🪄 MGC-X 🪄

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Motivation

- Observe several time series evolving simultaneously
- Analyzing each component individually can lead to wrong inference.

Independence Testing

Let $(X_i, Y_i) \sim F_{XY}, i \in [n] = \{1, 2, \dots, n\}$

We want to test:

$$H_0 : F_{XY} = F_X F_Y$$
$$H_A : F_{XY} \neq F_X F_Y$$

Indepdence Testing in Time-series data

- Let $(X_t, Y_t) \sim F_{XY},\ t \in [n] = \{1, 2, \ldots, n\}$
- Maximum lag M.

$$H_0 : F_{X_t, Y_{t-j}} = F_{X_t} F_{Y_{t-j}} \text{ for each } j \in \{0, 1, \ldots, M\}$$
$$H_A : F_{X_t, Y_{t-j}} \neq F_{X_t} F_{Y_{t-j}} \text{ for some } j \in \{0, 1, \ldots, M\}$$

- Assymetric test

Stationary Process

A time series $\{X_t\}$ is **strictly stationary** if the joint distribution of $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ is the same as that of $(X_{t_{1+\tau}}, X_{t_{2+\tau}}, \ldots, X_{t_{n+\tau}})$.

- $t, \tau \in \mathbb{R}, n \in \mathbb{N}_{>0}$

Consquently,

- $\mathbb{E}(X_t)$ is constant, does not depend on t
- If $\mathbb{E}(X_t^2) < \infty$, $Cov(X_t, X_{t-\tau})$ only depends on τ

Above two conditions form the **weakly stationary**.

Auto-covariance/correlation (ACVF/ACF)

Let $\{X_i\}_1^t$, $X_i \in \mathbb{R}$ be stationary time-series.

- Auto-covariance function (ACVF)
  - Time dependent covariance with itself

$$ACVF(j) = Cov(X_t, X_{t-j})$$

- Auto-correlation function (ACF)
  - Time dependent correlation with itself

$$\begin{aligned} ACF(j) &= \frac{Cov(X_t, X_{t-j})}{\sqrt{Var(X_t)Var(X_{t-j})}} \\ &= \frac{ACVF(j)}{ACVF(0)} \qquad \text{(Stationarity of } X_t\text{)} \end{aligned}$$

Cross-covariance/correlation (CCVF/CCF)

Let $\{(X_i, Y_i)\}_1^t$, $X_i, Y_i \in \mathbb{R}$ stationary time-series.

- Cross-covariance function (CCVF)
  - Time dependent covariance between two time series

$$CCVF(j) = Cov(X_t, Y_{t-j})$$

- Cross-correlation function (CCF)
  - Time dependent correlation between two time series

$$\begin{aligned} CCF(j) &= \frac{Cov(X_t, Y_{t-j})}{\sqrt{Var(X_t)Var(Y_{t-j})}} \\ &= \frac{CCF(j)}{\sqrt{ACVF_{X_t}(0)ACVF_{Y_t}(0)}} \end{aligned}$$

Ljung-Box Test (pronounced Yoong)

- Max lag M
- Originally for autocorrelated time-series data.

$$T = n(n+2)\sum_{j=1}^{M} \frac{ACF^2(j)}{n-j}$$

- Modified for cross-correlated time-series data.

$$T = n(n+2)\sum_{j=1}^{M} \frac{CCF^2(j)}{n-j}$$

Distance Based Test

- Max lag M
- Distance cross correlation

$$DCorr(j) := DCorr(X_t, Y_{t-j})$$

$$DCorrX = \sum_{j=1}^{M} \frac{n-j}{n} \cdot DCorr(j)$$

- Multiscale graph cross correlation

$$MGC(j) := MGC(X_t, Y_{t-j})$$

$$MGCX = \sum_{j=1}^{M} \frac{n-j}{n} \cdot MGC(j)$$

Optimal Lag

- Lag j that exhibits the strongest dependence.

$$\hat{M}^* = \arg\max_j \left(\frac{n-j}{n}\right) \cdot T(j)$$

Block Permutation Test

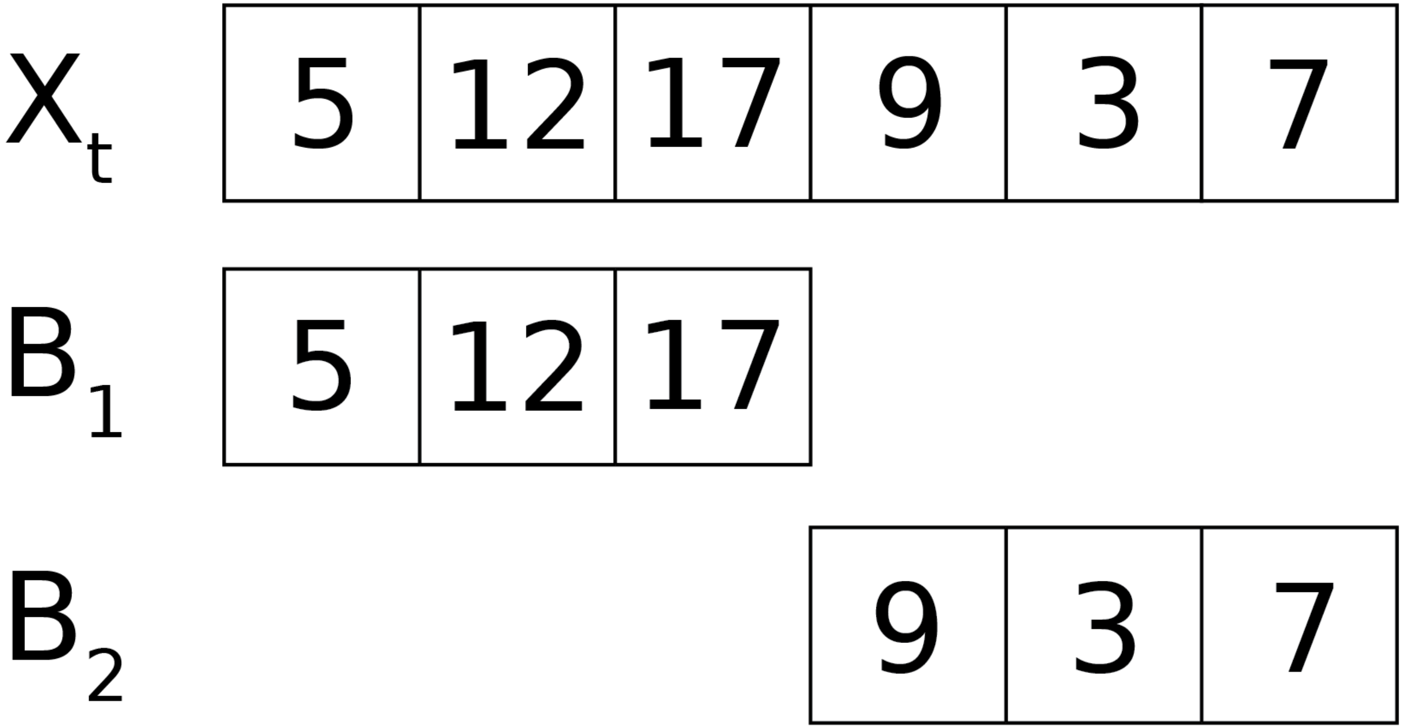
- Dependent data → cannot use standard permutation.

Procedure: Given number of blocks size $b < n$,

1. For each block $i \in 1, 2, \ldots, \lceil n/b \rceil$, produce block:
$$B_i = (Y_{b(i-1)+1}, Y_{b(i-1)+2}, \ldots, Y_{b(i-1)+b})$$
2. Choose $\lceil n/b \rceil$ blocks from $B_0, B_1, \ldots, B_{\lceil n/b \rceil}$ with replacement.
3. Concatenate the chosen blocks to form a new time series Y'.
4. Compute $T_n^{(r)}$ on the series $\{(X_t, Y_t')\}_1^n$. Repeat r times.
5. Compute p-value as:
$$p = \frac{1}{r} \sum_{i=1}^{r} \mathbb{I}\{T_n^{(r)} \geq T_n\}$$

Block Permutation Visualization

- $n = 6, b = 3$



Simulations

1. Independent data

$$\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} \phi & 0 \\ 0 & \phi \end{bmatrix} \begin{bmatrix} X_{t-1} \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_t \\ \eta_t \end{bmatrix}$$

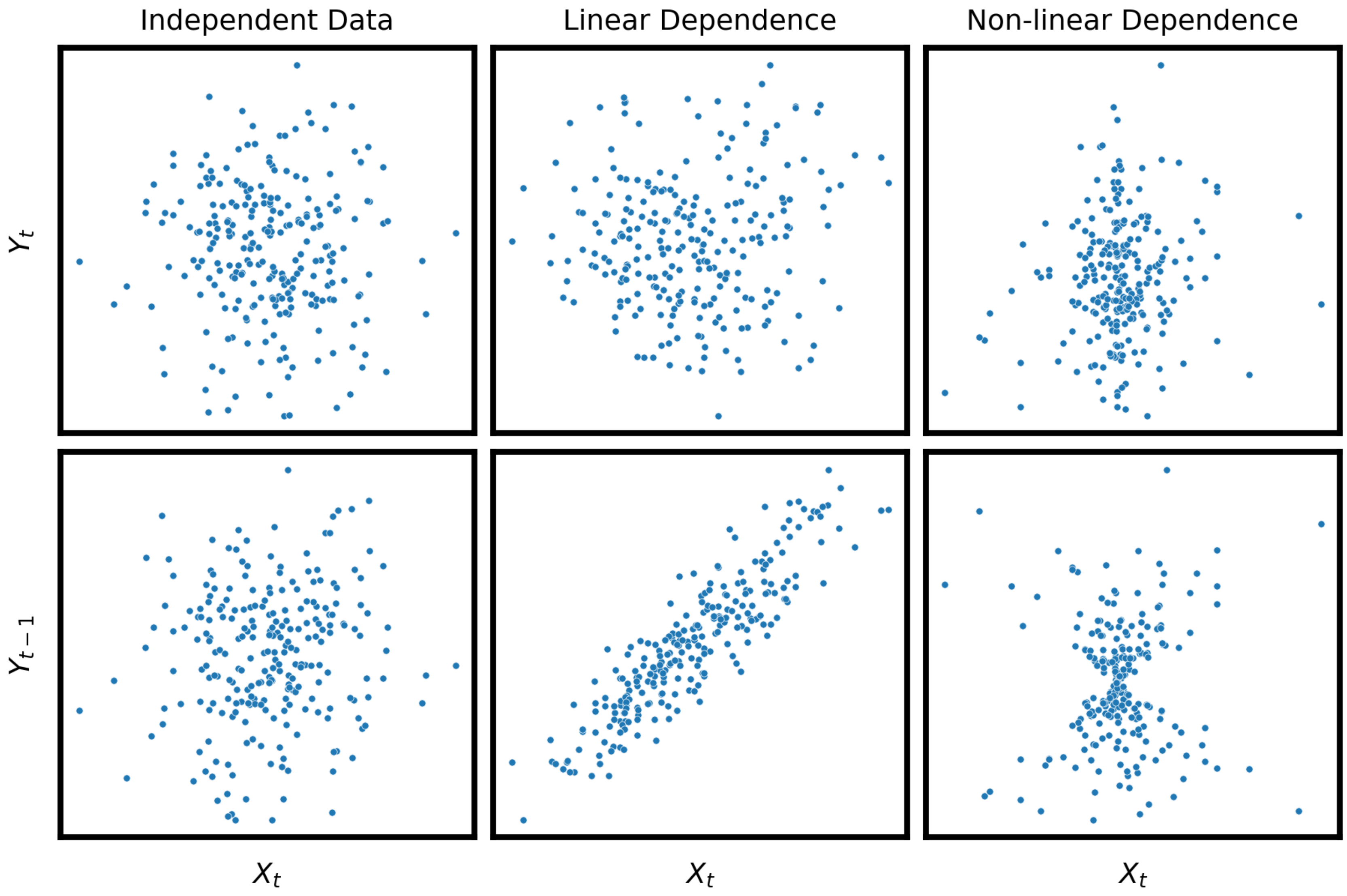
2. Linear dependence

$$\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} 0 & \phi \\ \phi & 0 \end{bmatrix} \begin{bmatrix} X_{t-1} \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_t \\ \eta_t \end{bmatrix}$$

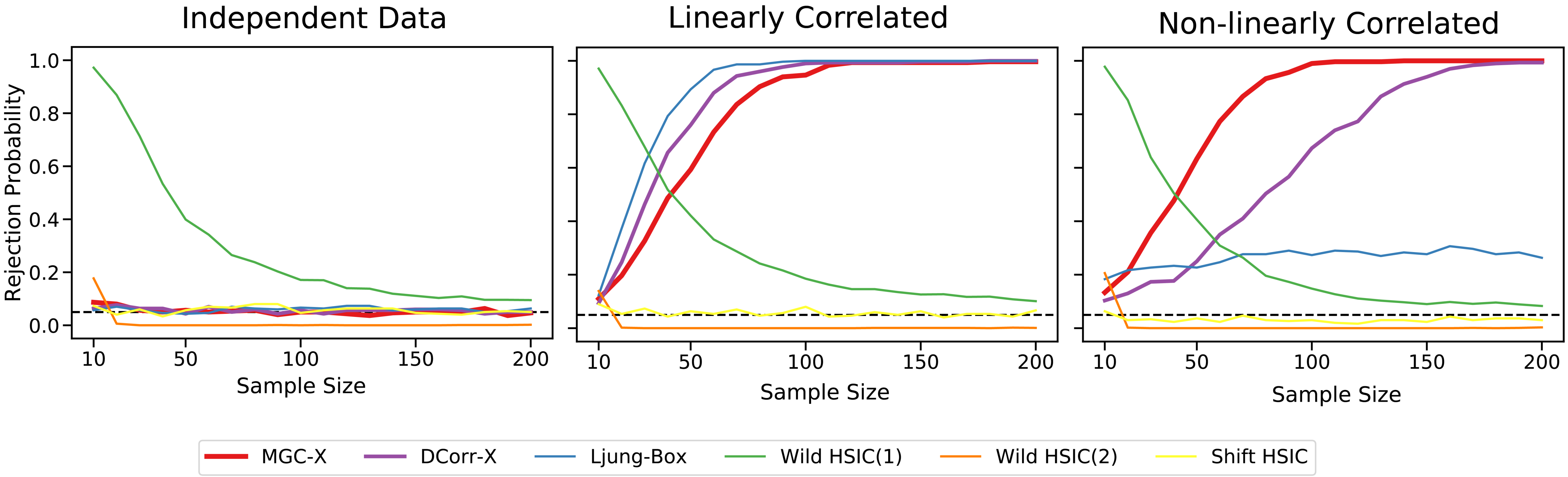
3. Non-linear dependence

$$\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} \epsilon_t Y_{t-1} \\ \eta_t \end{bmatrix}$$

Visualizations



Results

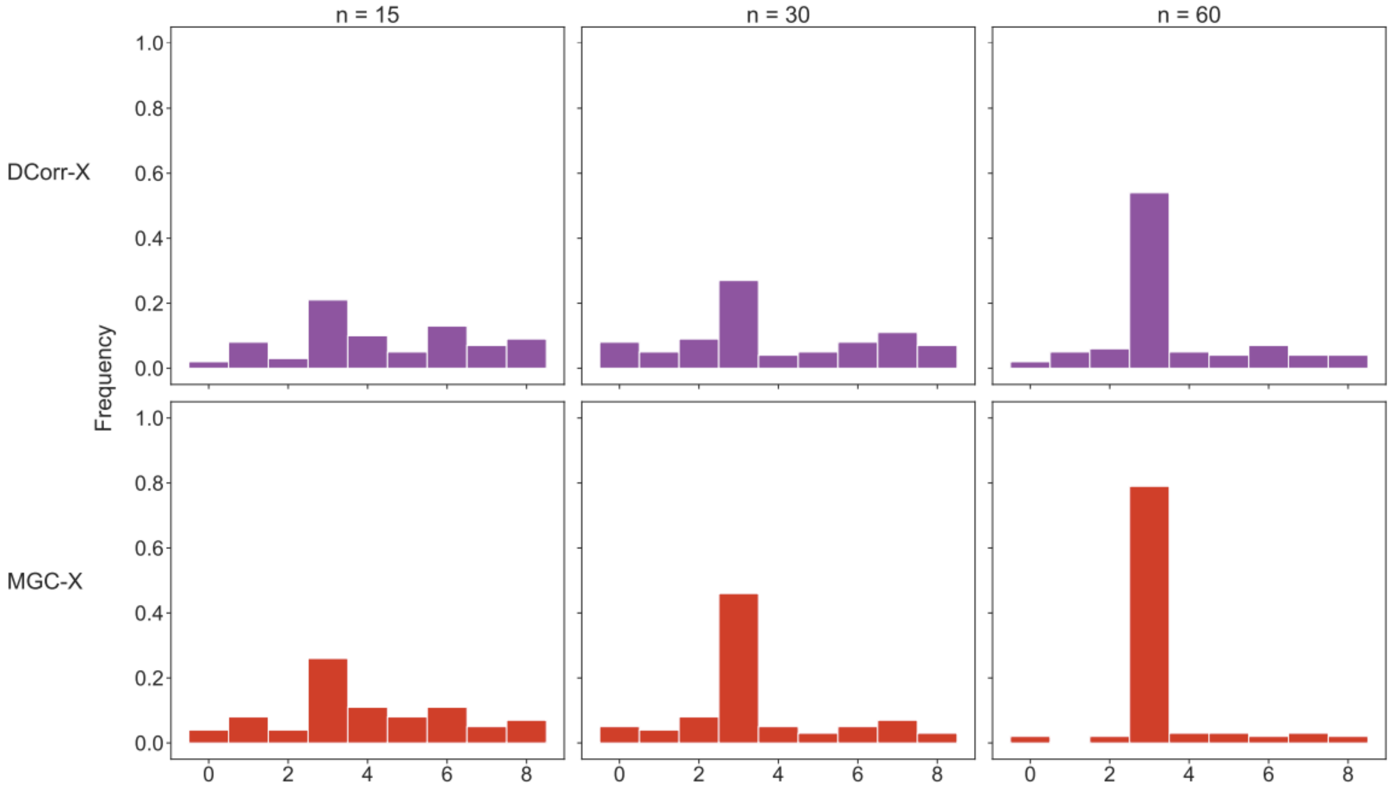


Independent Data
Linearly Correlated
Non-linearly Correlated
Rejection Probability
1.0
0.8
0.6
0.4
0.2
0.0
10
50
100
150
200
Sample Size
MGC-X
DCorr-X
Ljung-Box
Wild HSIC(1)
Wild HSIC(2)
Shift HSIC

Optimal lag estimation

- Data dependent on lag 3.

$$\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} \epsilon_t Y_{t-3} \\ \eta_t \end{bmatrix}$$



Real world data

- Resting state(?) fMRI data
- n=1200 frames, $\approx$ 0.75 second per frame.
- 22 brain regions, each part of some large-scale brain network.

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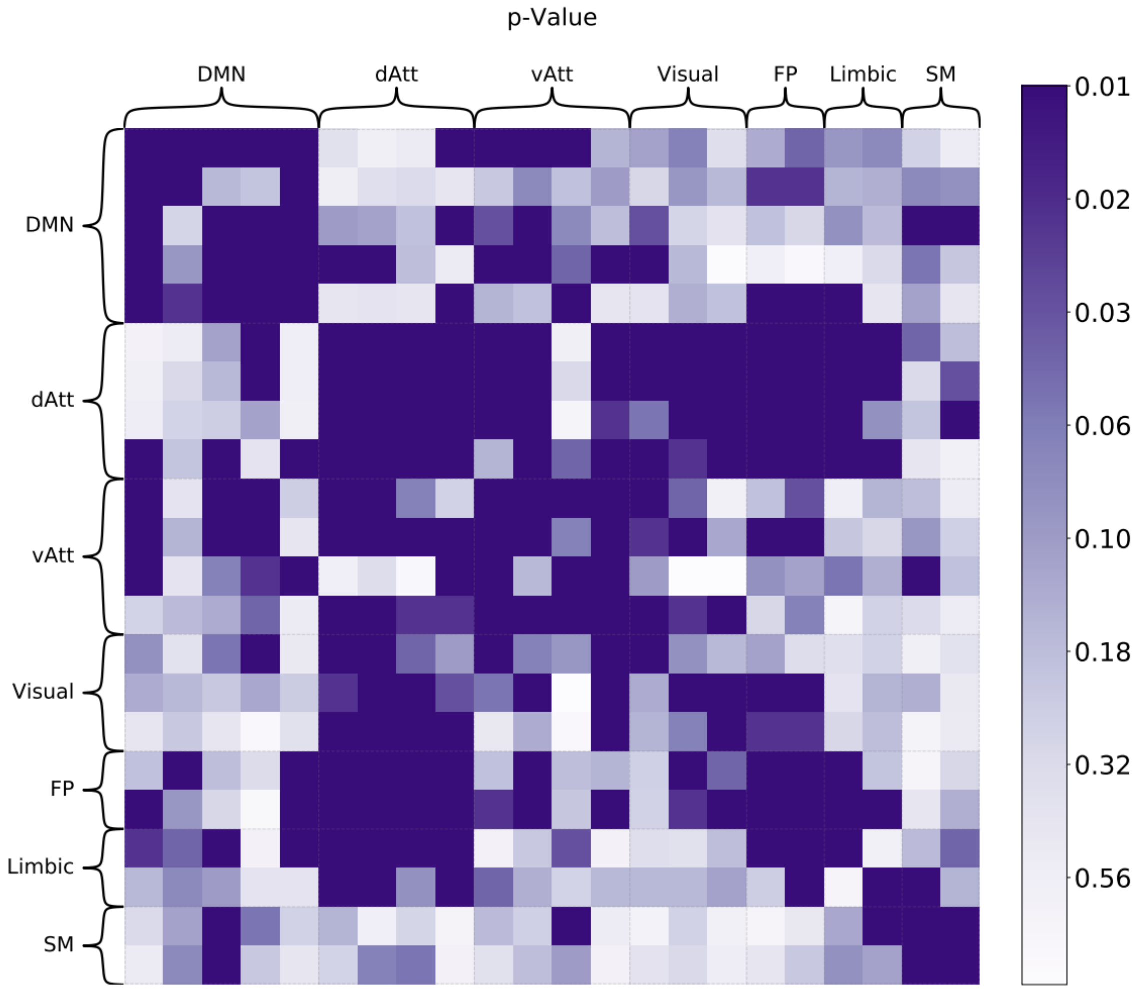
- Default Mode Network (DMN)
- Dorsal Attention Network (dAtt)
- Ventral Attention Network (vAtt)
- Visual Network (Visual)
- FrontoParietal Network (FP)
- Limbic Network (Limbic)
- Somatomotor Network (SM)
```

Network	Shorthand	Activation?
Default Mode	DMN	When resting
Dorsal Attention	dAtt	Selecting stimuli relevant to a goal
Ventral Attention	vAtt	Detecting and redirecting attention to relevant stimuli
Visual	Visual	Analyzing the various components of the visual scene
FrontoParietal	FP	Making decisions in the context of goal-driven behaviour
Limbic	Limbic	Emotion, consciousness, motivation and long-term memory
Somatomotor	SM	Detecting somatosensory stimuli, movement



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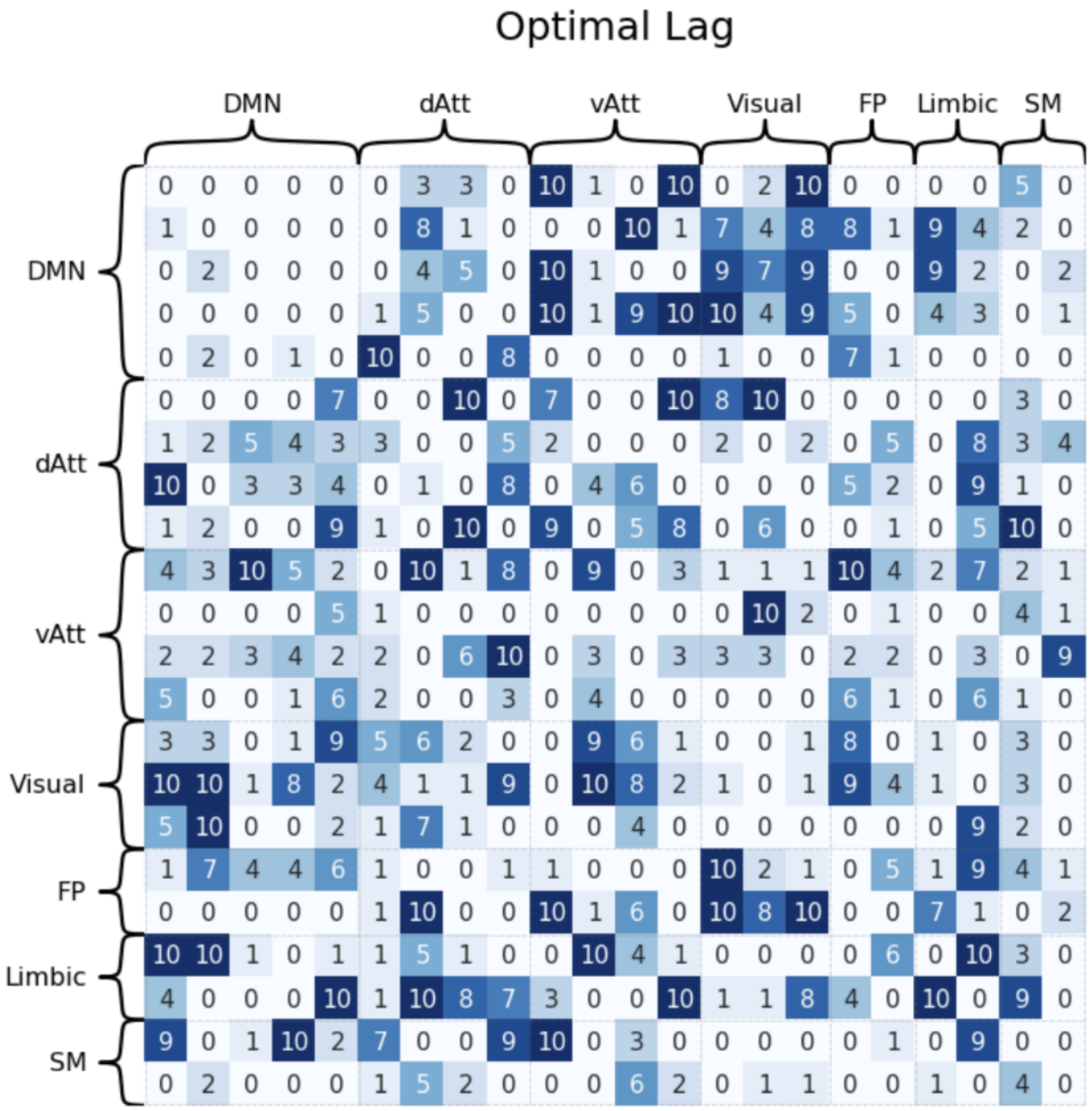

TODO

- Multivariate simulations