# MATH 127 Calculus for the Sciences

Lecture 6

# Today's lecture

**Office hours/Question session**
Monday: 1430-1520 MC5331
Wednesday and Friday after class
**Tutoring center**
MC3022 or online.

**Last time** Anti-derivatives

**Today**
**Course note coverage** Section 1.2.3, 2.1.1
Solving easy differential equations
Piecewise defined functions

**For Friday**
We will have a lecture of examples and applications, but you need to learn some theory by yourself at home.

## Anti-derivative recap

- If $y = f(x)$ is a function, then

$$\int f(x)dx, \quad \text{or} \quad \int y dx$$

is the *family* of anti-derivatives of $f(x)$.

- If $\boxed{F(x)}$ is *one* anti-derivative of $f(x)$,

$$\boxed{\int f(x)dx = F(x) + C}$$

where $C$ is constant of integration.

**Applications of anti-derivatives**

(1) Math: area under curves is the integral of the curve
–estimate cost of making a certain shape

(2) Physics: distance is the integral of speed
–estimate position of a planet

(3) Chemistry: concentration is the integral of rate of change
–estimate sugar content in blood after meal

# Solving a DE with integral

**Question** If a water tank receives water at a rate of $2e^{2t}$ $\text{m}^3/\text{s}$ at time $t$, then give a formula $V(t)$ for the volume of water at time $t$.

**(1)** The total amount is the $\boxed{\text{anti-derivative}}$ of the rate of change.

$$\boxed{V(t) = \boxed{\int 2e^{2t}\,dt}}$$

This is our usual integration problem.

**(2)** The rate of change is the $\boxed{\text{derivative}}$ of the total amount.

$$\boxed{V'(t) = \boxed{2e^{2t}}.}$$

This is a $\boxed{\text{differential}}$ equation with unknown function $\boxed{V(t)}$.

**Conclusion:**

To solve the DE **(2)**, we just need to find the integral **(1)**.

# Examples

**Example** Find the solutions to the first order differential equation

$$y' = \cos(u)$$

$\hookrightarrow$ first derivative of $y$.

This is a differential equation with unknown function $y$ and variable $u$.

$$y = \int \cos(u)\,du \qquad (\sin(u))' = \cos(u)$$

$$= \sin(u) + C$$

where $C$ is the constant of integration

The family $y = \sin(u) + C$ is a family of solutions for $y' = \cos(u)$.

$\int \cos(2u)\,du$

$= \frac{1}{2}\sin(2u) + C$

$\left(\frac{1}{2}\sin(2u)\right)' = \frac{1}{2}\cos(2u)\cdot(2u)'$

$= \frac{1}{2}\,2\cos(2u)$

$= \cos(2u)$

# Initial value problem

**Question** Suppose a water tank contains $V(t)$ $\mathrm{m}^3$ of water, satisfying first order differential equation

$$V'(t) = 2e^{2t}$$

Assume at time 0, we know the tank already holds 2 $\mathrm{m}^3$ of water. Find $V(t)$.

We have seen

$$V(t) = \int 2e^{2t}dt = \boxed{e^{2t} + C}$$

$$(e^{2t})' = e^{2t} \cdot (2t)' = 2e^{2t}$$

for some $C$. Now we have the additional information

$$V(0) = 2.$$

Such information is called **initial condition**. Substituting we get

$$2 = V(0) = \boxed{\underbrace{e^0}_{=1} + C}, \quad \text{hence } C = \boxed{1}.$$

Therefore $V(t) = e^{2t} + 1$

# Example

**Example** Solve the initial value problem

$$\frac{dy}{dx} = 24x^5, \quad y(1) = 3$$

$$y = \int 24x^5\,dx \qquad (4\cdot x^6)' = 4\cdot 6\cdot x^5$$

$$= \boxed{4\cdot x^6 + C}$$

where $C$ is constant of integration

$$3 = y(1) = 4\cdot 1^6 + C$$

$$C = -1$$

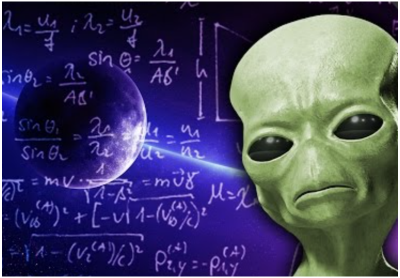
Therefore $y = 4x^6 - 1$

# Piecewise defined functions

Suppose an alien comes from a planet where the absolute value sign does not exist. How do we talk to them about the function $f(x) = |x|$?

We could write

$$f(x) = \begin{cases} \boxed{x}, & \text{if } x \geq 0; \\ \boxed{-x}, & \text{if } x < 0. \end{cases}$$

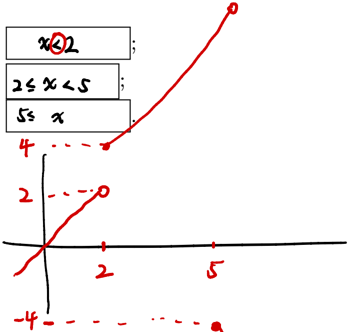


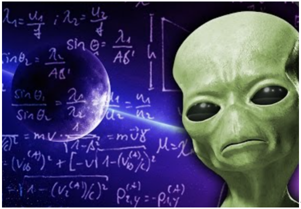
# Piecewise defined functions

Suppose an alien uses the symbol $\Diamond(x)$ to represent the function that is $x$ when $x < 2$, and $x^2$ when $2 \le x < 5$, and $1 - x$ when $x \ge 5$.

How does he tell us about this function without using the symbol $\Diamond$?

$$f(x) = \begin{cases} \boxed{x}, & \text{if } \boxed{x < 2}; \\ \boxed{x^2}, & \text{if } \boxed{2 \le x < 5}; \\ \boxed{1-x}, & \text{if } \boxed{5 \le x}. \end{cases}$$



< thanks, now I can do math.

# Piecewise defined functions

$$\boxed{f(x) = \begin{cases} x, & \text{if } x < 2; \\ x^2, & \text{if } 2 \leq x < 5; \\ 1 - x, & \text{if } 5 < x. \end{cases}}$$

Functions like this, defined by splitting $x$-axis into pieces of intervals, are called **piecewise defined functions**.

This is a good way to construct non-continuous functions for us humans.

$$f(x) = \begin{cases} x^2, & x < 0 \\ x^2, & 0 \leq x < 1 \\ x^2, & 1 \leq x. \end{cases}$$

is "trivially" a piecewise defined function.

# Example

**Why do we need piecewise defined functions?**

The tax rate jumps as income reaches a certain threshold:

- If your income is below \$10 (inclusive), then you pay 10% tax.
- If your income is above \$10, but below \$100 (inclusive), then you pay 20% tax.
- If your income is above \$100, but below \$1000 (inclusive), then you pay 30% tax.
- If your income is above \$1000, then you pay 120% tax.

What is the function of tax you should pay given your income?

Let $x$ represent income.
Let $T$ represent tax.

$$T(x) = \begin{cases} 0.1x, & x \le 10 \\ 0.2x, & 10 < x \le 100 \\ 0.3x, & 100 < x \le 1000 \\ 1.2x & 1000 < x \end{cases}$$