Stochastic Compartment Modeling

ODE-SDE recap and Simulations

Jaewon Choi[1, 2]

[1]Graduate Internship Program
Purdue IE

[2]Dept. of Industrial and Management Engineering
POSTECH

Summer Graduate Internship, 2025

Table of Contents

Table of Contents

Derelict Dynamics — jump $\ell = -1$

$$
\begin{aligned}
\Pr\Big(\hat{X}_{V,D}(t+h) = D-1 \mid \hat{X}_V(t) = (S,D,N)\Big) &= \binom{D}{1}\Big(\sigma_{DN} v \tfrac{N}{V} h\Big)\Big(1 - \sigma_{DN} v \tfrac{N}{V} h\Big)^{D-1} \\
&\quad + \binom{D}{1}\Big(\alpha\sigma_{DS} v \tfrac{S}{V} h\Big)\Big(1 - \alpha\sigma_{DS} v \tfrac{S}{V} h\Big)^{D-1} \\
&\quad + \Big(\tfrac{v}{d} D\Big) h + o(h) \\
&= Vh\Big(\sigma_{DN} v \tfrac{N}{V}\tfrac{D}{V} + \alpha\sigma_{DS} v \tfrac{S}{V}\tfrac{D}{V}\Big) + \tfrac{v}{d} D\, h + o(h).
\end{aligned}
$$

Jump size $= -1$: Derelict–Debris, Derelict–Satellite collisions, and drag-out removal.

S, D, N: object counts (satellite, derelict, debris)
α: proportion of encounters that satellite fails to avoid
$\sigma_{ij} = (r_i + r_j)^2$: squared impact parameter
v: relative velocity (km/s)
V: shell volume
h: small time increment
d: shell thickness
$\frac{v}{d} D$: drag rate to lower shell

Derelict Dynamics — jump $\ell = +1$

$$\begin{aligned}
\Pr\Big(\hat{X}_{V,D}(t+h) = D+1 \mid \hat{X}_V(t) = (S,D,N)\Big) &= \binom{S}{1}\Big(\delta\sigma_{SN} v \, \tfrac{N}{V} h\Big)\Big(1 - \delta\sigma_{SN} v \, \tfrac{N}{V} h\Big)^{S-1} \\
&\quad + \binom{S}{1}\Big(\delta\sigma_{DS} v \, \tfrac{D}{V} h\Big)\Big(1 - \delta\sigma_{DS} v \, \tfrac{D}{V} h\Big)^{S-1} \\
&\quad + \Big(\tfrac{v^+}{d^+} D^+\Big) h + o(h) \\
&= Vh\Big(\delta\sigma_{SN} v \, \tfrac{N}{V} \tfrac{S}{V} + \delta\sigma_{DS} v \, \tfrac{D}{V} \tfrac{S}{V}\Big) + \tfrac{v^+}{d^+} D^+ h + o(h).
\end{aligned}$$

Jump size $= +1$: degenerating S–N, S–D collisions, and derelicts drifting in from the upper shell.

S, D, N: object counts (satellite, derelict, debris)
α: proportion of encounters that satellite fails to avoid
δ: proportion of encounters that end up as *disabling*
$\sigma_{ij} = (r_i + r_j)^2$: squared impact parameter
v: relative velocity (km/s)
V: shell volume
h: small time increment
d: shell thickness
$\frac{v^+}{d^+} D^+$: drag rate from upper shell

Derelict Dynamics - jump $\ell = -2$

$$\Pr\Big(\hat{X}_{V,D}(t+h) = D-2 \;\Big|\; \hat{X}_V(t) = (S,D,N)\Big) = \binom{D}{1}\left(\sigma_{DD}\, v\, \tfrac{D}{V} h\right)\left(1 - \sigma_{DD}\, v\, \tfrac{D}{V} h\right)^{D-1}$$
$$= Vh\left(\sigma_{DD} v \frac{D}{V}\frac{D}{V}\right) + o(h)$$

Jump size (-2): Derelict–Derelict catastrophic collision

Derelict Evolution: Markov Jump Process $\to$ ODE

Markov jump process

$$\hat{X}_{V,D}(t) = \hat{X}_{V,D}(0) + \sum_{i\in\{-2,-1,+1\}} e_i\, N_i\Big(V\int_0^t \lambda_i\Big(\frac{\hat{X}_{V(u)}}{V}\Big)\,du\Big),$$

N_i are independent unit-rate Poisson clocks; e_i the jump size.

Pre-limit drift $(V < \infty)$

$$\frac{dX_{V,D}}{dt} = -V\Big(\sigma_{DN}v\,\frac{D}{V}\frac{N}{V} + \alpha\sigma_{DS}v\,\frac{D}{V}\frac{S}{V}\Big) + V\Big(\delta\sigma_{SN}v\,\frac{S}{V}\frac{N}{V} + \delta\sigma_{SD}v\,\frac{S}{V}\frac{D}{V}\Big) - 2V\sigma_{DD}v\left(\frac{D}{V}\right)^2 - \frac{v}{d}D + \frac{v^+}{d^+}D^+.$$

Limit ODE $(V \to \infty)$

$$\boxed{\frac{dx_D}{dt} = -\sigma_{DN}v\,x_Dx_N - \alpha\sigma_{DS}v\,x_Dx_S + \delta\sigma_{SN}v\,x_Sx_N + \delta\sigma_{SD}v\,x_Sx_D - 2\sigma_{DD}v\,x_D^2 - \frac{v}{d}x_D + \frac{v^+}{d^+}x_D^+}$$

where x_d is the population density of Derelicts

Table of Contents

Satellite Dynamics — jump $\ell = -1$

$$
\begin{aligned}
\Pr\Big(\hat{X}_{V,S}(t+h) = S-1 \;\Big|\; \hat{X}_V(t) = (S,D,N)\Big) = \\
&\underbrace{\delta\binom{S}{1}\Big(\sigma_{NS}v\tfrac{N}{V}h\Big)\Big(1-\sigma_{NS}v\tfrac{N}{V}h\Big)^{S-1}}_{\text{disabling (S–N)}} + \underbrace{\delta\binom{S}{1}\Big(\sigma_{DS}v\tfrac{D}{V}h\Big)\Big(1-\sigma_{DS}v\tfrac{D}{V}h\Big)^{S-1}}_{\text{disabling (S–D)}} \\
&+ \underbrace{\alpha\binom{S}{1}\Big(\sigma_{NS}v\tfrac{N}{V}h\Big)\Big(1-\sigma_{NS}v\tfrac{N}{V}h\Big)^{S-1}}_{\text{lethal (S–N)}} + \underbrace{\alpha\binom{S}{1}\Big(\sigma_{DS}v\tfrac{D}{V}h\Big)\Big(1-\sigma_{DS}v\tfrac{D}{V}h\Big)^{S-1}}_{\text{lethal (S–D)}} \\
&= Vh\Big(\delta\,\sigma_{NS}v\tfrac{N}{V}\tfrac{S}{V} + \delta\,\sigma_{DS}v\tfrac{D}{V}\tfrac{S}{V} + \alpha\,\sigma_{NS}v\tfrac{N}{V}\tfrac{S}{V} + \alpha\,\sigma_{DS}v\tfrac{D}{V}\tfrac{S}{V}\Big) + o(h).
\end{aligned}
$$

Jump size (-1): Satellite–Debris and Satellite–Derelict lethal/degenerating collisions

Raw encounter rate $\zeta_j = \sigma_{Sj}v\,\dfrac{S}{V}\dfrac{X_j}{V}$.

α: proportion of encounters that both satellites fails to avoid

δ: proportion of encounters that end up as *disabling*

$\alpha\zeta_j$ = lethal intensity

$\delta\zeta_j$ = disabling intensity

$(1-\alpha-\delta)\zeta_j$ = no effect

Satellite Dynamics — jump $\ell = -2$

$$\Pr\Big(\hat{X}_{V,S}(t+h) = S{-2} \;\Big|\; \hat{X}_V(t) = (S,D,N)\Big) = \binom{S}{1}\left(\alpha_a \sigma_{SS}\, v\, \tfrac{S}{V} h\right)\left(1 - \alpha_a \sigma_{SS}\, v\, \tfrac{S}{V} h\right)^{S-1}$$

$$= Vh\left(\alpha_a \sigma_{SS} v \frac{S}{V}\frac{S}{V}\right) + o(h)$$

Jump size (-2): Satellite–Satellite collision

Raw encounter rate $\zeta_j = \sigma_{Sj} v \dfrac{S}{V}\dfrac{X_j}{V}$.

α: proportion of encounters that both satellites fails to avoid

Satellite Dynamics — launch (jump $\ell = +1$)

$$\Pr\Big(\hat{X}_{V,S}(t+h) = S+1 \;\Big|\; \hat{X}_V(t) = (S,D,N)\Big) = V\,\lambda\,h \;+\; o(h).$$

- Each launch provider is modeled by an independent Poisson clock.
- λ: aggregate launch rate per unit volume.

Jump size $= +1$: insertion of a brand-new active satellite.

Satellite Evolution: Markov Jump Process and ODE Drift

Markov jump process

$$\hat{X}_{V,S}(t) = \hat{X}_{V,S}(0) + \sum_{i \in \{-2,-1,+1\}} e_i \, N_i\Big(V \int_0^t \lambda_i\Big(\tfrac{\hat{X}_V(u)}{V} \Big) du \Big).$$

Pre-limit drift

$$\frac{dX_{V,S}}{dt} = -V\Big((\delta+\alpha)\sigma_{NS} v \, \tfrac{N}{V} \tfrac{S}{V} + (\delta+\alpha)\sigma_{DS} v \, \tfrac{D}{V} \tfrac{S}{V} \Big) - 2V\alpha_a \sigma_{SS} v \, \tfrac{S}{V} \tfrac{S}{V} + \lambda.$$

Limit ODE

$$\frac{dx_S}{dt} = \lambda - (\delta+\alpha)\sigma_{NS} v \, x_N x_S - (\delta+\alpha)\sigma_{DS} v \, x_D x_S - 2\alpha_a \sigma_{SS} v \, x_S^2.$$

Table of Contents

Debris Dynamics — catastrophic jump $+n_{f,c}$

$$\Pr\Big(\hat{X}_{V,N}(t+h) = N + n_{f,c} \ \Big|\ \hat{X}_V(t) = (S,D,N)\Big) = \binom{S}{1}\Big(\alpha\,\sigma_{DS}\,v\,\tfrac{D}{V}h\Big)\Big(1-\alpha\,\sigma_{DS}\,v\,\tfrac{D}{V}h\Big)^{S-1}$$
$$+\binom{S}{1}\Big(\alpha_a\,\sigma_{SS}\,v\,\tfrac{S}{V}h\Big)\Big(1-\alpha_a\,\sigma_{SS}\,v\,\tfrac{S}{V}h\Big)^{S-1}$$
$$+\binom{D}{1}\Big(\sigma_{DD}\,v\,\tfrac{D}{V}h\Big)\Big(1-\sigma_{DD}\,v\,\tfrac{D}{V}h\Big)^{D-1}$$
$$= Vh\Big(\alpha\,\sigma_{DS}v\tfrac{D}{V}\tfrac{S}{V} + \alpha_a\,\sigma_{SS}v\tfrac{S}{V}\tfrac{S}{V} + \sigma_{DD}v\tfrac{D}{V}\tfrac{D}{V}\Big) + o(h)$$

Jump size $(+n_{f,c})$: Catastrophic S–D, S–S, or D–D collision.

Debris Dynamics — non-catastrophic jump $+n_{f,nc}$

$$\begin{aligned}
\Pr\Big(\hat{X}_{V,N}(t+h) = N + n_{f,nc} \;\Big|\; \hat{X}_V(t) = (S, D, N)\Big) &= \binom{D}{1}\left(\sigma_{DN}\, v\, \tfrac{N}{V} h\right)\left(1 - \sigma_{DN}\, v\, \tfrac{N}{V} h\right)^{D-1} \\
&\quad + \binom{S}{1}\left(\alpha\, \sigma_{NS}\, v\, \tfrac{N}{V} h\right)\left(1 - \alpha\, \sigma_{NS}\, v\, \tfrac{N}{V} h\right)^{S-1} \\
&\quad + \binom{N}{1}\left(\sigma_{NN}\, v\, \tfrac{N}{V} h\right)\left(1 - \sigma_{NN}\, v\, \tfrac{N}{V} h\right)^{N-1} \\
&= Vh\Big(\sigma_{DN} v \tfrac{D}{V} \tfrac{N}{V} + \alpha\, \sigma_{NS} v \tfrac{S}{V} \tfrac{N}{V} + \sigma_{NN} v \tfrac{N}{V} \tfrac{N}{V}\Big) + o(h)
\end{aligned}$$

Jump size ($+n_{f,nc}$): Non-catastrophic collisions.

Debris Evolution: Markov Jump Process and ODE Drift

Markov jump process

$$\hat{X}_{V,N}(t) = \hat{X}_{V,N}(0) + \sum_{i \in \{+n_{f,c},\, +n_{f,nc},\, -1\}} e_i\, N_i\Big(V \int_0^t \lambda_i\Big(\frac{\hat{X}_{V(u)}}{V}\Big)\, du\Big).$$

Pre-limit drift

$$\begin{aligned}\frac{dX_{V,N}}{dt} &= n_{f,nc}\Big(\sigma_{DN} v\, \frac{D}{V}\frac{N}{V} + \alpha\sigma_{NS} v\, \frac{N}{V}\frac{S}{V} + \sigma_{NN} v\, \frac{N}{V}\frac{N}{V}\Big) \\ &\quad + n_{f,c}\Big(\alpha\sigma_{DS} v\, \frac{D}{V}\frac{S}{V} + \alpha_a\sigma_{SS} v\, \frac{S}{V}\frac{S}{V} + \sigma_{DD} v\, \frac{D}{V}\frac{D}{V}\Big) - V\eta\, \frac{N}{V} + V\eta^+ \frac{N^+}{V}.\end{aligned}$$

Limit ODE

$$\begin{aligned}\frac{dx_N}{dt} &= n_{f,nc}\big(\sigma_{DN} v\, x_D x_N + \alpha\sigma_{NS} v\, x_N x_S + \sigma_{NN} v\, x_N^2\big) \\ &\quad + n_{f,c}\big(\alpha\sigma_{DS} v\, x_D x_S + \alpha_a\sigma_{SS} v\, x_S^2 + \sigma_{DD} v\, x_D^2\big) - \eta\, x_N + \eta^+ x_N^+.\end{aligned}$$

Table of Contents

Pre-limit & Law of Large Numbers

Population density $X_{V,Q} = \frac{\hat{X}_{V,Q}}{V}$ of species Q with volume V:

$$X_{V,Q}(t) = X_{V,Q}(0) + \sum_{\ell} \frac{1}{V}\, \tilde{Y}_{\ell,Q}\Big(V \int_0^t \beta_{\ell,Q}\big(X_V(u)\big)\, du \Big) + \int_0^t F_Q\big(X_V(u)\big)\, du.$$

where $\tilde{Y}_{\ell,Q}(u) = Y_{\ell,Q}(u) - u$: Poisson process centered at its expectation
Kurtz's Theorem: As $V \to \infty$,

$$X_{V,Q}(t) \longrightarrow X_Q(t) = X_Q(0) + \int_0^t F_Q\big(X(u)\big)\, du.$$

where

$$F_Q(x) = \sum_{\ell} \ell\, \beta_{\ell,Q}(x).$$

$F_Q(x)$: net change in the population
ℓ: jump size
$\beta_{\ell,Q}(x)$: rate at which an event occurs

Finite-volume Deviation $H_{V,Q}$

Ethier&Kurtz: Define the $\sqrt{V}$-scaled deviation:

$$H_{V,Q}(t) = \sqrt{V}\,\big[X_{V,Q}(t) - X_Q(t)\big].$$

$$H_{V,Q} \to H_Q \quad \text{as} \quad V \to \infty$$

where the limit Gaussian process $H_Q(t)$ satisfies the integral relation

$$H_Q(t) = H_Q(0) + \sum_{\ell} \ell\, W_{\ell,Q}\Big(\int_0^t \beta_{\ell,Q}\big(X(u)\big)\,du\Big) + \int_0^t \partial F_Q\big(X(u)\big)^T\, H_Q(u)\,du.$$

where $W_{\ell,Q}$ are independent Brownian motions

Approximate Path $Z_{V,Q}(t)$

First-order approximation:

$$\tilde{Z}_{V,Q}(t) = X_Q(t) + \frac{1}{\sqrt{V}} H_Q(t).$$

Captures $\mathcal{O}(V^{-1/2})$ fluctuation around the deterministic path.

We approximate population density with $Z_{V,Q}$, asymptotically equivalent to $\tilde{Z}_{V,Q}$:

$$Z_{V,Q}(t) = X_{V,Q}(0) + \frac{1}{\sqrt{V}} \sum_{\ell} \ell \int_0^t \beta_{\ell,Q}^{1/2}(X(u))\, dW_{\ell,Q}(u)$$

$$+ \int_0^t F_Q\big(X(u)\big)\, du.$$

Adding Drag and Launch Terms

Extend $Z_{V,Q}$ by replacing β inside the integrals:

- **Collisions:** $\beta_{\ell,Q}(\cdot)$
- **Drag:** $\beta_{\mathrm{drag},Q}(\cdot)$ produces deterministic term $-\int_0^t \beta_{\mathrm{drag},Q}(Z(u))\,du$ and noise $-\int_0^t \beta_{\mathrm{drag},Q}^{1/2}(Z(u))\,dW_{\mathrm{drag},Q}(u)$
- **Launch:** rate λ adds $\int_0^t \lambda\,du$ and $\int_0^t \sqrt{\lambda}\,dW_{\mathrm{launch},S}(u)$
- **Post-Mission disposal:** rate $\frac{X_S(u)}{\Delta t}$ adds $\int_0^t \frac{X_S(u)}{\Delta t}\,du$ and $\int_0^t \sqrt{\frac{X_S(u)}{\Delta t}}\,dW_{\mathrm{pmd},S}(u)$

The full integral path captures collision, drag, launch, and PMD

Table of Contents

Shell Sizing

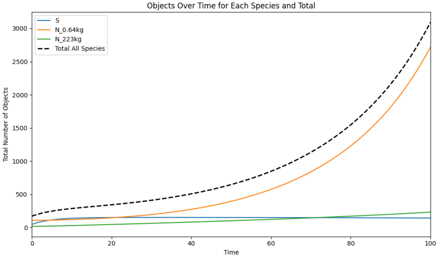
- **Academic research:** 10-50 km
- **Industry status:** 5-20km
- Lifson (Linares) paper: –5km
- Our research: – 2km
- Focusing on 500-700km altitude
- **Issues:** Initial population and launches

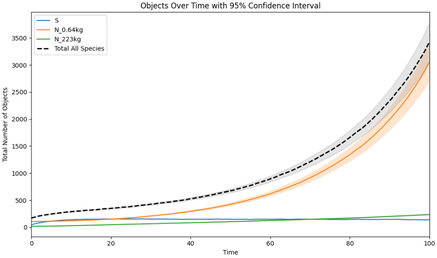
Table: Summary of Orbital Shell Strategies and References

Category	**Shell Interval**	**Altitude Range**
JASON Reports	Broad shell (700 km)	General LEO
MOCAT (D'Ambrosio et al.)	20-50 km	200-2000 km
SDC9 (Muciaccia et al.)	10 km, 50 km	625, 775, 850, 975, 1450 km
Fanto et al. (2023)	25 bins (unspecified)	500-2000 km
Lifson et al. (2024)	$\leq$ 5 km	600-630 km
Industry (Kuiper)	$\sim$ 20 km	590-630 km
SpaceX Starlink	5-20 km	340-360, 525-535, 604-614 km
Our Research	**–2 km**	Specific shells in LEO

Simulations (750m, SDE)

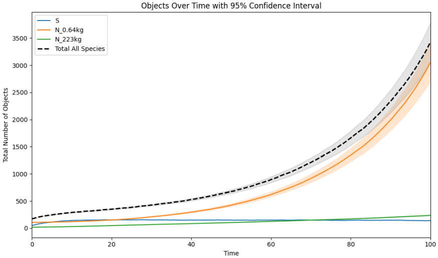
- **100-year simulation:** 100 year, 100000 timesteps ($\approx$ 9hrs)
- **3-shell case:** 2,500,000km^3 per shell ($\approx$ 750m between shells)
- Stochasticity accumulates in the SDE case, resulting in higher debris population

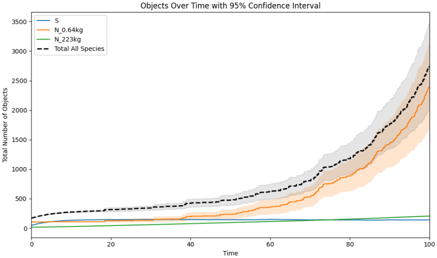




Simulations (750m, DES

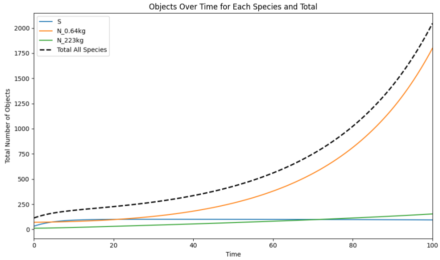
- **100-year simulation:** 100 year, 100000 timesteps ($\approx$ 9hrs)
- **3-shell case:** 2,500,000km^3 per shell ($\approx$ 750m between shells)
- DES sparks reactions after a lethal collision
- DES results in higher debris creation compared to ODE, but lower than SDE

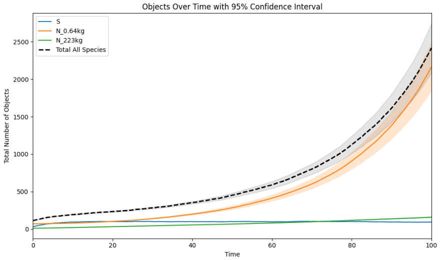




Simulations (500m, SDE)

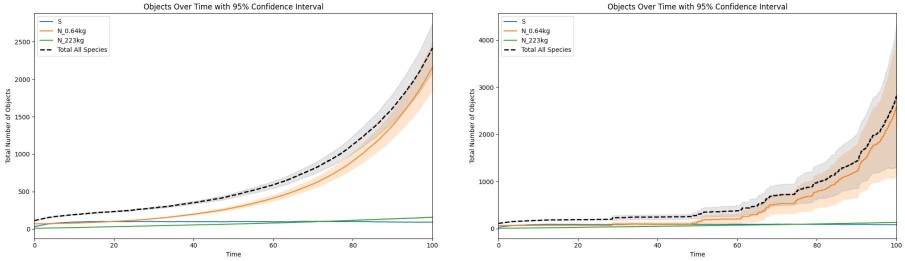
- **100-year simulation:** 100 year, 100000 timesteps ($\approx$ 9hrs)
- **3-shell case:** 1,650,000km^3 per shell ($\approx$ 500m between shells)
- SDE shows more variance in smaller V, and results in higher debris population than ODE
- SDE is closer to the DES case (continued...)





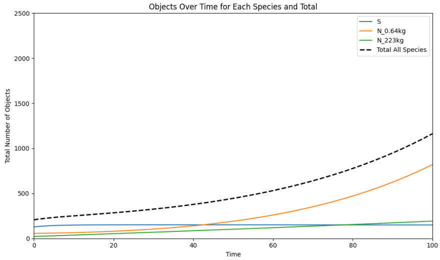
Simulations (500m, DES)

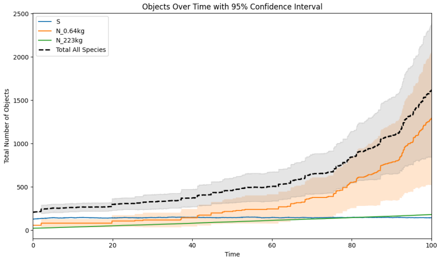
- **100-year simulation:** 100 year, 100000 timesteps ($\approx$ 9hrs)
- **3-shell case:** 1,650,000km^3 per shell ($\approx$ 500m between shells)
- Events occur when there are sufficient number of objects in the system
- Interactions rarely occur in $<$ 50y period, but 100y term is long enough to capture collisions and accumulation of debris
- DES results shows more variance in smaller V



Simulations (10 shells, ODE vs DES)

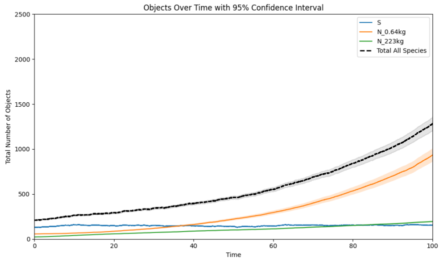
- **100-year simulation:** 100 year, 100000 timesteps ($\approx$ 9hrs)
- **10-shell case:** $1{,}650{,}000 km^3$ per shell ($\approx$ 500m between shells)
- Altitude range 515km - 520km
- Constant launch rate, about 3 satellites per shell – lower than previous simulations (30 satellites/shell)

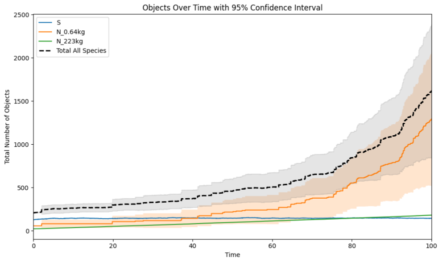




Simulations (10 shells, SDE vs DES)

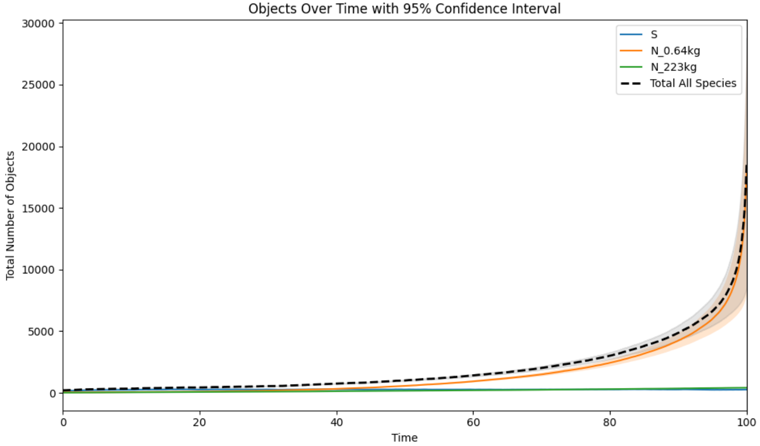
- **10-shell case:** 1,650,000km^3 per shell ($\approx$ 500m between shells)
- Altitude range 515km - 520km
- Constant launch rate, about 3 satellites per shell – lower than previous simulations (30 satellites/shell)
- SDE results show higher population than ODE, but it has lower population than DES





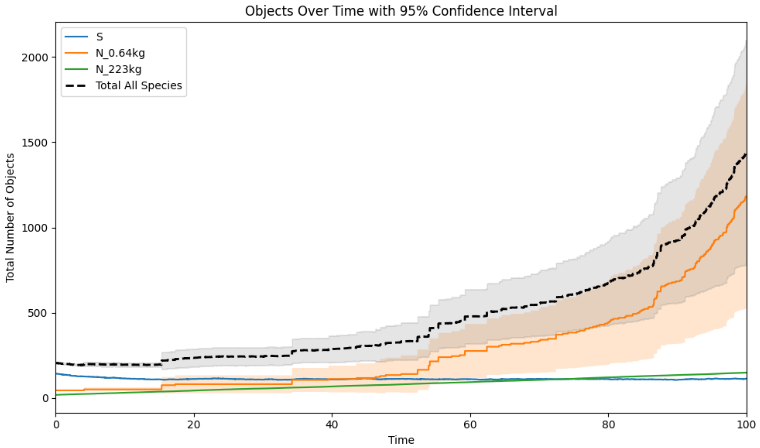
500 m $\times$ 20 shells case

- Population increase was so large that DES became extremely slow.



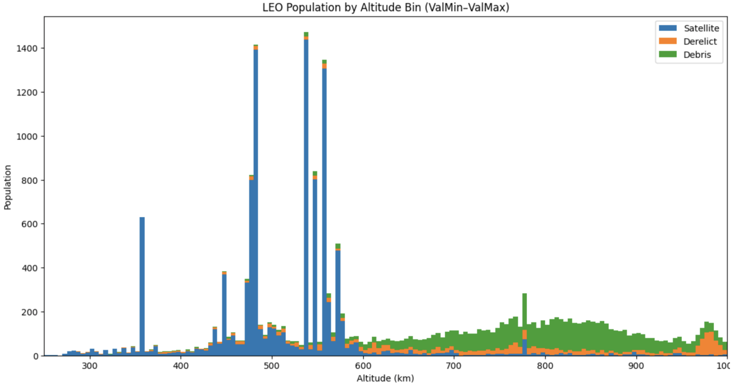
Currently running: **15-shell case (250 m per shell)**

- DES results show large fluctuations.
- SDE results are pending, expected to slightly underestimate compared to DES.



Initial Population of the Model

- For the initialization of the model, I referred to the dataset at:
  - `https://planet4589.org/space/stats/active.html`
- Extracted active objects in the range of 200–1000 km altitude.
- Plotted the population distribution by altitude.



Launch Rate: Current Issues

- Available information (FCC/ITU filings):
  - Provides capacity and sometimes target milestones.
  - Example:
    - Starlink: FCC approved 7,500 (Gen2) + filed 29,988 (no official milestones).
    - Kuiper: 3,236 (FCC approved), with 50% by 2026 and 100% by 2029 required.
- However, these are capacity/milestone numbers, not **launch rates**.
- In our simulations:
  - We have been using an *arbitrary constant rate*, e.g. $\sim$6 satellites per (km, year).
  - This rate is not derived from filings.
- Open Questions:
  - How to translate capacity + milestone data into a time-dependent launch rate?
  - Which assumptions best serve our goal (SDE vs. ODE vs. DES comparison)?

Table of Contents

Model Dynamics and Approximation

- $\hat{X}_{V,Q}(t)$: population of generic species Q in volume V at time t
- $X_{V,Q}(t) = \frac{\hat{X}_{V,Q}(u)}{V}$: density of Q in V at time t

$$\hat{X}_{V,Q}(t) = \hat{X}_{V,Q}(0) + \sum_{\ell} \ell \cdot Y_{\ell,Q}\left(V \int_0^t \beta_{\ell,Q}(\frac{\hat{X}_V(t)}{V})du\right)$$

where $\hat{X}_V(t) = (\hat{X}_{V,S}(t), \hat{X}_{V,D}(t), \hat{X}_{V,N}(t))$
for $Q \in \{S, D, N\}$

- Set $F_Q(X) = \sum_{\ell} \ell \beta_{\ell,Q}(X)$

$$X_{V,Q}(t) = X_{V,Q}(0) + \sum_{\ell} \ell \frac{1}{V} \tilde{Y}_{\ell,Q}\left(V \int_0^t \beta_{\ell,Q}(X_V(u))du\right) + \int_0^t F_Q(X_V(u))du$$

where $\tilde{Y}_{\ell,Q}(u) = Y_{\ell,Q}(u) - u$ (i.e., divide byV, apply definition of $\tilde{Y}$)

Pre-Limit and Limit Equations

- Suppose $X_{V,Q}$ satisfies

$$X_{V,Q}(t) = X_{V,Q}(0) + \sum_{\ell} \ell \cdot \frac{1}{V} \tilde{Y}_{\ell,Q}\left(V \int_0^t \beta_{\ell,Q}(X_V(u))du \right) + \int_0^t F_Q(X_V(u))du$$

and $\lim_{V\to\infty} X_V(0) = x_0$ (Pre-limit)

- Suppose X_Q satisfies

$$X_Q(t) = x_0 + \int_0^t F_Q(X(u))du$$

For every $t \geq 0$, $\lim_{V\to\infty} P(\sup_{0\leq s\leq t} |X_{V,Q}(s) - X_Q(s)|) = 0$ (Limit)

Scaling and Approximation

- Set $W^{(V)}_{\ell,Q}(u) = \frac{1}{\sqrt{V}}\tilde{Y}_{\ell,Q}(Vu)$ (scale $\tilde{Y}_\ell$ by $\frac{1}{\sqrt{V}}$)
- $W^{(V)}_{\ell,Q}(u) \to W_{\ell,Q}$
- Let $H_{V,Q}(t) = \sqrt{V}(X_{V,Q}(t) - X_Q(t))$
- Then $H_{V,Q}(t) = H_{V,Q}(0) + \sum_\ell \ell W^{(V)}_{\ell,Q}\left(\int_0^t \beta_{\ell,Q}(X_V(u))du\right) + \int_0^t \sqrt{V}\,(F_Q(X_V(u)) - F_Q(X(u)))\,du$ (Pre-limit)
- If we set $X_{V,Q} = X_Q + \frac{1}{\sqrt{V}}H_{V,Q}$,
- The limiting equation:

$$H_Q(t) = H_Q(0) - U_Q(t) + \int_0^t \partial F_Q(X(u))^T H_Q(u)du \qquad (\mathit{Limit})$$

Limit and Approximation of X

- $\tilde{Z}_{V,Q} = X_Q + \frac{1}{\sqrt{V}} H_Q$ (Approximation of X_V)

$$\tilde{Z}_{V,Q}(t) = X_Q(0) + \int_0^t F_Q(X(u))du$$

$$+ \frac{1}{\sqrt{V}} \left(H_Q(0) + U_Q(t) + \int_0^t \partial F_Q(X(u)) H_Q(u) du \right) \text{ by def}$$

$$= X_Q(0) + \frac{1}{\sqrt{V}} H_Q(0) + \frac{1}{\sqrt{V}} \sum_\ell \ell W_{\ell,Q} \left(\int_0^t \beta_{\ell,Q}(X(u))du \right)$$

$$+ \int_0^t \left(F_Q(X(u)) + \frac{1}{\sqrt{V}} (\partial F_Q(X(u)) \cdot \sqrt{V} (\tilde{Z}_{V,Q}(u) - X_Q(u))) \right) du$$

$$= X_Q(0) + \frac{1}{\sqrt{V}} H_Q(0) + \frac{1}{\sqrt{V}} \sum_\ell \ell \int_0^t \beta_{\ell,Q}^{1/2}(X(u)) dW_{\ell,Q}(u)$$

$$+ \int_0^t \left(F_Q(X(u)) + \partial F_Q(X(u))(\tilde{Z}_{V,Q}(u) - X_Q(u)) \right) du \quad \left(\frac{1}{\sqrt{V}} H(0) \to 0 \right)$$

Equivalent Approximations for XV

- $\tilde{Z}_{V,Q}(t) = X_Q(0) + \frac{1}{\sqrt{V}} \sum_\ell \ell \int_0^t \beta_{\ell,Q}^{1/2}(X(u))dW_{\ell,Q}(u) + \int_0^t \Big(F_Q(X(u)) + \partial F_Q(X(u))(\tilde{Z}_{V,Q}(u) - X_Q(u)) \Big) du$
- $Z_{V,Q}(t) = X_{V,Q}(0) + \frac{1}{\sqrt{V}} \sum_\ell \ell \int_0^t \beta_{\ell,Q}^{1/2}(Z_V(u))dW_{\ell,Q}(u) + \int_0^t F_Q(Z_V(u))du$
- (Theorem 3.2) $\tilde{Z}_{V,Q}(t)$ and $Z_{V,Q}(t)$ are asymptotically equivalent.

Ito Isometry $\rightarrow$ Quadratic Variations?

- $W_\ell\left(\int_0^t \beta_\ell(X(u))du\right) \rightarrow \int_0^t \beta_\ell^{1/2}(X(u))dW_\ell(u)$
- Suppose $\beta_\ell^{1/2}(X(u))$ is locally square integrable deterministic function, then $\int_0^t \beta_\ell^{1/2}(X(u))dW_\ell(u)$ is an Ito integral
- applying Ito isometry,
  i.e.,$\left(\int_0^t E[\beta_\ell^{1/2}(X(u))]dW_\ell(u)\right)^2 = E[\int_0^t \beta_\ell(X(u))du]$, and since $\int_0^t \beta_\ell(X(u))du$ is deterministic,
- $Var[\int_0^t \beta_\ell^{1/2}(X(u))dW_\ell(u)] =$

  $E[\left(\int_0^t \beta_\ell^{1/2}(X(u))dW_\ell(u)\right)^2] - \left(E[\int_0^t \beta_\ell^{1/2}(X(u))dW_\ell(u)]\right)^2$

  $= \int_0^t \beta_\ell(X(u))du$
- with mean 0 and variance $\int_0^t \beta_\ell(X(u))du$,
  $W_\ell\left(\int_0^t \beta_\ell(X(u))du\right)$ and $\int_0^t \beta_\ell^{1/2}(X(u))dW_\ell(u)$ are equal in distribution

Derelict Dynamics

- $\hat{X}_{V,D}(t)$: population of D in V at time t
- $X_{V,D}(t) = \frac{\hat{X}_{V,D}(t)}{V}$: density of D in V at time t
- $H_{V,D}(t) = \sqrt{V} \cdot (X_{V,D}(t) - X_D(t))$
- $\tilde{Z}_{V,D} = X_D + \frac{1}{\sqrt{V}} H_D$ (Approximation of $X_{V,D}$)
- We can approximate $X_{V,D}$ with $Z_{V,D}$ (Theorem 3.1)
- $Z_{V,D}(t) =$
  $Z_{V,D}(0) + V^{-1/2} \sum_\ell \int_0^t \beta_{\ell,D}^{1/2}(Z_V(s))\, dW_{\ell,D}(s) + \int_0^t F_D(Z_V(s))\, ds$
- $dZ_{V,D} = V^{-1/2} \sum_\ell \beta_{\ell,D}^{1/2}(Z_V(t)) dW_{\ell,D}(t) + F_D(Z_V(t))dt$
  where $l \in \{-2, -1, 1\}$, $F_D(x) = \sum_\ell \ell \beta_{\ell,D}(x)$
  $\beta_{(-2),D}(Z_V) = \sigma_{DD} v Z_D Z_D$
  $\beta_{(-1),D}(Z_V) = \sigma_{DN} v Z_D Z_N + \alpha \sigma_{DS} v Z_D Z_S$
  $\beta_{(+1),D}(Z_V) = \delta \sigma_{SN} v Z_S Z_N + \delta \sigma_{SD} v Z_S Z_D$

Collision and Atmospheric Drag on Species Q

Considering collision and atmospheric drag:

$$\hat{X}_{V,Q}(t) = \hat{X}_{V,Q}(0) + \sum_{\ell} \ell \cdot Y_{\ell,Q}\left(V\int_0^t \beta_{\ell,Q}(\frac{\hat{X}_V(t)}{V})du\right)$$

$$-Y^{+}_{\text{drag}}\left(V\int_0^t \beta^{+}_{\text{drag},Q}(\frac{\hat{X}_{V,Q}(u)}{V})du\right) + Y^{-}_{\text{drag}}\left(V\int_0^t \beta^{-}_{\text{drag},Q}(\frac{\hat{X}_{V,Q}(u)}{V})du\right)$$

$$X_{V,Q}(t) = X_{V,Q}(0) + \sum_{\ell} \frac{1}{V}\ell Y_{\ell,Q}\left(V\int_0^t \beta_{\ell,Q}(X_V(u))du\right)$$

$$-\frac{1}{V}Y^{+}_{\text{drag}}\left(V\int_0^t \beta^{+}_{\text{drag},Q}(X_{V,Q}(u))du\right) + \frac{1}{V}Y^{-}_{\text{drag}}\left(V\int_0^t \beta^{-}_{\text{drag},Q}(X_{V,Q}(u))du\right)$$

where:

- $Y_{\ell,Q}(t)$: Unit-rate Poisson process of collisions
- $Y_{\text{drag}}(t)$: Poisson process of atmospheric drag.
- $\beta_{\ell,Q}(x)$: Collision rate, $\beta_{\text{drag},Q}(x)$: Drag rate $(= \frac{v}{d}x)$

Limit Process for $X_Q(t)$

$V \to \infty$,

$$X_Q(t) = X_Q(0) + \int_0^t F_Q(X(u))du$$

$$- \int_0^t \beta^{+}_{\text{drag},Q}(X_Q(u))du + \int_0^t \beta^{-}_{\text{drag},Q}(X_Q(u))du$$

where:

- $F_Q(X) = \sum_\ell \ell \beta_{\ell,Q}(X)$: Deterministic change due to collisions
- $\beta_{\text{drag},Q}(X_Q(u))$: change of species due to atmospheric drag.

$H_{V,Q}(t)$ and $H_Q(t)$

- Define $H_{V,Q}(t) = \sqrt{V}\,(X_{V,Q}(t) - X_Q(t))$
- The limit process $H_Q(t)$:

$$H_Q(t) = H_Q(0) + \sum_{\ell} \ell W_{\ell,Q}\left(\int_0^t \beta_{\ell,Q}(X(u))du\right)$$

$$-W_{drag,Q}\left(\int_0^t \beta^+_{drag,Q}(X_Q(u))du\right) + W_{drag,Q}\left(\int_0^t \beta^-_{drag,Q}(X_Q(u))du\right)$$

$$-\int_0^t \left(\partial F_Q(X(u)) - \partial\beta^+_{\mathrm{drag},Q}(X_Q(u)) + \partial\beta^-_{\mathrm{drag},Q}(X_Q(u))\right) H_Q(u)du$$

- where $W^{(V)}_{\ell,Q}(u) = \frac{1}{\sqrt{V}}\tilde{Y}_{\ell,Q}(Vu)$, $W^{(V)}_{drag,Q}(u) = \frac{1}{\sqrt{V}}\tilde{Y}_{drag,Q}(Vu)$
- $W^{(V)}_{\ell,Q}(u) \to W_{\ell,Q}$, $W^{(V)}_{drag,Q}(u) \to W_{drag,Q}$

Approximating $X_{V,Q}(t)$ with $Z_{V,Q}(t)$

- The approximation $\tilde{Z}_{V,Q}(t)$: $\tilde{Z}_{V,Q}(t) = X_Q(t) + \frac{1}{\sqrt{V}} H_Q(t)$
- Equivalent approximation $Z_{V,Q}(t)$:

$$Z_{V,Q}(t) = X_Q(0) + \frac{1}{\sqrt{V}}\Big(\sum_{\ell} \ell \cdot W_{\ell,Q}\left(\int_0^t \beta_{\ell,Q}(X(u))\,du\right)$$

$$-W^{+}_{\text{drag}}\left(\int_0^t \beta^{+}_{\text{drag},Q}(X_Q(u))\,du\right) + W^{-}_{\text{drag}}\left(\int_0^t \beta^{-}_{\text{drag},Q}(X_Q(u))\,du\right)\Big)$$

$$+\int_0^t \left(F_Q(X(u)) - \beta^{+}_{\text{drag},Q}(X_Q(u)) + \beta^{-}_{\text{drag},Q}(X_Q(u))\right)\,du$$

Approximating $X_{V,Q}(t)$ with $Z_{V,Q}(t)$ ctd.

- The approximation considering collisions and drag:

$$Z_{V,Q}(t) = X_{V,Q}(0) + \frac{1}{\sqrt{V}} \sum_{\ell} \ell \int_0^t \beta_{\ell,Q}^{1/2}(Z_V(u)) dW_{\ell,Q}(u)$$

$$+\frac{1}{\sqrt{V}} \int_0^t \left(-(\beta^{+}_{drag,Q}(Z_{V,Q}(u)))^{1/2} + (\beta^{-}_{drag,Q}(Z_{V,Q}(u)))^{1/2} \right) dW_{drag,Q}(u)$$

$$+ \int_0^t \left(F_Q(Z_V(u)) - \beta^{+}_{\text{drag},Q}(Z_{V,Q}(u)) + \beta^{-}_{\text{drag},Q}(Z_{V,Q}(u)) \right) du$$

Discrete-Event Simulations

- Events spark only after lethal collisions S-S ,S-D occur, creating a large number of Debris.
- Shell volume 1,000 km^3 (2m between shells): rarely experiences collisions – active satellite accumulates.
- Shell

Discrete-Event Simulations: Larger V

- Events spark only after lethal collisions S-S ,S-D occur, creating a large number of Debris.
- Shell volume 100,000 km^3 (2km between shells): Debris steeply grows after a lethal collision.

Discrete-Event Simulations: Larger V

- Shell volume 100,000 km^3 (2km between shells):
- Differs from SDE