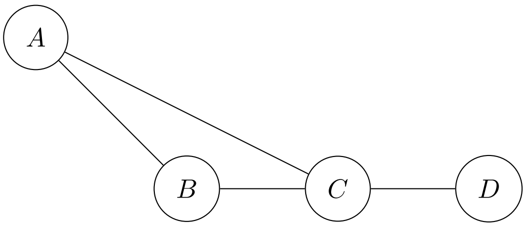
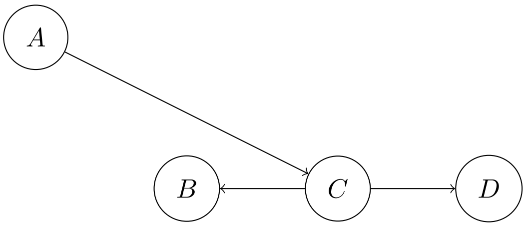
2 A Reduction Warm-up

It is incorrect. It is true that if the longest path in the DAG has length $|V| - 1$ then there is a Rudrata path in G. However, to prove a reduction correct, **you have to prove both directions.**
That is, if you have reduced problem A to problem B by transforming instance I to instance I' then you should prove that I has a solution **if and only if** I' has a solution. In the above "reduction," one direction does not hold. Specifically, if G has a Rudrata path then the DAG that we produce does not necessarily have a path of length $|V| - 1$–it depends on how we choose directions for the edges.
For a concrete counterexample, consider the following graph:



It is possible that when traversing this graph by DFS, node C will be encountered before node B and thus the DAG produced will be



which does not have a path of length 3 even though the original graph did have a Rudrata path.

3 Reduction to 3-Coloring

(a) It is easier to show the equivalent statement that if all the gray vertices on the left are assigned the color 0, then the gray vertex on the right must be assigned the color 0 as well. Consider the triangle on the left. Since all the gray vertices are assigned 0, the two left points must be assigned the colors 1 and 2, and so the right point in this triangle must be assigned 0 in any valid coloring. We can repeat this logic with the triangle on the right, to conclude that the gray vertex on the right must be assigned 0 in any valid coloring.

(b) We add the edges $(x_i, \neg x_i)$, $(x_i, Base)$ and $(\neg x_i, Base)$. Since x_i, $\neg x_i$ are both adjacent to Base they must be assigned a different color than Base, i.e. they both are assigned either the color of True or the color of False. Since we added an edge between x_i and $\neg x_i$, they can't be assigned the same color, i.e. one is assigned the same color as True and one the same color as False.

(c) Given a 3-SAT instance, we create the three special vertices and edges described in the problem statement. As in part a, we create vertices x_i and $\neg x_i$ for each variable x_i , and add the edges we gave in the answer to part a. For clause j, we add a vertex C_j and edges $(C_j, False)$, $(C_j, Base)$. Lastly, for clause j we add vertices and edges to create a gadget where the three gray vertices on the left of the gadget are the vertices of three literals in the clause, and the gray vertex on the right is the vertex C_j (All black vertices in the gadget are only used in this clause's gadget).
If there is a satisfying 3-SAT assignment, then there is a valid 3-coloring in this graph as follows. Assign False the color 0, True the color 1, and Base the color 2; assign xi the color 1 if x_i is True and 0 if x_i is False (vice-versa for $\neg x_i$). Assign each C_j the color 1. Lastly, fix any gadget. Since the 3-SAT assignment is satisfying, in each gadget at least one of the gray vertices on the left is assigned 1, so by the observation (ii) in the problem statement the gadget can be colored.
If there is a valid 3-coloring, then there is a satisfying 3-SAT assignment. By symmetry, we can assume False is colored 0, True is colored 1, and Base is colored 2. Then for each literal where x_i is color 1, that literal is true in the satisfying assignment. By part a, we know that exactly one of x_i, $\neg x_i$ is colored 1, so this produces a valid assignment. By observation (i), we also know every node C_j must be colored 1. All literal nodes are colored 0 or 1, so by part b, this implies that for every clause, one of the gray literal nodes in the clause gadget is colored 1, i.e. the clause will be satisfied in the 3-SAT assignment.