MATH 102 (SP21, Eggers)

Homework 5

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13 May 2021

1. *Orthogonal matrix*

   (a) *Absolute eigenvalue value*

   Proof. Let λ be an eigenvalue of Q orthogonal and let $\mathbf{v}$ be the associated eigenvector. Then

   $$Q\mathbf{v} = \lambda\mathbf{v}$$

   Recall that by properties of orthogonality, for any vector $\mathbf{x}$ we have that $||\mathbf{x}|| = ||Q\mathbf{x}||$ as a consequence of that orthogonal matrices represent isometric transformations. Therefore we must have

   $$||\mathbf{v}|| = ||Q\mathbf{v}|| = ||\lambda\mathbf{v}||$$

   and by bilinearity of the inner product

   $$||\mathbf{v}|| = |\lambda| \cdot ||\mathbf{v}||$$

   which is only satisfied for $|\lambda| = 1$. Then all eigenvalues of orthogonal matrices are necessarily of absolute value 1.

   □

   (b) *Determinant of one*

   Proof. Let Q be orthogonal. Then $Q^{-1} = Q^T$. Additionally, recall that for all matrices, $\det M = \det M^T$. Furthermore, note that

   $$\begin{aligned} \det I &= 1 \\ \det(QQ^{-1}) &= 1 \\ \det(QQ^T) &= 1 \\ \det(Q) \cdot \det(Q) &= 1 \\ |\det(Q)| &= 1 \end{aligned}$$

   as required.

   □

2. *Eigenvectors of a matrix and its transpose*

   Proof. By definition, we have

   $$A\mathbf{x} = \lambda_1\mathbf{x}$$

Taking the transpose yields

$$\mathbf{x}^T A^T = \lambda_1 \mathbf{x}^T$$

Right-multiplying by $\mathbf{y}$ yields

$$\mathbf{x}^T A^T \mathbf{y} = \lambda_1 \mathbf{x}^T \mathbf{y}$$

Similarly, we also have

$$\begin{aligned} A^T \mathbf{y} &= \lambda_2 \mathbf{y} \\ \mathbf{x}^T A^T \mathbf{y} &= \lambda_2 \mathbf{x}^T \mathbf{y} \end{aligned}$$

and equating obtains

$$\begin{aligned} \lambda_1 \mathbf{x}^T \mathbf{y} &= \lambda_2 \mathbf{x}^T \mathbf{y} \\ \lambda_1 \mathbf{x}^T \mathbf{y} - \lambda_2 \mathbf{x}^T \mathbf{y} &= 0 \\ \mathbf{x}^T \mathbf{y}(\lambda_1 - \lambda_2) &= 0 \end{aligned}$$

Since $\lambda_1 \neq \lambda_2$ it follows that $\mathbf{x}^T \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle = 0$, so $\mathbf{x}$ and $\mathbf{y}$ are orthogonal. □

3. *Square matrices*

   (a) *Nonzero eigenvalue*

   Proof. Let λ be an eigenvalue of AB. We will assume, that since the products AB and BA are defined, that both A and B are $n \times n$ matrices. By definition, we must have, for some vector $\mathbf{v}$

   $$\begin{aligned} AB\mathbf{v} &= \lambda \mathbf{v} \\ BAB\mathbf{v} &= \lambda B\mathbf{v} \end{aligned}$$

   from which it follows that the vector $B\mathbf{v}$ is an eigenvector corresponding to the eigenvalue λ in the matrix BA, as required.

   □

   (b) *Zero eigenvalue*

   Proof. If $\lambda = 0$ is an eigenvalue of AB, then by the invertible matrix theorem AB is singular. Then

   $$\det(AB) = 0 = \det(A) \cdot \det(B) = \det(B) \cdot \det(A) = \det(BA)$$

   so BA is likewise singular and has a zero eigenvalue, as required.

   □

4. *Initial value problems*

   (a) $y_1' = -y_1 + 2y_2,\ y_2' = 2y_1 - y_2,\ y_1(0) = 3,\ y_2(0) = 1$

   A differential equation of the form

   $$y' = \alpha y$$

   is solved by

   $$y = ce^{\alpha t}$$

for a fixed scalar α and any scalar c. Thus it follows that to find all such α we need only find eigenvalues of the coefficient matrix, as the exponential is an eigenfunction under differentiation:

$$\begin{aligned}
\det\left(\begin{bmatrix}-1 & 2\\ 2 & -1\end{bmatrix} - \lambda I\right) &= 0\\
&= \det\begin{bmatrix}-1-\lambda & 2\\ 2 & -1-\lambda\end{bmatrix}\\
&= (-1-\lambda)^2 - 4\\
\lambda^2 + 2\lambda - 3 &= 0\\
\lambda &= -3, 1
\end{aligned}$$

The associated eigenvectors are given by

$$\begin{aligned}
\lambda = 1: \quad & \begin{bmatrix}-1-1 & 2\\ 2 & -1-1\end{bmatrix}\mathbf{v} = \mathbf{0}\\
& \mathbf{v} = c_1\begin{bmatrix}1\\1\end{bmatrix}\\
\lambda = -3: \quad & \begin{bmatrix}-1+3 & 2\\ 2 & -1+3\end{bmatrix}\mathbf{v} = \mathbf{0}\\
& \mathbf{v} = c_2\begin{bmatrix}1\\-1\end{bmatrix}
\end{aligned}$$

therefore

$$\mathbf{y} = c_1e^t\begin{bmatrix}1\\1\end{bmatrix} + c_2e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix} \quad \text{where} \quad y_1 = c_1e^t + c_2e^{-3t} \text{ and } y_2 = c_1e^t - c_2e^{-3t}a$$

Substituting the initial conditions and solving, we find that

$$\begin{aligned}
3 &= c_1 + c_2\\
1 &= c_1 - c_2\\
&\Rightarrow c_1 = 2 \qquad c_2 = 1
\end{aligned}$$

so

$$\boxed{\mathbf{y} = 2e^t\begin{bmatrix}1\\1\end{bmatrix} + e^{-3t}\begin{bmatrix}1\\-1\end{bmatrix} \qquad \equiv \qquad y_1 = 2e^t + e^{-3t} \text{ and } y_2 = 2e^t - e^{-3t}}$$

(b) $y_1' = y_2 - 2y_2,\ y_2' = 2y_1 + y_2,\ y_1(0) = 1,\ y_2(0) = -2$

As the above, we shall find all eigenvalues of the coefficient matrix:

$$\begin{aligned}
\det\left(\begin{bmatrix}1 & -2\\ 2 & 1\end{bmatrix} - \lambda I\right) &= 0\\
&= \det\begin{bmatrix}1-\lambda & -2\\ 2 & 1-\lambda\end{bmatrix}\\
&= (1-\lambda)^2 + 4\\
&= 1 - 2\lambda + \lambda^2 + 4\\
&= \lambda^2 - 2\lambda + 5
\end{aligned}$$

and applying the quadratic equation gives

$$\begin{aligned}\lambda &= \frac{2 \pm \sqrt{2^2 - 4(5)}}{2} \\ &= \frac{2 \pm 4i}{2} \\ \lambda &= 1 \pm 2i\end{aligned}$$

The associated eigenvectors are therefore complex conjugates of one another and it suffices to find just one (from which the second will follow). Thus we must find the null space of $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} - (1 + 2i)I$, ie.

$$\begin{aligned}\left[\begin{array}{cc|c} 1-(1+2i) & -2 & 0 \\ 2 & 1-(1+2i) & 0 \end{array}\right] &= \left[\begin{array}{cc|c} 1-1-2i & -2 & 0 \\ 2 & 1-1-2i & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \\ &\Rightarrow \mathbf{v} = c_1 \begin{bmatrix} 1 \\ -i \end{bmatrix}\end{aligned}$$

from which it follows that a solution is (note that c_1, c_2 change)

$$\begin{aligned}\mathbf{y} &= c_1 e^{t(1+2i)} \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 e^{t(1-2i)} \begin{bmatrix} 1 \\ i \end{bmatrix} \\ &= c_1 e^t (\cos 2t + i \sin 2t) \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 e^t (\cos 2t - i \sin 2t) \begin{bmatrix} 1 \\ i \end{bmatrix} \\ &= c_1 e^t \begin{bmatrix} \cos 2t + i \sin 2t \\ \sin 2t - i \cos 2t \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos 2t - i \sin 2t \\ \sin 2t + i \cos 2t \end{bmatrix} \\ &= c_1 e^t \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix} + i c_2 e^t \begin{bmatrix} \sin 2t \\ -\cos 2t \end{bmatrix} \\ &= c_1 e^t \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin 2t \\ -\cos 2t \end{bmatrix}\end{aligned}$$

To solve for the constants, we'll use the initial values to find

$$\begin{aligned}\begin{bmatrix} 1 \\ -2 \end{bmatrix} &= c_1 \begin{bmatrix} -\sin 0 \\ \cos 0 \end{bmatrix} + c_2 \begin{bmatrix} \cos 0 \\ \sin 0 \end{bmatrix} \\ &= c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &\Rightarrow c_1 = -2, \quad c_2 = 1\end{aligned}$$

and we have

$$\boxed{\mathbf{y} = -2e^t \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix} + e^t \begin{bmatrix} \sin 2t \\ -\cos 2t \end{bmatrix} \qquad \equiv \qquad y_1 = e^t\left(-2\cos 2t + \sin 2t\right) \text{ and } y_2 = e^t\left(-2\sin 2t - \cos 2t\right)}$$

5. *Initial value problem redux*

   (a) *Initial value as a linear combination*

Proof. Since $\mathbf{Y}(0) + \mathbf{Y}_0$ it follows that

$$\begin{aligned}\mathbf{Y}_0 &= c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + \ldots + c_n e^{\lambda_n t}\mathbf{x}_n \Big|_{t=0} \\ &= c_1 e^{\cancelto{0}{\lambda_1 t}}\mathbf{x}_1 + c_2 e^{\cancelto{0}{\lambda_2 t}}\mathbf{x}_2 + \ldots + c_n e^{\cancelto{0}{\lambda_n t}}\mathbf{x}_n \\ &= c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_n\mathbf{x}_n\end{aligned}$$

as required.

□

(b) *Solving the differential equation with the inverse*

Proof. Observe that since $\alpha e^{\beta 0} = 1$ for any choice of scalar constants α and β we must have that

$$\mathbf{Y}_0 = \mathbf{Y}(0) = X\mathbf{c}$$

because the exponentials in $\mathbf{Y}$ go to unity at zero. Then because X has n linearly independent columns, it is full rank, and therefore invertible. It follows that

$$\mathbf{Y}_0 = X\mathbf{c} \quad \iff \quad \mathbf{c} = X^{-1}\mathbf{Y}_0$$

as required.

□

6. *Differential equations and matrix methods*

(a) *nth order equation matrix methods*

Letting $y_1 = y$, $y_2 = y_1'$, $y_3 = y_2'$ and so on yields

$$\begin{aligned} y_1 &= y & y_1' &= y_2 \\ y_2 &= y' = y_1' & y_2' &= y_3 \\ y_3 &= y'' = y_2' & y_3' &= y_4 \\ &\vdots & & \\ y_{n-1} &= y^{(n-1)} = y_{n-2}' & y_{n-1}' &= a_1 y_1 + a_2 y_2 + \ldots + a_{n-1}y_{n-1}\end{aligned}$$

such that

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-2} \\ y_{n-1}\end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 & \ldots & a_{n-2} & a_{n-1}\end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-2} \\ y_{n-1}\end{bmatrix}$$

where the middle matrix is $n-1 \times n-1$ as required.

(b) *Characteristic polynomial*

Let A denote the coefficient matrix and A_k denote the lower-right $k \times k$ submatrix of the coefficient matrix. The characteristic polynomial is given by

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & -\lambda & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & -\lambda & 1 & \ldots & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -\lambda & 1 \\ a_1 & a_2 & a_3 & a_4 & \ldots & a_{n-2} & a_{n-1} - \lambda\end{bmatrix}\right)$$

Note that

$$\begin{aligned}
\det(A_1 - \lambda I_1) &= a_{n-1} - \lambda \\
\det(A_2 - \lambda I_2) &= \lambda^2 - \lambda a_{n-1} - a_{n-2} \\
\det(A_3 - \lambda I_3) &= -\lambda^3 + \lambda^2 a_{n-1} + \lambda a_{n-2} + a_{n-3} \\
\det(A_4 - \lambda I_4) &= \lambda^4 - \lambda^3 a_{n-1} - \lambda^2 a_{n-2} - \lambda a_{n-3} - \lambda_{n-4}
\end{aligned}$$

This pattern continues inductively and so we obtain the characteristic polynomial

$$\boxed{p(\lambda) = (-1)^{n-1}\lambda^{n-1} + (-1)^{n-2}\sum_{k=1}^{n-1} a_k \lambda^{n-1-k}}$$

as required.