MATH 127 Calculus for the Sciences

Lecture 7

Today's lecture

Last time
Solving easy differential equations
Piecewise defined functions

This time
Course note coverage Section 2.1.2
Absolute value functions

Quiz 3 is today:
6 multiple choice and 4 long answers.

Motivation

Absolute value functions is so easy, it's just

$$|x| = \begin{cases} x, & x \geq 0; \\ -x & x < 0. \end{cases}$$

But do you know how to express

$$f(x) = x|x^2 - 1| + 5|2x - 3|$$

to an alien who does not use of the absolute value symbol?

Expressing functions with absolute values

Question How did we get the expression

$$|x| = \begin{cases} x, & x \geq 0; \\ -x & x < 0. \end{cases}$$

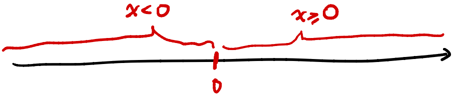
in the first place?

Step 1: Where did the function change behavior?
Answer:

$$x = \boxed{0}$$

This is called a **transition point**.

Step 2: Split the x-axis using the transition points and deal with each piece.



Expressing functions with absolute values

Question Express

$$f(x) = x|x^2 - 1| + 5|2x - 3|$$

without the use of the absolute value symbol.

Step 1: What are potential transition points? Transition points occur whenever one of the absolute value sign is 0. Therefore

$$\boxed{\text{Transition points happen when } \boxed{x^2-1} = 0 \text{ or } \boxed{2x-3} = 0}$$

So the transition points are

$$x = \boxed{1, -1, \tfrac{3}{2}}$$

Expressing functions with absolute values

Question Express

$$f(x) = x|x^2 - 1| + 5|2x - 3|$$

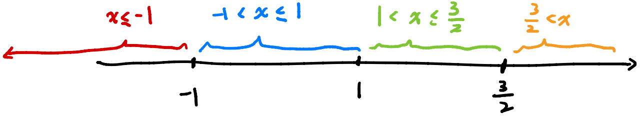
without the use of the absolute value symbol.

The transition points are

$$x = -1, 1, \frac{3}{2}.$$

Step 2: Cut the domain of the function using the transition points, and describe the functions on each domain.
For this example, we have a few cases:

$$\boxed{x \le -1}, \quad \boxed{-1 < x \le 1}, \quad \boxed{1 < x \le \frac{3}{2}}, \quad \boxed{\frac{3}{2} < x}$$

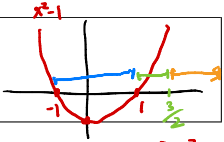


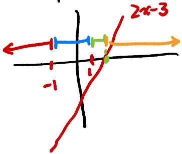
Expressing functions with absolute values

Question Express

$$f(x) = x|x^2 - 1| + 5|2x - 3|$$

without the use of the absolute value symbol.





Case 1: $x \leq -1$. Then $x^2 - 1 \geq 0$ and $2x - 3 \leq 0$, so

$$f(x) = x \cdot \boxed{(x^2-1)} + 5 \cdot \boxed{(-(2x-3))}$$

Case 2: $-1 < x \leq 1$. Then $x^2 - 1 \leq 0$ and $2x - 3 \leq 0$, so

$$f(x) = x \cdot \boxed{(-(x^2-1))} + 5 \cdot \boxed{(-(2x-3))}$$

Case 3: $1 < x \leq \frac{3}{2}$. Then $x^2 - 1 \geq 0$ and $2x - 3 \leq 0$,

$$f(x) = x \cdot \boxed{(x^2-1)} + 5 \cdot \boxed{(-(2x-3))}$$

Case 4: $\frac{3}{2} < x$. Then $x^2 - 1 \geq 0$ and $2x - 3 \geq 0$, so

$$f(x) = x \cdot \boxed{(x^2-1)} + 5 \cdot \boxed{(2x-3)}$$

Details

$$f(x) = x|x^2-1| + 5|2x-3|$$

If $1 < x \le \frac{3}{2}$ then $x^2 - 1 \ge 0$

$$|x^2-1| = x^2-1$$

so $f(x) = x(x^2-1) + 5|2x-3|$

also, $2x-3 \le 0$

$$|2x-3| = -(2x-3)$$

$$f(x) = x(x^2-1) + 5 \cdot (-(2x-3))$$

Expressing functions with absolute values

Question Express

$$f(x) = x|x^2 - 1| + 5|2x - 3|$$

without the use of the absolute value symbol.

Finally, combine all cases and get

$$f(x) = \begin{cases} x(x^2-1) - 5(2x-3), & x \le -1; \\ -x(x^2-1) - 5(2x-3), & -1 < x \le 1; \\ x(x^2-1) - 5(2x-3), & 1 < x \le \frac{3}{2}; \\ x(x^2-1) + 5(2x-3), & \frac{3}{2} < x. \end{cases}$$

Expressing errors with absolute values

Remark If $|x - y| \leq z$ for some number z, that means the difference between x and y is at most z. This idea is used to express a range for error.

Example Suppose I measured my height h to be 175cm, but my ruler has an error of at most 5cm. This means my height could be anything between 170cm to 180cm.

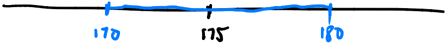
Which one below correctly records this error?

✗ 1. $|h - 175| = 5$. → This means the difference between h and 175 is exactly 5 so h must be 170, or 180.

✓ 2. $|h - 175| \leq 5$. → This means the difference between h and 175 is at most 5

3. $|h + 175| \leq 5$. → ?? does not mean anything.

✓ 4. $|175 - h| \leq 5$.



Derivative of absolute value

$\sqrt{2^2} = 2 = |2|$

$\sqrt{x^2} = x$ if $x > 0$

$\sqrt{x^2} = -x$ if $x < 0$

$\sqrt{(-2)^2} = 2 = |-2|$

Recall that

$$\boxed{|x| = \sqrt{x^2}}$$

so to find the derivative of the absolute value, we can use the chain rule.

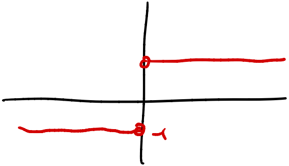
Before we do that, let us also recall that $|x|$ does not exist when $\boxed{x = 0}$, so whatever result we get should reflect that.

$$(|x|)' = \left(\sqrt{x^2}\right)' = \frac{1}{2\sqrt{x^2}} \cdot (x^2)', \text{ because } \boxed{(\sqrt{u})' = \frac{1}{2\sqrt{u}}}$$

$$= \frac{2x}{2\sqrt{x^2}}$$

$$= \frac{x}{|x|}$$

$$= \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$



Example

Question Find the derivative of

$$f(x) = |e^x - \sin(x)|.$$

$$(|u|)' = \frac{u}{|u|}$$

$$f'(x) = \frac{e^x - \sin(x)}{|e^x - \sin(x)|} \cdot (e^x - \sin x)'$$

$$= \frac{e^x - \sin(x)}{|e^x - \sin(x)|} (e^x - \cos x).$$

The derivative does not exist if $e^x - \sin x = 0$

Challenge Find the transition points of

$$f(x) = |e^x - \sin(x)|.$$

Homework for Friday

On Friday, we have an example focused class, but this means you need to do some preparations for the thoery to be able to understand the methods.

- Read page 47 (10 minutes)
- Read page 49 (up to the end of Example 4) (10 minutes)
- Read Page 51 (up to the end of Example 7) (5 minutes)
- Read page 52 (only Theorem 1 at the bottom) (3 minutes)
- Read page 53 (up to right before Example 12) (5 minutes)