MATH 127 Calculus for the Sciences

Lecture 3

September 8, 2025

Today's lecture

Last time
Tangent line approximation, differential approximation.

This time
Course note coverage Section 1.1.4, 1.1.5
Differentials recap
Differential equations: definition, order, solution

Differentials recap

Recall from last time: we have the approximation

$$\Delta y \approx f'(a)\Delta x$$

which is more and more accurate when Δx becomes smaller, but is not perfect unless Δx is exactly 0.

To capture this infinitesimal behaviour, we say the **differentials** dy, dx satisfy

$$\boxed{dy = f'(x)dx.}$$

Given $y = f(x)$

$$dy = f'(x)\,dx$$

Example

Example Suppose $u = sin(m)$. Find du.

Given $u = \sin(m)$

$du = \sin'(m)\,dm$

$= \cos(m)\,dm$

Example

Question For what function $f(x)$ do we have

$$f'(x) = f(x)?$$

If $f(x) = e^x$

then $f'(x) = e^x$. therefore $f'(x) = f(x)$.

If $f(x) = \cancel{1}\ 0$

$f'(x) = 0$ therefore $f'(x) = f(x)$

If $f(x) = 2 \cdot e^x$

$f'(x) = 2 \cdot (e^x)' = 2e^x$ therefore $f'(x) = f(x)$

Differential equations

A **differential equation** is an equation involving an unknown function, its variable, and its derivatives.

Example The equation

$$f'(x) = f(x)$$

is a differential equation, because it involves the unknown function $f(x)$, the variable x, and the derivative $f'(x)$.

Example The equation

$$\frac{d^2y}{dx^2} = x$$

is a differential equation, because it involves the unknown function $\boxed{y}$, the variable $\boxed{x}$, and the (second) derivative of $\boxed{y}$.

Order of a differential equation

Remark The equation

$$\frac{g^{(2)}(x)}{x} = f'(x) + g(3)^2 + 1$$

has two functions, g and f.
So to say that it is a differential equation, we should specify: regarding to which unknown function and (sometimes) which variable.

Definition Given a differential equation with respect to an unknown function $f(x)$ in the variable x, the $\boxed{\textbf{order}}$ of the differential equation is the order of the highest derivative of $f(x)$ involved in the equation.

Example The equation

$$\underbrace{f''(x)}^{2^{nd}} + \overbrace{f'(x)}^{1st} = \overbrace{f(x)}^{0th}$$

has order 2, because the highest derivative it involves is the *second* derivative $f''(x)$.

Examples

k is regarded as constant
because k is independent of f, x.

Example The equation

$$k + f''(x) = 2$$

viewed as a differential equation in the unknown function $f(x)$ and variable x, has degree 2.

Example The equation

$$f'''(3) + f''(x) = 2$$

is a differential equation
in the function f(x) and variable x.

has order $\boxed{2}$, because although $f'''(3)$ appears, it is evaluated at 3.
which makes this term a constant

Example The equation

$y'' = z'''$

$$\frac{d^2y}{dx^2} = \frac{d^3z}{dx^3}$$

has order $\boxed{2}$ when viewed as a differential equation in the unknown function y, and has order $\boxed{3}$ when viewed as a differential equation in the unknown function z,

Solution of a differential equation

A function that satisfies a differential equation is called a **solution** of that differential equation. As we have seen before,

$$f(x) = e^x, f(x) = 2e^x$$

are both solutions to the differential equation

$$f'(x) = f(x).$$

Example

$$g(x) = \frac{dy}{dx} + C$$

is a solution for any constant value C to the differential equation

LHS RHS

$$g'(x) = \frac{d^2y}{dx^2} \cdot \left(\frac{dy}{dx} + C\right)' = \frac{d^2y}{dx^2} + 0 \overset{\checkmark}{=} \frac{d^2y}{dx^2}$$

when the equation is viewed in regards to the unknown function $\boxed{g}$ and the variable $\boxed{x}$.

Family of solutions

$(x+C)' = 1+0=1$

Consider the differential equation

$$f'(x) = 1.$$

A solution of this is $f(x) = x$. Another solution is $f(x) = x + 1$.
More generally, any solution of the above can be written in the form

$$f(x) = x + C$$

for some constant value C. We say C is a **parameter** of the **family of solutions** $f(x) = x + C$.

Example $f(x) = x^2 + C_1 x + C_2$ is a family of solutions with parameters C_1, C_2 of the differential equation $f''(x) = 2$.

LHS = RHS

$(x^2 + C_1 \cdot x + C_2)''$

$= (2x + C_1)'$

$= 2$

2

therefore $f(x) = x^2 + C_1 x + C_2$ is a solution to $f''(x) = 2$

Exercise

Question The function $y = u^2 + f(x) + C$ is a family of solutions of which of the following differential equations (considered in regards to the unknown function y and the variable u)?

① $y = u^2 + f(x) + 1$; ② $\dfrac{dy}{du} = 2u$; ③ $\dfrac{dy}{du} = 2u + f'(x)$; ④ $\dfrac{d^2y}{du^2} = 2$.

① LHS $u^2 + f(x) + C$ RHS $u^2 + f(x) + 1$

not equal if $C \neq 1$. so

$y = u^2 + f(x) + C$ is not a family of solutions to ①.

LHS

$\dfrac{d}{du}\left(u^2 + f(x) + C\right)$

$= 2u + 0 + 0$

$= 2u$

RHS

② $2u$

③ $2u + f'(x)$

Therefore it is a solution to ② but not ③

Exercise

Question The function $y = u^2 + f(x) + C$ is a family of solutions of which of the following differential equations (considered in regards to the unknown function y and the variable u)?

$$y = u^2 + f(x) + 1; \quad \frac{dy}{du} = 2u; \quad \frac{dy}{du} = 2u + f'(x); \quad 4 \quad \frac{d^2y}{du^2} = 2.$$

4 LHS = RHS

$$\frac{d^2}{du^2}\left(u^2 + f(x) + C\right) \qquad 2$$

$$= \frac{d}{du}(2u)$$

$$= 2$$

Therefore $y = u^2 + f(x) + C$ is a solution to $\frac{d^2y}{du^2} = 2$