ORTHOGONAL MATRIX

A square matrix with real numbers or elements is said to be an orthogonal matrix if its transpose is equal to its inverse matrix. Or we can say when the product of a square matrix and its transpose gives an identity matrix, then the square matrix is known as an orthogonal matrix.

A Square matrix ‘A’ is orthogonal if

$$A^T = A^{-1}$$

(OR)

$$AA^T = A^TA = I$$, where

- A^T = Transpose of A
- A^{-1} = Inverse of A
- I = Identity matrix of same order as ‘A’

The value of the determinant of an orthogonal matrix is always ±1.

Example 1:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{A^T = A^{-1}}$$

$$\boxed{AA^T = A^TA = I}$$

Example 2:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 3:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

Example 4:

Q) Prove that given matrix is Orthogonal. $A = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$

Orthogonal and orthonormal vectors

Definition. The two vectors are said to be orthogonal if they are perpendicular to each other. i.e. the dot product of the two vectors is zero.

Definition. The set of vectors {v1, v2, ..., vn} are mutually orthogonal if every pair of vectors is orthogonal. i.e. vi . vj = 0, for all i ≠ j.

EXAMPLE. The set of vectors $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$ is mutually orthogonal.

$$\begin{aligned} (1,0,-1).(1,\sqrt{2},1) &= 0 \\ (1,0,-1).(1,-\sqrt{2},1) &= 0 \\ (1,\sqrt{2},1).(1,-\sqrt{2},1) &= 0 \end{aligned}$$

Definition. A set of vectors S is orthonormal if every vector in S has magnitude one and the set of vectors are mutually orthogonal.

An orthogonal set of non-zero vectors is linearly independent.

EXAMPLE. We just checked that the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

are mutually orthogonal. The vectors however are not normalized (this term is sometimes used to say that the vectors are not of magnitude 1). Let

$$\begin{aligned}
\vec{u}_1 &= \frac{\vec{v}_1}{|\vec{v}_1|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \\
\vec{u}_2 &= \frac{\vec{v}_2}{|\vec{v}_2|} = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ \sqrt{2}/2 \\ 1/2 \end{pmatrix} \\
\vec{u}_3 &= \frac{\vec{v}_3}{|\vec{v}_3|} = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -\sqrt{2}/2 \\ 1/2 \end{pmatrix}
\end{aligned}$$

The set of vectors $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is orthonormal.

ORTHONORMAL VECTOR

- ***Definition.*** If v1,v2,......Vn whose ||v1||=||v2||=......||Vn||=1 then v1,v2,......Vn is a orthonormal vector.

- ***Definition.*** An orthogonal set in which each vector is a unit vector is called **orthonormal**.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subseteq V$$

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Example

$u_1 = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right]$

$u_2 = [0, 0, 1, 0]$

$u_3 = \left[\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0, 0\right]$

$u_4 = [0, 0, 0, 1]$

Orthonormal Set

- A set of vectors is called an orthonormal set if it is an orthogonal set, and the norm of all the vectors is 1

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \right\}$$

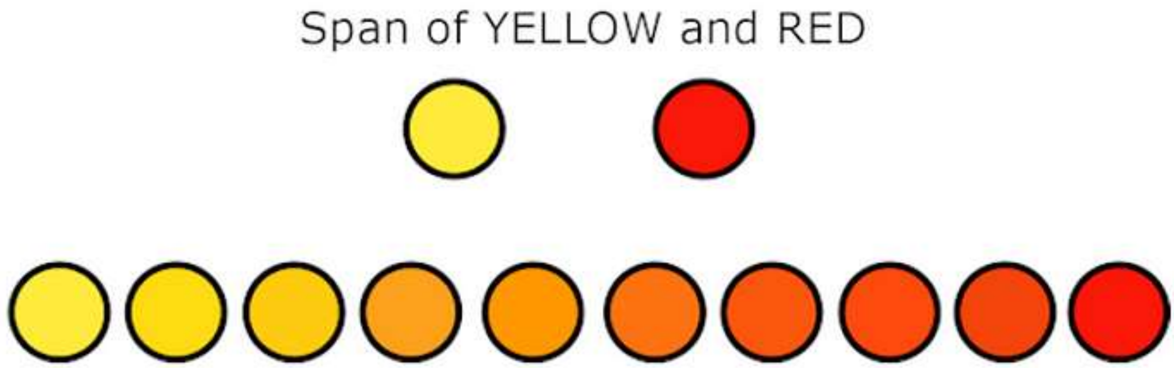
Is orthonormal set independent?

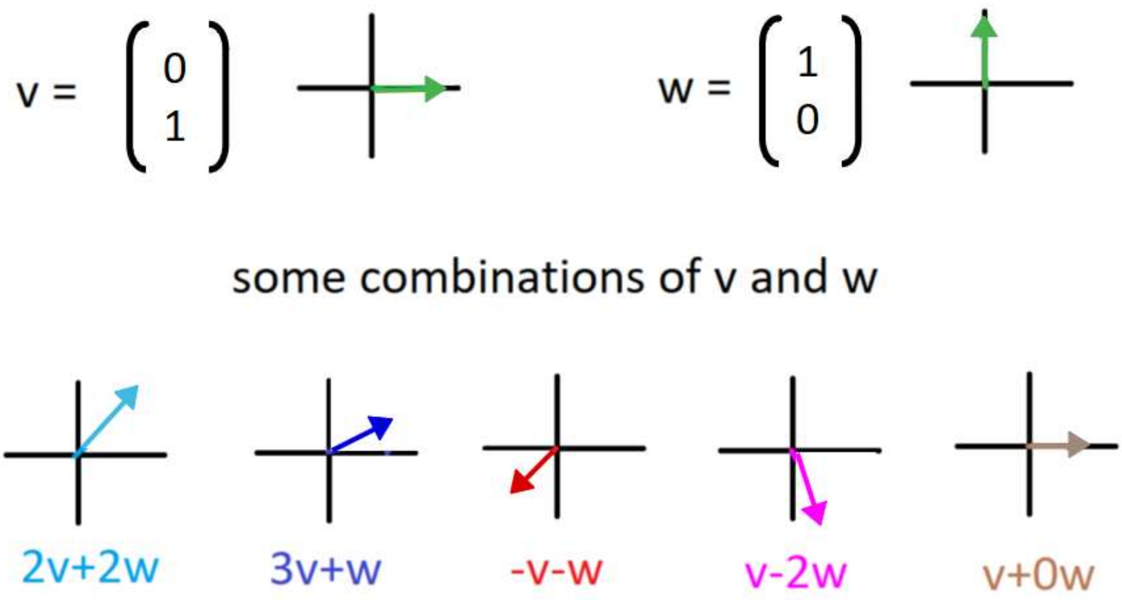
Yes

$$\frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \frac{1}{\sqrt{42}}\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

A vector that has norm equal to 1 is called a unit vector.

Span

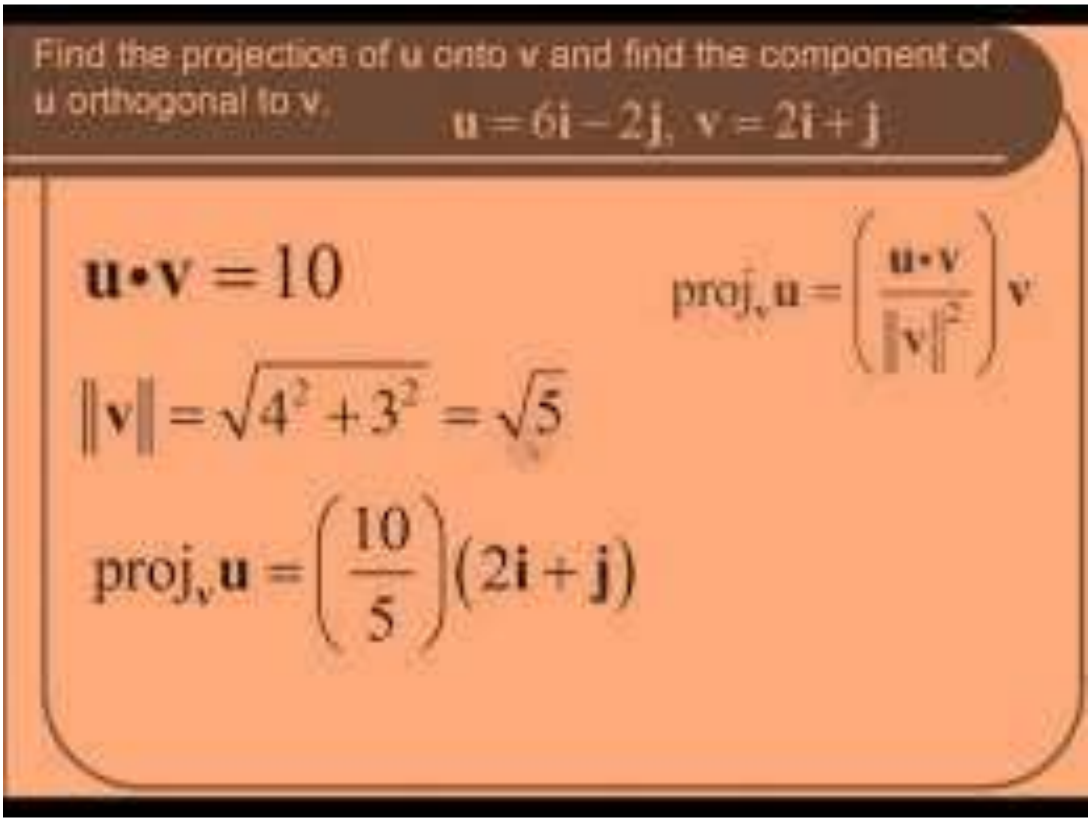


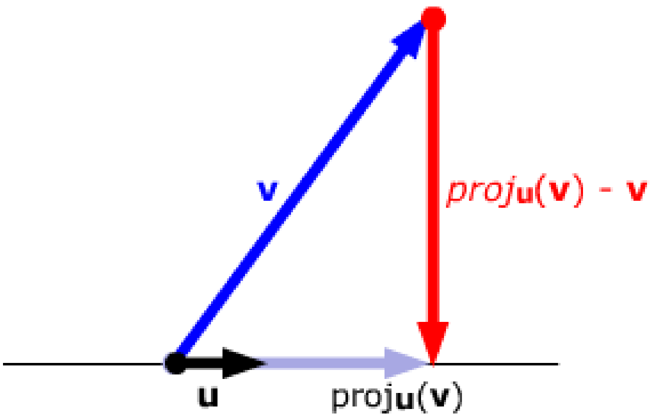


Each of these linear combinations, on their own, can be thought of as $c_1v_1 + c_2v_2 + c_3v_3$ where each *c* is a real number. The set of all of these linear combinations is called the *span* of (v_1, v_2, v_3) and is sometimes just written as Span(v_1, v_2, v_3).

DEFINITION. *Let $\vec{u}$ and $\vec{v}$ be two vectors. The projection of the vector $\vec{v}$ on $\vec{u}$ is defined as folows:*

$$Proj_{\vec{u}}\vec{v} = \frac{(\vec{v}.\vec{u})}{|\vec{u}|^2}\vec{u}.$$





EXAMPLE. Consider the two vectors $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
These two vectors are linearly independent.
However they are not orthogonal to each other. We create an orthogonal vector in the following manner:

$$\begin{aligned} \vec{v}_1 &= \vec{v} - (\text{Proj}_{\vec{u}}\vec{v}) \\ \text{Proj}_{\vec{u}}\vec{v} &= \frac{(1)(1) + (1)(0)}{(\sqrt{1^2 + 0^2})^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \vec{v}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - (1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

$\vec{v}_1$ thus constructed is orthogonal to $\vec{u}$.

Gram-Schmidt process

The Gram-Schmidt process (or procedure) is a sequence of operations that allow us to transform a set of linearly independent vectors into a set of orthonormal vectors that span the same space spanned by the original set.

The aim of the Gram-Schmidt process

Start with	**Find**
linearly independent vectors	orthonormal vectors
$s_1, \dots, s_K$	$u_1, \dots, u_K$

Given a set of linearly independent vectors, it is often useful to convert them into an orthonormal set of vectors.

Orthonormal sets of vectors are useful because they are almost equivalent to linearly independent sets. They are also more convenient because they have an additional inner product structure.

Orthonormal vectors are linearly independent, which means that no vector in the set can be written as a linear combination of the others. This property is important for spanning vector spaces without redundancy.

The Gram-Schmidt Algorithm:

Let $v_1, v_2, \ldots, v_n$ be a set of n linearly independent vectors in $\mathcal{R}^n$. Then we can construct an orthonormal set of vectors as follows:

Step 1. Let $\vec{u}_1 = \vec{v}_1$. $\qquad \vec{e}_1 = \frac{\vec{u}_1}{|\vec{u}_1|}$.

Step 2. Let $\vec{u}_2 = \vec{v}_2 - \mathrm{Proj}_{\vec{u}_1}\vec{v}_2$. $\qquad \vec{e}_2 = \frac{\vec{u}_2}{|\vec{u}_2|}$.

Step 3. Let $\vec{u}_3 = \vec{v}_3 - \mathrm{Proj}_{\vec{u}_1}\vec{v}_3 - \mathrm{Proj}_{\vec{u}_2}\vec{v}_3$. $\qquad \vec{e}_3 = \frac{\vec{u}_3}{|\vec{u}_3|}$.

Step 4. Let $\vec{u}_4 = \vec{v}_4 - \mathrm{Proj}_{\vec{u}_1}\vec{v}_4 - \mathrm{Proj}_{\vec{u}_2}\vec{v}_4 - \mathrm{Proj}_{\vec{u}_3}\vec{v}_4$. $\qquad \vec{e}_4 = \frac{\vec{u}_4}{|\vec{u}_4|}$.

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EXAMPLE. We will apply the Gram-Schmidt algorithm to orthonormalize the set of vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

To apply the Gram-Schmidt, we first need to check that the set of vectors are linearly independent.

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1(0-1) - 1((-1)(2) - (1)(1)) + 1((-1)(1) - 0) = 1 \neq 0.$$

Therefore the vectors are linearly independent.
Gram-Schmidt algorithm:

Step 1. Let

$$\begin{aligned} \vec{u}_1 &= \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ \vec{e}_1 &= \frac{\vec{u}_1}{|\vec{u}_1|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}. \end{aligned}$$

Step 2. Let

$$\begin{aligned} \vec{u}_2 &= \vec{v}_2 - \mathrm{Proj}_{\vec{u}_1} \vec{v}_2 \\ \mathrm{Proj}_{\vec{u}_1} \vec{v}_2 &= \frac{(1,0,1).(1,-1,1)}{1^2 + (-1)^2 + 1^2} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ \vec{u}_2 &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix} \\ \vec{e}_2 &= \frac{\vec{u}_2}{|\vec{u}_2|} = \frac{3}{\sqrt{6}} \begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix}. \end{aligned}$$

Step 3. Let

$$
\begin{aligned}
\vec{u}_3 &= \vec{v}_3 - \mathrm{Proj}_{\vec{u}_1}\vec{v}_3 - \mathrm{Proj}_{\vec{u}_2}\vec{v}_3 \\
\mathrm{Proj}_{\vec{u}_1}\vec{v}_3 &= \frac{(1,1,2).(1,-1,1)}{1^2+(-1)^2+1^1}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \frac{2}{3}\begin{pmatrix}1\\-1\\1\end{pmatrix} \\
\mathrm{Proj}_{\vec{u}_2}\vec{v}_3 &= \frac{(1,1,2).(1/3,2/3,1/3)}{(1/3)^2+(2/3)^2+(1/3)^2}\begin{pmatrix}1/3\\2/3\\1/3\end{pmatrix} = \frac{5}{2}\begin{pmatrix}1/3\\2/3\\1/3\end{pmatrix} \\
\vec{u}_3 &= \begin{pmatrix}1\\1\\2\end{pmatrix} - \frac{2}{3}\begin{pmatrix}1\\-1\\1\end{pmatrix} - \frac{5}{2}\begin{pmatrix}1/3\\2/3\\1/3\end{pmatrix} \\
&= \begin{pmatrix}-1/2\\0\\1/2\end{pmatrix} \\
\vec{e}_3 &= \frac{\vec{u}_3}{|\vec{u}_3|} = \sqrt{2}\begin{pmatrix}-1/2\\0\\1/2\end{pmatrix}.
\end{aligned}
$$

Working Procedure

1. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for an inner product space V.

2. Let $B' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$, where

$$
\begin{aligned}
\mathbf{w}_1 &= \mathbf{v}_1 \\
\mathbf{w}_2 &= \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle}\mathbf{w}_1 \\
\mathbf{w}_3 &= \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2\rangle}{\langle \mathbf{w}_2, \mathbf{w}_2\rangle}\mathbf{w}_2 \\
&\vdots \\
\mathbf{w}_n &= \mathbf{v}_n - \frac{\langle \mathbf{v}_n, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_n, \mathbf{w}_2\rangle}{\langle \mathbf{w}_2, \mathbf{w}_2\rangle}\mathbf{w}_2 - \cdots - \frac{\langle \mathbf{v}_n, \mathbf{w}_{n-1}\rangle}{\langle \mathbf{w}_{n-1}, \mathbf{w}_{n-1}\rangle}\mathbf{w}_{n-1}.
\end{aligned}
$$

Then B' is an *orthogonal* basis for V.

3. Let $\mathbf{u}_i = \dfrac{\mathbf{w}_i}{\|\mathbf{w}_i\|}$. Then $B'' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an *orthonormal* basis for V.

Also, $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \mathrm{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ for $k = 1, 2, \ldots, n$.

Example: Consider a square matrix

$$\mathbf{A}=\begin{bmatrix}1 & 1 & 2\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}$$

- Replace the basis $S=\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}=\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\1\\0\end{bmatrix}\right\}$ with an orthonormal basis $\{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\}$

$$\mathbf{v}_1=\mathbf{u}_1\Rightarrow\mathbf{w}_1=\mathbf{v}_1/\|\mathbf{v}_1\|=\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}$$

$$\mathbf{v}_2=\mathbf{u}_2-(\mathbf{w}_1,\mathbf{u}_2)\mathbf{w}_1=\begin{bmatrix}1\\0\\0\end{bmatrix}-\frac{1}{\sqrt{2}}\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}=\begin{bmatrix}1/2\\0\\-1/2\end{bmatrix}$$

$$\mathbf{w}_2=\mathbf{v}_2/\|\mathbf{v}_2\|=\begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}$$

$$\mathbf{v}_3=\mathbf{u}_3-(\mathbf{u}_3,\mathbf{w}_1)\mathbf{w}_1-(\mathbf{u}_3,\mathbf{w}_2)\mathbf{w}_2$$

$$=\begin{bmatrix}2\\1\\0\end{bmatrix}-\sqrt{2}\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}-\sqrt{2}\begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

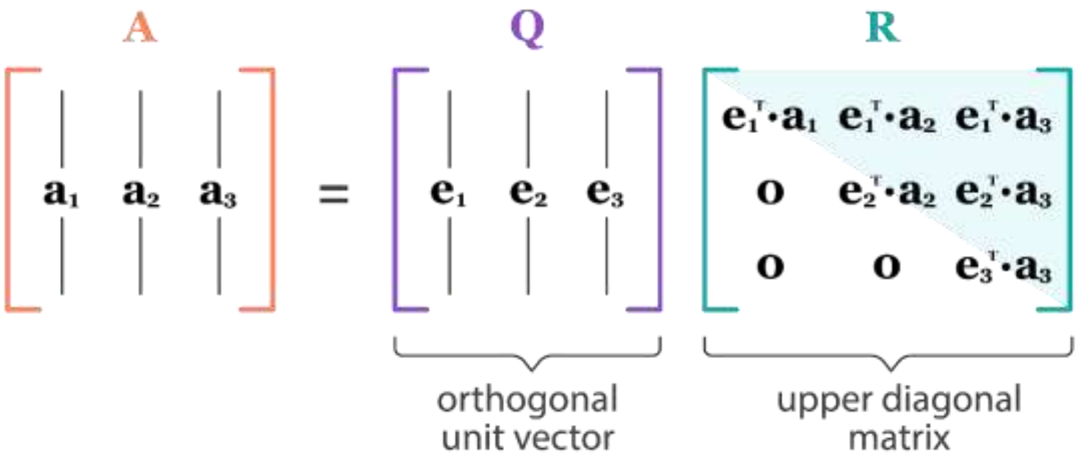
$$\mathbf{w}_3=\mathbf{v}_3/\|\mathbf{v}_3\|=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Orthonormal set is $\left\{\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix},\begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

$$\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\ 0 & 0 & 1\\ 1/\sqrt{2} & -1/\sqrt{2} & 0\end{bmatrix}$$

QR Decomposition with Gram-Schmidt

The QR decomposition (also called the QR factorization) of a matrix is a decomposition of the matrix into an orthogonal matrix and a triangular matrix. A QR decomposition of a real square matrix A is a decomposition of A as A = QR, where Q is an orthogonal matrix (i.e., $Q^TQ = I$) and R is an upper triangular matrix. If A is non-singular, then this factorization is unique.



There are several methods for actually computing the QR decomposition. One of such method is the Gram-Schmidt process.

Consider the above example:

- In the Gram-Schmidt example, the basis

$$\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\1\\0\end{bmatrix}\right\} \text{ is transformed to } \left\{\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix},\begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$$

Interpreting these vectors as column vectors of matrices, the following result holds

$$\mathbf{A}=\begin{bmatrix}1&1&2\\0&0&1\\1&0&0\end{bmatrix}=\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}&0\\0&0&1\\1/\sqrt{2}&-1/\sqrt{2}&0\end{bmatrix}\begin{bmatrix}\sqrt{2}&1/\sqrt{2}&\sqrt{2}\\&1/\sqrt{2}&\sqrt{2}\\&&1\end{bmatrix}=\mathbf{QR}$$

This is called the QR-Factorization of **A**

Python Code

```python
import numpy as np

# Define the matrix
A = np.array([[1, 1, 2], [0, 0, 1], [1, 0, 0]])

# Perform QR factorization
Q, R = np.linalg.qr(A)

print("Q matrix:\n", Q)
print("R matrix:\n", R)
```

```
Q matrix:
 [[-0.70710678 -0.70710678  0.        ]
 [-0.          0.          1.        ]
 [-0.70710678  0.70710678  0.        ]]
R matrix:
 [[-1.41421356 -0.70710678 -1.41421356]
 [ 0.         -0.70710678 -1.41421356]
 [ 0.          0.          1.        ]]
```