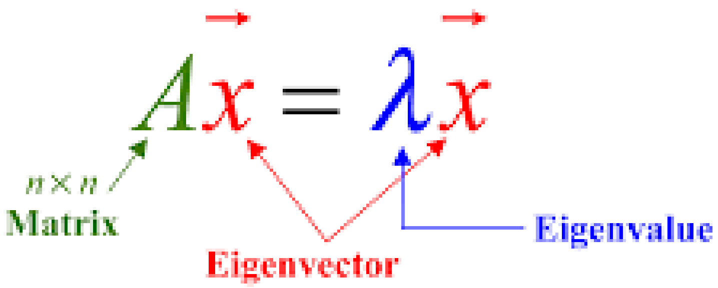
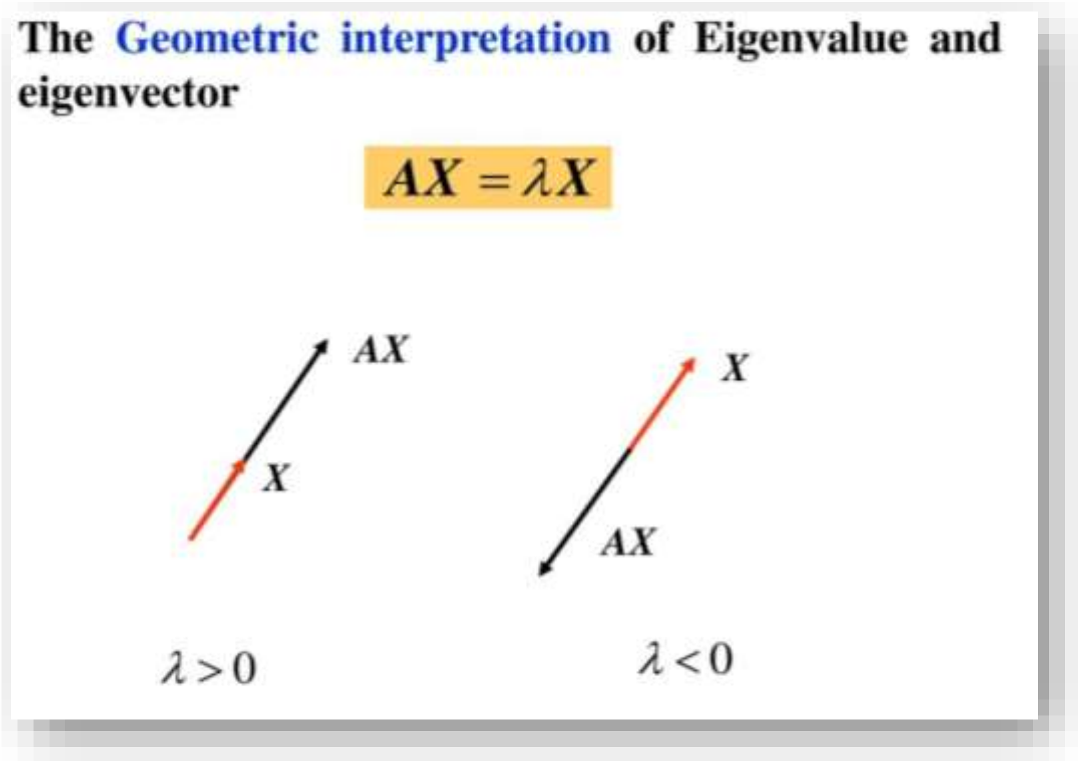
EIGEN VALUES AND EIGEN VECTORS

Eigenvalues and Eigenvectors are the scalar and vector quantities associated with Matrix used for linear transformation. The vector that does not change even after applying transformations is called the Eigenvector and the scalar value attached to Eigenvectors is called Eigenvalue. Eigenvectors are the vectors that are associated with a set of linear equations. For a matrix, eigenvectors are also called characteristic vectors and we can find the eigenvector of only square matrices. Eigenvectors are very useful in solving various problems of matrices and differential equations.

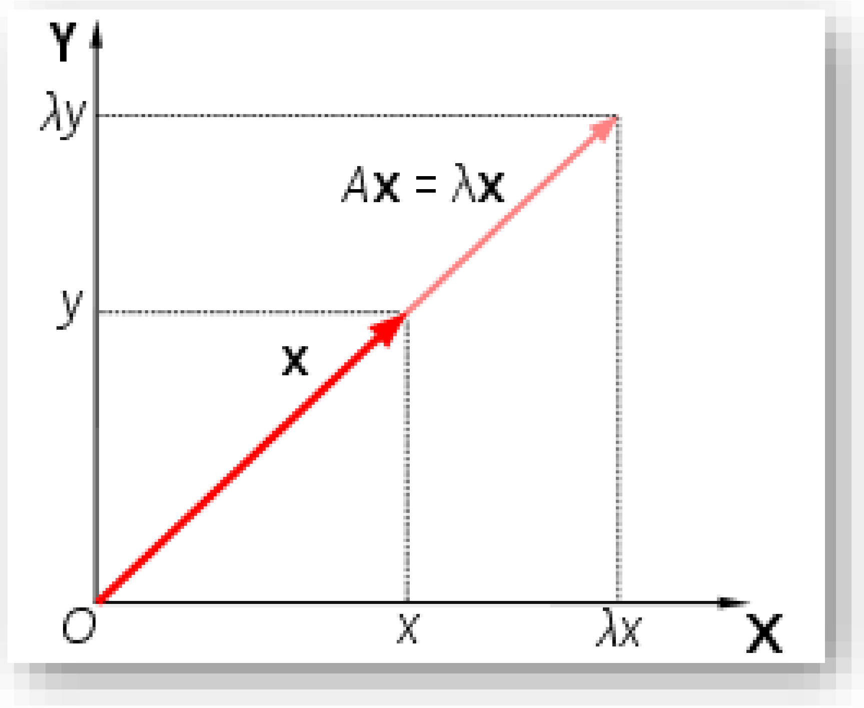
Eigenvalues are the scalar values associated with the eigenvectors in linear transformation. The word 'Eigen' is of German Origin which means 'characteristic'. Hence, these are the characteristic value that indicates the factor by which eigenvectors are stretched in their direction. It doesn't involve the change in the direction of the vector except when the eigenvalue is negative. When the eigenvalue is negative the direction is just reversed. The equation for eigenvalue is given by

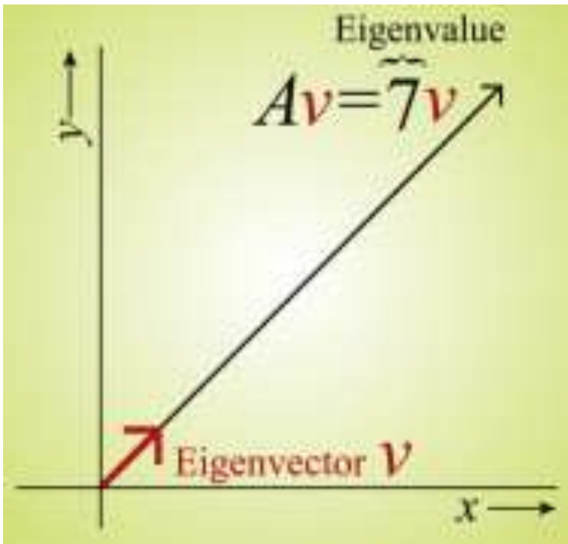
$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

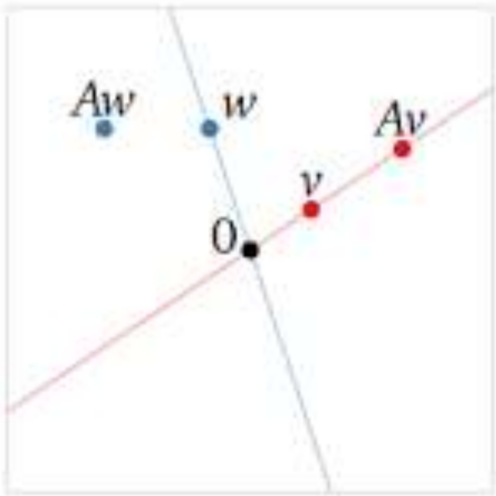




Vector X and Image AX are overlapping vectors, so that the Eigenvalue is positive. Vector X and Image AX are in the opposite direction, so that the Eigenvalue is negative.

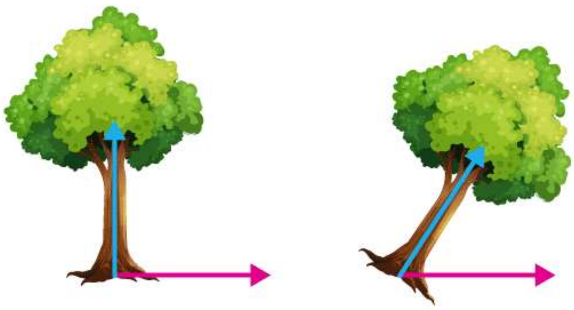

Y
λy
Ax = λx
y
x
O
x
λx
X


Eigenvalue
Av=7v
y
Eigenvector v
x

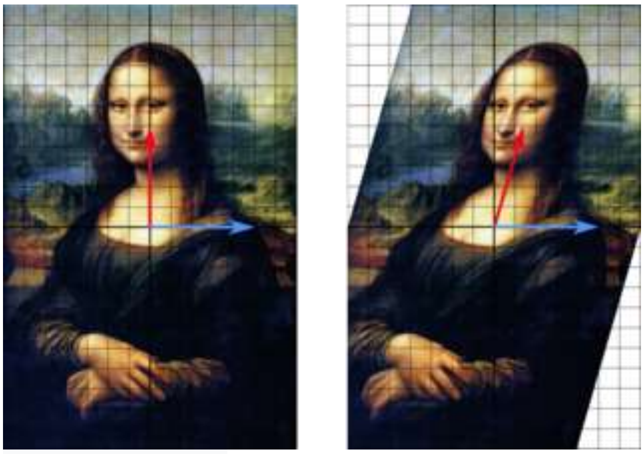


v is an eigenvector

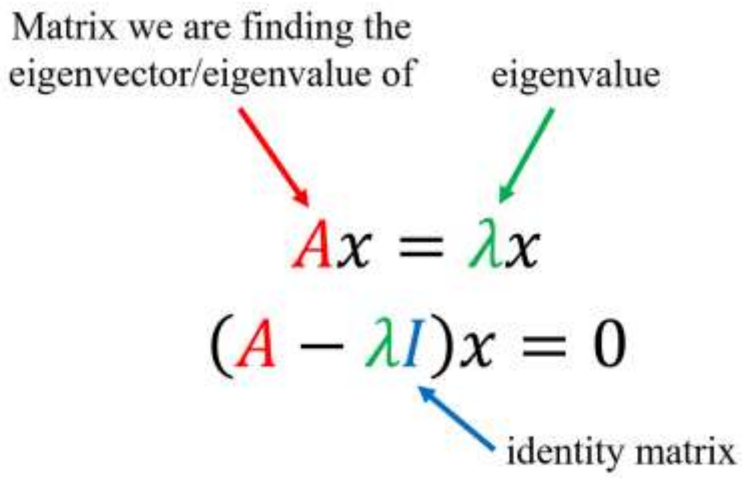
w is not an eigenvector



In this shear mapping, the blue arrow changes direction, whereas the pink arrow does not. Here, the pink arrow is an eigenvector because it does not change direction. Also, the length of this arrow is not changed; its eigenvalue is 1.



In this shear mapping, the red arrow changes direction, but the blue arrow does not. The blue arrow is an eigenvector of this shear mapping because it does not change direction, and since its length is unchanged, its eigenvalue is 1.



Definition

Let A be a square matrix of order n, λ be a scalar and I be the unit matrix of order n then the equation

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$

is called characteristic equation or Eigen equation of A.

On expanding the determinant, the characteristic equation takes the form

$$(-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \cdots + k_n = 0.$$

The roots of this equation are called characteristic roots or Eigen values of the matrix A.

Suppose there exists a scalar λ(real or complex) and a non-zero column matrix X of order n such that $AX = \lambda X$, then λ is called Eigen value of A and X is called an Eigen vector of the matrix A corresponding to an Eigen value λ.

$i.e.,$ $AX = \lambda X \Rightarrow (A - \lambda I)X = 0$ gives the system of homogeneous linear equations having a non-zero solution $X = (x_1, x_2, \dots, x_n)$ which is called an eigen vector.

Note: *The characteristic equation of a third order square matrix A can be obtained without expanding* $|A - \lambda I| = 0$ by using the rule

$$\lambda^3 - \left(\sum d\right)\lambda^2 + \left(\sum M_{ii}\right)\lambda - |A| = 0$$

where $\sum d =$ sum of the diagonal elements of A and

$\sum M_{ii} =$ sum of the minors of the diagonal elements of A

WORKED EXAMPLES

Example 1. Find all the Eigen values and the corresponding Eigen vectors of the matrix

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

Solution:

The characteristic equation is $|A - \lambda I| = 0$

$$i.e., \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-5-\lambda)(-2-\lambda) - 4 = 0$$

$$\Rightarrow \lambda^2 + 7\lambda + 6 = 0$$

$$\Rightarrow \lambda = -1, -6 \text{ are the eigen values of } A$$

Consider, $(A - \lambda I)X = 0$

$$\begin{aligned} (-5-\lambda)x + 2y &= 0 \\ 2x + (-2-\lambda)y &= 0 \end{aligned} \qquad \ldots..(1)$$

$Case - 1$: Putting $\lambda = -1$ in (1), we get

$-4x + 2y = 0, 2x - y = 0 \Rightarrow y = 2x$

Let $x = k$, then $y = 2k$

Hence the eigen vector corresponding to $\lambda = -1$ is $[k, 2k]^T = k[1,2]^T$

$Case - 2$: Putting $\lambda = -6$ in (1), we get

$x + 2y = 0, 2x + 4y = 0 \Rightarrow x = -2y$

Let $y = k$, then $x = -2k$

Hence the eigen vector corresponding to $\lambda = -6$ is $[-2k, k]^T = k[-2,1]^T$

$\therefore$ The two eigen vectors corresponding to two eigen values $\lambda = -1, -6$ are

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Example 2. Find all the Eigen values and the corresponding Eigen vectors of the matrix

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{3} \end{bmatrix}$$

Solution:

The characteristic equation is $|A - \lambda I| = 0$

$$i.e., \begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda) = 0$$

$$\Rightarrow \lambda = 1, 3 \text{ are the eigen values of } A$$

Consider, $(A - \lambda I)X = 0$

$$\begin{aligned} (1-\lambda)x + 2y &= 0 \\ 0x + (3-\lambda)y &= 0 \end{aligned} \qquad \ldots\ldots..(1)$$

$Case-1$: Putting $\lambda = 1$ in (1), we get

$0x + 2y = 0, 0x + 2y = 0 \Rightarrow y = 0$

Let $x = k$,

Hence the eigen vector corresponding to $\lambda = 1$ is $[k, 0]^T = k[1,0]^T$

$Case-2$: Putting $\lambda = 3$ in (1), we get

$-2x + 2y = 0, 0x + 0y = 0 \Rightarrow x = y$

Let $y = k$, then $x = k$

Hence the eigen vector corresponding to $\lambda = 3$ is $[k, k]^T = k[1,1]^T$

$\therefore$ The two eigen vectors corresponding to two eigen values $\lambda = 1, 3$ are

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example 3. Find all the Eigen values and the corresponding Eigen vectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution:

The characteristic equation is $|A - \lambda I| = 0$

$$i.e., \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\text{We have,} \quad \lambda^3 - \left(\sum d\right)\lambda^2 + \left(\sum M_{ii}\right)\lambda - |A| = 0 \qquad \ldots\ldots..(1)$$

$$\text{where } \sum d = 1 + 5 + 1 = 7$$

$$\sum M_{ii} = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 4 + (-8) + 4 = 0$$

$$|A| = 1(5-1) - 1(1-3) + 3(1-15) = 4 + 2 - 42 = -36$$

$\therefore$ Expression (1) becomes,

$$\lambda^3 - 7\lambda^2 + 36 = 0$$

By solving this equation we get, $\lambda = -2, 3, 6$

These are eigen values of A.

Consider, $(A - \lambda I)X = 0$

$$\begin{aligned} (1-\lambda)x + y + 3z &= 0 \\ x + (5-\lambda)y + z &= 0 \\ 3x + y + (1-\lambda)z &= 0 \end{aligned} \qquad \ldots\ldots\ldots\ldots..(2)$$

$Case - 1$: Putting $\lambda = -2$ in (2), we get

$3x + y + 3z = 0, x + 7y + z = 0, 3x + y + 3z = 0$

From first and second equations, we have ($\because$ first and third are same)

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix}} \Rightarrow \frac{x}{-20} = \frac{-y}{0} = \frac{z}{20} \text{ or } \frac{x}{-1} = \frac{-y}{0} = \frac{z}{1}$$

Hence the eigen vector corresponding to $\lambda = -2$ is $[-1,0,1]^T$

$Case - 2$: Putting $\lambda = 3$ in (2), we get

$-2x + y + 3z = 0, x + 2y + z = 0, 3x + y - 2z = 0$

From first and second equations, we have

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix}} \Rightarrow \frac{x}{-5} = \frac{-y}{-5} = \frac{z}{-5} \text{ or } \frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$$

Hence the eigen vector corresponding to $\lambda = 3$ is $[1, -1, 1]^T$

$Case - 3$: Putting $\lambda = 6$ in (2), we get

$-5x + y + 3z = 0, x - y + z = 0, 3x + y - 5z = 0$

From first and second equations, we have

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -5 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -5 & 1 \\ 1 & -1 \end{vmatrix}} \Rightarrow \frac{x}{4} = \frac{-y}{-8} = \frac{z}{4} \text{ or } \frac{x}{1} = \frac{y}{2} = \frac{z}{1}$$

Hence the eigen vector corresponding to $\lambda = 6$ is $[1,2,1]^T$

$\therefore$ The three eigen vector are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Example 4. Find all the Eigen values and the corresponding Eigen vectors of the following matrix

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{8} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-6} & \mathbf{7} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-4} & \mathbf{3} \end{bmatrix}$$

Solution:

The characteristic equation is $|A - \lambda I| = 0$

$$i.e., \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

We have, $\lambda^3 - \left(\sum d\right)\lambda^2 + \left(\sum M_{ii}\right)\lambda - |A| = 0 \qquad \dots \dots (1)$

where $\sum d = 8 + 7 + 3 = 18$

$$\sum M_{ii} = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 5 + 20 + 20 = 45$$

$|A| = 8(21 - 16) + 6(-18 + 8) + 2(24 - 14) = 40 - 60 + 20 = 0$

$\therefore$ Expression (1) becomes, $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

By solving this equation we get, $\lambda = 0, 3, 15$ are the eigen values of A.

Consider, $(A - \lambda I)X = 0$

$$\begin{aligned} (8-\lambda)x - 6y + 2z &= 0 \\ -6x + (7-\lambda)y - 4z &= 0 \\ 2x - 4y + (3-\lambda)z &= 0 \end{aligned} \qquad \dots \dots \dots (2)$$

$Case - 1$: Putting $\lambda = 0$ in (2), we get

$8x - 6y + 2z = 0, -6x + 7y - 4z = 0, 2x - 4y + 3z = 0$

From first and second equations, we have

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}} \Rightarrow \frac{x}{10} = \frac{-y}{-20} = \frac{z}{20} \text{ or } \frac{x}{1} = \frac{y}{2} = \frac{z}{2}$$

Hence the eigen vector corresponding to $\lambda = 0$ is $[1,2,2]^T$

$Case - 2$:Putting $\lambda = 3$ in (2), we get

$5x - 6y + 2z = 0, -6x + 4y - 4z = 0, 2x - 4y + 0z = 0$

From first and second equations, we have

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}} \Rightarrow \frac{x}{16} = \frac{-y}{-8} = \frac{z}{-16} \text{ or } \frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

Hence the eigen vector corresponding to $\lambda = 3$ is $[2,1,-2]^T$

$Case - 3$: Putting $\lambda = 15$ in (2), we get

$-7x - 6y + 2z = 0, -6x - 8y - 4z = 0, 2x - 4y - 12z = 0$

From first and second equations, we have

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -7 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}} \Rightarrow \frac{x}{40} = \frac{-y}{40} = \frac{z}{20} \text{or} \frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$$

Hence the eigen vector corresponding to $\lambda = 15$ is $[2,-2,1]^T$

$\therefore$ The three eigen vector are $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

PYTHON CODE TO FIND EIGENVALUES AND EIGENVECTORS OF THE MATRIX

PYTHON SYNTAX

```
# Calculate eigenvalues and eigenvectors
 eigenvalues, eigenvectors = np.linalg.eig(A)
```

eigenvalues is a 1D array containing the eigenvalues of A.

eigenvectors is a 2D array whose columns are the eigenvectors corresponding to the eigenvalues in eigenvalues.

Example

Find all the Eigen values and the corresponding Eigen vectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

```
import numpy as np

# Define the matrix
A = np.array([[1, 1, 3], [1, 5, 1], [3, 1, 1]])

# Calculate eigenvalues and eigenvectors
eigenvalues, eigenvectors = np.linalg.eig(A)

print(f"Eigenvalues: {eigenvalues}")
print(f"Eigenvectors: {eigenvectors}")
```

```
Eigenvalues: [-2.  3.  6.]
Eigenvectors: [[ 7.07106781e-01 -5.77350269e-01 -4.08248290e-01]
 [-2.21146215e-17  5.77350269e-01 -8.16496581e-01]
 [-7.07106781e-01 -5.77350269e-01 -4.08248290e-01]]
```