A Brief Introduction to Half-Iteration and Null-Iteration, from Set Theory to Continuous Functions

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The purpose of this paper is to provide a first step in a series of systematic descriptions of the problem of fractional iteration, a generalization of iterated functions to non-integer quantities of iteration. The three equations traditionally used to describe a broader set of related functional-analytic problems, the Abel equation, Schröder's equation, and Böttcher's equation, are not discussed here, but will be referenced in further papers exploring solutions using niche approaches.

This paper will begin with a basic definition of the half-iterate of a function, using first functions on finite sets. It will then extend this to countable infinite sets, then comment on the larger problem of half-iteration of functions on uncountable infinite sets, including how half-iterates can be constructed for discontinuous functions. Finally, the notion of a null-iterate will be discussed.

Set-Theoretic Definition of a Half-Iterate

Suppose that the nth iteration (or iterate) of a function f is denoted by f^n,

$$f^n(x) = f\left(f^{n-1}(x)\right)$$

with additional properties

$$f^a f^b = f^{a+b}$$

$$(f^a)^b = f^{ab}$$

where $f^a f^b$ is a function of x defined by $f^a\left(f^b(x)\right)$, known as the composition of f^a on f^b.

For a function that maps a finite set onto itself, it is easy to determine the second iterate $n = 2$. Since a function can be defined as a set of ordered pairs, an example looks like this

$$f: S \mapsto S$$

$$S = \{1, 2, 3, 4\}$$

$$f(1) = 2, f(2) = 3, f(3) = 4, f(4) = 1$$

therefore

$$f = \{(1,2), (2,3), (3,4), (4,1)\}$$

taking

$$g = f^2$$

$$g(1) = f\left(f(1)\right) = f(2) = 3$$

etc. yields

$$g = \{(1,3), (2,4), (3,1), (4,2)\}$$

$$g: S \mapsto S$$

In this example, f could be called the half-iterate of g, since f is only "half" an iteration of g.

Suppose instead of deriving the second iterate f^2 we were to derive the half-iterate (also known as the functional square root) $h = f^{\frac{1}{2}}$, and we were to assume $h: S \mapsto S$. Since $h(h(x)) = f(x)$, we know that for each ordered pair (a, b) in f corresponding to $f(a) = b$, there must be ordered pairs (a, c) and (c, b) in h, since $f(a) = h(h(a)) = h(c) = b$. For h to map S onto itself, c must be in S.

In this case, $h(h(1)) = 2$, and since making $h(1) = 1$ or $h(2) = 2$ would require that $f(1) = 1$ or $f(2) = 2$, $h(1)$ cannot be either 1 or 2 and must be either 3 or 4. The choice here is arbitrary, so let's take $h(1) = 4$. Then we know $f(1) = h(h(1)) = h(4) = 2$, so now we have two ordered pairs already for h:

$$h = \{(1,4), (4,2), \dots\}$$

Now, we also further know $f(4) = h(h(4)) = h(2) = 1$, so we can add (2,1).

$$h = \{(1,4), (4,2), (2,1), \dots\}$$

By now, a problem is apparent. If $h(1) = 4$ and $h(2) = 1$, then $h(h(2)) = h(1) = 4$, but $f(2) = 3$, meaning that h cannot be the half-iterate of f (it is trivial to show that the same problem would arise if $h(1) = 3$ were chosen instead). The constraint of $h: S \mapsto S$ therefore is impossible.

A clue to why this is can be found in the second iterate, $g = f^2$, from before. Recall that

$$g = \{(1,3), (2,4), (3,1), (4,2)\}$$

It is clear that, in fact, g maps disjoint subsets $S_1 = \{1,3\}$ and $S_2 = \{2,4\}$ only onto themselves:

$$g: S_1 \mapsto S_1$$

$$g: S_2 \mapsto S_2$$

And, in fact,

$$f: S_1 \mapsto S_2$$

$$f: S_2 \mapsto S_1$$

Performing one more iteration, one can see that

$$g^2 = f^4 = \{(1,1), (2,2), (3,3), (4,4)\}$$

What this illustrates is for bijective functions on finite sets with an even number of elements, iteration of that function eventually yields an identity function $f(x) = x$. This holds for finite sets with an odd number of elements, as shown below.

$$S = \{1,2,3\}$$

$$f = \{(1,2), (2,3), (3,1)\}$$

$$f^2 = \{(1,3), (2,1), (3,2)\}$$

$$f^3 = \{(1,1), (2,2), (3,3)\}$$

$$f = f^4 = f^7 \ldots$$

$$f^2 = f^5 = f^8 \ldots$$

$$f^{1/2} = f^2$$

Since f^3 is an identity function, $f = f^3 = f^4$. Furthermore, since $f = f^{1+0} = f^1 f^0$, we automatically know f^0 is an identity function. Therefore, for this system, $f^3 = f^0$.

Recall the first function described

$$f = \{(1,2), (2,3), (3,4), (4,1)\}$$

and its iterates

$$f^2 = \{(1,3), (2,4), (3,1), (4,2)\}$$

$$(f^2)^2 = f^4 = \{(1,1), (2,2), (3,3), (4,4)\}$$

Notice how

$$f: \{1,2,3,4\} \mapsto \{1,2,3,4\}$$

$$f^2: \{1,3\} \mapsto \{1,3\}$$

$$f^2: \{2,4\} \mapsto \{2,4\}$$

$$f^4: \{a\} \mapsto \{a\} \text{ for } a \in \{1,2,3,4\}$$

For each iteration, f^n maps at least one set S_n onto itself as a successor function on S_n. S_n is half the size of S_{n-1}. Suppose a function μ can be defined as

$$\mu(f^n) = |S_n|$$

This gives $\mu(f) = 4$, $\mu(f^2) = 2$, and $\mu(f^4) = 1$. For these functions, we then see $\mu(f^n) = 2\mu(f^{2n})$.

Remembering that $f^{1/2}$ is a set where for each ordered pair $(a, b) \in f$ there are ordered pairs (a, c) and (c, b) in $f^{1/2}$. However, this is not true for odd-cardinality sets, as can be seen in the other previous example

$$f = \{(1,2), (2,3), (3,1)\}$$

$$f^2 = \{(1,3), (2,1), (3,2)\}$$

$$\mu(f) = \mu(f^2)$$

Extension to Countable Infinite Sets

Suppose $\mathbb{N}$ is the set of natural numbers $0,1,2,3,4, \ldots$. There is one obvious successor function on $\mathbb{N}$ that can be represented as $s(x) = x + 1$. Since this is an infinite set, the successor function cannot "wrap" back around once it reaches the end of the set like it could in earlier examples. Therefore, if $s^2(x) = x + 2$, starting with 0, s^2 will never reach any odd number (s^2 would instead be the successor function on the set of even natural numbers). Assume that there is a half-iterate $s^{1/2}: \mathbb{N} \mapsto \mathbb{N}$. For each $(n, n+1)$ in s, there must be $(n, m), (m, n+1) \in s^{1/2}$ with $m \in \mathbb{N}$. Knowing $(m, m+1) \in s$,

$$(n, m), (m, n+1), (n+1, m+1) \in s^{1/2}$$

Suppose A is the sequence generated by iteratively applying $s^{1/2}$, starting at 0.

$$A = (a_i) = \left(0, a_{1,} 1, a_{3,} 2, a_{5,} 3, a_7, \dots\right)$$

By just observing every other element, it is clear that $(a_{2i}) = (0,1,2, \dots), i \in \mathbb{N}$. In other words, $\{a_{2i}\} = \mathbb{N}, i \in \mathbb{N}$. Therefore, the other elements $a_{2i+1}, i \in \mathbb{N}$ cannot also be natural numbers, or else they would appear multiple times in the sequence, preceded and succeeded by different numbers in each instance. Since $(a_1, a_3), (a_3, a_5)$ ... $\in s$, the sequence (a_{2i+1}) must be the same sequence as (a_{2i}) but starting from a different element. Take the example

$$(0,4,1,5,2,6,3,7,4,8,5, \dots)$$

There are two appearances of the element 4. For the first, $s^{1/2}(4) = 1$ but for the second $s^{1/2}(4) = 8$. This violates the constraint that the successor function is bijective, and therefore it is impossible to construct such a half-iterate. Each a_{2i+1} must therefore be from another set of the same cardinality. This proves that for countably infinite sets,

$$\mu(f) = 2\mu(f^2) \text{ (countably infinite sets)}$$

However, consider a function g that, starting at 0, generates the sequence

$$(0, -1, 1, -2, 2, -3, 3, -4, 4, -5, 5, \dots)$$

Clearly, this is a possible half-iterate of s, and furthermore it can be shown that a bijective function g can sequentially map each element in the sequence of natural numbers to this new sequence, by taking

$$g(x) = \begin{cases} x+1, & x = 0 \\ -x, & x > 0 \\ -(x+1), & x < 0 \end{cases}$$

In other words, these sets must have the same cardinality, even though the measure μ for their successor functions is different. To avoid any ambiguities, let's say $\mu(f)$ is not generally (i.e. for infinite sets) equal to the cardinality of the set S on which f is a successor function.

Continuous Real-Valued Half-Iterates of $f(x) = x$

Since the identity function for real numbers is represented by a set of ordered pairs $f = \{(x, x) | x \in \mathrm{R}\}$, its half-iterate is any function $f^{1/2} = \{(a, b), (b, a) | a \in R\}$. In other words, for any point (a, b) on the graph of the function $f^{1/2}$, its reflection about the line $y = x$, (b, a), must also be on the graph. This means any function symmetric about $y = x$ is a half-iterate of $f(x) = x$. Examples include $-x$, $\frac{1}{x}$, $\frac{x+1}{x-1}$, and the identity function x itself.

Half-Iterate Properties of the Negation Function

Consider the set $S_1 = \{0,1,2\}$ and its power set $\mathfrak{P}(S_1) = \{\{\}, \{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{0,2\}, \{0,1,2\}\}$. Suppose there is a negation function $\daleth_S$ that, for a given set S, returns the largest subset of that set that is disjoint to the argument. For example, $\daleth_{S_1}(\{0\}) = \{1,2\}$, $\daleth_{S_1}(\{1,2\}) = \{0\}$, and $\daleth_{S_1}(\{0,1,2\}) = \{\}$. By observing that each ordered pair $\left(x, \daleth_{S_1}(x)\right)$ has elements in $\mathfrak{P}(S_1)$, it is clear

$$\daleth_S : \mathfrak{P}(S) \mapsto \mathfrak{P}(S)$$

Furthermore, since $\daleth_S(a) = b$ implies $\daleth_S(b) = a$, $\daleth_S^2(a) = a$ is the identity function

$$\daleth_S^2 = \mathrm{id}_{\mathfrak{P}(S)}$$

$$\mu(\daleth_S) = 2$$

$$\mu\left(\mathrm{id}_{\mathfrak{P}(S)}\right) = 1$$

$$\mu(\daleth_S) = 2\mu\left(\mathrm{id}_{\mathfrak{P}(S)}\right)$$

This was shown by only accepting as a proposition the existence of S_1 and its power set $\mathfrak{P}(S_1)$. Countably infinite sets have uncountable power sets, so this is a possible example of a half-iterate defined in a set-theoretic way on an uncountable set, for example $\mathfrak{P}(\mathbb{N})$.

Half-Iteration of the Dirichlet Function

Consider the Dirichlet function on the real numbers

$$d(x) = \mathbf{1}_{\mathbb{Q}}(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

This function is the indicator function for the set of rational numbers $\mathbb{Q}$. Since $0,1 \in \mathbb{Q}$, $d^2(x) = 1$ for all real values and therefore $d^{\frac{1}{2}} \neq d$. A valid half-iterate h of the Dirichlet function must obey the rules

$$h: \mathbb{Q} \mapsto A$$

$$h: \neg\mathbb{Q} \mapsto B$$

$$h: A \mapsto \{1\}$$

$$h: B \mapsto \{0\}$$

It is first apparent that $A \cap B = \{\}$. A solution can be found by mapping $\neg\mathbb{Q}$ onto a subset B of $\mathbb{Q}$ that maps onto $\{0\}$ and mapping a subset A of $\mathbb{Q}$ disjoint to B onto $\{1\}$. A requirement of this solution is that $h(1) = 1$ and that $0 \in A$. A possible formulation therefore is

$$h: \mathbb{Q} \mapsto \{1\}$$

$$h: \neg\mathbb{Q} \mapsto \mathbb{Q} \cap \neg\{0\}$$

$$h: \{1\} \mapsto \{1\}$$

$$h: \mathbb{Q} \cap \neg\{0\} \mapsto \{0\}$$

However, though this is the most intuitive solution to the system earlier described, it does not produce the easiest explicit function, because it requires a function to map all nonzero real numbers onto nonzero rational numbers, or “rounding” to the nearest (or at least a corresponding) rational, hitting every rational given every real number. A much easier solution is found by redefining A and B; instead of B including all nonzero rational numbers, set B as the set of all integers greater than 1, and A therefore as all rational numbers except integers greater than 1. The slice of h on irrational arguments is therefore possibly defined as

$$h_{\neg\mathbb{Q}}(x) = \lceil |x| + 1 \rceil$$

Finally leading to a solution

$$h(x) = \mathbf{1}_{\mathbb{Q}}{}^{1/2}(x) = \begin{cases} 1, & x \in \mathbb{Q}, x \notin \mathbb{Z} \\ 1, & x \in \mathbb{Z}, x \leq 1 \\ 0, & x \in \mathbb{Z}, x > 1 \\ \lceil |x| + 1 \rceil, & x \notin \mathbb{Q} \end{cases}$$

An interesting note about the relationship between this function and the Dirichlet function can be discussed by observing their respective Lebesgue integrals. Since the rational numbers are countable and the real numbers are uncountable, the Dirichlet function necessarily has a Lebesgue integral on any interval of 0. However, notice that for $h(x)$, the value of the function on the irrational numbers is not 0 but instead $\lceil |x| + 1 \rceil$, which is nonzero for all real numbers. Therefore, its Lebesgue integral is nonzero for any interval on the real line.

Definition of a Null-Iterate

The null-iterate is the function f^0 described earlier. Specifically, for a function $f: R \mapsto S$,

$$f^0 = \{(x, x) | x \in S\} \cup \{(x, y) | x \notin S\}$$

i.e. it is the identity function on the function's image, and may take any other value for arguments outside the image. This definition is necessary to preserve the rule $f^n = f^{n+0} = f^n f^0$. By this definition, f^0 is not unique and specifically the identity function on the real numbers $\mathrm{id}_{\mathbf{R}}$ is always a possible choice.

Suppose a function f has image S for the domain of all real numbers such that $f: \mathbb{R} \mapsto S$. Its null-iterate is any function f^0 such that

$$f^0(x) = \begin{cases} x, & x \in S \\ g(x), & x \notin S \end{cases}$$

where $g(x)$ is any real-valued function. Suppose f_S is the function f only defined on the set S. There is some function $f_{\mathbb{R}}^0$ such that

$$f_S f_{\mathbb{R}}^0 = f_{\mathbb{R}}$$

where $f_{\mathbb{R}}$ is f defined for all real numbers. Since we are assuming only real values, take $f_{\mathbb{R}} \equiv f$. Note that composition is not commutative, as illustrated by the fact that the domain of the composition is defined by the image of the inner (right) function. While the domain of $f_S f_{\mathbb{R}}^0$ is $\mathbb{R}$, the domain of $f_{\mathbb{R}}^0 f_S$ is S. Therefore,

$$f_{\mathbb{R}}^0 f_S = f_S$$