Anti-Coordination Games and Stable Graph Colorings

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Abstract. Motivated by understanding non-strict and strict pure strategy equilibria in network anti-coordination games, we define notions of stable and, respectively, strictly stable colorings in graphs. We characterize the cases when such colorings exist and when the decision problem is NP-hard. These correspond to finding pure strategy equilibria in the anti-coordination games, whose price of anarchy we also analyze. We further consider the directed case, a generalization that captures both coordination and anti-coordination. We prove the decision problem for non-strict equilibria in directed graphs is NP-hard. Our notions also have multiple connections to other combinatorial questions, and our results resolve some open problems in these areas, most notably the complexity of the strictly unfriendly partition problem.

1 Introduction

Anti-coordination games form some of the basic payoff structures in game theory. Such games are ubiquitous; miners deciding which land to drill for resources, company employees trying to learn diverse skills, and airplanes selecting flight paths all need to mutually anti-coordinate their strategies in order to maximize their profits or even avoid catastrophe.

Two-player anti-coordination is simple and well understood. In its barest form, the players have two actions, and payoffs are symmetric for the players, paying off 1 if the players choose different actions and 0 otherwise. This game has two strict pure-strategy equilibria, paying off 1 to each player, as well as a (non-strict) mixed-strategy equilibrium paying off $1/2$ to each player.

In the real world, however, coordination and anti-coordination games are more complex than the simple two-player game. People, companies, and even countries play such multi-party games simultaneously with one another. One straightforward way to model this is with a graph, whose nodes correspond to agents and whose edges capture their pairwise interactions. A node then chooses one of k strategies, trying to anti-coordinate with all its neighbors simultaneously. The payoff of a node is the sum of the payoffs of its games with

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its neighbors – namely the number of neighbors with which it has successfully anti-coordinated. It is easy to see that this model naturally captures many applications. For example countries may choose commodities to produce, and their value will depend on how many trading partners do not produce that commodity.

In this paper we focus on finding pure strategies in equilibrium, as well as their associated social welfare and price of anarchy, concepts we shall presently define. We look at both strict and non-strict pure strategy equilibria, as well games on directed and undirected graphs. Directed graphs characterize the case where only one of the nodes is trying to anti-coordinate with another. The directed case turns out to not only generalize the symmetric undirected case, but also captures coordination in addition to anti-coordination.

These problems also have nice interpretations as certain natural graph coloring and partition problems, variants of which have been extensively studied. For instance, a pure strategy equilibrium in an undirected graph corresponds to what we call a stable k-coloring of the graph, in which no vertex can have fewer neighbors of any color different than its own. For $k = 2$ colors this is equivalent to the well-studied *unfriendly partition* or *co-satisfactory partition* problem. The strict equilibrium version of this problem (which corresponds to what we call a strictly stable k-coloring) generalizes the *strictly unfriendly partition problem*. We establish both the NP-hardness of the decision problem for strictly unfriendly partitions and the case for higher k.

1.1 Previous work

In an early work on what can be seen as a coloring game, Naor and Stockmeyer [12] define a *weak k-coloring* of a graph to be one in which each vertex must have at least one neighbor of a color different than its own. They give a locally distributed algorithm that, under certain conditions, weakly 2-colors a graph in constant time. Interestingly, the lower bound is no longer constant when the condition is strengthened even to two colors.

Then, in an influential experimental study of anti-coordination in networks, Kearns et al. [10] propose a true graph coloring game, in which each participant controlled the color of a node, with the goal of coloring a graph in a distributed fashion. The players receive a reward only when a proper coloring of the graph was found. The theoretical properties of this game were further studied by Chaudhuri et al. [6] who prove that in a graph of maximum degree d, if players have $d + 2$ colors available they will w.h.p. converge to a proper coloring rapidly using a greedy local algorithm. Our work is also largely motivated by the work of Kearns et al., but for a somewhat relaxed version of proper coloring.

Bramoullé et al. [3] also study a general anti-coordination game played on networks. In their formulation, nodes can choose to form links, and the payoffs of two anti-coordinated strategies may not be identical. They go on to characterize the strict equilibria of such games, as well as the effect of network structure on the behavior of individual agents. We, on the other hand, consider an arbitrary number of strategies but with a simpler payoff structure.

This paper also has strong relationship to the concept of *unfriendly partitions* in graph theory. An unfriendly partition of a graph is one in which each vertex has at least as many neighbors in other partitions as in its own. This topic has been extensively studied, especially in the combinatorics community [1, 4, 7, 14]. One relevant result is that every locally finite graph has a 2-friendly partition (uncountable graphs, however, may not [14]).

Friendly (the natural counterpart) and unfriendly partitions are also studied under the names *max satisfactory* and *min co-satisfactory partitions* by Bazgan et al. [2], who focus on partitions of size greater than 2. They characterize the complexity of determining whether a graph has a k-friendly partition and asked about characterizing the $k > 2$ case explicitly as an open problem. Our notion of stable colorings captures unfriendly partitions, and we solve the $k > 2$ case by giving an algorithm that finds a k-unfriendly partition for all k.

Furthermore, a natural strengthening of the notion above yields *strictly unfriendly partitions*, first defined by Shafique and Dutton [13]. A strictly unfriendly partition requires that each node has strictly more neighbors outside its partition than inside it. Shafique and Dutton are able to characterize a weakening of this notion, called *alliance-free partition*, but leave characterizing strictly unfriendly partitions open. Our notion of strictly stable coloring captures alliance-free partitions, giving some of the first results on this problem. Finally a related problem originating from sociological research, called the *matching pennies game* or the *fashion game* is studied, among others, by Cao and Yang [5]. In their version of the game, some nodes try to coordinate and others try to anti-coordinate. They prove that it is NP-hard to decide whether such a game has a pure strategy equilibrium. Our work on the directed case generalizes their notion (something they suggested as a research direction). Among our results we give a simpler proof of their hardness result for $k = 2$ and also tackle higher k, settling one of their open questions.

1.2 Results

Our main results are as follows.

1. *$\forall k \geq 2$, every undirected graph has a stable k-coloring, and such a coloring can be found in polynomial time.*
   Our notion of stable k-colorings is a strengthening of the notion of k-unfriendly partitions of Bazgan et al. [2], solving their open problem number 15.
2. *For undirected graphs, the price of anarchy for stable k-colorings is bounded by $\frac{k}{k-1}$, and this bound is tight.*
3. *In undirected graphs, $\forall k \geq 2$, determining whether a graph has a strictly stable k-coloring is NP-hard.*
   For $k = 2$, this notion is equivalent to the notion that was defined by Shafique and Dutton [13], but left unsolved.
4. *$\forall k \geq 2$, determining whether a directed graph has even a non-strictly stable k-coloring is NP-hard.*

Because directed graphs also capture coordination, this solves two open problems of Cao and Yang [5], namely generalizing the coin matching game to more than two outcomes and considering the directed case.

2 Preliminaries

For an unweighted undirected graph $G = (V, E)$, let

$$C = \{f | f : V \to \{1, \ldots k\}\}.$$

We call a function $c \in C$ a coloring.

We study the following anti-coordination game played on a graph $G = (V, E)$. In the game, all vertices simultaneously choose a color, which induces a coloring $c \in C$ of the graph. In a given coloring c, an agent v's **payoff**, $\mu_c(v)$, is the number of neighbors choosing colors different from v's, namely

$$\mu_c(v) := \sum_{\{v,w\} \in E} \mathbf{1}_{\{c(v) \neq c(w)\}}.$$

Note that in this game higher degree vertices have higher potential payoffs.

We also have a natural generalization to directed graphs. That is, if $G = (V, E)$ is a directed graph and c is a coloring of V, we can define the payoff $\mu_c(v)$ of a vertex $v \in V$ analogously as the sum over outgoing edges:

$$\mu_c(v) := \sum_{(v,w) \in E} \mathbf{1}_{\{c(v) \neq c(w)\}}$$

Here a directed edge from v to w is interpreted as "v cares about w." We can then define the social welfare and cost of anarchy for directed graphs identically using this payoff function.

Given a graph G, we define the **social welfare** of a coloring c to be

$$W(G, c) := \sum_{v \in V} \mu_c(v).$$

We say a coloring c is **stable**, or in equilibrium, if no vertex can improve its payoff by changing its color from $c(v)$ to another color. We define Q to be the set of colorings in equilibrium.

We call a coloring function c **strictly stable**, or in strict equilibrium, if every vertex would decrease its payoff by changing its color from $c(v)$ to another color. If a coloring function is stable and at least one vertex can change its color without decreasing its payoff, then the coloring is **non-strict**.

We define the **price of anarchy** for a graph G to be

$$\mathrm{PoA}(G) := \frac{\max_{c' \in C} W(G, c')}{\min_{c \in Q} W(G, c)}.$$

Mixed and pure strategies It is natural to consider both pure and mixed strategies for the players in our network anti-coordination game. A pure strategy solution does not in general exist for every 2 player game, while a mixed strategy solution will. However, in this coloring game an analysis of pure strategies suffices as not only will a pure strategy solution always exist, but for any mixed strategy solution there is a pure strategy equilibrium solution which achieves a social welfare at least as good, and where each player's payoff is identical with its expected payoff under the mixed strategy.

Strict and non-strict stability It is worthwhile to note that a strictly stable coloring c need not provide the maximum social welfare. In fact, it is not difficult to construct a graph for which a strictly stable coloring exists yet the maximum social welfare is achieved by a non-strictly stable coloring, as shown in Figure 1.

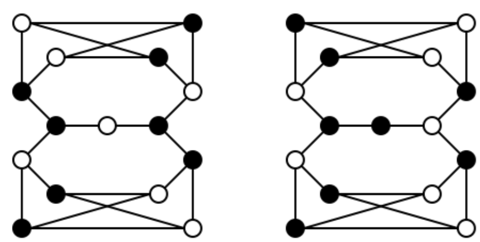

Fig. 1. The strictly stable 2-coloring on the left attains a social welfare of 40 while the non-strictly stable coloring on the right achieves social welfare of 42, which can be verified is the maximum social welfare for a 2-coloring of this graph.

3 Stable colorings

First we consider the problem of finding stable colorings in graphs. For the case $k = 2$, this is equivalent to the solved unfriendly partition problem. For $k = 2$, the algorithm we give in the proof is also equivalent to the well-studied local algorithm for the MAX-CUT problem [8, 11], where the goal is to divide a graph into two parts maximizing the number of edges crossing the cut. To the best of our knowledge, no such generalization has been shown for higher k, but our argument is a variant of standard arguments for MAX-CUT.

Theorem 1. $\forall k \geq 2$, *every finite graph* $G = (V, E)$ *admits a stable* k*-coloring. Moreover, a stable* k*-coloring can be found in polynomial time.*

Proof. First we observe that for any graph G,

$$\max_{c \in C} W(G, c) \leq 2|E|.$$

In a given coloring, we call a vertex v *unhappy* if v has more neighbors of its color than of some other color. We now run the following process: while any unhappy vertex exists, change its color to the color

$$c'(u) = \operatorname*{argmin}_{m\in\{1,\ldots,k\}} \sum_{v\in N(u)} \mathbf{1}_{\{c(v)=m\}}. \tag{1}$$

Each time such a switch is made the new coloring has strictly higher social welfare because not only v's payoff increases, but also v's switch increases the payoff of more neighbors than it decreases. This means the overall social welfare will increase by at least 2. Because social welfare is bounded from above by $2|E|$, eventually no vertex will be unhappy, which by definition means the coloring is stable. Hence, a stable coloring is achieved after at most $|E|$ such flips. □

We note that because, in the case of $k = 2$, maximizing the social welfare of a stable coloring is equivalent to finding the MAX-CUT of the same graph, which is known to be NP-hard [9], we cannot hope to find the optimal solution. However, we can ask about the price of anarchy, for which we obtain a tight bound.

Theorem 2. *The price of anarchy of the k-coloring anti-coordination game is at most $\frac{k}{k-1}$, and this bound is tight.*

Proof. By the pigeonhole principle, each vertex can always achieve a $\frac{k-1}{k}$ fraction of its maximum payoff by choosing its color according to Equation 1. Hence, if some vertex does not achieve this payoff then the coloring is not stable. This implies that the price of anarchy is at most $\frac{k}{k-1}$.

To see that this bound is tight take two copies of K_k on vertices $v_1, ..., v_k$ and $v_{k+1}, ..., v_{2k}$ respectively. Add an edge joining v_i with v_{i+k} for $i \in \{1, ..., k\}$. If each vertex v_i and v_{i+k} is given color i this gives a stable k-coloring of the graph, as each vertex has 1 neighbor of each of the k colors attaining the low bound of $2\frac{k-1}{k}|E|$. If, however, the vertices v_{i+k} takes color i+1 for $i \in \{1, ..., k-1\}$ and v_{2k} takes color 1, the graph achieves the maximum social welfare of $2|E|$. This is illustrated for $k = 5$ in Figure 2. □

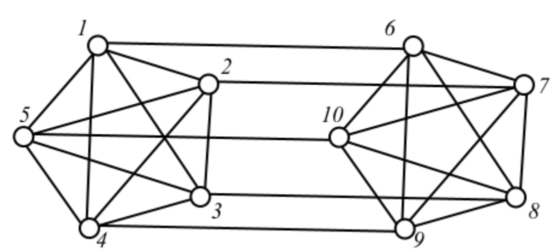

Fig. 2. The construction of a graph achieving PoA of $\frac{5}{4}$, for k=5

4 Strictly Stable Colorings

In this section we show the problem of finding a strictly stable equilibrium with any fixed number $k \geq 2$ of colors is NP-complete. We give NP-hardness reductions first for $k \geq 3$ and then for $k = 2$. The $k = 2$ case is equivalent to the strictly unfriendly 2-partition problem [13], whose complexity we settle.

Theorem 3. *For all $k \geq 2$, determining whether a graph has a strictly stable k-coloring is NP-complete.*

Proof. This problem is clearly in NP. We now analyze the hardness in two cases. *1)* $k \geq 3$: For this case we reduce from classical k-coloring. Given a graph G, we produce a graph G' as follows.

Start with $G' = G$, and then for each edge $e = (v, w)$ in G, add a copy H_e of the complete graph K_{k-2} to G' and enough edges so that the induced subgraph of G' on the vertex set $V(H_e) \cup \{v, w\}$ is the complete graph on k vertices. Figure 3 illustrates this construction.

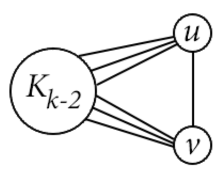

Fig. 3. The gadget added for each edge in G.

Now supposing that G is k-colorable, we construct a strictly stable equilibrium in G' as follows. Fix any coloring φ of G. Color each vertex in G' which came from G (which is not in any H_e) using φ. For each edge $e = (v, w)$ we can trivially assign the remaining $k - 2$ colors among the vertices of H_e to put the corresponding copy of K_k in a strict equilibrium. Doing this for every such edge results in a strictly stable coloring. Indeed, this is a proper k-coloring of G' in which every vertex is adjacent to vertices of all other $k - 1$ colors.

Conversely, suppose G' has a strictly stable equilibrium with k colors. Then no edge e originally coming from G can be monochromatic. If it were, then there would be $k - 1$ remaining colors to assign among the remaining $k - 2$ vertices of H_e. No matter the choice, some color is unused and any vertex of H_e could change its color without being penalized, contradicting that G' is in a strict equilibrium.

The only issue is if G originally has an isolated vertex (G' will have an isolated vertex, and hence will not have a strict equilibrium). In this case, augment the reduction to attach a copy of K_{k-1} to the isolated vertex, and the proof remains the same.

2) $k = 2$: We reduce from 3-SAT. Let $\varphi = C_1 \wedge \cdots \wedge C_k$ be a boolean formula in 3-CNF form. We construct a graph G by piecing together gadgets as follows.

For each clause C_i construct an isomorphic copy of the graph shown in Figure 4. We call this the *clause gadget* for C_i. In Figure 4, we label certain vertices to show how the construction corresponds to a clause. We call the two vertices labeled by the same literal in a clause gadget a *literal gadget.* In particular, Figure 4 would correspond to the clause $(x \vee y \vee \bar{z})$, and a literal assumes a value of true when the literal gadget is monochromatic. Later in the proof we will force literals to be consistent across all clause gadgets, but presently we focus on the following key property of a clause gadget.

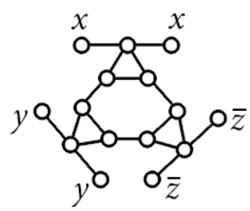

Fig. 4. The clause gadget for $(x \vee y \vee \bar{z})$. Each literal corresponds to a pair of vertices, and a literal being satisfied corresponds to both vertices having the same color.

Lemma 1. *Any strictly stable 2-coloring of a clause gadget has a monochromatic literal gadget. Moreover, any coloring of the literal gadgets which includes a monochromatic literal extends to a strictly stable coloring of the clause gadget (excluding the literal gadgets).*

Proof. This last qualification will be resolved later by the high-degree of the vertices in the literal gadgets. Up to symmetries of the clause gadget (as a graph) and up to swapping colors, the proof of Lemma 1 is illustrated in Figure 5. The first five graphs show the cases where one or more literal gadgets are monochromatic, and the sixth shows how no strict equilibrium can exist if all literal gadgets are not monochromatic. Using the labels in Figure 5, whatever the choice of color for the vertex v_1, its two uncolored neighbors must have the same color (or else v_1 is not in equilibrium). Call this color a. For v_2, v_3, use the same argument and call the corresponding colors b, c, respectively. Since there are only two colors, one pair of a, b, c must agree. Without loss of generality suppose $a = b$. But then the two vertices labeled by a and c which are adjacent are not in strict equilibrium. □

Using Lemma 1, we complete the proof of the theorem. We must enforce that any two identical literal gadgets in different clause gadgets agree (they are both monochromatic or both not monochromatic), and that any negated literals disagree. We introduce two more simple gadgets for each purpose.

The first is for literals which must agree across two clause gadgets, and we call this the *literal persistence gadget.* It is shown in Figure 6. The choice of

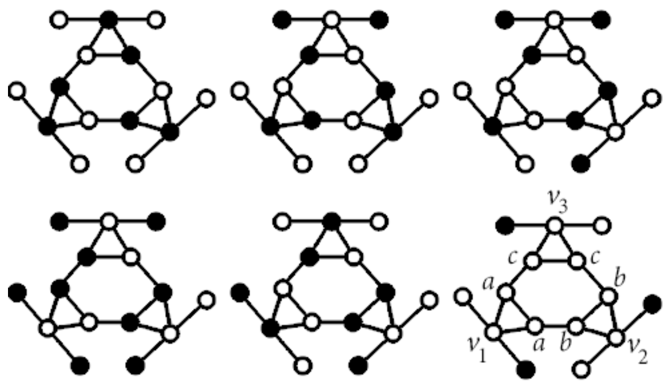

Fig. 5. The proof of the clause gadget lemma. The first five figures show that a coloring with a monochromatic literal gadget can be extended to a strict equilibrium. The sixth (bottom right) shows that no strict equilibrium can exist if all the literals are not monochromatic.

colors for the literals on one side determines the choice of colors on the other, provided the coloring is strictly stable. In particular, this follows from the central connecting vertex having degree 2. A nearly identical argument applies to the second gadget, which forces negated literals to assume opposite truth values. We call this the *literal negation gadget*, and it is shown in Figure 6.

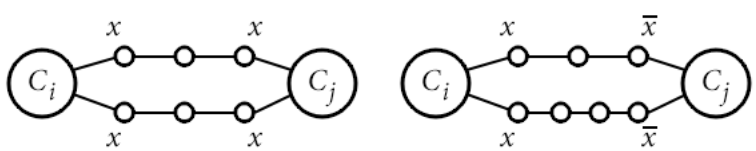

Fig. 6. The literal persistence gadget (left) and literal negation gadget (right) connecting two clause gadgets C_i and C_j. The vertices labeled x on the left are part of the clause gadget for C_i, and the vertices labeled x on the right are in the gadget for C_j.

The reduction is proved in a straightforward way. If φ is satisfiable, then monochromatically color all satisfied literal gadgets in G. We can extend this to a stable 2-coloring: all connection gadgets and unsatisfied literal gadgets are forced, and by Lemma 1 each clause gadget can be extended to an equilibrium. By attaching two additional single-degree vertices to each vertex in a literal gadget, we can ensure that the literal gadgets themselves are in strict equilibrium and this does not affect any of the forcing arguments in the rest of the construction.

Conversely, if G has a strictly stable 2-coloring, then each clause gadget has a monochromatic literal gadget. This immediately provides an assignment of the variables which satisfies φ. All of the gadgets have a constant number of vertices, and so the construction is polynomial in the size of φ. This completes the reduction and proves the theorem. □

5 Stable colorings in directed graphs

In this section we turn to directed graphs. The directed case clearly generalizes the undirected, as each undirected edge can be replaced by two directed edges. Moreover, directed graphs can capture coordination. If vertex u wants to coordinate with vertex v, we can create a vertex u' and edges (u, u') and (u', v). Hence, this model is quite general.

Unlike in the undirected graph case, a vertex updating its color according to Equation 1 does not necessarily improve the overall social welfare. In fact, we cannot guarantee that a pure strategy equilibrium even exists – e.g. a directed 3-cycle has no stable coloring, a fact that we will use in this section.

We now turn to the problem of determining if a directed graph has an equilibrium with k colors and prove it is NP-hard. Indeed, for strictly stable colorings the answer is immediate by reduction from the undirected case. Interestingly enough, it is also NP-hard for non-strict k-colorings for any $k \geq 2$.

Theorem 4. *For all $k \geq 2$, determining whether a directed graph has a stable k-coloring is NP-complete.*

Proof. This problem is in clearly in NP. We again separate the analysis into two parts: $k = 2$ and $k \geq 3$.
1) $k = 2$: We reduce from the balanced unfriendly partition problem. A balanced 2-partition of an undirected graph is called unfriendly if each vertex has at least as many neighbors outside its part as within. Bazgan et al. proved the decision problem for balanced unfriendly partitions is NP-complete [2]. Given an undirected graph G as an instance of balanced unfriendly partition, we construct a directed graph G' as follows.

Start by giving G' the same vertex set as G, and replace each undirected edge of G with a pair of directed edges in G'. Add two vertices u, v to G', each with edges to the other and to all other vertices in G'. Add an additional vertex w with an edge (w, v), and connect one vertex of a 3-cycle to u and w, as shown in Figure 7.

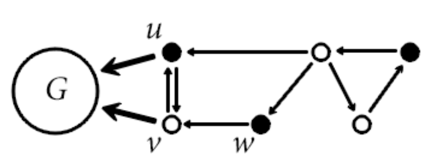

Fig. 7. The construction from balanced unfriendly partition to directed stable 2-coloring. Here u and v "stabilize" the 3-cycle. A bold arrow denotes a complete incidence from the source to the target.

If G has a balanced unfriendly partition, then this corresponds to a two-coloring of G in which the two colors occur equally often. Partially coloring G' in this way, we can achieve stability by coloring u, v opposite colors, w the same color as u, and using this to stabilize the 3-cycle, as shown in Figure 7.

Conversely, suppose G does not have an unbalanced friendly partition, fix a stable 2-coloring of G'. Without loss of generality suppose G has an even number of vertices and suppose color 1 occurs more often among the vertices coming from G. Then u, v must both be colored by color 2, and hence w is colored by 1. Since u, w have different colors, the 3-cycle will not be stable. This completes the reduction.

2) $k \geq 3$: We reduce from the case of $k = 2$. The idea is to augment the construction G' above by disallowing all but two colors to be used in the G' part. We call the larger construction G''.

Formally, start with $G'' = G'$ add two new vertices x, y to G'' which are adjacent to each other. In a stable coloring, x and y will necessarily have different colors (in our construction they will not be the tail of any other edges). We will force these two colors to be used in coloring G'. Specifically, let n be the number of vertices of G', and construct $(k-2)n^3$ copies of K_{k-2}. For each vertex v in one of these copies, add the edges $(v, x), (v, y)$. Figure 8 shows this construction visually.

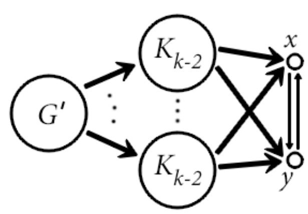

Fig. 8. Reducing k colors to two colors. A bold arrow indicates complete incidence from the source subgraph to the target subgraph.

Now in a stable coloring any vertex from a copy of K_{k-2} must use a different color than both x, y, and the vertex set of a copy of K_{k-2} must use all possible remaining $k-2$ colors. Now we add all edges (a, b) where $a \in G'$ and b comes from a copy of K_{k-2}. Each $a \in G'$ will have exactly n^3 neighbors of each of the $k-2$ colors, forcing the vertices of G' to use the remaining two colors. $\square$

6 Discussion and open problems

In this paper we defined new notions of graph coloring. Our results elucidated anti-coordination behavior, and solved some open problems in related areas.

Many interesting questions remain. For instance, one can consider alternative payoff functions. For players choosing colors i and j, the payoff $|i - j|$ is related to the *channel assignment problem* [15]. In the cases when the coloring problem is hard, as in our problem and the example above, we can find classes of graphs in which it is feasible, or study random graphs in which we conjecture colorings should be possible to find. Another variant is to study weighted graphs, perhaps with weights, as distances, satisfying a Euclidian metric.

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