# Jacobian matrix

The **Jacobian matrix** is a matrix composed of the first-order partial derivatives of a multivariable function.

The formula for the Jacobian matrix is

$$f(x_1, x_2, \ldots, x_n) = (f_1, f_2, \ldots, f_m)$$

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Therefore, Jacobian matrices will always have as many rows as vector components $(f_1, f_2, \ldots, f_m)$, and the number of columns will match the number of variables $(x_1, x_2, \ldots, x_n)$ of the function.

## Example of Jacobian matrix

1. Find the Jacobian matrix of the function $f(x, y) = (x^2 y, 5x + \sin y)$.

$$f_1(x, y) = x^2 y$$

$$f_2(x, y) = 5x + \sin y$$

$$\mathbf{J_f}(x, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy & x^2 \\ 5 & \cos y \end{bmatrix}$$

2. Find the Jacobian matrix at the point (1, 2) of the following function:

$$f(x,y) = (x^4 + 3y^2x \ , \ 5y^2 - 2xy + 1)$$

First, we find all the first-order partial derivatives of the given function:

$$\frac{\partial f_1}{\partial x} = 4x^3 + 3y^2 \qquad \frac{\partial f_1}{\partial y} = 6yx$$

$$\frac{\partial f_2}{\partial x} = -2y \qquad \frac{\partial f_2}{\partial y} = 10y - 2x$$

$$J_f(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 4x^3 + 3y^2 & 6yx \\ -2y & 10y - 2x \end{pmatrix}$$

we evaluate it at the point (1,2)

$$J_f(1,2) = \begin{pmatrix} 4 \cdot 1^3 + 3 \cdot 2^2 & 6 \cdot 2 \cdot 1 \\ -2 \cdot 2 & 10 \cdot 2 - 2 \cdot 1 \end{pmatrix}$$

The Jacobian matrix J of the given function is

$$J_f(1,2) = \begin{pmatrix} 16 & 12 \\ -4 & 18 \end{pmatrix}$$

3. Find the Jacobian matrix of the function $f(x,y) = (x^2 + y^2 - 1, x^2y + \sin xy^2)$

$$\mathbf{J_f}(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$$J = \begin{bmatrix} 2\mathrm{x} & 2y \\ 2\mathrm{xy} + \mathrm{y}^2\cos(\mathrm{xy}^2) & \mathrm{x}^2 + 2x\mathrm{y}\cos(\mathrm{xy}^2)) \end{bmatrix}$$

4. Find the Jacobian matrix of $F(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1^2 + 5x_2 - 5 \\ \sin(x_1 x_2) + {x_2}^2 x_1 \end{pmatrix}$

$$J = \begin{pmatrix} 2x_1 & 5 \\ x_2\cos(x_1x_2) + x_2^2 & x_1\cos(x_1x_2) + 2x_1x_2 \end{pmatrix}$$

**PYTHON CODE TO FIND JACOBIAN MATRIX**

```
PYTHON SYNTAX

# Create a vector of the functions
F = Matrix([f1, f2])

# Create a vector of the variables
X = Matrix([x, y])

# Compute the Jacobian matrix
J = F.jacobian(X)
```

**Examples**

Find the Jacobian matrix of the function $f(x, y) = (x^2 + y^2 - 1, x^2y + xy^2)$

```python
from sympy import symbols, Matrix

# Define the variables
x, y = symbols('x y')

# Define the functions
f1 = x**2 + y**2 - 1
f2 = x**2*y + x*y**2

# Create a vector of the functions
F = Matrix([f1, f2])

# Create a vector of the variables
X = Matrix([x, y])

# Compute the Jacobian matrix
J = F.jacobian(X)

print("The Jacobian matrix is:")
print(J)
```

```
The Jacobian matrix is:
Matrix([[2*x, 2*y], [2*x*y + y**2, x**2 + 2*x*y]])
```

Find the Jacobian matrix of the function $f(x, y) = (x^2 + y^2 - 1, x^2 y + \sin xy^2)$

```python
from sympy import symbols, Matrix, sin

# Define the variables
x, y = symbols('x y')

# Define the functions
f1 = x**2 + y**2 - 1
f2 = x**2*y + sin(x*y**2)

# Create a vector of the functions
F = Matrix([f1, f2])

# Create a vector of the variables
X = Matrix([x, y])

# Compute the Jacobian matrix
J = F.jacobian(X)

print("The Jacobian matrix is:")
print(J)
```

```
The Jacobian matrix is:
Matrix([[2*x, 2*y], [2*x*y + y**2*cos(x*y**2), x**2 + 2*x*y*cos(x*y**2)]])
```