GAUSS ELIMINATION METHOD

This method is used to solve a system of linear equations. In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which the unknowns are found by back substitution.

Working Procedure:

Consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

The above system of equations can be written in matrix form as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$i.e., AX = B$$

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Consider the augmented matrix

$$[A:B] = \begin{bmatrix} a_1 & b_1 & c_1 & : & d_1 \\ a_2 & b_2 & c_2 & : & d_2 \\ a_3 & b_3 & c_3 & : & d_3 \end{bmatrix}$$

By using the element $a_1 (\neq 0)$ to make the elements a_2 and a_3 zero by elementary

row transformations. This reduces to the following form

$$[A:B] \sim \begin{bmatrix} a_1 & b_1 & c_1 & : & d_1 \\ 0 & {b_2}' & {c_2}' & : & {d_2}' \\ 0 & {b_3}' & {c_3}' & : & {d_3}' \end{bmatrix}$$

Again, by using the element $b_2' (\neq 0)$ to make the element b_3' zero by elementary row transformation. This reduces to the following form

$$[A:B] \sim \begin{bmatrix} a_1 & b_1 & c_1 & : & d_1 \\ 0 & b_2' & c_2' & : & d_2' \\ 0 & 0 & c_3'' & : & d_3'' \end{bmatrix}$$

This form is called upper triangular form.

From this, we get the new system of equations as

$$a_1x + b_1y + c_1z = d_1$$

$$b_2'y + c_2'z = d_2'$$

$$c_3''z = d_3''$$

We get the value of z from the last equation and by back substitution we get y and x

This method of finding the solution of the given system of linear equations is called *Gauss elimination method*.

This method can be generalized to system of n equations in n unknowns.

Example 1. Apply Gauss elimination method to solve the system of equations

$$\mathbf{2x + y + 4z = 12}$$

$$\mathbf{4x + 11y - z = 33}$$

$$\mathbf{8x - 3y + 2z = 20}$$

Solution:

The augmented matrix associated with the given system equations is

$$[A:B] = \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 4 & 11 & -1 & : & 33 \\ 8 & -3 & 2 & : & 20 \end{bmatrix}$$

operating $R_2 - 2R_1, R_3 - 4R_1$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 9 & -9 & : & 9 \\ 0 & -7 & -14 & : & -28 \end{bmatrix}$$

operating $9R_3 + 7R_2$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 9 & -9 & : & 9 \\ 0 & 0 & -189 & : & -189 \end{bmatrix}$$

From this we have,

$$2x + y + 4z = 12$$
$$9y - 9z = 9$$
$$-189z = -189$$

From the last equation we have, $z = 1$

By back substitution, $9y - 9(1) = 9 \Rightarrow 9y = 9 + 9 \Rightarrow y = 2$

and $2x + 2 + 4(1) = 12 \Rightarrow 2x = 12 - 4 - 2 \Rightarrow x = 3$

$\therefore$ The required solution of the given system of equations is

$$x = 3, y = 2, z = 1$$

Example 2. Apply Gauss elimination method to solve the system of equations

$$\mathbf{2x - 3y + z = -1}$$
$$\mathbf{x + 4y + 5z = 25}$$
$$\mathbf{3x - 4y + z = 2}$$

Solution:

The augmented matrix associated with the given system equations is

$$[A:B] = \begin{bmatrix} 2 & -3 & 1 & : & -1 \\ 1 & 4 & 5 & : & 25 \\ 3 & -4 & 1 & : & 2 \end{bmatrix}$$

operating $2R_2 - R_1, 2R_3 - 3R_1$

$$[A:B] \sim \begin{bmatrix} 2 & -3 & 1 & : & -1 \\ 0 & 11 & 9 & : & 51 \\ 0 & 1 & -1 & : & 7 \end{bmatrix}$$

operating $11R_3 - R_2$

$$[A:B] \sim \begin{bmatrix} 2 & -3 & 1 & : & -1 \\ 0 & 11 & 9 & : & 51 \\ 0 & 0 & -20 & : & 26 \end{bmatrix}$$

From this we have,

$$2x - 3y + z = -1$$
$$11y + 9z = 51$$

$$-20z = 26$$

From the last equation we have, $z = -\dfrac{13}{10}$

By back substitution, $11y + 9\left(-\dfrac{13}{10}\right) = 51 \Rightarrow 11y = 51 + \left(\dfrac{117}{10}\right) \Rightarrow y = \dfrac{57}{10}$

and $2x - 3\left(\dfrac{57}{10}\right) + \left(-\dfrac{13}{10}\right) = -1 \Rightarrow 2x = -1 + \left(\dfrac{13}{10}\right) + \left(\dfrac{171}{10}\right) \Rightarrow x = \dfrac{87}{10}$

∴ The required solution of the given system of equations is

$$x = \frac{87}{10}, y = \frac{57}{10}, z = \frac{-13}{10}$$

Example 3. Apply Gauss elimination method to solve the system of equations

$$\mathbf{2x + y + z = 10}$$

$$\mathbf{3x + 2y + 3z = 18}$$

$$\mathbf{x + 4y + 9z = 16}$$

Solution:

The augmented matrix associated with the given system equations is

$$[A:B] = \begin{bmatrix} 2 & 1 & 1 & : & 10 \\ 3 & 2 & 3 & : & 18 \\ 1 & 4 & 9 & : & 16 \end{bmatrix}$$

$$operating 2R_2 - 3R_1, 2R_3 - R_1$$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 1 & : & 10 \\ 0 & 1 & 3 & : & 6 \\ 0 & 7 & 17 & : & 22 \end{bmatrix}$$

$$operating R_3 - 7R_2$$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 1 & : & 10 \\ 0 & 1 & 3 & : & 6 \\ 0 & 0 & -4 & : & -20 \end{bmatrix}$$

From this we have,

$$2x + y + z = 10$$

$$y + 3z = 6$$

$$-4z = -20$$

From the last equation we have, $z = 5$

By back substitution, $y + 3(5) = 6 \Rightarrow y = 6 - 15 \Rightarrow y = -9$

and $2x + (-9) + 5 = 10 \Rightarrow 2x = 10 - 5 + 9 \Rightarrow x = 7$

$\therefore$ The required solution of the given system of equations is

$$x = 7, y = -9, z = 5$$

PYTHON CODE

Python Syntax to solve system of equations AX = B

X = np.linalg.solve(A, B)

Example

Solve the following system

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$$

In [1]:
```
import numpy as np

A = np.array([[1,-1,1], [2,1,-3],[1,1,1]])
B = np.array([4,0,2])
X = np.linalg.solve(A, B)

print(X)
```

OUTPUT

```
[ 2. -1.  1.]
```

Application Problems

1. Mr. Johns sells Mango, Apple and Peach. The price of a kg of Mango, 3 kgs of Apple, and a kg of Peach is Rs 145. The price of 3 kgs of Mango, 4 kgs of Apple, and a kg of Peach is Rs 280. The price of 2 kgs of Apple, and a kg of Peach is Rs 65. Find out the price of a kg of each fruit.

This is a system of linear equations problem. We can represent the problem as follows:

Let's denote:

- x as the price of a kg of Mango
- y as the price of a kg of Apple
- z as the price of a kg of Peach

We can then form the following equations from the problem:

$$x + 3y + z = 145$$

$$3x + 4y + z = 280$$

$$2y + z = 65$$

The augmented matrix associated with the given system equations is

$$[A:B] = \begin{bmatrix} 1 & 3 & 1 & : & 145 \\ 3 & 4 & 1 & : & 280 \\ 0 & 2 & 1 & : & 65 \end{bmatrix}$$

operating $R_2 - 3R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 3 & 1 & : & 145 \\ 0 & -5 & -2 & : & -155 \\ 0 & 2 & 1 & : & 65 \end{bmatrix}$$

operating $5R_3 + 2R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 3 & 1 & : & 145 \\ 0 & -5 & -2 & : & -155 \\ 0 & 0 & 1 & : & 15 \end{bmatrix}$$

From this we have,

$$x + 3y + z = 145$$

$$-5y - 2z = -155$$

$$z = 15$$

From the last equation we have, $z = 15$

By back substitution, $-5y - 2(15) = -155 \Rightarrow 5y = -30 + 155 \Rightarrow y = 25$

and $x + 3(25) + 15 = 145 \Rightarrow x = 55$

$\therefore$ The required solution of the given system of equations is

$$x = 55, y = 25, z = 15$$

Price per kg: Mango = Rs.55, Apple = Rs.25 and Peach = Rs.15

We can solve this system of equations using Python's numpy library.

Here's how you can do it:

```
import numpy as np

# Coefficients of the equations
A = np.array([[1, 3, 1], [3, 4, 1], [0, 2, 1]])

# Right-hand side of the equations
B = np.array([145, 280, 65])

# Solve the system of equations
x = np.linalg.solve(A, B)

print(f"Price per kg: Mango = Rs {x[0]}, Apple = Rs {x[1]}, Peach = Rs {x[2]}")
```

```
Price per kg: Mango = Rs 54.99999999999999, Apple = Rs 25.000000000000007, Peach = Rs 14.99999999999999
```

2. Mr. Murgan sells 3 different products. He sells products X, Y & Z. If he sells one unit of X, 5 units of Y and a unit of Z he makes a profit of Rs.1080. If he sells one of Y and a unit of Z he makes a profit of Rs. 540. If he sells 2 units of X and buys two units of Y and a unit of Z from another seller same as his selling price, he incurs a loss of 180 rupees. Find out the price of product X, Y, & Z?

This is a system of linear equations problem. We can represent the problem as follows:

Let's denote:

- x as the price of product X
- y as the price of product Y
- z as the price of product Z

We can then form the following equations from the problem:

$$x + 5y + z = 1080$$

$$y + z = 540$$

$$2x - 2y - z = -180$$

The augmented matrix associated with the given system equations is

$$[A:B] = \begin{bmatrix} 1 & 5 & 1 & : & 1080 \\ 0 & 1 & 1 & : & 540 \\ 2 & -2 & -1 & : & -180 \end{bmatrix}$$

operating $R_3 - 2R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 5 & 1 & : & 1080 \\ 0 & 1 & 1 & : & 540 \\ 0 & -12 & -3 & : & -2340 \end{bmatrix}$$

$$\text{operating } R_3 + 12R_2$$

$$[A:B] \sim \begin{bmatrix} 1 & 5 & 1 & : & 1080 \\ 0 & 1 & 1 & : & 540 \\ 0 & 0 & 9 & : & 4140 \end{bmatrix}$$

From this we have,

$$x + 5y + z = 1080$$
$$y + z = 540$$
$$9z = 4140$$

From the last equation we have, $z = 460$

By back substitution, $y + 460 = 540 \Rightarrow y = 80$

and $x + 5(80) + 460 = 1080 \Rightarrow x = 220$

$\therefore$ The required solution of the given system of equations is

$$x = 220, y = 80, z = 460$$

We can solve this system of equations using Python's numpy library.

Here's how you can do it:

```
import numpy as np

# Coefficients of the equations are represented as a 2D array
A = np.array([[1, 5, 1], [0, 1, 1], [2, -2, -1]])

# Constants on the right side of the equations are represented as a 1D array
B = np.array([1080, 540, -180])

# Use np.linalg.solve() function to solve for [x, y, z]
solution = np.linalg.solve(A, B)

print(f"The solution is {solution}")
```

```
The solution is [220.  80. 460.]
```