MATH 127 Calculus for the Sciences

Lecture 12

Today's lecture

Last time
Exponential and Logarithmic functions

1. Definition
2. Rules
3. their derivatives

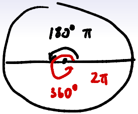
$(\ln x)' = \frac{1}{x}$

$(\ln|x|)' = \frac{1}{x}$

This time
Trigonometric functions
Course note coverage Section 2.5.1

1. Radian vs degree
2. sin and cos of special angles, how to compute
3. Compute sin and cos given tan
4. Trigonometric identities
5. Graphing trig functions

Radian to degree



Radian and degree conversion is just a conversion of scaling. Follow these steps:

Rad to deg: How many degrees is x rad?

$x = 0.99$ radian $= \boxed{\frac{0.99}{\pi}} \cdot \pi$ so $\boxed{\frac{0.99}{\pi}} \cdot 180$ degrees.

1. How many times of π is x?

2.1. If x is half of π radian, then the answer is half of 180 degrees, so 90;

2.2. If x is twice of π radian, then the answer is twice of 180 degrees, so 360;

2. If x is $\boxed{y}$ times of π radian, then the answer is $\boxed{y \cdot 180}$.

Deg to rad: How many radian is x degrees?

1. How many times of 180 is x?

2.1. If x is half of 180 degrees, then the answer is half of π radian, so $\pi/2$;

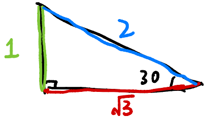
2.2. If x is twice of 180 degrees, then the answer is twice of π radian, so 2π;

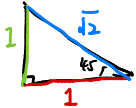
2. If x is y times of 180 degrees, then the answer is $\boxed{y \cdot \pi}$.

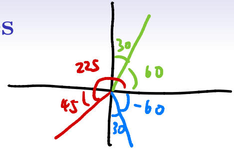
$\theta = \frac{1}{e}$ degree $= \boxed{\frac{1}{e \cdot 180}} \cdot 180$

so $\theta = \frac{1}{\theta \cdot 180} \cdot \pi$ radian

tan of special angles







The first three special angles are $0^\circ, 30^\circ$ and 45°. In radians, they are are 0, $\boxed{\frac{\pi}{6}}$ and $\boxed{\frac{\pi}{4}}$.

Other special angles are these three added to or subtracted by multiples of 90°.

More secrets they don't teach you: If you know the following special lengths, you can find $\sin, \cos, \tan$ of any special angles.

$$\text{adjacent} = \begin{cases} 1, & \text{if angle is } 0^\circ; \\ \sqrt{3}, & \text{if angle is } 30^\circ; \\ 1, & \text{if angle is } 45^\circ; \end{cases}, \quad \text{opposite} = \begin{cases} 0, & \text{if angle is } 0^\circ; \\ 1, & \text{if angle is } 30^\circ; \\ 1, & \text{if angle is } 45^\circ; \end{cases}$$

$\sin$ and $\cos$ and $\tan$ of special angles

Use the following steps to find sin or cos or tan of special angles:

1. Draw the angle on a circle. A special angle would be $0^\circ, 30^\circ$, or 45° away from one of x or y axis.
2. Starting from where the angle touches the circle, draw the perpendicular line to the x or y-axis, whichever one is closer.
3. From where this perpendicular line touches the axis, connect it to the origin.
4. You have drawn a triangle whose angle at the origin is $0^\circ, 30^\circ$, or 45°. The horizontal edge is x, and the vertical edge is y.
5. Check which of x, y is adjacent and which is opposite to the angle, then write

   opposite

   $$\text{adjacent} = \begin{cases} 1, & \text{if angle is } 0^\circ; \\ \sqrt{3}, & \text{if angle is } 30^\circ; \\ 1, & \text{if angle is } 45^\circ; \end{cases}, \qquad \cancel{\text{adjacent}} = \begin{cases} 0, & \text{if angle is } 0^\circ; \\ 1, & \text{if angle is } 30^\circ; \\ 1, & \text{if angle is } 45^\circ; \end{cases}$$

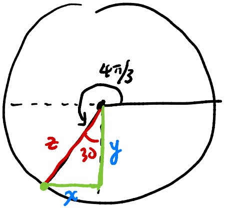
   But with correct signs based on whether x is on left or right of origin, and whether y is on top of bottom of origin.
6. From this, you can use the formulas

   $$\tan\theta = \frac{y}{x}, \quad \sin(\theta) = \frac{y}{z} \quad \cos(\theta) = \frac{x}{z}, \quad \text{where} \quad z^2 = x^2 + y^2$$

Example

Example Follow the previous method to find

$$\sin(4\pi/3), \quad \cos(4\pi/3), \quad \tan(4\pi/3).$$



Say $x = -1$

$y = -\sqrt{3}$

$z = 2$

$\sin\frac{4\pi}{3} = \frac{y}{z} = \frac{-\sqrt{3}}{2}$

$\cos\frac{4\pi}{3} = \frac{x}{z} = \frac{-1}{2}$

$\tan\frac{4\pi}{3} = \frac{y}{x} = \frac{-\sqrt{3}}{-1} = \sqrt{3}$

Find sin, cos if given tan

Suppose you are given a value $\tan\theta = Q$ for some Q. Find $\cos(\theta)$.

> **Remark** Usually you will be told what domain θ belongs to. Because otherwise there could be multiple possible values of $\cos\theta$ and $\sin\theta$.

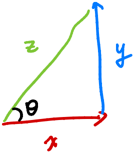
1. Draw a horizontal line and label it x.
2. At the end of horizontal line, draw a vertical line, going up and label it y.
3. Label the angle at the origin θ. If you are given $\tan\theta$, then set x, y to values so that $\tan\theta = \dfrac{y}{x}$, with appropriate signs.
4. From this, you can use the formulas

$$\tan\theta = \frac{y}{x}, \quad \sin(\theta) = \frac{y}{z} \quad \cos(\theta) = \frac{x}{z}, \quad \text{where} \quad z^2 = x^2 + y^2$$

> **Remark** If you are given sin or cos and asked to find tan, it's the same story. Draw a triangle in the appropriate quadrant, and label x, y, z accordingly.

Example

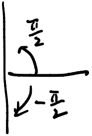
Example Given $\tan\theta = \dfrac{\alpha}{\widehat{wow}}$, find $\sin\theta$ if we know θ is in $(-\pi/2, \pi/2)$.



$\alpha > 0$

$\widehat{wow} > 0$

$\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$



$$\tan\theta = \frac{\alpha}{\widehat{wow}} = \frac{y}{x}$$

① $y = \alpha$, $x = \widehat{wow}$; 2α, $2\widehat{wow}$; ② $-\alpha$, $-\widehat{wow}$

$$z = \sqrt{\alpha^2 + \widehat{wow}^2}$$

$$\sin\theta = \frac{y}{z} = \frac{\alpha}{\sqrt{\alpha^2 + \widehat{wow}^2}} \text{ or } \frac{-\alpha}{\sqrt{\alpha^2 + \widehat{wow}^2}}$$

$$\cos\theta = \frac{x}{z} = \frac{\widehat{wow}}{\sqrt{\alpha^2 + \widehat{wow}^2}} \text{ or } \frac{-\widehat{wow}}{\sqrt{\alpha^2 + \widehat{wow}^2}}$$

Trigonometric identities

I don't like memorization. If I can let you take a cheat sheet for these things I would. But you should remember the following

1. $\sin^2(x) + \cos^2(x) = 1$
2. $\sin(-x) = -\sin(x)$
3. $\cos(-x) = \cos(x)$
4. $\sin(x + y) = \sin x \cos y + \cos x \sin y$
5. $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

$\sin(2x) = \sin(x+x)$

$= \sin x \cos x + \cos x \sin x$

$= 2 \cdot \sin x \cos x$

Example

Example Given $\sin(x)\cos(x) = -\frac{\sqrt{3}}{\cancel{2}\ 4}$, find all possible values of x in $[0, 2\pi]$.

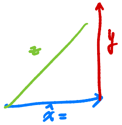
$$\sin(x)\cos(x) = -\frac{\sqrt{3}}{4}$$

$$2\sin(x)\cos(x) = -\frac{\sqrt{3}}{2}$$

$$\sin(2x) = \frac{-\sqrt{3}}{2}.$$

Let $\theta = 2x$.

$$\sin(\theta) = -\frac{\sqrt{3}}{2}$$

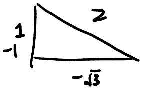


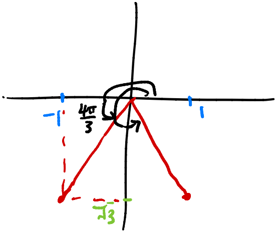
$$\sin\theta = \frac{y}{z} = -\frac{\sqrt{3}}{2}$$

$y = -\sqrt{3}$

$z = 2$

$\hat{x} = 1$ or -1





$\theta = \frac{4\pi}{3} + 2\pi n$, n integer,

or $\frac{5\pi}{3} + 2\pi n$, n integer

$\theta = 2x$, $x = \frac{2\pi}{3} + \pi \cdot n$

or $\frac{5\pi}{6} + \pi \cdot n$,

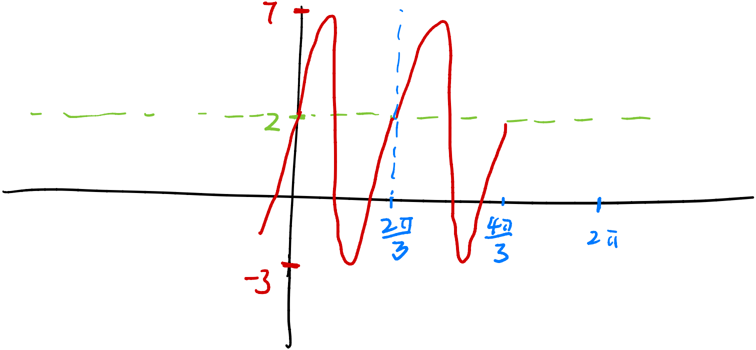
Since $x \in [0, 2\pi]$

$n = 0$ or 1.

Example

Example Graph the function $y = 5\sin(3x) + 2$. What is its maximum? What is the distance between a maximum and the subsequent minimum?

Remark More generally, you might be asked to figure out the period, range, where it is increasing/decreasing, where it is concave up/down, given a function. Or vise versa: given these data, figure out the function. You should be able to graph the function and retrieve these data and vice versa.



Example

Example Graph the function $y = 5\sin(Ax) + C$ for some constant A, C. If we know the period is 3π, what is A? If we know the maximum is 7, what is C?