Contents

I Chapter 1

I.1 x

$(1+x)$ is a unit because $1 = (1+x^n) = (1+x)(1-x+x^2-x^3+...)$. If u is a unit say $uv = 1$, then $v(u+x) = 1+vx$ is a unit so $u+x$ is a unit.

I.2 x

(i) $\Leftarrow$ is by Ex 1.

$\Rightarrow$ Suppose $g = b_0 + ... + b_m x^m$ is the inverse of f, then $a_n b_m = 0$ (unless $n = 0$). Suppose $a_n^r b_{m-r+1} = 0$. Note that the $(r+1)n+m-r$ degree term $f^{r+1}g$ is 0, its coefficeint is the sum $a_{i_1}...a_{i_{r+1}}b_k$ where $\sum i_j + k = (r+1)n + m - r$. If $k = m - r$, then i_j must all be n. If $k = m - r + 1$, then one of the i_j can be $n-1$, and the rest must all be n. In general, we obsreve that if $k = m - r + \ell$, then at least $n - \ell$ of the i_j must be n, so by induction hypothesis, those products are all 0 for $k > m - r$. Leaving us $a_n^{r+1} b_{m-r} = 0$. Now apply this to $r = m$, and note that $a_0 b_0 = 1$, so $a_n^{m+1} = a_n^{m+1} b_0 a_0 = 0 a_0 = 0$. Thus a_n is nilpotent, and by an induction on n we see a_i are all nilpotent for all $i > 0$. Hence f is a unit by Ex 1.

(ii) $\Leftarrow$ by expanding powers of f.

$\Rightarrow$ if $f^n = 0$, then compare coefficients and see $a_0^n = 0$. So a_0 nilpotent. Now $f+1$ is a unit so by (i) all the a_i are nilpotent too for $i > 0$.

(iii) Suppose f is a zero divisor with $fg = 0$ where g has minimal degree. Then $a_n g f = 0$ and $a_n g$ has degree $m-1$, so by minimality $a_n g = 0$. An induction (we have showed base case and basically decreased the degree of f by 1) shows $a_i g = 0$ for all $i \geq 0$. Hence $a_i b_0 = 0$ for all i and $b_0 f = 0$.

(iv) $\Rightarrow$ Say $\sum_k r_k \sum_{i+j=k} a_i b_j = 1$, then this gives a relation on a_i, and b_j, showing $(a_1, ..., a_n) = (1), (b_1, ..., b_m) = (1)$.

$\Leftarrow$ (Gauss's Lemma) Suppose fg not primitive, say the coefficients contained in maximal ideal $\mathfrak{m}$. Sicne the coefficients of f, g are not all in $\mathfrak{m}$, let a_s, b_t be the minimal ones not contained in m, then the degree $s+t$ term of fg has coefficient $\sum i + j = s + t a_i b_j$. The terms are all in $\mathfrak{m}$ except $a_s b_t$, so the sum is not in $\mathfrak{m}$, which is a contradiction.

I.3 x

Let $f \in R[x_1, ..., x_n]$. Write $f = g_0 + g_1 x_n + ... + g_m x_n^m$ where $g_i \in R[x_1, ..., x_{n-1}]$.

By 2(i) f is a unit iff g_0 is a unit and g_i are nilpotent for $i \geq 1$. Observe g_0 now has one less variable, and again, an induction on n shows that f is a unit iff the constant term is a unit and other coefficients over R are nilpotent.

(ii), (iii) can also be done by induction on number of indeterminates.

(iv) is the same proof, when we take minimal element, we take it with respect to lexicographic order.

I.4 x

Since nilradical is contained in Jacobson radical, we show the other inclusion. Suppose f is in the Jacobson radical, so $1 + fg$ is a unit for all g by prop 1.9. Take $g = x$ and get $1 + f$ is a unit, so by 2(i) $a_0, a_1, ..., a_n$ are all nilpotent. and by 2(ii) f is nilpotent.

I.5 x

(i) $\Rightarrow$ If $fg = 1$ then $a_0 b_0 = 1$ and a_0 is a unit.

$\Leftarrow$ Suppose a_0 is a unit, WLOG say $a_0 = 1$. Then use the identity $(1-t)(1+t+t^2+...) = 1$ via substituting $t = -(a_1 x + a_2 x^2 + ...)$.

(ii) Say $f^n = 0$, then $a_0^n = 0$. Consider the degree n term, its coefficeints is a sum of multiples of a_0 and a_1^n, and therefore a_1^n is in the nilradical, and since nilradical is radical, a_1 is nilpotent. An induction with similar procedure shows the rest of the a_i are all nilpotent.

Converse is not true. Take $A = k[x_1, x_2, ...]/(x_1^2, x_2^2, ...)$ and $a \in R$ non-zero nilpotent. Let $f = \sum x_i x^i$. Then f^n always has term $x_1...x_n x^{1+2+...+n}$ not canceled out.

(iii) If f in Jacobson radical, then $1+fg$ unit for all g. Take g constants in A and by (i), a_0 is in Jacobson radical. Conversely, if $1 + a_0 b$ unit for all b, then $1 + fg$ unit for all g by (i).

(iv) By (i), adding multiples of x to a non-unit is non-unit. Thus $\mathfrak{m} + (x)$ is not (1), and therefore by maximality $\mathfrak{m} + (x) = \mathfrak{m}$ and $x \in \mathfrak{m}$. Now the contraction, again by (i), is not (1), so it is contained in some maximal ideal of A, adding this maximal ideal to $\mathfrak{m}$ we see we get an ideal that is not (1), so it must be the contraction of $\mathfrak{m}$, and we can conclude $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and x.

(v) Consider the ideal generated by the prime ideal and x which we see is prime.

I.6 x

Suppose Jacobson radical is not nilradical, then by assumption there is a non-trivial idempotent e in it. Now $1 - e$ is a unit but $e(1 - e) = e - e^2 = 0$, a contradiction.

I.7 x

Suffices to show an integral domain with only idempotent element is a field. This is because $x^n = x$ implies $x^{n-1} = 1$ or $x = 0$, so x is a unit or $x = 0$.

I.8 x

Follows from Zorn's lemma by observing that intersection of a chain of ideal is an ideal.

I.9 x

$\Rightarrow$ Follows from $\sqrt{\mathfrak{a}} = \cap_{\alpha \subseteq \mathfrak{p}} \mathfrak{p}$.

$\Leftarrow$ Suppose $\alpha = \cap \mathfrak{p}$ and $x^n \in \mathfrak{a}$, then $x \in \mathfrak{p}$ for each $\mathfrak{p}$ and so $x \in \mathfrak{a}$.

I.10 x

(i)$\Rightarrow$(ii) Only one prime ideal means only one maximal ideal, and it is exactly the nilradical. So the ring is local and elements are either units or nilpotent.

(ii)$\Rightarrow$(iii) We have all the non-units form an ideal, the nilradical. So the ring is local and it is maximal. Hence quotient is field.

(iii)$\Rightarrow$(i) If it has more than one prime ideals, then the nilradical must not be maximal, so the quotient is not a field.

I.11 x

(i) Follows from $(x + 1)^2 = x + 1$.

(ii) $A/\mathfrak{p}$ is an integral domain where every element is idempotent. Since $x^2 = x$ implies $x = 0$ or 1, this is a field with two elements and $\mathfrak{p}$ is maximal.

(iii) Given two generators a, b, we can replace them by $a + b$ because $(a + b)(a - b) = a^2 - b^2 = a - b$, which gives us a and b.

I.12 x

If e idempotent, then $e(1 - e) = 0$ so $e, 1 - e$ are non-units, and must both be in $\mathfrak{m}$, a contradiction.

I.13 construction of an algebraic closure of a field

Suppose $\sum g_i f_i(x_{f_i}) = 1$ where g_i is in terms of x_f. Consider the degree > 1 terms, note that they will cancel out only if $g_i = 0$ for all i, a contradiction.

I.14 x

Has maximal elements by Zorn's lemma. Suppose $\mathfrak{p}$ is maximal element, and $xy \in \mathfrak{p}$. Then $axy = 0$ for some $a \in A$. Now if $x, y \notin \mathfrak{p}$, then $(x) + \mathfrak{p}$, $(y) + \mathfrak{p}$ contain non-zero-divisors f, g, and fg is not a zero divisor, but $fg \in (xy) + (x)\mathfrak{p} + (y)\mathfrak{p} + \mathfrak{p}^2 \subseteq (xy) + \mathfrak{p} = \mathfrak{p}$, a contradiction. Thus $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

I.15 the prime spectrum of a ring

(i) If $x^n \in \mathfrak{a} \subseteq \mathfrak{p}$ then $x \in \mathfrak{p}$ so $\sqrt{\mathfrak{a}} \subseteq \mathfrak{p}$. Hence $V(E) = V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.

(ii) All primes contain 0, no prime contain 1.

(iii) A prime contains all E_i iff it contains each E_i.

(iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\sqrt{\mathfrak{a} \cap \mathfrak{b}}) = V(\sqrt{\mathfrak{ab}}) = V(\mathfrak{ab})$ by 1.13(iii). A prime contains $\mathfrak{ab}$ implies it either contains $\mathfrak{a}$ or some $a \in \mathfrak{a}$ is not in $\mathfrak{p}$, then $a\mathfrak{b} \subseteq \mathfrak{p}$ implies $\mathfrak{p}$ contains $\mathfrak{b}$, so $V(\mathfrak{ab}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

I.16 x

I.17 x

If $\mathfrak{p}$ is not in $V(\mathfrak{a})$, then $\mathfrak{p}$ does not contain some element $f \in \mathfrak{a}$ and $D(f)$ is a neighbourhood around $\mathfrak{p}$ in the complement of $V(\mathfrak{a})$, so $D(f)$ form a basis of open sets.

(i) Prime $\mathfrak{p}$ does not contain f and g iff it does not contain fg.

(ii) If f is in all $\mathfrak{p}$ then f in nilradical.

(iii) If f is in no prime ideal then it is in no maximal ideal, so it is a unit.

(iv) $\sqrt{(f)} = \sqrt{(g)}$ because they are intersections of elements in $V(f) = V(g)$ by Prop1.14.

(v) Say X covered by $D(f_i)$ (can decompose arbitrary cover into this form because $D(f)$ form a basis), then $(f_i)_{i \in I} = (1)$ because otherwise the maximal ideal containing $(f_i)_{i \in I}$ is in X but not in any $D(f_i)$. Write $r_1 f_1 + ... + r_n f_n = 1$ for some $r_i \in A$, then $\{D(f_i)\}_{i=1}^n$ covers X because a prime can not contain all of $f_1, ..., f_n$.

(vi) This is because $X_f = \operatorname{Spec} A_f$. Alternatively, given cover $D(f_i)$, since all primes containing $\mathfrak{a}$ must contain f, we have f is nilpotent in $X/\mathfrak{a}$ so $f \in \sqrt{\mathfrak{a}}$. Take $f_1, ..., f_n$ such that $(f_1, ..., f_n)$ contains a power of f, then $D(f_1), ..., D(f_n)$ covers X_f.

(vii) An open set is a union of $D(f)$ since they form a basis, if it were quasi-compact, then it is a finite union.

I.18 x

(i) By (ii), $\{x\}$ is closed iff x is the only prime ideal containing itself, iff it is maximal.

(ii) If $x \in V(\mathfrak{a})$, then $\mathfrak{a} \subseteq x$, so the smallest closed set containg x corresponds to the largest ideal inside x which is x itself, so closure is $V(x)$.

(iii) $y \in \{x\}^-$ iff $y \in V(x)$ iff $x \subseteq y$.

(iv) If x, y are two distinct points, then either $x \not\subseteq y$ or $y \not\subseteq x$, so by (iii) either $x \notin \{y\}^-$ or $y \notin \{x\}^-$, so either there is a neighbourhood around x not containing y (take $X - \{y\}^-$), or the other way around.

I.19 x

Since intersection of open sets contain $D(f) \cap D(g) = D(fg)$, it suffices to show $D(fg) \neq \phi$ for any non-empty $D(f), D(g)$. By Ex17(ii), we know f, g are not nilpotent, so fg is not nilpotent for all such f, g iff the nilradical is prime.

I.20 x

(i) If $\overline{Y}$ were a union of two closed proper subsets, then Y is a union of the corresponding closed subsets, and they are proper since closure of each of them can not be all of $\overline{Y}$.

(ii) Follows from Zorn's lemma, union of a chain of irreducible subspaces is irreducible (check non-empty open sets must intersect).

(iii) Singletons are irreducible, so extend them to maximal irreducible subspaces and get a cover. For Hausdorff spaces, if the space has more than 1 point, then we can split it into two proper closed subsets, making it reducible, so irreducible components are singletons.

(iv) By (i) irreducible components must be closed, since maximal closed subspace correspond to minimal prime ideals, and that $V(\mathfrak{p}) = \operatorname{Spec} X/\mathfrak{p}$ where $X/\mathfrak{p}$ is integral domain, so nilradical is 0 and prime, meaning $V(\mathfrak{p})$ is irreducible, we know the irreducible components are extactly $V(\mathfrak{p})$ where $\mathfrak{p}$ minimal prime.

I.21 x

(i) $(\varphi^*)^{-1}(D(f))$ is the set of ideals $\mathfrak{q}$ such that $\varphi^{-1}(\mathfrak{q})$ does not contain f. This happens iff $\mathfrak{q}$ does not contain $\varphi(f)$. So $(\varphi^*)^{-1}(D(f)) = D(\varphi(f))$.

(ii) $(\varphi^*)^{-1}(V(\mathfrak{a}))$ is the set of ideals $\mathfrak{q}$ such that $\varphi^{-1}(\mathfrak{q})$ contains $\mathfrak{a}$. This is the set of ideals that contains $\varphi(\mathfrak{a})$, which are those that contain $V(\mathfrak{a}^e)$. So $(\varphi^*)^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.

(iii) Trivially $\overline{\varphi^*(V(\mathfrak{b}))} \subseteq V(\mathfrak{b}^c)$.

$$\bigcap_{\mathfrak{p}\in\varphi^*(V(\mathfrak{b}))} \mathfrak{p} = \bigcap_{\mathfrak{q}\in V(\mathfrak{b})} \varphi^{-1}(\mathfrak{q}) = f^{-1}(\sqrt{b}) = \sqrt{b^c}$$

where last equality is by 1.18. This means $\overline{\varphi^*(V(\mathfrak{b}))} = V(\mathfrak{a})$ for some $\mathfrak{a} \subseteq \sqrt{b^c}$ and implies the other inclusion.

(iv) Since surjective, φ gives a one to one correspondence between Y and $V(\ker(\varphi))$ by prop1.1. In particular, it sends closed sets to closed sets, so homeomorphism.

(v) If injective then by (iii), $\overline{\varphi^*(Y)} = V(\varphi^{-1}(0)) = V(0) = X$. In general, it is dense if f

$$\overline{\varphi^*(Y)} = V(\varphi^{-1}(0)) = V(\ker\varphi) = X$$

which happens iff $\ker\varphi$ is contained in nilradical.

(vi) follows from $(\psi\circ\varphi)^{-1} = \varphi^{-1}\circ\psi^{-1}$.

(vii) B has two prime ideals $(0, K)$ and $(A/\mathfrak{p}, 0)$. The inverse image of them are $\mathfrak{p}$ and 0. Thus φ^* is bijective. It is not homoemorphism because the singletons in $\operatorname{Spec} B$ are closed, but $\{(0)\}$ is not closed in $\operatorname{Spec} A$.

I.22 x

Let $e_i = (..., 0, 1, 0, ...)$, then $e_i e_j = 0$ is in all prime ideals. So if $\mathfrak{p} \subseteq A$ were to not contain e_i, then it must contain all other e_j. Thus $\mathfrak{p}$ is of form $A_1 \times ... \times \mathfrak{p}' \times ... \times A_n$, and can be identified with prime ideal $\mathfrak{p}' \subseteq A_i$. In particular, X is the disjoint union of $\operatorname{Spec} A_i$.

(i)$\Rightarrow$(iii) Suppose there are closed sets $V(\mathfrak{a})\cap V(\mathfrak{b}) = \phi$, $V(\mathfrak{a})\cup V(\mathfrak{b}) = \operatorname{Spec} A$. This means $\mathfrak{a}\cap\mathfrak{b} \subseteq \operatorname{nil}(A)$ and $\mathfrak{a}+\mathfrak{b} = (1)$. Take $a\in\mathfrak{a}, b\in\mathfrak{b}$ such that $(a, b) = (1)$ and $(ab)^n = 0$. Since any maximal ideal containing (a^n, b^n) will contain (a, b), we know $(a^n, b^n) = (1)$. Say $fa^n + gb^n = 1$. Replace a with fa^n and b with gb^n and get $a + b = 1, ab = 0$. This means $a(1-a) = 0$ and $a\in\mathfrak{a}, b\in\mathfrak{b}$ is idempotent, not 1 and not 0 (if one is 0, the other has to be 1).

(iii)$\Rightarrow$(ii) Consider $\varphi : A \to A/(a) \times A/(1-a)$ where a idempotent. Injective because $\ker\varphi = (a)\cap(1-a)$ and if $a|r, 1-a|r$, then $r = r(a + (1-a)) = ra + r(1-a) = r_1(1-a)a + r_2 a(1-a) = 0$. Surjective because $(1-a)s + ar \mapsto (r, s)$ for any $r, s \in A$.

(ii)$\Rightarrow$(i) as shown before.

I.23 x

(i) Since $f(1-f)=0$, a prime contains exactly one of f and $1-f$, so $D(f) \cup D(1-f) = X$ and they are both closed and open.

(ii) $\mathfrak{p}$ does not contain $1-(1-f)(1-g)$ iff it contains $(1-f)(1-g)$, which means it either does not contain g or f. So $D(f) \cup D(g) = D(1-(1-f)(1-g)) = D(1-1+f+g-fg) = D(fg-f-g)$. Since $1 = (-1)^2 = -1$, this is the same as $D(f+g+fg)$.

(iii) See hint.

(iv) Hausdorff because $D(f)$ form a basis. Compact because quasi-compact.

I.24 x

Additive inverse for a is a', multiplicative identity is 1, additive identity is 0. Check distribution of multiplication over addition, etc.

Given a, b suppose $a = ac$ and $b = bc$, then $c(a+b+ab) = ca+cb+cab = a+b+ab$, so $c \geq (a+b+ab)$. Thus $a \vee b = a+b+ab$. Suppose $c = ca, c = cb$, then $cab = c$, so $c \leq ab$. Thus $a \wedge b = ab$. Check complement is given by $a' = 1-a$.

I.25 x

Suffices to show the lattice from a boolean ring correspond to the lattice of clopen sets in its spectrum, i.e. those of form $D(f)$. This is because $D(f) \cup D(g) = D(g+f+fg)$ and $D(f) \cap D(g) = D(fg)$ as shown before, and $D(f) \subseteq D(g)$ implies a prime ideal either contains $1-g$, or it does not contain $1-g$ which implies it is not in $D(g)$, so not in $D(f)$, so it contains f. Therefore $f(1-g)$ is in all the prime ideals. Since there is no nontrivial nilpotent elements, $f(1-g) = 0$ and $f = fg$.

I.26 x

(iii) $U_f = f^{-1}(\mathbb{R} - \{0\})$ is open, and form a basis by Urysohn's lemma (can find f with $f(x) \neq 0$ for $x \notin U$). Now $x \in U_f$ iff $f \notin \mathfrak{m}_x$ iff $\mathfrak{m}_x \in \tilde{U}_f$. So $\mu(U_f) = \tilde{U}_f$. Varify $\tilde{U}_f$ form a basis similar as before.

I.27 affine algebraic varieties

I.28 x

We need an inverse to the above identification. Suppose we have a k-algebra homomorphism $\varphi : A(Y) \to A(X)$. Let $f_i = \varphi(y_i) \in A(X)$. Now define $f : X \to Y$ by $f(x) = (f_1(x), ..., f_m(x))$ is regular. $g \circ f$ is equal to $\varphi(g)$ because $g(f(x)) = g(f_1(x), ..., f_m(x)) = g(\varphi(y_1)(x), ..., \varphi(y_m)(x)) = \varphi(g)(x)$ (since φ is algebra homomorphism). The image lies inside Y because $g \circ f = \varphi(g) = 0$ for all $g \in I(Y)$, so $f(x) \in V(I(Y)) = Y$.

II Chapter 2

II.1 x

If m, n are coprime, we can divide by n in $\mathbb{Z}/m\mathbb{Z}$, so $x \otimes y = (x/n) \otimes (ny) = (x/n) \otimes 0 = 0$ for all $x \otimes y \in \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$.

II.2 x

Tensor $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ with M and get $\mathfrak{a} \otimes M \to A \otimes M \to A/\mathfrak{a} \otimes M \to 0$. Suffices to show the image of $\mathfrak{a} \otimes M \to A \otimes M \cong M$ is $\mathfrak{a}M$. This is because am is given by $a \otimes m$.

II.3 x

Since $M \otimes N = 0$, by 2.15 and Ex2, we have

$$0 = k \otimes_k k \otimes_A M \otimes_A N = (A/\mathfrak{m} \otimes_A M) \otimes_k (A/\mathfrak{m} \otimes_A N) = M/\mathfrak{m}M \otimes N/\mathfrak{m}N$$

Since the dimension of $M/\mathfrak{m}M \otimes N/\mathfrak{m}N$ is $\dim M/\mathfrak{m}M \dim N/\mathfrak{m}N$, one of them must be 0, giving $M = \mathfrak{m}M$ or $N = \mathfrak{m}N$, and by Nakayama $M = 0$ or $N = 0$.

II.4 x

Consider $f : \oplus N_i \to \oplus N_i'$ where each component N_i is mapped to N_i'. Then this is injective iff it is injective componentwise (use independentness of the components of the direct sum). Thus by Prop 2.14(iii) and Prop 2.19 M is flat iff each M_i is.

II.5 x

Consider $A[x] = \oplus x^i A$, note each $x^i A$ is isomorphic to A via $ax^i \mapsto a$, so each component is flat since A is flat. By Ex4, $A[x]$ is flat.

II.6 x

Define $A[x] \times M \to M[x]$ by $f \otimes m \to mf$ is bilinear, so we get a map $A[x] \otimes M \to M[x]$ where $x \otimes m \mapsto mx$. Note it is surjective. From argument in Ex4, we know since on each degree, $x^i \otimes m \mapsto mx^i$ is injective, the map is injective. Hence $A[x] \otimes M \cong M[x]$.

II.7 x

Consider $0 \to \mathfrak{p} \to A \to A/\mathfrak{p} \to 0$ and since $A[x]$ flat, $0 \to \mathfrak{p} \otimes A[x] \to A[x] \to A/\mathfrak{p}[x] \to 0$. By Ex6 $\mathfrak{p} \otimes A[x] = \mathfrak{p}[x]$. Since $A/\mathfrak{p}[x]$ is an integral domain, $\mathfrak{p}[x]$ is a prime ideal.

Take $A = k$ field, 0 is maximal ideal, but $0[x] = 0$ is not maximal in $k[x]$.

II.8 x

(i) Given $f : M_1 \to M_2$ injective, we have $f \times 1 : M_1 \otimes M \to M_2 \otimes M$ injective and therefore $f \otimes 1 \otimes 1 : M_1 \otimes M \otimes N \to M_2 \otimes M \otimes N$. Since $f \otimes 1 \otimes 1 = f \otimes (1 \otimes 1)$, $M \otimes N$ is flat.

(ii) Same proof with Exercise 2.15 as "associativity".

II.9 x

Let $m_1', ..., m_m'$ generators of M', corresponding to $m_1, ..., m_m$ in M. Let $m_1'', ..., m_n''$ generators of M'' corresponding to some pre-image $m_{m+1}, ..., m_{m+n}$ in M. Consider

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^m & \longrightarrow & R^{m+n} & \longrightarrow & R^n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

then by snake lemma (Prop 2.10) we see the middle verticle arrow is surjective, so M is finitely generated.

II.10 x

The image of $M \otimes A/\mathfrak{a} \to N \otimes A/\mathfrak{a}$ by $u \otimes 1$ is simply $\text{im}(u) \otimes A/\mathfrak{a}$. Since surjective, we have $\text{im}(u) \otimes A/\mathfrak{a} = N \otimes A/\mathfrak{a}$. We have exact sequence $0 \to \text{im}(u) \to N \to \text{coker}(u) \to 0$, and therefore

$$N \otimes A/\mathfrak{a} = \text{im}(u) \otimes A/\mathfrak{a} \to N \otimes A/\mathfrak{a} \to \text{coker}(u) \otimes A/\mathfrak{a} \to 0$$

This means $\text{coker}(u) \otimes A/\mathfrak{a} = 0$ and $\text{coker}(u) = \mathfrak{a}\,\text{coker}(u)$ and by Nakayama $\text{coker}(u) = 0$ and u surjective.

II.11 x

$(A/\mathfrak{m}) \otimes A^m = k^m \to (A/\mathfrak{m}) \otimes A^n = k^n$ is isomorphism iff $m = n$ for $\mathfrak{m}$ maximal, k field. If $A^m \to A^n$ surjective, then since tensoring is right exact, the vector space map is surjective, so $m \geq n$.

Suppose $f : A^m \to A^n$ is injective but $m > n$. Consider A^n as submodule $\mathfrak{a}A^m$ where $\mathfrak{a}$ is the ideal with only first n cooridnates non-zero. By Prop 2.4 (Caley Hamilton), $f^n + a_1 f^{n-1} + ... + a_n = 0$. Sicne f injective, $a_n \neq 0$ (otherwise $f^{n-1} + ... + a_{n-1}$ must have zero image). But evaluate at $(0, ..., 0, 1)$ and get $(..., a_n)$ a contradiction.

II.12 x

Pick u_i such that $\varphi(u_i) = e_i$. Then u_i generate $M/\ker\varphi$ since φ is surjective. M is direct sum of u_i and $\ker\varphi$ because e_i are independent. Let v_j be generators for $\ker\varphi$. Let $\mathfrak{m}$ be maximal ideal. Since M finitely generated, we can pick a finite basis of $M/\mathfrak{m}M$ from $\{u_i, v_j\}$, then we get a finite set of generators for M by Nakayama (Prop 2.8) and as a result, the finite subset of v_j must generate $\ker\varphi$.

II.13 x

Define $p : N_B \to N$ by $p(b \otimes y) = by$ (induced by corresponding bilinear map). Then $p \circ g$ is the identity. So g is injective. Let $b \otimes y \in N_B$. Then $b \otimes y - 1 \otimes by \in \ker p$, so $N_B = \text{im}(g) + \ker p$. Suppose $m \in \text{im}(g) \cap \ker p$, then m is of form $1 \otimes y$, and $p(m) = 0$ means $y = 0$, but then $m = 0$. Thus this sum is direct.

II.14 direct limits

$\varinjlim$

II.15 x

Given an element of form $x = \sum_{j=1}^n x_{i_j}$ where $x_{i_j} \in M_{i_j}$, pick k such that $i_j \leq k$ for all j, then x has the same image as $x' = \sum_{j=1}^n \mu_{i_j,k}(x_{i_j}) \in M_k$, and $\mu(x) = \mu_k(x')$ is of the required form.

Suppose $\mu_i(x_i) = 0$, then

$$x_i = \sum_{j=1}^n x_{i_j} - \mu_{i_j,k_j}(x_{i_j}) \in M_i$$

Since this lies in M_i, all the M_k components are 0 for $k \neq i$, so we can find K such that $K \geq k_j, i_j$ and apply $\mu_{k,K}$ to the M_k component, and does not affect x_i, so

$$x_i = \sum_j x_i^{(j)} - \mu_{i,K}(x_i^{(j)}) + \sum_{j=1}^n \mu_{i_j,K}(x_{i_j}) - \mu_{i_j,K}(x_{i_j}) = \sum x_i^{(j)} - \mu_{i,K}(x_i^{(j)})$$

where $\sum x_i^{(j)}$ is the M_i component of x_i, and as a result $0 = \sum_j \mu_{i,K}(x_i^{(j)}) = \mu_{i,K}(x_i)$ as required.

II.16 x

Define $\alpha : M \to N$ by $\mu_i(x_i) \mapsto \alpha_i(x_i)$. If $\mu_i(x_i) = 0$, then $\mu_{ij}(x_i) = 0$ for some j, and $\alpha_i(x_i) = \alpha_j\mu_{ij}(x_i) = \alpha_j(0) = 0$. So this is well-defined because if $\mu_i(x_i) = \mu_j(x_j)$, then $\mu_{ik}(x_i) = \mu_{ik}(x_j)$ as elements in M, so $\mu_k(\mu_{ik}(x_i) - \mu_{jk}(x_j)) = 0$ and $\alpha_i(x_i) = \alpha_k\mu_{ik}(x_i) = \alpha_k\mu_{jk}(x_j) = \alpha_j(x_j)$.

Uniqueness because it is determined by how it acts on M_i, which has to be α_i.

II.17 x

Consider $\alpha_i : M_i \to \cup M_i$ by inclusion, and get $\alpha : M \to \cup M_i$. Surjective because $x_i \in M_i$ is sent from $\mu_i(x_i)$. If $\mu_i(x_i)$ is sent to $\alpha_i(x_i) = 0$, then $x_i = 0$ because α_i are injective. Thus α is an isomorphism. Hence any A-module is the direct limit of its finitely generated submodules because every element is contaiend in the submodule generated by itself.

II.18 x

Define $\varphi(\mu_i(x_i)) = \nu_i(\varphi_i(x_i))$. Check similar to Ex16.

II.19 x

Suppose $\varphi(\nu_i(x_i)) = \rho_i(\varphi_i(x_i)) = 0$, then there is some j, $\rho_{ij}(\varphi_i(x_i)) = 0$ and so $\varphi_j(\nu_{ij}(x_i)) = 0$. So $\nu_{ij}(x_i)$ is in the image of $\psi : M_j \to N_i$, say $\psi_i(x_i')$. Now $\nu_j(\nu_{ij}(x_i)) = \nu_j(\psi_i(x_i')) = \psi(\mu_j(x_j))$, so $\nu_i(x_i) = \nu_j(\nu_{ij}(x_i))$ is in the image of ψ.

Conversely, $\varphi\psi\mu_i x_i = \varphi\nu_i\psi_i x_i = \nu_i\varphi_i\psi_i x_i = \nu_i(0) = 0$. Hence sequence is exact.

II.20 tensor product commutes with direct limits

Define $g : M_i \times N \to M_i \otimes N$ by $(x_i, y) \mapsto x_i \otimes y$ bilinear. Pass to limit and get $g : M \times N \to P$ by identifying $\mu_i \times 1(x_i, n)$ with $(\mu_i(x_i), n)$. Now $(\mu_i(x_i), n)$ is sent to $\nu_i(x_i \otimes n)$, so it is bilinear, and g extends to $\varphi : M \otimes N \to P$ and $\mu_i(x_i) \otimes n \mapsto \nu_i(x_i \otimes n)$. Since ψ is by Ex16 $\nu_i(x_i \otimes n) \mapsto \mu_i \otimes 1(x_i \otimes n) = \mu_i(x_i) \otimes n$, ψ and φ are inverses, thus isomorphism.

II.21 x

Define multiplication maps $A \times A \to A$ by $\alpha_i(a_i)\alpha_j(b_j) = \alpha_k(\alpha_{ik}(a_i)\alpha_{jk}(b_j))$ for $k \geq i, j$. Check well defined using Ex15. (if $\alpha_i(x_i) = \alpha_j(x_j)$, then for some k, $\alpha_{ik}(x_i) = \alpha_{jk}(x_j)$)

Now mappings $A_i \to A$ where $a_i \mapsto \alpha_i(a_i)$ are ring homomorphisms because $a_i a_j \mapsto \alpha_i(a_i a_j) = \alpha_i(a_i)\alpha_i(a_j)$.

If $A = 0$, then take $\mu_i(1) = 0$ for some i, so $\mu_{ij}(1) = 0$ for some j, which means $1 = 0$ in A_j and it is the 0 ring.

II.22 x

Consider $\varphi_i : \mathrm{nil}(A_i) \to A_i$ inclusion. Since the image of nilradical is inside nilradical, we can restrict α_{ij} to get maps on directed system $\mathrm{nil}(A_i)$. Now $\varphi_i \circ \alpha_{ij} = \alpha_{ij} \circ \varphi_j$ because nilradical of image is the image of nilradical (surjective morphisms perserve prime ideals). Thus we get an embedding $\varinjlim \mathrm{nil}(A_i) \to \varinjlim A_i$.

Suppose $\alpha_i(x_i)^n = \alpha_i(x_i^n) = 0$, then there is some j such that $\alpha_{ij}(x_i^n) = \alpha_{ij}(x_i)^n = 0$, so $\alpha_{ij}(x_i) \in \mathrm{nil}\, A_j$. Now $\alpha_i(x_i) = \alpha_j(\alpha_{ij}(x_i))$ is in $\varinjlim \mathrm{nil}(A_i)$. Conversely, given $\alpha_i(x_i)$ where $x_i \in \mathrm{nil}(A_i)$, we have $\alpha_i(x_i)^n = \alpha_i(x_i^n) = \alpha_i(0) = 0$.

If each A_i integral domain, then $\alpha_i(x)\alpha_j(y) = 0$ implies $\alpha_{kK}(\alpha_{ik}(x_i)\alpha_{jk}(y_j)) = 0 = \alpha_{iK}(x)\alpha_{jK}(y)$ in some A_K. Then $\alpha_{iK}(x) = 0$ so $\alpha_i(x) = 0$ or $\alpha_{jK}(y) = 0$ so $\alpha_j(y) = 0$.

II.23 x

Similar to Ex21. Since all the maps perserve structure on A, those ring homomorphisms can be viewed as A-algebra homomorphisms.

II.24 flatness and tor

(i)⇒(ii) Take free resolution of N and tensor with M, since flat, resulting sequence is exact and homology is 0 at $n > 0$.

(iii)⇒(i) Given $0 \to N' \to N \to N'' \to 0$, consdier long exact sequence

$$0 = \mathrm{Tor}_1^A(M, N'') \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$$

so M is flat.

II.25 x

Consider long exact sequence

$$0 = \mathrm{Tor}_2^A(M, N'') \to \mathrm{Tor}_1^A(M, N') \to \mathrm{Tor}_1^A(M, N) \to \mathrm{Tor}_1^A(M, N'') = 0$$

So $\mathrm{Tor}_1^A(M, N') = 0$ iff $\mathrm{Tor}_1^A(M, N) = 0$ and by Ex24 N is flat iff N' is.

II.26 x

By Prop 2.19(iv) and proof of Ex24, N is flat if $\mathrm{Tor}_1(M, N) = 0$ for all finitely generated M. Let $x_1, ..., x_n$ generate M, take M_i generated by $x_1, ..., x_i$. Then we have $0 \to M_{i-1} \to M_i \to M_i/M_{i-1}$. From proof of Ex25, to show $\mathrm{Tor}_1(M, N) = 0$, it suffices to show $\mathrm{Tor}_1(M_n/M_{n-1}, N) = 0$ and $\mathrm{Tor}_1(M_{n-1}, N) = 0$, and to show $\mathrm{Tor}_1(M_{n-1}, N) = 0$, it suffices to show $\mathrm{Tor}_1(M_{n-1}/M_{n-2}, N) =)$ and $\mathrm{Tor}_1(M_{n-2}, N) = 0$. Now since M_1 is cyclic and each M_i/M_{i-1} is cyclic (generated by image of x_i), an induction tells us we only need to show $\mathrm{Tor}_1(M, N) = 0$ for all M cyclic, say generated by x with annihilator $\mathfrak{a}$, so $M = A/\mathfrak{a}$. Thus N is flat iff $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$ iff $0 \to \mathfrak{a} \otimes N \to A \otimes N \to A/\mathfrak{a} \otimes N \to 0$ is exact. A similar argument as proof of Prop 2.19 shows we can reduce this to $\mathfrak{a}$ finitely generated.

II.27 absolutely flat

(i)⇒(ii) Let $x \in A$, then since $A/(x)$ is flat, we have

$$0 \to (x) \otimes A/(x) \to A \otimes A/(x) \to A/(x) \otimes A/(x) \to 0$$

but since the injection $(x) \otimes A/(x) \to A \otimes A/(x)$ sends $x \otimes a$ to $x \otimes a = 1 \otimes xa = 0$, we have $(x) \otimes A/(x) = 0$. By Ex2, $(x) \otimes A/(x) = (x)/(x^2)$, so $(x) = (x^2)$.

(ii)⇒(iii) Since $(x) = (x^2)$, $x = ax^2$ for some a, so $ax = a^2x^2$ is idempotent and $(ax) = (x)$. Given ideal (e, f) where e, f idempotent, we have $(e, f) = (e + f - ef)$ because $e(e + f - ef) = e, f(e + f - ef) = f$. Thus every finitely generated ideal is principal with idempotent generator, and $A = (e) \oplus (1 - e)$ so we are done.

(iii)⇒(i) By Ex26, suffices to show $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$ for all finitely generated $\mathfrak{a}$ for all N. That is to show $A/\mathfrak{a}$ is flat. By Ex4, $A/\mathfrak{a}$ is flat because $A = \mathfrak{a} \oplus \mathfrak{b} = A/\mathfrak{a} \oplus A/\mathfrak{b}$ and A is flat.

II.28 x

Boolean ring absolutely flat by Ex27(ii). Ring from Ex1.7 is flat because $x^n = x$ implies $x \in (x^2)$. Image of absolutely flat ring is flat because $f(x) = f(ax^2) = f(a)f(x)^2$. If a local ring is absolutely flat, then let x be a non-unit, and $x = ax^2$, then ax is idempotent, so it must be 0, which means $x = ax^2 = 0$, and the local ring has no non-units, so a field.

Given $x = ax^2$, we have $x(1 - ax) = 0$, and since $1 - ax \neq 0$ for x non-unit, x must be a zero-divisor.

III Chapter 3

III.1 x

Suppose $S^{-1}M = 0$ and M generated by $m_1, ..., m_n$, then there is some s_i such that $s_i(m_i - 0) = 0$. Now take $s = \prod s_i$ and $sM = 0$.

III.2 x

Suffices to show that $\mathfrak{a}$ is contained in all maximal ideals that do not meet $S = 1 + \mathfrak{a}$. Suppose $\mathfrak{m}$ is such maximal ideal and it does not contain some $a \in \mathfrak{a}$. Then $\mathfrak{m} + (a) = (1)$ so $m = 1 - ra \in S$ for some $r \in A$, a contradiction.

Suppose $M = \mathfrak{a}M$, then $S^{-1}M = S^{-1}\mathfrak{a}S^{-1}M$ and by Nakayama, $S^{-1}M = 0$, so there is some $1+a \in 1+\mathfrak{a}$ such that $(1+a)M = 0$ by Ex1. isn't this argument circular because Nak uses Cor 2.5?

III.3 x

Define $A \to U^{-1}(S^{-1}A)$ by $a \mapsto (a/1)/1$. If $st \in ST$, then since s is a unit in $S^{-1}A$ and $t/1$ is a unit in $U^{-1}(S^{-1}A)$, we have $st/1$ is a unit. Hence this extends to $(ST)^{-1}A \to U^{-1}(S^{-1}A)$. Now given $(a/s)/t$, it is the image $a/(st) \mapsto (a/1)/(st/1) = (a/s)/t$ by definition, so surjective. Suppose $a/(st) \mapsto 0$, then $(a/1)/1 = 0$ so there is some $t'/1 \in U$ where $t' \in T$ such that $(t'/1)(a/1) = t'a/1 = 0 \in S^{-1}A$ and then there is some s' so that $t's'a = 0$. Therefore $a/(st) = 0$ in $(ST)^{-1}A$.

III.4 x

Define $S^{-1}B \to T^{-1}B$ by

$$b/s \mapsto \frac{b}{f(s)}$$

which is well defined because if $b/s = c/t$, say there is some $u \in S$, $u(sc - tb) = 0$, then by definition $f(u)(f(s)c - f(t)b) = 0$ and $b/f(s) = c/f(t)$ in $T^{-1}B$. Surjective because $\frac{1}{s} \otimes b \mapsto b/f(s)$. Suppose $b/s \mapsto 0$, then there is some $f(t)$ such that $f(t)b = 0$, which means $t \cdot b = 0$ and so $b/s = 0$ in $S^{-1}B$. Hence injective.

III.5 x

If $A_{\mathfrak{p}}$ has no nilpotent, then by Cor 3.12, $\text{nil}(A)_{\mathfrak{p}} = \text{nil}(A_{\mathfrak{p}}) = 0$ for all $\mathfrak{p}$, and by Prop 3.8, $\text{nil}(A) = 0$.

Consider a boolean ring like $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since local rings have no non-trivial idempotent, its localization must be $\{0, 1\}$ at all prime ideals, so integral domain. But the (non-trivial) boolean ring is not an integral domain.

III.6 x

Suppose S is maximal, then if $A - S$ is not an ideal, say $t \in A - S$ such that $at \in S$. Then we can add $tS, t^2S, ...$ into S because if $t^n s = 0$ for some $s \in S$, then $(at)^n s = 0 \in S$, a contradiction. Now since S is multiplicatively closed, $A - S$ is a prime ideal, so maximal S correspond to minimal primes, which exist.

III.7 saturation

(i) Suppose $A - S$ union of prime ideals and $x \notin S$ or $y \notin S$, then $xy \notin S$. Suppose S is saturated. Then $1 \in S$ because $1s = s \in S$ for any $s \in S$. So any unit is in S. Assume S does not contain 0 because otherwise it is just A. Let $x \in A - S$. If S meets all prime ideals containing x, then $S^{-1}A$ must have no prime ideals, so it is the zero ring and therefore S contains 0 (by Ex1), a contraidiction. Thus there is some prime ideal containing x, take the union of those prime ideals.

(ii) The complement of union of primes not meeting S gives $\overline{S}$. To show uniqueness and minimality, Observe that complement of any saturated set containing S would be a union of prime ideals, and they necessarily not intersect S.

Let $S = 1 + \mathfrak{a}$. For each $x \in A$, if the image of x in $A/\mathfrak{a}$ is not a unit, then it is in some maximal ideal $\mathfrak{m}'$ corresponding to some maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$, so $\mathfrak{m}$ contains x and since it contains $\mathfrak{a}$, it does not intersect S. Thus $x \notin \overline{S}$. Conversely, if the image of x is a unit, then $xy + a = 1$ for some $a \in \mathfrak{a}$ and $x \in \overline{S}$. Hence $\overline{S}$ is elements whose image in $A/\mathfrak{a}$ are units.

III.8 x

(i)⇒(ii) φ is bijective implies for each t, $\varphi(a/s) = 1/t$ in $T^{-1}A$. Now this means a/s is a unit in $T^{-1}A$ and therefore a unit in $S^{-1}A$.

(ii)⇒(iii) Suppose $t/1$ is a unit in $S^{-1}A$, say with inverse a/s, then $at = s \in S$ in $S^{-1}A$.

(iii)⇒(iv) Follows from definition.

(iv)⇒(v) By Ex7, the complement of T contains all the prime ideals which do not meet S. The contrapositive says if a prime ideal meets T, then it must meet S.

(v)⇒(i) Suppose φ is not bijective, then there is some a/t not in the image, so $1/t$ is not in the image. Therefore t is not a unit in $S^{-1}A$. Thus there is a prime ideal of $S^{-1}A$ which contains t, and corresponds to a prime ideal of A which contains t and does not meet S, a contradiction.

III.9 x

By Ex4 suffices to show all maximal multiplicatively closed subsets contain S_0. Suppose not, say S maximal but does not contain $t \in S_0$. Then we can add in $tS, t^2S, ...$ and get a bigger multiplicatively closed set (that does not contain 0), because if $t^n s = 0$, then t would have to be a 0 divisor, which it is not.

(i) $a/1$ is sent to 0 if $xa = 0$ for some $x \notin S$. Thus x is a 0 divisor, and we see $A \to S_0^{-1}A$ is injective, and anything larger than S_0 contains some zero-divisor a with $xa = 0$, therefore inducing a homomorphism not injective.

(ii) Say we have a/s. If a is in S_0, then it is a unit with inverser s/a. If $a \notin S_0$, then it is a zero-divisor with $xa = 0$ and then $xa/s = 0$. So everything in $S_0^{-1}A$ is a unit or a zero-divisor.

(iii) If every non-unit is a zero-divisor, then S_0 is the set of all units, and therefore $A \to S_0^{-1}A$ is bijective by Ex8 because S_0 is the saturation of $\{1\}$.

III.10 x

(i) By II.27 suffices to show every principal ideal is idempotent. We know it is true for A since it is absolutely flat, so $x = ax^2$ for some a. Now $x/s = as(x/s)^2$, so $(x/s) = (x/s)^2$ as ideals and we are done.

(ii) ⇒ If $x/1$ is a non-unit, then $x = ax^2$ for some a and $x(1 - ax) = 0$. Since $A_\mathfrak{m}$ is local, both x and $1 - ax$ are non-units, and contained in $\mathfrak{m}A_\mathfrak{m}$ but then $1 \in \mathfrak{m}$, a contradiction.

⇐ First note that fields are absolutely flat. By Prop 2.19, $M_\mathfrak{m}$ is flat over $A_\mathfrak{m}$ for all $\mathfrak{m}$, so M is flat over A for all M and A is absolutely flat.

III.11 x

(i)⇒(ii) Suppose $A/\operatorname{nil}(A)$ is absolutely flat, then for any maximal ideal, $(A/\operatorname{nil}(A))_\mathfrak{m}$ is a field, so $A_\mathfrak{m}/\operatorname{nil}(A)_\mathfrak{m} = A_\mathfrak{m}/\operatorname{nil}(A_\mathfrak{m})$ is a field, so $\operatorname{nil}(A_\mathfrak{m})$ is maximal. This means there is no prime ideal strictly contained inside $\mathfrak{m}$, so every prime ideal is maximal.

(ii)⇒(iii) Every point $\mathfrak{p}$ has closure $V(\mathfrak{p}) = \{\mathfrak{p}\}$.

(iii)⇒(iv) Given two prime ideals, one can not contain the other because each point is closed. So pick $f \in \mathfrak{p} - \mathfrak{q}$ and $g \in \mathfrak{q} - \mathfrak{p}$, and we have $\mathfrak{q} \in D(f), \mathfrak{p} \in D(g)$. Now since $A_\mathfrak{p}$ has only one prime ideal, f is nilpotent in it, and $f^n x = 0$ for some $x \notin \mathfrak{p}$. Since fx is nilpotent, $fx \in \mathfrak{q}$, so $x \in \mathfrak{q}$. Replace g with x and we see fg is nilpotent in A. Thus we can assume fg nilpotent and $D(f) \cap D(g) = D(fg) = \phi$.

(iv)⇒(i) $B = A/\operatorname{nil}(A)$ is a reduced ring where all primes are maximal (Hausdorff implies T_1). $B_\mathfrak{m}$ is a local ring with only one prime ideal and no nilpotents, which means its only maximal ideal is 0, so it is a field, and therefore B is absolutely flat.

Suppose these conditions are satisfied. Recall $\operatorname{Spec}(A)$ is quasi-compact, so it is compact because Hausdorff. Given a set with more than two points, say contains $\mathfrak{p}, \mathfrak{q}$, then pick $f \in \mathfrak{p} - \mathfrak{q}$. Now $D(f)$ is quasi-compact. quasi-compact subset of Hausdorff space is closed: Let $K \subseteq X$ quasi-compact, $x \notin K$. Pick open sets U_y, V_y for each $y \in K$ that separate x and y. V_y covers K, so there is a finite subcover V_{y_i}. Now $\cap U_{y_i}$ is a neighbourhood of x outside of K. Thus K closed. So $D(f)$ is both open and closed, making the set A disconnected.

III.12 x

(i) Suppose $ax = 0$ for some $x \in M/T(M)$, then $ax \in T(M)$ so $bax = 0$ for some $b \neq 0 \in A$, but this means $ba \in \operatorname{Ann}(x)$, so $x = 0$ or $ba = 0$. Since A integral domain, $a = 0$. Thus $\operatorname{Ann}(x) = 0$ for non-zero elements of $M/T(M)$, so torsion-free.

(ii) If $ax = 0$ then $af(x) = f(ax) = 0$. So $f(T(M)) \subseteq T(N)$.

(iii) If M' injects into M, then $T(M')$ injects into $T(M)$ by the same embedding. Suppose $g(x) = 0$ for some $x \in T(M)$, then $x = f(y)$ for some $y \in M'$. Say $ax = 0$, then $f(ay) = 0$ and since f injection, $ay = 0$, so $y \in T(M')$. Hence sequence is exact.

(iv) Suppose $x \in T(M)$ with $ax = 0$ for some $a \neq 0$, then $1 \otimes x = 1/a \otimes ax = 1/a \otimes 0 = 0$.

Conversely, consider K as direct limit of submodules $A\xi$, $\xi \in K$ by II.17 (since $A(a/s)+A(b/t) \subseteq A(1/st)$). So $K \otimes A = \varinjlim(A\xi \otimes K)$ and by II.15, if $1 \otimes x = 0$, then $\mu_0(1 \otimes x) = 0$ which means $\mu_{0i}(1 \otimes x) = 0$, i.e. $1 \otimes x$ is 0 in some $A(1/a) \otimes M$. Since A is integral domain, $A(1/a) \cong A$, so $A(1/a) \otimes M \cong A \otimes M \cong M$ by $b/a \otimes m \mapsto bm$, so $1 \otimes m$ is sent to am which is 0. Hence $am = 0$ and $m \in T(M)$.

III.13 x

Suppose $(a/s)(m/t) = 0$ for some $a/s \neq 0$, then this means $xam = 0$ for some $x \in S$, so $m \in T(M)$ and $m/t \in S^{-1}TM$. Conversely suppose $m/t \in S^{-1}TM$, then $am/t = 0$ so $m/t \in TS^{-1}M$.

This means $T(M)_\mathfrak{p} = T(M_\mathfrak{p})$ for all $\mathfrak{p} \subseteq A$, and by Prop 3.8, we see torsion-free is a local property.

III.14 x

We need to show $M/\mathfrak{a}M = 0$. $M_\mathfrak{m} = 0$ implies $sM = 0$ for some $s \notin M$ by Ex1. So there is some $s \notin \mathfrak{m}$ in $A/\mathfrak{a}$ such that $sM/\mathfrak{a}M = 0$. Since this is true for all maximal ideals containing $\mathfrak{a}$, $M/\mathfrak{a}M = 0$ and $M = \mathfrak{a}M$.

III.15 x

Consider $\varphi(e_i) = x_i$. Assume A local, $k = A/\mathfrak{m}$. By II.24 since $Tor(A/\mathfrak{m}, F) = 0$, $0 \to k \otimes \ker\varphi \to k \otimes F \to k \otimes F \to 0$ is exact. Since $K \otimes F = k^n$ is vector space, $k \otimes \ker\varphi = \ker\varphi/\mathfrak{m}\ker\varphi = 0$. Since $\ker\varphi$ finitely generated II.12, $\ker\varphi = 0$ by Nakayama.

If we were to have a set of generators with less than n elements, then we can add in 0 and get a basis, a contradiction.

III.16 x

(i)⇒(ii) Let $\varphi : A \to B$, $\mathfrak{p} \subseteq A$, then $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}^e)$. So $\operatorname{Spec}\varphi$ is surjective.

(ii)⇒(iii) Let $\mathfrak{m} \subseteq A$, say $\mathfrak{m} = \varphi^{-1}(\mathfrak{m}')$. Then $\mathfrak{m}^e \subseteq \mathfrak{m}' \neq (1)$.

(iii)⇒(iv) Let $x \in M$ non-zero and $M' = Ax$, then since B flat, suffices to show M'_B is non-zero (then injects into M_B). Since $M' = A/\mathfrak{a}$ for some $\mathfrak{a}$, $M'_B = B \otimes_A A/\mathfrak{a} = B/\mathfrak{a}B = B/\mathfrak{a}^e$. Since $\mathfrak{a}^e \neq (1)$ by assumption, $M'_B \neq 0$.

(iv)⇒(v) Let M' be kernel, then $0 \to M' \to M \to M_B$ where M_B viewed as A-module. Since B flat, $0 \to M'_B \to M_B \to (M_B)_B$. Since $M_B \to (M_B)_B$ is injective by II.13, $M'_B = 0$. So $M' = 0$ and the original map is injective.

(v)⇒(i) Take $M = A/\mathfrak{a}$, then $A/\mathfrak{a} \to B \otimes A/\mathfrak{a} = B/\mathfrak{a}^e$ by $a \mapsto \varphi(a)$ injective. This means no element outside of $\mathfrak{a}$ is mapped into $\mathfrak{a}^e$ by φ, so $\varphi^{-1}(\mathfrak{a}^e) = \mathfrak{a}$.

III.17 x

Let $\varphi : M \to M'$ injective, then since $g \circ f$ is flat, $M \otimes_A C \to M' \otimes_A C$ injective. Let N be the kernel of $M \otimes B \to M' \otimes B$. Since g is flat, we have $0 \to N_C \to M \otimes_A B \otimes_B C \to M' \otimes_A B \otimes_B C$, which is of course the same map (use canonical isomorphism) as $M \otimes_A C \to M' \otimes_A C$, so $N_C = 0$. Now since g is faithful, $N = 0$ by Ex16(iv), so B is flat.

III.18 x

See hint. $B_\mathfrak{q}$ is localization of $B_\mathfrak{p}$ at $\mathfrak{q}B_\mathfrak{p}$ (prime by Prop 3.11(iv)), is flat over $B_\mathfrak{p}$ by Cor 3.6. Note the image of the maximal ideal $\mathfrak{p}A_\mathfrak{p}$ is inside $\mathfrak{q}B_\mathfrak{q}$, $(\mathfrak{p}A_\mathfrak{p})^e \neq (1)$, so $B_\mathfrak{q}$ satisfies Ex16(iii).

III.19 x

(i) By Prop 3.8.

(ii) $(A/\mathfrak{a})_\mathfrak{p} = A_\mathfrak{p}/\mathfrak{a}A_\mathfrak{p}$ is 0 iff $\mathfrak{a} = (1)$ by Nakayama, iff $\mathfrak{a}$ meets $A - \mathfrak{p}$ (since $A - \mathfrak{p}$ saturated, $a/1$ is not a unit for $a \in \mathfrak{p}$). So its support contains ideals inside $\mathfrak{p}$, namely $V(\mathfrak{a})$.

(iii) Localization is exact (3.3), so $0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$ and $M_\mathfrak{p}$ is non-zero iff $M'\mathfrak{p} \neq 0$ or $M''_\mathfrak{p} \neq 0$. So $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$.

(iv) Since $M_\mathfrak{p} = \sum (M_i)_\mathfrak{p}$.

(v) By III.1.

(vi) Since $(M \otimes N)_\mathfrak{p} \cong M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p}$ and use II.3.

(vii) $M/\mathfrak{a}M = A/\mathfrak{a} \otimes M$, so $\mathrm{Supp}(M/\mathfrak{a}M) = \mathrm{Supp}(M) \cap \mathrm{Supp}(A/\mathfrak{a}) = V(\mathrm{Ann}\, M) \cap V(\mathfrak{a}) = V(\mathrm{Ann}\, M + \mathfrak{a})$.

(viii) $(B \otimes_A M)_\mathfrak{q} = B_\mathfrak{q} \otimes_B B \otimes_A M = B_\mathfrak{q} \otimes_A M = B_q \otimes_{A_\mathfrak{p}} A_\mathfrak{p} \otimes_A M = B_\mathfrak{q} \otimes_{A_\mathfrak{p}} M_\mathfrak{p}$. So $(B \otimes_A M)_\mathfrak{q} = 0$ iff $B_\mathfrak{q} = 0$ or $M_\mathfrak{p} = 0$ iff $M_\mathfrak{p} = 0$ where $\mathfrak{p} = f^*\mathfrak{q}$ iff $f^*(\mathfrak{q})$ contains $\mathrm{Ann}(M)$. So $\mathrm{Supp}(B \otimes_A M) = f^{*-1}(V(\mathrm{Ann}\, M)) = f^{*-1}(\mathrm{Supp}\, M)$.

III.20 x

(i) By definition of f^*.

(ii) Suppose $\mathfrak{p} \neq \mathfrak{q}$, say $\mathfrak{p} = (\mathfrak{p}')^e, \mathfrak{q} = (\mathfrak{q}')^e$, then $\mathfrak{p}^c = (\mathfrak{p}')^{ec} \supseteq \mathfrak{p}', \mathfrak{q}^c = (\mathfrak{q}')^{ec} \supseteq \mathfrak{q}'$. This means $(\mathfrak{p}')^e \subseteq \mathfrak{p}^{ce} \subseteq \mathfrak{p}$, so $\mathfrak{p}^{ce} = \mathfrak{p}$ and similarly $\mathfrak{q}^{ce} = \mathfrak{q}$. Thus $f^*(\mathfrak{p}) = \mathfrak{p}^c \neq \mathfrak{q}^c = f^*(\mathfrak{q})$.

Consider $k \to k[x]/(x^2)$, the induced map is bijective as each only has one prime ideal, but (x) is not an extended ideal (from (0)).

III.21 fiber

(i) Follows from the correspondence between prime ideals of $S^{-1}A$ and prime ideals of A not meeting S.

(ii) A prime of $f(S)^{-1}B$ corresponds to a prime of B not intersecting $f(S)$, which is sent to a prime ideal of A that does not meet S and corresponds to a prime of $S^{-1}A$. Conversely, if a prime ideal of B is sent to $S^{-1}X$, then it can not intersect $f(S)$ and correspond to an element of $S^{-1}Y$, so $S^{-1}Y = f^{*-1}(S^{-1}X)$.

(iii) Follows from correspondence between primes containing $\mathfrak{a}$ and primes of $A/\mathfrak{a}$ (and for B and $B/\mathfrak{b}$).

(iv) We are interested in the inverse image of $\mathfrak{p}$. Since $\mathfrak{p}$ is contained in $S^{-1}X$, by (ii), we can restrict our map to $S^{-1}f^* : \mathrm{Spec}(B_\mathfrak{p}) \to \mathrm{Spec}(A_\mathfrak{p})$. By (iii), since $\mathfrak{p}$ is in the image of $\mathrm{Spec}(A_\mathfrak{p}/\mathfrak{p})$, we can restirct our map to $\overline{S^{-1}f}^* : \mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) \to \mathrm{Spec}(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p})$. Now since $\mathrm{Spec}(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p})$ has one point $\mathfrak{p}$, we know the inverse image is homeomorphic to $\mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) = \mathrm{Spec}(k(p) \otimes_A B)$.

III.22 x

The canonical image of $\mathrm{Spec}(A_{\mathfrak{p}})$ is the set of primes contained inside $\mathfrak{p}$. If $D(f)$ is a neighbourhood around $\mathfrak{p}$, and $\mathfrak{q} \subseteq \mathfrak{p}$, then we know $f \notin \mathfrak{q}$ so $\mathfrak{q} \in D(f)$. Conversely, if $\mathfrak{q}$ is a prime that does not contain anything that $\mathfrak{p}$ does not contain, then $\mathfrak{q} \subseteq \mathfrak{p}$. Thus $\mathrm{Spec}(A_{\mathfrak{p}})$ corresponds to the intersection of all open neighbourhoods of $\mathfrak{p}$.

III.23 x

(i) Suppose $D(f) = D(g)$. Then $D(fg) = D(f) \cap D(g) = D(f) = D(g)$, so we need to show $A_f \cong A_{fg}$, observe that $(A_f)_g = A_{fg}$ by $a/(fg)^n \mapsto (a/f^n)/g^n$. Thus we have reduced the problem to showing that if the image of $D(f) = \mathrm{Spec}\, A_f$ is $\mathrm{Spec}\, A$, then $A_f \cong A$. Note that this means f is not in any prime ideal, so f is a unit, but we know localizing at a unit does nothing to the ring, so we are done.

(ii) If $U' \subseteq U$, then $V((f)) \subseteq V((g))$, and any prime ideal that contains f must contain g, the image of g in $A/(f)$ is in its nilradical, and $g^n = uf$ for some u, n. To see ρ is not dependent on f, g, we use the identification from part (i) (check details).

(iii) Take $g = f$, and use the fact it is independent of the choice of g.

(iv) If $g^n = uf$, $h^m = vg$, then $h^{mn} = v^n uf$ and $a/f^\ell \mapsto a(v^n u)^\ell / h^{\ell mn}$ factors through $a/f^\ell \mapsto au^\ell / g^{\ell n} \mapsto au^\ell v^{\ell n}/h^{\ell nm}$. So the diagram commutes.

(v) If $\mathfrak{p} \subseteq D(f)$, then $f \notin \mathfrak{p}$ and we can define $A_f \mapsto A_{\mathfrak{p}}$ by $a/f \mapsto a/f$. Extend to $\varinjlim A(U) \to A_{\mathfrak{p}}$ by II.16. We have surjection because elements of $A_{\mathfrak{p}}$ are of form a/f where $f \notin \mathfrak{p}$. Suppose $a/f = 0$ in $A_{\mathfrak{p}}$, then $ag = 0$ for some $g \notin \mathfrak{p}$, and then $ag/fg = 0$ in A_{fg}, so the image of a/f is 0 in the direct limit. Hence the map is a isomorphism.

III.24 x

Since $\mathrm{Spec}\, A$ is quasi-compact for all A, we can prove this for a finite subcover, then the glueing works for any cover because when we restrict to any open set, the finite cover of that open set $\mathrm{Spec}\, A_f$ is the required element by uniqueness. Say $s_i = a_i / f_i^{n_i}$, then since restirctions are the same, we have $a_i f_j^{n_i}/(f_i f_j)^{n_i} = a_j f_i^{n_j}/(f_i f_j)^{n_j}$ in $A_{f_i f_j}$ and

$$(f_i f_j)^n (a_i f_j^{n_j+n_i} f_i^{n_i} - a_i f_i^{n_i+n_j} f_j^{n_j}) = 0$$

$$(f_i f_j)^N (a_i f_j^{n_j} - a_i f_i^{n_i}) = 0$$

Take N large enough to work all finitely many pairs of i, j, replace f_i with $f_i^{N+n_i}$ and a_i with $a_i f_i^N$, then $s_i = a_i/f_i$ and

$$a_i f_j = a_j f_i$$

Take $\sum b_i f_i = 1$ to form our subcover. Let $s = \sum b_i a_i$, then

$$f_j s = \sum b_i a_i f_j = \sum b_i f_i a_j = a_j$$

Thus $s = a_j/f_j$ in U_j and is the required element in A. It is unique because if $s = t$ in all A_{f_i}, then $f_i^N(s-t) = 0$ for some N big enough so it holds for all i. Since $(f_1, ..., f_n) = (1)$, we know $(f_1^n, ..., f_n^N) = (1)$, so $s = t$ in A.

III.25 x

See hint.

III.26 x

See hint. Basically $\mathfrak{p} \in f^*(X)$ iff $f^{*-1}(\mathfrak{p}) \neq \varphi$ iff $\mathrm{Spec}(B \otimes_A k(\mathfrak{p})) \neq \phi$ iff $B \otimes_A k(\mathfrak{p}) \neq 0$.

III.27 x

(i) Let B_J denote finite tensor $\otimes_{\alpha \in J} B_\alpha$ for finite subset J. By II.23, the tensor product of B_α is the direct limit of B_J, so by Ex25 and Ex26,

$$f^*(\operatorname{Spec} B) = \bigcap f_J^*(\operatorname{Spec} B_J) = \bigcap_J \bigcap_{\alpha \in J} f_\alpha^*(\operatorname{Spec} B_\alpha) = \bigcap_\alpha f_\alpha^*(\operatorname{Spec} B_\alpha)$$

(ii) $\mathfrak{p} \in f^*(\operatorname{Spec} B)$ iff $B \otimes_A k(\mathfrak{p}) = \oplus(B_\alpha \otimes k(\mathfrak{p})) \neq 0$ (since finite direct product is direct sum, which distributes over tensor) iff one of $B_\alpha \otimes k(\mathfrak{p}) \neq 0$ iff $\mathfrak{p} \in f^*(\operatorname{Spec} B_\alpha)$ for some α.

(iii) It contains the Zariski topology because $V(\mathfrak{a})$ is the image of $\operatorname{Spec}(A/\mathfrak{a})$.

(iv) Suffices to show if closed sets intersect to be empty, then finitely many of them intersect to the empty set. Say $\cap f_\alpha^*(\operatorname{Spec} B_\alpha) = \phi$, let B be the tensor product, then $f^*(\operatorname{Spec} B) = \phi$. By II.21, for some finite set J, $B_J \otimes_A k(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} A$ where $B_J = \otimes_{\alpha \in J} B_\alpha$. This means for each $\mathfrak{p}$, $B_\alpha \otimes k(\mathfrak{p}) = 0$ for some $\alpha \in J$, so $\mathfrak{p} \notin f^*(\operatorname{Spec} B_\alpha)$ and $\cap_{\alpha \in J} f^*(\operatorname{Spec} B_\alpha) = \phi$.

III.28 x

(i) $D(g)$ is image of $\operatorname{Spec} A_g$, $V((g))$ is image of $\operatorname{Spec} A/(g)$.

(ii) Let $\mathfrak{p}$, $\mathfrak{q}$ be distinct prime ideals. Without loss of generality assume there is some $f \in \mathfrak{p} - \mathfrak{q}$, then $\mathfrak{q} \in D(f)$, and $\mathfrak{p} \in V(f)$. So Hausdorff since $V(f), D(f)$ are both open and complement of each other.

(iii) Identity map is continous because the generators of C' are all in C. Since X_C is compact, closed subsets are mapped to compact subsets, which are closed in Hausdorff space (red in Ex11). Hence homeomorphism.

(iv) Follows from (iii) and totally disconnected by the decomposition given in (ii).

III.29 x

As shown in Ex28(iii) continous maps are closed from compact space to Hausdorff space. Continuous because $f^{*-1}(V(\mathfrak{a})) = V(f(\mathfrak{a}))$.

III.30 x

By Ex11, Zariski is same as constructible then X Hausdorff then $A/\operatorname{nil}(A)$ absolutely flat.

If absolutely flat, then X Hausdorff so $D(f)$ (quasi-compact) is closed and open, so Zariski is constructible by minimality of definition in Ex28(ii).

IV Chapter 4

IV.1 x

The irreducible components of $\operatorname{Spec}(A/\mathfrak{a})$ correspond to minimal prime ideals of $A/\mathfrak{a}$ by I.20. Since $\mathfrak{a}$ has a primary decomposition, there are only finitely many minimal primes containing $\mathfrak{a}$, so finitely many irreducible components.

IV.2 x

$$\mathfrak{a} = r(\mathfrak{a}) = r(\bigcap_{i=1}^n \mathfrak{q}_i) = \bigcap_{i=1}^n r(\mathfrak{q}_i) = \bigcap_{i=1}^n \mathfrak{p}_i$$

Thus $\bigcap_{i=1}^n \mathfrak{p}_i$ is a primary decomposition of $\mathfrak{a}$, by removing the embedded ones, we can get a minimal primary decomposition, and by Theorem 4.5 (1st uniqueness theorem), $\mathfrak{a}$ does not have embedded primes to begin with.

IV.3 x

Since A is absolutely flat, $A_\mathfrak{m}$ is a field for all $\mathfrak{m}$ by III.10, so by correspondence of primary ideals in Prop 4.8, there is no primary ideal strictly contained in $\mathfrak{m}$, so all primary ideals are maximal ideals.

IV.4 x

Since $2^2 \in (4,t)$ and $t \in (4,t)$, $(2,t) \subseteq r(4,t)$, and since $(2,t)$ is maximal, $r(4,t) = (2,t)$, so by Prop 4.2, $(4,t)$ is $(2,t)$-primary, but not a power of $(2,t)$.

IV.5 x

$\alpha = \mathfrak{p}_1\mathfrak{p}_2 = (x^2, xz, xy, yz)$, which are multiples of x of degree at least 2, together with multiples of yz. Thus $\alpha = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a reduced (all distinct and $\mathfrak{m}^2$ does not contain $\mathfrak{p}_1 \cap \mathfrak{p}_2$) primary decomposition. $\mathfrak{p}_1, \mathfrak{p}_2$, are isolated, and $\mathfrak{m}$ is not be cause it contains $\mathfrak{p}_1, \mathfrak{p}_2$.

IV.6 x

Given a prime ideal, let $V(\mathfrak{p})$ be the set of common zeroes of functions in $\mathfrak{p}$, if $V(\mathfrak{p})$ is the union of two proper closed subsets U, V, then $0 = gh$ where g vanishes $X - V$ and h vanishes on $X - U$, so $g, h \notin \mathfrak{p}$, a contradiction, thus $V(\mathfrak{p})$ would be irreducible. Now since the irreducible components of a Hausdorff space are singletons, the prime ideals of $C(X)$ are contained in a unique maximal ideal. Since $r(\mathfrak{p})$ is prime for any primary $\mathfrak{p}$ and is also the intersection of all primes containing $\mathfrak{p}$, we know any primary ideal is contained in a unique maximal ideal.

Suppose $(0) = \cap\mathfrak{q}_i$, then by Prop 1.11, all maximal ideals of $C(X)$ must contain some q_i. Thus the intersection can not be finite since we have infinitely many maximal ideals of $C(X)$, corresponding to the points of X (by I.26).

IV.7 x

(i) Follows from definition.

(ii) $A[x]/\mathfrak{p}[x] \cong (A/\mathfrak{p})[x]$ (by the map $A[x] \to A/\mathfrak{p}[x]$) is an integral domain.

(iii) By I.2, if f is a zero-divisor in $(A/\mathfrak{q})[x]$, then $af = 0$ for some $a \in A/\mathfrak{q}$, so coefficients of f are zero-divisors of $A/\mathfrak{q}$, which are nilpotent, again by I.2, f is nilpotent. Thus $\mathfrak{q}[x]$ is primary. Also, since the nilpotent elements correspond to $r(\mathfrak{q})[x] = \mathfrak{p}[x]$, we have $\mathfrak{q}[x]$ is $\mathfrak{p}[x]$-primary.

(iv) $\mathfrak{a}[x] = \cap_{i=1}^n \mathfrak{q}_i[x]$ follows from definition (compare coefficient-wise). In general, suppose $\mathfrak{a} \not\supseteq \mathfrak{b}$, then we see $\mathfrak{a}[x] \not\supseteq \mathfrak{b}[x]$ (compare constant terms). Therefore this is a minimal primary decomposition.

(v) Suppose $\mathfrak{q} \subseteq \mathfrak{p}[x]$ prime, then its contraction must be $\mathfrak{p}$ since $\mathfrak{p}$ is minimal. By Prop 1.17 $\mathfrak{q} \supseteq \mathfrak{q}^{ce} = \mathfrak{p}^e = \mathfrak{p}[x]$ and $\mathfrak{q} = \mathfrak{p}[x]$. So $\mathfrak{p}[x]$ minimal.

IV.8 x

Since $(x_1, ..., x_i)$ is maximal in $k[x_1, ..., x_i]$, its powers are primary. Its extension in $k[x_1, ..., x_n]$ which is $(x_1, ..., x_i)$, is prime and its powers are again primary by IV.7.

IV.9 x

Suppose x zero divisor with $xa = 0$, then $x \in (0 : a)$ and is in some minimal $\mathfrak{p}$ containing $(0 : a)$, so $\mathfrak{p} \in D(A)$. Suppose $x \in \mathfrak{p}$ for some $\mathfrak{p} \in D(A)$, say $\mathfrak{p}$ is minimal containing $(0 : a)$. We would like to show all elements of $\mathfrak{p}$ are zero-divisors. It suffices to do it for $A_\mathfrak{p}$. By I.14, the set of zero-divisors is a union of prime ideals, and by Prop 1.11, $(0 : a)$ is in one of those prime ideals, but by minimality of $\mathfrak{p}$, we know that prime ideal must be $\mathfrak{p}$, and therefore every element of $\mathfrak{p}$ is a zero-divisor.

Let $\mathfrak{p}$ be a prime ideal containing $(0 : a/s)$ that is minimal, then $\mathfrak{p}$ corresponds to a prime $\mathfrak{p}'$ that does not meet S. We see $(0 : a) \subseteq \mathfrak{p}'$ as ideals in A because $(0 : a/s)^c$ contains $(0 : a)$. Now if $\mathfrak{p}'$ were not minimal, say $\mathfrak{q}'$ inside $\mathfrak{p}'$ also contains $(0 : a) = \mathrm{Ann}((a))$, then $\mathfrak{q} = S^{-1}\mathfrak{q}'$ contains $S^{-1}\,\mathrm{Ann}((a)) = \mathrm{Ann}(S^{-1}(a))$ by Prop 3.14, which contains $\mathrm{Ann}((a/s)) = (0 : a/s)$. (alternatively if $(x/s')(a/s) = 0$, then $xat = 0$ for some $t \in S$, giving $xt \in \mathfrak{q}'$ and $x \in \mathfrak{q}'$, meaning $x/s' \in \mathfrak{q}$.) Contradicting minimality of $\mathfrak{p}$. Thus $\mathfrak{p}'$ is minimal and $D(S^{-1}A) = D(A) \cap \mathrm{Spec}(S^{-1}A)$.

By Theorem 4.5, primes belonging to 0 are of form $r(0 : a)$ for some $a \in A$, making them minimal so inside $D(A)$. Conversely suppose $\mathfrak{p} \in D(A)$, say $\mathfrak{p}$ minimal containing $(0 : a)$, $a \neq 0$. From proof of 4.5, $r(0 : a) = \cap_{a \notin \mathfrak{q}_j} \mathfrak{p}_j$. Since $\mathfrak{p}$ contains $r(0 : a)$, by Prop 1.11, $\mathfrak{p}$ contains some $\mathfrak{p}_j$. Now $(0 : a) \subseteq \mathfrak{p}_j \subseteq \mathfrak{p}$ and by minimality, $\mathfrak{p} = \mathfrak{p}_j$ for some j.

IV.10 x

(i) Suppose $a/1 = 0$ in $A_{\mathfrak{p}}$, then $aq = 0$ for some $q \notin \mathfrak{p}$, meaning $a \in \mathfrak{p}$.

(ii) Note if $\mathfrak{q}$ is any prime ideal inside $\mathfrak{p}$, then $S_{\mathfrak{p}}(0) \subseteq \mathfrak{q}$ by the same argument. So $r(S_{\mathfrak{p}}(0))$ (intersection of all primes containing it) is $\mathfrak{p}$ iff there is no prime strictly inside $\mathfrak{p}$.

(iii) As shown above.

(iv) Suppose $a \in \cap_{\mathfrak{p} \in D(A)} S_{\mathfrak{p}}(0)$. If $a \neq 0$, then for each $\mathfrak{p} \in D(A)$, $aq_{\mathfrak{p}} = 0$ for some $q_{\mathfrak{p}} \notin \mathfrak{p}$. Now take prime $\mathfrak{q} \in D(A)$ that contains $(0 : a)$. We can not find element $q_{\mathfrak{q}}$, meaning $(0 : a)$ must be (1), so $a = 0$.

IV.11 x

When $\mathfrak{p}$ is minimal, $A_{\mathfrak{p}}$ has only one prime ideal, meaning every non-unit is nilpotent and so 0 is the smallest $\mathfrak{p}_{\mathfrak{p}}$-primary ideal in $A_{\mathfrak{p}}$. By correspondence of primary ideals, $S_0(\mathfrak{p}) = (0)^c$ is the smallest $\mathfrak{p}$-primary ideal.

$\mathfrak{a}$ is contained in intersection of all minimal ideals, which is the nilradical.

If not, then the intersection of the primary ideals corresponding to the isolated ideals would be non-zero. Those primary ideals would be $S_{\mathfrak{p}}(0)$ because they are uniquely determined by Cor 4.11, and we can manually pick them to be the smallest, thus their intersection would be $\mathfrak{a} \neq 0$.

IV.12 x

(i) (ii) Exercise 1.18.

(iii) Prop 3.11(ii): $S(\mathfrak{a}) = (1)$ iff $S^{-1}\mathfrak{a} = (1)$ iff $\mathfrak{a}$ meets S.

(iv) Observe $a \in S(\mathfrak{a})$ iff $as \in \mathfrak{a}$ for some $s \in S$. So $a \in (S_1S_2)(\mathfrak{a})$ iff $as_1s_2 \in \mathfrak{a}$ iff $as_1 \in S_2(\mathfrak{a})$ iff $a \in S_1(S_2(\mathfrak{a}))$.

Say $\mathfrak{a} = \cap_{i=1}^n \mathfrak{q}_i$. Then $S(\mathfrak{a}) = \cap_{i=1}^n S(\mathfrak{q}_i)$. By correspondence of primary ideals, $S(\mathfrak{q}_i) = \mathfrak{q}_i$ if $\mathfrak{q}_i$ does not meet S. So each term in the intersection can only be (1) or $\mathfrak{q}_i$, resulting in finitely many possibilities.

IV.13 nth symbolic power

(i) By correspondence, suffices to show $(\mathfrak{p}^n)_{\mathfrak{p}}$ is $\mathfrak{p}_{\mathfrak{p}}$ primary, this is because $(\mathfrak{p}^n)_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}^n$ (simply compare elements) and $\mathfrak{p}_{\mathfrak{p}}$ is maximal (prop 4.2).

(ii) Suppose $\mathfrak{p}^n = \cap q_i$, then $(\mathfrak{p}^n)_{\mathfrak{p}} = \cap(\mathfrak{q}_i)_{\mathfrak{p}} = \cap(\mathfrak{q}_i')_{\mathfrak{p}}$, where $\mathfrak{q}_i'$ are those $\mathfrak{p}_i$-primary components such that $\mathfrak{p}_i \subseteq \mathfrak{p}$. However $\mathfrak{p} = r(\mathfrak{p}^n) \subseteq r(\mathfrak{q}_i') = \mathfrak{p}_i \subseteq \mathfrak{p}$, so in fact $\mathfrak{p}_i = \mathfrak{p}$ and this $\mathfrak{q}_i$ is unique, so $(\mathfrak{p}^n)_{\mathfrak{p}} = (\mathfrak{q}_i)_{\mathfrak{p}}$. Now $S_{\mathfrak{p}}(\mathfrak{p}^n) = (\mathfrak{q}_i)_{\mathfrak{p}}^c = \mathfrak{q}_i$ by correspondence of primary ideals. This $\mathfrak{q}_i$ exists because $\mathfrak{p}_{\mathfrak{p}}^n \neq (1)$.

(iii) $(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)})_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}^{(m)}\mathfrak{p}_{\mathfrak{p}}^{(n)} = \mathfrak{p}_{\mathfrak{p}}^m\mathfrak{p}_{\mathfrak{p}}^n = \mathfrak{p}_{\mathfrak{p}}^{m+n}$. SO repeat the above argument and see $\mathfrak{p}_{\mathfrak{p}}^{(m+n)}$ is the $\mathfrak{p}$-primary component.

(iv) Follows from correspondence. Typo in question: should be $\mathfrak{p}^n$ is primary.

IV.14 x

Suppose $y, z \notin \mathfrak{p} = (\mathfrak{a} : x)$, but $yz \in \mathfrak{p}$. Then $yzx \in \mathfrak{a}$ and $y \in (\mathfrak{a} : zx)$. This means $\mathfrak{p} \subsetneq (y) + \mathfrak{p} \subseteq (\mathfrak{a} : zx)$ where $zx \notin \mathfrak{a}$, contradicting maximality. Hence $\mathfrak{p} = (a : x) = r(a : x)$ prime and it belongs to $\mathfrak{a}$ by Theorem 4.5.

IV.15 x

Let $\mathfrak{a} = \bigcap \mathfrak{q}_i$, then $S^{-1}\mathfrak{a} = \bigcap S^{-1}\mathfrak{q}_i$ where $S^{-1}\mathfrak{q}_i = S^{-1}A$ iff $\mathfrak{q}_i$ meets S_f iff $\mathfrak{p}_i$ meets S_f iff $\mathfrak{p}_i \notin \Sigma$. Thus $S^{-1}\mathfrak{a} = \bigcap_\Sigma S^{-1}\mathfrak{q}_i = S^{-1}\mathfrak{q}_\Sigma$. So $S_f(\mathfrak{a}) = \bigcap_\Sigma (\mathfrak{q}_i)^{ec} = \bigcap_\Sigma \mathfrak{q}_i = \mathfrak{q}_\Sigma$.

Suppose $a \in S_f(\mathfrak{a})$, then $(af^m - b)f^n = 0$ for some $b \in \mathfrak{a}$, so $a \in S_f(\mathfrak{a})$ iff $af^n \in \mathfrak{a}$ for some n dependent on a. Thus $(a : f^n) \subseteq S_f(\mathfrak{a}) \subseteq \cup_n (a : f^n)$ and so $S_f(\mathfrak{a}) = \cup_n (a : f^n)$ This is Prop 3.11(ii). Since $(a : f^n)$ is a increasing sequence of ideals, it suffices to show that it stabilizes to conclude that $S_f(\mathfrak{a}) = (a : f^n)$ for large n. Note that $(a : f^n) = \cap (q_i : f^n)$, so it suffices to do it for $\mathfrak{q}_i$ and then pick a large enough n.

Suppose $f \in \mathfrak{p}_i$, then $f^n \in \mathfrak{q}_i$ for some n, so for large n, $(q_i : f^n)$ is (1). On the other hand, if $f \notin \mathfrak{p}_i$, then we know $f \notin r(q_i)$, so $af^n \in \mathfrak{q}_i$ implies $a \in \mathfrak{q}_i$, and $(\mathfrak{q}_i : f^n)$ stabilizes to $\mathfrak{q}_i$ (which agrees with our previous assertion that $S_f(\mathfrak{a}) = \mathfrak{q}_\Sigma$).

IV.16 x

The contraction of any ideal of $S^{-1}A$ has primary decomposition, giving a primary decomposition of itself (take extension and pass through intersections).

IV.17 x

Apply Ex11 to $A/\mathfrak{a}$ and get $\mathfrak{q}_1 = S_{\mathfrak{p}_1}(\mathfrak{a})$ is $\mathfrak{p}_1$-primary. Then $\mathfrak{a} = \mathfrak{q}_1 \cap (\mathfrak{a} + (x))$ because if $a + fx \in \mathfrak{q}_1$ for some $a \in \mathfrak{a}$, then $fx \in \mathfrak{q}_1$ so $f \in \mathfrak{q}_1$. Since $\mathfrak{q}_i = (\mathfrak{a} : x)$, $fx \in \mathfrak{a}$, so $a + fx \in \mathfrak{a}$. See hint for rest of argument.

IV.18 x

(i)⇒(ii) Let $\mathfrak{a} = \cap \mathfrak{q}_i$, then $S_n(\mathfrak{a}) = \cap S_n(\mathfrak{q}_i)$. Now as the sets S_n get smaller, they either stop meeting $\mathfrak{q}_i$ at some point, or they always meet $\mathfrak{q}_i$. In the former case we have $S_n(\mathfrak{q}_i) = S_{n+1}(\mathfrak{q}_i) = ... = \mathfrak{q}_i$ (by correspondence of primary), and in the latter, $S_n(\mathfrak{q}_i) = (1)$. Pick n big enough to work for all i and we have $S_n(\mathfrak{a}) = S_{n+1}(\mathfrak{a}) = ...$ as required.

Suppose $\mathfrak{p}$ is a prime, then $S_\mathfrak{p}(\mathfrak{a}) = \cap S_\mathfrak{p}(\mathfrak{q}_i) = \cap_{\mathfrak{q}_i \subseteq \mathfrak{p}} \mathfrak{q}_i$. Pick $f \in \cap_{\mathfrak{q}_i \notin \Sigma_\mathfrak{p}} \mathfrak{q}_i - \cap_{\mathfrak{q}_i \in \Sigma_\mathfrak{p}} \mathfrak{q}_i$ which exists because decomposition is irredundant and $\Sigma_\mathfrak{p} = \{\mathfrak{q}_i \subseteq \mathfrak{p}\}$ is isolated set of primes. By Ex15 $(\mathfrak{a} : f^n) = S_f(\mathfrak{a}) = \cap_{\mathfrak{q}_i \subseteq \mathfrak{p}} \mathfrak{q}_i = S_\mathfrak{p}(\mathfrak{a})$ for some n.

(ii)⇒(i) From the proof of Ex17, note $\mathfrak{a}_n$ strictly contains $\mathfrak{a}_{n-1}$, so S_n meets $\mathfrak{a}_n$, giving $S_n(\mathfrak{a}) = \mathfrak{q}_1 \cap ... \cap \mathfrak{q}_n = S_{n+1} = \mathfrak{q}_1 \cap ... \cap \mathfrak{q}_{n+1}$. Take contraction and $\mathfrak{q}_1 \cap ... \cap \mathfrak{q}_n \cap \alpha_n = \mathfrak{q}_1 \cap ... \cap \mathfrak{q}_{n+1} = \mathfrak{q}_1 \cap ... \cap \mathfrak{q}_{n+1} \cap (1)$ so α_{n+1} is chosen to be (1), and the process ends.

IV.19 x

Let $\mathfrak{q}$ be $\mathfrak{p}$-primary, then $a \in S_\mathfrak{p}(0)$ iff $aq = 0$ for some $q \notin \mathfrak{p} = r(\mathfrak{q})$ so $a \in \mathfrak{q}$.

See hint. $\mathfrak{a} = \mathfrak{q}_1 \cap ... \cap \mathfrak{q}_n$, $\mathfrak{b} = \mathfrak{q}_1 \cap ... \cap \mathfrak{q}_{n-1} \not\subseteq \mathfrak{q}_n$. So there is some $x \in \mathfrak{b} - \mathfrak{q}_n$ and by Lemma 4.4 $r(\mathfrak{a}, x) = \cap r(\mathfrak{q}_i, x) = \mathfrak{p}_n$, so by Theorem 4.5 $\mathfrak{p}_n$ belongs to $\mathfrak{a}$. For each $i \neq n$, by induction we can pick $x \in \cap_{j \neq i,n} \mathfrak{q}_j - \mathfrak{q}_i$, then take $y \in \mathfrak{p}_n - \mathfrak{p}_i$ because $\mathfrak{p}_n$ is maximal, and get $xy^m \in \mathfrak{q}_j$ for $j \neq i$ for some m and $xy^m \notin \mathfrak{q}_i$. So apply the above argument and see $r(\mathfrak{a}, xy^m) = \mathfrak{p}_i$ and $\mathfrak{p}_i$ belongs to $\mathfrak{a}$.

IV.20 primary decomposition of modules

$r_M(N) = r(N : M) = r(\operatorname{Ann}(M/N))$ by definition and Exercise 2.2.

(i) $r_M(N) \supseteq \operatorname{Ann}(M/N)$.

(ii) $r(r_M(N)) = r_M(N)$.

(iii) $r_M(N \cap P) = r_M(N) \cap r_M(P)$. $r_M(\mathfrak{a}N) = r_M(\mathfrak{a}M) \cap r_M(N)$ because if x^n sends M to $\mathfrak{a}M$, x^m sends M to N, then x^{n+m} sends M to $\mathfrak{a}N$. If x^n sends M to $\mathfrak{a}N$, then it sends M to $\mathfrak{a}M$ and N.

(iv) $r_M(N) = (1) \Leftrightarrow N = M$.

(v) $r_M(N + P) \subseteq r(r_M(N) + r_M(P))$ because if $x^n = y + z$ where $y \in r_M(N), z \in r_M(P)$, then $x^{nm} = (y+z)^m$ sends M to $N + P$.

(vi) $r_M(\mathfrak{p}^n M) = r_M(\mathfrak{p}M)$. In case M is finitely generated, we want to show $r_M(\mathfrak{p}M) = \mathfrak{p}$. It suffices to show $\operatorname{Ann}(M/\mathfrak{p}M) = 0$ when viewed as an $A/\mathfrak{p}$-module. This is because its image when localized at $\mathfrak{p} = (0)$ (in $A/\mathfrak{p}$) is $\operatorname{Ann}(M \otimes k(\mathfrak{p})) = 0$ as it is a $k(\mathfrak{p})$-vector space, so it is an ideal contained in (0).

IV.21 x

Suppose $xyM \subseteq Q$ and $y \notin (Q : M)$. That is $\varphi_x \varphi_y = 0$ and $\varphi_y \neq 0$ as functions on M/Q. This means φ_x is a zero-divisor and thus nilpotent. Therefore $x^n M \subseteq Q$ for some n, so $x^n \in (Q : M)$. Hence $(Q : M)$ is primary.

If Q_i are $\mathfrak{p}$-primary, then $Q = \cap_{i=1}^n Q_i$ is $\mathfrak{p}$-primary.

Proof: $r_M(Q) = \cap r_M(Q_i) = \mathfrak{p}$. Suppose φ_x is not injective, say $mx \in Q$, then $mx \in Q_i$ for all i, so it is not injective on M/Q_i, so $x^{n_i} m \in Q_i$ for all m. Pick n large enough and we see $x^n M \subseteq Q$, so φ_x is nilpotent.

Let Q be $\mathfrak{p}$-primary. If $m \in Q$, then $(Q : m) = (1)$.

If $m \notin Q$, then $(Q : m)$ is $\mathfrak{p}$-primary

Proof: If $m \notin Q$, then $(Q : m)$ is primary by same proof as above. If $x \in (Q : m)$, then φ_x not injective, so φ_x nilpotent and $x^n \in (Q : M)$ and $x \in \mathfrak{p}$, so $\mathfrak{p} = r(Q : M) \subseteq r(Q : m) \subseteq \mathfrak{p}$.

If $x \notin \mathfrak{p}$, then $(Q : x) = Q$

Proof: x is not nilpotent, so not zero-divisor, so injective on M/Q.

IV.22 x

$\mathfrak{p}_i$ are of form $r(N : m)$, $m \in M$

Proof: $r(N : m) = \cap r(Q_i : m) = \cap_{m \notin Q_i} \mathfrak{p}_i$. So if $r(N : m)$ is prime, then it is equal to some $\mathfrak{p}_i$. Conversely, for each $\mathfrak{p}_i$ we can pick some m in $\cap_{j \neq i} Q_j - Q_i$ since irredundant.

IV.23 x

By correspondence theorem, can assume $N = 0$. 4.6: Minimal elements of primes belonging to N are minimal prime ideals containing $(N : M)$. Corresponds to minimal primes of $A/\operatorname{Ann}(M)$ when $N = 0$.

4.7: $0 = \cap Q_i$ minimal primary decomposition. $\cup \mathfrak{p}_i = \{x \in M : (0 : x) \neq 0\} = \{\text{zero-divisors of } M\} =: D$

Proof: $x \in D$ if there is some $xm = 0$, observe that $D = \cup_{m \neq 0} r(0 : m)$. Now $r(0 : m) = \cap_{m \notin Q_i} \mathfrak{p}_i$, so $D \subseteq \cup \mathfrak{p}_i$. From Ex22 we know $\mathfrak{p}_i = r(0 : m)$ for some m, so $D = \cup \mathfrak{p}_i$.

4.8: Let Q be $\mathfrak{p}$-primary. If $S \cap \mathfrak{p} \neq \phi$, then $S^{-1}Q = S^{-1}M$

Proof: Let $s \in S \cap \mathfrak{p}$, then $s^n \in (Q : M)$, so $S^{-1}M/S^{-1}Q = S^{-1}(M/Q) = 0$ because a unit of $S^{-1}A$ kills everything.

4.8: If $S \cap \mathfrak{p} = \phi$, then $S^{-1}Q$ is $S^{-1}\mathfrak{p}$-primary

Proof: $r_{S^{-1}M}(S^{-1}Q) = r(\operatorname{Ann}(S^{-1}(M/Q))) = S^{-1} r_M(Q) = S^{-1}\mathfrak{p}$. Suppose $(x/s)(m/t) = 0$, then $xms' = 0$ for some $s' \in S$. Since $\mathfrak{p}$ contains $\operatorname{Ann}(M/Q)$. $s'm \neq 0$ and x is nilpotent, so x/s is nilpotent.

4.9: $S^{-1}N = \cap S^{-1}Q_i$ where Q_i are the ones that do not meet S

4.10, 4.11 similar.

V Chapter 5

V.1 x

Let $\mathfrak{a}$ be an ideal of B. Then $f^{-1}(\mathfrak{q})$ is a prime containing $f^{-1}(\mathfrak{a})$. Suppose $\mathfrak{p}$ contains $f^{-1}(\mathfrak{a})$, then since $f : A \to f(A)$ is surjective and surjective maps perserve prime ideals, $f(\mathfrak{p})$ is prime in $f(A)$, and by Theorem 5.10 there is some $\mathfrak{q}$ lying over $f(\mathfrak{p})$, giving $f^{-1}(\mathfrak{q}) = f^{-1}(f(\mathfrak{p}))$. Note that $f^{-1}(f(\mathfrak{p})) = \mathfrak{p} + \ker f$ because if $f(a) = f(p)$ for some $p \in \mathfrak{p}$, then $a - p \in \ker f$, but since $\mathfrak{p}$ contains $f^{-1}(\mathfrak{a})$ contains $\ker f$, we know $\mathfrak{p} = f^{-1}(f(\mathfrak{p}))$. Hence $f^*(V(\alpha)) = V(f^{-1}(\mathfrak{a}))$ and f is closed.

V.2 x

Let $\mathfrak{p} = \ker f$ and $\mathfrak{q}$ be a prime of B lying over $\mathfrak{p}$, and we have an injection $f : A/\mathfrak{p} \to \Omega$. By Prop 5.6, $B/\mathfrak{q}$ is integral over $A/\mathfrak{p}$. By Theorem 5.21 there is a valuation ring C over $A/\mathfrak{p}$ extending f, of the field of fraction of $B/\mathfrak{q}$. By Prop 5.18 C is integrally closed, so it contains $B/\mathfrak{q}$. Thus restricting to $B/\mathfrak{q}$ we get a homomorphism from $B/\mathfrak{q}$ to Ω and $B \to B/\mathfrak{q} \to \Omega$ required homomorphism.

V.3 x

Since integral elements closed under addition and multiplication, it suffices to show $b \otimes c$ is integral for $b \in B'$. Suppose $b^n + u_1 b^{n-1} + ... + u_n = 0$ for $u \in B$. Then

$$(b \otimes c)^n + (u_1 \otimes c)(b \otimes c)^{n-1} + ... + u_n \otimes c^n = 0 \otimes c^n = 0$$

So $f \otimes 1$ is integral (coefficients are all elements of $B \otimes C$).

V.4 x

Consdier subring $k[x^2 - 1]$ of $k[x]$, integral because $x^2 - (x^2 - 1) - 1 = 0$. Let $\mathfrak{n} = (x - 1)$, $\mathfrak{m} = (x^2 - 1)$. Now suppose $1/(x + 1)$ is integral over $k[x^2 - 1]_{(x^2-1)}$, then

$$\frac{1}{(x+1)^n} + \frac{a_1}{(x+1)^{n-1}} + ... + a_n = 0$$

$$a_1(x + 1) + ... + a_n(x + 1)^n = -1$$

in $B_{\mathfrak{n}}$. The denominators of $a_i \in k[x^2 - 1]_{(x^2-1)}$ are of form $f_i(x^2 - 1)$ where f_i is a polynomial with nonzero constant term. Clear denominator and we will get a non-zero constant term and the other terms in the equation are all divisible by $(x + 1)$, a contradiction.

V.5 x

(i) If $x^{-1} \in B$, then since integral, $x^{-n} + ... + a_n = 0$, so $x^{-1} + a_1 + ... + a_n x^{n-1} = 0$ so $x^{-1} \in A$.

(ii) Since every maximal is a contraction of maximal by Cor 5.8 and Thm 5.10, so contraction of Jacobson radical is Jacobson radical.

V.6 x

Suppose $(b_1, ..., b_n) \in \prod_{i=1}^n B_i$, with $f_i(b_i) = 0$ where $f_i \in A[x]$. Then $\prod f_i(b_1, ..., b_n) = 0$ so integral over A.

V.7 x

Suppose $x^n + ... + a_n = 0$, then $x(x^{n-1} + ... + a_{n-1}) = x^n + ... + a_{n-1}x \in A$, so $x^{n-1} + ... + a_{n-1} \in A$ because $B - A$ multiplicatively closed. Thus by induction $x \in A$.

V.8 x

(i) See hint.

(ii) We create a bigger ring containing B such that f, g splits similar to constructing a splitting field. Let $h \in B[x]$ monic, then take $B' = B[x]/(h)$, so x is a root of h in B' and $B \to B'$ is injective because no constant is a multiple of h. Now split f, g and repeat argument from (i).

V.9 x

Suppose $f^m + ... + g_m = 0$ for $g \in A[x]$. Let $f_1 = f - x^r$, then

$$(f_1 + x^r)^m + ... + g_m = 0$$

$$f_1^m + h_1 f_1^{m-1} + ... + h_m = 0$$

for some h_i combinations of x^r and g_j, so in $A[x]$, and $h_m = (x^r)^m + ... + g_m \in A[x]$. Apply V.8 to this and see either $f_1 \in C[x]$ or $f_1^{m-1} + h_1 f_1^{m-2} + ... + h_{m-1} \in C[x]$. If the latter, then use induction and eventually, $f_i = f - x^{r_1} - ... - x^{r_i} \in C[x]$ so $f \in C[x]$.

V.10 x

(i) (a)$\Rightarrow$(b) Given $\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq A$ and $\mathfrak{q}_1 \subseteq B$ over $\mathfrak{p}_1$, need $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ over $\mathfrak{p}_2$. Since f^* closed mapping, it sends $V(\mathfrak{q}_1)$ to some $V(\mathfrak{a})$ which contains $\mathfrak{p}_1$, so contains $\mathfrak{p}_2$. Thus f^* sends some $\mathfrak{q}_2 \in V(\mathfrak{q}_1)$ to $\mathfrak{p}_2$, as required.

(b)$\Rightarrow$(c) Given any prime in A containing $\mathfrak{p}$, it is contraction of some prime containing $\mathfrak{q}$ by going up property, so $f^* : \operatorname{Spec} B/\mathfrak{q} \to \operatorname{Spec} A/\mathfrak{p}$ is surjective.

(c)$\Rightarrow$(b) Take inverse image of $\mathfrak{p}_2$ in $B/\mathfrak{q}$ and get $\mathfrak{q}_2$ using the surjective map $f^* : \operatorname{Spec} B/\mathfrak{q}_1 \to \operatorname{Spec} A/\mathfrak{p}_1$.

(ii) (a)$\Rightarrow$(c) See hint. Note that an open neighbourhood of $\mathfrak{p}$ must contain all primes inside $\mathfrak{p}$.

(b)$\Leftrightarrow$(c) Similar as above.

V.11 x

By III.18. and V.10(ii).

V.12 x

If $x \in A$, then x is a root of the polynomial $\prod_{\sigma \in G}(t - \sigma x)$ which is G-invariant, so in $A^G[x]$. Thus A integral over A^G.

$\sigma(a/s) = \sigma(a)/\sigma(s)$ is a group action. We have $(S^G)^{-1}A^G \to (S^{-1}A)^G$ by $a/s \mapsto a/s$. Suppose a/s is in the kernel where $a \in A^G$, $s \in S^G$, then $ta = 0$ for some $t \in S$. Then $(\prod \sigma(t))a = 0$ and since $\prod \sigma(t) \in S^G$, we have $a/s = 0$, so injective.

Suppose $\sigma(a/s) = a/s$ for all $\sigma \in G$, then

$$\frac{a \prod_{\sigma \neq 1} \sigma(s)}{\prod \sigma(s)} = a/s$$

so we can assume $s \in S^G$. So $\sigma(a)/s = a/s$ for all $\sigma \in G$, and $st(\sigma(a) - a) = 0$ for some $t \in S$. This means $\prod \tau(t) s(\sigma(a) - a) = 0$. Write $s'(\sigma(a) - a) = 0$ where $s' \in S^G$. Now observe that $s'a = s'\sigma(a) = \sigma(s'a)$, so $s'a \in A^G$, and $a/s = s'a/(s's) \in (S^G)^{-1}A^G$ is of the required form.

V.13 x

Let $\mathfrak{p}_1, \mathfrak{p}_2 \in P$. Let $x \in \mathfrak{p}_1$, then $\prod \sigma(x) \in \mathfrak{p}_1 \cap \mathfrak{p} = \mathfrak{p}_2$, so $\sigma(x) \in \mathfrak{p}_2$ for some σ. Therefore $\mathfrak{p}_1 \subseteq \cup \sigma(\mathfrak{p}_2)$ so $\mathfrak{p}_1 \subseteq \sigma(\mathfrak{p}_2)$ for some σ because G is finite, by (1.11). By (5.9), $\mathfrak{p}_1 = \sigma(\mathfrak{p}_2)$. Thus G acts transitively.

V.14 x

Let $x \in B$, then x has minimal polynomial f over K with $f \in A[x] \subseteq K[x]$ by (5.15). Since $\sigma(f(x)) = f(\sigma(x)) = 0$, $\sigma(x)$ is integral over A, so $\sigma(B) \subseteq B$. Similarly, $\sigma^{-1}(B) \subseteq B$ so $B = \sigma(\sigma^{-1}B) \subseteq \sigma(B)$.

Suppose x is fixed by G, then since the fixed field of Galois group is just the base field we know $x \in K \cap B$. Since A is integrally closed, $K \cap B = A$. Therefore $A = B^G$.

V.15 x

Suppose L' is an extension of L and C is the integral closure of B, then it is also the integral closure of A in L' by (5.4). Any prime $\mathfrak{q} \subseteq B$ that contracts to $\mathfrak{p} \subseteq A$ would lift to some $\mathfrak{q}' \subseteq C$ by (5.10),so it suffices to show the result for L'. Therefore we can take any extension of L we want. Also, note the property is transitive, as in that if $\mathfrak{p}$ lifts to finitely many $\mathfrak{q}$, and each $\mathfrak{q}$ lifts to finitely many $\mathfrak{q}'$, then $\mathfrak{p}$ lifts to finitely many $\mathfrak{q}'$.

Let E be the separable closure of K in L, then E/K is separable and L/E is purely inseparable (because if $\alpha \in L - E$ with minimal polynomial f over E, then $f = g(x^p)$ because $f' = 0$. So α^p has degree strictly smaller than α and by induction, $\alpha^{p^n} \in E$. Therefore the f factors as $(x-\alpha)^{p^n}$ and L is purely inseparable over E). So by previous parargraph, we can assume L is separable or purely inseparable.

If L is separable, take its Galois closure, then apply V.13 and V.14.

If L is purely inseparable, say $\mathfrak{q} \cap A = \mathfrak{p}$, then elements of $\mathfrak{q}$ have minimal polynomial $x^{p^n} - a$ for some $a \in A \cap \mathfrak{q} = \mathfrak{p}$. Conversely, if $\alpha^n \in \mathfrak{p}$, then $\alpha^n \in \mathfrak{q}$ and $\alpha \in \mathfrak{q}$. Hence $\mathfrak{q}$ is exactly the set of elements of B that has some power lying in $\mathfrak{p}$, so $\mathfrak{q}$ is unique, and in particular, $\operatorname{Spec} B \to \operatorname{Spec} A$ is bijective.

V.16 noether's normalization lemma

Consider $f(x_1' + \lambda_1 x_n, ..., x_{n-1}' + \lambda_{n-1} x_n, x_n)$ whose homogeneous part of highest degree (in terms of $x_1', ..., x_{n-1}', x_n$) is $F(x_1' + \lambda_1 x_n, ..., x_{n-1}' + \lambda_{n-1} x_n, x_n)$ because we are replacing variables with homogeneous polynomials. Now evaluate at $x_i' = 0, x_n = 1$ we get $F(\lambda_1, ..., \lambda_{n-1}, 1) \neq 0$, so when viewed as a polynomial in $k[x_1', ..., x_{n-1}', x_n]$, it must have a term $a x_n^m$ for some $a \in k$. Therefore when viewed as a polynomial in x_n, it is monic and hence integral over $A' = k[x_1', ..., x_{n-1}']$. By induction hypothesis, A' is integral over some $k[y_1, ..., y_r]$, so A is integral over it too by transitivity.

From Hartshrone II.3.5 we know a finite morphism has finite fiber. So $A = k[y_1, ..., y_r] \hookrightarrow B = k[x_1, ..., x_n]$, gives a linear map, and induces $\operatorname{Spec} B \to \operatorname{Spec} A$ surjective and has finite fiber (since maximal maps to maximal by (5.8), the map between varieties is finite to 1?).

V.17 weak nullstellensatz

Let $A = k[x_1, ..., x_n]/I(X)$, then $A \neq 0$ and by geometric interpretation of Noether's normalization lemma there is a surjection from X to a linear subspace of dimension $r \geq 0$, so it is non-empty.

If I ix maximal, then $V(I)$ has some point $(a_1, ..., a_n)$ meaning I is contained in and therefore must be the maximal ideal $(x_1 - a_1, ..., x_n - a_n)$.

V.18 x

V.19 x

Let $B = k[x_1, ..., x_n]/I(X)$ and if I is maximal, B is a field, so it is a finite field extension of k. Since k is algebraically closed, $B = k$. Thus $x_i + f_i = a_i$ for some $a_i \in k, f_i \in I$. So $x_i - a_i \in I$ for each i, and I must be the maximal ideal $(x_1 - a_1, ..., x_n - a_n)$.

V.20 x

Let $K = S^{-1}A$ where $S = A - \{0\}$, then by Noether's normalization lemma B is integral over some $K[y_1, ..., y_n]$. Say B is generated by $z_1, ..., z_m$, then $f_i(z_i) = 0$ for some $f_i \in K[y_1, ..., y_n][x]$. Take $s \in A$ to be product of all demonimators of all the f_i, then we see sz_i is integral over $B' = A[y_1, ..., y_n]$. Therefore B_s is integral over B'_s because $z_i = sz_i/s$ would be integral over B'_s.

V.21 x

Since y_i algebraically independent over A, we can define $B' \to \Omega$ by sending y_i to 0. Since $f(s) \neq 0$, we can define $B'_s \to \Omega$. Then extend using V.2 to $B_s \to \Omega$. Now we have $B \to B_s \to \Omega$ required homomorphism.

V.22 x

Let $v \neq 0$ in B. Pick s from V.21 applied to B_v with subring A. Let $\mathfrak{m}$ not contain s, and $f : A \to A/\mathfrak{m} \to \Omega$ where Ω is algebraic closure of $A/\mathfrak{m}$. Then $f(s) \neq 0$ and f can be extended to $B_v \to \Omega$. Since v is a unit, $f(v) \neq 0$. Let $\mathfrak{n} = B \cap \ker f$, then $B/\mathfrak{n}$ injects into Ω and contains $A/\mathfrak{m}$. Elements of $B/\mathfrak{n}$ are algebraic over the field $A/\mathfrak{m}$, meaning $B/\mathfrak{n}$ is integral over $A/\mathfrak{m}$. Since kernel of a map into integral domain is prime, $B/\mathfrak{n}$ is integral domain, so B is a field by (5.7). Hence $\mathfrak{n}$ is maximal and does not contain v. So the Jacobson radical of B is 0.

V.23 jacobson ring

(i)$\Leftrightarrow$(ii) If primes are intersections of maximal ideals, then the nilradical of any homomorphic image, which is intersection of certain prime ideals can be replaced by an intersection of maximal ideals, so it would contain the Jacobson radical of that image, thus they are equal.

Conversely given a prime, the nilradical of $A/\mathfrak{p}$ is 0, and if it is the Jacobson radical, then $\mathfrak{p}$ would be intersection of the corresponding maximal ideals (all maximal ideals contianing $\mathfrak{p}$).

(i)$\Rightarrow$(iii) Any prime would be intersections of maximal, and if itself is not maximal, then those maximal ideals would contain it strictly.

(iii)$\Rightarrow$(i) Suppose (i) is false, so there is a prime that is not intersection of maximal. Replace A by $A/\mathfrak{p}$ then assume A is integral domain, with non-zero Jacobson radical. Let $f \neq 0$ in Jacobson radical, then A_f has a maximal ideal that contracts to $\mathfrak{p} \subseteq A$ which does not contain f. However, $\mathfrak{p}$ is not maximal because it is strictly contained in Jacobson radical, and every prime ideal that strictly contains $\mathfrak{p}$ must also contain f by maximality, so (iii) does not hold.

V.24 x

(i) Let $\mathfrak{q} \subseteq B$, then $B/\mathfrak{q}$ is integral over $A/\mathfrak{p}$. By V.5(ii), the Jacobson radical of $B/\mathfrak{q}$ contracts to the Jacobson radical of $A/\mathfrak{p}$, which is 0 since $A/\mathfrak{p}$ is Jacobson ring by V.23(ii). Assume A, B integral domain, and we shall show $(0) \subseteq B$ is the Jacobson radical, then by V.23(i), the original result is proven.

Suppose $f \neq 0$ is in the Jacobson radical J, where $f^n + a_1 f^{n-1} + ... + a_n = 0$ for $a_i \in A$. Then $f(f^{n-1} + ... + a_{n-1}) = -a_n \in A \cap J = (0)$, so $f^{n-1} + ... + a_{n-1} = 0$ (integral domain). Induction then show $f = 0$, a contradiction.

(ii) Let $\mathfrak{q} \subseteq B$, then $B/\mathfrak{q}$ is finitely generated over $A/\mathfrak{p}$. By V.22, the Jacobson radical of $B/\mathfrak{q} = (0)$ since $A/\mathfrak{p}$ is Jacobson ring. Hence $\mathfrak{q}$ is intersection of maximal ideals.

Since $\mathbb{Z}$ is Jacobson (every prime is maximal), and fields are Jacobson ring, finitely generated ring over $\mathbb{Z}$ or a field is Jacobson ring.

V.25 x

(i)$\Rightarrow$(ii) Let B be A-algebra with $\varphi : A \to B$. Since A is Jacobson, $A/\ker\varphi$ is. Also note that by taking the same generators, B is finitely generated $A/\ker\varphi$-algebra and is finite over A iff finite over $A/\ker\varphi$. Thus assume A is a subring of B. Apply V.21 and find $s \in A$. Since $s \neq 0$ and $A \subseteq B$ is integral domain, it is not in the nilradical of A, so not in the Jacobson radical of A. So there is some maximal ideal $\mathfrak{m}$ not containing s, then $A \to A/\mathfrak{m} = k$ extends to $B \to \Omega = \overline{k}$. Since B is a field, this is injective, so restricted to $A \to k$ must also be injective. Therefore $A = k$ is a field and $B \subseteq \Omega$ is alegbraic over A. Since B is finite algebraic k-algebra, it is finite dimensional over k.

(ii)$\Rightarrow$(i) Let $\mathfrak{p}$ be prime not maximal, $f \neq 0$ in $B = A/\mathfrak{p}$. Then B_f is finitely generated A-algebra, generated by $1/f$. If it is a field then it is finite over B, so integral over B. By (5.7) B would be a field which it is not. Thus B_f is not a field and has a non-trivial prime ideal that contracts to a prime ideal that strictly containing $\mathfrak{p}$, and not containing f. Hence the intersection of primes strictly containing $\mathfrak{p}$ is $\mathfrak{p}$, and by V.23(iii) A is Jacobson.

V.26 x

(1)$\Leftrightarrow$(2) Suppose $\overline{E \cap X_0} \neq E$, then there is a point of E with a neighbourhood U not intersecting $E \cap X_0$. Now $E \cap U$ is locally closed so must intersect X_0, a contradiction. Conversely if there is some open set such that $E \cap U$ does not intersect X_0, then $\overline{E \cap X_0} \subseteq E - U \subsetneq E$.

(1)$\Leftrightarrow$(3) Surjective by definition of induced topology. Let U, V be two open sets. Then $U - V = U \cap (X - V)$ is locally closed, so intersects X_0. Therefore the image of U contains something not in V, so the map is injective. Conversely, if $U \cap X_0 = V \cap X_0$, then $U - V$ does not intersect X_0.

(i)$\Leftrightarrow$(ii) Given closed set $V(\mathfrak{a})$, the closure of $V(\mathfrak{a}) \cap X_0$ is $V(\mathfrak{b})$ where $\mathfrak{b}$ is the largest ideal that contained in all the primes that contains $\mathfrak{a}$, corresponding to the Jacobson radical of $A/\mathfrak{a}$, which is the same as its nilradical since A is Jacobson, so $\mathfrak{b} = \sqrt{\mathfrak{a}}$ and $V(\mathfrak{a}) = V(\mathfrak{b})$, so X_0 is very dense by (2). Conversely if $A/\mathfrak{a}$ has Jacobson radical different from nilradical, then the closure $V(\mathfrak{b}) \neq V(\sqrt{\mathfrak{a}}) = V(\mathfrak{a})$.

(i)$\Leftrightarrow$(iii) Suppose we have locally closed set $V(\mathfrak{a}) \cap D(\mathfrak{b})$ consisting of a single point, namely there is only one prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$ and not contain $\mathfrak{b}$. If $\mathfrak{p}$ is not maximal, then it is intersection of all primes strictlying containing it by V.23(iii), so some of it must also not contain $\mathfrak{b}$, a contradiction. Thus $\mathfrak{p}$ is maximal and the set is a closed point. Conversely suppose some prime $\mathfrak{p}$ not maximal, such that all primes containing $\mathfrak{p}$ must contain $f \notin \mathfrak{p}$, then $V(\mathfrak{p}) \cap D(f)$ is locally closed with only one point, its closure contains some maximal ideal, so it is not closed.

V.27 valuation rings and valuations

Has maximal element by Zorn's lemma. Suppose A is maximal, then consider the set of subrings of K with homomorphisms into $\Omega = \overline{A/\mathfrak{m}}$. Let (B, g) be a maximal element over (A, f) in this partial order. By (5.19) B is local with maximal ideal $\ker g$. Since g extends $f : A \to A/\mathfrak{m}$, $\ker g \cap A = \mathfrak{m}$, so B dominates A, and by maximality $A = B$. By (5.21) $B = A$ is a valuation ring.

Suppose A is a valuation ring. B local ring dominating it. Let $x \in B - A$, then A contains x^{-1} and x^{-1} is a non-unit in A, so it must be a non-unit in B since $\mathfrak{m} \subseteq \mathfrak{n}$, a contradiction.

V.28 x

(1)$\Rightarrow$(2) Let $a, b \in A$, then either $a/b \in A$ or $b/a \in A$, so either $a \in (b)$ or $b \in (a)$. Now if $a \in \mathfrak{a} - \mathfrak{b}$, then for all $b \in \mathfrak{b}$, $b \in (a)$, therefore $\mathfrak{b} \subseteq \mathfrak{a}$.

(2)$\Rightarrow$(1) Given $x = a/b \in K$, either $(a) \subseteq (b)$ or $(b) \subseteq (a)$, so either x or x^{-1} is in A.

By (5.18(ii)), $A_{\mathfrak{p}}$ is valuation ring because it contains A. $A/\mathfrak{p}$ is valuation ring because by ideal correspondence, it satisfies (2).

V.29 x

Let B be a subring of K containing A. Let S be the intersection of units of B and A. Now if $b \in B - A$, then b^{-1} is in A so b is a unit in B. Therefore $b^{-1} \in S$ and $b = 1/b^{-1} \in S^{-1}A$. Given $a/s \in S^{-1}A$, we have $a/s \mapsto as^{-1} \in B$, gives a bijection between B and $S^{-1}A$.

Since B contains A, B is a valuation ring by (5.18), so a local ring. Its maximal ideal contracts to $\mathfrak{p} \subseteq A$ and anything out of $\mathfrak{p}$ is a unit in B, thus $S = A - \mathfrak{p}$ and $B = A_{\mathfrak{p}}$.

V.30 x

Since $xy^{-1} \in A$ or $yx^{-1} \in A$, either $\xi \geq \eta$ or $\eta \geq \xi$. This is well-defined since it holds when x, y are replaced by their associates. Suppose $\xi \geq \eta$ and $\eta \geq \xi$, then $xy^{-1} \in U$, so $\xi\eta^{-1} = 1$ and $\xi = \eta$. Now if ω is represented by z, then $xy^{-1} \in A$ implies $xz(y^{-1}z^{-1}) \in A$, so this order is compatible with group structure.

Suppose $v(x) \geq v(y)$, then $xy^{-1} \in A$, so $(x+y)y^{-1} = xy^{-1} + 1 \in A$ and $v(x+y) \geq \min(v(x), v(y))$.

V.31 valuation ring

First we have $v(1) = v(1) + v(1)$, $v(1) = 0$. Then $v(x^{-1}x) = v(x) + v(x^{-1}) = 0$. If $v(x) \leq 0$ then $0 = v(x) + v(x^{-1}) \leq v(x^{-1})$ So either $v(x) \geq 0$ or $v(x^{-1}) \geq 0$. The set is closed under ring operations by (1) and (2).

V.32 value group

Note that every element of A has value at least 0. Suppose $\alpha \in v(A - \mathfrak{p})$ and $0 \leq \beta \leq \alpha$. Say α represented by x, β represented by $y \in A$ sinec $\beta \geq 0$, then $xy^{-1} \in A$. If $y \in \mathfrak{p}$, then $x = xy^{-1}y \in \mathfrak{p}$, which is not true, so $y \in A - \mathfrak{p}$ and $\beta \in v(A - \mathfrak{p})$. Also, $v(A - \mathfrak{p})$ is inside an isolated subgroup by adding in $v(x^{-1})$, since $v(x) + v(y) = v(xy)$ and $xy \in A - \mathfrak{p}$ for any $x, y \in A - \mathfrak{p}$.

Given two primes $\mathfrak{p}$, $\mathfrak{q}$, they would differ by some element that is not an associate of anything in $\mathfrak{p}$, or anything in $\mathfrak{q}$, and the image of that element would give that $v(A - \mathfrak{p}) \neq v(A - \mathfrak{q})$. For surjectivity, suppose Δ is an isolated set, then take $\mathfrak{p} = A - v^{-1}(\Delta)$. If $x, y \in v^{-1}(\Delta)$, then since $v(xy) = v(x) + v(y) \in \Delta$. Now if $x, y \in \mathfrak{p}$, then $x + y \in \mathfrak{p}$, $ax \in \mathfrak{p}$ because $v(x+y) \geq \min(v(x), v(y)), v(ax) \geq v(x)$. Thus $\mathfrak{p}$ is a prime ideal.

In $A_{\mathfrak{p}}$, we are making everything in $A - \mathfrak{p}$ a unit, so the value group of $A_{\mathfrak{p}}$ is K^*/U' where $U' = U + (A - \mathfrak{p}) + (A - \mathfrak{p})^{-1}$, so Γ/Δ where Δ is the subgroup found above for $v(A - \mathfrak{p})$.

Let Γ' be value group of $A/\mathfrak{p}$. Define $\Gamma' \to \Gamma$. If $\xi = v(x/y)$ where $x, y \in A/\mathfrak{p}$, then send ξ to $v(x) - v(y)$ mod E where $x \in A$. Well defined because if $x \in A - \mathfrak{p}$, then $v(x) \leq v(p)$ for all $p \in \mathfrak{p}$ for $v(A - \mathfrak{p})$ is isolated, and therefore $v(x + p) = \min(v(x), v(p)) = v(x)$. The image lies in $\Delta = v(A - \mathfrak{p})$. If $a \in A$ is a unit, then a is a unit in $A/\mathfrak{p}$, so $v(a) = 0$ in Γ', thus injective. So value group of $A/\mathfrak{p}$ is the subgroup Δ.

V.33 x

Consider $(\sum a_i g_i)(\sum b_i h_i)$, then observe the maximal term of the product is the product of the maximal terms, and will not be cancelled to 0, so A is an integral domain.

Check (1) and (2) hold for v_0. (omitted)

Define $v(x/y) = v_0(x) - v_0(y)$. Check (1) and (2) hold (omitted), so it is a valuation. It is surjective because x_α is sent to α. So value group is Γ.

V.34 x

Let $C = g(B)$. Let $\mathfrak{n}$ be maximal of C, let $\mathfrak{p} = g^{-1}\mathfrak{n}$, then since f^* is closed mapping, f has going-up property, so if $f^{-1}\mathfrak{p}$ is contained in maximal ideal $\mathfrak{m}$, then we can find $\mathfrak{q}$ such that $f^{-1}(\mathfrak{q}) = \mathfrak{m}$. Since $B \to g(B) \cong B/\ker g$ is surjective, $\mathfrak{p}$ is in fact maximal by correspondence, so $\mathfrak{p} = \mathfrak{q}$ and $\mathfrak{n} \cap A = f^{-1}g^{-1}\mathfrak{n} = \mathfrak{m}$. Thus $C_{\mathfrak{n}}$ dominates A, and by V.27, $C_{\mathfrak{n}} = A$. Thus $C \subseteq A$ and $C = A$.

V.35 x

Replace A by image and assume $A \subseteq B$. Let K be field of fraction of B and A' be the valuation ring containing A. Then $\operatorname{Spec}(B \otimes_A A') \to \operatorname{Spec} A'$ is closed. Define $B \otimes_A A' \to K$ by $b \otimes a = ba$, which is

A'-algebra homomorphism because for $a \in A'$, $a \mapsto 1 \otimes a \mapsto a$. Thus $g(B \otimes A') = A'$ and $B \subseteq A'$. Since this is true for all valuation rings that contain A, B is inside the integral closure of A by (5.22).

Suppose $\operatorname{Spec}(B \otimes_A C) \to \operatorname{Spec} C$ is closed. Consider $\mathfrak{p} \otimes C \to B \otimes C \to B/\mathfrak{p} \otimes C \to 0$ by right directness of tensor $\mathfrak{p} \to B \to B/\mathfrak{p}$. So $\operatorname{Spec} B/\mathfrak{p} \otimes C = \operatorname{Spec}(B \otimes C)/(\mathfrak{p} \otimes C) \to \operatorname{Spec} B \otimes C$ is a closed mapping. Therefore combined together we get $A \to B/\mathfrak{p}$ is integral for each minimal ideal $\mathfrak{p}$ of B. Hence $A \to \prod B/\mathfrak{p}_i$ integral by V.6 (finite product since B noetherian, minimal primes correspond to irreducible components of $\operatorname{Spec} B$, which is finite since noetherian). Now $B/\operatorname{nil}(B)$ injects into $\prod B/\mathfrak{p}_i$ because if $a = b \bmod \mathfrak{p}_i$ for all i, then $a - b \in \mathfrak{p}_i$ for all i and $a - b \in \operatorname{nil}(B)$ so $a = b \bmod \operatorname{nil}(B)$. Thus $A \to B/\operatorname{nil}(B)$ is integral. So for any $b \in B$, we can find monic polynomial p with $f(A)$ coefficients such that $p(b)$ is nilpotent. Take power and get $p^n(b) = 0$ so $A \to B$ is integral.

VI Chapter 6

VI.1 x

(i) Consider submodules $\ker(u^n)$ increasing chain of ideals, which stabilizes since M is noetherian. Thus we must have $\ker u = 0$, for otherwise $\ker u^{n+1} = (u^n)^{-1}(\ker u)$ would strictly contain $\ker u^n$ and keep getting bigger. So u is isomorphism.

(ii) Consider image $\operatorname{im}(u^n)$ which keeps getting smaller because u is injective, unless it is all of M. So u is isomorphism because M is artinian.

VI.2 x

Suppose M' is a submodule of M, then consider the set of finitely generated submodules of M'. It has a maximal element N. If $N \neq M'$, then we can add in some element of $M' - N$ and it is still finitely generated, contradicting maximality. So $N = M'$ is finitely generated and M is Noetherian by (6.2).

VI.3 x

Consider
$$0 \to N_2/(N_2 \cap N_1) \to M/(N_2 \cap N_1) \to M/N_2 \to 0$$
from (2.1(i)). From (2.1(ii)), $N_2/(N_2 \cap N_1) = (N_1 + N_2)/N_1$ is a submodule of M/N_1, which is Noetherian. By (6.3), $M/(N_1 \cap N_2)$ is Noetherian (Artinian by same argument).

VI.4 x

Since M noetherian it is finitely generated by $m_1, ..., m_n$. Let $A \to M^n$ by $a \mapsto (am_1, ..., am_n)$, which induces an injection $A/\mathfrak{a} \to M_n$. Therefore $A/\mathfrak{a}$ is a Noetherian A-module, hence a Noetherian $A/\mathfrak{a}$-module.

If we replace Noetherian by Artinian, consider $\mathbb{Q}/\mathbb{Z}$ which is Artinian, faithful $\mathbb{Z}$-module, but $\mathbb{Z}$ itself is not Artinian.

VI.5 x

Chains of open sets in subspace are induced by chains of open sets in X, so any subspace with induced topology is Noetherian.

Given an open cover, consider the set of finite unions, which has a maximal element, which must be all of X (otherwise we can add something from the cover and contradict maximality). Thus we get a finite subcover and X is quasi-compact.

VI.6 x

(i)⇒(ii) By VI.5.

(ii)⇒(iii) Given an open cover U of subspace Y, U is a cover for the open subspace $\cup U$, so has finite subcover, which is a subcover for Y.

(iii)⇒(i) Given a chain of open subsets, they form a cover for their union, so there is a finite subcover, meaning the chain will stabilize after finitely many steps.

VI.7 x

Consider the set of closed subsets of X which are not finite union of irreducibles, if non-empty then it has minimal element. It is not irreducible, so it is union of two closed subsets, which by minimality are finite unions of irreducibles, a contradiction. Hence X is a finite union of irreducible subspaces and therefore only finitely many irreducible components exist.

VI.8 x

Let $D(\mathfrak{a}) = \operatorname{Spec} A - V(\mathfrak{a})$ be open subspace. Since A noetherian, $\mathfrak{a} = (a_1, ..., a_n)$ finitely generated. Now $D(\mathfrak{a})$ is covered by finitely many $D(a_i)$. Since $D(a_i) \cong \operatorname{Spec} A_{a_i}$, it is quasi-compact, so $D(\mathfrak{a})$ is quasi-compact. and by VI.6(ii), $\operatorname{Spec} A$ is Noetherian.

Let $A = k[x_1, ..., x_n]/(x_1^2, ..., x_n^2)$, then reduced ring is A, $\operatorname{Spec} A = \operatorname{Spec} A_{\text{red}}$ is singleton, so noetherian.

VI.9 x

Given ideal $\mathfrak{a}$, if it is not prime, then let $\mathfrak{p}_0, \mathfrak{p}_\alpha$ be minimal primes over $\mathfrak{a}$, then $V(\mathfrak{p}_0)$ is a proper closed subset because it does not contain $\mathfrak{p}_\alpha$ by minimality. $V(\cap_\alpha \mathfrak{p}_\alpha)$ is proper closed because it does not contain $\mathfrak{p}_0$ by (1.11(ii)). Hence $V(\mathfrak{a}) = V(\mathfrak{p}_0) \cup V(\cap_\mathfrak{a} \mathfrak{p}_\alpha)$ is reducible. So the irreducible components, which are maximal irreducible sets, must correspond to minimal prime ideals, and there are only finitely many of them.

VI.10 x

By III.19(v), $\operatorname{Supp}(M) = V(\operatorname{Ann}(M)) \cong \operatorname{Spec} A/\operatorname{Ann}(M)$. So by VI.4, it is a closed noetherian subspace.

VI.11 x

From V.10, only need to show one direction.

Suppose f has going-up property. Let $C = f(A)$ subring of B, $i : C \to B$ inclusion, $f : A \to C$ surjection. Then f^* is closed by I.21(vi). So it suffices to show i is closed. Thus we simply assume $f : A \to B$ is injective.

Let $V(\mathfrak{b})$ be a closed set of $\operatorname{Spec} B$. Since $\operatorname{Spec} B$ is noetherian, there are only finitely many minimal primes $\mathfrak{q}_i$ over $\mathfrak{b}$. Let $\mathfrak{a} = \cap f^{-1}(\mathfrak{q}_i)$. Then note that $f^*(V(\mathfrak{b})) \subseteq V(\mathfrak{a})$ because any prime in $V(\mathfrak{b})$ must contain some $\mathfrak{q}_i$. Suppose $\mathfrak{p}$ contains $\cap f^{-1}(\mathfrak{q}_i)$, then by (1.11) it contains some $\mathfrak{p}_i = f^{-1}(\mathfrak{q}_i)$, so by going-up property, there is some $\mathfrak{q} \supseteq \mathfrak{q}_i$ such that $f^{-1}(\mathfrak{q}) = f^{-1}(\mathfrak{q} \cap f(A)) = f^{-1}(f(\mathfrak{p})) = \mathfrak{p}$ (since f is injective). Therefore $\mathfrak{p} \in f^*(V(\mathfrak{b}))$ and hence $f^*(V(\mathfrak{b})) = V(\mathfrak{a})$ is closed.

Noetherian is not necessary? see Hartshrone solution II.3.5

VI.12 x

Since distinct prime ideals correspond to distinct closed subspaces, an ascending chain of primes correspond to a descending chain of closed subspaces, so ascending chain condition is satisfied for primes when $\operatorname{Spec} A$ is noetherian.

Converse is not true. Consider $k[x_1, x_2, ...]/(x_i x_j : i \neq j)$. Then all (non-zero) prime ideals must contain all x_i except some x_j. In particular, the minimal (non-zero) primes are $(x_i : i \neq j)$. Say $\mathfrak{p}$ contains $(x_2, x_3, ...)$, and it also contains some polynomial that involves x_1, then by subtracting the terms with $x_2, \ldots$ in them, we get a polynomial in x_1 which must be linear because $\mathfrak{p}$ is prime and k is algebraically closed. Thus maximal ideals are of form $(x_i - a, x_j : j \neq i)$. Hence any chain of prime can only have 3 terms at most. However, $\operatorname{Spec} A$ is not noetherian because of the chain $V(x_1) \subset V(x_1, x_2) \subset \ldots$.

VII Chapter 7

VII.1 x

Find maximal element using Zorn's lemma. Let $\mathfrak{a}$ be maximal, say $xy \in \mathfrak{a}$ and $x, y \notin \mathfrak{a}$, then $(x)+\mathfrak{a}, (y)+\mathfrak{a}$ are finitely generated. Say $(x)+\mathfrak{a}$ generated by $a_i x + b_i$ where $a_i \in A, b_i \in \mathfrak{a}$. Let $\mathfrak{a}_0 = (b_1, \ldots, b_n) \subseteq \mathfrak{a}$, then $\mathfrak{a}_0 + (x) = \mathfrak{a} + (x)$. Let $ax + b \in (x) + \mathfrak{a}$, for any $b \in \mathfrak{a}$, then $ax + b = cx + d \in (x) + \mathfrak{a}_0$, so $b = (c-a)x + d$ where $d \in \mathfrak{a}_0$ and $(c-a)$ must be in $(\mathfrak{a} : x)$. Conversely if $a \in (\mathfrak{a} : x)$ and $b \in \mathfrak{a}_0$, then $ax + b \in \mathfrak{a}$. Thus $\mathfrak{a} = x(\mathfrak{a} : x) + \mathfrak{a}_0$, is finitely generated, a contradiction. Thus x or $y \in \mathfrak{a}$ and $\mathfrak{a}$ is prime.

VII.2 x

By (7.15) Nilradical is nilpotent, so if each a_n is nilpotent, then there is some m such that any product of m of the a_n would be 0, and $f^m = 0$.

Suppose $f^m = 0$, we shall show that $a_n \in \mathfrak{p}$ for all prime $\mathfrak{p} \subseteq A$. Pass to $A/\mathfrak{p}$ and we can assume A is a domain and show $a_n = 0$ for all n. Note that $a_0^m = 0$ so $a_0 = 0$ (compare constant term of $f^m = 0$). Then $a_1^m + a_0(\ldots) = 0$ (compare terms of degree m), so $a_1 = 0$. An induction shows $a_n = 0$ for all n.

VII.3 x

(i)$\Rightarrow$(ii) Suppose $\mathfrak{a}$ is primary. If $a \in \mathfrak{a} \cap S \neq \phi$, then $(S^{-1}\mathfrak{a})^c = A = (\mathfrak{a} : a)$. Say $\mathfrak{a} \cap S = \phi$, then by (4.8) $(S^{-1}\mathfrak{a})^c = \mathfrak{a}$. By (3.11) $(S^{-1}\mathfrak{a})^c = \cup_{s \in S}(\mathfrak{a} : s)$. Therefore for all $s \in S$, we have $\mathfrak{a} \subseteq (\mathfrak{a} : s) \subseteq \mathfrak{a}$

(ii)$\Rightarrow$(iii) If $x^n \in \mathfrak{a}$ for some n, then $(\mathfrak{a} : x^n) = A$ is stationary. Suppose $x^n \notin \mathfrak{a}$ for all n, let $S = \{x^n\}$, then $(S^{-1}\mathfrak{a})^c = (\mathfrak{a} : x^n)$ for some n by assumption. Now by (3.11), $(S^{-1}\mathfrak{a})^c = (\mathfrak{a} : x^n) \subseteq (\mathfrak{a} : x^{n+1}) \subseteq (S^{-1}\mathfrak{a})^c$, so $(\mathfrak{a} : x^n)$ is stationary.

(iii)$\Rightarrow$(i) Suppose $xy \in \mathfrak{a}$, and say $(\mathfrak{a} : x^n) = (\mathfrak{a} : x^{n+1})$, then $(x^n) \cap (y) = \mathfrak{a}$ because if $a \in (y)$ then $ax \in \mathfrak{a}$, and if $a = bx^n$, then $ax = bx^{n+1} \in \mathfrak{a}$ and $b \in (\mathfrak{a} : x^{n+1}) = (\mathfrak{a} : x^n)$ so $a = bx^n \in \mathfrak{a}$. Since $\mathfrak{a}$ irreducible, $\mathfrak{a} = (x^n)$ or $\mathfrak{a} = (y)$. So $\mathfrak{a}$ primary.

Alternative: pass to $A/\mathfrak{a}$ and assume $\mathfrak{a} = 0$, and use the same proof from (7.12).

VII.4 x

(i) Let S be polynomials with no roots on $|z| = 1$, multiplicatively closed. Then the ring of rational functions with no poles on $|z| = 1$ is $S^{-1}\mathbb{C}[x]$ is Noetherian by (7.5) and (7.3).

(ii) Consdier I_n the ideal of power series whose radius of convergence is at least n. This is an ideal because radius of convergence of products can only increase

(iii) First observe this is a local ring (local at 0) with maximal ideal (x) because everything with non-zero constant term has inverse (which is a power series of the inverse of the holomorphic function represented by the original power series, on a disk around 0). Now since everything is a unit away from some x^n, we know all the ideals are of form (x^n), so this is a discrete valuation ring (thus a PID), so noetherian.

(iv) Consider the ideal of holomorphic functions with at least one zero inside disk of radius n. This gives a chain of increasing ideals so not Noetherian.

(v) Such polynomials is of form $zf(z,w)+a$ where $a \in \mathbb{C}$. So this is the ring $k[z, zw, zw^2, \ldots]$. Now consider the chain of ideals $(z), (z, zw), (z, zw, zw^2), \ldots$. This is strictly increasing because $z(zf(z,w)+a) = zw$, then $zf(z,w)+a = w$, a contradiction, so $(z) \subsetneq (z, zw)$ (similar for general case using divisibility of w). So not noetherian.

VII.5 x

We have $A \subseteq B^G \subseteq B$ and B is integral over B^G by V.12, so by (7.8) it is finitely generated A-algebra.

VII.6 x

If K has characteristic 0, then $\mathbb{Z} \subseteq \mathbb{Q} \subseteq K$ and by (7.9), it is finitely generated $\mathbb{Q}$-module, so by (7.8), $\mathbb{Q}$ is finitely generated over $\mathbb{Z}$, which is not true. So K has characteristic p and is a finitely generated $\mathbb{Z}/(p)$-algebra, so by (7.9) it is finite over $\mathbb{Z}/(p)$, so a finite field.

VII.7 x

Since $k[x_1, ..., x_n]$ Noetherian, the ideal generated by $f_\mathfrak{a}$ is finitely genertaed, so take those finitely many generators and get the same variety.

VII.8 x

Yes, because every ideal $\mathfrak{a}$ of A is a contraction of ideal $\mathfrak{a}[x]$ in $A[x]$ by IV.7(i). (or just $A = A[x]/(x)$).

VII.9 x

Let $\mathfrak{a}$ be an ideal of A, then $S^{-1}\mathfrak{a}$ is finitely generated for all maximal ideals $\mathfrak{m}$ where $S = A - \mathfrak{m}$. Pick $x_0 \in \mathfrak{a}$, then a maximal ideal either contains $\mathfrak{a}$, call those $m_1, ..., m_r$ (finite by assumption (2)), or it contains x_0 but not some $y_j \in \mathfrak{a}$, call those $m_j (1 \leq j \leq s)$, or it does not contain x_0. If $\mathfrak{m}$ contains $\mathfrak{a}$, say those $S^{-1}\mathfrak{a}$ are generated by the images of $x_1, ..., x_R$ (by assumption (1)). Then we note the image of $(x_0, x_1, ..., x_R, y_1, ..., y_s)$ generate $S^{-1}\mathfrak{a}$ for any $S = A - \mathfrak{m}$. Therefore $(Ax_0 + ... + Ay_s)_\mathfrak{m} \to \mathfrak{a}_\mathfrak{m}$ is a surjection for all $\mathfrak{m}$, so it gives surjection $(x_0, ..., y_s) \to \mathfrak{a}$ and $\mathfrak{a}$ is finitely generated.

VII.10 x

By II.6 $M[x] = A[x] \otimes_A M$, say M is generated by $m_1, ..., m_n$, then we have $M[x]$ is generated by $1 \otimes m_1, ..., 1 \otimes m_n$ over $A[x]$, so Noetherian by (7.5) and (6.5).

VII.11 x

Consider $R[x_1, x_2, ...]$ where $R = \mathbb{Z}_6$. Now a prime ideal either contains 2 or 3 but not both, so localizing would make either 2 or 3 a unit, but since they are zero divisors, the localizaiton would just be 0. Thus this is not noetherian but localization at primes are.

VII.12 x

A is noetherian because every ideal $\mathfrak{a}$ of A is a contraction of ideal $\mathfrak{a}^e$ in B by III.16. (use ascending chain condition).

VII.13 x

Fibers of f^* are of form $\mathrm{Spec}(k(\mathfrak{p}) \otimes_A B)$. Note that B is finite dimensional $k(\mathfrak{p})$-vector space because if B is generated by $b_1, ..., b_n$ over A, then $\sum c_i \otimes a_i b_i = \sum \overline{a_i c_i}(1 \otimes b_i)$ where $\overline{a_i c_i}$ is its image in $k(\mathfrak{p})$, so genertaed by $1 \otimes b_i$. Therefore $k(\mathfrak{p}) \otimes_A B$ is Noetherian by (7.7) and the spectrum is a Noetherian space by VI.8.

VII.14 x

Let $f \notin r(\mathfrak{a})$. Then there is a prime ideal containing $\mathfrak{a}$ with $f \notin \mathfrak{p}$. Let $B = A/\mathfrak{p}$ and $C = B_f$, and $\mathfrak{m}$ maximal ideal of C. Since C is finitely generated k-algebra, $C/\mathfrak{m} \cong k$ by weak Nullstellensatz. Let a_i be the images of x_i, then we get a point $\alpha = (a_1, ..., a_n)$. Now $f(\alpha) \neq 0$ because $f \notin \mathfrak{m}$. On the other hand, if $g \in \mathfrak{a}$, then $g \in \mathfrak{p}$ and $g \in \mathfrak{m}$ so $g(\alpha) = 0$, so $\alpha \in V(\mathfrak{a})$ and therefore $f \notin I(V(\mathfrak{a}))$.

VII.15 x

(i)⇒(ii) If M is free, then it is flat by II.4.

(ii)⇒(iii) Since $\mathfrak{m} \to A$ injective.

(iii)⇒(iv) By the exact sequence $\mathrm{Tor}_1(k, M) \to \mathfrak{m} \otimes M \hookrightarrow A \otimes M \to k \otimes M$.

(iv)⇒(i) Let $x_1, ..., x_n$ be a basis of $M/\mathfrak{m}M$, then they generate M by (2.8). Let $F = A^n$ with $\varphi : F \to M$ with kernel E. So we get

$$0 = \mathrm{Tor}_1(k, M) \to E \otimes k \to F \otimes k \to M \otimes k \to 0$$

Since $F \otimes k$ and $M \otimes k$ have the same dimension, $E \otimes k = E/\mathfrak{m}E = 0$, so $E = 0$ by Nakayama (since $\mathfrak{m}$ is inside Jacobson as A is local), E is finitely generated because it is submodule of F and A Noetherian.

VII.16 x

Follows from VII.15 and (3.10).

VII.17 x

Define "irreudcible submodule" to be those that are not intersection of two strictly bigger submodules. Then proof of (7.11) works.

For (7.12), suppose Q is irreducible, and x is a zero-divisor of M/Q, namely $xm = 0 \in M/Q$ for some $m \neq 0 \in M/Q$. Then consider the chain of submodules $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(x^2) \subseteq ...$ which is stationary since M/Q is noetherian. So $x^n(M/Q) \cap Am = 0$ in M/Q because if $am \in Am$ then $axm = 0$ and if $am = x^n m'$ then $x^{n+1}m' = 0$ so $x^n m' = 0$ and $am = 0$. Hence $(x^n M + Q) \cap (Am + Q) = 0$ and since $m \notin Q$, $x^n m \in Q$ for all m, meaning x is nilpotent over M/Q, so Q is primary.

VII.18 x

(i)⇒(ii) By definition, $\mathfrak{p} = r_M(Q_i) = r(Q_i : M)$. Since A is noetherian, $\mathfrak{p}^m \subseteq (Q_i : M)$ for some n and $A \cap \mathfrak{p}^m M = \cap_{j \neq i} Q_j \cap \mathfrak{p}^m M \subseteq \cap Q_i = 0$. Let $m \geq 1$ to be the smallest integer such that $A \cap \mathfrak{p}^m M = 0$, let x be non-zero in $\mathfrak{a}_i \mathfrak{p}^{m-1}$, then $\mathfrak{p}_i x = 0$, so $\mathrm{Ann}(x) \supseteq \mathfrak{p}_i$. By IV.22, $\mathfrak{p} = r(\mathrm{Ann}(x))$ for any $x \in A$, so we have $\mathfrak{p} = \mathrm{Ann}(x)$.

(ii)⇒(iii) Take Ax.

(iii)⇒(i) $r(0 : x) = \mathfrak{p}$ so belongs to 0 by IV.22.

Given M_i, consider M/M_i finitely generated and let $\mathfrak{p}$ belong to 0 in M/M_i, then find a submodule M_{i+1}/M_i isomorphic to $A/\mathfrak{p}$ giving us M_{i+1}.

VII.19 x

Take quoteitn and we can assume $\mathfrak{a} = 0$ then A injects into $A/\mathfrak{b}_1 \times ... \times A/\mathfrak{b}_r$. Let $I_j = \mathfrak{b}_1 \cap ... \cap \mathfrak{c}_j \cap ... \cap \mathfrak{b}_r$, its image in A/b_k is 0 for $k \neq j$, so I_j injects into $0 \times ... A/\mathfrak{b}_i \times ... 0 = A/\mathfrak{b}_i$. Since $\cap I_j = 0$, the intersection of their image is 0 in A/β_i (image are ideals since we can multiply by elements of $A/\mathfrak{b}_i$ using elements of A). Since $\mathfrak{b}_i$ is irreducible, one of I_j has image 0, so $\mathfrak{a} \subseteq I_j \subseteq \mathfrak{b}_i = I_j \cap \mathfrak{b}_i \subseteq \mathfrak{a}$, meaning $I_j = \mathfrak{a}$. Now repeat and replace each b_i by c_j, and get $r = s$.

Since irreducible ideals are primary, by primary decomposition $r(\mathfrak{b}_i) = \mathfrak{p}_i$ are fixed prime ideals. Say $r(\mathfrak{b}_1) = ... = r(\mathfrak{b}_n) = r(\mathfrak{c}_1) = ... = r(\mathfrak{c}_m) = \mathfrak{p}$ where $\mathfrak{p}$ is a minimal prime, then by (4.11), $\cap_{i=1}^n \mathfrak{b}_i = \cap_{i=1}^m \mathfrak{c}_i = \mathfrak{q}$ is uniquely determined, so by the result, $m = n$.

Now assume we have dealt with all minimal primes, we go up one step. Suppose $\mathfrak{p}'$ a prime ideal belonging to $\mathfrak{a}$ that only contain minimal primes $\mathfrak{p}_1, ...\mathfrak{p}_k$, and $r(\mathfrak{b}_n) = ... = r(\mathfrak{b}_{n'}) = r(\mathfrak{c}_n) = ... = r(\mathfrak{c}_{m'}) = \mathfrak{p}'$ with $r(\mathfrak{b}_i), r(\mathfrak{c}_i) \in \{\mathfrak{p}_1, ..., \mathfrak{p}_k\}$. Then $\cap_{i=1}^{n'} \mathfrak{b}_i = \cap_{i=1}^{m'} \mathfrak{c}_i = \mathfrak{q}_1 \cap ... \cap \mathfrak{q}_k \cap \mathfrak{q}'$. Then since $\{\mathfrak{p}', \mathfrak{p}_1, ..., \mathfrak{p}_k\}$ is an isolated set of prime ideals, by (4.10), it is uniquely determined in the primary decomposition. Therefore $n' = m'$ and so $n' - n = m' - n$ and the number of ideals $\mathfrak{b}_i$ and $\mathfrak{c}_i$ with radical $\mathfrak{p}'$ are the same. Continue with induction (on the size of maximal chains of primes belonging to $\mathfrak{a}$ we can form, beginning form $\mathfrak{p}'$ going down).

VII.20 x

(i) If direction follows from definition. Now we need to show the set of finite union of sets of form $U \cap C$ is closed under finite union (by definition) and complements. This is because $X - U \cap C = (X \cap (X - U)) \cup ((X - C) \cap X)$ is of the required form.

(ii) If E contains non-empty open set, then since X is irreducible (so open sets are dense), it is dense.

Suppose E is dense, say $E = \cup(U_i \cap C_i)$. Since closure of union is union of closure, $X = \cup\overline{U_i \cap C_i}$, which means $\overline{U_i \cap C_i} = X$ for some i for X is irreducible. Therefore $C_i = X$ and $U_i \subseteq E$ is non-empty open set.

VII.21 constructible sets

If $E \in \mathscr{F}$, then intersect with X_0 and apply VII.20(ii).

Suppose $E \notin \mathscr{F}$. Consider the set of closed subsets X' such that $E \cap X' \notin \mathscr{F}$ (non-empty because it contains X). Since X noetherian there is minimal X_0. If $X_0 = Y \cup Z$ is reducible, then $E \cap X_0 = (E \cap Y) \cup (E \cap Z)$ is constructible by minimality, a contradiction, so X_0 is irreducible.

In the case $C := \overline{E \cap X_0} \neq X_0$, we have $E \cap X_0 \subseteq C$, so $E \cap X_0 \subseteq E \cap C \subseteq E \cap X_0$, but then $E \cap X_0 = E \cap C \in \mathscr{F}$ by minimality of X_0, a contradiction.

In the case $E \cap X_0$ contains a non-empty open subset U of X_0, if $U = X_0$, then $E \cap X_0 = X_0$ is constructible. If $U \neq X_0$, then $E \cap (X_0 - U)$ is constructible by minimality of X_0, which means $E \cap X_0 = U \cup (E \cap (X_0 - U))$ is constructible, a contradiction.

VII.22 x

One direction is by definition. Suppose E is not open. Consider the set of X' such that $E \cap X'$ is not open. Similar as before the mimimal element X_0 is irreducible. Since $E \cap X_0$ is not open, it is non-empty. Suppose it contains open set U of X_0, then $E = U \cup (E \cap (X_0 - U))$ is open, a contradiction.

VII.23 x

It suffices to show $E = U \cap C$ subsets are mapped to constructible sets. Since C closed is of form $\mathrm{Spec}(B/\mathfrak{a})$, induced by $A \to B \to B/\mathfrak{a}$, we can assume $E = U$ is open. Since B Noetherian, open sets are quasi-compact covered by finitely many $D(g) = \mathrm{Spec}(B_g)$ so it suffices to show $f^*(Y)$ is constructible.

Let X_0 be an irreducible subset of X and $f^*(Y) \cap X_0$ dense in X_0. We have $f^*(Y) \cap X_0 = f^*((f^*)^{-1}(X_0))$. From III.21(iii), we know $(f^*)^{-1}(X_0) = \mathrm{Spec}(B/\mathfrak{p}B) = \mathrm{Spec}(A/\mathfrak{p} \otimes_A B)$ where $X_0 = V(\mathfrak{p}) = \mathrm{Spec}\, A/\mathfrak{p}$. Since X_0 irredcuible, $\mathrm{nil}(A/\mathfrak{p})$, which corresponds to $\sqrt{\mathfrak{p}}$ is prime, so we can replace $\mathfrak{p}$ by $\sqrt{\mathfrak{p}}$ and assume it is prime. Now we can replace B with $A/\mathfrak{p} \otimes B$ and A with $A/\mathfrak{p}$ an integral domain and $f : A/\mathfrak{p} \to B/\mathfrak{p}^e$.

If f is not injective, then $f^*(Y)$ would be contained in the closed set $V(\ker f)$ which does not contain the 0 ideal which is prime because A is integral domain, so f must be injective because $f^*(Y)$ is dense. It suffices to show the image of one of irreducible subsets of Y contains a non-empty open subset of X, so we can assume $\mathrm{Spec}\, B$ is irreducible, and similar as before we can assume B an integral domain. Now we have to show if $f : A \to B$ injective of finite type of Noetherian integral domains, then $f^*(Y)$ contains a non-empty open set of X.

Apply V.21 to A and find $s \neq 0$ in A. Suppose $\mathfrak{p}$ does not contains s. Consider $\varphi : A \to A/\mathfrak{p} \to k(\mathfrak{p})$ with kernel $\mathfrak{p}$ and the image of s is not zero, so it extends to some $\psi : B \to \Omega$ where Ω is algebraic closure of $k(\mathfrak{p})$. Now $\mathfrak{q} := \ker \psi$ is a prime ideal because the image is in an integral domain, and $f^*(\mathfrak{q}) = \ker \psi \cap A = \ker \varphi = \mathfrak{p}$. Hence $f^*(Y)$ contains open set $D(s)$ which is non-empty because the intersection of all primes is the nilpotent which is 0, and $s \neq 0$. So we are done.

VII.24 x

Suppose f has going-down property. Since open sets are union of $\mathrm{Spec}(B_g)$, it suffices to prove $E = f^*(Y)$ is open. Going-down property means if $\mathfrak{p}' \subseteq \mathfrak{p} \in E$, then $\mathfrak{p}' \in E$. If $E \cap X_0 \neq \phi$ for some irreducible closed subset $X_0 \subseteq X$, say they meet at $\mathfrak{p}$ and $X_0 = V(\mathfrak{p}')$, then $\mathfrak{p}' \in E$ so E is dense in X_0. By VII.23, E is constructible, so by VII.20(ii), $E \cap X_0$ contaisn a non-empty open set in X_0, and by VII.22, this means E is open.

VII.25 x

Follows from V.11 and VII.24.

VII.26 grothendieck groups

(i) Because additive function maps elements of D to 0 by definition.

(ii) Given $M \in F(A)$, get chain

$$0 = M_0 \subseteq M_1 \subseteq ... \subseteq M_r = M$$

where $M_i/M_{i-1} = A/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i$. So we have chains $0 \to M_{i-1} \to M_i \to A/\mathfrak{p}_i \to 0$ and $\gamma(A\mathfrak{p}_i) = \gamma(M_i) - \gamma(M_{i-1})$. So $\sum \gamma(A/\mathfrak{p}_i) = \gamma(M) - \gamma(0) = \gamma(M)$ and $K(A)$ is generated by elements of form $\gamma(A/\mathfrak{p})$.

(iii) If A PID, then any non-zero ideal of A is isomorphic to A as A-module, so $\gamma(A/\mathfrak{p}) = \gamma(A) - \gamma(\mathfrak{p}) = 0$. Therefore $K(A)$ is generated by $\gamma(A)$. Note that $\gamma(A^n) = n\gamma(A)$,

Observe that $(M \oplus N)/(A \oplus B) = (M/A) \oplus (N/B)$, so any element $\sum (M_i') - (M_i) + (M_i'')$ in C can be represented by some short exact sequence $0 \to \oplus M_i' \to \oplus M_i \to \oplus M_i'' \to 0$, so

$$\sum (M_i') - (M_i) + (M_i'') = (\oplus M_i') - (\oplus M_i) + (\oplus M_i'')$$

So elements of C all correspond to short exact sequences.

Suppose $n(N) \in C$ for some $n \in \mathbb{Z}^+$, then $n(N) = (M') - (M) + (M'')$ which means $(M'), (M), (M'')$ are all multiples of (N), so we would have $M' = N^{m_1}, M = N^{m_2}, M'' = N^{m_3}$ and giving a short exact sequence

$$0 \to N^{m_1} \to N^{m_2} \to N^{m_3} \to 0$$

making $m_1 + m_3 = m_2$ (look at image of standard bases), and therefore $n(N) = 0$ making $(N) = (0)$. Hence $\gamma(A^n) = n\gamma(A) \neq 0$ for all n, so $K(A) \cong \mathbb{Z}$.

(iv) Let λ map (M) to $\gamma(M_A)$ by restriction of scalars. Suppose $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of B-modules, then it is a short exact sequence when viewed as A-modules, so $\lambda(M' - M + M'')$ is 0 in $K(A)$. Hence by (i) defines $f_! : K(B) \to K(A)$ with $f_!(\gamma(M)) = \lambda(M) = \gamma(M)$. $(g \circ f)_! = f_! \circ g_!$ because restricting from C to A is the same as restricting first from C to B then from B to A.

VII.27 x

(i) Use the fact $\gamma(M) + \gamma(N) = \gamma(M \oplus N)$ and that tensor distributes over sum, and that tensoring with flat modules perserve short exact sequences.

(ii) Similar to (i).

(iii) If A is Noetherian local then flat implies free by VII.15, so everything is generated by $\gamma_1(A)$, and similar as before, this means $K_1(A) = \mathbb{Z}$.Noetherian is not necessary by Matsumura?

(iv) Similar to 26(iv). If M is flat and finitely generated over A, then by (2.20) $B \otimes_A M$ is flat and finitely generated over B. $(g \circ f)^! = g^! \circ f^!$ because extending to C is the same as extending to B then to C by (2.15).

(v) Can assume $x = \gamma_1(M), y = \gamma(N)$ because $f_!$, $f^!$ are homomorphisms. Then

$$f_!(f^!(x)y) = f_!(\gamma_B((B \otimes_A M) \otimes_B N)) = \gamma_A((B \otimes_A M) \otimes_B N)$$

$$= \gamma_A(M \otimes_A B \otimes_B N) = \gamma_A(M \otimes_A N) = \gamma_1(M)\gamma_A(N) = xf_!(y)$$

VIII Chapter 8

VIII.1 x

Suppose $\mathfrak{q}$ is $\mathfrak{p}$-primary, so $\sqrt{\mathfrak{q}} = \mathfrak{p}$. Since A is Noetherian, $\mathfrak{p}^n \subseteq \mathfrak{q}$ for some n. Now suppose $S_\mathfrak{p}(\mathfrak{p}^n)$ contains $x = p/a \in A$ where $p \in \mathfrak{p}^n$, $a \notin \mathfrak{p}$. Suppose $x \notin \mathfrak{q}$, then since $xa \in \mathfrak{q}$, $a^m \in \mathfrak{q}$ for some m, but this means $a \in \sqrt{\mathfrak{q}} = \mathfrak{q}$, a contradiction. Therefore $x \in \mathfrak{q}$ and so $S_\mathfrak{p}(\mathfrak{p}^n) \subseteq \mathfrak{q}$.

If $\mathfrak{q}_i$ is isolated primary component, then $\mathfrak{p}_i$ is a minimal prime ideal, so $A_{\mathfrak{p}_i}$ is Noetherian and with dimension 0, so it is Artinian by (8.5). Thus $(\mathfrak{p}_i A_{\mathfrak{p}_i})^r = 0$ for large r. By (4.9) we have $S^{-1}(0) = S^{-1}\mathfrak{q}_i$, so contraction (using correspondence of primary ideals) gives us $\mathfrak{q}_i = \mathfrak{p}_i^{(r)}$.

If $\mathfrak{q}_i$ is embedded primary component then $A_{\mathfrak{p}_i}$ is not Artinian because it has prime ideals that are not maximal. So $(\mathfrak{p}_i A_{\mathfrak{p}_i})^r$ are all distinct and so $\mathfrak{p}_i^{(r)}$ are all distinct because of correspondence of primary ideals ($\mathfrak{p}_i^{(r)}$ are $\mathfrak{p}_i$-primary by VI.13(i). Hence for large r we can replace $\mathfrak{q}_i$ by $\mathfrak{p}_i^{(r)}$ and get infinitely many distinct decompositions of 0 which differ in the $\mathfrak{p}_i$ component only.

VIII.2 x

(i)⇒(ii) If A is Artinian, then by (8.1), every point of $\operatorname{Spec} A$ is closed. By (8.3), there are only finitely many points, so every subset is closed (and therefore open) and $\operatorname{Spec} A$ is discrete.

(ii)⇒(iii) Clear.

(iii)⇒(i) By (8.5) it suffices to show $\dim A = 0$. Suppose A has a prime ideal that is not maximal. Say $\mathfrak{p} \subseteq \mathfrak{m}$, then $\{\mathfrak{m}\}$ is not open because $\operatorname{Spec} A - \mathfrak{m}$ contains $\mathfrak{p}$ and if it were closed, then it must contain $\mathfrak{m}$, so $\operatorname{Spec} A$ is not discrete.

VIII.3 x

(i)⇒(ii) By (8.7), we can assume A is local. By Nullstelensatz, $A/\mathfrak{m}$ is a finite extension of k. Let A be generated by $x_1, ..., x_n$ over k. Then the image of x_i in $A/\mathfrak{m}$ satisfies some polynomial in $k[x]$, say $f_i(x_i) \in \mathfrak{m}$, then since $\mathfrak{m}^n = 0$ for some n by (8.6), $f_i^n(x_i) = 0$. Now observe that A is generated by $\{x_i^{m_j} : 0 \le m_j \le n \deg f_i\}$.

(ii)⇒(i) Ideals of A are sub-k-spaces, so satisfy descending chain condition.

VIII.4 x

(i)⇒(ii) Suppose f is finite. The fibers are of form $\operatorname{Spec}(k(\mathfrak{p}) \otimes_A B)$. Since $k(\mathfrak{p}) \otimes_A B$ is a finite $k(\mathfrak{p})$-algebra by (2.17). By VIII.3, it is Artinian, so by VIII.2, the spectrum is discrete.

(ii)⇔>(iii) By VIII.2,3, spectrum discrete iff $B \otimes_A k(\mathfrak{p})$ is Artinian iff a finite $k(\mathfrak{p})$-algebra (it is finitely generated $k(\mathfrak{p})$-algebra because B finite type).

(iii)⇒(iv) Again use VIII.2,3.

Integral + finite fiber does not imply finite. Consider $\mathbb{Q} \to \overline{\mathbb{Q}}$, the spectrum map is just single point map, but $\overline{\mathbb{Q}}$ is not finitely generated over $\mathbb{Q}$.

VIII.5 x

The map is induced by $A = k[y_1, ..., y_r] \to B = k[x_1, ..., x_n]$ where B is integral over A. Also note that this is finite type, so by Remark (5.3) the map is finite, and therefore has finite fibers by VIII.4. This number can not be bigger than the number of maximal ideals of $B \otimes_A k(\mathfrak{p})$. Since ideals are $k(\mathfrak{p})$-subspaces, it can not exceed its dimension. This dimension is bounded by the sum of degrees of the generators of B over A (for example if $f(b) = 0$ with A-coefficients, then passing to $k(\mathfrak{p})$ we have $f(b \otimes 1) = 0$).

VIII.6 x

Quotient by $\mathfrak{q}$ and assume $\mathfrak{q} = 0$ and $\mathfrak{p}$ is nilradical (correspondence theorem for primary ideals). Localize at $\mathfrak{p}$ and assume $\mathfrak{p}$ maximal. Note that $\mathfrak{p}$ is minimal since it is nilradical, so by (8.5) A is Artin. Thus A satisies a.c.c and d.c.c, so chains have bounded length. Since $\mathfrak{p}$ is nilpotent, it is in fact the radical of any ideal of A, so by (4.2) every ideal is primary. Thus maximal chains of primary ideals is simply composition series, which have the same length by (6.7).