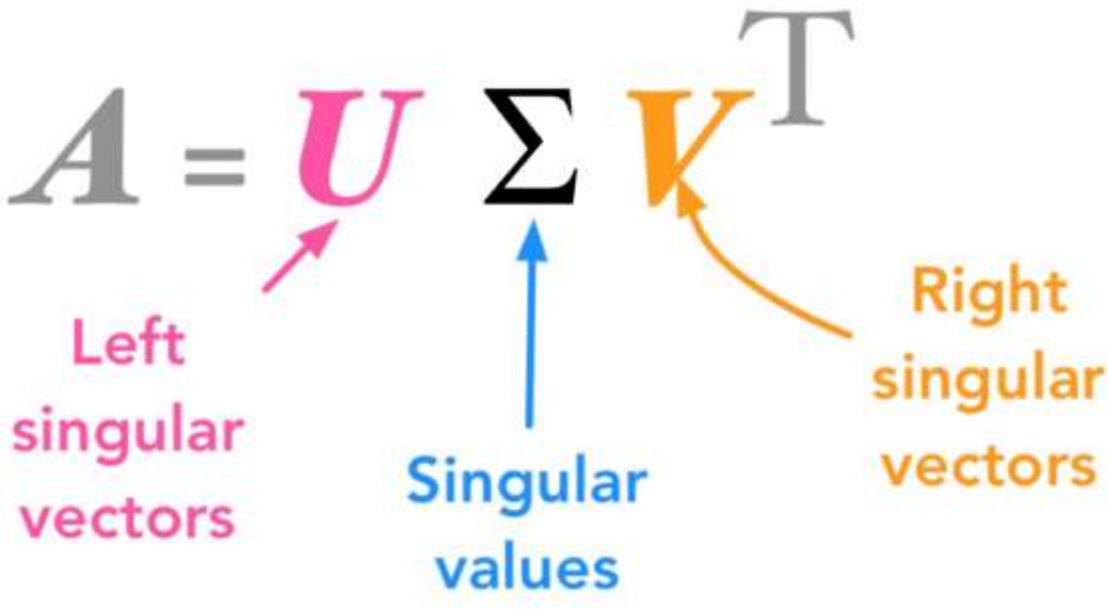
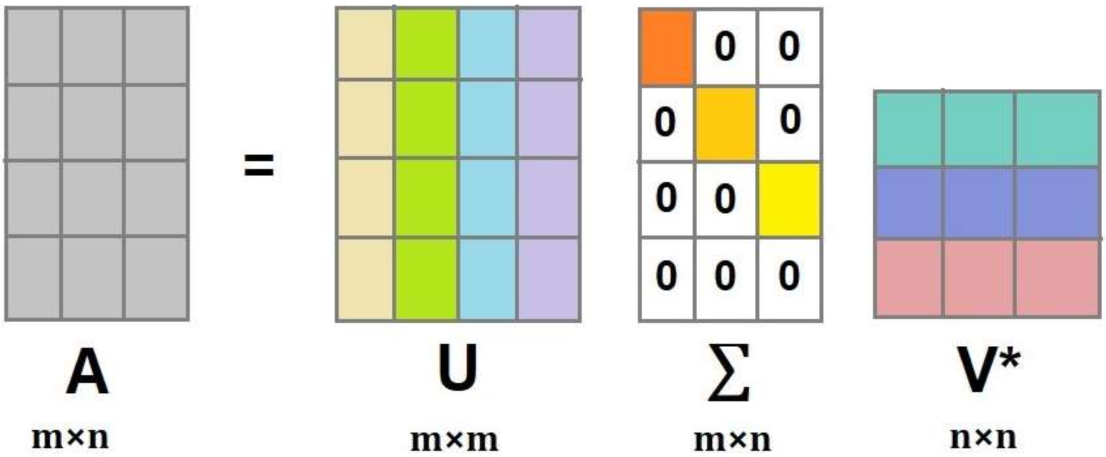
Singular Value Decomposition (SVD)

Singular Value Decomposition (SVD) is a fundamental matrix factorization technique used in linear algebra and data analysis. It decomposes a matrix into three simpler matrices, allowing for various applications such as dimensionality reduction, data compression, and solving linear equations. It plays a crucial role in understanding the structure and relationships within data matrices and is a foundational tool in Data Analysis and Machine Learning.

Let A be a m × n matrix. Then there exists an m × n matrix $\sum$ for which the diagonal entries in D are the singular values of A, $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_r > 0$ and there exist an m × m orthogonal matrix U and an n × n orthogonal matrix V such that A = U $\sum$ V^T





Working Procedure to Find Singular Value Decomposition (SVD)

The mathematical procedure for SVD involves the following steps:

Given, a matrix A of order m x n, where m is the number of rows and n is the number of columns.

Matrix Decomposition

- SVD decomposes A into three matrices: U, $\sum$ (Sigma) and V^T.
- U is an (m x m) orthogonal matrix.
- $\sum$ is a diagonal (m x n) matrix with non-negative real numbers on the diagonal, called singular values, arranged in descending order.
- V^T is an (n x n) orthogonal matrix.

Step 1: Singular Value Calculation

First, we need to compute the singular values by finding eigenvalues of AA^T (or A^TA). The singular values of A are the positive square roots of the eigenvalues of AA^T (or A^TA). i.e., if λ_1, λ_2 and λ_3 are the non-zero eigenvalues then the singular values are $\sigma_1 = \sqrt{\lambda_1}\,, \sigma_2 = \sqrt{\lambda_2}$ and $\sigma_3 = \sqrt{\lambda_3}$.

These singular values are then sorted in descending order.

Step 2: Orthogonal matrix U

Find the left singular vectors by finding the eigenvectors of AA^T. Normalize these eigenvectors to form the columns of U.

Step 3: Orthogonal matrix V^T

Find the right singular vectors by finding the eigenvectors of A^TA. Normalize these eigenvectors to form the rows of V^T.

Step 4: Construct Σ

Σ is a diagonal matrix where the singular values are placed in descending order. Fill the rest of Σ with zeros to match the dimensions of A. It represents the scaling of the singular vectors.

Step 5: Reconstruction

The original matrix A can be reconstructed using the three decomposed matrices U, Σ and V^T. i.e., $A = U \Sigma V^T$

NOTE:

1. If the order of the matrix A is $m \times n$ and $m < n$ then the singular values can be determined from AA^T. Normalize the eigenvectors of AA^T to form the columns of U and normalize the eigenvectors of A^TA to form the rows of V^T.

2. If the order of the matrix A is $m \times n$ and $m > n$ then the singular values can be determined from A^TA. Normalize the eigenvectors of A^TA to form the rows of V^T and normalize the eigenvectors of AA^T to form the columns of U.

Examples

1. Find the singular value decomposition of a matrix $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

Solution:

Singular value decomposition of a matrix A is A = U $\sum$ V^T

Given matrix is $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$, which is of order 2 × 3 (m × n)

Here m < n, so the singular values can be determined from AA^T.

$$AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

The characteristic equation for the above matrix is $|AA^T - \lambda I| = 0$

$$\begin{vmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{vmatrix} = 0$$

$$(17-\lambda)(17-\lambda) - 64 = 0$$

$$\lambda^2 - 34\lambda + 225 = 0$$

$$\lambda = 25, 9$$

The Eigenvalues are 25 and 9.

So, our singular values are $\sigma_1 = 5$ and $\sigma_2 = 3$

Consider, $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(17-\lambda)x + 8y = 0$$

$$8x + (17-\lambda)y = 0$$

$\mathbf{Case - 1}$: Putting $\lambda = 25$, we get

$$-8x + 8y = 0 \Rightarrow \mathrm{x} = \mathrm{y}$$

Let $x = k$, then $y = k$

Hence the eigen vector corresponding to $\lambda = 25$ is $[k, k]^T = k[1, 1]^T$

$\mathbf{Case - 2}$: Putting $\lambda = 9$, we get

$$8x + 8y = 0 \Rightarrow \mathrm{x} = -\mathrm{y}$$

Let $x = k$, then $y = -k$

Hence the eigen vector corresponding to $\lambda = 9$ is $[k, -k]^T = k[1, -1]^T$

$\therefore$ The two eigen vectors corresponding to two eigen values $\lambda = 25, 9$ are

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Normalize these eigenvectors to form the columns of U.

$$\therefore U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

To find V^T,

$$A^T A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

The characteristic equation for the above matrix is $|A^T A - \lambda I| = 0$

$$\begin{vmatrix} 13-\lambda & 12 & 2 \\ 12 & 13-\lambda & -2 \\ 2 & -2 & 8-\lambda \end{vmatrix} = 0$$

We have, $\lambda^3 - \left(\sum d\right)\lambda^2 + \left(\sum M_{ii}\right)\lambda - |A| = 0 \qquad \ldots\ldots(1)$

where $\sum d = 13 + 13 + 8 = 34$

$$\sum M_{ii} = \begin{vmatrix} 13 & -2 \\ -2 & 8 \end{vmatrix} + \begin{vmatrix} 13 & 2 \\ 2 & 8 \end{vmatrix} + \begin{vmatrix} 13 & 12 \\ 12 & 13 \end{vmatrix} = 100 + 100 + 25 = 225$$

$$|A| = 13(104 - 4) - 12(96 + 4) + 2(-24 - 26) = 1300 - 1200 - 100 = 0$$

$\therefore$ Expression (1) becomes, $\lambda^3 - 34\lambda^2 + 225\lambda = 0$

$$\lambda = 25, 9, 0$$

Consider, $(A - \lambda I)X = 0$

$$\begin{gathered} (13-\lambda)x + 12y + 2z = 0 \\ 12x + (13-\lambda)y - 2z = 0 \\ 2x - 2y + (8-\lambda)z = 0 \end{gathered} \qquad \ldots\ldots(2)$$

Case – 1: Putting $\lambda = 25$, we get

$$-12x + 12y + 2z = 0, 12x - 12y - 2z = 0, 2x - 2y - 17z = 0$$

From first and third equations, we have

$$\frac{x}{\begin{vmatrix} 12 & 2 \\ -2 & -17 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -12 & 2 \\ 2 & -17 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -12 & 12 \\ 2 & -2 \end{vmatrix}} \Rightarrow \frac{x}{-200} = \frac{-y}{200} = \frac{z}{0} \text{ or } \frac{x}{1} = \frac{y}{1} = \frac{z}{0}$$

Hence the eigen vector corresponding to $\lambda = 25$ is $[1, 1, 0]^T$

Case – 2: Putting $\lambda = 9$, we get

$$4x + 12y + 2z = 0, 12x + 4y - 2z = 0, 2x - 2y - z = 0$$

From first and second equations, we have

$$\frac{x}{\begin{vmatrix} 12 & 2 \\ 4 & -2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 4 & 2 \\ 12 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 4 & 12 \\ 12 & 4 \end{vmatrix}} \Rightarrow \frac{x}{-32} = \frac{-y}{-32} = \frac{z}{-128} \text{ or } \frac{x}{1} = \frac{y}{-1} = \frac{z}{4}$$

Hence the eigen vector corresponding to $\lambda = 9$ is $[1, -1, 4]^T$

Case – 3: Putting $\lambda = 0$, we get

$$13x + 12y + 2z = 0, 12x + 13y - 2z = 0, 2x - 2y + 8z = 0$$

From first and second equations, we have

$$\frac{x}{\begin{vmatrix} 12 & 2 \\ 13 & -2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 13 & 2 \\ 12 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 13 & 12 \\ 12 & 13 \end{vmatrix}} \Rightarrow \frac{x}{-50} = \frac{-y}{-50} = \frac{z}{25} \text{ or } \frac{x}{-2} = \frac{y}{2} = \frac{z}{1}$$

Hence the eigen vector corresponding to $\lambda = 0$ is $[-2, 2, 1]^T$

$\therefore$ The three eigen vectors corresponding to eigen values $\lambda = 25, 9, 0$ are

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

Normalize these eigenvectors to form the rows of V^T.

$$\therefore V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{18}} & \frac{-1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \\ \frac{-2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Diagonal matrix $\sum = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

Hence, our final SVD equation becomes

$$A = U\sum V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{18}} & \frac{-1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \\ \frac{-2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

2. Find the singular value decomposition of a matrix $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$

Solution:

Singular value decomposition of a matrix A is A = $U\sum V^T$

Given matrix is $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$, which is of order 3 × 2 (m × n)

Here m > n, so the singular values can be determined from A^TA.

$$A^TA = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

The characteristic equation for the above matrix is $|AA^T - \lambda I| = 0$

$$\begin{vmatrix} 9-\lambda & -9 \\ -9 & 9-\lambda \end{vmatrix} = 0$$

$$(9-\lambda)(9-\lambda) - 81 = 0$$

$$\lambda^2 - 18\lambda = 0$$

$$\lambda = 18, 0$$

The Eigenvalues are 18 and 0.

So, our non-zero singular value is $\sigma_1 = \sqrt{18}$

Consider, $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 9-\lambda & -9 \\ -9 & 9-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(9-\lambda)x - 9y = 0$$

$$-9x + (9-\lambda)y = 0$$

$\mathbf{Case-1}$: Putting $\lambda = 18$, we get

$$-9x - 9y = 0 \Rightarrow \mathrm{x} = -\mathrm{y}$$

Let $x = k$, then $y = -k$

Hence the eigen vector corresponding to $\lambda = 18$ is $[k, -k]^T = k[1, -1\,]^T$

$\mathbf{Case-2}$: Putting $\lambda = 0$, we get

$$9x - 9y = 0 \Rightarrow \mathrm{x} = \mathrm{y}$$

Let $x = k$, then $y = k$

Hence the eigen vector corresponding to $\lambda = 0$ is $[k, k]^T = k[1, 1]^T$

∴ The two eigen vectors corresponding to two eigen values $\lambda = 18, 0$ are

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Normalize these eigenvectors to form the rows of V^T.

$$\therefore\ \mathrm{V^T} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

To find U,

$$AA^T = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 4 \\ -4 & 8 & -8 \\ 4 & -8 & 8 \end{bmatrix}$$

The characteristic equation for the above matrix is $|AA^T - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -4 & 4 \\ -4 & 8-\lambda & -8 \\ 4 & -8 & 8-\lambda \end{vmatrix} = 0$$

We have, $\lambda^3 - \left(\sum d\right)\lambda^2 + \left(\sum M_{ii}\right)\lambda - |A| = 0 \qquad \dots\dots(1)$

where $\sum d = 2 + 8 + 8 = 18$

$$\sum M_{ii} = \begin{vmatrix} 8 & -8 \\ -8 & 8 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 4 & 8 \end{vmatrix} + \begin{vmatrix} 2 & -4 \\ -4 & 8 \end{vmatrix} = 0 + 0 + 0 = 0$$

$|A| = 2(0) + 4(0) + 4(0) = 0$

$\therefore$ Expression (1) becomes, $\lambda^3 - 18\lambda^2 = 0$

$$\lambda = 18, 0, 0$$

Consider, $(A - \lambda I)X = 0$

$$\begin{aligned} (2-\lambda)x - 4y + 4z &= 0 \\ -4x + (8-\lambda)y - 8z &= 0 \\ 4x - 8y + (8-\lambda)z &= 0 \end{aligned} \qquad \ldots\ldots..(2)$$

Case – 1: Putting $\lambda = 18$, we get

$$-16x - 4y + 4z = 0, -4x - 10y - 8z = 0, 4x - 8y - 10z = 0$$

From first and second equations, we have

$$\frac{x}{\begin{vmatrix} -4 & 4 \\ -10 & -8 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -16 & 4 \\ -4 & -8 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -16 & -4 \\ -4 & -10 \end{vmatrix}} \Rightarrow \frac{x}{72} = \frac{-y}{144} = \frac{z}{144} \text{ or } \frac{x}{1} = \frac{y}{-2} = \frac{z}{2}$$

Hence the eigen vector corresponding to $\lambda = 18$ is $[1, -2, 2]^T$

Case – 2: Putting $\lambda = 0$, we get

$$2x - 4y + 4z = 0, -4x + 8y - 8z = 0, 4x - 8y + 8z = 0$$

From these equations, we have $x = 2y - 2z$

Let $z = k_1$ and $y = k_2$ then $x = 2k_2 - 2k_1$

Let $k_1 = k_2 = 1$ then $x = 0, y = 1, z = 1$

Let $k_1 = 0$ and $k_2 = 1$ then $x = 2, y = 1, z = 0$

Hence the eigen vector corresponding to $\lambda = 0$ are

$[0, 1, 1]^T$ and $[2, 1, 0]^T$

$\therefore$ The three eigen vectors corresponding to eigen values $\lambda = 18, 0, 0$ are

$$\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Normalize these eigenvectors to form the columns of U.

$$\therefore U = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{\sqrt{5}} \\ \frac{-2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Diagonal matrix $\sum = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Hence, our final SVD equation becomes

$$A = U \sum V^T = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{\sqrt{5}} \\ \frac{-2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

3. Find the singular value decomposition of a matrix $A = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix}$

Solution:

Singular value decomposition of a matrix A is A = U $\sum$ V^T

Given matrix is $A = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix}$, which is of order 2 × 2 (m × n)

$$AA^T = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ -7 & 4 \end{bmatrix} = \begin{bmatrix} 65 & -32 \\ -32 & 17 \end{bmatrix}$$

The characteristic equation for the above matrix is $|AA^T - \lambda I| = 0$

$$\begin{vmatrix} 65-\lambda & -32 \\ -32 & 17-\lambda \end{vmatrix} = 0$$

$$(65-\lambda)(17-\lambda) - 1024 = 0$$

$$\lambda^2 - 82\lambda + 81 = 0$$

$$\lambda = 81, 1$$

The Eigenvalues are 81 and 1.

So, our singular values are $\sigma_1 = 9$ and $\sigma_2 = 1$

Consider, $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 65-\lambda & -32 \\ -32 & 17-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(65-\lambda)x - 32y = 0$$

$$-32x + (17-\lambda)y = 0$$

$\mathbf{Case-1}$: Putting $\lambda = 81$, we get

$$-16x - 32y = 0 \Rightarrow \mathrm{x} = -2\mathrm{y}$$

Let $y = k$, then $x = -2k$

Hence the eigen vector corresponding to $\lambda = 81$ is $[-2k, k]^T = k[-2, 1]^T$

$\mathbf{Case-2}$: Putting $\lambda = 1$, we get

$$64x - 32y = 0 \Rightarrow \mathrm{y} = 2\mathrm{x}$$

Let $x = k$, then $y = 2k$

Hence the eigen vector corresponding to $\lambda = 1$ is $[k, 2k]^T = k[1, 2]^T$

$\therefore$ The two eigen vectors corresponding to two eigen values $\lambda = 81, 1$ are

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Normalize these eigenvectors to form the columns of U.

$$\therefore U = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

To find V^T,

$$A^T A = \begin{bmatrix} -4 & 1 \\ -7 & 4 \end{bmatrix} \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 32 \\ 32 & 65 \end{bmatrix}$$

The characteristic equation for the above matrix is $|AA^T - \lambda I| = 0$

$$\begin{vmatrix} 17-\lambda & 32 \\ 32 & 65-\lambda \end{vmatrix} = 0$$

$$(65-\lambda)(17-\lambda) - 1024 = 0$$

$$\lambda^2 - 82\lambda + 81 = 0$$

$$\lambda = 81, 1$$

The Eigenvalues are 81 and 1.

Consider, $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 17-\lambda & 32 \\ 32 & 65-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(17-\lambda)x + 32y = 0$$

$$32x + (65-\lambda)y = 0$$

$\mathbf{Case-1}$: Putting $\lambda = 81$, we get

$$-64x + 32y = 0 \Rightarrow \mathrm{y} = 2\mathrm{x}$$

Let $x = k$, then $y = 2k$

Hence the eigen vector corresponding to $\lambda = 81$ is $[k, 2k]^T = k[1, 2\,]^T$

$\mathbf{Case-2}$: Putting $\lambda = 1$, we get

$$16x + 32y = 0 \Rightarrow \mathrm{x} = -2\mathrm{y}$$

Let $y = k$, then $x = -2k$

Hence the eigen vector corresponding to $\lambda = 1$ is $[-2k, k]^T = k[-2, 1]^T$

$\therefore$ The two eigen vectors corresponding to two eigen values $\lambda = 81, 1$ are

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Normalize these eigenvectors to form the rows of V^T.

$$\therefore\ \mathrm{V}^{\mathrm{T}} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

Diagonal matrix $\sum = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$

Hence, our final SVD equation becomes

$$A = U \sum V^T = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

or

$$\begin{bmatrix} -0.89442719 & 0.4472136 \\ 0.4472136 & 0.89442719 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.4472136 & 0.89442719 \\ -0.89442719 & 0.4472136 \end{bmatrix}$$

4. Find the singular value decomposition of a matrix $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$

Solution:

Singular value decomposition of a matrix A is A = U $\sum$ V^T

Given matrix is $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$, which is of order 2 × 2 (m × n)

$$AA^T = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 12 \\ 12 & 41 \end{bmatrix}$$

The characteristic equation for the above matrix is $|AA^T - \lambda I| = 0$

$$\begin{vmatrix} 9-\lambda & 12 \\ 12 & 41-\lambda \end{vmatrix} = 0$$

$$(9-\lambda)(41-\lambda) - 144 = 0$$

$$\lambda^2 - 50\lambda + 225 = 0$$

$$\lambda = 45, 5$$

The Eigenvalues are 45 and 5.

So, our singular values are $\sigma_1 = \sqrt{45}$ and $\sigma_2 = \sqrt{5}$

Consider, $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 9-\lambda & 12 \\ 12 & 41-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(9-\lambda)x + 12y = 0$$

$$12x + (41 - \lambda)y = 0$$

$\textbf{Case} - \mathbf{1}$: Putting $\lambda = 45$, we get

$$-36x + 12y = 0 \Rightarrow \mathrm{y} = 3\mathrm{x}$$

Let $x = k$, then $y = 3k$

Hence the eigen vector corresponding to $\lambda = 45$ is $[k, 3k]^T = k[1, 3\ ]^T$

$\textbf{Case} - \mathbf{2}$: Putting $\lambda = 5$, we get

$$4x + 12y = 0 \ \Rightarrow \ \mathrm{x} = -3\mathrm{y}$$

Let $y = k$, then $x = -3k$

Hence the eigen vector corresponding to $\lambda = 5$ is $[-3k, k]^T = k[-3, 1]^T$

$\therefore$ The two eigen vectors corresponding to two eigen values $\lambda = 45, 5$ are

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Normalize these eigenvectors to form the columns of U.

$$\therefore U = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$$

To find V^T,

$$A^T A = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}$$

The characteristic equation for the above matrix is $|AA^T - \lambda I| = 0$

$$\begin{vmatrix} 25 - \lambda & 20 \\ 20 & 25 - \lambda \end{vmatrix} = 0$$

$$(25 - \lambda)(25 - \lambda) - 400 = 0$$

$$\lambda^2 - 50\lambda + 225 = 0$$

$$\lambda = 45, 5$$

The Eigenvalues are 45 and 5.

Consider, $(A-\lambda I)X=0$

$$\Rightarrow \begin{bmatrix} 25-\lambda & 20 \\ 20 & 25-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(25-\lambda)x+20y=0$$

$$20x+(25-\lambda)y=0$$

$\mathbf{Case-1}$: Putting $\lambda=45$, we get

$$-20x+20y=0 \Rightarrow \mathrm{x}=\mathrm{y}$$

Let $x=k$, then $y=k$

Hence the eigen vector corresponding to $\lambda=45$ is $[k,k]^T=k[1,1\,]^T$

$\mathbf{Case-2}$: Putting $\lambda=5$, we get

$$20x+20y=0 \Rightarrow \mathrm{y}=-\mathrm{x}$$

Let $x=k$, then $y=-k$

Hence the eigen vector corresponding to $\lambda=5$ is $[k,-k]^T=k[1,-1]^T$

$\therefore$ The two eigen vectors corresponding to two eigen values $\lambda=45,5$ are

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Normalize these eigenvectors to form the rows of V^T.

$$\therefore\ \mathrm{V}^{\mathrm{T}}=\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

Diagonal matrix $\sum=\begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$

Hence, our final SVD equation becomes

$$\mathrm{A}=\mathrm{U}\sum\mathrm{V}^{\mathrm{T}}=\begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}\begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

or

$$\begin{bmatrix} 0.31622777 & -0.9486833 \\ 0.9486833 & 0.31622777 \end{bmatrix} \begin{bmatrix} 6.708204 & 0 \\ 0 & 2.236068 \end{bmatrix} \begin{bmatrix} 0.707068 & 0.707068 \\ 0.707068 & -0.707068 \end{bmatrix}$$

Python Code to Find SVD

```
import numpy as np

# Create your matrix A
A = np.array([[   ], [   ]])

# Calculate the SVD
U, S, VT = np.linalg.svd(A)

# You can also reconstruct the original matrix A using U, S, and VT
# This is an approximation of the original matrix
A_reconstructed = U @ np.diag(S) @ VT

# Print the results
print("U (Left Singular Vectors):")
print(U)
print("\nS (Singular Values):")
print(S)
print("\nVT (Transpose of Right Singular Vectors):")
print(VT)
print("\nA (Reconstructed):")
```

```
print(A_reconstructed)
```

Examples

1. **Find the singular value decomposition (SVD) of the matrix** $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

```python
import numpy as np

# Define the matrix A
A = np.array([[1, 2],[3, 4],[5, 6]])

# Calculate the SVD
U, S, VT = np.linalg.svd(A)

# Print the results
print("U (Left Singular Vectors):")
print(U)
print("\nS (Singular Values):")
print(S)
print("\nVT (Transpose of Right Singular Vectors):")
print(VT)
```

```
U (Left Singular Vectors):
[[-0.2298477   0.88346102  0.40824829]
 [-0.52474482  0.24078249 -0.81649658]
 [-0.81964194 -0.40189603  0.40824829]]

S (Singular Values):
[9.52551809 0.51430058]

VT (Transpose of Right Singular Vectors):
[[-0.61962948 -0.78489445]
 [-0.78489445  0.61962948]]
```

2. **Find the singular value decomposition (SVD) of the matrix** $A = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix}$

```python
import numpy as np

# Create your matrix A
A = np.array([[-4, -7],[1, 4]])

# Perform SVD
U, S, VT = np.linalg.svd(A)

# You can also reconstruct the original matrix A using U, S, and VT
# This is an approximation of the original matrix
A_reconstructed = U @ np.diag(S) @ VT

# Print the results
print("U (Left Singular Vectors):")
print(U)
print("\nS (Singular Values):")
print(S)
print("\nVT (Transpose of Right Singular Vectors):")
print(VT)
print("\nA (Reconstructed):")
print(A_reconstructed)
```

```
U (Left Singular Vectors):
[[-0.89442719  0.4472136 ]
 [ 0.4472136   0.89442719]]

S (Singular Values):
[9. 1.]

VT (Transpose of Right Singular Vectors):
[[ 0.4472136   0.89442719]
 [-0.89442719  0.4472136 ]]

A (Reconstructed):
[[-4. -7.]
 [ 1.  4.]]
```

3. **Find the singular value decomposition of a matrix $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$**

```python
import numpy as np

# Create your matrix A
A = np.array([[3, 2, 2],[2, 3, -2]])

# Perform SVD
U, S, VT = np.linalg.svd(A)

# Print the results
print("U (Left Singular Vectors):")
print(U)
print("\nS (Singular Values):")
print(S)
print("\nVT (Transpose of Right Singular Vectors):")
print(VT)
```

```
U (Left Singular Vectors):
[[ 0.70710678 -0.70710678]
 [ 0.70710678  0.70710678]]

S (Singular Values):
[5. 3.]

VT (Transpose of Right Singular Vectors):
[[ 7.07106781e-01  7.07106781e-01  3.67439059e-16]
 [-2.35702260e-01  2.35702260e-01 -9.42809042e-01]
 [-6.66666667e-01  6.66666667e-01  3.33333333e-01]]
```

Applications of Singular Value Decomposition (SVD)

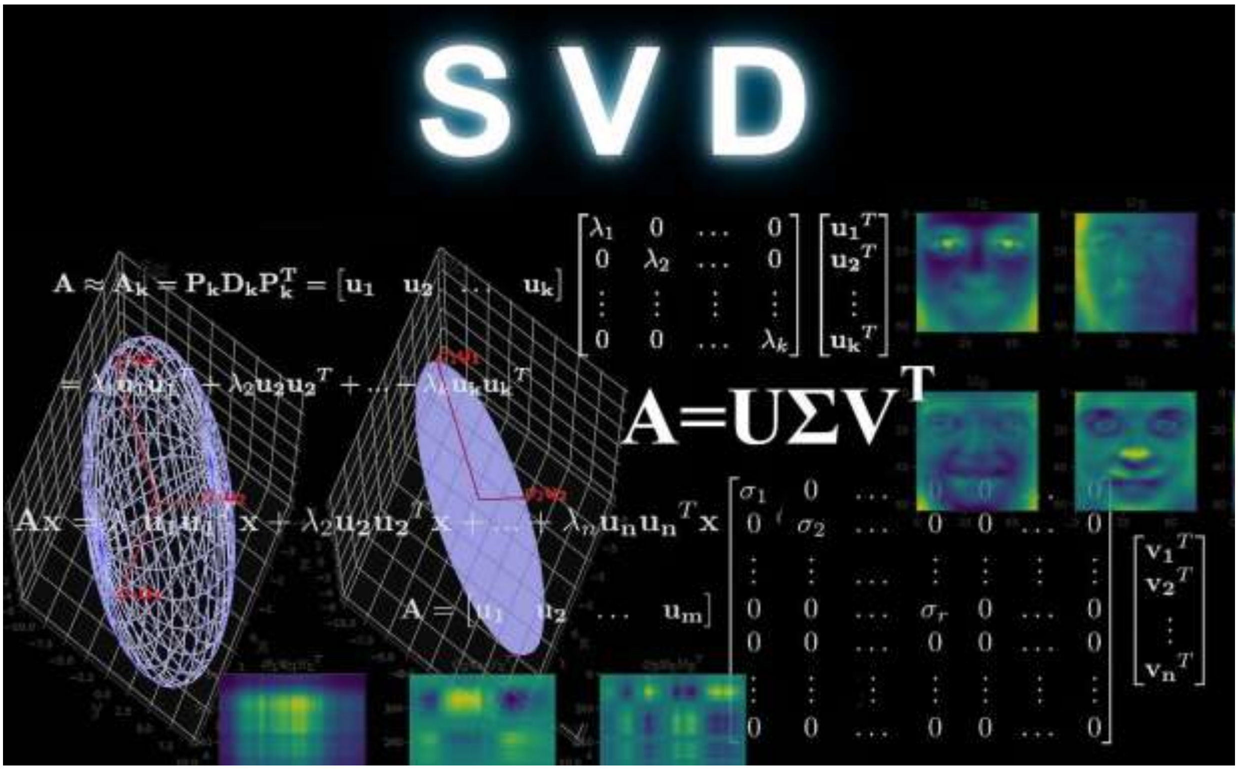
Singular Value Decomposition (SVD) has a wide range of applications across various fields due to its ability to extract important information from data and represent it in a more structured and simplified form.

Here are some key applications of SVD:

1. **Dimensionality Reduction**: SVD is widely used for reducing the dimensionality of large datasets. It helps eliminate noise and redundant information while retaining essential patterns, making it valuable in data compression and storage.
2. **Image Compression**: SVD is employed in image compression techniques like JPEG. By representing images using fewer singular values and vectors, SVD reduces the storage space required while preserving image quality.
3. **Recommendation Systems**: SVD plays a crucial role in collaborative filtering-based recommendation systems. It can uncover latent factors in user-item interaction matrices, enabling more accurate and personalized recommendations.

4. **Natural Language Processing (NLP)**:
    - **Latent Semantic Analysis (LSA)**: SVD is used to perform topic modeling and document clustering in text analysis.
    - **Word Embeddings**: Techniques like Word2Vec and GloVe utilize SVD to create dense vector representations of words.
5. **Principal Component Analysis (PCA)**: PCA, a dimensionality reduction technique, relies on SVD to find the principal components of data. It helps in data visualization, noise reduction, and feature selection.
6. **Face Recognition**: SVD is used in face recognition systems to extract eigenfaces, which are the principal components of facial images. These eigenfaces can be used for facial feature recognition.
7. **Biomedical Data Analysis**: SVD is employed in analysing gene expression data, medical imaging, and DNA sequencing to identify relevant patterns, reduce noise, and improve the interpretation of complex biological data.
8. **Signal Processing**: In signal processing, SVD is used for filtering, noise reduction, and feature extraction. It is especially useful in speech and audio processing.
9. **Collaborative Filtering**: SVD is used to analyse user-item interaction data in recommendation systems and collaborative filtering models, such as the Netflix movie recommendation algorithm.
10. **Eigenvalue Problems**: SVD can be applied to solve eigenvalue problems, making it valuable in physics, engineering, and structural mechanics.
11. **Low-Rank Matrix Approximation**: SVD can be used to find the best low-rank approximation of a matrix. This is beneficial in data compression and approximation tasks.
12. **Chemometrics**: SVD is used in chemometrics for data analysis in fields like spectroscopy, chromatography, and mass spectrometry.
13. **Quality Control and Anomaly Detection**: SVD can identify anomalies or outliers in data, making it useful for quality control and anomaly detection in various industries.
14. **Data Denoising**: SVD can help remove noise from data, improving the accuracy of various data analysis tasks.
15. **Machine Learning**: SVD is occasionally used as a preprocessing step or feature engineering technique in machine learning algorithms.

These applications demonstrate the versatility and importance of Singular Value Decomposition across a wide range of domains, making it a fundamental tool in data analysis and modelling.


SVD
$\mathbf{A} \approx \mathbf{A_k} = \mathbf{P_k D_k P_k^T} = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} & \dots & \mathbf{u_k} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_k \end{bmatrix} \begin{bmatrix} \mathbf{u_1}^T \\ \mathbf{u_2}^T \\ \vdots \\ \mathbf{u_k}^T \end{bmatrix}$
$= \lambda_1 \mathbf{u_1 u_1}^T + \lambda_2 \mathbf{u_2 u_2}^T + \dots + \lambda_k \mathbf{u_k u_k}^T$
$\mathbf{A=U\Sigma V^T}$
$\mathbf{Ax} = \lambda_1 \mathbf{u_1 u_1}^T \mathbf{x} + \lambda_2 \mathbf{u_2 u_2}^T \mathbf{x} + \dots + \lambda_n \mathbf{u_n u_n}^T \mathbf{x}$
$\mathbf{A} = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} & \dots & \mathbf{u_m} \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \sigma_r & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v_1}^T \\ \mathbf{v_2}^T \\ \vdots \\ \mathbf{v_n}^T \end{bmatrix}$