MATH 100A (FA21, Alberts)

Homework 6

Jiahong Long

[redacted]

11 November 2021

Discussed approaches to problem (1) with students at Libman's office hours. Pick's theorem is restated from Artin, p. 411 ex. M.4. Consulted Watkins' *Fundamentals of Matrix Computations*, the 170 series textbook for the formulation of Householder reflectors. Solutions are my own.

1. *If G is the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by (a, b) and (c, d), prove that the index of G in $\mathbb{Z} \times \mathbb{Z}$ is given by $|ad - bc|$ if this number is nonzero, and infinite otherwise. (If you get stuck, try first proving the easier statement that $G = \mathbb{Z} \times \mathbb{Z}$ if and only if $ad - bc = \pm 1$.)*

   Proof. Let $|ad - bc|$ be zero. In particular,

   $$\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = 0$$

   so the matrix whose columns are the basis of G is not full rank; in particular we must have that

   $$(a, b) = k(c, d)$$

   for some integer[1] k. So linear combinations of the two form points along a line in the direction of the vector (a, b), and G is the lattice composed of points along that line. Suppose (a, b) is nontrivial (ie. not the identity), then such a line clearly has infinite order: cosets of it by any $(x, 0) \in \mathbb{Z}^2$ will shift the x-intercept of the line by x; in particular there are infinitely many choices for the x-intercept and so there are infinitely many cosets thereof. A similar argument holds for the trivial case, ie. $(a, b) = (c, d) = (0, 0)$: the coset by any $(x, y) \in \mathbb{Z}^2$ is the singleton set containing exactly (x, y). There are infinitely many lattice points in $\mathbb{Z}^2$ so there are infinitely many cosets in this case too.

   Let $|ad - bc|$ be nonzero. We restate Pick's theorem in full:

   Theorem. *Let L be the integer lattice $\mathbb{Z}^2$. Let P be a polygon in the plane with vertices in L. Then the area of P is equal to $a + \frac{b}{2} + 1$ where a is the number of lattice points in the interior of P and B is the number of lattice points on the boundary of P.*

   With the notation in the theorem, let P denote the parallelogram swept out by the basis vectors (a, b) and (c, d) such that we have

   $$\Delta P = \left| \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right| = |ad - bc|$$

   Let R denote the *open set* corresponding to this parallelogram, ie. the *interior* of P; in additon let ∂R denote its boundary in the usual manner. In particular, write

   $$\partial R = L_{ab} \cup L_{cd} \cup L'_{ab} \cup L'_{cd}$$

[1]since all vectors we consider are integer-valued

where L_{ab} is the segment from the origin to (a, b) and likewise for L_{cd}; L'_{ab} is the segment parallel to L_{ab} translated upwards by the vector (c, d) (that is, L'_{ab} is one of the top edges of the parallelogram) and likewise for L'_{cd}:

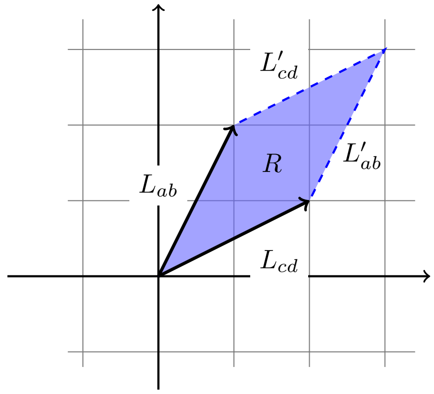

Figure 1: Illustration of P, its interior, and its boundary segments.

A coset of G in $\mathbb{Z}^2$ is equivalent to translating a copy of the lattice formed by the basis vectors of G. In particular, we can fully characterise a coset (up to redundancy) by where the origin is sent under the translation. We claim that every unique coset is the result of sending the origin to some point contained within P:

Proof. Suppose the origin is sent to a point *outside* of P. Call this point $p = (x, y)$. $p \notin G$ because cosets by elements of the subgroup are the subgroup itself. So it follows that we can write

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}_{p_0} + \underbrace{\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}}_{p'}$$

where

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \left\lfloor \begin{bmatrix} a & c \\ b & d \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \right\rfloor \qquad \text{and} \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

where the matrix is invertible since it has nonzero determinant. Note that p_0 is the element-wise floor of the the matrix product; in particular because p' is the difference between $\begin{bmatrix} x_0 & y_0 \end{bmatrix}^T$ and $\begin{bmatrix} x & y \end{bmatrix}^T$, p' is some fractional linear combination of the two basis vectors

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = w \begin{bmatrix} a \\ b \end{bmatrix} + z \begin{bmatrix} c \\ d \end{bmatrix}$$

for $0 \leq w, z < 1$ which follows from the definition of the floor function.

We now show that $pG = p'G$. In the first direction, let $p + x \in pG$ for $x \in G$. Then $p + x = p_0 + p' + x \Rightarrow p' + (x + p_0) = p + x$. Note $(x + p_0) \in G$ because p_0 is a linear combination of the basis vectors of G. It follows that $p + x \in p'G$. The opposite containment follows in a similar fashion.

We've defined p' to be a fractional linear combination of the basis vectors $p' = w(a, b) + z(c, d)$ for $0 \leq w, z < 1$. It follows that the vector $w \begin{bmatrix} a & b \end{bmatrix}^T$ coincides with L_{ab} and is of strictly less norm; likewise $z \begin{bmatrix} c & d \end{bmatrix}^T$ with L_{cd}. It follows immediately that their sum is in the interior of P. $\square$

The number of lattice points in P can be partitioned into two classes: those on ∂R and those in R (the open region). We now consider cosets induced by translation of the origin to lattice points on ∂R.

With the notation from the theorem, suppose there are b lattice points on ∂R. First, note that there are four lattice points on ∂R which correspond to the vertices of the parallelogram. The remainder lie strictly on the open line segments comprising the boundary. Note that we have redundancy by a factor of two: suppose we translate the origin to some point (e, f) along L_{cd}. The same coset is induced by translating the origin to the point $(e, f) + (a, b)$ because they differ by a basis vector. The same holds for translations to points along L_{ab} which correspond to translations to points along L'_{ab} for the same reason. So one-half of the lattice points on the boundary corresponds to exactly two vertices, and one-half of the points along the open segments. However, translation to any vertex gives rise to the same lattice because of closure of G, so we subtract one to remove a vertex point. It follows that there are $\frac{b}{2} - 1$ distinct cosets induced by translating the origin to a point along the boundary.

We now consider cosets induced by the translation to points in the interior. There are a many points; furthermore there is no redundancy within these points because all of them are separated by a distance less than reachable by integral linear combinations of the basis vectors. So there are a distinct such cosets.

It follows that there are, in total, $a + \frac{b}{2} - 1$ many cosets. So the index of G in $\mathbb{Z}^2$ is equal to this quantity. Furthermore, the area of the parallelogram spanned by the two basis vectors is exactly given by the magnitude of the determinant, here $|ad - bc|$. By Pick's theorem we have

$$|ad - bc| = [G : \mathbb{Z}^2]$$

as required. □

2. *6, 3.4*

Prove that a conjugate of a glide reflection in M is a glide reflection, and that the glide vectors have the same length.

Proof. It follows from Theorem 6.3.4 that:

- composing two nonparallel reflections yields a rotation about the intersection of the lines, and
- composing two parallel reflections yields a translation by a vector orthogonal to the lines of reflection.

So an *arbitrary* isometry can be generated by choosing a number of reflections and composing them. Hence it suffices to show that conjugation of a glide reflection by a reflection yields a glide reflection.

Take $g \in M$ to be a glide reflection and assume, without loss of generality, that the line of reflection is the horizontal axis. Thus g can be written as the matrix-vector operation on a point a

$$a \mapsto \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} a + \begin{bmatrix} t \\ 0 \end{bmatrix} \right)$$

where t is the length of the translation vector. Now, let $r_\ell \in M$ be an arbitrary reflection; in particular assume without loss that it is a reflection about a line through the origin. Let $u = [u_x \ u_y]^T$ be the unit vector orthogonal to ℓ in the plane. The orthogonal Householder reflector matrix[2] corresponding to reflection about ℓ is given by

$$\begin{aligned} Q = I - 2uu^T &= I - 2 \begin{bmatrix} u_x \\ u_y \end{bmatrix} \begin{bmatrix} u_x & u_y \end{bmatrix} \\ &= I - 2 \begin{bmatrix} u_x^2 & u_x u_y \\ u_x u_y & u_y^2 \end{bmatrix} \\ Q &= \begin{bmatrix} 1 - u_x^2 & -2u_x u_y \\ -2u_x u_y & 1 - 2u_y^2 \end{bmatrix} \end{aligned}$$

[2] Watkins p. 198

We can also rewrite the original reflection about e_1 as a Householder reflector:

$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = I - 2e_2e_2^T$$

Conjugation of g by r_ℓ can now be written as the matrix expression acting on the point a:

$$\begin{aligned}r_\ell g r_\ell^{-1}(a) &= r_\ell g(Qa)\\&= r_\ell\left(\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}Qa + \begin{bmatrix}t\\0\end{bmatrix}\right)\\&= Q^{-1}\left(\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}Qa + \begin{bmatrix}t\\0\end{bmatrix}\right)\end{aligned}$$

where Q is invertible because it is orthogonal and hence full rank. In particular, Q represents a reflection, so it is involutary such that $Q^{-1} = Q$, so

$$\begin{aligned}r_\ell g r_\ell^{-1}(a) &= Q\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}Qa + Q\begin{bmatrix}t\\0\end{bmatrix}\\&= Q(I - 2e_2e_2^T)Qa + Q\begin{bmatrix}t\\0\end{bmatrix}\end{aligned}$$

Observe that $Q(I - 2e_2e_2^T)Q$ is a composition of three reflections. In particular, $Q(I - 2e_2e_2^T)$ is a composition of reflections about intersecting (possibly coincident) lines and so is therefore a rotation r_γ, with γ possibly zero. Then $r_\gamma Q$ is a composition of a rotation with a reflection, from which it follows that it must be a *new* reflection. Call the matrix corresponding to this reflection R, that

$$r_\ell g r_\ell^{-1}(a) = Ra + Q\begin{bmatrix}t\\0\end{bmatrix}$$

So it follows that the conjugation of a glide reflection with a reflection is also a glide reflection. In addition, because Q is an orthogonal matrix, its product with a vector preserves norm, so $Q\begin{bmatrix}t & 0\end{bmatrix}^T$ has equal norm as $\begin{bmatrix}t & 0\end{bmatrix}^T$, ie. t. So the translation vectors are the same length. To show that the *glide vectors* are the same length, it suffices to show that the reflection R is about the line in the direction of $Q\begin{bmatrix}t & 0\end{bmatrix}^T$, ie. the reflection of $\begin{bmatrix}t & 0\end{bmatrix}^T$ about ℓ. We know that R is a reflection, and the eigenvectors of reflection matrices lie along the line of reflection and have eigenvalue 1; that is, if R reflects along ℓ then $R(Q\begin{bmatrix}t & 0\end{bmatrix}^T)$ should equal exactly $Q\begin{bmatrix}t & 0\end{bmatrix}^T$. By inspection:

$$\begin{aligned}R(Q\begin{bmatrix}t & 0\end{bmatrix}^T) &= Q(I - 2e_2e_2^T)\cancelto{I}{QQ}\begin{bmatrix}t & 0\end{bmatrix}^T\\&= Q(I - 2e_2e_2^T)\begin{bmatrix}t & 0\end{bmatrix}^T\end{aligned}$$

Note that $\begin{bmatrix}t & 0\end{bmatrix}^T$ lies along the e_1 axis, which is the axis of reflection represented by the reflector $I - 2e_2e_2^T$. Therefore

$$= Q\begin{bmatrix}t & 0\end{bmatrix}^T$$

with equality as required. It follows therefore that R reflects along ℓ, and we have shown that the glide vectors are of equal norm.

Since any isometry can be written as a product of reflections, it follows that conjugation of a glide reflection by any isometry yields a glide reflection with an equal-length translation. $\square$

3. *6, 3.6*

   (a) *Let s be the rotation of the plane with angle $\pi/2$ about the point $(1,1)^t$. Write the formula for s as a product $t_a\rho_\theta r$.*

Proof. Lemma 6.3.5 informs us that a rotation through θ about a point p can be written as

$$t_a p_\theta \qquad \text{where} \begin{bmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{bmatrix} p = a$$

We are given p; it follows that

$$\begin{bmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = a$$
$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = a \quad \Rightarrow \quad a = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

To verify, rotation of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ through $\frac{\pi}{2}$ is given by

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and then translation through a yields

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

as required. In particular, we expect that a point (a, b) should be mapped to the point resulting from shifting it by $(-1, -1)$, rotating through $\frac{\pi}{2}$, and shifting back by $(1, 1)$. Verification yields

$$\begin{aligned} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a-1 \\ b-1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-b \\ a-1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 2-b \\ a \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= t_{(2,0)} \rho_{\pi/2}(a, b) \end{aligned}$$

as required. So rotation by $\frac{\pi}{2}$ about $(1, 1)$ is given by $t_{(2,0)}\rho_{\pi/2}$. ☐

(b) *Let s denote reflection of the plane about the vertical axis $x = 1$. Find an isometry g such that $grg^{-1} = s$, and write s in the form $t_a\rho_\theta r$.*

Proof. Let $g = t_{(1,0)}$, the translation along the horizontal axis of one unit. To verify that such a g works, pick a point (a, b). We expect it to be mapped to $(-(a-1)+1, b) = (2-a, b)$: note that

$$\begin{aligned} grg^{-1}(a, b) &= gr(a-1, b) = g(1-a, b) \\ &= (2-a, b) \end{aligned}$$

as required. Using the formal manipulation rules (listing 6.3.3), we have that

$$\begin{aligned} s &= grg^{-1} \\ &= t_{(1,0)} r t_{(-1,0)} \\ &= t_{(1,0)} t_{(1,0)} r \\ &= t_{(2,0)} r \end{aligned}$$

so $s = t_{(2,0)}r$. ☐

4. *6, 4.3*

(a) *Compute the left cosets of the subgroup $H = \{1, x^5\}$ in the dihedral group D_{10}.*

Proof. Note that the coset of H by x is $\{x, x^6\}$, which is equivalent to the coset by x^6, ie. $\{x^6, x^{11} = x\}$. By inspection this pattern occurs for the pairs $(x, x^6), (x^2, x^7)$, etc. up to $(x^5, x^{10} = 1)$. In fact, the same is true for the pairs $(yx, yx^6), (yx^2, yx^7)$, etc. up to $(yx^5, yx^{10} = y)$ where

$$yxH = \{yx, yx^6\} = \{yx^6, yx\} = yx^6H$$

There are twenty elements of D_{10}, corresponding to 10 such pairs, and we have the left cosets

Coset by	**Elements**
$1, x^5$	$\{1 = x^{10}, x^5\} = H$
x, x^6	$\{x, x^6\}$
x^2, x^7	$\{x^2, x^7\}$
x^3, x^8	$\{x^3, x^8\}$
x^4, x^9	$\{x^4, x^9\}$
y, yx^5	$\{y = yx^{10}, yx^5\}$
yx, yx^6	$\{yx, yx^6\}$
yx^2, yx^7	$\{yx^2, yx^7\}$
yx^3, yx^8	$\{yx^3, yx^8\}$
yx^4, yx^9	$\{yx^4, yx^9\}$

where each left coset groups elements of D_{10} by residue class modulo 5 of the rotation exponent with and without reflections. □

(b) *Prove that H is normal and that D_{10}/H is isomorphic to D_5.*

Proof. Let $g \in D_{10}$. Either $g = x^a$ or $g = yx^a$ for $0 \le a < 10$. In the former case, $gH = Hg$ as $H \subset \langle x \rangle$ and powers of x commute. In the latter case, note that

$$\begin{aligned} Hg &= \{yx^a, x^5yx^a\} = \{yx^a, yx^{-5}x^a\} \\ &= \{yx^a, yx^5x^a\} = \{yx^a, yx^{a+5}\} = gH \end{aligned}$$

with normality as required. To show isomorphism, define $\varphi : D_{10} \longrightarrow D_5$ where

$$\varphi(y^a x^b) = y^a x^{b \bmod 5}$$

To show this is a homomorphism, pick $y^a x^b, y^c x^d \in D_{10}$ and note that

$$\varphi(y^a x^b)\varphi(y^c x^d) = y^a x^{b \bmod 5} y^c x^{d \bmod 5}$$

Assume $y^c = 1$, ie. $c \bmod 2 \equiv 0$ implying no reflection, and it follows that

$$\begin{aligned} &= y^a x^{b \bmod 5} x^{d \bmod 5} = y^a x^{(b+d \bmod 5)} \\ &= \varphi(y^a x^{b+d}) = \varphi(y^a x^b \cdot y^c x^d) \end{aligned}$$

Where $y^c = y$, ie. $c \bmod 2 \equiv 1$ implying a reflection, note that

$$\begin{aligned} \varphi(y^a x^b)\varphi(y^c x^d) = y^a(x^{b \bmod 5} y^c)x^{d \bmod 5} &= y^a y x^{-b \bmod 5} x^{d \bmod 5} = y^{a+c} x^{d-b \bmod 5} \\ &= \varphi(y^{a+c} x^{d-b}) = \varphi(y^a y x^{-b} x^d) = \varphi(y^a x^b \cdot y^c x^d) \end{aligned}$$

as required. Also note that H is the kernel of φ:

$$\varphi(1) = \varphi(x^{10}) = x^{10 \bmod 5} = x^0 = 1 \qquad \text{and} \qquad \varphi(x^5) = x^{5 \bmod 5} = x^0 = 1$$

Finally, φ is also surjective: the elements $x^0, \ldots, x^4$ correspond to the respective rotations in D_5, while the elements $yx^0, \ldots, yx^4$ correspond to the reflections of D_5. It follows from the first isomorphism theorem that $D_{10}/H \cong D_5$ as required. □

(c) *Is D_{10} isomorphic to $D_5 \times H$?*

Proof. We propose that the two are isomorphic. Define $\varphi : D_5 \times H \longrightarrow D_{10}$ as the map

$$\varphi(y^a x^b, x^c) = y^a x^{b+c}$$

To show homomorphism, let $(y^a x^b, x^{5c}), (y^d x^e, x^{5f}) \in D_5 \times H$ and note that

$$\begin{aligned}
\varphi(y^a x^b, x^{5c})\varphi(y^d x^e, x^{5f}) &= y^a x^{b+5c} y^d x^{e+5f} = y^a y^d x^{e+5f-b-5c} \\
&= y^a y^d x^{e-b} x^{5f-5c} = y^a y^d x^{e-b} x^{5f} x^{-5c} = y^a y^d x^{e-b} x^{5f} x^{5c} \\
&= \varphi(y^a y^d x^{e-b}, x^{5c} x^{5f}) = \varphi(y^a x^b y^d x^e, x^{5c} x^{5f}) \\
&= \varphi((y^a x^b, x^{5c}) \cdot (y^d x^e, x^{5f}))
\end{aligned}$$

as required. To show bijectivity, it suffices to find an inverse. Define $\varphi^{-1}(y^a x^b)$ piecewise to be

$$\varphi^{-1}(y^a x^b) = \begin{cases} (y^a x^b, 1) & b < 5 \\ (y^a x^{b-5}, x^5) & \text{otherwise} \end{cases}$$

and note that

$$\varphi^{-1}\varphi(y^a x^b, 1) = \varphi^{-1}(y^a x^{b+0}) = \varphi^{-1}(y^a x^b) = y^a x^b$$

since $0 \leq b < 5$ by the nature of D_5. The only remaining elements in the Cartesian product are of the form $(y^a x^b, x^5)$ and note that

$$\varphi^{-1}(\varphi(y^a x^b, x^5)) = \varphi^{-1}(y^a x^{b+5}) = (y^a x^{(b+5)-5}, x^5) = (y^a x^b, x^5)$$

since $0 \leq b < 5 \Rightarrow 5 > b + 5$. So the inverse exists and is well-defined, so φ is a bijection and therefore an isomorphism. □

5. *6, 5.11*

If S and S' are subsets of $\mathbb{R}^n$ with $S \subset S'$, then S is dense *in S' if for every element $s' \in S'$, there are elements of S arbitrarily near to s'.*

(a) *Prove that a subgroup Γ of $\mathbb{R}^+$ is either dense in $\mathbb{R}$, or else discrete.*

Proof. It suffices to show that if Γ is not dense, then it is discrete. Let Γ be not dense in $\mathbb{R}^+$. Pick $x \in \Gamma$ to be an element such that

$$\exists \varepsilon > 0 \in \mathbb{R} \quad \forall x' \in \Gamma \setminus x \quad (|x - x'| \not< \varepsilon)$$

Pick ε that the above is true. Because Γ is a subgroup, $|x - x'| \in \Gamma$ for any choice of x'. Assume towards a contradiction that there are two elements $a, b \in \mathbb{R}^+$ that $|a - b| < \varepsilon$. Then it follows that $x + |a - b| \in \Gamma$, and in particular,

$$|x - (x + |a - b|)| = |a - b| < \varepsilon$$

which contradicts nondenseness about x. So it follows that there can be no elements $a, b \in \Gamma$ whose difference is less than ε, making Γ discrete by definition. □

(b) *Prove that the subgroup of $\mathbb{R}^+$ generated by 1 and $\sqrt{2}$ is dense in $\mathbb{R}^+$.*

Proof. Assume towards a contradiction that $\langle 1, \sqrt{2} \rangle$ is discrete. Note that $\mathbb{R}^+$ is a subgroup of $\mathbb{R}^{2+}$, so $\langle 1, \sqrt{2} \rangle$ is also a subgroup of $\mathbb{R}^{2+}$: it follows by Theorem 6.5.5 that $\langle 1, \sqrt{2} \rangle = \mathbb{Z}a$ for $a \in \mathbb{R}$. This is because $\langle 1, \sqrt{2} \rangle$ is clearly nontrivial, and also cannot be generated by two linearly independent vectors because it lies on the real line. Then it follows that $1 = an$ and $\sqrt{2} = am$ for $n, m \in \mathbb{Z}$. In particular,

$$\sqrt{2} = \frac{\sqrt{2}}{1} = \frac{am}{an} = \frac{m}{n} \in \mathbb{Q}$$

which contradicts that $\sqrt{2}$ is irrational. So the generated subgroup is not discrete, and by part (a) must be dense. □

(c) *Let H be a subgroup of the group G of angles. Prove that H is either a cyclic subgroup of G or else it is dense in G.*

Proof. It suffices to show that if H is acyclic then it is dense. Let H be acyclic. Assume towards a contradiction that it is discrete. So there is a θ such that ρ_θ is the minimal rotation, ie. there are no rotations through γ where $0 < \gamma < \theta$. Furthermore, acyclicality implies that there is no $n \in \mathbb{Z}$ that $\rho_\theta^n = \rho_0 = \rho_{2\pi}$. Then it follows that we can take

$$\frac{2\pi}{\theta} = \underbrace{\left\lceil \frac{2\pi}{\theta} \right\rceil}_{z \in \mathbb{Z}} - \underbrace{z_0}_{0 < z_0 < 1}$$

(note the strictly nonzero remainder). In particular, observe that

$$2\pi = z\theta - z_0\theta$$

where $0 < z_0\theta < \theta$. By closure, $\rho_z \in H$; also $\rho_z \rho_{-z_0} = \rho_{2\pi} = \rho_0$, so ρ_{z_0} is the inverse of ρ_z and by closure, $\rho_{z_0} \in H$ as well. Yet $z_0 < \theta$, violating minimality of θ. It follows that H cannot be discrete; in particular it must be dense. $\square$