Determinant of the Matrix

The determinant of a matrix is a number that is specially defined only for square matrices. It is very useful to find the solution of systems of linear equations.

For every square matrix $A = [a_{ij}]_{n \times n}$ of order n, we can associate a number (real or complex) called determinant of the square matrix A, where a_{ij} = (i, j)th element of A.

Evaluating Determinants

(1) Order One:

$$A = [\, a \,]$$

$$|A| = |\, a \,| = a$$

(2) Order Two:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

(3) Order Three:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

1. Find the determinant of the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{2} \\ \mathbf{2} & \mathbf{3} & \mathbf{5} \\ \mathbf{-2} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

$$|A\,| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{vmatrix} = (1)\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} + (2)\begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix}$$

$$= 3 + 12 + 12 = 27$$

2. Find the determinant of the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{4} & \mathbf{5} \\ \mathbf{6} & \mathbf{1} & \mathbf{3} \\ \mathbf{4} & \mathbf{0} & \mathbf{7} \end{bmatrix}$

Singular and Non-singular Matrix

A square matrix A is said to be singular if |A| = 0.

For example, the determinant of matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$ is zero.

Hence A is a singular matrix.

A square matrix A is said to be non-singular if |A| ≠ 0

For example, the determinant of matrix $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is not equal to zero.

Hence B is a non-singular matrix.

Adjoint of a Matrix

The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is denoted by $adj\ A$ and is defined as the transpose of the cofactor matrix $\left[C_{ij}\right]_{\mathrm{n \times n}}$, where C_{ij} is the cofactor of the element a_{ij} which is obtained by using the formula

$$C_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the Minor of the element a_{ij} which is obtained by omitting the i^{th} row and the j^{th} column.

The cofactor of any element is found by taking its minor and imposing a place sign according to the following rule

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{pmatrix}$$

Find the adjoint of the matrix $A = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{2} \\ \mathbf{2} & \mathbf{3} & \mathbf{5} \\ \mathbf{-2} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

$$adj\ A = \begin{bmatrix} +\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} \\ +\begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} 3 & -12 & 6 \\ 1 & 5 & 2 \\ -11 & -1 & 5 \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$$

The inverse of a Matrix

The **inverse** of a square $n \times n$ matrix A, is another $n \times n$ matrix denoted by A^{-1} such that

$$AA^{-1} = A^{-1}A = I$$

where I is the $n \times n$ identity matrix. That is, multiplying a matrix by its inverse produces an identity matrix. Not all square matrices have an inverse matrix. If the determinant of the matrix is zero, then it will not have an inverse, and the matrix is said to be **singular**. Only non-singular matrices have inverses.

Given any non-singular matrix A, its inverse can be found from the formula

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

1. Find the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{vmatrix} = (1)\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} + (2)\begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix}$$

$$= 3 + 12 + 12 = 27 \neq 0$$

A is non-singular. Therefore, A^{-1} exists.

$$A^{-1} = \frac{adj\ A}{|A|}$$

$$adj\ A = \begin{bmatrix} +\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} \\ +\begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} 3 & -12 & 6 \\ 1 & 5 & 2 \\ -11 & -1 & 5 \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{27}\begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$$

2. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 5 & 7 \\ 2 & 6 & 0 \\ 3 & 5 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 5 & 7 \\ 2 & 6 & 0 \\ 3 & 5 & 1 \end{vmatrix} = (1)\begin{vmatrix} 6 & 0 \\ 5 & 1 \end{vmatrix} - (5)\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} + (7)\begin{vmatrix} 2 & 6 \\ 3 & 5 \end{vmatrix}$$

$$= 6 - 10 - 56 = -60 \neq 0$$

A is non-singular. Therefore, A^{-1} exists.

$$A^{-1} = \frac{adj\ A}{|A|}$$

$$adj\ A = \begin{bmatrix} +\begin{vmatrix} 6 & 0 \\ 5 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 6 \\ 3 & 5 \end{vmatrix} \\ -\begin{vmatrix} 5 & 7 \\ 5 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 7 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 5 \\ 3 & 5 \end{vmatrix} \\ +\begin{vmatrix} 5 & 7 \\ 6 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 7 \\ 2 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} 6 & -2 & -8 \\ 30 & -20 & 10 \\ -42 & 14 & -4 \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} 6 & 30 & -42 \\ -2 & -20 & 14 \\ -8 & 10 & -4 \end{bmatrix}$$

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{-1}{60}\begin{bmatrix} 6 & 30 & -42 \\ -2 & -20 & 14 \\ -8 & 10 & -4 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{30}\begin{bmatrix} 3 & 15 & -21 \\ -1 & -10 & 7 \\ -4 & 5 & -2 \end{bmatrix}$$

3. Find the inverse of the matrix $\boldsymbol{A = \begin{bmatrix} 2 & 1 & 2 \\ -3 & 4 & 5 \\ 6 & 1 & 0 \end{bmatrix}}$

Solution of system of linear equations using inverse of a matrix

Consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

The above system of equations can be written in matrix form as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$i.e., AX = B$$

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

The solution of the system AX = B is given by

$$X = A^{-1}\mathrm{B}$$

This matrix equation provides unique solution for the given system of equations as inverse of a matrix is unique. This method of solving system of equations is known as Matrix Method.

1. Solve the following system of equations by matrix method.

3x – 2y + 3z = 8, 2x + y – z = 1, 4x – 3y + 2z = 4

The system of equations can be written in the form AX = B, where

$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$

The solution of the system AX = B is given by $X = A^{-1}\mathrm{B}$

$$|A| = \begin{vmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{vmatrix} = -17 \neq 0$$

A is non-singular. Therefore, A^{-1} exists.

$$A^{-1} = \frac{adj\ A}{|A|}$$

$$adj\ A = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{-1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$X = A^{-1}\mathrm{B} = \frac{-1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, x = 1, y = 2 and z = 3.

2. The sum of three numbers is 6. If we multiply third number by 3 and add second number to it, we get 11. By adding first and third numbers, we get double of the second number. Represent it algebraically and find the numbers using matrix method.

Let first, second and third numbers be denoted by x, y and z, respectively. Then, according to given conditions, we have

x + y + z = 6, y + 3z = 11, x + z = 2y or x – 2y + z = 0

This system can be written as A X = B, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

The solution of the system AX = B is given by $X = A^{-1}\mathrm{B}$

$$|A\ | = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{vmatrix} = 9 \neq 0$$

A is non-singular. Therefore, A^{-1} exists.

$$A^{-1} = \frac{adj\ A}{|A|}$$

$$adj\ A = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

$$X = A^{-1}\mathrm{B} = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, x = 1, y = 2 and z = 3.

3. Mr. Johns sells Mango, Apple and Peach. The price of a kg of Mango, 3 kgs of Apple, and a kg of Peach is Rs 145. The price of 3 kgs of Mango, 4 kgs of Apple, and a kg of Peach is Rs 280. The price of 2 kgs of Apple, and a kg of Peach is Rs 65. Find out the price of a kg of each fruit.

This is a system of linear equations problem. We can represent the problem as follows: Let's denote:

- x as the price of a kg of Mango
- y as the price of a kg of Apple
- z as the price of a kg of Peach

We can then form the following equations from the problem:

$$\mathrm{x} + 3\mathrm{y} + \mathrm{z} = 145,\ 3\mathrm{x} + 4\mathrm{y} + \mathrm{z} = 280,\ 2\mathrm{y} + \mathrm{z} = 65$$

This system can be written as A X = B, where

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 4 & 1 \\ 0 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 145 \\ 280 \\ 65 \end{bmatrix}$$

The solution of the system AX = B is given by $X = A^{-1}\mathrm{B}$

$$|A| = \begin{vmatrix} 1 & 3 & 1 \\ 3 & 4 & 1 \\ 0 & 2 & 1 \end{vmatrix} = -1 \neq 0$$

A is non-singular. Therefore, A^{-1} exists.

$$A^{-1} = \frac{adj\ A}{|A|}$$

$$adj\ A = \begin{bmatrix} 2 & -1 & -1 \\ -3 & 1 & 2 \\ 6 & -2 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{-1}\begin{bmatrix} 2 & -1 & -1 \\ -3 & 1 & 2 \\ 6 & -2 & -5 \end{bmatrix}$$

$$X = A^{-1}\text{B} = \begin{bmatrix} -2 & 1 & 1 \\ 3 & -1 & -2 \\ -6 & 2 & 5 \end{bmatrix} \begin{bmatrix} 145 \\ 280 \\ 65 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 55 \\ 25 \\ 15 \end{bmatrix}$$

∴ The required solution of the given system of equations is

$$x = 55, y = 25, z = 15$$

Price per kg: Mango = Rs.55, Apple = Rs.25 and Peach = Rs.15

4. Mr. Murgan sells 3 different products. He sells products X, Y & Z. If he sells one unit of X, 5 units of Y and a unit of Z he makes a profit of Rs.1080. If he sells one of Y and a unit of Z he makes a profit of Rs. 540. If he sells 2 units of X and buys two units of Y and a unit of Z from another seller same as his selling price, he incurs a loss of 180 rupees. Find out the price of product X, Y, & Z?

This is a system of linear equations problem. We can represent the problem as follows:

Let's denote:

- x as the price of product X
- y as the price of product Y
- z as the price of product Z

We can then form the following equations from the problem:

$$x + 5y + z = 1080, \quad y + z = 540, 2x - 2y - z = -180$$

This system can be written as A X = B, where

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & 1 \\ 2 & -2 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1080 \\ 540 \\ -180 \end{bmatrix}$$

The solution of the system AX = B is given by $X = A^{-1}\text{B}$

$$|A| = \begin{vmatrix} 1 & 5 & 1 \\ 0 & 1 & 1 \\ 2 & -2 & -1 \end{vmatrix} = 9 \neq 0$$

A is non-singular. Therefore, A^{-1} exists.

$$A^{-1} = \frac{adj\ A}{|A|}$$

$$adj\ A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -3 & -1 \\ -2 & 12 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{9}\begin{bmatrix} 1 & 3 & 4 \\ 2 & -3 & -1 \\ -2 & 12 & 1 \end{bmatrix}$$

$$X = A^{-1}\mathrm{B} = \frac{1}{9}\begin{bmatrix} 1 & 3 & 4 \\ 2 & -3 & -1 \\ -2 & 12 & 1 \end{bmatrix}\begin{bmatrix} 1080 \\ 540 \\ -180 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 220 \\ 80 \\ 460 \end{bmatrix}$$

∴ The required solution of the given system of equations is

$$x = 220, y = 80, z = 460$$

5. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is Rs.60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is Rs.90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is Rs.70. Find cost of each item per kg by matrix method.

The system can be represented as follows:

4x + 3y + 2z = 60, 2x + 4y + 6z = 90, 6x + 2y + 3z = 70

where:

x is the cost of onions per kg,

y is the cost of wheat per kg,

z is the cost of rice per kg.

This system can be written as A X = B, where

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

The solution of the system AX = B is given by $X = A^{-1}\mathrm{B}$

$$|A| = \begin{vmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{vmatrix} = 50 \neq 0$$

A is non-singular. Therefore, A^{-1} exists.

$$A^{-1} = \frac{adj\ A}{|A|}$$

$$adj\ A = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$X = A^{-1}\text{B} = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$\therefore$ The required solution of the given system of equations is

$$x = 5, y = 8, z = 8$$

Cost per kg: Onions = Rs 5, Wheat = Rs 8, Rice = Rs 8