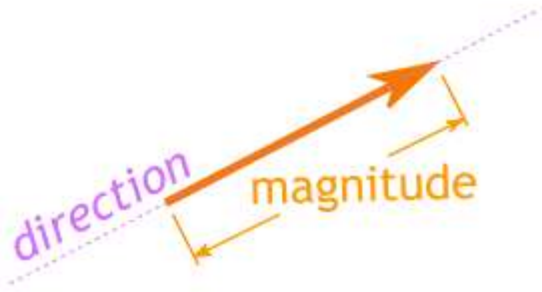
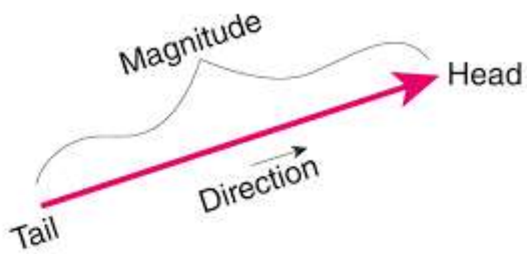
Scalars

A physical quantity that has only magnitude is called a scalar.

Vectors



A physical quantity that has magnitude as well as direction is called a vector.

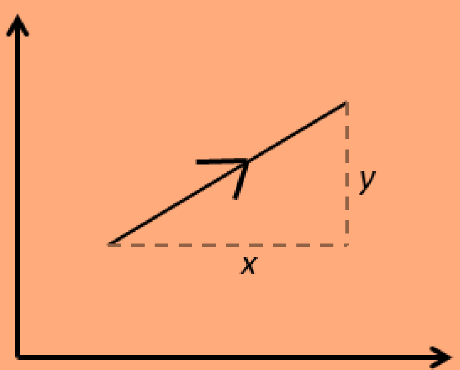


Magnitude of a Vector

The magnitude of vector $\vec{a}$ is written as $|\vec{a}|$.

The magnitude of vector $\overrightarrow{AB}$ is written as $|\overrightarrow{AB}|$.

If $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ then the magnitude $|\vec{a}| = \sqrt{x^2 + y^2}$ (using Pythagoras theorem)

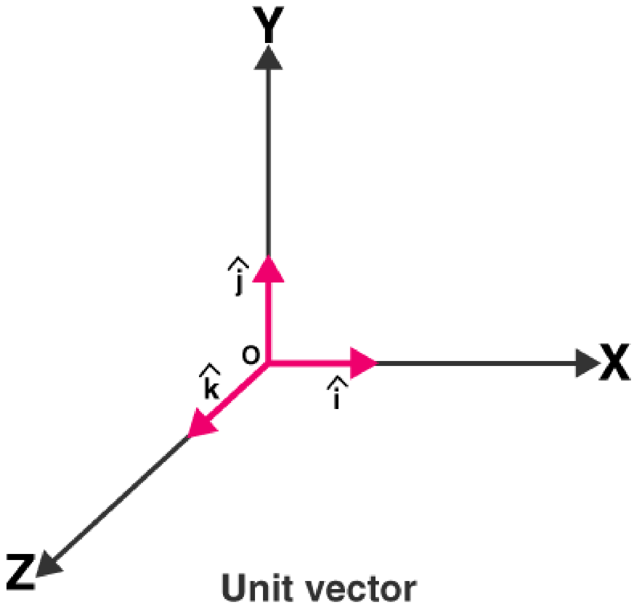


Unit Vector

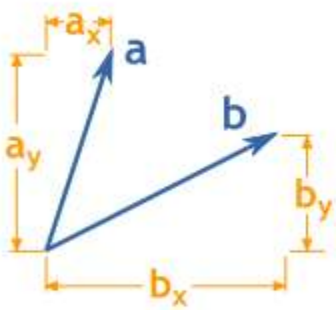
A vector whose magnitude is unity (i.e., 1 unit) is called a unit vector.

The unit vector in the direction of $\vec{a}$ given vector is denoted by $\hat{a}$.

$$\hat{a} = \frac{a}{|a|}$$



Scalar Product or Dot Product of Two Vectors



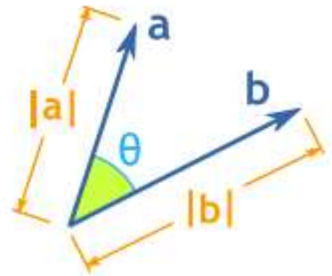
The scalar product of two vectors **a** and **b** for two dimensional is given by

$$a \cdot b = a_x b_x + a_y b_y$$

So, we multiply the x's, multiply the y's, then add.

The scalar product of two vectors **a** and **b** for three dimensional is given by

$$a \cdot b = a_x b_x + a_y b_y + a_z b_z$$

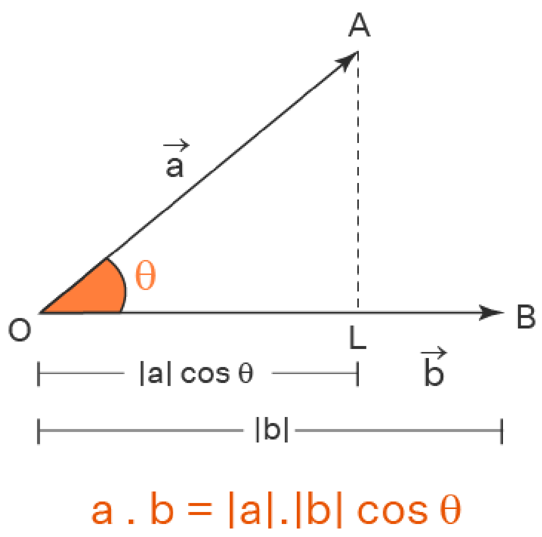


The scalar product of two vectors **a** and **b** of magnitude **|a|** and **|b|** is given as

a . b=|a||b| cos θ,

where θ represents the angle between the vectors **a** and **b** taken in the direction of the vectors.

Geometrical meaning of Dot Product



Example 1: Let there be two vectors [6, 2, -1] and [5, -8, 2]. Find the dot product of the vectors.

Solution:

Given vectors: [6, 2, -1] and [5, -8, 2] be a and b respectively.

a.b = (6)(5) + (2)(-8) + (-1)(2)

a.b = 30 - 16 - 2

a.b = 12

Example 2: Let there be two vectors |a|=4 and |b|=2 and θ = 60°. Find their dot product.

Solution:

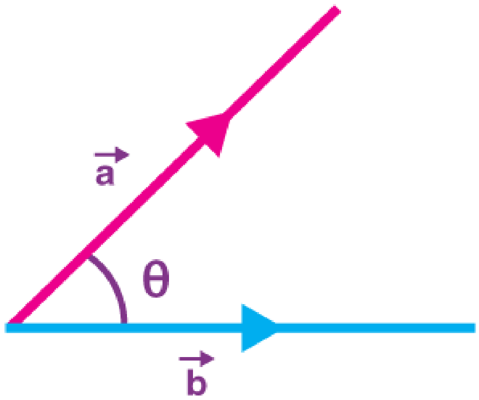
a.b = |a||b|cos θ

a.b = 4.2 cos 60°

a.b = 4.2 × (1/2)

a.b = 4

Angle Between Two Vectors



The angle between two vectors a and *b* is given by

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\theta = cos^{-1}\left(\frac{a \cdot b}{|a||b|}\right)$$

Working Procedure

Step 1. Find the dot product of the vectors

Step 2. Find the magnitude of the vectors

Step 3. Substitute in the angle formula

Step 4. Use the inverse cosine for the angle.

Example - 1

Find the angle between $\binom{3}{-2}$ and $\binom{1}{7}$

1. Find the dot product

$$a \cdot b = a_x b_x + a_y b_y$$

$$a \cdot b = (3)(1) + (-2)(7)$$

$$a \cdot b = -11$$

2. Find the magnitude of the vectors

$$|a| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$$

$$|b| = \sqrt{1^2 + 7^2} = \sqrt{50}$$

3. Substitute into the angle formula

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{-11}{\sqrt{13}\sqrt{50}}$$

4. Use inverse cosine for the angle

$$\theta = \cos^{-1}\left(\frac{-11}{\sqrt{13}\sqrt{50}}\right)$$

$$\theta = 116°$$

Example - 2

Find the angle between $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

1. Find the dot product

$$a \cdot b = a_x b_x + a_y b_y + a_z b_z$$

$$a \cdot b = (2)(2) + (-1)(0) + (3)(1)$$

$$a \cdot b = 7$$

2. Find the magnitude of the vectors

$$|a| = \sqrt{2^2 + (-1)^2 + (3)^2} = \sqrt{14}$$

$$|b| = \sqrt{2^2 + (0)^2 + (1)^2} = \sqrt{5}$$

3. Substitute into the angle formula

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{7}{\sqrt{14}\sqrt{5}}$$

4. Use inverse cosine for the angle

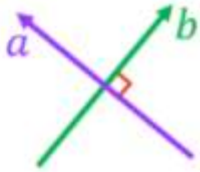
$$\theta = \cos^{-1}\left(\frac{7}{\sqrt{14}\sqrt{5}}\right)$$

$$\theta = 33.2°$$

Two vectors are perpendicular

If their dot product is equal to zero

$$\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$



$$\cos\theta = \frac{0}{|\boldsymbol{a}||\boldsymbol{b}|}$$

$$\theta = \cos^{-1}(0)$$ Inverse cosine of 0 is 90°

$$\theta = 90°$$

Example

Show that $\binom{3}{4}$ and $\binom{-8}{6}$ are perpendicular

1. Show that the dot product is zero

$$\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{a}_x\boldsymbol{b}_x + \boldsymbol{a}_y\boldsymbol{b}_y$$

$$\boldsymbol{a}\cdot\boldsymbol{b} = (\mathbf{3})(-\mathbf{8}) + (\mathbf{4})(\mathbf{6})$$

$$\boxed{\boldsymbol{a}\cdot\boldsymbol{b} = 0}$$

2. Inverse cosine of zero is 90°

$$\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

$$\cos\theta = \frac{0}{|\boldsymbol{a}||\boldsymbol{b}|}$$

$$\theta = \cos^{-1}(0)$$

$$\theta = 90°$$

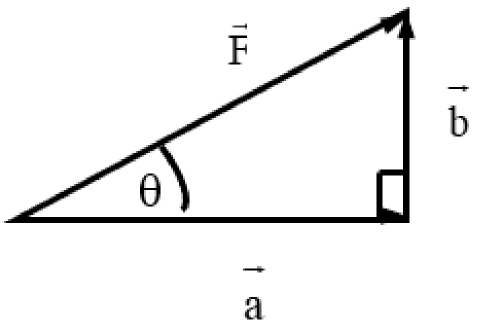
Projections

There are two types of vector projection:

1. Scalar projection
2. Vector projection

For scalar projection, we calculate the length (a scalar quantity) of a vector in a particular direction.

For vector projection we calculate the vector component of a vector in a given direction.



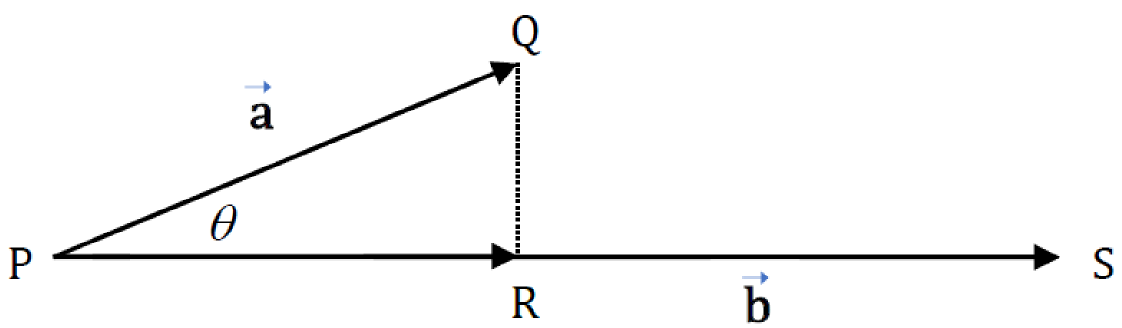
Often, in Physics, Engineering and Mathematics courses you are asked to resolve a vector into two component vectors that are perpendicular to one another. As an example, in the diagram a vector $\vec{a}$ is the projection of $\vec{F}$ in the horizontal direction while $\vec{b}$ is the projection of $\vec{F}$ in the vertical direction.

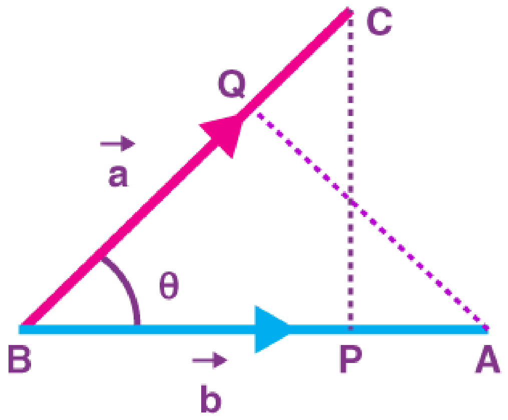
You can project a vector in any direction, not only horizontally and vertically.

Scalar Projection

Scalar Projection of $\vec{a}$ onto $\vec{b}$

$$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$$

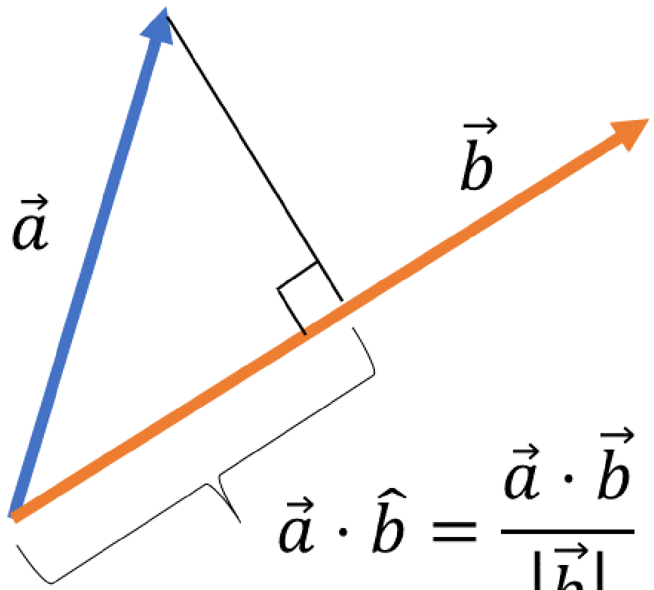




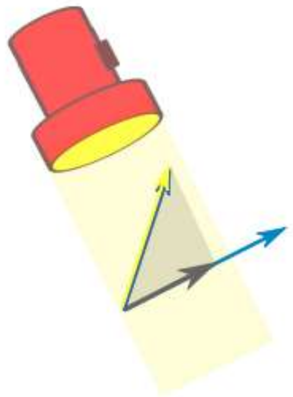
BP is known to be the projection of a vector **a** on vector **b** in the direction of vector **b** given by **|a|** cos θ.

Similarly, the projection of vector **b** on a vector **a** in the direction of the vector **a** is given by **|b|** cos θ.

Projection of vector **a** in direction of vector **b** is expressed as



$$\vec{a} \cdot \hat{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$



Like shining a light to see where the shadow lies

Example

Find the scalar projection of the vector $\vec{a}$ = (2, 3, 1) in the direction of vector $\vec{b}$ = (5, −2, 2)

Solution:

The dot product of $\vec{a}$ and $\vec{b}$ is

$$\vec{a}.\vec{b} = (2)(5) + (3)(-2) + (1)(2) = 10 - 6 + 2 = 6$$

The magnitude of $\vec{b}$ is

$$||\ \vec{b}\ || = \sqrt{5^2 + (-2)^2 + 2^2} = \sqrt{33}$$

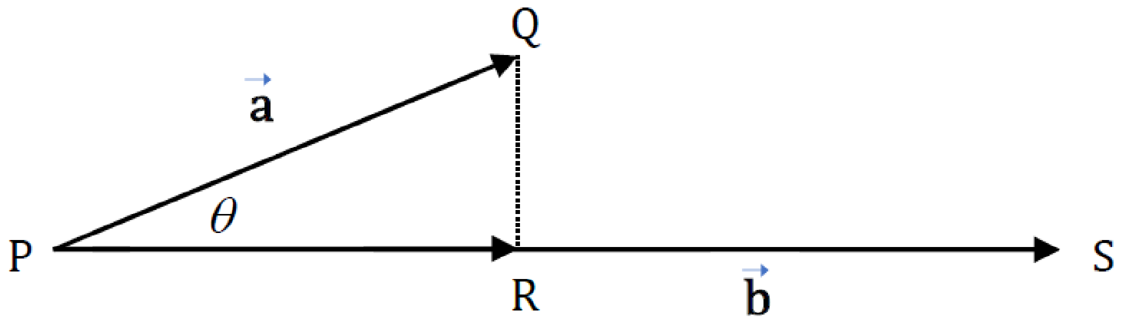
So, the scalar projection of a⃗ in the direction of b⃗ is

$$\frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{6}{\sqrt{33}}$$

Vector Projection

$$\boxed{\begin{array}{c}\textbf{Vector Projection of } \vec{a} \textbf{ onto } \vec{b} \\ \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|^2}\vec{\boldsymbol{b}}\end{array}}$$

The vector projection of a vector $\vec{a}$ in the direction of vector $\vec{b}$ is a vector in the direction of $\vec{b}$ with magnitude equal to the length of the straight line PR or $|\overrightarrow{PR}|$ as shown below.



Therefore, the vector projection of $\vec{a}$ in the direction of $\vec{b}$ is the scalar projection multiplied by a unit vector in the direction of $\vec{b}$

The vector projection of vector $\vec{a}$ in the direction of vector $\vec{b}$ is

$$(\vec{a}.\hat{b})\hat{b} = \frac{(\vec{a}.\vec{b})\vec{b}}{|\vec{b}|^2}$$

Example

Find the vector projection of the vector $\vec{a}$ = (2, 3, 1) in the direction of vector $\vec{b}$ = (5, −2, 2)

Solution:

The dot product of $\vec{a}$ and $\vec{b}$ is

$$\vec{a}.\vec{b} = (2)(5) + (3)(-2) + (1)(2) = 10 - 6 + 2 = 6$$

The magnitude of $\vec{b}$ is

$$\|\vec{b}\| = \sqrt{5^2 + (-2)^2 + 2^2} = \sqrt{33}$$

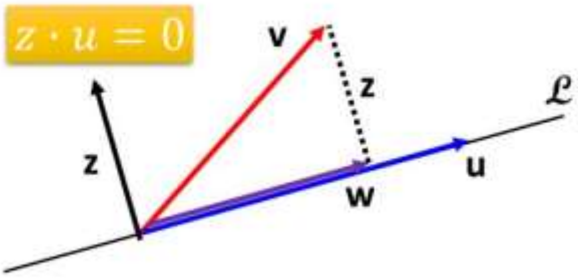
The vector projection of vector $\vec{a}$ in the direction of vector $\vec{b}$ is

$$(\vec{a}.\hat{b})\hat{b} = \frac{(\vec{a}.\vec{b})\vec{b}}{|\vec{b}|^2} = \frac{6(5\hat{\imath} - 2\hat{\jmath} + 2\hat{k})}{33}$$

$$= \frac{2}{11}(5\hat{\imath} - 2\hat{\jmath} + 2\hat{k})$$

Orthogonal Projection on a line

- Orthogonal projection of a vector on a line



v: any vector
u: any nonzero vector on $\mathcal{L}$
w: orthogonal projection of **v** onto $\mathcal{L}$, **w** = c**u**
z: **v** – **w**

$$(v - w) \cdot u = (v - cu) \cdot u = v \cdot u - cu \cdot u = v \cdot u - c\|u\|^2 = 0$$

$$c = \frac{v \cdot u}{\|u\|^2} \quad w = cu = \frac{v \cdot u}{\|u\|^2} u$$

Distance from tip of **v** to $\mathcal{L}$: $\|z\| = \|v - w\| = \left\| v - \frac{v \cdot u}{\|u\|^2} u \right\|$

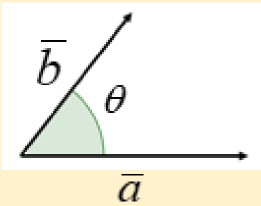
Summary

Dot Product

If $\bar{a} = \langle a_1, a_2, a_3 \rangle$ and $\bar{b} = \langle b_1, b_2, b_3 \rangle$

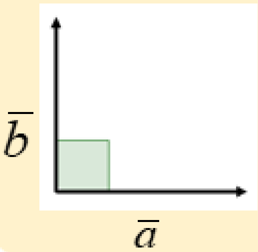
then the dot product is

$$\bar{a} \bullet \bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$



If θ is the angle between $\bar{a}$ and $\bar{b}$ then

$$\bar{a} \bullet \bar{b} = |\bar{a}||\bar{b}|\cos\theta$$



$\bar{a} \bullet \bar{b}$ are orthogonal (perpendicular)

if and only if $\bar{a} \bullet \bar{b} = 0$