MATH 102 (SP21, Eggers)

# Homework 6


Jiahong Long

[redacted]


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1. *Real-valued matrix*

   (a) *Linearly independent eigenvectors*

   *Proof.* Using the sesquilinear form of the inner product, we observe that

$$\begin{aligned}
\langle \mathbf{x} + i\mathbf{y}, \mathbf{x} - i\mathbf{y} \rangle &= \langle \mathbf{x}, \mathbf{x} - i\mathbf{y} \rangle + \langle i\mathbf{y}, \mathbf{x} - i\mathbf{y} \rangle \\
&= \langle \mathbf{x}, \mathbf{x} \rangle + i\langle \mathbf{x}, \mathbf{y} \rangle - i\langle \mathbf{y}, \mathbf{x} \rangle + i^2\langle \mathbf{x}, \mathbf{y} \rangle \\
&= \langle \mathbf{x}, \mathbf{x} \rangle + \cancelto{0}{i\langle \mathbf{x}, \mathbf{y} \rangle - i\langle \mathbf{y}, \mathbf{x} \rangle} - \langle \mathbf{x}, \mathbf{y} \rangle \\
&= \cancelto{0}{\langle \mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{x}, \mathbf{y} \rangle} \\
&= 0
\end{aligned}$$

   and so the two are orthogonal and necessarily linearly independent.

   $\square$

   (b) *Linearly independent components*

   *Proof.* We begin by showing $\mathbf{y} \neq \mathbf{0}$. Observe that this statement is equivalent to stating that the eigenvector corresponding to a complex eigenvalue has a nonzero imaginary component. Suppose now towards the contrary that $\mathbf{z}_1$ is purely real, such that $\mathbf{y} = 0$ and

$$\begin{aligned}
A\mathbf{z}_1 &= \lambda_1 \mathbf{z}_1 \\
A(\mathbf{x} + i\mathbf{y}) &= (a + bi)(\mathbf{x} + i\mathbf{y}) \\
A\mathbf{x} + \cancelto{\mathbf{0}}{iA\mathbf{y}} &= a\mathbf{x} + bi\mathbf{x} + \cancelto{0}{ai\mathbf{y} - b\mathbf{y}} \\
A\mathbf{x} &= a\mathbf{x} + bi\mathbf{x}
\end{aligned}$$

   But since $\mathbf{x} \in \mathbb{R}^n$, we have that a *purely real transformation* applied to a *purely real vector* yields a *complex vector*, which is not possible. Then we must have $\mathbf{y} \neq \mathbf{0}$.

   To show $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, it suffices to follow a similar structure as the above. Towards a contradiction, we will denote, for some scalar $k \in \mathbb{R}$:

$$\mathbf{y} = k\mathbf{x}$$

such that

$$\begin{aligned} A\mathbf{z}_1 &= \lambda_1 \mathbf{z}_1 \\ A(\mathbf{x} + ki\mathbf{x}) &= (a+bi)(\mathbf{x} + ki\mathbf{x}) \\ A\mathbf{x} + kiA\mathbf{x} &= (a+bi)(1+ki)\mathbf{x} \\ (1+ki)A\mathbf{x} &= (a+bi)(1+ki)\mathbf{x} \\ A\mathbf{x} &= (a+bi)\mathbf{x} \\ &= a\mathbf{x} + bi\mathbf{x} \end{aligned}$$

where once again a contradiction arises as we cannot have complex vectors as the product of a purely real matrix and a purely real vector. Therefore $\mathbf{x}$ and $\mathbf{y}$ need be linearly independent.

□

2. *Outer product matrix*

   (a) *Zero eigenvalue*

   *Proof.* Given the outer product $A = \mathbf{x}\mathbf{y}^T$ it follows that the $i$th row of $A$ for $1 \leq i \leq n$ is given by $x_i\mathbf{y}$ where $x_i$ is the $i$th entry of $\mathbf{x}$. Since $\dim \mathbb{R}^n = n$, pick any $n-1$ linearly independent vectors $v_2, \ldots, v_n$ such that

   $$\{\mathbf{y}, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$$

   is a basis for $\mathbb{R}^n$. Using Gram-Schmidt, we can then orthonormalise this basis to obtain the orthonormal basis

   $$\mathcal{B} = \{\hat{\mathbf{x}}, \hat{\mathbf{v}'}_2, \ldots, \hat{\mathbf{v}'}_n\}$$

   Then, each of $\hat{\mathbf{v}'}_i$ for $2 \leq i \leq n$ is orthogonal to $\mathbf{x}$. We can therefore write

   $$A\hat{\mathbf{v}'}_i = \mathbf{x}\mathbf{y}^T\hat{\mathbf{v}'}_i$$

   By orthogonality, we have

   $$\begin{aligned} &= 0\mathbf{x} \\ A\hat{\mathbf{v}'}_i &= 0\hat{\mathbf{v}'}_i \end{aligned}$$

   Since this holds for all $n-1$ linearly independent vectors $\hat{\mathbf{v}'}_i$, we have found $n-1$ linearly independent eigenvectors corresponding to the eigenvalue 0, as required.

   □

   (b) *Single eigenvalue*

   *Proof.* It suffices to show that $(\langle \mathbf{x}, \mathbf{y} \rangle, \mathbf{x})$ is an eigenpair of $A$. Algebraically,

   $$\begin{aligned} A\mathbf{x} &= \mathbf{x}\mathbf{y}^T\mathbf{x} \\ &= \mathbf{x}\langle \mathbf{y}, \mathbf{x} \rangle \\ &= \langle \mathbf{x}, \mathbf{y} \rangle \mathbf{x} \end{aligned}$$

   and since the inner product is a scalar, it follows immediately that $\mathbf{x}^T\mathbf{y}$ is an eigenvalue corresponding to the eigenvector $\mathbf{x}$ of $A$.

   □

   (c) *Diagonalisability*

   *Proof.* From part (a) it is evident that $A$ has $n$ linearly independent eigenvectors. Using the same notation, we may then express

   $$A\begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{v}'}_2 & \hat{\mathbf{v}'}_3 & \cdots & \hat{\mathbf{v}'}_n \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{v}'}_2 & \hat{\mathbf{v}'}_3 & \cdots & \hat{\mathbf{v}'}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Observe that from part (a), $\lambda_1 = \mathbf{x}^T\mathbf{y}$ and $\lambda_2 = \lambda_3 = \ldots = \lambda_n = 0$, such that we can form a diagonal matrix $\Lambda$ that is all zero except for the $\lambda_1$ entry. Then, using $P$ to denote the eigenvector matrix, we may write

$$AP = P\Lambda$$

We know from part (a) that $P$ has linearly independent columns, by explicit construction of an orthonormal basis. Hence, $P$ is invertible and we may write

$$P^{-1}AP = \Lambda$$

and thus $A$ is diagonalisable.

□

3. *Nonsingularity of the matrix exponential*

*Proof.* Let $A$ be an arbitrary $n \times n$ matrix. Then it has $n$ eigenvalues (counting multiplicities). Suppose $\lambda_i$ is one such eigenvalue, and let $\mathbf{v}_i$ be an associated eigenvector such that

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

Suppose we multiply the matrix exponential into the eigenvector, to obtain

$$\begin{aligned} e^A \mathbf{v}_i &= \sum_{k=0}^{\infty} \frac{1}{k!} A^k \mathbf{v}_i \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_i^k \mathbf{v}_i \end{aligned}$$

But this is equivalent to the Maclaurin expansion of the exponential, so we have

$$e^A \mathbf{v}_i = e^{\lambda_i} \mathbf{v}_i$$

It follows that all eigenvectors of $A$ are eigenvectors of $e^A$. But the associated eigenvalues are of the form $e^{\lambda_i}$, which is never zero. Since $\det e^A = \prod_{i=0}^{n} e^{\lambda_i}$ and $e^{\lambda_i} \neq 0$, the determinant is nonzero, and so the matrix exponential is nonsingular for *any* matrix $A$.

□

4. *Hermitian matrix*

*Proof.* Observe that

$$\begin{aligned} \mathbf{x}^H A \mathbf{x} &= \left(\mathbf{x}^H A\right) \mathbf{x} && (1) \\ &= \left(A^H \mathbf{x}\right)^H \mathbf{x} && (2) \\ &= (A\mathbf{x})^H \mathbf{x} && (3) \\ &= \langle \mathbf{x}, A\mathbf{x} \rangle && (4) \end{aligned}$$

Associating terms differently yields

$$\begin{aligned} \mathbf{x}^H A \mathbf{x} &= \mathbf{x}^H (A\mathbf{x}) && (5) \\ &= \langle A\mathbf{x}, \mathbf{x} \rangle && (6) \\ \langle A\mathbf{x}, \mathbf{x} \rangle &= \overline{\langle \mathbf{x}, A\mathbf{x} \rangle} && (7) \end{aligned}$$

Equating with (4) yields

$$\langle \mathbf{x}, A\mathbf{x} \rangle = \overline{\langle \mathbf{x}, A\mathbf{x} \rangle} \qquad (8)$$

but for a quantity to be equal to its complex conjugate requires it to be purely real, as required.

□

5. *Unitary matrix*

   (a) *Normality*

   *Proof.* A matrix $U$ is unitary if and only if we have

   $$U^{-1} = U^H$$

   but because matrices commute with their inverses, we have

   $$U^H U = U U^H = I$$

   so $U$ is normal. □

   (b) *Preservation of the norm*

   *Proof.* Let $U$ be unitary. Then we may write, for $\mathbf{x} \in \mathbb{C}^n$

   $$\begin{aligned}\langle U\mathbf{x}, U\mathbf{x}\rangle &= (U\mathbf{x})^H U\mathbf{x} \\ &= \mathbf{x}^H U^H U \mathbf{x}\end{aligned}$$

   Since $U$ is unitary, the middle terms go to the identity, and we have

   $$\begin{aligned}&= \mathbf{x}^H \mathbf{x} \\ &= \langle \mathbf{x}, \mathbf{x}\rangle\end{aligned}$$

   as required. □

   (c) *Eigenvalue magnitude*

   *Proof.* Take $\lambda$ to be an eigenvalue of the unitary matrix $U$. Let $\mathbf{x}$ be an associated eigenvector, such that

   $$U\mathbf{x} = \lambda\mathbf{x}$$

   Taking the inner product of both sides, we obtain

   $$\begin{aligned}\langle U\mathbf{x}, U\mathbf{x}\rangle &= \langle \lambda\mathbf{x}, \lambda\mathbf{x}\rangle \\ &= |\lambda|^2 \langle \mathbf{x}, \mathbf{x}\rangle\end{aligned}$$

   Per part (b), transformations encoded by unitary matrices preserve length, such that we now have

   $$\begin{aligned}\langle U\mathbf{x}, U\mathbf{x}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle &= |\lambda|^2 \langle \mathbf{x}, \mathbf{x}\rangle \\ 1 &= |\lambda|^2 \\ |\lambda| &= 1\end{aligned}$$

   as required. □

6. *Eigenvalues of Hermitian matrices*

   *Proof.* Given that $A$ is Hermitian, the spectral theorem applies and it is diagonalised by a unitary matrix $U$ that $UTU^H = A$. Furthermore, we can write

   $$\begin{aligned}A &= UTU^H \\ AU &= UT\end{aligned}$$

The $i$th column of $AU$ is equal to the $i$th column of $UT$, and since $T$ is diagonal, we may write, using $\mathbf{u}_i$ to denote the $i$th column in $U$,

$$A\mathbf{u}_i = \mathbf{u}_i t_{ii}$$

from which it follows that the columns of $U$ are eigenvectors of $A$, and the diagonal entries of $T$ are the associated eigenvalues. Therefore, we may rewrite the diagonalisation as an eigendecomposition:

$$\begin{aligned}
A &= UTU^H \\
&= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^H \\ \mathbf{u}_2^H \\ \vdots \\ \mathbf{u}_n^H \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 \mathbf{u}_1 & \lambda_2 \mathbf{u}_2 & \cdots & \lambda_n \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^H \\ \mathbf{u}_2^H \\ \vdots \\ \mathbf{u}_n^H \end{bmatrix}
\end{aligned}$$

The column matrix on the right can be rewritten as a matrix sum, and distributivity of matrix multiplication applies to yield

$$\begin{aligned}
&= \sum_{i=1}^{n} \begin{bmatrix} 0 & \cdots & \lambda_i \mathbf{u}_i & \cdots & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^H \\ \mathbf{u}_2^H \\ \vdots \\ \mathbf{u}_n^H \end{bmatrix} \\
&= \sum_{i=1}^{n} \left(0 + 0 + \cdots + \lambda_i \mathbf{u}_i \mathbf{u}_i^H + 0 + \cdots + 0\right) \\
&= \sum_{i=1}^{n} \lambda_i \mathbf{u}_i \mathbf{u}_i^H \\
&= \lambda_1 \mathbf{u}_1 \mathbf{u}_1^h + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^H + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^H
\end{aligned}$$

so $A$ can be expressed as a linear combination of the outer products of its eigenvectors, as required. $\square$