ORTHOGONAL MATRIX

A square matrix with real numbers or elements is said to be an orthogonal matrix if its transpose is equal to its inverse matrix. Or we can say when the product of a square matrix and its transpose gives an identity matrix, then the square matrix is known as an orthogonal matrix.

A Square matrix 'A' is orthogonal if

$$A^T = A^{-1}$$

(OR)

$$AA^T = A^TA = I \text{ , where}$$

- A^T = Transpose of A
- A^{-1} = Inverse of A
- I = Identity matrix of same order as 'A'

The value of the determinant of an orthogonal matrix is always ±1.

Gram-Schmidt process

The Gram-Schmidt process (or procedure) is a sequence of operations that allow us to transform a set of linearly independent vectors into a set of orthonormal vectors that span the same space spanned by the original set.

The aim of the Gram-Schmidt process

Start with	Find
linearly independent vectors	orthonormal vectors
$s_1, \dots, s_K$	$u_1, \dots, u_K$

1. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for an inner product space V.

2. Let $B' = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$, where

$$\mathbf{w}_1 = \mathbf{v}_1$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2$$

$$\vdots$$

$$\mathbf{w}_n = \mathbf{v}_n - \frac{\langle \mathbf{v}_n, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_n, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2 - \cdots - \frac{\langle \mathbf{v}_n, \mathbf{w}_{n-1} \rangle}{\langle \mathbf{w}_{n-1}, \mathbf{w}_{n-1} \rangle}\mathbf{w}_{n-1}.$$

Then B' is an *orthogonal* basis for V.

3. Let $\mathbf{u}_i = \frac{\mathbf{w}_i}{\|\mathbf{w}_i\|}$. Then $B'' = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an *orthonormal* basis for V.
Also, $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ for $k = 1, 2, \dots, n$.

Example: Consider a square matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- Replace the basis $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ with an orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$

$$\mathbf{v}_1 = \mathbf{u}_1 \Rightarrow \mathbf{w}_1 = \mathbf{v}_1 / \|\mathbf{v}_1\| = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{u}_2 - (\mathbf{w}_1, \mathbf{u}_2)\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}$$

$$\mathbf{w}_2 = \mathbf{v}_2 / \|\mathbf{v}_2\| = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{u}_3 - (\mathbf{u}_3, \mathbf{w}_1)\mathbf{w}_1 - (\mathbf{u}_3, \mathbf{w}_2)\mathbf{w}_2$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} - \sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{w}_3 = \mathbf{v}_3 / \|\mathbf{v}_3\| = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

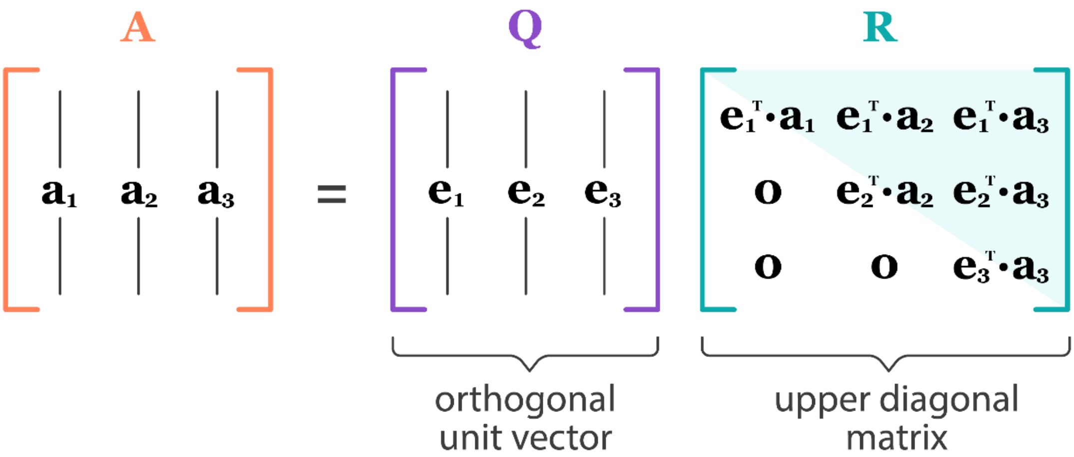
$$\text{Orthonormal set is} \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Orthonormal basis is

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

QR Decomposition with Gram-Schmidt

The QR decomposition (also called the QR factorization) of a matrix is a decomposition of the matrix into an orthogonal matrix and a triangular matrix. A QR decomposition of a real square matrix A is a decomposition of A as A = QR, where Q is an orthogonal matrix (i.e., $Q^TQ = I$) and R is an upper triangular matrix. If A is non-singular, then this factorization is unique.



There are several methods for actually computing the QR decomposition. One of such method is the Gram-Schmidt process.

Consider the above example:

- In the Gram-Schmidt example, the basis

$$\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\1\\0\end{bmatrix}\right\} \text{ is transformed to} \left\{\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix},\begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$$

Interpreting these vectors as column vectors of matrices, the following result holds

$$\mathbf{A}=\begin{bmatrix}1&1&2\\0&0&1\\1&0&0\end{bmatrix}=\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}&0\\0&0&1\\1/\sqrt{2}&-1/\sqrt{2}&0\end{bmatrix}\begin{bmatrix}\sqrt{2}&1/\sqrt{2}&\sqrt{2}\\&1/\sqrt{2}&\sqrt{2}\\&&1\end{bmatrix}=\mathbf{QR}$$

This is called the QR-Factorization of **A**

Python Code

```
import numpy as np

# Define the matrix
A = np.array([[1, 1, 2], [0, 0, 1], [1, 0, 0]])

# Perform QR factorization
Q, R = np.linalg.qr(A)

print("Q matrix:\n", Q)
print("R matrix:\n", R)
```

```
Q matrix:
 [[-0.70710678 -0.70710678  0.        ]
 [-0.          0.          1.        ]
 [-0.70710678  0.70710678  0.        ]]
R matrix:
 [[-1.41421356 -0.70710678 -1.41421356]
 [ 0.         -0.70710678 -1.41421356]
 [ 0.          0.          1.        ]]
```