MATH 127 Calculus for the Sciences

Lecture 9

Today's lecture

Last time

Integral Functions:

- Definition
- Derivative of an integral function
- Interpret what they represent physically

$$\left(\int_0^x f(t)\,dt\right)' = f(x)$$

$$\int_0^x f(t)\,dt$$

This time

Course note coverage Section 2.3

Inverse functions

- Definition
- How to find them?
- Do they even exist?
- Cancellation rules
- Secret knowledge

You know what to do

https://chatgpt.com/share/68cdd77d-658c-8006-b6b0-03524a20d5d2

Notations on your quizzes

The graders for your quizzes will be especially picky on your notations. Make sure what you write down is mathematically and notationally correct.

Example Which of the following is valid?

1. $x = f(y)$; ✓
2. $\Big|_a^b f(x)dx$; ✗ $F(x)\Big|_a^b$ $\int_a^b f(x)dx$
3. $\int x^2 \cdot 5x$; $\int x^2 \cdot 5x\,dx$
4. $dy = \sin x + dx$; $dy = \cdots dx$
5. $h(u)\Big|_3^5 = h(5) - h(3)$
6. $5x$ is the anti-derivative of 5.

an

Motivation

Suppose you are given the function

$$y = f(x)$$

and you want to express x in terms of y. You would need the "inverse" to the function f.

Definition Suppose f is a function. If there exists a function g such that

$$\boxed{y = f(x) \quad \text{if and only if} \quad x = g(y)}$$

then g is *the* **inverse function** of f. We write

$$g = f^{-1}$$

Remark

1. The notation f^{-1} means inverse function. It is different from the reciprocal function $\dfrac{1}{f}$.
2. Inverse function is **unique**. So we say "*the* inverse function" instead of "*an* inverse function".

Inverse functions

Example

$$y = x + 1 \quad \text{if and only if} \quad x = y - 1$$

Therefore the *inverse function* of $f(x) = x + 1$ is

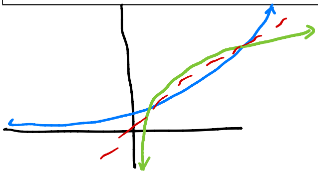
$$g(y) = \boxed{y-1}.$$

Example

$$y = e^x \quad \text{if and only if} \quad x = \ln y.$$

Therefore the *inverse function* of $f(x) = e^x$ is

$$g(y) = \boxed{\ln y}.$$



Domain of e^x : $(-\infty, \infty)$ of $\ln y$ $(0, \infty)$

Range of e^x : $(0, \infty)$ of $\ln y$ $(-\infty, \infty)$

Finding inverse functions

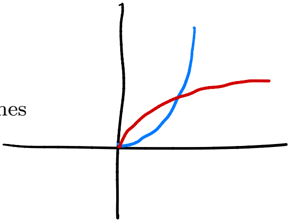
Question What is the inverse function of $f(x) = x^2$?

We want to find a function g such that

$$y = x^2 \quad \text{if and only if} \quad x = g(y).$$

A natural attempt is to take $g(y) = \sqrt{y}$. So the above becomes

$$y = x^2 \quad \text{if and only if} \quad x = \sqrt{y}.$$



But is this guess right? Let us check it:

(1) If $x = \sqrt{y}$, must we have $y = x^2$? ✓

(2) If $y = x^2$, must we have $x = \sqrt{y}$? ✗

Hence $\sqrt{y}$ is not the inverse of x^2

(1) $x = \sqrt{y}$ square both sides

$x^2 = (\sqrt{y})^2$

$x^2 = y$

(2) $y = x^2$ square root both sides

$\sqrt{y} = \sqrt{x^2}$

$\sqrt{y} = |x| \neq x$ if $x < 0$

Existence of inverse function

The problem we encounter is due to the following

> For each value y, there are two values $x = \pm\sqrt{y}$ such that $y = x^2$, so we have two candidates for the inverse function, but the inverse function is **UNIQUE**.

> **Question** What is the inverse function of $f(x) = x^2$, with respect to the domain $[0, \infty)$?

We set $y = x^2$ and solve for x in terms of y: take square root on both sides

$$\sqrt{y} = \sqrt{x^2}$$

$$\sqrt{y} = |x|$$

But because x belong to $[0, \infty)$, we have

$$\sqrt{y} = \boxed{x}, \quad \text{hence} \quad f^{-1}(y) = \boxed{\sqrt{y}}$$

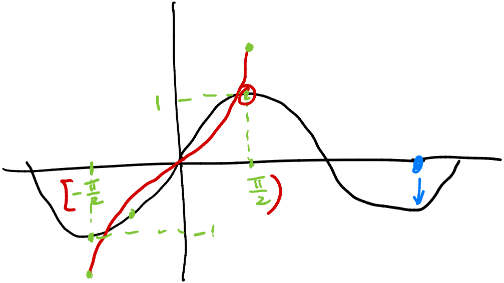
The horizontal line test

Example On which interval does sin(x) admit an inverse? What are the domain and range of the inverse if it exists?

(A) $[0, \pi/2)$; ✓ domain [0,1) Range $[0, \frac{\pi}{2})$

(B) $[0, \pi)$; ← X

(C) $[-\pi/2, \pi/2)$. ← ✓ domain [-1,1) Range $[-\frac{\pi}{2}, \frac{\pi}{2})$



Invertibility

With the horizontal line test, we can check whether a function admits an inverse.

Definition
If a function $f(x)$ admits an inverse $f^{-1}(x)$, we say $f(x)$ is **invertible**.

Cancellation rule

Sometimes give $y = f(x)$, it is hard to rewrite it as $x = g(y)$. So is there another way to check whether something is an inverse?

Proposition (Cancellation property)
Suppose we have functions $\boxed{f}$ *and* $\boxed{g}$. *Then*

$$\underline{g = f^{-1}}$$

if and only if

$$\boxed{g(f(x)) = x} \quad \text{and} \quad \boxed{f(g(y)) = y.}$$

Remark The above only works over valid domains and range. If f has domain $[3, 5)$, and g is the inverse with respect to this domain, then we can only guarentee

$$g(f(x)) = x$$

for x inside $[3, 5)$.

Example

Example Is the function $g(u) = -\sqrt{u}$ ~~an~~ the inverse of $f(t) = t^2$ over $\boxed{(-\infty, 0]}$?

We just need to check

$$(1) \quad f(g(y)) = (g(y))^2 = (-\sqrt{y})^2 \overset{\text{is this true?}}{=} y;$$

$$(2) \quad g(f(x)) = -\sqrt{f(x)} = \boxed{-\sqrt{x^2}} \overset{\text{is this true?}}{=} x.$$

$$= -|x|$$

$$= -(-x)$$

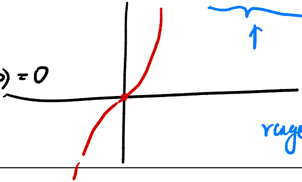
$$= x$$

Therefore $g(x)$ is the inverse of $f(x)$

Example

Example We know $\tan(x)$ is invertible on the interval $\underbrace{(-\pi/2, \pi/2)}$. Let us call this inverse $\tan^{-1}$.

1. What is $\tan^{-1}(0)$? $=0$
2. What is $\tan^{-1}(\tan(0))$? $= \tan^{-1}(0) = 0$
3. What is $\tan(\tan^{-1}(123))$?
4. What is $\tan^{-1}(\tan(2\pi))$? $=0$



range $\tan^{-1} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Secret knowledge Think of it like this

$$\tan^{-1}(123) \textbf{ means the } \tan \textbf{ of this thing is } 123.$$

More generally,

$$f^{-1}(a) \textbf{ means the } f \textbf{ of this thing is } a.$$

$\tan(*) = \tan(2\pi)$

$\tan(0) = 0$

3. tan of $\tan^{-1}(123)$ is 123, so

$$\boxed{\tan(\tan^{-1}(123)) = 123.}$$

4. $\tan^{-1}(\tan(2\pi))$ is something such that the tan of it is $\tan 2\pi$, but does that mean that thing has to be 2π?