Homework 2

ECE 45 (Franceschetti)

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1 Homework 2

1. Are the following systems linear? Are they time-invariant?

(a) $x(t) \to [\text{System } (a)] \to 2x(t-3)$

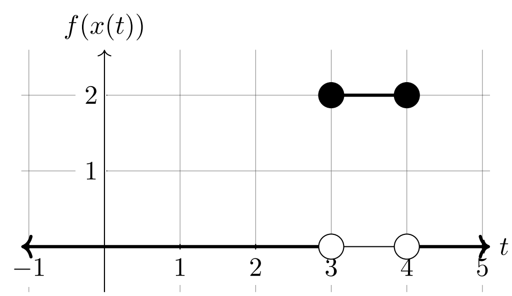
The system is linear and time-invariant. Denote the system by $f(x(t), t)$. Then note that superposition applies:

$$\begin{aligned} f(x_1(t) + x_2(t), t) &= f(x_1(t), t) + f(x_2(t), t) \\ 2\left(x_1(t-3) + x_2(t-3)\right) &= 2x_1(t-3) + 2x_2(t-3) \end{aligned}$$

Hence the system is linear. Also note that time-delaying the input creates the same effect as time-delaying the output:

$$\begin{aligned} f(x(t+\delta), t) &= f(x(t), t)(t+\delta) \\ 2x(t-3+\delta) &= 2x(t-3+\delta) \end{aligned}$$

Hence the system is time-invariant. Plotted:



(b) $x(t) \to [\text{System } (b)] \to x(t) + t$

The system is neither linear nor time-invariant. Define the system as

$$f(x(t), t) = x(t) + t$$

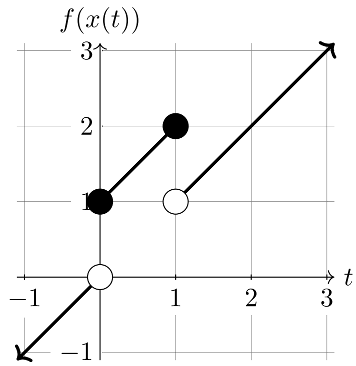
Observe that $f(x_1(t), t) = x_1(t) + t$ and that $f(x_2(t), t) = x_2(t) + t$. Superposition therefore does not apply, as

$$\begin{aligned} f(x_1(t) + x_2(t), t) &\neq f(x_1(t), t) + f(x_2(t), t) \\ x_1(t) + x_2(t) + t &\neq x_1(t) + x_2(t) + 2t \end{aligned}$$

Hence the system is not linear. Note now that the system is also not time-invariant: if we time-delay the input $x(t)$ by some δ to obtain a new input $x(t+\delta)$, the output thereof is not equal to the output of $x(t)$ time-delayed by δ:

$$\begin{aligned} f(x(t+\delta),t) &\neq f(x(t),t)(t+\delta) \\ x(t+\delta)+t &\neq x(t+\delta)+t+\delta \end{aligned}$$

Hence the system is time-variant. Plotted:



(c) $x(t) \to$ [System (c)] $\to (x(t)+1)^2$

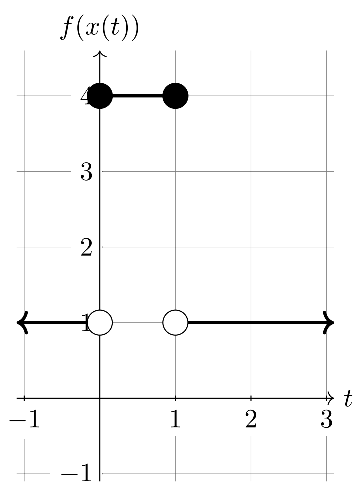
The system is not linear, but is time-invariant. Denote the system as a function of a signal, and of time $f(x(t),t)$. Then, observe that superposition does not apply:

$$\begin{aligned} f(x_1(t)+x_2(t),t) &\neq f(x_1(t),t)+f(x_2(t),t) \\ (x_1(t)+x_2(t)+1)^2 &\neq (x_1(t)+1)^2+(x_2(t)+1)^2 \end{aligned}$$

which is not true for all x_1, x_2. Thus the system is nonlinear. However, time-delaying the input has the same effect as time-delaying the output:

$$\begin{aligned} f(x(t+\delta),t) &= f(x(t),t)(t+\delta) \\ (x(t+\delta)+1)^2 &= (x(t+\delta)+1)^2 \end{aligned}$$

Thus the system is time-invariant. Plotted:



(d) $x(t) \to [\text{System } (d)] \to \cos(x(t))$

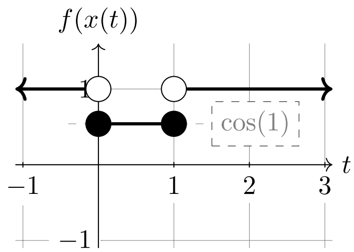
The system is not linear, but is time-invariant. Denote the system as a function of a signal, and of time $f(x(t), t)$. Then, observe that superposition does not apply:

$$f(x_1(t) + x_2(t), t) \neq f(x_1(t), t) + f(x_2(t), t)$$
$$\cos(x_1(t) + x_2(t)) \neq \cos(x_1(t)) + \cos(x_2(t))$$

Thus the system is nonlinear. However, time-delaying the input has the same effect as time-delaying the output:

$$f(x(t + \delta), t) = f(x(t), t)(t + \delta)$$
$$\cos(x(t + \delta)) = \cos(x(t + \delta))$$

Thus the system is time-invariant. Plotted:



(e) $x(t) \to [\text{System } (e)] \to \int_{-\infty}^{t} x(\tau) d\tau$

The system is both linear and time-invariant. Let $X(t)$ denote the antiderivative of $x(t)$. Then, the output can be rewritten as

$$\int_{-\infty}^{t} x(\tau) d\tau = X(t) - \lim_{\tau \to -\infty} X(\tau)$$

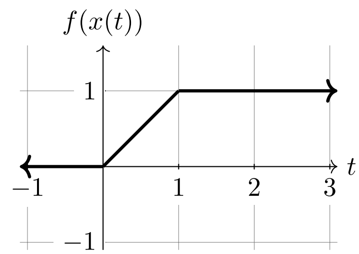
Denote the system as a function of an input signal and of time, $f(x(t), t)$. Observe that superposition applies:

$$f(x_1(t) + x_2(t), t) = f(x_1(t), t) + f(x_2(t), t)$$
$$\int_{-\infty}^{t} x_1(\tau) + x_2(\tau) d\tau = \int_{-\infty}^{t} x_1(\tau) d\tau + \int_{-\infty}^{t} x_2(\tau) d\tau$$

which is true by linearity of the integral. Then, the system is linear. Observe now that applying a time delay to the signal is equivalent to applying a time delay to the output:

$$f(x(t + \delta), t) = f(x(t), t)(t + \delta)$$
$$\int_{-\infty}^{t+\delta} x(\tau) d\tau = X(t + \delta) - \lim_{\tau \to -\infty} X(\tau)$$
$$X(t + \delta) - \lim_{\tau \to -\infty} X(\tau) = X(t + \delta) - \lim_{\tau \to -\infty} X(\tau)$$

Therefore the system is time-invariant. Plotted:



(f) $x(t) \to$ [System (f)] $\to t$

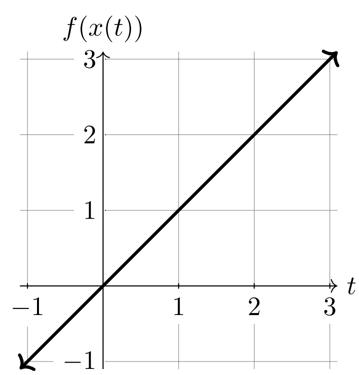
The system is neither linear nor time-invariant. Denote the system as a function of a signal and time $f(x(t), t)$. Superposition does not apply:

$$f(x_1(t) + x_2(t), t) \neq f(x_1(t), t) + f(x_2(t), t)$$
$$t \neq 2t$$

Then the system is nonlinear. Additionally, time-delaying the signal is not the same as time-delaying the output:

$$f(x(t+\delta), t) \neq f(x(t), t)(t+\delta)$$
$$t \neq t + \delta$$

Then the system is time-variant. Plotted:



2. Find the Bode plot (magnitude and phase) and label all critical points of the transfer function

$$H(\omega) = \frac{225(\omega^2 - 2500j\omega - 10^6)(1 + 10^{-6}j\omega)}{9\left(\frac{\omega^2}{9} - \frac{20000j\omega}{3} - 10^8\right)(25 + 5j\omega)}$$

Rewriting in standard form:

$$\begin{aligned}
H(\omega) &= \frac{225(\omega^2 - 2500j\omega - 10^6)(1 + 10^{-6}j\omega)}{9\left(\frac{\omega^2}{9} - \frac{20000j\omega}{3} - 10^8\right)(25 + 5j\omega)} \\
&= \frac{(\omega^2 - 2500j\omega - 10^6)(1 + \frac{j\omega}{10^6})}{\frac{1}{9}(-30000j)^2\left(1 + \frac{j\omega}{30000}\right)^2(1 + \frac{j\omega}{5})} \\
&= \frac{9\left(\omega^2 - 2500j\omega - 10^6\right)\left(1 + \frac{j\omega}{10^6}\right)}{-3\cdot 10^4 j\left(1 + \frac{j\omega}{3\cdot 10^4}\right)^2\left(1 + \frac{j\omega}{5}\right)} \\
&= \frac{9\left(\omega - j10^3\right)^2\left(1 + \frac{j\omega}{10^6}\right)}{-3\cdot 10^4 j\left(1 + \frac{j\omega}{3\cdot 10^4}\right)^2\left(1 + \frac{j\omega}{5}\right)} \\
&= \frac{-9\cdot 10^3 j\left(1 + \frac{j\omega}{10^3}\right)^2\left(1 + \frac{j\omega}{10^6}\right)}{-30\cdot 10^3 j\left(1 + \frac{j\omega}{3\cdot 10^4}\right)^2\left(1 + \frac{j\omega}{5}\right)} \\
&= \frac{3\left(1 + \frac{j\omega}{10^3}\right)^2\left(1 + \frac{j\omega}{10^6}\right)}{10\left(1 + \frac{j\omega}{3\cdot 10^4}\right)^2\left(1 + \frac{j\omega}{5}\right)}
\end{aligned}$$

$$\omega^2 - 2500j\omega - 10^6 \approx \left(\omega - j10^3\right)^2$$

Then there are breakpoints at

- $\omega = 5$, multiplicity 1 in the denominator
- $\omega = 10^3$, multiplicity 2 in the numerator
- $\omega = 3\cdot 10^4$, multiplicity 2 in the denominator
- $\omega = 10^6$, multiplicity 1 in the numerator

with an initial value of $\lim_{\omega\to 0}|H(\omega)| = \frac{1}{100}$. Plotted:

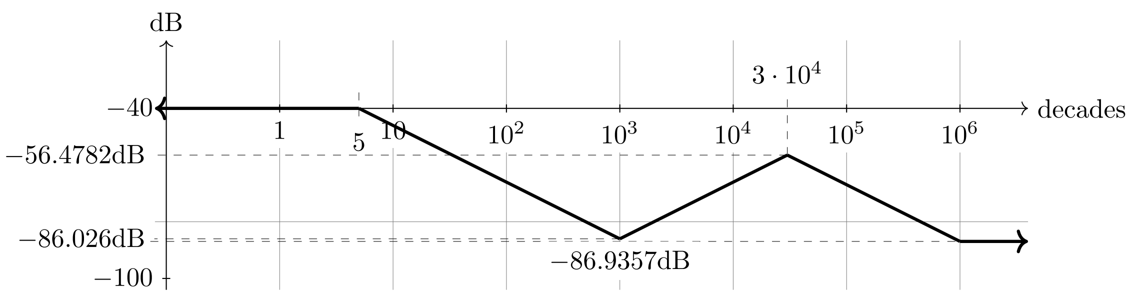


Figure 1: Bode Plot (Magnitude) for $H(\omega)$

And for frequency:

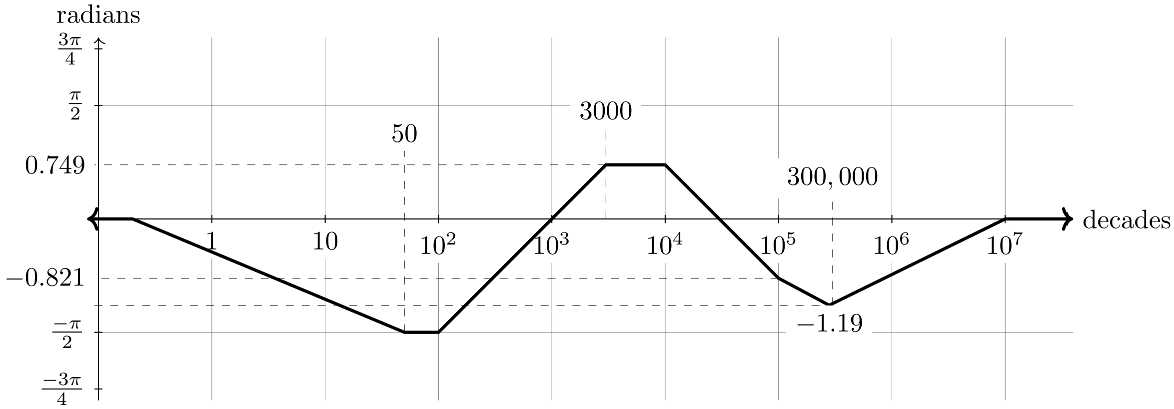


Figure 2: Bode Plot (Phase) for $H(\omega)$

The actual Bode plots are:

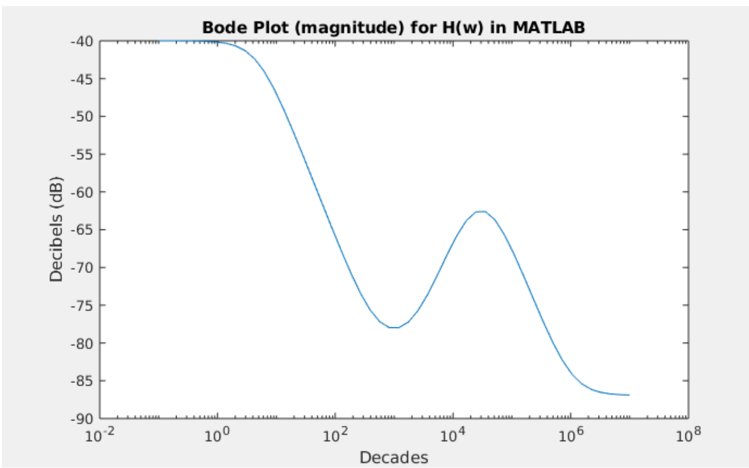


Figure 3: MATLAB Bode Plot (Magnitude) for $H(\omega)$

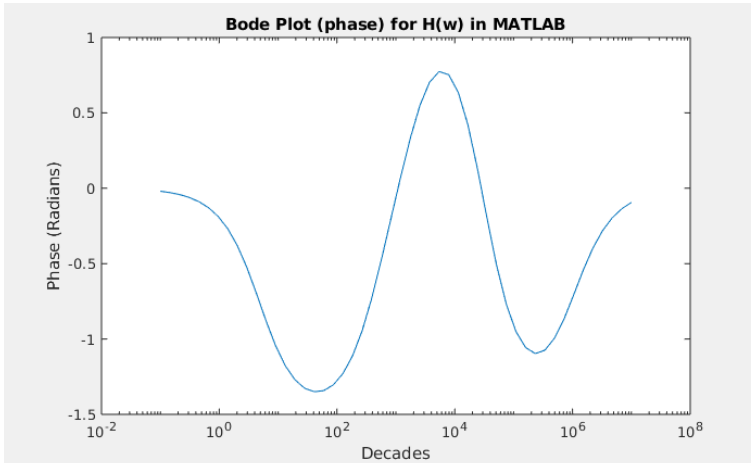


Figure 4: MATLAB Bode Plot (Phase) for $H(\omega)$

The actual Bode plots appear to have the same maxima and minima as the approximations, but are smooth, continuously differentiable functions.

3. Are the following functions periodic? If so, find the period and fundamental frequency.

(a) $f_1(t) = \cos^2(10t)$

Yes. Using the half-angle formula, rewrite the function as

$$f_1(t) = \cos^2(10t) = \frac{1+\cos(20t)}{2} = \frac{1}{2} + \frac{1}{2}\cos(20t)$$

Then, we have $\omega = 20$ and $T = \frac{2\pi}{\omega} = \frac{\pi}{10}$.

(b) $f_2(t) = \sum_{n=-\infty}^{\infty} x(t-n)$ where $x(t) = t$ for $0 \leq t < 1$ and $x(t) = 0$ otherwise.

No. Clearly the function is not periodic, as it expands to

$$\sum_{n=-\infty}^{\infty} x(t-n) = \sum_{n=-\infty}^{\infty} t-n = \lim_{n\to-\infty}(t-n)\ldots+(t+1)+(t-0)+(t-1)+(t-2)\ldots+\lim_{n\to\infty}(t-n) = \infty$$

which a diverging sum.

(c) $f_3(t) = \tan(t)$

Yes. The tangent function is periodic with $w = \omega$ and $T = \pi$.

(d) $f_4(t) = \cos(t) + \cos(\pi t)$

No. Consider $f_4(t)$ as two terms in a Fourier series. Then it follows that for some integer k, we have $\Re\{c_k e^{jk\omega_0 t}\} = \cos(t)$ and $\Re\{c_k e^{j(k+l)\omega_0 t}\} = \cos(\pi t)$. Then, we have

$$\begin{aligned}
k\omega_0 &= 1 \\
(k+l)\omega_0 &= \pi \\
k\omega_0 + l\omega_0 &= \pi && \text{substituting} \\
1 + l\omega_0 &= \pi \\
l\omega_0 &= \pi - 1 \\
\omega_0 &= \frac{\pi - 1}{l}
\end{aligned}$$

implying

$$\begin{aligned}
k\omega_0 &= 1 \\
k\frac{\pi - 1}{l} &= 1 \\
k &= \frac{l}{\pi - 1}
\end{aligned}$$

But both k and l must be integers, and since there is an irrational number in the expressions with equality to k k is *not actually* an integer. Therefore it is impossible for the above function to be periodic.

4. Determine $H(\omega)$ and use the approximations from the Bode plot to determine the output of the system for the following inputs.

We are given the breakpoints

$$\{1, 10, 10^4, 10^5, 10^6\}$$

At low frequences (prior to the first breakpoint) there is a +20dB/decade slope. Then there is a $j\omega$ term in the numerator. Where $\omega = 0$, the gain-intercept is -60dB, suggesting there is a DC gain k where

$$-60 = 20 \log k$$

Then there is a term $k = 10^{-3}$ in the numerator. After the first breakpoint, there is a +40dB/decade slope, implying there is a term $(1 + \frac{j\omega}{1})$ term in the numerator

$$H(\omega) = \frac{j\omega(1 + j\omega)}{10^3}?$$

After the second breakpoint, the slope returns to +20dB/decade. Then there is a $(1 + \frac{j\omega}{10})$ term in the denominator

$$H(\omega) = \frac{j\omega(1 + j\omega)}{10^3(1 + \frac{j\omega}{10})}?$$

After the third breakpoint, the slope zeroes, so we add a term to the denominator

$$H(\omega) = \frac{j\omega(1 + j\omega)}{10^3(1 + \frac{j\omega}{10})(1 + \frac{j\omega}{10^4})}?$$

After the fourth breakpoint, the slope increases to +40dB/decade, so we add two terms to the numerator

$$H(\omega) = \frac{j\omega(1 + j\omega)(1 + \frac{j\omega}{10^5})^2}{10^3(1 + \frac{j\omega}{10})(1 + \frac{j\omega}{10^4})}?$$

After the fifth breakpoint, the slope zeroes, and we add two terms to the denominator to arrive at the final equation for the transfer function

$$H(\omega) = \frac{j\omega(1 + j\omega)(1 + \frac{j\omega}{10^5})^2}{10^3(1 + \frac{j\omega}{10})(1 + \frac{j\omega}{10^4})(1 + \frac{j\omega}{10^6})^2}$$

(a) $x_1(t) = \cos(0.1t)$

Here, we have $\omega_0 = 0.1 = 10^{-1}$. Then the gain is equal to -80dB, and the phase is equal to $90°$. Since $H(\omega) = \frac{\mathbf{Y}}{\mathbf{X}}$, we have

$$\mathbf{Y} = 10^{-80/20}e^{j90°}e^{j0} = 10^{-4}e^{j90°}$$

yielding

$$y(t) = 10^{-4}\cos(0.1t + \frac{\pi}{2})$$

(b) $x_2(t) = \cos(8t)$

Here, we have $\omega_0 = 8 \approx< 10^1$. Then the gain is approximately -18dB, and the phase is approximately $120°$. Since $H(\omega) = \frac{\mathbf{Y}}{\mathbf{X}}$, we have

$$\mathbf{Y} = 10^{-18/20}e^{j120°}e^{j0} = 10^{-0.9}e^{j120°}$$

yielding

$$y(t) = 0.127\cos(8t + \frac{2\pi}{3})$$

(c) $x_3(t) = \cos(30t)$

Here, we have $\omega_0 = 30$. Then the gain is approximately $-20 + 20/2 = -10$dB, and the phase is approximately $135°$. Since $H(\omega) = \frac{\mathbf{Y}}{\mathbf{X}}$, we have

$$\mathbf{Y} = 10^{-10/20}e^{j135°}e^{j0} = 10^{-0.5}e^{j135°}$$

yielding

$$y(t) = 3.162\cos(30t + \frac{3\pi}{4})$$

(d) $x_4(t) = \cos(2000t)$

Here, we have $\omega_0 = 2000$. Then the gain is approximately $-20 + 20 \cdot 2.3 = 26$dB, and the phase is approximately $90 - 0.3 * 45 = 76.5°$. Since $H(\omega) = \frac{\mathbf{Y}}{\mathbf{X}}$, we have

$$\mathbf{Y} = 10^{26/20}e^{j76.5°}e^{j0} = 10^{13/10}e^{j76.5°}$$

yielding

$$y(t) = 19.95\cos(2000t + \frac{76.5\pi}{180})$$

(e) $x_5(t) = \cos(50000t)$

Here, we have $\omega_0 = 50000$. Then the gain is approximately $-20 + 20 \cdot 3 = 40$dB, and the phase is approximately $45 + 45 * 0.7 = 76.5°$. Since $H(\omega) = \frac{\mathbf{Y}}{\mathbf{X}}$, we have

$$\mathbf{Y} = 10^{40/20}e^{j76.5°}e^{j0} = 10^{2}e^{j76.5°}$$

yielding

$$y(t) = 100\cos(50000t + \frac{76.5\pi}{180})$$

5. Suppose $f(t)$ is a periodic function with period $T = 2$ and Fourier series components

$$F_n = \frac{1}{2n^2} \text{ for all integers } n \neq 0 \text{ and } F_0 = 1$$

$f(t)$ is the input to an LTI system with transfer function

$$H(\omega) = \cos(\omega) + j\sin(\omega)$$

Find the output $y(t)$ of the system in terms of only real numbers (all imaginary components should cancel).

Ignoring F_0, the Fourier expansion of the function $f(t)$ is given by

$$f(t) = \sum_{k=-\infty}^{\infty} \frac{1}{2k^2} e^{jk\omega_0 t}$$

It follows that the system response is given by the transfer function times the input, and

$$\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} \frac{1}{2k^2} e^{jk\omega_0 t}\left(cos(k\omega_0) + j\sin(k\omega_0)\right) \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{2k^2} e^{jk\omega_0 t} e^{jk\omega_0} \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{2k^2} e^{jk\omega_0 (1+t)} \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{2k^2} \left(\cos(k\omega_0 t + k\omega_0) + j\sin(k\omega_0 t + k\omega_0)\right) \\
&= F_0 + \sum_{k=1}^{\infty} \frac{1}{2k^2} \left(\cos(k\omega_0 t + k\omega_0) + j\sin(k\omega_0 t + k\omega_0)\right) + \sum_{k=-1}^{-\infty} \frac{1}{2k^2} \left(\cos(k\omega_0 t + k\omega_0) + j\sin(k\omega_0 t + k\omega_0)\right) \\
&= F_0 + \sum_{k=1}^{\infty} \frac{1}{2k^2} \left(\cos(k\omega_0 t + k\omega_0) + j\sin(k\omega_0 t + k\omega_0)\right) + \sum_{k=1}^{\infty} \frac{1}{2k^2} \left(\cos(-k\omega_0 t - k\omega_0) + j\sin(-k\omega_0 t - k\omega_0)\right) \\
&= F_0 + \sum_{k=1}^{\infty} \frac{1}{2k^2} \Big(\cos(k\omega_0 t + k\omega_0) + \xcancel{j\sin(k\omega_0 t + k\omega_0)} + \cos(-k\omega_0 t - k\omega_0) + \xcancel{j\sin(-k\omega_0 t - k\omega_0)}\Big) \\
&= F_0 + \sum_{k=1}^{\infty} \frac{1}{k^2} \cos(k\omega_0 t + k\omega_0)
\end{aligned}$$

Since $\sin(-f(t)) = -\sin(f(t))$, the imaginary components cancel and we are left with a purely real response. It follows that we can define the response as

$$y(t) = 1 + \sum_{k=1}^{\infty} \frac{1}{k^2} \cos(k\omega_0(t+1))$$

6. Find the Fourier series components F_n of $f(t) = \sin^4(t)$.

Observe that

$$\begin{aligned}
\sin^4(t) &= \sin^2(t)\cdot\sin^2(t) \\
&= \frac{1-\cos(2t)}{2}\cdot\left(1-\cos^2(t)\right) \\
&= \frac{1-\cos(2t)}{2} - \frac{1-\cos(2t)}{2}\frac{1+\cos(2t)}{2} \\
&= \frac{1-\cos(2t)}{2} - \frac{1+\cos(4t)}{8} \\
&= \frac{5}{8} - \frac{1}{2}\cos(2t) + \frac{1}{8}\cos(8t)
\end{aligned}$$

Then, we define the Fourier series expansion coefficients piecewise specifically for the frequencies in the linear combination of sinusoids specified above such that the imaginary parts cancel, for $\omega_0 = 1$:

- $F_0 = \frac{5}{8}$
- $F_2 = -\frac{1}{4}$
- $F_{-2} = -\frac{1}{4}$
- $F_8 = \frac{1}{16}$
- $F_{-8} = \frac{1}{16}$
- All other coefficients are zero.

To demonstrate these coefficients work, observe that

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} c_k e^{jkt} &= F_{-8}e^{-8jt} + F_{-2}e^{-2jt} + F_0 + F_2 e^{2jt} + F_8 e^{2jt} \\
&= \frac{1}{16}\cos(8t) - \cancel{\frac{1}{16}\sin(-8t)} - \frac{1}{4}\cos(2t) + \cancel{\frac{1}{4}\sin(2t)} + \frac{5}{8} \\
&\quad - \frac{1}{4}\cos(2t) - \cancel{\frac{1}{4}\sin(2t)} + \frac{1}{16}\cos(8t) + \cancel{\frac{1}{16}\sin(8t)} \\
&= \frac{5}{8} - \frac{1}{2}\cos(2t) + \frac{1}{8}\cos(8t)
\end{aligned}$$

7. Determine the Fourier series coefficients F_n and find the average power in a period of the function

$$\sum_{n=-\infty}^{\infty} x(t-4n) \quad \text{where} \quad x(t) = \begin{cases} 1 & 0 \le t < 2 \\ 2 & 2 \le t < 4 \end{cases}$$

The average power of a signal is given by

$$E_{avg}[x(t)] = \frac{1}{T_0}\int_{T_0} |x(t))|^2\, dt$$

Since x is piecewise defined, we split the integral and obtain

$$\begin{aligned}
E_{avg}[x(t)] &= \frac{1}{T_0}\left(\int_0^2 1dt + \int_2^4 4dt\right) \\
&= \frac{1}{T_0}\left(t\Big|_0^2 + 4t\Big|_2^4\right) \\
&= \frac{1}{T_0}(2+16-8) \\
&= \frac{1}{4}(10) \\
&= \frac{5}{2}
\end{aligned}$$

$F_0 = c_0$ is calculated as $\frac{1}{T}\int_{T_0} x(t)dt = \frac{1}{4} = \frac{3}{2}$. The remaining Fourier coefficients are given (similarly) by a split integral

$$
\begin{aligned}
c_k &= \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt \\
&= \frac{1}{T_0}\left(\int_0^2 e^{-jk\omega_0 t}dt + \int_2^4 2e^{-jk\omega_0 t}dt\right) \\
&= \frac{1}{T_0}\left(\frac{-e^{-jk\omega_0 t}}{jk\omega_0}\bigg|_0^2 + \frac{-2e^{-jk\omega_0 t}}{jk\omega_0}\bigg|_2^4\right) \\
&= \frac{1}{jk\omega_0 t_0}\left(-e^{-jk\omega_0 t}\Big|_0^2 + -2e^{-jk\omega_0 t}\Big|_2^4\right) \\
&= \frac{1}{jk2\pi}\left(-\left(e^{-2jk\omega_0} - 1\right) - 2\left(e^{-4jk\omega_0} - e^{-2jk\omega_0}\right)\right) \\
&= \frac{1}{jk2\pi}\left(-e^{-2jk\omega_0} + 1 - 2e^{-4jk\omega_0} + 2e^{-2jk\omega_0}\right) \\
&= \frac{1}{jk2\pi}\left(1 - 2e^{-4jk\omega_0} + 1e^{-2jk\omega_0}\right) \qquad\qquad \omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2} \\
&= \frac{1}{jk2\pi}\left(1 - 2e^{-2\pi jk} + 1e^{-\pi jk}\right).
\end{aligned}
$$

8. Find the Fourier series components of $f(t)$ and $g(t)$ and write $f(t)$ and $g(t)$ as purely real sums of sine and/or cosine functions.

Begin with $f(t)$. Observe that the function has even symmetry, with $T_0 = 1$ and $\omega_0 = 2\pi$. We can split the integral:

$$
\begin{aligned}
c_k &= \frac{1}{T_0}\int_{T_0} f(t)e^{-jk\omega_0 t}dt \\
&= \int_1 f(t)e^{-jk\omega_0 t}dt \\
&= \int_{-1/2}^{0} f(t)e^{-jk\omega_0 t}dt + \int_0^{1/2} f(t)e^{-jk\omega_0 t}dt \\
&= \int_{-1/2}^{0} (2t+1)e^{-jk\omega_0 t}dt + \int_0^{1/2} (-2t+1)e^{-jk\omega_0 t}dt \\
&= \int_{1/2}^{0} 2te^{-jk\omega_0 t}dt + \int_0^{1/2} -2te^{-jk\omega_0 t}dt + \int_{-1/2}^{1/2} e^{-jk\omega_0 t} \qquad \text{by parts} \\
&= \frac{1}{jk\omega_0}\left[-2te^{-jk\omega_0 t}\Big|_{-1/2}^{0} + 2\int_{-1/2}^{0} e^{-jk\omega_0 t}dt + 2te^{-jk\omega_0 t}\Big|_0^{1/2} - 2\int_0^{1/2} e^{-jk\omega_0 t}dt - e^{-jk\omega_0 t}\Big|_{-1/2}^{1/2}\right] \\
&= \frac{1}{jk\omega_0}\left[\cancel{-e^{jk\omega_0/2}} + \cancel{e^{-jk\omega_0/2}} - \cancel{e^{-jk\omega_0/2}} + \cancel{e^{jk\omega_0/2}} + 2\int_{-1/2}^{0} e^{-jk\omega_0 t}dt - 2\int_0^{1/2} e^{-jk\omega_0 t}dt\right] \\
&= \frac{-2}{(k\omega_0)^2}\left[-e^{-jk\omega_0 t}\Big|_{-1/2}^{0} + e^{-jk\omega_0 t}\Big|_0^{1/2}\right] \\
&= \frac{-2}{(k\omega_0)^2}\left[-1 + e^{jk\omega_0/2} + e^{-jk\omega_0/2} - 1\right] \\
&= \frac{-2}{(k\omega_0)^2}\left[2\cos\left(\frac{k\omega_0}{2}\right) - 2\right].
\end{aligned}
$$

Define $F_0 = \frac{1}{2}$, the average value of the function over one period. Let these coefficients be denoted by F_n. Observe that $F_n = F_{-n}$ by the even symmetry of the composition (since cosine is even, $\frac{1}{n^2}$ is even, etc.), which implies that the sine components cancel out in the sum due to the odd nature of the sine wave. Hence we have purely real coefficients and purely real terms for the Fourier series expansion of $f(t)$:

$$F_n = \begin{cases} \frac{-2}{(k\omega_0)^2}\left[2\cos\left(\frac{k\omega_0}{2}\right) - 2\right] & n \neq 0 \\ \frac{1}{2} & n = 0 \end{cases}$$

Since $g(t)$ is given by

$$g(t) = 8f\left(t - \frac{1}{4}\right) - 4 = 8\left(\sum_{n=-\infty}^{\infty} c_k e^{jn\omega_0 t}\right) - 4$$

we can say the Fourier series expansion of $g(t)$ has coefficients given by

$$G_n = \begin{cases} 8F_n & n \neq 0 \\ \frac{1}{T}\int_{T_0} g(t)dt = 0 & n = 0 \end{cases}.$$

Note that for $f(t)$, we have $T = 1$ and thus $\omega_0 = 2\pi$. Therefore we have

$$f(t) = \frac{1}{2} + 2\sum_{k=1}^{\infty} \frac{-2}{(2k\pi)^2}\left(2\cos\left(\frac{2k\pi}{2}\right) - 2\right)\cos(2k\pi t)$$

and

$$g(t) = 16\sum_{k=1}^{\infty} \frac{-2}{(2k\pi)^2}\left(2\cos\left(\frac{2k\pi}{2}\right) - 2\right)\cos(2k\pi t)$$

9. Find the Fourier series components of $f(t)$. Using the properties of the Fourier series, find the Fourier series coefficients of $g(t)$.

We first find the Fourier series expansion for $g(t)$. The DC term is as follows:

$$\begin{aligned} c_0 &= \frac{1}{T}\int_{T_0} y(t)dt \\ &= \frac{1}{4}\int_{-2}^{2} \frac{t^2}{4}dt \\ &= \frac{1}{16}\int_{-2}^{2} t^2 dt \\ &= \frac{1}{16}\left[\frac{1}{3}t^3\right]_{-2}^{2} \\ &= \frac{1}{3}. \end{aligned}$$

The remaining terms are

$$
\begin{aligned}
c_k &= \frac{1}{T}\int_{T_0} y(t)e^{-jk\omega_0 t}dt \\
&= \frac{1}{4}\int_{-2}^{2} \frac{t^2}{4}e^{-jk\omega_0 t}dt \\
&= \frac{1}{16}\int_{-2}^{2} t^2 e^{-jk\omega_0 t}dt \\
&= \frac{1}{16}\left[\frac{-t^2e^{-jk\omega_0 t}}{jk\omega_0} + \frac{2}{jk\omega_0}\int te^{-jk\omega_0 t}dt\right]_{-2}^{2} \\
&= \frac{1}{16}\left[\frac{-t^2e^{-jk\omega_0 t}}{jk\omega_0} + \frac{2}{jk\omega_0}\left(\frac{-te^{-jk\omega_0 t}}{jk\omega_0} + \frac{1}{jk\omega_0}\int e^{-jk\omega_0 t}dt\right)\right]_{-2}^{2} \\
&= \frac{1}{16}\left[\frac{-t^2e^{-jk\omega_0 t}}{jk\omega_0} + \frac{2}{jk\omega_0}\left(\frac{-te^{-jk\omega_0 t}}{jk\omega_0} + \frac{1}{jk\omega_0}\frac{-e^{-jk\omega_0 t}}{jk\omega_0}\right)\right]_{-2}^{2} \\
&= \frac{1}{16jk\omega_0}\left[e^{-jk\omega_0 t}\left(-t^2 + \frac{2}{jk\omega_0}\left(-t - \frac{1}{jk\omega_0}\right)\right)\right]_{-2}^{2} \\
&= \frac{1}{16jk\omega_0}\left[e^{-jk\omega_0 2}\left(-4 + \frac{2}{jk\omega_0}\left(-2 - \frac{1}{jk\omega_0}\right)\right) - e^{jk\omega_0 2}\left(-4 + \frac{2}{jk\omega_0}\left(2 - \frac{1}{jk\omega_0}\right)\right)\right] \\
&= \frac{1}{16jk\omega_0}\left[e^{-jk\omega_0 2}\left(\underbrace{-4 + \frac{2}{(k\omega_0)^2}}_{a}\underbrace{+\frac{4}{k\omega_0}j}_{+bj}\right) - e^{jk\omega_0 2}\left(\underbrace{-4 + \frac{2}{(k\omega_0)^2}}_{a}\underbrace{-\frac{4}{k\omega_0}j}_{-bj}\right)\right] \\
&= \frac{-1}{16k\omega_0}\left[2a\sin(2k\omega_0) - 2b\cos(2k\omega_0)\right] \\
&= \frac{-4}{16\pi k}\left[a\sin(k\pi) - b\cos(k\pi)\right] \\
&= \frac{-4}{16\pi k}\left[b\cos(k\pi)\right] \\
&= \frac{-4}{16\pi k}\left[\frac{-8}{\pi k}\cos(k\pi)\right] \\
&= \frac{2}{\pi^2k^2}\cos(k\pi).
\end{aligned}
$$

Observe that the postive and negative sine components cancel out due to the odd symmetry of sine and the even symmetry of the coefficients, as $c_n = c_{-n}$, and $\sin(-n\pi) = -\sin(n\pi)$. Therefore, we have found the Fourier coefficients for $g(t)$. Observe now that by linearity of the derivative:

$$\begin{aligned}
\frac{dg(t)}{dt} &= \sum_{n=-\infty}^{\infty} \frac{dy(t-4n)}{dt} \\
&= \sum_{n=-\infty}^{\infty} \begin{cases} \frac{t-4n}{2} & -2 \leq t-4n < 2 \\ 0 & \text{otherwise} \end{cases} \\
\frac{dg(\tau)}{d\tau} &= \sum_{n=-\infty}^{\infty} \begin{cases} \frac{\tau-4n}{2} & -2 \leq \tau-4n < 2 \\ 0 & \text{otherwise} \end{cases} \\
&\text{Let } \tau = t-2. \\
\frac{dg(\tau)}{d\tau}(t-2) &= \sum_{n=-\infty}^{\infty} \begin{cases} \frac{t}{2}-1-2n & 0 \leq t < 4 \\ 0 & \text{otherwise} \end{cases} \\
2\frac{dg(\tau)}{d\tau}(t-2) &= \sum_{n=-\infty}^{\infty} \begin{cases} t-2-4n & 0 \leq t < 4 \\ 0 & \text{otherwise} \end{cases} = \sum_{n=-\infty}^{\infty} x(t-4n) \\
2+2\frac{dg(\tau)}{d\tau}(t-2) &= 2+\sum_{n=-\infty}^{\infty} \begin{cases} t-2-4n & 0 \leq t < 4 \\ 0 & \text{otherwise} \end{cases} = 2+\sum_{n=-\infty}^{\infty} x(t-4n) = f(t)
\end{aligned}$$

All that remains is to find the DC term in the expansion of $f(t)$, which is given by $\frac{1}{4}\int_0^4 t dt = \frac{1}{4}8 = 2 = c_0 = F_0$. Therefore, we have

$$g(t) = \frac{1}{3} + 2\sum_{n=1}^{\infty} \frac{2}{\pi^2 k^2} \cos(k\pi) \cos\left(\frac{k\pi}{2}t\right)$$

and by extension, $f(t) = 2 + 2\frac{dg}{dt}(t-2)$ and

$$f(t) = 2 + 4\sum_{n=1}^{\infty} \frac{\cos(\pi k)}{\pi k} \sin\left(\frac{\pi k}{2}t + \pi k\right)$$

10. Suppose $f(t)$ is the input to an LTI system with transfer function $H(\omega)$ where

$$f(t) = |\sin(t)| \quad \text{and} \quad H(\omega) = 1 - 4\left(\frac{\omega}{2\pi}\right)^2$$

Find the output $y(t)$ and write it as a purely real sum of sines and/or cosines.

Observe that $f(t)$ can be rewritten in terms of $x(t)$ where

$$x(t) = \begin{cases} \sin(t) & 0 \leq t < \pi \\ 0 & \text{otherwise} \end{cases}$$

such that

$$f(t) = \sum_{n=\infty}^{\infty} x(t-n\pi)$$

Then we can model $f(t)$ with a Fourier series, where $T = \pi$ and $\omega_0 = 2$. The DC term is given by

$$c_0 = \frac{1}{T}\int_{T_0} f(t)dt = \frac{1}{\pi}\int_0^{\pi} \sin(t)dt = \frac{2}{\pi}$$

All other terms are given by the integral $c_k = \frac{1}{T}\int_{T_0} \sin(t)e^{-jk\omega_0 t}dt$. The antiderivative is

$$\begin{aligned}
C_k(t) = \int \sin(t)e^{-jk\omega_0 t}dt &= \frac{-\sin(t)e^{-jk\omega_0 t}}{jk\omega_0} - \frac{-1}{jk\omega_0}\int \cos(t)e^{-jk\omega_0 t}dt \\
&= \frac{-\sin(t)e^{-jk\omega_0 t}}{jk\omega_0} - \frac{-\cos(t)e^{-jk\omega_0 t}}{(jk\omega_0)^2} - \frac{1}{jk\omega_0}\int \sin(t)e^{-jk\omega_0 t}dt \\
&= \frac{1}{1+\frac{1}{(jk\omega_0)^2}}\left(\frac{-\sin(t)e^{-jk\omega_0 t}}{jk\omega_0} + \frac{-\cos(t)e^{-jk\omega_0 t}}{(jk\omega_0)^2}\right) \\
&= \frac{-jk\omega_0}{(jk\omega_0)^2+1}\left(e^{-jk\omega_0 t}\left(\sin(t) + \frac{\cos(t)}{jk\omega_0}\right)\right)
\end{aligned}$$

It follows that the actual integral is given by

$$\begin{aligned}
\frac{1}{T}\int_{T_0} \sin(t)e^{-jk\omega_0 t}dt &= C_k(\pi) - C_k(0) \\
&= \frac{-jk\omega_0}{\pi(-jk\omega_0)^2+\pi}\left(e^{jk\omega_0\pi}\left(0 - \frac{1}{jk\omega_0}\right) - e^0\left(0 + \frac{1}{jk\omega_0}\right)\right) \\
&= \frac{-1}{\pi(jk\omega_0)^2+\pi}\left(-e^{-jk\omega_0\pi} - 1\right) \\
&= \frac{-1}{\pi(2jk)^2+\pi}\left(-e^{-2jk\pi} - 1\right) \\
&= \frac{-1}{\pi(2jk)^2+\pi}\left(-1 - \cos(2\pi k) + j\sin(2\pi k)\right) \\
&= \frac{-1}{\pi(2jk)^2+\pi}(-2) \\
&= \frac{2}{-4\pi k^2+\pi}
\end{aligned}$$

Then the Fourier series expansion is

$$\begin{aligned}
f(t) &= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{2}{-4\pi n^2+\pi}e^{-2jnt} + \frac{2}{-4\pi(-n)^2+\pi}e^{2jnt} \\
f(t) &= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{-4\pi n^2+\pi}\cos(2nt)
\end{aligned}$$

And in the phasor domain:

$$f(t) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{-4\pi n^2+\pi}e^{j0}$$

It follows that the output phasor for a given frequency $\omega = 2n$ is given by

$$\begin{aligned}
\mathbf{Y} &= \mathbf{F}H(\omega) \\
&= \frac{4}{-4\pi n^2+\pi}\left(1 - 4\left(\frac{n}{\pi}\right)^2\right)
\end{aligned}$$

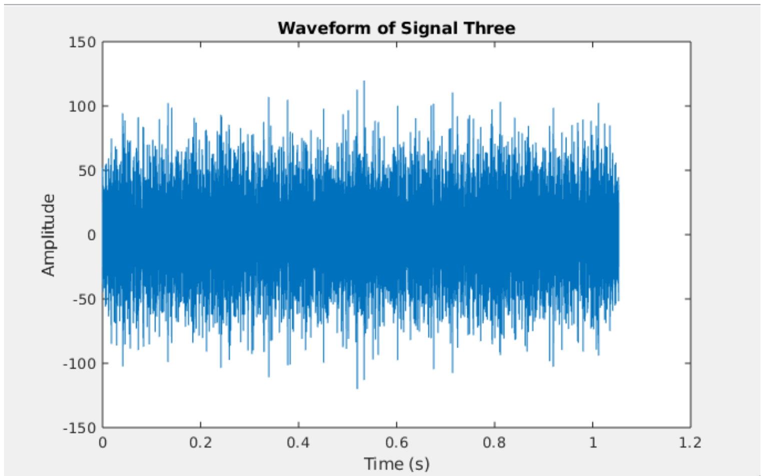
This quantity is purely real, so there is no phase shift. Then, the output $y(t)$ in the time domain is given by

$$y(t) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4 - 16\left(\frac{n}{\pi}\right)^2}{-4\pi n^2+\pi}\cos(2nt)$$

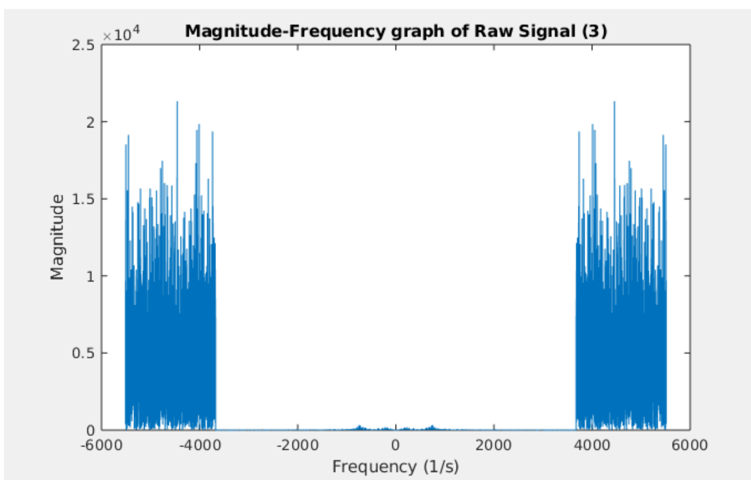
2 MATLAB Homework

3 The signal with audio embedded is signal **three**. The contents are the words, "I'll be back".

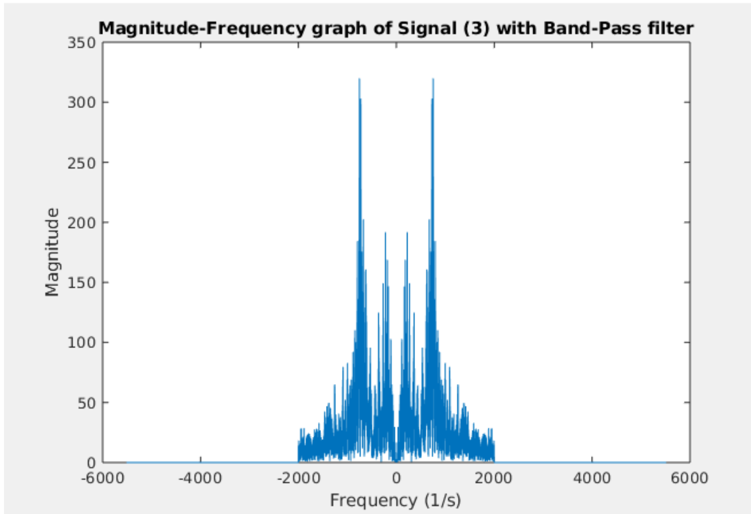
In the time-domain, the signal is given by



In the frequency domain, the signal is given by



And with a band-pass filter passing frequencies in the range $[0, 2000]$, the signal in the frequency domain is given by



The MATLAB commands used to decode the signal and generate the plot are

```
N = 11613
Fs = 11025
t = (0 : N-1) / Fs
f = (-Fs / 2 : Fs / (N-1) : Fs / 2)
load three.mat
% Generating waveform plot
plot(t, three)
xlabel("Time (s)")
ylabel("Amplitude")
title("Waveform of Signal Three")
% Fast Discrete Fourier transform of signal three
Three = fft(three)
% Generating unfiltered magnitude-frequency plot
plot(f, abs(fftshift(Three)))
xlabel("Frequency (1/s)")
ylabel("Magnitude")
title("Magnitude-Frequency graph of Raw Signal (3)")
% Band-pass filter
Z = Three .* HW2_Filter(f, 0, 2000)
% Generating filtered magnitude-frequency plot
plot(f, abs(fftshift(Z)))
xlabel("Frequency (1/s)")
ylabel("Magnitude")
title("Magnitude-Frequency graph of Signal (3) with Band-Pass filter")
% Returning to time domain
z = real(ifft(Z))
% Playing sound
sound(z, Fs)
```