w + y)\right) + b(5x + 3z) \\ -a\left(2w + \frac{2}{3}y\right) &= b(5x + 3z) \end{aligned}$$

Since this must hold for arbitrary a and b, we obtain the system

$$\begin{aligned} 5x + 3z = 0 \\ 6w + 2y = 0 \end{aligned}$$

Thus we can rewrite $f \in \text{Span}(1, t)^{\perp}$ with two free variables as

$$\begin{aligned} f &= w + xt - 3wt^2 - \frac{5}{3}xt^3 \\ &= w(1 - 3t^2) + x(t - \frac{5}{3}t^3) \end{aligned}$$

from which follows the basis $\boxed{\{1 - 3t^2, t - \frac{5}{3}t^3\}}$ as required.

2. *Inner product space properties*

   (a) *Extensions of bilinearity*

   Proof. Note that by bilinearity:

$$\begin{aligned}
||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 &= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle + \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle \\
&= \langle \mathbf{u}, \mathbf{u}+\mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}+\mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{u}-\mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}-\mathbf{v}\rangle \\
&= \langle \mathbf{u}, \mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{u}\rangle - \langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle \\
&= (\langle \mathbf{u}, \mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{u}\rangle) + (\langle \mathbf{v}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle) + (\langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}\rangle) + (\langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}\rangle) \\
&= 2\langle \mathbf{u}, \mathbf{u}\rangle + 2\langle \mathbf{v}, \mathbf{v}\rangle \\
&= 2||\mathbf{u}||^2 + 2||\mathbf{v}||^2
\end{aligned}$$

   as required. □

   (b) *Implications of orthogonality on the inner product*

   Proof. Note that by bilinearity:

$$\begin{aligned}
||\mathbf{u}-\mathbf{v}||^2 &= \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle \\
&= \langle \mathbf{u}, \mathbf{u}-\mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}-\mathbf{v}\rangle \\
&= \langle \mathbf{u}, \mathbf{u}\rangle - \langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle \\
&= ||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle
\end{aligned}$$

   It follows that we have equality if and only if $2\langle \mathbf{u}, \mathbf{v}\rangle = 0$, from which we have that $\langle \mathbf{u}, \mathbf{v}\rangle = 0$ and as such $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, as required. □

3. *Projection*

   (a) *Projection matrix*

   To begin, denote the first column of A by $\mathbf{a}_1$ and the second column by $\mathbf{a}_2$. Observe that the columns are A are unit vectors

$$\sqrt{\frac{4}{2^2}} = 1$$

   and that they are orthogonal

$$-\frac{1}{2^2} - \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^2} = 0$$

   so it follows that A is an orthogonal matrix spanning some subspace of $\mathbb{R}^4$. Thus for any given vector $\mathbf{v} \in \mathbb{R}^4$, its projection onto $\mathbf{a}_1$ is therefore given by

$$\mathbf{p}_1 = \mathbf{a}_1\langle \mathbf{a}_1, \mathbf{v}\rangle$$

   and analogously for $\mathbf{a}_2$, so it follows that the projection of $\mathbf{v}$ into $R(A)$ is given by

$$A\begin{bmatrix}\langle \mathbf{a}_1, \mathbf{v}\rangle \\ \langle \mathbf{a}_2, \mathbf{v}\rangle\end{bmatrix} \quad = \quad AA^Tv$$

   and as such $P = AA^T$ and

$$\boxed{P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}}$$

   as required.

(b) *Orthonormal basis for the orthogonal complement*

To find the kernel of A^T, it suffices to row-reduce $[A^T \mid \mathbf{0}]$. Using Mathematica, we obtain

$$[A^T \mid \mathbf{0}] \quad = \quad \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Reading off entries it becomes immediately evident that $\mathbf{x} \in N(A^T)$ can be written as a linear combination, for scalars a and b

$$\mathbf{x} \in N(A^T) \quad = \quad a \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Normalising the two vectors (which both have length $\sqrt{1+1} = \sqrt{2}$), we obtain the orthonormal basis

$$\boxed{\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}}$$

as required.

(c) *Projection matrix into the complement*

Analogously to part (a), given an orthonormal basis, the projection matrix is simply given by the square matrix obtained by that basis multiplied by its transpose. Therefore we have the projection matrix

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}^T \quad = \quad \boxed{\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}}$$

as required.

4. *Projection matrices*

(a) *Into the range*

Proof. We're given that P is the projection matrix $\mathbb{R}^m \to R(A)$, ie. for any $\mathbf{v} \in \mathbb{R}^m$ its projection into $R(A)$ is given by

$$P\mathbf{v} \in R(A)$$

Recall that if $\mathbf{p}$ is the projection of $\mathbf{v}$ into a subspace, then $\mathbf{v} - \mathbf{p}$ is in the *orthogonal complement* of that subspace. Furthermore, $\mathbf{v} - \mathbf{p}$ is also *the closest vector* to $\mathbf{v}$ in the orthogonal complement since the residual is in the orthogonal complement *of the orthogonal complement*. It therefore follows here that $\mathbf{v} - \mathbf{p}$ is the projection of $\mathbf{v}$ into $R(A)^\perp$. By the fundamental subspaces theorem, this is equivalent to the projection into $N(A^T)$. Thus the projection from $\mathbb{R}^m$ into $N(A^T)$ is given by

$$\begin{gathered} \mathbf{v} - \mathbf{p} \\ \mathbf{v} - P\mathbf{v} \\ I\mathbf{v} - P\mathbf{v} \\ (I - P)\mathbf{v} \end{gathered}$$

and by definition $I - P$ is the projection matrix from $\mathbb{R}^m \to N(A^T)$.

□

(b) *Into the kernel*

Proof. Let $E = A^T$. By the fundamental subspaces theorem, it follows that $N(A^T) = R(A)^\perp$ and therefore $N(E) = R(E^T)^\perp$. Then the proof is analogous to that presented in part (a); we need only replace P by Q, m with n and vice versa, and A with E. That is to say, the proof follows from that for part (a) by the symmetric dual of the fundamental subspaces theorem.

□

5. *Preservation of the norm*

Proof. By definition,

$$\mathbf{v} = \mathbf{p} + \mathbf{z} \qquad \text{where } \mathbf{z} \in S^\perp$$

and so it follows by bilinearity that

$$\begin{aligned}\langle \mathbf{p}, \mathbf{v}\rangle &= \langle \mathbf{p}, \mathbf{p} + \mathbf{z}\rangle \\ &= \langle \mathbf{p}, \mathbf{p}\rangle + \langle \mathbf{p}, \mathbf{z}\rangle\end{aligned}$$

As $\mathbf{z} \in S^\perp$, the inner product of it with $\mathbf{p}$ must vanish, so we have

$$\begin{aligned}&= \langle \mathbf{p}, \mathbf{p}\rangle \\ &= ||\mathbf{p}||^2\end{aligned}$$

as required.

□

6. *Orthonormal basis of continuous functions*

We shall use Gram-Schmidt to find an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. To begin, fix $\mathbf{v}_1$ in the direction 1. Note that

$$||1||^2 = \int_{-1}^{1} 1 \cdot 1 \; dx = 1\Big|_{-1}^{1} = 2$$

and so we normalise to $\mathbf{v}_1 = \frac{1}{\sqrt{2}}$. We'll let $\mathbf{v}_2$ be in the direction of x without the 1 component, ie.

$$x - \frac{1}{\sqrt{2}}\int_{-1}^{1} x \; dx = x - \frac{1}{\sqrt{2}}\left(\frac{x^2}{2}\Big|_{-1}^{1}\right) = x$$

which has a length of

$$||x||^2 = \int_{-1}^{1} x^2 \; dx = \frac{2}{3}$$

so we normalise to $\mathbf{v}_2 = \frac{\sqrt{2}}{\sqrt{3}}x$. Finally, $\mathbf{v}_3$ may be found to be in the direction

$$x^2 - \left(\frac{\sqrt{2}}{\sqrt{3}}\int_{-1}^{1} x^3 \; dx\right)x - \left(\frac{1}{\sqrt{2}}\int_{-1}^{1} x^2 \; dx\right) = x^2 - \left(\frac{1}{3}t^3\Big|_{-1}^{1}\right) - 0 = x^2 - \frac{1}{3}$$

which has a squared length of

$$\int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)^2 \; dx = \frac{x^5}{5} - \frac{2x^3}{9} + \frac{x}{9}\Bigg|_{-1}^{1} = \frac{8}{45}$$

and thus $\mathbf{v}_3 = \frac{2\sqrt{2}}{3\sqrt{5}}\left(x^2 - \frac{1}{3}\right)$ and we have the orthonormal basis

$$\boxed{\left\{\frac{1}{\sqrt{2}},\ \frac{\sqrt{3}}{\sqrt{2}}x,\ \frac{2\sqrt{2}}{3\sqrt{5}}\left(x^2 - \frac{1}{3}\right)\right\}}$$

as required.

7. *QR factorisation*

We begin by finding an orthogonal matrix using A. The first column of Q is given by the first column of A, normed to a unit, and thus we have

$$\mathbf{q}_1 = \frac{1}{\sqrt{2}}[1,0,1,0]^T$$

Similarly, the second column of Q is given by the second column of A projected into the orthogonal complement of the span of $\mathbf{q}_1$:

$$\mathbf{q}_2 = \text{norm}\left([1,1,1,1]^T - \langle \mathbf{q}_1, \mathbf{a}_2\rangle \mathbf{q}_1\right) = \frac{1}{\sqrt{2}}[0,1,0,1]^T$$

Finally, the third column is given the projection of the third column of A into the span of $\mathbf{q}_1, \mathbf{q}_2$:

$$\mathbf{q}_3 = \text{norm}\left([0,0,1,1]^T - \langle \mathbf{q}_1, \mathbf{a}_3\rangle \mathbf{q}_1 - \langle \mathbf{q}_2, \mathbf{a}_3\rangle \mathbf{q}_2\right) = \frac{1}{2}[-1,-1,1,1]^T$$

Thus we have

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

Then R is given by $Q'A$. Using Mathematica, this yields

$$R = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 \end{bmatrix}$$

and we have

$$A = QR \qquad \Rightarrow \qquad \boxed{A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 \end{bmatrix}}$$

as required.

8. *Orthogonality of subspaces*

Proof. To begin, by definition, the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is pairwise orthogonal. That is, for any choice i, j where $i \neq j$ we must have $\langle \mathbf{x}_i, \mathbf{x}_j\rangle = 0$. Now, take any $\mathbf{s}_1 \in S_1$ such that

$$\mathbf{s}_1 = m_1\mathbf{x}_1 + m_2\mathbf{x}_2 + \ldots + m_k\mathbf{x}_k$$

Similarly, take any $\mathbf{s}_2 \in S_2$ where

$$\mathbf{s}_2 = m_{k+1}\mathbf{x}_{k+1} + m_{k+2}\mathbf{x}_{k+1} + \ldots + m_n\mathbf{x}_n$$

The inner product of $\mathbf{s}_1$ with $\mathbf{s}_2$ is given by

$$\langle \mathbf{s}_1, \mathbf{s}_2 \rangle = \langle m_1\mathbf{x}_1 + m_2\mathbf{x}_2 + \ldots + m_k\mathbf{x}_k, \quad m_{k+1}\mathbf{x}_{k+1} + m_{k+2}\mathbf{x}_{k+2} + \ldots + m_n\mathbf{x}_n \rangle$$

Leveraging bilinearity, we have

$$\begin{aligned} &= \sum_{i=1}^{k} m_i \langle \mathbf{x}_i, \quad m_{k+1}\mathbf{x}_{k+1} + m_{k+2}\mathbf{x}_{k+2} + \ldots + m_n\mathbf{x}_n \rangle \\ &= \sum_{i=1}^{k} m_i \sum_{j=k+1}^{n} m_j \ \langle \mathbf{x}_i, \mathbf{x}_j \rangle \end{aligned}$$

Because the i and j summations are over disjoint vectors in the orthonormal basis, every single one of the inner products vaniches, since vectors in the basis are pairwise orthgonal. Thus we have

$$\langle \mathbf{s}_1, \mathbf{s}_2 \rangle = 0$$

for *arbitrary* vectors $\mathbf{s}_1 \in S_1$ and $\mathbf{s}_2 \in S_2$. Thus it follows by definition that the two subspaces are orthogonal.

$\square$