Transformation Matrix

Transformation Matrix is a matrix that transforms one vector into another vector by the process of matrix multiplication. The transformation matrix alters the cartesian system and maps the coordinates of the vector to the new coordinates. The transformation matrix T of order m x n on multiplication with a vector A of n components represented as a column matrix transforms it into another matrix representing a new vector B.

$$TA = B$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Examples

Example 1: Find the new vector formed for the vector 5i + 4j with the help of the transformation matrix $\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$.

The given transformation matrix is $T = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$

The given vector A = 5i + 4j is written as a column matrix as $A = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

Let the new matrix after transformation be B and we have the transformation formula as TA = B

$$B = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 13 \end{bmatrix}$$

Therefore, the new vector on transformation is -2i + 13j.

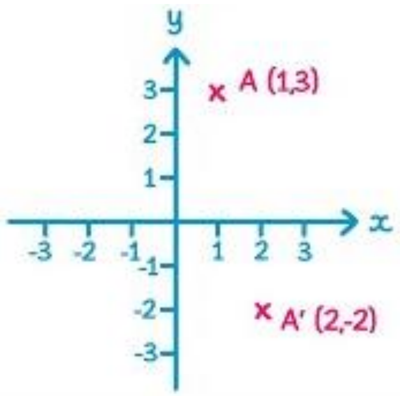
Example 2: Find the new vector formed for the vector i + 3j with the help of the transformation matrix $\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$.

The given transformation matrix is $T = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

The given vector A = i + 3j is written as a column matrix as $A = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Let the new matrix after transformation be B and we have the transformation formula as TA = B

$$\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$



Therefore, the new vector on transformation is 2i – 2j.

Types of Transformation Matrix

The transformation matrix transforms a vector into another vector, which can be understood geometrically in a 2-dimensional or a 3-dimensional space. The frequently used transformations are

- **Stretching**
- **Squeezing**
- **Rotation**
- **Reflection**
- **Scaling**

Let us learn about some of these transformations in detail.

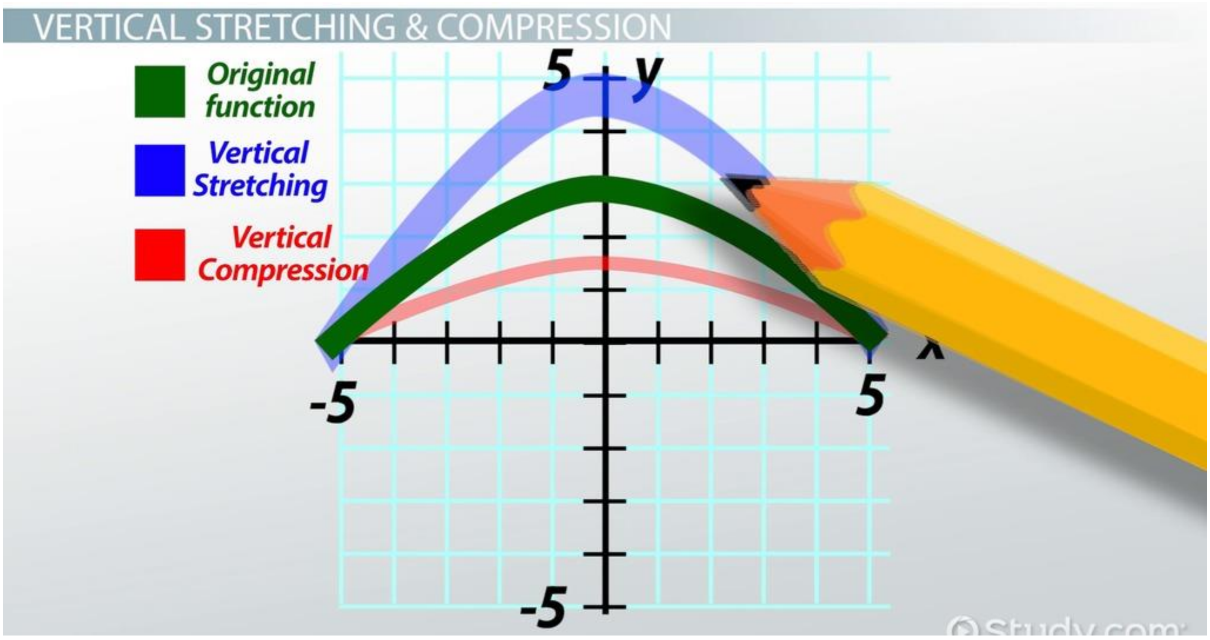
1. Stretching

The linear transformation enlarges the distance in the xy plane by a constant value. Here the distance is enlarged or compressed in a particular direction with reference to only one of the axis and the other axis is kept constant. A stretch along the x-axis by keeping the y-axis the same is x' = kx, and y' = y. Here k is a constant numeric value and for k > 1 it is a stretch, k < 1 it is a compression, and for k = 1 it is the same point. The transformation matrix for a stretch along the x-axis is as follows.

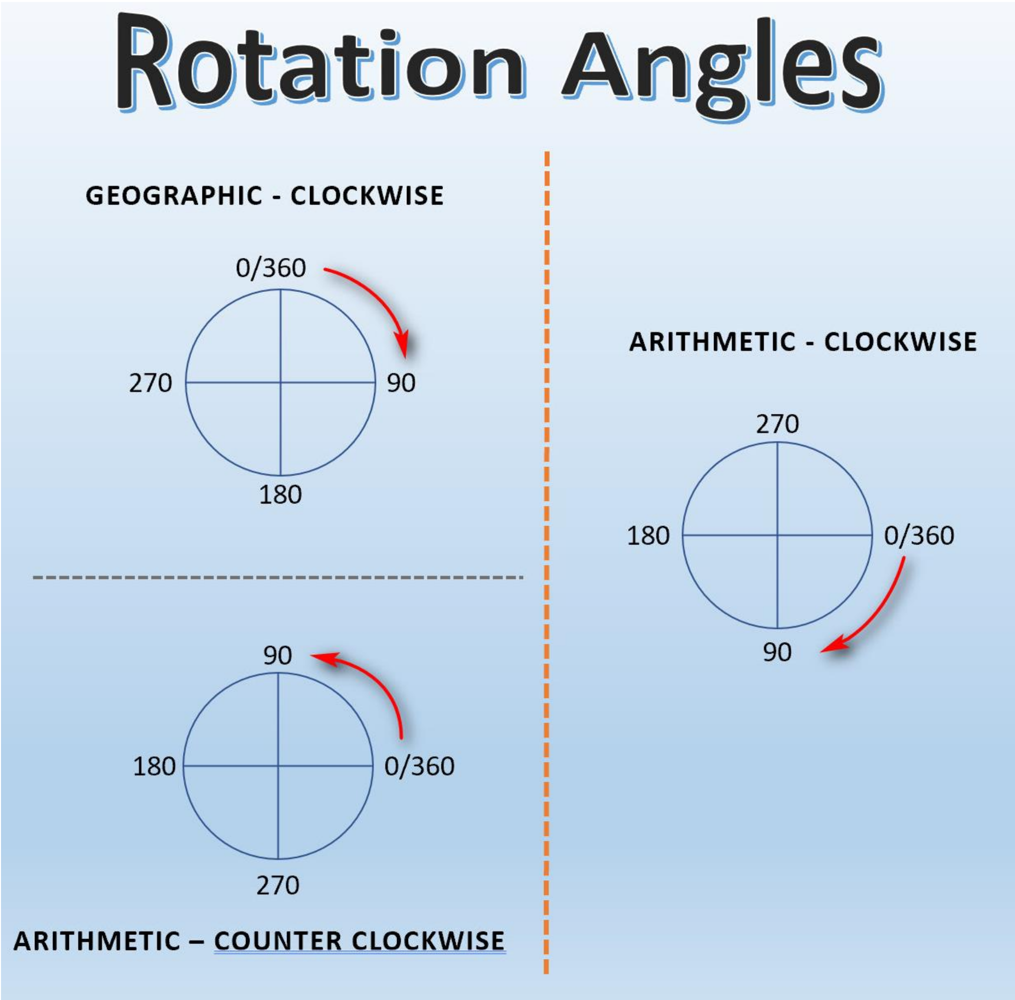
$$T_x = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

Also, we can stretch along the y-axis to obtain x' = x, and y' = ky. The transformation matrix for a stretch along the y-axis is as follows.

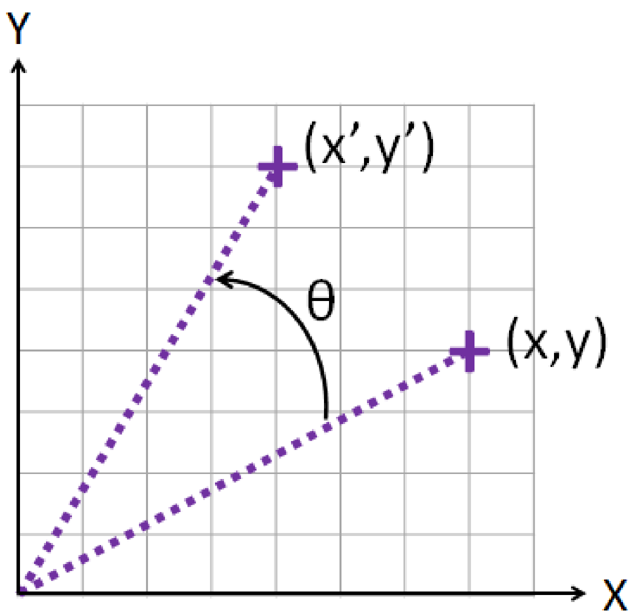
$$T_x = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$



2. Rotation



Rotation Angles
GEOGRAPHIC - CLOCKWISE
0/360
270
90
180
ARITHMETIC - CLOCKWISE
270
180
0/360
90
90
180
0/360
270
ARITHMETIC – COUNTER CLOCKWISE



Y
(x',y')
θ
(x,y)
X

The transformation matrix helps to rotate the vector in an anticlockwise (counter clockwise) direction at an angle θ. The transformation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ transforms the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x' \\ y' \end{bmatrix}$ counter clockwise, which is represented as follows.

$$R(\theta) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

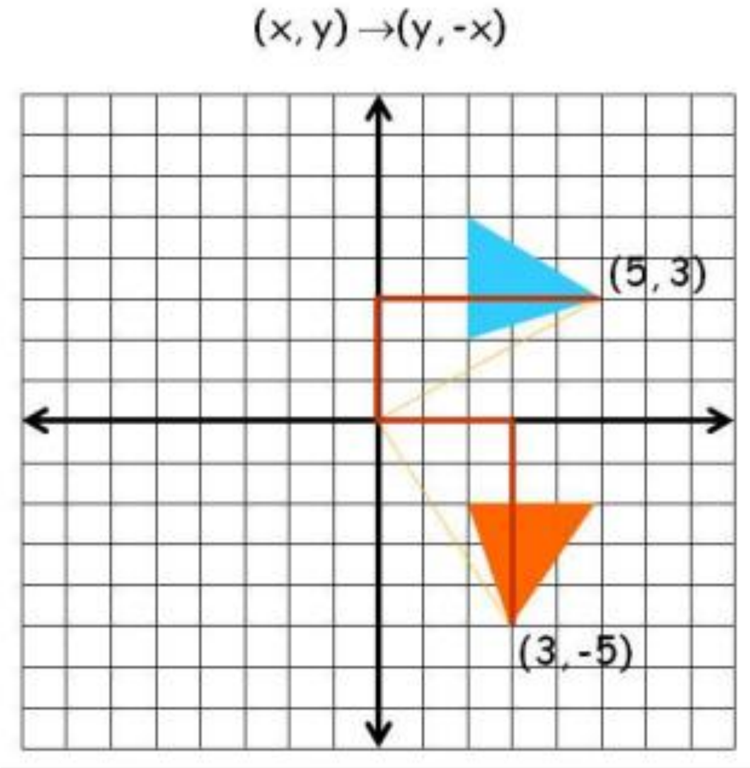
The transformation matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ transforms the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x' \\ y' \end{bmatrix}$ clockwise, which is represented as follows.

$$R(-\theta) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

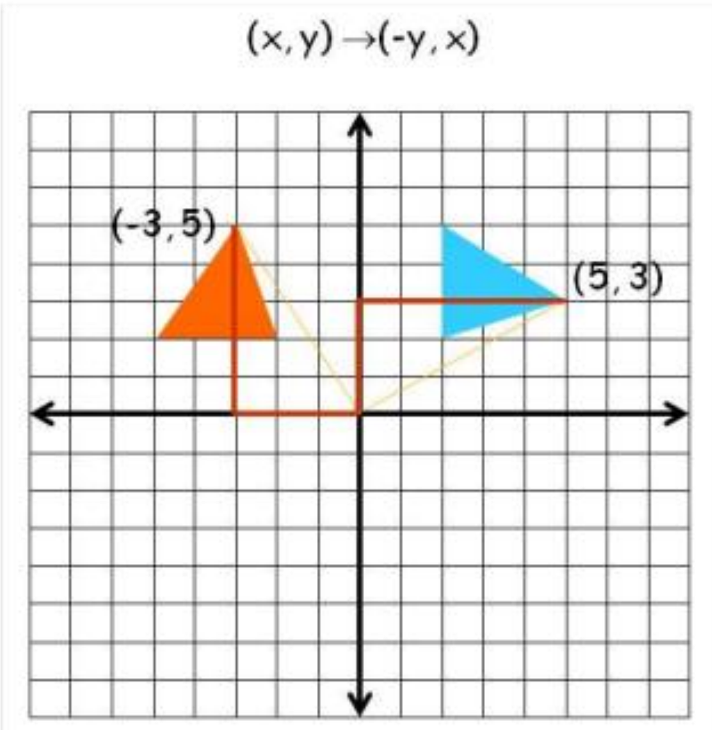
TYPE OF ROTATION	Matrix to be multiplied
Rotation of 90° (clock wise)	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
Rotation of 90° (counter clock wise)	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
Rotation of 180° (clock wise & counter clock wise)	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

TYPE OF ROTATION	Matrix to be multiplied
Rotation of 270° (clock wise)	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
Rotation of 270° (counter clock wise)	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

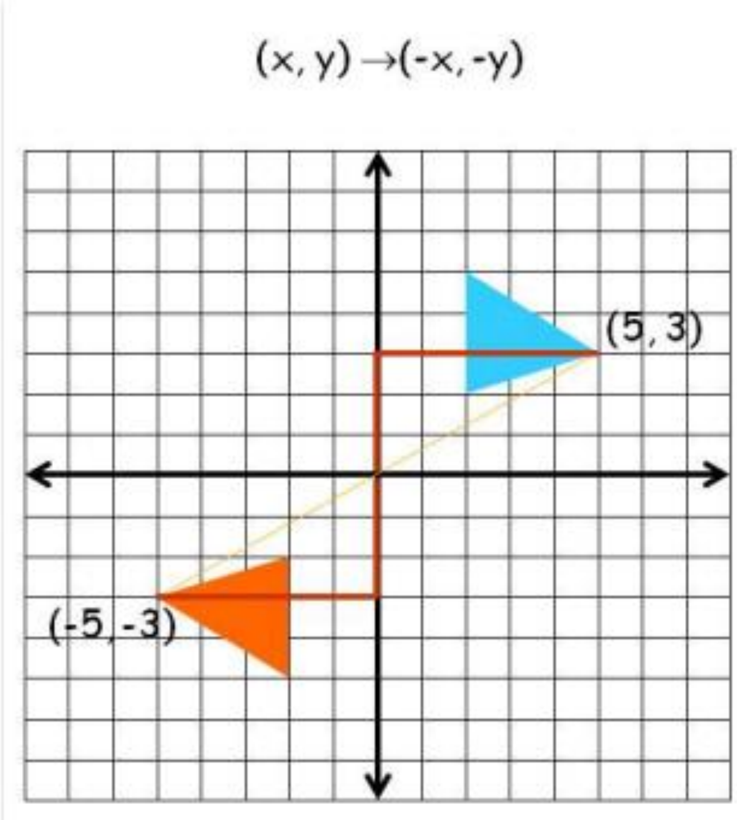
90° Rotation (Clock Wise)



90° Rotation (Counter Clock Wise)



180° Rotation (Clock Wise and Counter Clock Wise)



Problems

1. Rotate the vector v = 2i + 9j about 50° counter clockwise.

The given vector A = 2i + 9j is written as a column matrix as $A = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$

The transformation matrix is $T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Let the new matrix after transformation be B and we have the transformation formula as TA = B

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 50^{\circ} & -\sin 50^{\circ} \\ \sin 50^{\circ} & \cos 50^{\circ} \end{bmatrix} \begin{bmatrix} 2 \\ 9 \end{bmatrix} = \begin{bmatrix} -5.61 \\ 7.32 \end{bmatrix}$$

After rotation, the new vector is v' = –5.61i + 7.32j

2. Let A (-2, 1), B (2, 4) and C (4, 2) be the three vertices of a triangle. If this triangle is rotated about 90° counter clockwise, find the vertices of the rotated image A'B'C' using matrices.

First, we have to write the vertices of the given triangle ABC in matrix form as given below.

$$\begin{bmatrix} -2 & 2 & 4 \\ 1 & 4 & 2 \end{bmatrix}$$

Since the triangle ABC is rotated about 90° counter clockwise, to get the rotated image, we have to multiply the above matrix by the matrix given below.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Now, let us multiply the two matrices.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} -2 & 2 & 4 \\ 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -2 \\ -2 & 2 & 4 \end{bmatrix}$$

Now we can get vertices of the rotated image A'B'C' from the resultant matrix.

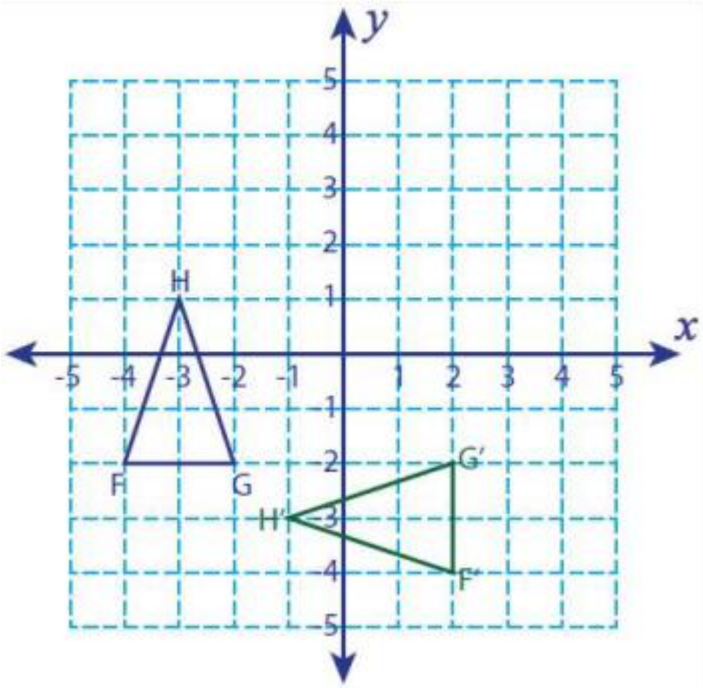
Vertices of the rotated image are

A'(-1, -2), B'(-4, 2) and C'(-2, 4)

3. Let F (-4, -2), G (-2, -2) and H (-3, 1) be the three vertices of a triangle. If this triangle is rotated 90° counter clockwise, find the vertices of the rotated figure and graph.

Vertices of the rotated figure are

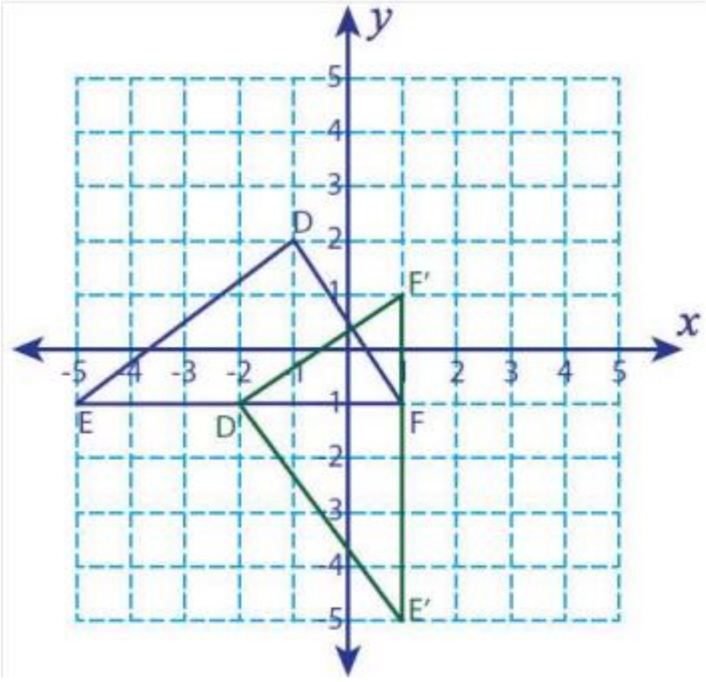
F'(2, -4), G'(2, -2) and H'(-1, -3)



4. Let D (-1, 2), E (-5, -1) and F (1, -1) be the vertices of a triangle. If the triangle is rotated 90° counterclockwise, find the vertices of the rotated figure and graph.

Vertices of the rotated figure are

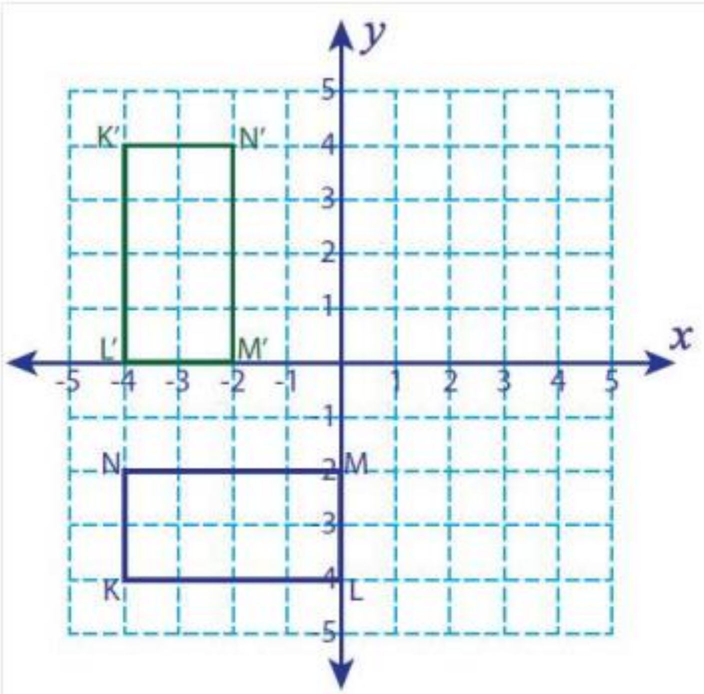
D'(-2, -1), E'(1, -5) and F'(1, 1)



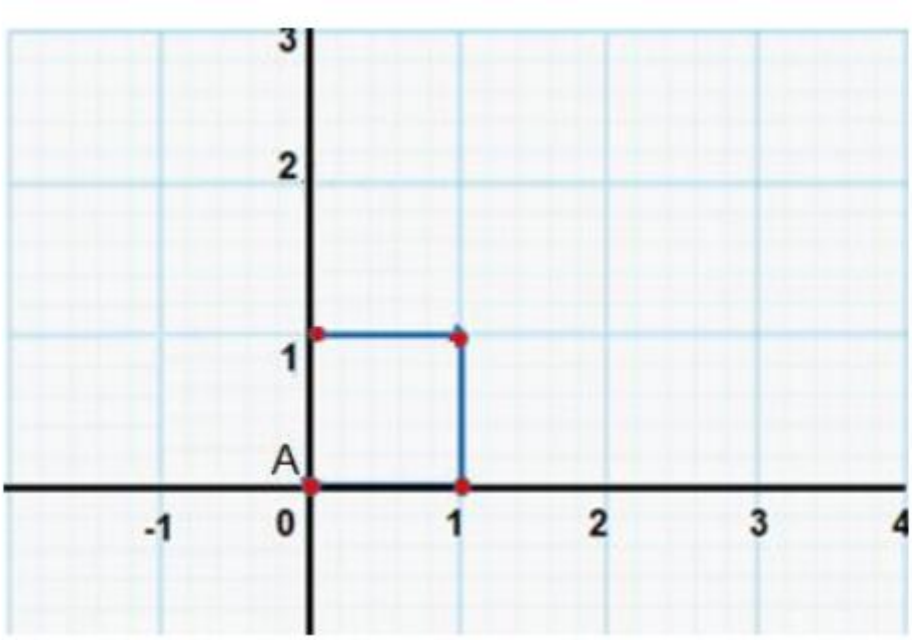
5. Let K (-4, -4), L (0, -4), M (0, -2) and N (-4, -2) be the vertices of a rectangle. If this rectangle is rotated 90° clockwise, find the vertices of the rotated figure and graph.

Vertices of the rotated figure are

K'(-4, 4), L'(-4, 0), M'(-2, 0) and N'(-2, 4)



6. The following box was rotated at an angle of 60° counter Clockwise around the origin (or point A). Find out the distance between the coordinate point of the box which passes through the y axis (excluding the origin or the point A) after the transformation and the point A.



The coordinates of the box are A (0, 0), B (1, 0), C (1, 1), D (0, 1)

First, we have to write the vertices of the given box ABCD in matrix form as given below.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

To find the new coordinates of a point after a rotation about the origin, the transformation matrix is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(60^{\circ}) & -\sin(60^{\circ}) \\ \sin(60^{\circ}) & \cos(60^{\circ}) \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.5 & -0.366 & -0.866 \\ 0 & 0.866 & 1.366 & 0.5 \end{bmatrix}$$

The new coordinates after the rotation are given by:

A (0, 0), B' (0.5, 0.866), C' (-0.366, 1.366), D' (-0.866, 0.5)

Now, we have to find the distance between each of these points and point A.

The distance formula between two points (x_1, y_1) and (x_2, y_2) is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

1. Distance between A and the new coordinates of B'

$$d = \sqrt{(0.5 - 0)^2 + (0.866 - 0)^2} = 0.99997 \approx 1$$

2. Distance between A and the new coordinates of C'

$$d = \sqrt{(-0.366 - 0)^2 + (1.366 - 0)^2} = 1.4142$$

3. Distance between A and the new coordinates of D'

$$d = \sqrt{(-0.866 - 0)^2 + (0.5 - 0)^2} = 0.99997 \approx 1$$

3. Shearing

The shearing changes the vector in such a way that the square boxes in the coordinate axes are deformed into parallelograms. The transformation matrix for shearing along the x-axis is $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ and it transforms the vector A = xi + yj to A' = x'i + y'j such that x' = x + ky, y' = y. This transformation can be understood from the below multiplication of matrices.

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Also, the transformation matrix for shearing along the y-axis is $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ and it changes the vector such that x' = x, and y' = y + kx. The matrix multiplication for this transformation is as follows.

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

4. **Reflection Matrix**

It is also called a flip matrix. Reflection is the mirror image of the original object. In other words, we can say that it is a rotation operation with 180°. In reflection transformation, the size of the object does not change.

It can be represented in matrix form as follows:

The matrix for Reflection along the X- axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The matrix for Reflection along the Y- axis is given by

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

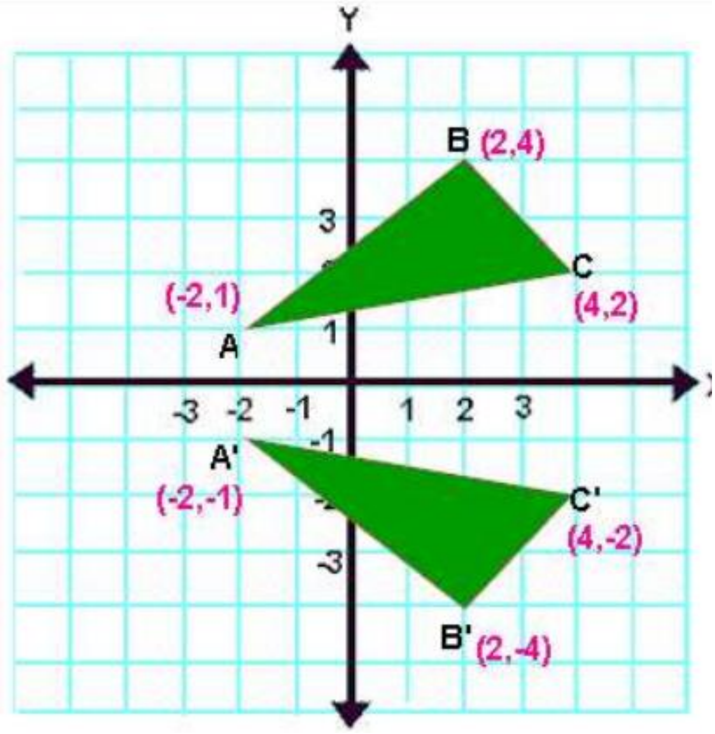
The matrix for Reflection along the x = y is given by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

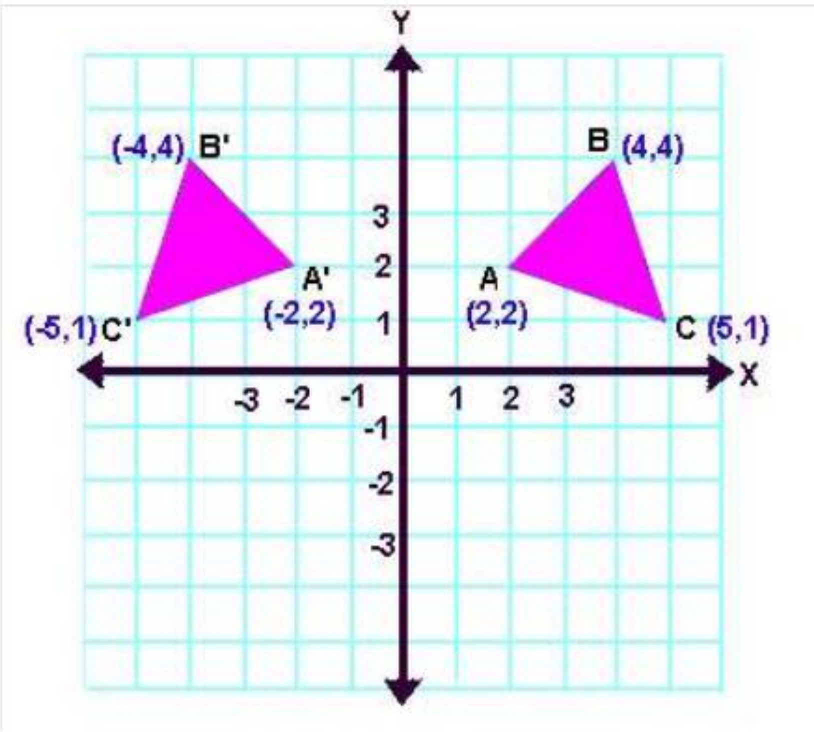
TYPE OF REFLECTION	Matrix to be multiplied
Reflection about the x- axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection about the y-axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflection about the line y = x	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Reflection about the line y = -x	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
Reflection about the origin	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

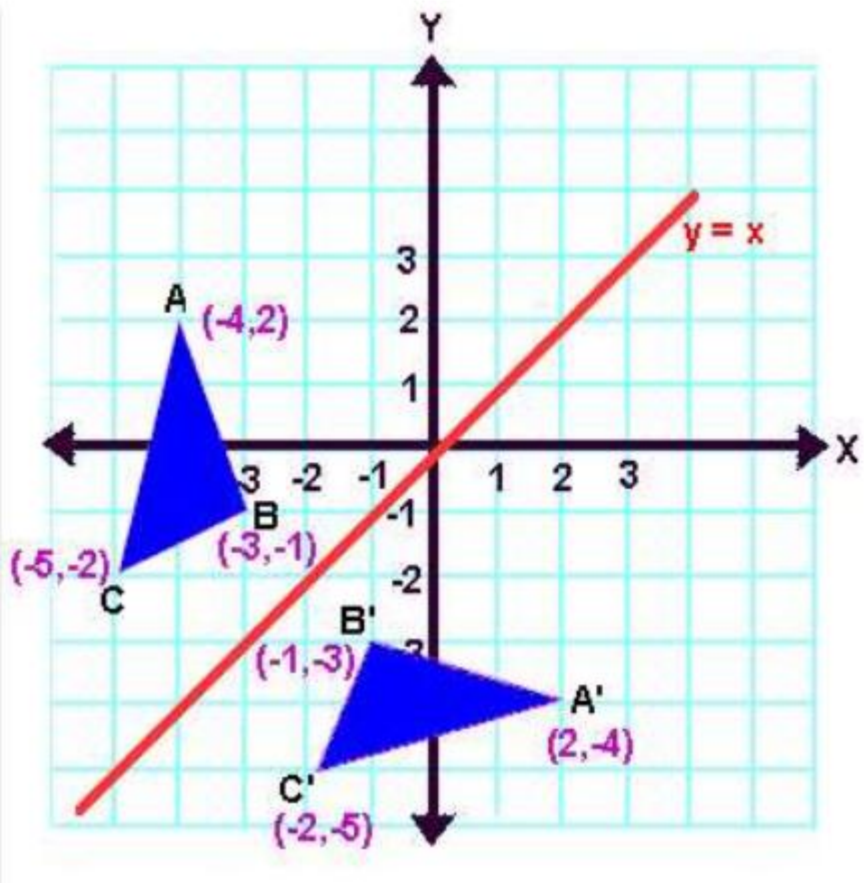
Reflection about the x-axis



Reflection about the y-axis



Reflection about the line y = x



Examples

Example 1: Reflect the $A = \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}$ matrix along X- axis.

The given matrix is $A = \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}$

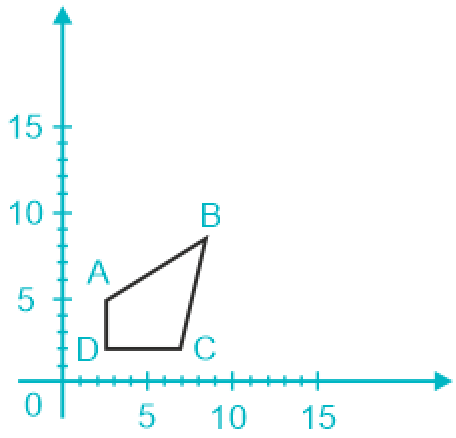
The matrix for Reflection along the X- axis is given by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

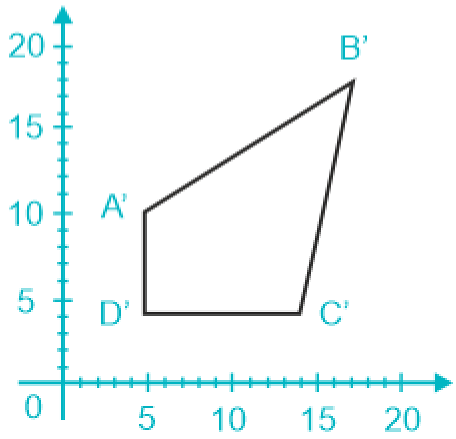
Let the new matrix after transformation be B, and we have the transformation formula as TA = B

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -2 & 2 \end{bmatrix}$$

5. Scaling Matrix

A scaling transform changes the size of an object by expanding or contracting all voxels or vertices along the three axes by three scalar values specified in the matrix. When we're scaling a vector, we are increasing the length of the arrow by the amount we'd like to scale, keeping its direction the same. Scaling can be achieved by multiplying the original coordinates of the object with the scaling factor to get the desired result.





Let us assume that the original coordinates are X, Y, the scaling factors are (S_X, S_Y), and the produced coordinates are X', Y'. This can be mathematically represented as shown below

$$X' = X \cdot S_X \text{ and } Y' = Y \cdot S_Y$$

The scaling factor S_X, S_Y scales the object in X and Y direction respectively. The above equations can also be represented in matrix form as below

$$\begin{bmatrix} S_X & 0 \\ 0 & S_Y \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X' \\ Y' \end{bmatrix}$$

Examples

1. Magnify a triangle placed at A (0, 0), B (1, 1) and C (5, 2) to twice its size.

The coordinates of the box are A (0, 0), B (1, 1), C (5, 2)

First, we have to write the vertices of the given triangle ABC in matrix form as given below.

$$\begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

Since, we have to magnify the triangle to twice its size, we have to take the scaling factor as $S_X = 2$, $S_Y = 2$.

We will multiply the scaling matrix with the triangle matrix to get the coordinates after transformation.

$$\begin{bmatrix} S_X & 0 \\ 0 & S_Y \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X' \\ Y' \end{bmatrix}$$

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 10 \\ 0 & 2 & 4 \end{bmatrix}$$

The new coordinates after scaling are given by (0, 0), (2, 2), (10, 4).

Application of Transformation Matrix

The transformation matrix has numerous applications in vectors, linear algebra, matrix operations. The following are some of the important applications of the transformation matrix.

- Vectors represented in a two or three-dimensional frame are transformed to another vector.
- Linear Combinations of two or more vectors through multiplication are possible through a transformation matrix.
- The linear transformations of matrices can be used to change the matrices into another form.
- Matrix multiplication is the transformation of one matrix into another matrix.
- Determinants can be solved using the concepts of the transformation matrix.
- Inverse Space also use matrix transformations.
- Dot Product and Cross Product of Vectors
- Change of Basis of vectors is possible through transformations
- Eigen Vectors and Eigen Values involve matrix and matrix transformation.
- Abstract Vector Spaces also use the concepts of the transformation matrix.