Proof that z^a is analytic

John Giesbrecht (j21marshall@gmail.com)

The product and composition of analytic functions are analytic. Using this, it can be shown z^a is analytic for any constant complex number a using the Cauchy-Riemann equations. If z^a can be shown to be analytic for any real a, and for any imaginary a, then it can be shown for any complex a since $z^a = z^{\mathrm{Re}(a)} z^{i\mathrm{Im}(a)}$.

Begin by solving for real a:

$$f(z) = z^a = |z|^a e^{ia\mathrm{Arg}(z)} = |z|^a\big(\cos\big(a\mathrm{Arg}(z)\big) + i\sin\big(a\mathrm{Arg}(z)\big)\big)$$

$$u(z) = \mathrm{Re}(f)(z) = |z|^a \cos\big(a\mathrm{Arg}(z)\big)$$

$$v(z) = \mathrm{Im}(f)(z) = |z|^a \sin\big(a\mathrm{Arg}(z)\big)$$

$$\frac{\partial u}{\partial x} = \frac{\partial(|z|^a)}{\partial x}\cos\big(a\mathrm{Arg}(z)\big) + \frac{\partial\big(\cos\big(a\mathrm{Arg}(z)\big)\big)}{\partial x}|z|^a$$

$$\frac{\partial(|z|^a)}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2)^{a/2} = ax(x^2+y^2)^{a/2-1}$$

$$\frac{\partial\big(\cos\big(a\mathrm{Arg}(z)\big)\big)}{\partial x} = \frac{ay\sin(a\mathrm{Arg}(z))}{|z|^2}$$

$$\frac{\partial u}{\partial x} = ax(x^2+y^2)^{a/2-1}\cos\big(a\mathrm{Arg}(z)\big) + \frac{ay\sin(a\mathrm{Arg}(z))}{|z|^2}|z|^a$$

Now, additional conversions to exponentials are helpful

$$\cos\big(a\mathrm{Arg}(z)\big) = \frac{e^{ia\mathrm{Arg}(z)}}{2} + \frac{e^{-ia\mathrm{Arg}(z)}}{2}$$

$$\sin\big(a\mathrm{Arg}(z)\big) = \frac{ie^{-ia\mathrm{Arg}(z)}}{2} - \frac{ie^{ia\mathrm{Arg}(z)}}{2}$$

To get a longer form of $\frac{\partial u}{\partial x}$

$$\frac{\partial u}{\partial x} = ax(x^2+y^2)^{a/2-1}\left(\frac{e^{ia\mathrm{Arg}(z)}}{2} + \frac{e^{-ia\mathrm{Arg}(z)}}{2}\right) + ay|z|^{a-2}\left(\frac{ie^{-ia\mathrm{Arg}(z)}}{2} - \frac{ie^{ia\mathrm{Arg}(z)}}{2}\right)$$

Noticing that $(x^2+y^2)^{a/2-1} = |z|^{a-2}$,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{(ax|z|^{a-2} - iay|z|^{a-2})e^{ia\mathrm{Arg}(z)}}{2} + \frac{(ax|z|^{a-2} + iay|z|^{a-2})e^{-ia\mathrm{Arg}(z)}}{2} \\ &= \frac{(ax-iay)|z|^{a-2}e^{ia\mathrm{Arg}(z)}}{2} + \frac{(ax+iay)|z|^{a-2}e^{-ia\mathrm{Arg}(z)}}{2}\end{aligned}$$

Moving on to $\frac{\partial v}{\partial y}$,

$$\frac{\partial v}{\partial y} = \frac{\partial(|z|^a)}{\partial y}\sin\big(a\mathrm{Arg}(z)\big) + \frac{\partial\big(\sin\big(a\mathrm{Arg}(z)\big)\big)}{\partial y}|z|^a$$

$$\frac{\partial(|z|^a)}{\partial y} = ay(x^2+y^2)^{a/2-1}$$

$$\frac{\partial\big(\sin\big(a\mathrm{Arg}(z)\big)\big)}{\partial y} = \frac{axsin(aArg(z))}{|z|^2}$$

$$\frac{\partial v}{\partial y} = ay(x^2+y^2)^{a/2-1}\sin\big(a\mathrm{Arg}(z)\big) + \frac{axcos(aArg(z))}{|z|^2}|z|^a$$

$$= ay(x^2+y^2)^{a/2-1}\left(\frac{ie^{-ia\mathrm{Arg}(z)}}{2} - \frac{ie^{ia\mathrm{Arg}(z)}}{2}\right) + ax|z|^{a-2}\left(\frac{e^{ia\mathrm{Arg}(z)}}{2} + \frac{e^{-ia\mathrm{Arg}(z)}}{2}\right)$$

$$= \frac{(ax|z|^{a-2} - iay|z|^{a-2})e^{ia\mathrm{Arg}(z)}}{2} + \frac{(ax|z|^{a-2} + iay|z|^{a-2})e^{-ia\mathrm{Arg}(z)}}{2}$$

$$= \frac{(ax-iay)|z|^{a-2}e^{ia\mathrm{Arg}(z)}}{2} + \frac{(ax+iay)|z|^{a-2}e^{-ia\mathrm{Arg}(z)}}{2}$$

Therefore, the first of the Cauchy-Riemann equations has been solved:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Moving to the next equation, the above patterns can be reused to move more quickly:

$$\frac{\partial v}{\partial x} = \frac{\partial(|z|^a)}{\partial x}\sin\big(a\mathrm{Arg}(z)\big) + \frac{\partial\big(\sin\big(a\mathrm{Arg}(z)\big)\big)}{\partial x}|z|^a = ax|z|^{a-2}\sin\big(a\mathrm{Arg}(z)\big) - ay|z|^{a-2}\cos\big(a\mathrm{Arg}(z)\big)$$

$$= ax|z|^{a-2}\left(\frac{ie^{-ia\mathrm{Arg}(z)}}{2} - \frac{ie^{ia\mathrm{Arg}(z)}}{2}\right) - ay|z|^{a-2}\left(\frac{e^{ia\mathrm{Arg}(z)}}{2} + \frac{e^{-ia\mathrm{Arg}(z)}}{2}\right)$$

$$= \frac{(-ay-iax)|z|^{a-2}e^{ia\mathrm{Arg}(z)}}{2} + \frac{(-ay+iax)|z|^{a-2}e^{-ia\mathrm{Arg}(z)}}{2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial(|z|^a)}{\partial y}\cos\big(a\mathrm{Arg}(z)\big) + \frac{\partial\big(\cos\big(a\mathrm{Arg}(z)\big)\big)}{\partial y}|z|^a$$

$$= ay|z|^{a-2}\cos\big(a\mathrm{Arg}(z)\big) - ax|z|^{a-2}\sin\big(a\mathrm{Arg}(z)\big) = ay|z|^{a-2}\left(\frac{e^{ia\mathrm{Arg}(z)}}{2} + \frac{e^{-ia\mathrm{Arg}(z)}}{2}\right)$$

$$- ax|z|^{a-2}\left(\frac{ie^{-ia\mathrm{Arg}(z)}}{2} - \frac{ie^{ia\mathrm{Arg}(z)}}{2}\right) = \frac{(ay+iax)|z|^{a-2}e^{ia\mathrm{Arg}(z)}}{2} + \frac{(ay-iax)|z|^{a-2}e^{-ia\mathrm{Arg}(z)}}{2}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Now, both Cauchy-Riemann equations are shown to be true and z^a to be analytic for real a (except at zero since $\mathrm{Arg}(z)$ has an essential singularity there). The only thing that remains is to show the same result for imaginary a. Notice that this can be shown by taking $z^a = \left(z^b\right)^i$, where b is a real number. We already know z^b is analytic, and if z^i is analytic, z^a is the composition of analytic functions, and therefore analytic. This is proven by showing $\left(z^i\right)^i = z^{-1}$ is analytic:

$$f(z) = z^{-1} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = u(z) + iv(z)$$

$$\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x}\square$$