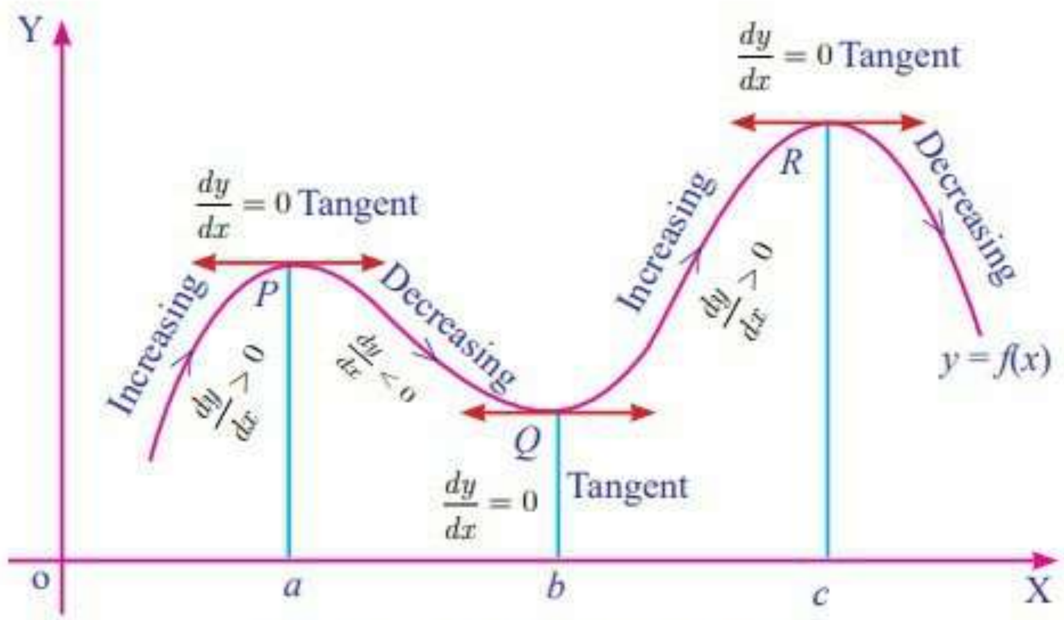
Applications of Derivatives



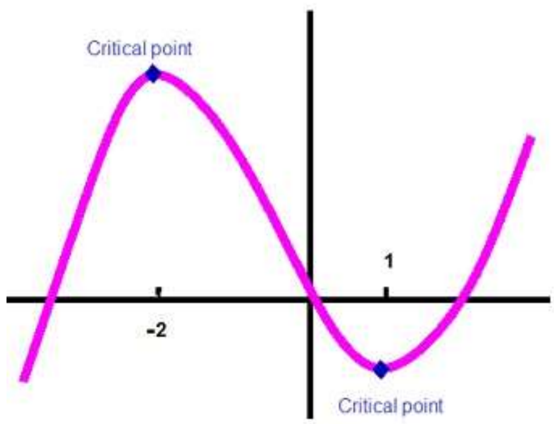
P, Q, R are called Stationary Points

Critical points or Turning points or Stationary points of a Function

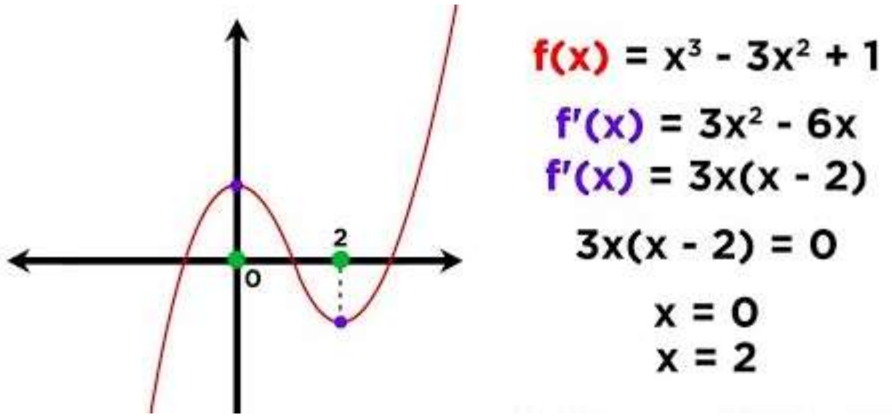
To find the critical points of a function, you need to find the values of $f(x)$ where the derivative of the function is equal to zero. Critical points can occur at points where the derivative is zero (stationary points) or where it is undefined.

Step 1: Find the derivative of $f(x)$ with respect to x

Step 2: Set the derivative equal to zero and solve for $f'(x) = 0$



Example 1: Find the critical points of the function $f(x) = x^3 - 3x^2 + 1$



So, the critical points of the function $f(x) = x^3 - 3x^2 + 1$ are $x = 0, x = 2$

Example 2: Find the critical points of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Solution:

Given, $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Differentiate w.r.t. x

$$f'(x) = 5x^4 - 20x^3 + 15x^2$$

Solve $f'(x) = 0$

$$5x^4 - 20x^3 + 15x^2 = 0$$

$$5x^2 (x^2 - 4x + 3) = 0$$

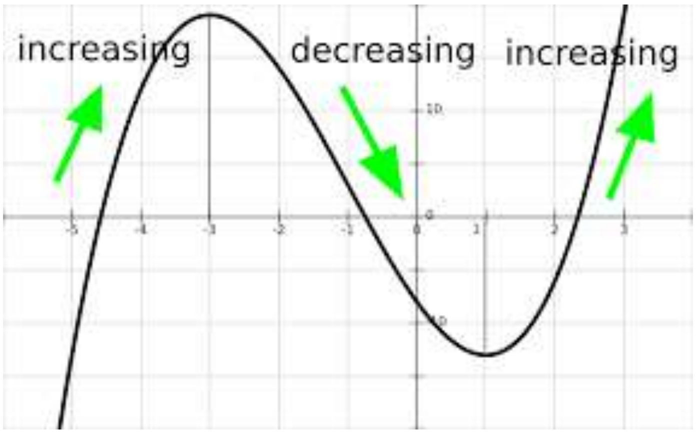
$$5x^2 (x - 1)(x - 3) = 0$$

$$x = 0, x = 1, x = 3$$

So, the critical points of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$ are

$$x = 0, x = 1, x = 3$$

Increasing or Decreasing Function



Analyse the sign of $f'(x)$ to determine whether the function $f(x)$ is increasing or decreasing:

- If $f'(x) > 0$ for all X, then $f(x)$ is increasing.
- If $f'(x) < 0$ for all X, then $f(x)$ is decreasing.

Examples:

Example 1: Find out whether the function is increasing or decreasing f(x) = –8x^2 + 15

Solution:

Given, $f(x) = -8x^2 + 15$

Differentiate w.r.t. x

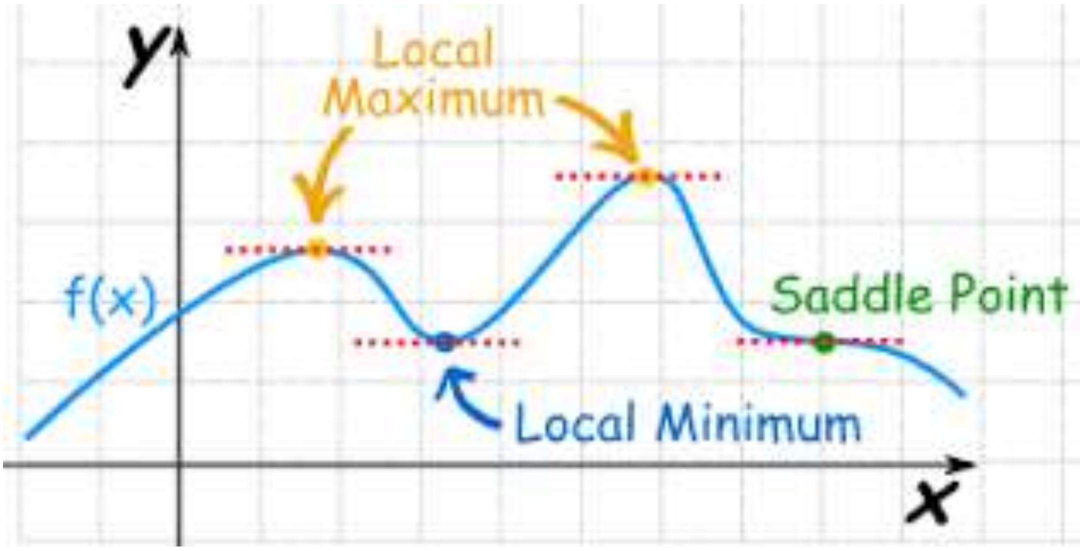
$$f'(x) = -16x$$

The sign of $f'(x)$ depends on the sign of x.

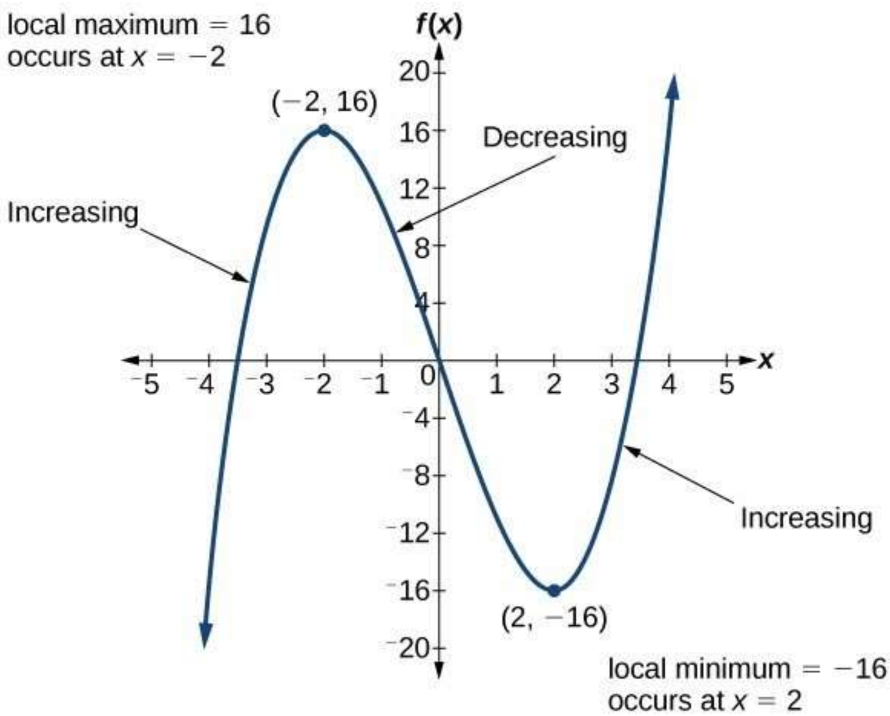
- When x > 0, $f'(x)$ is negative, so the function is decreasing.
- When x < 0, $f'(x)$ is positive, so the function is increasing.

Therefore, the function $f(x) = -8x^2 + 15$ is decreasing for x > 0 and increasing for x < 0.

Maxima and Minima



Example:



First Derivative Test: Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then

(i) If f ′(x) changes sign from positive to negative as x increases through c, i.e., if f ′(x) > 0 at every point sufficiently close to and to the left of c, and f ′(x) <

0 at every point sufficiently close to and to the right of c, then c is a point of local maxima.

(ii) If f ′(x) changes sign from negative to positive as x increases through c, i.e., if f ′(x) < 0 at every point sufficiently close to and to the left of c, and f ′(x) > 0 at every point sufficiently close to and to the right of c, then c is a point of local minima.

(iii) If f ′(x) does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called point of inflection.

Second Derivative Test: Let f be a function defined on an interval I and c ∈ I. Let f be twice differentiable at c. Then

(i) x = c is a point of local maxima if f ′(c) = 0 and f ″(c) < 0.
The value f(c) is local maximum value of f.

(ii) x = c is a point of local minima if f ′(c) = 0 and f ″(c) > 0
In this case, f (c) is local minimum value of f.

(iii) The test fails if f ′(c) = 0 and f ″(c) = 0. In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.

Example 1: Find the point of inflection, local maxima & minima for the function $f(x) = 5x^3 + 2x^2 - 3x$ in the interval, (-2, 3)

Solution:

Given, $f(x) = 5x^3 + 2x^2 - 3x$

Differentiate w.r.t. x

$$f'(x) = 15x^2 + 4x - 3$$

Differentiate again w.r.t. x

$$f''(x) = 30x + 4$$

Solve $f'(x) = 0$ to find critical points

$$15x^2 + 4x - 3 = 0$$

$$(3x - 1)(5x + 3) = 0$$

$$x = \frac{1}{3}, x = \frac{-3}{5}$$

(i) At $x = \frac{1}{3}$,

$$f''\left(\frac{1}{3}\right) = 30\left(\frac{1}{3}\right) + 4 = 14 > 0$$

$f(x)$ attains Local minimum at $x = \frac{1}{3}$ and the minimum value is $f\left(\frac{1}{3}\right) =$ $5\left(\frac{1}{3}\right)^3 + 2x\left(\frac{1}{3}\right)^2 - 3\left(\frac{1}{3}\right) = -\frac{16}{27}$

(ii) At $x = \frac{-3}{5}$,

$$f''\left(\frac{-3}{5}\right) = 30\left(\frac{-3}{5}\right) + 4 = -14 < 0$$

$f(x)$ attains Local maximum at $x = \frac{-3}{5}$ and the maximum value is $f\left(\frac{-3}{5}\right) =$ $5\left(\frac{-3}{5}\right)^3 + 2x\left(\frac{-3}{5}\right)^2 - 3\left(\frac{-3}{5}\right) = \frac{36}{25}$

Evaluate, $f(x)$ at the endpoints of the given interval $(-2, 3)$

$$f(-2) = 5(-2)^3 + 2(-2)^2 - 3(-2) = -40 + 8 + 6 = -26$$

$$f(3) = 5(3)^3 + 2(3)^2 - 3(3) = 135 + 18 - 9 = 144$$

To find point of inflection, consider $f''(x) = 0$

$$30x + 4 = 0$$

$$x = -\frac{2}{15}$$

$x = -\frac{2}{15}$ is the point of inflection.

Example 2: Find the maxima or minima of the function $f(x) = x^3 - 2x$ in the interval (-5, -2)

Solution:

Given, $f(x) = x^3 - 2x$

Differentiate w.r.t. x

$$f'(x) = 3x^2 - 2$$

Differentiate again w.r.t. x

$$f''(x) = 6x$$

Solve $f'(x) = 0$ to find critical points

$$3x^2 - 2 = 0$$

$$3x^2 = 2$$

$$x = \pm\sqrt{\frac{2}{3}}$$

Here, the only point $x = -\sqrt{\frac{2}{3}}$ lies in the given interval (-5, -2).

At $x = -\sqrt{\frac{2}{3}}$,

$$f''(x) = 6\left(-\sqrt{\frac{2}{3}}\right) = -\sqrt{24} < 0$$

$f(x)$ attains Local maximum at $x = -\sqrt{\frac{2}{3}}$ and the maximum value is $f\left(-\sqrt{\frac{2}{3}}\right) = \left(-\sqrt{\frac{2}{3}}\right)^3 - 2\left(-\sqrt{\frac{2}{3}}\right) = \frac{4}{3}\sqrt{\frac{2}{3}}$

Example 3: Find the minimum value of f(x) when x < 5, where f(x) = $3x^6 + 5x^4 + 1$

Solution:

Given, $f(x) = 3x^6 + 5x^4 + 1$

Differentiate w.r.t. x

$$f'(x) = 18x^5 + 20x^3$$

Differentiate again w.r.t. x

$$f''(x) = 90x^4 + 60x^2$$

Solve $f'(x) = 0$ to find critical points

$$18x^5 + 20x^3 = 0$$

$$2x^3(18x^2 + 10) = 0$$

$x = 0$ is the only real point.

At $x = 0,\ f''(x) = 0$

$f(x)$ has neither maximum nor minimum at x = 0 and this point is called point of inflection.

Example 4: Find the minimum value of f(x) when x < 5, where f(x) = $x^4 + x^2 + 1$

Solution:

Given, $f(x) = x^4 + x^2 + 1$

Differentiate w.r.t. x

$$f'(x) = 4x^3 + 2x$$

Differentiate again w.r.t. x

$$f''(x) = 12x^2 + 2$$

Solve $f'(x) = 0$ to find critical points

$$4x^3 + 2x = 0$$

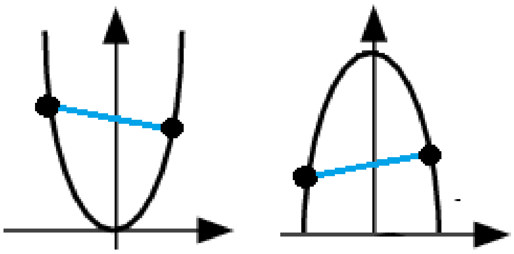
$$2x(2x^2 + 1) = 0$$

$x = 0$ is the only real point.

At $x = 0,\ f''(x) = 2 > 0$

$f(x)$ attains minimum at x = 0 and the minimum value is $f(0) = 1$.

Convex or Concave Function



To determine whether a function is concave or convex, you can look at the second derivative of the function. Specifically:

- If the second derivative is positive for all x in the domain, then the function is convex.
- If the second derivative is negative for all x in the domain, then the function is concave.

Examples:

Example 1: Verify the function $f(x) = -x^2 - 7x$ is concave function or convex function in the interval (-5, -2)

Solution:

Given, $f(x) = -x^2 - 7x$

Differentiate w.r.t. x

$$f'(x) = -2x - 7$$

Differentiate again w.r.t. x

$$f''(x) = -2$$

The second derivative is a constant −2 and it is negative.

Since the second derivative is negative for all x, (including the interval (−5, −2) the function is concave everywhere in this interval.

In conclusion, on the interval (−5, −2), the function $f(x) = -x^2 - 7x$ is concave.

Example 2: Find out whether the function f(x) = –8x² + 15 is concave or convex.

Solution:

Given, $f(x) = -8x^2 + 15$

Differentiate w.r.t. x

$$f'(x) = -16x$$

Differentiate again w.r.t. x

$$f''(x) = -16$$

The second derivative is a constant −16 and it is negative.

Since the second derivative is negative for all x, the function is concave.

Derivatives as a Rate Measure

If a quantity y is dependent on and varies in relation to a quantity x, then the rate of change of y with respect to x is denoted by the symbol

$$\frac{dy}{dx}$$

For example, the rate of change of pressure p in relation to height h is denoted by the symbol

$$\frac{dp}{dh}$$

Examples

Example 1: Find the rate of change of the area of a circle per second with respect to its radius r when r = 5 cm.

Solution:

The area A of a circle with radius r is given by $A = \pi r^2$.

Therefore, the rate of change of the area A with respect to its radius r is given by

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$$

When $r = 5$ cm, $\frac{dA}{dr} = 10\pi$.

Thus, the area of the circle is changing at the rate of 10π cm^2 /s.

Example 2: A stone is dropped into a quiet lake and waves move in circles at a speed of 4cm per second. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?

Solution:

The area A of a circle with radius r is given by $A = \pi r^2$.

Therefore, the rate of change of area A with respect to time t is

$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2) = \frac{d}{dr}(\pi r^2) \cdot \frac{dr}{dt} = 2\pi r \frac{dr}{dt}$$

It is given that $\frac{dr}{dt} = 4\text{cm/s}$

Therefore, when $r = 10$ cm, $\frac{dA}{dt} = 2\pi(10)(4) = 80\pi$

Thus, the enclosed area is increasing at the rate of 80π cm^2 /s, when r = 10 cm.

Example 3: The length x of a rectangle is decreasing at the rate of 3 cm/minute and the width y is increasing at the rate of 2cm/minute. When x =10cm and y = 6cm, find the rates of change of (a) the perimeter and (b) the area of the rectangle.

Solution:

Since the length x is decreasing and the width y is increasing with respect to time, we have

$$\frac{dx}{dt} = -3\ cm/\min \quad and \quad \frac{dy}{dt} = 2\ cm/min$$

(a) The perimeter P of a rectangle is given by P = 2 (x + y)

$$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2\,(-3+2) = -2 \text{ cm/min}$$

(b) The area A of the rectangle is given by A = x.y

$$\frac{dA}{dt} = \frac{dx}{dt}\cdot y + x\cdot\frac{dy}{dt}$$

$$= -3(6) + 10(2) \quad (\text{as } x = 10 \text{ cm and } y = 6 \text{ cm})$$

$$= 2 \text{ cm}^2\text{/min}$$

Example 4: The volume of a cube is increasing at a rate of 9 cubic centimetres per second. How fast is the surface area increasing when the length of an edge is 10 centimetres?

Solution:

Let x be the length of a side, V be the volume and S be the surface area of the cube. Then, $V = x^3$ and $S = 6x^2$, where x is a function of time t.

Now $$\frac{dV}{dt} = 9\text{cm}^3\text{/s (Given)}$$

Therefore $$9 = \frac{dV}{dt} = \frac{d}{dt}(x^3) = \frac{d}{dx}(x^3)\cdot\frac{dx}{dt} \quad \text{(By Chain Rule)}$$

$$= 3x^2\cdot\frac{dx}{dt}$$

or $$\frac{dx}{dt} = \frac{3}{x^2} \quad \text{... (1)}$$

Now $$\frac{dS}{dt} = \frac{d}{dt}(6x^2) = \frac{d}{dx}(6x^2)\cdot\frac{dx}{dt} \quad \text{(By Chain Rule)}$$

$$= 12x\cdot\left(\frac{3}{x^2}\right) = \frac{36}{x} \quad \text{(Using (1))}$$

Hence, when $x = 10$ cm, $\frac{dS}{dt} = 3.6 \text{ cm}^2\text{/s}$

Marginal Revenue

Marginal Revenue (MR) is the additional revenue generated from producing and selling one more unit of a good or service. It represents the change in total revenue resulting from the sale of an additional unit. Mathematically, marginal revenue is often defined as the derivative of the total revenue function with respect to the quantity of units sold.

Examples

1. The revenue generation function of an IT company is $3000x - 20x^2 + 200$ rupees where x is the number of employees. Find out the marginal revenue generation when 10 employees are hired.

Solution:

The marginal revenue (MR) is the additional revenue generated from selling one more unit of a product or, in this case, hiring one more employee. To find the marginal revenue, you need to find the derivative of the revenue function with respect to the number of employees $R(x)$, and then evaluate it at the given point.

Given the revenue function: $R(x) = 3000x - 20x^2 + 200$

Differentiate w.r.t. x

$$R'(x) = 3000 - 40x$$

Evaluate $R'(x)$ at x =10 (number of employees):

$$R'(10) = 3000 - 40 \times 10 = 3000 - 400 = 2600$$

Marginal revenue when 10 employees are hired = Rs.2600

2. The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue, when x = 5, where by marginal revenue we mean the rate of change of total revenue with respect to the number of items sold at an instant.

Solution: Since marginal revenue is the rate of change of total revenue with respect to the number of units sold, we have

Marginal Revenue (MR) = 3(2x) + 36(1) = 6x + 36

When x = 5, MR = 6(5) + 36 = 66

Hence, the required marginal revenue is Rs.66.

Marginal Cost

Marginal Cost (MC) is the additional cost incurred by producing one more unit of a good or service. It represents the change in total cost resulting from the production of an additional unit. Marginal cost is a fundamental concept in economics and business decision-making, helping businesses determine the optimal level of production. Mathematically, marginal cost is often defined as the derivative of the total cost function with respect to the quantity of units produced.

Example:

The total cost C(x) in Rupees, associated with the production of x units of an item is given by $C(x) = 0.005\, x^3 - 0.02\, x^2 + 30x + 5000$. Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.

Solution:

Since marginal cost is the rate of change of total cost with respect to the output, we have

Marginal cost (MC) = $0.005(3x^2) - 0.02(2x) + 30$

When x = 3, MC = 0.135 – 0.12 + 30 = 30.015

Hence, the required marginal cost is Rs. 30.02 (nearly).

Summary

	$f(x)$	
$f'(x) > 0$	$\nearrow$	increasing
$f'(x) < 0$	$\searrow$	decreasing
$f''(x) > 0$	$\cup$	concave up
$f''(x) < 0$	$\cap$	concave down