

# Binary Search Techniques

Andre Kessler

December 10, 2010

## 1 Knowing the Answer is Helpful

Suppose we have a difficult problem labeled  $A$ . Specifically

Prob A: Find the answer to problem  $P$ .

Prob B: Given  $X$ , is this answer less than or equal to  $X$ ?

It is very often the case that solving problem  $B$  is much easier than solving problem  $A$ , so we can solve  $A$  by binary searching on the answer and applying the solution of problem  $B$  at each step. As a warm-up, solve the following problems:

1. Given an integer  $N$ , compute  $\lfloor \sqrt{N} \rfloor$ .
2. Find the minimum bottleneck shortest paths in a graph; that is, the set of paths whose maximum edge weight is as small as possible.
3. Given an array  $A[1 \dots 100,000]$  as input, find a subarray  $A[i \dots j]$  whose length is greater than 100 having maximum average.
4. I'm thinking of a positive integer and I will tell you if your guess is too high or too low. I do not give you an upper bound on the possibilities. Please determine my number.
5. I'm thinking of a positive real  $x$  between 1 and  $N$ . Determine the number I'm thinking of to within one decimal place in  $O(\log \log N)$  time.

## 2 Gold Binary Search Problems

Here is a brief analysis of the some problems involving binary search that have appeared on USACO. Try to determine what to binary search on and then how to solve the “inner problem” in the binary search.

1. (USACO FEB05, Jan Kuipers 2004) Farmer John has built a new long barn, with  $N$  ( $2 \leq N \leq 100,000$ ) stalls. The stalls are located along a straight line at positions  $x_1, \dots, x_N$  ( $0 \leq x_i \leq 1,000,000,000$ ). His  $C$  ( $2 \leq C \leq N$ ) cows don't like this barn layout and become aggressive towards each other once put into a stall. To prevent the cows from hurting each other, FJ want to assign the cows to the stalls, such that the minimum distance between any two of them is as large as possible. What is the largest minimum distance?

Analysis: Binary search on answer, cost function evaluation.

2. (USACO DEC10, Michael Cohen) Farmer John continues his never-ending quest to keep the cows fit by having them exercise on various cow paths that run through the pastures. These cow paths can be represented as a set of vertices connected with bidirectional edges so that each pair of vertices has exactly one simple path between them. In the abstract, their layout bears a remarkable resemblance to a tree. Surprisingly, each edge (as it winds its way through the pastures) has the same length. For any given set of cow paths, the canny cows calculate the longest possible distance between any pair of vertices on the set of cowpaths

and call it the pathlength. If they think this pathlength is too large, they simply refuse to exercise at all. Farmer John has mapped the paths and found  $V$  ( $2 \leq V \leq 100,000$ ) vertices, conveniently numbered from  $1..V$ . In order to make shorter cowpaths, he can block the path between any two vertices, thus creating more sets of cow paths while reducing the pathlength of both cowpath sets. Starting from a single completely connected set of paths (which have the properties of a tree), FJ can block  $S$  ( $1 \leq S \leq V - 1$ ) paths, creating  $S + 1$  sets of paths. Your goal is to compute the best paths he can create so that the largest pathlength of all those sets is minimized.

Analysis: Binary search on the answer  $D$ , DFS to process node information, greedily cut subtrees.

3. (USACO JAN08, Paul Christiano 2007) Farmer John wants to set up a telephone line at his farm. Unfortunately, the phone company is uncooperative, so he needs to pay for some of the cables required to connect his farm to the phone system. There are  $N$  ( $1 \leq N \leq 1,000$ ) forlorn telephone poles conveniently numbered  $1..N$  that are scattered around Farmer John's property; no cables connect any them. A total of  $P$  ( $1 \leq P \leq 10,000$ ) pairs of poles can be connected by a cable; the rest are too far apart. The  $i$ -th cable can connect the two distinct poles  $A_i$  and  $B_i$ , with length  $L_i$  ( $1 \leq L_i \leq 1,000,000$ ) units if used. The input data set never names any  $\{A_i, B_i\}$  pair more than once. Pole 1 is already connected to the phone system, and pole  $N$  is at the farm. Poles 1 and  $N$  need to be connected by a path of cables; the rest of the poles might be used or might not be used. As it turns out, the phone company is willing to provide Farmer John with  $K$  ( $0 \leq K < N$ ) lengths of cable for free. Beyond that he will have to pay a price equal to the length of the longest remaining cable he requires (each pair of poles is connected with a separate cable), or 0 if he does not need any additional cables. Determine the minimum amount that Farmer John must pay.

Analysis: Construct a graph, binary search on edges used with weight greater than a limit, Dijkstra.

4. Farmer John has partitioned the pasture into a matrix of  $N \times N$  ( $2 \leq N \leq 100$ ) small squares and measured the altitude  $A_{i,j}$  of each square ( $0 \leq A_{i,j} \leq 2,000,000,000$ ). FJ is currently positioned at square  $1,1$  and wishes to visit Bessie who, in no small coincidence, is at square  $N,N$ . FJ can travel north, south, east, and west (i.e., in a rectilinear manner) and cannot travel diagonally. Ever aware of altitude sickness, FJ wants to minimize the maximum height he encounters. Determine a path for FJ to get from grid coordinate  $1,1$  to  $N,N$  that minimizes the height of the highest piece of land encountered along that route.

Analysis: Binary search on answer, floodfill to check connectivity.

5. Farmer John's cows are going white-water rafting in the Cowlorado River. The river is parallel to the  $x$ -axis and of length  $L$  ( $0 < L <= 1000000$ ) and width  $W$  ( $0 < W <= L$ ) with  $N$  ( $1 <= N <= 1000$ ) circular rocks of radius  $R_i$  ( $0 <= R_i <= W/2$ ) with centers at  $X_i$  ( $0 <= X_i <= L$ ),  $Y_i$  ( $0 <= Y_i <= W$ ) placed throughout the river. None of the rocks intersect each other, though some rocks may touch. The banks of the Cowlorado River are the straight lines  $y = 0$  and  $y = W$ . The cows are starting from somewhere on the line  $x = 0$  and want to get to the line  $x = L$ . The cows would like to know the radius of the largest circular raft they can fit through the river without going over rocks, to within 0.001 (the raft may touch rocks, but may not intersect them).

Analysis: Binary search on answer, expand rafts by test radius, check bank connectivity.

6. (USACO NOV08, Chen Hu 2006) Bessie's birthday is coming up, and she wishes to celebrate for the next  $D$  ( $1 \leq D \leq 100,000$ ; 70% of testdata has  $1 \leq D \leq 500$ ) days. Cows have short

attention spans so Bessie wants to provide toys to entertain them. She has calculated that she will require  $T_i (1 \leq T_i \leq 50)$  toys on day  $i$ . Bessie's kindergarten provides many services to its aspiring bovine programmers, including a toy shop which sells toys for  $T_c (1 \leq T_c \leq 60)$  dollars. Bessie wishes to save money by reusing toys, but Farmer John is worried about transmitting diseases and requires toys to be disinfected before use. (The toy shop disinfects the toys when it sells them.) The two disinfectant services near the farm provide handy complete services. The first one charges  $C1$  dollars and requires  $N1$  nights to complete; the second charges  $C2$  dollars and requires  $N2$  nights to complete ( $1 \leq N1 \leq D; 1 \leq N2 \leq D; 1 \leq C1 \leq 60; 1 \leq C2 \leq 60$ ). Bessie takes the toys to the disinfectors after the party and can pay and pick them back up the next morning if one night service is rendered, or on later mornings if more nights are required for disinfecting. Being an educated cow, Bessie has already learned the value of saving her money. Help her find the cheapest way she can provide toys for her party.

Analysis: Definition of a cost function, prove convexity of cost function, ternary search on cost function (alternative, binary search on derivative of slope function).