MATH 170A (WI21, Dumitriu)

Homework 2

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1. Let L_1, L_2, and L_3 be the matrices below:

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & 0 & 1 & 0 \\ l_{41} & 0 & 0 & 1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \\ 0 & l_{42} & 0 & 1 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & l_{43} & 1 \end{bmatrix}$$

Calculate $L_1 L_2 L_3$ and show it is equal to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}$$

We begin by calculating $L_1 L_2$:

$$L_1 L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & 0 & 1 & 0 \\ l_{41} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \\ 0 & l_{42} & 0 & 1 \end{bmatrix}$$

By right-multiplying against identity vectors in the first, third, and fourth columns, the first, third, and fourth columns of the result are the same as those in L_1. Then, the second column of the result is given by $L_1 \begin{bmatrix} 0 & 1 & l_{32} & l_{42} \end{bmatrix}^T$.

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 \cdot l_{21} & 1 & 0 & 0 \\ 1 \cdot l_{31} & 1 \cdot l_{32} & 1 & 0 \\ 1 \cdot l_{41} & 1 \cdot l_{42} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & 0 & 1 \end{bmatrix}$$

And then we find $(L_1 L_2) L_3$:

$$(L_1 L_2) L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & l_{43} & 1 \end{bmatrix}$$

A similar phenomenon occurs. We are right-multiplying (L_1L_2) against identity vectors in the first, second, and fourth columns, so those columns are the same as in (L_1L_2) as in the final result. The third column, therefore, is given simply by $(L_1L_2)\begin{bmatrix}0 & 0 & 1 & l_{43}\end{bmatrix}^T$.

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & 1 \cdot l_{43} & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}}$$

Hence we have shown that the answer is as expected.

2. Let A be an $n \times n$ matrix. We apply the elementary transformation of type 1 to A: "add m times row j to row i," where m is a non-zero constant and $j < i$.

   (a) Show that this transformation is encoded as a matrix multiplication LA where L is the identity matrix except for a single entry m below the diagonal.

   Let A' denote the matrix A after the elementary row operation. Let a_k denote the kth row of A. Suppose that the matrix L is given as the identity matrix, except for a single entry m in row i, column j. We remark that by assumption, $j < i$, and as such the entry in row i, column j is necessarily below the diagonal.

   Using a'_{xy} to denote the xyth entry in A', recall that a'_{xy} can be written as

$$a'_{xy} = \sum_{k=1}^{n} l_{xk} a_{ky}$$

   For all $x \neq i$, we have that $l_{xk} = 0$ where $k \neq x$, and $l_{xk} = 1$ where $k = x$. Hence, we have

$$\text{For } x \neq i: \ a'_{xy} = a_{xy}$$

   which implies that *all rows other than row i are unchanged.* Then, suppose $x = i$. In this case, $l_{ik} = 1$ for $k = i$, $l_{ik} = m$ for $k = j$, and $l_{ik} = 0$ otherwise. Therefore, we have

$$\text{For } x = i: \ a'_{iy} = a_{iy} + m a_{jy}$$

   Therefore, the yth entry of row i is equal to $a_{iy} + m a_{jy}$, ie. it is equal to itself plus m times the corresponding entry in row j. Therefore, m times row j has been added to row i.

   This is as required, and hence we have shown that a matrix of in the form of L correctly encodes this elementary row operation. ■

   (b) Write down a closed form for the matrix L^{-1}. Multiplying with L^{-1} from the left also corresponds to an elementary transformation of type 1, what is this transformation?

   The inverse of L^{-1} can be found rather simply using Gaussian elimination on L right-augmented with the identity:

$$\begin{bmatrix} L \mid I_n \end{bmatrix} \sim \begin{bmatrix} I_n \mid L^{-1} \end{bmatrix}$$

   Since L is exactly the identity except for an m at row i, columns j where $j < i$, all we need to do to make the augmented matrix into reduced row-echelon form is to subtract m times the row with a 1 in column j (which turns out to be row j) from row i. Hence, the matrix L^{-1} must simply be $\boxed{\text{the matrix } L \text{ with a } -m \text{ at row } i\text{, column } j.}$ We can Using the results from part (a), it is plainly true that this matrix encodes the elementary transformation of $\boxed{\text{subtracting } m \text{ times row } j \text{ from row } i,}$ which is consistent with the notion of the inverse.

3. Prove the following property of lower/upper triangular matrices:

   (a) Let L_1, L_2 be two lower triangular $n \times n$ matrices. Show that L_1L_2 is also lower triangular.

   To show that L_1L_2 is lower triangular, we will show that all entries above the diagonal must vanish. Denote $L_1L_2 = R$, and recall that the entry r_{ij} is given by

$$r_{ij} = \sum_{k=1}^{n} (L_1)_{ik}(L_2)_{kj}$$

   where $(L_1)_{ik}$ and $(L_2)_{kj}$ are the entries of L_1 at ik, and of L_2 and kj, respectively. Since L_1 is lower triangular, entries at a row x and column y where $x < y$ vanish. Hence we can adjust the bounds of summation

$$r_{ij} = \sum_{k=1}^{i} (L_1)_{ik}(L_2)_{kj}$$

   Suppose that $i < j$. Then, the sum ranges from 1 to some value less than j. However, this means that in each term of the sum, we will be multiplying by $(L_2)_{kj}$ where $k < j$. Since L_2 is lower triangular, all such values vanish. Hence, the summation vanishes.

   We have therefore shown that when multiplying two lower triangular matrices, all entries in the result where the row index is less than the column index will vanish. It follows immediately that the product must be lower triangular. ■

   (b) Conclude that the product of two upper triangular $n \times n$ matrices is also upper triangular.

   A proof follows in exactly the same format as above; we need only flip the direction of every inequality:

   > To show that L_1L_2 is **upper** triangular, we will show that all entries **below** the diagonal must vanish. Denote $L_1L_2 = R$, and recall that the entry r_{ij} is given by
   >
   > $$r_{ij} = \sum_{k=1}^{n} (L_1)_{ik}(L_2)_{kj}$$
   >
   > where $(L_1)_{ik}$ and $(L_2)_{kj}$ are the entries of L_1 at ik, and of L_2 and kj, respectively. Since L_1 is **upper** triangular, entries at a row x and column y where $x > y$ vanish. Hence we can adjust the bounds of summation
   >
   > $$r_{ij} = \sum_{k=i}^{n} (L_1)_{ik}(L_2)_{kj}$$
   >
   > Suppose that $i > j$. Then, the sum ranges from i to n. However, this means that in each term of the sum, we will be multiplying by $(L_2)_{kj}$ where $k > j$. Since L_2 is **upper** triangular, all such values vanish. Hence, the summation vanishes.
   >
   > We have therefore shown that when multiplying two **upper** triangular matrices, all entries in the result where the row index is **greater** than the column index will vanish. It follows immediately that the product must be **upper** triangular. ■

4. Prove that the LU factorisation is unique by considering the possibility that $LU = \tilde{L}\tilde{U}$ for some L, $\tilde{L}$ lower triangular matrices and U, $\tilde{U}$ upper triangular matrices, such that $L \neq \tilde{L}$ and $U \neq \tilde{U}$.

Suppose we have two LU factorisations for a matrix A, that is

$$LU = A = \tilde{L}\tilde{U}$$

Assume towards a contradiction that $L \neq \tilde{L}$ and $U \neq \tilde{U}$. We will show, via induction, that this assumption is incorrect. As the base case, we will show that the first row of U and and of $\tilde{U}$ are equal and uniquely determined, as are the first column of L and of $\tilde{L}$. By definition, the diagonal entries of L and $\tilde{L}$ are uniformly 1. With this, we can express the $1j$ entry of A as

$$a_{1j} = \cancelto{1}{l_{11}} u_{1j} + \cancelto{0}{l_{12}u_{2j} + \ldots} = u_{1j}$$

and similarly

$$a_{1j} = \cancelto{1}{\tilde{l}_{11}} \tilde{u}_{1j} + \cancelto{0}{\tilde{l}_{12}\tilde{u}_{2j} + \ldots} = \tilde{u}_{1j}$$

Hence, the values of u_{1j} and $\tilde{u}_{1j}$ are equal and uniquely determined by a_{1j}. We now consider the $i1$ entry of A:

$$a_{i1} = l_{i1}u_{11} + \cancelto{0}{l_{i2}u_{21} + \ldots} = l_{i1}u_{11}$$

and similarly

$$a_{i1} = \tilde{l}_{i1}\tilde{u}_{11} + \cancelto{0}{\tilde{l}_{i2}\tilde{u}_{21} + \ldots} = \tilde{l}_{i1}\tilde{u}_{11}$$

Since $u_{11} = \tilde{u}_{11}$ was shown to be uniquely determined, we have that l_{i1} is equal to $\tilde{l}_{i1}$, and both are uniquely determined by $\frac{a_{i1}}{u_{11}}$. Thus, in the base case, we have shown that the first rows of U and $\tilde{U}$ are equal and uniquely determined, as are the first columns of L and $\tilde{L}$.

We will now prove that, for $1 < k \leq n$, row k of U is uniquely determined and equal to that of $\tilde{U}$, as is column k of L and $\tilde{L}$. Assume, as the inductive hypothesis, that rows $1 \ldots k-1$ of U and $\tilde{U}$ are equal and uniquely determined, and the same for columns $1 \ldots k-1$ of L and $\tilde{L}$. Then, we concern ourselves with entry a_{kj}:

$$a_{kj} = \sum_{x=1}^{n} l_{kx}u_{xj}$$

Since L is lower triangular, entries l_{kx} where $x > k$ must be zero. Then

$$a_{kj} = \sum_{x=1}^{k} l_{kx}u_{xj} = l_{k1}u_{1j} + l_{k2}u_{2j} + \ldots + \cancelto{1}{l_{kk}} u_{kj}$$

and because rows 1 through $k-1$ of U (and corresponding columns of L) are unique and known, we can solve for u_{kj} as

$$u_{kj} = a_{kj} - \sum_{x=1}^{k-1} l_{kx}u_{xj}$$

In addition, since rows 1 through $k-1$ of U are equal to those of $\tilde{U}$ (and corresponding columns of L and $\tilde{L}$), the right-hand side does not change if values from U are swapped for those from $\tilde{U}$ (likewise for L and $\tilde{L}$). Hence

$$u_{kj} = \tilde{u}_{kj}$$

for all possible j. Then row k of U and $\tilde{U}$ are equal and uniquely determined. We now demonstrate the same for column k of L and $\tilde{L}$:

$$a_{jk} = \sum_{x=1}^{n} l_{jx}u_{xk}$$

Again, since L is lower triangular, all entries l_{xk} for $x > k$ vanish. Then we can state

$$a_{jk} = \sum_{x=1}^{k} l_{jx}u_{xk} = l_{j1}u_{1k} + l_{j2}u_{2k} + \ldots + l_{jk}u_{kk}$$

We already know the identity of columns 1 through $k-1$ of L, and we know that of rows 1 through k of U. Thus, we can solve for l_{jk}:

$$l_{jk} = \frac{\sum_{x=1}^{k-1} l_{jx}u_{xk}}{u_{kk}}$$

Because it was shown that rows 1 through k of U are equal to those of $\tilde{U}$ (and that columns 1 through $k-1$ of L are equal to those of $\tilde{L}$), we can simply replace all instances of U and L on the right-hand side with the tilde equivalents, and we obtain

$$l_{jk} = \tilde{l}_{jk}$$

which is true for all possible j. Hence we have shown that column k of L is equal to that of $\tilde{L}$, and that they are uniquely determined. ■

5. Write a MATLAB code that does Gaussian elimination with partial pivoting. The input should be

- an $n \times n$ matrix A
- a column vector b of size $n \times 1$

and the output should be the solution x to $Ax = b$.

The code is:

```
notfunction x = pge_solve(A, b)
if size(A, 2) ~= size(b)
    error("Size mismatch, aborting");
end
if size(A, 1) ~= size(A, 2)
    error("A is singular");
end

n = size(A, 1);

% Main loop
for i = 1:n
    % Swap in the row with the absolute largest coefficient
    row = i;
    for r = i:n
        if abs(A(r, i)) > abs(A(row, i))
            row = r;
        end
    end
    if row ~= i
        % Swap b
        tmp = b(i);
        b(i) = b(row);
        b(row) = tmp;

        % Swap main matrix
        for j = i:n
```

```
                tmp = A(i, j);
                A(i, j) = A(row, j);
                A(row, j) = tmp;
            end
        end

        for j = (i+1):n
            l = A(j, i) / A(i, i);
            for k = (i+1):n
                A(j, k) = A(j, k) - l * A(i, k);
            end
            b(j) = b(j) - l * b(i);
        end

    end

    x = uppertriangsolve(A, b);
```

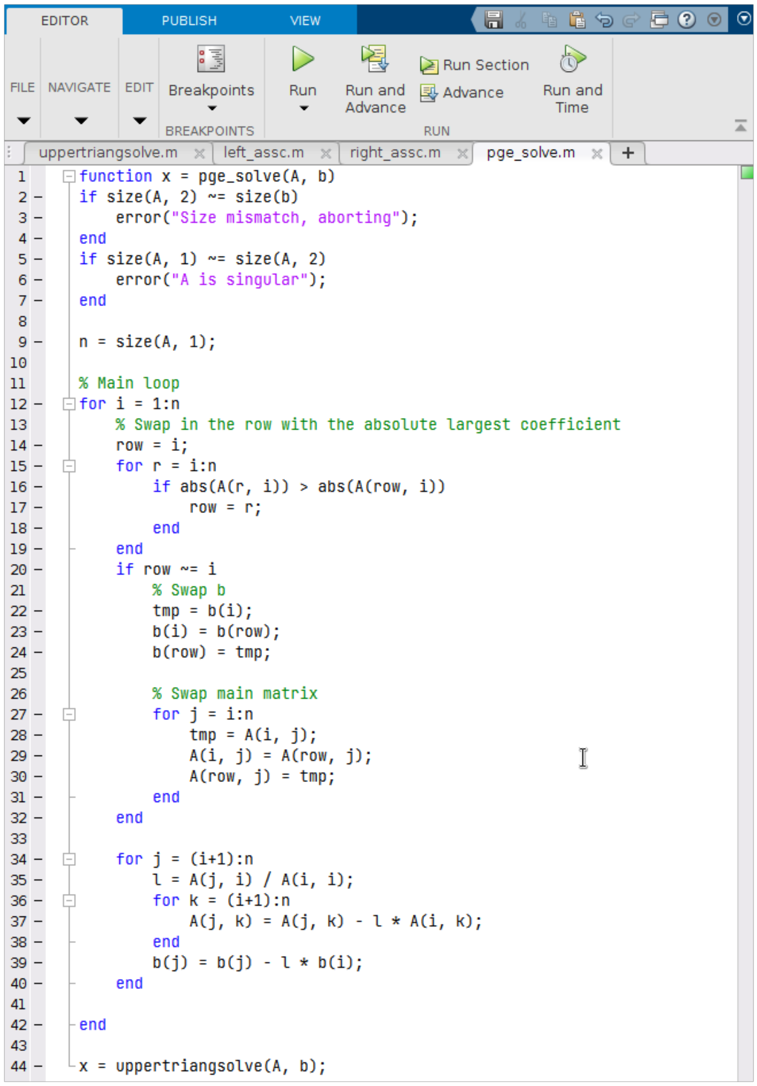

Figure 1: Script solving systems by Gaussian elimination with partial pivoting