MATH 127 Calculus for the Sciences

Lecture 11

Today's lecture

Last time
Exponential and logarithmic functions

- Real life application of exp.
- Simplify things using exp and log rules.
- Derivative of exp and log function.

This time
Course note coverage

- Integrate $\frac{1}{x}$. (Section 2.4.4)
- Riemann sum approximation of ln. (part of Section 2.4.3: this could be on Quiz 4)
- Logarithmic scale (real life application of log). (section 2.4.6)

Quiz 4 is next Wednesday!

Coverage:

1. Topics that were on the coverage of Quiz 2-3 might show up in Quiz 4.
2. Integral functions.
3. Inverse function.
4. Exp and log functions.
5. More on log functions (will cover today).

Derivatives of exp/log functions

Example What is the derivative of $\log_{2x} 3^x$? What is its domain?

$$\log_{2x} 3^x = \frac{\ln 3^x}{\ln 2x} = \boxed{\frac{x \ln 3}{\ln 2x}}$$

$$(\log_{2x} 3^x)' = \left(\frac{x \ln 3}{\ln 2x}\right)' = \left(x \ln 3 \cdot \frac{1}{\ln 2x}\right)'$$

$$= \ln 3 \cdot \frac{1}{\ln 2x} + x \ln 3 \cdot (\ln 2x)'$$

$$(\ln u)' = \frac{1}{u}$$

$$= \ln 3 \cdot \frac{1}{\ln 2x} + x \ln 3 \cdot \frac{1}{2x} \cdot (2x)'$$

$$= \ln 3 \cdot \frac{1}{\ln 2x} + x \ln 3 \cdot \frac{1}{2x} \cdot 2$$

Domain Denominator can not be 0, so

so $\ln 2x \neq 0$, $2x \neq 1$, $x \neq \frac{1}{2}$

Input of ln can not be negative

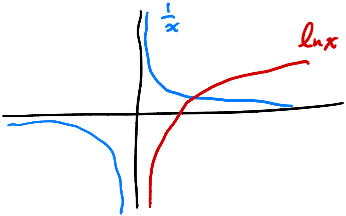
so $2x > 0$, $x > 0$

Easier examples

Example What is the derivative of $\ln(x)$? What is its domain?

$$(\ln(x))' = \frac{1}{x}$$

Domain: $x > 0$



Example What is the derivative of $\ln|x|$?

$$(\ln|x|)' = \frac{1}{|x|}\cdot(|x|)'$$

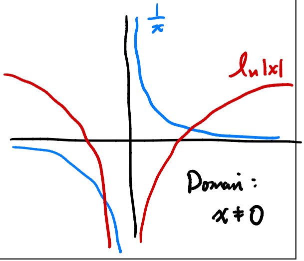
$$= \frac{1}{|x|}\cdot\frac{x}{|x|}$$

$$= \frac{1}{|x|}\cdot\frac{|x|}{x}$$

$$= \frac{1}{x}$$

$$\frac{x}{|x|} = \begin{cases} \frac{x}{x} = 1, & x > 0 \\ \frac{x}{-x} = -1, & x < 0 \end{cases}$$

$$\frac{|x|}{x} = \begin{cases} \frac{x}{x} = 1, & x > 0 \\ \frac{-x}{x} = -1 & x < 0 \end{cases}$$



Domain: $x \neq 0$

Approximating $\ln x$

Now that we know $\boxed{\ln|x|}$ is the anti-derivative of $\frac{1}{x}$. In particular, for $\boxed{x > 0}$

$\ln|x| = \ln x$

$$\int_1^x \frac{1}{t}\,dt = \ln|t|\Big|_1^x = \ln x - \ln 1 = \ln x.$$

That is

$$\boxed{\ln x = \int_1^x \frac{1}{t}\,dt.}$$

$(\ln x)' =$

$\left(\int_1^x \frac{1}{t}dt\right)' = \frac{1}{x}$, Domain : $x > 0$

In real life, this formula allows us to estimated $\ln x$. Because knowing that $\ln x$ is an integral, we can use

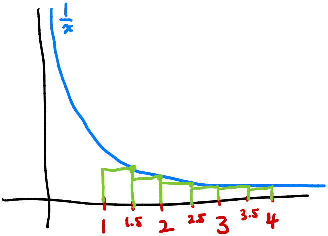
left/right Riemann sum approximation

to approximate it!

Approximating $\ln x$

Example Approximate $\ln 4$ using $\boxed{\text{right}}$ Riemann approximation by dividing a interval into $\boxed{6}$ pieces.

$$\ln 4 = \underbrace{\int_1^4 \frac{1}{t}\,dt}$$



$\ln 4 \approx$ sum of area of rectangles

$$= 0.5 \times \frac{1}{1.5} + 0.5 \times \frac{1}{2} + 0.5 \times \frac{1}{2.5} + 0.5 \times \frac{1}{3} + 0.5 \times \frac{1}{3.5} + 0.5 + \frac{1}{4}$$

Logarithmic scale

Logarithmic scale would be a good motivation for why we need logarithmic functions.

- Earthquake scale is measured on a logarithmic scale:
  an earthquake of level 6 is 10 times stronger than level 5.
- pH value is measured on a logarithmic scale:
  a pH value of 3 is 100 times ~~less~~ more acidic than a pH value of 5.
- Volume is a bit different:
  a sound that is 130 dB is 1000 times louder than 100 dB.

Logarithmic scale

Definition We say a function is a **logarithmic scale of base** a if

adding b to the $\boxed{\text{output}}$ means the $\boxed{\text{input}}$ has to be multiplied by $\boxed{a^b}$

By how is this description in any sense logarithmic? Here's why:

If $f(x)$ is a logarithmic scale of base a, then we know

$$f(x) = \log_a \left(\frac{x}{A}\right)$$

for some number A.

For the above, we see a needs to satisfy the condition

$$\boxed{a > 0, a \neq 1,}\ .$$

$A \neq 0$

Logarithmic scale

Definition We say a function is a **logarithmic scale of base** a if

adding $\boxed{b}$ to the output means the input has to be multiplied by $\boxed{a^b}$.

Example The earthquake scale is on a base-$\boxed{10}$ logarithmic scale. The input is intensity and the output is the magnitude.

1. If you add $\boxed{3}$ to the magnitude, then the intensity needs to be multiplied by a factor of

$$\boxed{10^3}$$

2. If one earthquake has magnitude 4 and another has magnitude 6.5, then the intensity of the second one is

$$\boxed{10^{2.5} \text{ times stronger}}$$

than the first one.

Logarithmic scale

Definition We say a function is a **logarithmic scale of base** a if

adding b to the output means the input has to be multiplied by $\boxed{a^b}$.

Example Some aliens measure earthquake on a base-9 logarithmic scale. The input is badness and the output is the level.

1. If one earthquake is level $\boxed{3}$ and another has level $\boxed{6.5}$, then the badness of the second one is

$$\boxed{9^{3.5} \text{ times more bad}}$$

than the first one.

2. We can compute this value using exp rules

$$\boxed{9^{3.5}} = \boxed{9^3 \cdot 9^{0.5}} = \boxed{729 \cdot 3} = \boxed{2187}$$

Logarithmic scale

Definition We say a function is a **logarithmic scale of base** a if

adding b to the output means the input has to be multiplied by a^b.

Example The volume of sound is a bit weird. The output is volume/decible and input is intensity. It satisfies

$$D(I) = 10 \log_{10}\left(\frac{I}{10^{-12}}\right)$$

And this is not a logarithmic scale. But if we divide the output by 10, then we do get a logarithmic scale of base-10.

1. If sound is 25 dB higher in volume than another sound, then its intensity is

$$\boxed{10^{2.5} \text{ times stronger}}$$

$3^2 = 9$

$(?)^2 = 10$

$4^2 = 16$

2. Let us try to estimate this value

$$\boxed{10^{2.5}} = \boxed{10^2 \cdot 10^{0.5}} \approx \boxed{10^2 \cdot 3} = \boxed{300}$$

3 questions on your opinions

I went to many teaching seminars where old people talk about how I should teach. Now I want to know what you think.