MATH 127 Calculus for the Sciences

Lecture 6

Today's lecture

Office hours/Question session
Monday: 1430-1520 MC5331
Wednesday and Friday after class
Tutoring center
MC3022 or online.

Last time Anti-derivatives

Today
Course note coverage Section 1.2.3, 2.1.1
Solving easy differential equations
Piecewise defined functions

For Friday
We will have a lecture of examples and applications, but you need to learn some theory by yourself at home.

Anti-derivative recap

- If $y = f(x)$ is a function, then

$$\int f(x)dx, \quad \text{or} \quad \int y dx$$

is the *family* of anti-derivatives of $f(x)$.

- If $F(x)$ is *one* anti-derivative of $f(x)$,

$$\boxed{\int f(x)dx = \qquad\qquad}$$

where C is $\boxed{\qquad\qquad}$.

Applications of anti-derivatives

(1) Math: area under curves is the integral of the curve
–estimate cost of making a certain shape

(2) Physics: distance is the integral of speed
–estimate position of a planet

(3) Chemistry: concentration is the integral of rate of change
–estimate sugar content in blood after meal

Solving a DE with integral

Question If a water tank receives water at a rate of $2e^{2t}$ m^3/s at time t, then give a formula $V(t)$ for the volume of water at time t.

(1) The total amount is the $\boxed{}$ of the rate of change.

$$V(t) = \boxed{}$$

This is our usual integration problem.

(2) The rate of change is the $\boxed{}$ of the total amount.

$$V'(t) = \boxed{}.$$

This is a $\boxed{}$ equation with unknown function $\boxed{}$.

Conclusion:

To solve the DE **(2)**, we just need to find the integral **(1)**.

Examples

Example Find the solutions to the first order differential equation

$$y' = \cos(u)$$

Initial value problem

Question Suppose a water tank contains $V(t)$ m^3 of water, satisfying first order differential equation

$$V'(t) = 2e^{2t}$$

Assume at time 0, we know the tank already holds 2 m^3 of water. Find $V(t)$.

We have seen

$$V(t) = \int 2e^{2t} dt = \boxed{}$$

for some C. Now we have the additional information

$$V(0) = 2.$$

Such information is called **initial condition**. Substituting we get

$$2 = V(0) = \boxed{ + C}, \quad \text{hence } C = \boxed{}.$$

Example

Example Solve the initial value problem

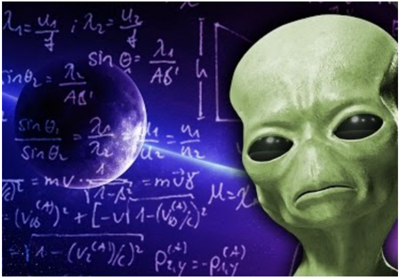
$$\frac{dy}{dx} = 24x^5, \quad y(1) = 3$$

Piecewise defined functions

Suppose an alien comes from a planet where the absolute value sign does not exist. How do we talk to them about the function $f(x) = |x|$?

We could write

$$f(x) = \begin{cases} \boxed{}, & \text{if } x \geq 0; \\ \boxed{}, & \text{if } x < 0. \end{cases}$$

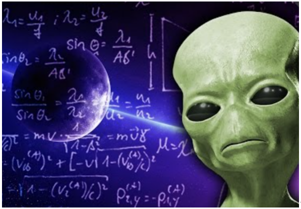


Piecewise defined functions

Suppose an alien uses the symbol $\Diamond(x)$ to represent the function that is x when $x < 2$, and x^2 when $2 \leq x < 5$, and $1 - x$ when $x \geq 5$.

How does he tell us about this function without using the symbol $\Diamond$?

$$f(x) = \begin{cases} \boxed{\quad\quad}, & \text{if } \boxed{\quad\quad\quad\quad\quad\quad}; \\ \boxed{\quad\quad}, & \text{if } \boxed{\quad\quad\quad\quad\quad}; \\ \boxed{\quad\quad\quad\quad}, & \text{if } \boxed{\quad\quad\quad\quad\quad\quad}. \end{cases}$$



$<$ $\boxed{\text{thanks, now I can do math.}}$

Piecewise defined functions

$$\boxed{f(x) = \begin{cases} x, & \text{if } x < 2; \\ x^2, & \text{if } 2 \leq x < 5; \\ 1 - x, & \text{if } 5 < x. \end{cases}}$$

Functions like this, defined by splitting x-axis into pieces of intervals, are called **piecewise defined functions**.

This is a good way to construct non-continuous functions for us humans.

Example

Why do we need piecewise defined functions?
The tax rate jumps as income reaches a certain threshold:

- If your income is below \$10 (inclusive), then you pay 10% tax.
- If your income is above \$10, but below \$100 (inclusive), then you pay 20% tax.
- If your income is above \$100, but below \$1000 (inclusive), then you pay 30% tax.
- If your income is above \$1000, then you pay 120% tax.

What is the function of tax you should pay given your income?