

Proof that z^a is analytic

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The product and composition of analytic functions are analytic. Using this, it can be shown z^a is analytic for any constant complex number a using the Cauchy-Riemann equations. If z^a can be shown to be analytic for any real a , and for any imaginary a , then it can be shown for any complex a since $z^a = z^{\operatorname{Re}(a)} z^{i\operatorname{Im}(a)}$.

Begin by solving for real a :

$$f(z) = z^a = |z|^a e^{ia\operatorname{Arg}(z)} = |z|^a (\cos(a\operatorname{Arg}(z)) + i \sin(a\operatorname{Arg}(z)))$$

$$u(z) = \operatorname{Re}(f)(z) = |z|^a \cos(a\operatorname{Arg}(z))$$

$$v(z) = \operatorname{Im}(f)(z) = |z|^a \sin(a\operatorname{Arg}(z))$$

$$\frac{\partial u}{\partial x} = \frac{\partial(|z|^a)}{\partial x} \cos(a\operatorname{Arg}(z)) + \frac{\partial(\cos(a\operatorname{Arg}(z)))}{\partial x} |z|^a$$

$$\frac{\partial(|z|^a)}{\partial x} = \frac{\partial}{\partial x} (x^2 + y^2)^{a/2} = ax(x^2 + y^2)^{a/2-1}$$

$$\frac{\partial(\cos(a\operatorname{Arg}(z)))}{\partial x} = \frac{a y \sin(a\operatorname{Arg}(z))}{|z|^2}$$

$$\frac{\partial u}{\partial x} = ax(x^2 + y^2)^{a/2-1} \cos(a\operatorname{Arg}(z)) + \frac{a y \sin(a\operatorname{Arg}(z))}{|z|^2} |z|^a$$

Now, additional conversions to exponentials are helpful

$$\cos(a\operatorname{Arg}(z)) = \frac{e^{ia\operatorname{Arg}(z)}}{2} + \frac{e^{-ia\operatorname{Arg}(z)}}{2}$$

$$\sin(a\operatorname{Arg}(z)) = \frac{ie^{-ia\operatorname{Arg}(z)}}{2} - \frac{ie^{ia\operatorname{Arg}(z)}}{2}$$

To get a longer form of $\frac{\partial u}{\partial x}$

$$\frac{\partial u}{\partial x} = ax(x^2 + y^2)^{a/2-1} \left(\frac{e^{ia\operatorname{Arg}(z)}}{2} + \frac{e^{-ia\operatorname{Arg}(z)}}{2} \right) + ay|z|^{a-2} \left(\frac{ie^{-ia\operatorname{Arg}(z)}}{2} - \frac{ie^{ia\operatorname{Arg}(z)}}{2} \right)$$

Noticing that $(x^2 + y^2)^{a/2-1} = |z|^{a-2}$,

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{(ax|z|^{a-2} - iay|z|^{a-2})e^{ia\operatorname{Arg}(z)}}{2} + \frac{(ax|z|^{a-2} + iay|z|^{a-2})e^{-ia\operatorname{Arg}(z)}}{2} \\ &= \frac{(ax - iay)|z|^{a-2}e^{ia\operatorname{Arg}(z)}}{2} + \frac{(ax + iay)|z|^{a-2}e^{-ia\operatorname{Arg}(z)}}{2} \end{aligned}$$

Moving on to $\frac{\partial v}{\partial y}$,

$$\frac{\partial v}{\partial y} = \frac{\partial(|z|^a)}{\partial y} \sin(a\operatorname{Arg}(z)) + \frac{\partial(\sin(a\operatorname{Arg}(z)))}{\partial y} |z|^a$$

$$\frac{\partial(|z|^a)}{\partial y} = ay(x^2 + y^2)^{a/2-1}$$

$$\begin{aligned}
\frac{\partial(\sin(a\text{Arg}(z)))}{\partial y} &= \frac{ax\sin(a\text{Arg}(z))}{|z|^2} \\
\frac{\partial v}{\partial y} &= ay(x^2 + y^2)^{a/2-1} \sin(a\text{Arg}(z)) + \frac{ax\cos(a\text{Arg}(z))}{|z|^2} |z|^a \\
&= ay(x^2 + y^2)^{a/2-1} \left(\frac{ie^{-ia\text{Arg}(z)}}{2} - \frac{ie^{ia\text{Arg}(z)}}{2} \right) + ax|z|^{a-2} \left(\frac{e^{ia\text{Arg}(z)}}{2} + \frac{e^{-ia\text{Arg}(z)}}{2} \right) \\
&= \frac{(ax|z|^{a-2} - iay|z|^{a-2})e^{ia\text{Arg}(z)}}{2} + \frac{(ax|z|^{a-2} + iay|z|^{a-2})e^{-ia\text{Arg}(z)}}{2} \\
&= \frac{(ax - iay)|z|^{a-2}e^{ia\text{Arg}(z)}}{2} + \frac{(ax + iay)|z|^{a-2}e^{-ia\text{Arg}(z)}}{2}
\end{aligned}$$

Therefore, the first of the Cauchy-Riemann equations has been solved:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Moving to the next equation, the above patterns can be reused to move more quickly:

$$\begin{aligned}
\frac{\partial v}{\partial x} &= \frac{\partial(|z|^a)}{\partial x} \sin(a\text{Arg}(z)) + \frac{\partial(\sin(a\text{Arg}(z)))}{\partial x} |z|^a = ax|z|^{a-2} \sin(a\text{Arg}(z)) - ay|z|^{a-2} \cos(a\text{Arg}(z)) \\
&= ax|z|^{a-2} \left(\frac{ie^{-ia\text{Arg}(z)}}{2} - \frac{ie^{ia\text{Arg}(z)}}{2} \right) - ay|z|^{a-2} \left(\frac{e^{ia\text{Arg}(z)}}{2} + \frac{e^{-ia\text{Arg}(z)}}{2} \right) \\
&= \frac{(-ay - iax)|z|^{a-2}e^{ia\text{Arg}(z)}}{2} + \frac{(-ay + iax)|z|^{a-2}e^{-ia\text{Arg}(z)}}{2} \\
\frac{\partial u}{\partial y} &= \frac{\partial(|z|^a)}{\partial y} \cos(a\text{Arg}(z)) + \frac{\partial(\cos(a\text{Arg}(z)))}{\partial y} |z|^a \\
&= ay|z|^{a-2} \cos(a\text{Arg}(z)) - ax|z|^{a-2} \sin(a\text{Arg}(z)) = ay|z|^{a-2} \left(\frac{e^{ia\text{Arg}(z)}}{2} + \frac{e^{-ia\text{Arg}(z)}}{2} \right) \\
&\quad - ax|z|^{a-2} \left(\frac{ie^{-ia\text{Arg}(z)}}{2} - \frac{ie^{ia\text{Arg}(z)}}{2} \right) = \frac{(ay + iax)|z|^{a-2}e^{ia\text{Arg}(z)}}{2} + \frac{(ay - iax)|z|^{a-2}e^{-ia\text{Arg}(z)}}{2} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{aligned}$$

Now, both Cauchy-Riemann equations are shown to be true and z^a to be analytic for real a (except at zero since $\text{Arg}(z)$ has an essential singularity there). The only thing that remains is to show the same result for imaginary a . Notice that this can be shown by taking $z^a = (z^b)^{i\pi/2}$, where b is a real number. We already know z^b is analytic, and if $z^{i\pi/2}$ is analytic as well, z^a is the composition of two analytic functions, and therefore analytic.

$$f(z) = z^{i\pi/2} = (x + iy)^{i\pi/2} = -y + ix$$

$$u(z) = \text{Re}(f)(z) = -y$$

$$v(z) = \text{Im}(f)(z) = x$$

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -1 = -\frac{\partial v}{\partial x} \square$$