MATH 127 Calculus for the Sciences

Lecture 18

Today's lecture

Last time
Midterm review, so this time we have to catch up with the schedule.

This time
Course note coverage Section 3.1.1, 3.1.2
Limits
Continuity

Midterm is today!

- Covers everything up to before the reading week.
- Check your seating and time on Odyssey. I will be proctoring in one of the rooms.
- No, there is no quiz today. Yes, there is class today.

Limit

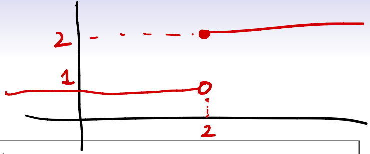
Limit is used to express a value that might not be reached, but can be recognized through a infinite process.

Example My bones weigh 15kg. So even if I keep losing weight by losing meat, I will never go below 15kg.
BUT I can not reach exactly 15kg because I am not a skeleton, so I could keep losing meat, but not all the meat:

$$\lim_{\text{meat}\to 0}(\text{myself}) = \text{bone} = 15$$

Remark Maybe I will also rant about that video about infinity by Vsauce.

Limit



Definition If the value of a function $f(x)$ approaches L as x approaches a from the ***left***, then we say the **left limit of $f(x)$ at a** is

$$\lim_{x\to a^-} f(x) = L$$

Definition If the value of a function $f(x)$ approaches L as x approaches a from the ***right***, then we say the **right limit of $f(x)$ at a** is

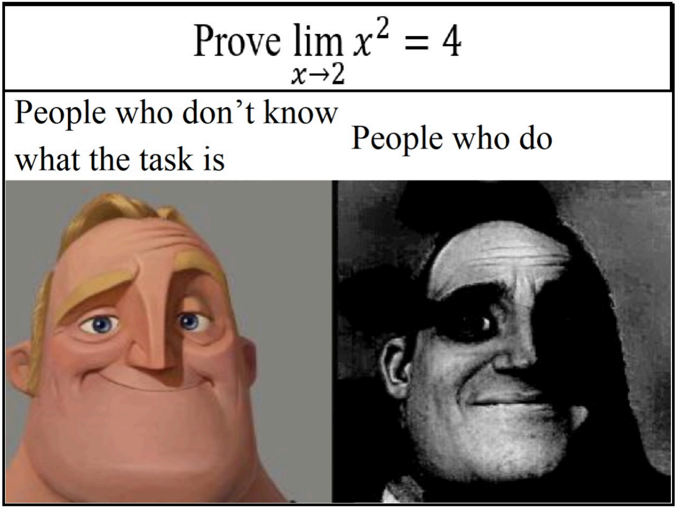
$$\lim_{x\to a^+} f(x) = L$$

Definition The limit for both left and right are L, then we write

$$\lim_{x\to a} f(x) = L$$

Limit

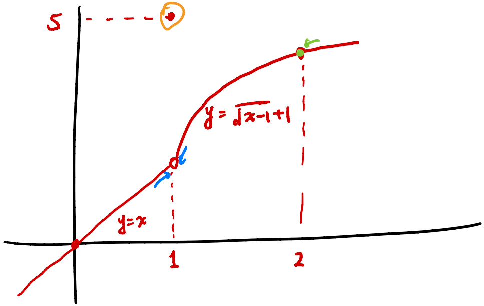
Remark This definition is super hand wavy. Have you heard of the ϵ-δ definitions?



Find limit from picture

Example Look at the following graph of $f(x)$. What are

- $\lim\limits_{x\to 1^+} f(x)$
- $\lim\limits_{x\to 1^-} f(x)$
- $\lim\limits_{x\to 2^+} f(x)$



$\lim\limits_{x\to 1^+} f(x) = 1$

$\lim\limits_{x\to 1^-} f(x) = 1$

$\lim\limits_{x\to 1} f(x) = 1$

$\boxed{f(1) = 5}$

$\lim\limits_{x\to 2^+} f(x) = \sqrt{2-1}+1 = 2$

Limit rules

There are a bunch of rules to know. You can replace $x \to a$ by $x \to a^+$ or $x \to a^-$ (but do it consistently!).

1. $\lim\limits_{x\to a} c = c$ for constant c.
2. $\lim\limits_{x\to a} x = \boxed{a}$
3. $\lim\limits_{x\to a} (f(x)+g(x)) = \begin{cases} \text{which is it...} \\ f(a)+g(a), \text{ or} \\ \boxed{\lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)}? \end{cases}$
4. $\lim\limits_{x\to a} (f(x)g(x)) = \boxed{\lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x)}$
5. $\lim\limits_{x\to a} f(x)^{\cancel{x}\,b} = \boxed{\left(\lim\limits_{x\to a} f(x)\right)^b}$, **assuming that** $\lim\limits_{x\to a} f(x)$ **is inside the** $\boxed{\text{domain of } (\cdot)^b}$.

Find limit from picture

Example Find

$$\lim_{x\to 3}\left(x^2+\frac{3}{x}\right)$$

using limit rules.

$$\lim_{x\to 3}\left(x^2+\frac{3}{x}\right) = \lim_{x\to 3} x^2 + \lim_{x\to 3}\frac{3}{x}$$

$$= \left(\lim_{x\to 3} x\right)^2 + \lim_{x\to 3} 3\cdot x^{-1}$$

$$= (3)^2 + \lim_{x\to 3} 3 \cdot \lim_{x\to 3} x^{-1}$$

$$= 3^2 + 3\cdot\left(\lim_{x\to 3} x\right)^{-1}$$

$$= 9 + 3\cdot 3^{-1} = 9+1 = 10.$$

Having zeroes in denominator

Exercise What about

$$\lim_{x\to 1^-} \frac{1}{x-1}$$

$$\lim_{x\to 1^-} (x-1)^{-1}$$

$$= \left(\lim_{x\to 1^-} x-1\right)^{-1}$$

← power rule does not apply

$$= \left(\lim_{x\to 1^-} x - \lim_{x\to 1^-} 1\right)^{-1}$$

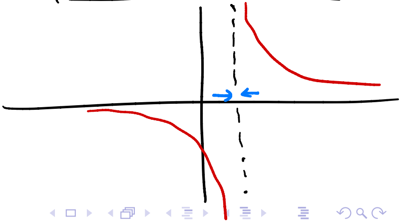
$$= (1-1)^{-1} = \cancel{0^{-1}}$$

As $x \to 1^-$

$\boxed{x-1}$ approaches 0

from negative side.

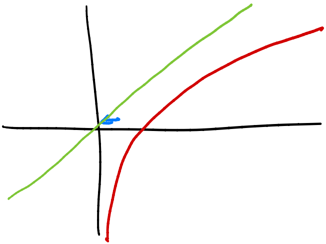
$\frac{1}{x-1}$ approaches $-\infty$



Having infinities show up

Example What is

$$\lim_{x \to 0^+} \frac{\ln(x)}{x}$$



As $x \to 0^+$, $\boxed{\ln x}$ approaches $-\infty$

As $x \to 0^+$, $\boxed{x}$ approaches 0

from the positive side.

So $\displaystyle\lim_{x\to 0^+} \frac{\ln x}{x} = -\infty$.

Continuity

Given a function $f(x)$, we can look at three things

(1) the value at a: $f(a)$

(2) the left limit: $\lim_{x \to a^-} f(x)$

(3) the right limit: $\lim_{x \to a^+} f(x)$

Definition

- If (1)=(2)=(3), i.e.
$$f(a) = \lim_{x \to a} f(x),$$
then f is
continuous at a.
- We say f is
discontinuous at a
if f is $\boxed{\text{not continuous.}}$.

Remark To talk about whether f is continuous at $x = a$, the input $x = a$ must be in the domain of f in the first place. Otherwise the term (1) doesn't even make sense.

Example

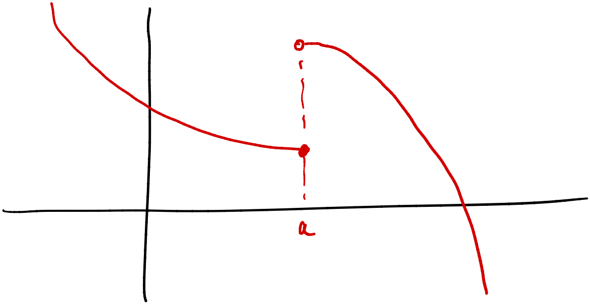
Given a function $f(x)$, we can look at three things

(1) the value at a: $f(a)$

(2) the left limit: $\lim\limits_{x \to a^-} f(x)$

(3) the right limit: $\lim\limits_{x \to a^+} f(x)$

Draw a graph such that (1)=(2) but (2)≠(3).



Example

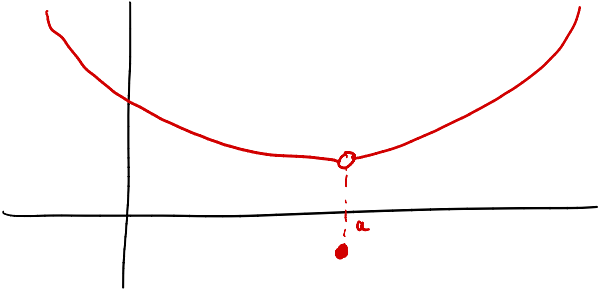
Given a function $f(x)$, we can look at three things

(1) the value at a: $f(a)$

(2) the left limit: $\lim_{x \to a^-} f(x)$

(3) the right limit: $\lim_{x \to a^+} f(x)$

Draw a graph such that (2)=(3) but (1)$\neq$(2).

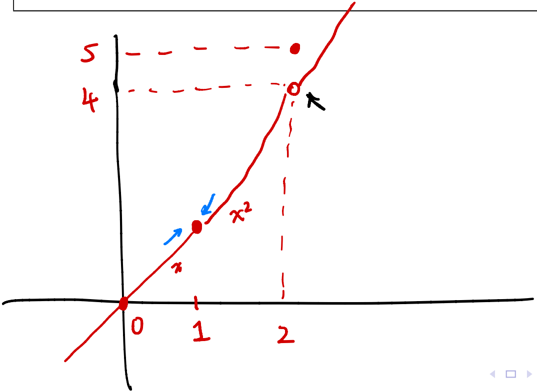


Example

Example Let

$$f(x) = \begin{cases} x, & x < 1; \\ x^2, & 1 \leq x < 2; \\ 5, & x = 2; \\ 2x, & 2 < x. \end{cases}$$

Is $f(x)$ continuous at $x = 0, 1, 2$?



$f(x)$ is continuous at 0;
continuous at 1;
discontinuous at 2.

Continuity rules

Suppose f, g are two functions continuous at $x = a$, then

1. $f + g$ is continuous at $x = a$;
2. cf is continuous at $x = a$ for any constant c;
3. fg is continuous at $x = a$;
4. $\frac{1}{f}$ is continuous at $x = a$, provided that $\boxed{f(a) \neq 0}$.

Remark Provided $g(a) \neq 0$, do we know $\frac{f}{g}$ is continuous at a?

5. If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Continuity rule example

Example Is the function

$$f(x) = x(x^2 - \sin x) + e^x$$

continuous at $x = 0$? Explain why.

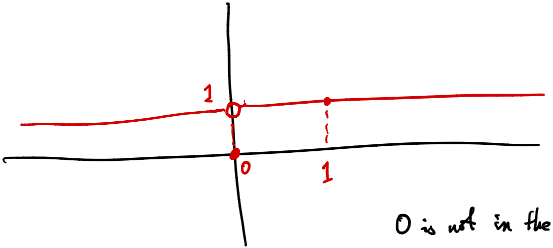
Continuity rule example

Example Is the function

$$f(x) = \frac{x}{x}$$

continuous at 1?

What about at $x = 0$?



0 is not in the domain of f

so it does not make sense

to talk about continuity at 0.

Continuity rule example

Example left as exercise? Determine the interval where

$$f(x) = \tan(\arccos(x))$$

is continuous.

Hint: Use continuity rule 5.

Reminder: set an alarm for 1850 for the midterm.

1. domain of arccos
2. continuity rule
3. domain of f.