$P = W$ CONJECTURE AND BRIDGELAND STABILITY

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Abstract. This is the notes taken from talks given by Ziyu Zhang, Junliang Shen, and Xiaolei Zhao at the algebraic geometry summer school of Fudan University in July 2023. These talks feature an introduction to derived category, the $P = W$ conjecture for the Hitchin moduli space on a curve, and Bridgeland moduli spaces for K3 categories. Basic knowledge about slope/Gieseker stability for sheaves is assumed.

Contents

1. Derived categories

1.1. **Motivation.** Let us recall the sheaf cohomology of a sheaf of abelian group F on a topological space X. We compute $H^i(X, F)$ as the i-th derived functor of $\Gamma(X, \cdot)$.

- Step 1: Take an injective resolution of F:
$$0 \to F \to I^0 \to I^1 \to \dots,$$
- Step 2: Apply $\Gamma(X, \cdot)$ to $I^\bullet$:
$$0 \to \Gamma(X, I^0) \to \Gamma(X, I^1) \to \dots,$$
- Step 3: Take cohomology of this sequence
$$H^i(X, F) = H^i(\Gamma(X, I^\bullet)).$$

Looking at this computation, one might ask the following questions:

(i) Why use an injective resolution?
(ii) When we take Γ of $I^\bullet$, why did we throw away the F term?

We first consider question (ii) by taking a different approach. View F as a complex and the injective resolution as a map f between complexes:

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots & = F^\bullet \arrow[d, "f"] \\
\dots \arrow[r] & 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \dots & = I^\bullet
\end{tikzcd}$$

Then f induces isomorphisms $H^i(F^\bullet) \cong H^i(I^\bullet)$. Such a morphism f is called a *quasi-isomorphism* (q.i.). To build on this new perspective, we would like to look for a theory which deals with complexes, and considers quasi-isomorphic complexes to be "the same".

Let $\mathcal{A}$ be an abelian category. We consider the Category of complexes:

$$\mathcal{C}(\mathcal{A}) = \begin{cases} \text{objects: } \{A^\bullet = [\dots \xrightarrow{d} A^{n-1} \xrightarrow{d} A^n \xrightarrow{d} A^{n+1} \xrightarrow{d} \dots] \text{ with } d^2 = 0\}, \\ \text{morphisms: } \{f^\bullet : A^\bullet \to B^\bullet \text{ compatible with } d.\} \end{cases}$$

The naive way to construct a category $D(\mathcal{A})$ where quasi-isomorphic complexes are isomorphic is by manually inverting the quasi-isomorphisms in $\mathcal{C}(\mathcal{A})$. That is setting

$$\mathrm{Hom}_D(A^\bullet, B^\bullet) := \mathrm{Hom}_{\mathcal{C}}(A^\bullet, B^\bullet)[\text{q.i.}^{-1}].$$

This way, the morphisms between $A^\bullet$ and $B^\bullet$ in this new categories are of form

$$A^\bullet \to C_1^\bullet \leftarrow C_2^\bullet \to \dots \leftarrow C_n^\bullet \to B^\bullet$$

for arrows in $\mathrm{Hom}_{\mathcal{C}}(A^\bullet, B^\bullet)$, and the left arrows are quasi-isomorphisms. This is not what we want as this chain of arrows can be arbitrarily long and is difficult to control.

1.2. **Homotopy categories.**

Definition 1.1. A morphism $f : A^\bullet \to B^\bullet$ in $\mathcal{C}(\mathcal{A})$ is *homotopic to* 0, write $f \sim 0$, if there exist $h^i : A^i \to B^{i-1}$, forming the following diagram (not necessarily commutative):

$$\begin{tikzcd}
\dots \arrow[r] & A^{n-1} \arrow[r] \arrow[d] \arrow[dl] & A^n \arrow[r] \arrow[d, dashed, "f^n"] \arrow[dl, dashed, "h^{n-1}"'] & A^{n+1} \arrow[r] \arrow[d] \arrow[dl, dashed, "h^{n+1}"'] & \dots \arrow[dl] \\
\dots \arrow[r] & B^{n-1} \arrow[r] & B^n \arrow[r] & B^{n+1} \arrow[r] & \dots
\end{tikzcd}$$

such that

$$f^n = h^{n+1} \circ d + d \circ h^n$$

for all n.

Definition 1.2. The *homotopy category* associated to an abelian category $\mathcal{A}$ is

$$K(\mathcal{A}) = \begin{cases} \text{objects: } \mathrm{obj}(\mathcal{C}(\mathcal{A})), \\ \text{morphisms: } \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(A^\bullet, B^\bullet)/(f \sim 0). \end{cases}$$

Definition 1.3. The *derived category* associated to an abelian category $\mathcal{A}$ is

$$D(\mathcal{A}) = \begin{cases} \text{objects: } \mathrm{obj}(\mathcal{C}(\mathcal{A})), \\ \text{morphisms: } \begin{cases} \text{equivalent classes of roofs } \begin{tikzcd} & C^\bullet \arrow[dl, "q.i."'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd} \end{cases} \end{cases}$$

where two roofs $C_1^\bullet, C_2^\bullet$ over $A^\bullet, B^\bullet$ are equivalent if there exists $C^\bullet \to C_1^\bullet, C_2^\bullet$ making the following diagram commuative in $K(\mathcal{A})$:

$$\begin{tikzcd}
& & C^\bullet \arrow[dl] \arrow[dr] & & \\
& C_1^\bullet \arrow[dl] \arrow[drrr] & & C_2^\bullet \arrow[dlll] \arrow[dr] & \\
A^\bullet & & & & B^\bullet
\end{tikzcd}.$$

The bounded derived categories $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(A)$ are defined by as the full subcategory whose objects are complexes with $H^i(\cdot) = 0$ for large, small, or all but finitely many i, respectively.

The original category $\mathcal{A}$ embeds into the derived category by viewing $A \in \mathcal{A}$ as a complex where we put it in the 0-th position, denote

$$A[0] = [\ldots \to 0 \to A \to 0 \to \ldots] \in D(\mathcal{A}).$$

In general, we define a *shift functor* $[n]$ that shifts complexes n slots to the left.

$$[n] : D(\mathcal{A}) \to D(A)$$

$$A^\bullet = [A^i] \mapsto [A^{i+n}].$$

We also have the *truncation functor* that cuts a complex off at the m-th position.

$$\tau_{\leq m} : D(\mathcal{A}) \to D(A)$$

$$A^\bullet \mapsto [\ldots \to A^{m-1} \to \ker(d : A^m \to A^{m+1}) \to 0 \to \ldots].$$

The truncation functor $\tau_{\geq m}$ is defined similarly with the left most term replaced by $\operatorname{coker}(d : A^m \to A^{m+1})$. We observe natural morphisms

$$\tau_{\leq m} A^\bullet \to A^\bullet, \quad A^\bullet \to \tau_{\geq m} A^\bullet.$$

Definition 1.4. Let $\mathcal{A}, \mathcal{B}$ be abelian categories, and $F : \mathcal{A} \to \mathcal{B}$ a left exact additive functor. Suppose $\mathcal{A}$ has enough injectives. Then the *right derived functor* $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is defined as

$$RF(A^\bullet) := F(I^\bullet)$$

where $I^\bullet$ consists of injective objects and is quasi-isomorphic to $A^\bullet$. Set $R^iF(A^\bullet) := H^i(RF(A^\bullet))$.

Recall the sheaf cohomology is the same as taking the derived functor of the global section functor. Back to the question we had before: do we have to take injective resolutions? We look at this computation of in another perspective. In the category $K^+(\mathcal{A})$, consider all quasi-isomorphisms $A^\bullet \to C^\bullet$. Then we may define

$$RF(A^\bullet) := \varinjlim_{A^\bullet \to C^\bullet} F(C^\bullet), \text{ if it exists.}$$

In the case when $\mathcal{A}$ has enough injectives, the resolution $A^\bullet \to I^\bullet$ is a final object among all quasi-isomorphisms, so the above limit exists and agrees with our previous definition.

1.3. **Perverse sheaves.** A *triangulated category* consists of a triplet $(D, ([n])_{n\in\mathbb{Z}}, \mathcal{T})$ such that

- D is an additive category.
- $[n] : D \to D$ are additive functors with $[n] \circ [m] = [n+m]$.
- $\mathcal{T}$ is the set of *exact triangles* of form
$$X \to Y \to Z \to X[1]$$
such that
  (1) Any triangle isomorphic to an exact triangle is exact.
  (2) $X \xrightarrow{\mathrm{id}} X \to 0 \to X[1]$ is exact.
  (3) For all $f : X \to Y$, there exists some exact triangle of form $X \to Y \to Z \to X[1]$.
  (4) $X \to Y \to Z \to X[1]$ is exact if and only if $Y \to Z \to X[1] \xrightarrow{-f[1]} Y[1]$ is exact.
  (5) Given two exact triangles and morphisms forming the following diagram, a dotted morphism exists making the whole diagram commute:
$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}.$$
  (6) Given exact triangles $X \to Y \to Q_1 \to X[1]$, $X \to Z \to Q_2 \to X[1]$, $Y \to Z \to Q_3 \to Y[1]$, the dotted triangle induced by the previous condition $Q_1 \to Q_2 \to Q_3 \to Q_1[1]$ is exact, with the following commutative diagram:
$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & Y \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d, dashed] & X[1] \\
 & Z \arrow[r] \arrow[d] & Q_2 \arrow[ur] \arrow[dl, dashed] & \\
 & Q_3 \arrow[d] \arrow[dl, dashed] & & \\
Q_1[1] & Y[1] & &
\end{tikzcd}.$$

Theorem 1.5. *The derived category $D(\mathcal{A})$ is a triangulated category.*

Proof. (Sketch) The exact triangles are of form $X^\bullet \xrightarrow{f} Y^\bullet \to C(f) \to X^\bullet[1]$, where $C(f)$ is the *cone* of f, defined by
$$C(f)^n = Y^n \oplus X^{n+1}$$
$$d^n : Y^n \oplus X^{n+1} \to Y^{n+1} \oplus X^{n+2}$$
$$(y, x) \mapsto \begin{bmatrix} d_Y^n & f^{n+1} \\ 0 & d_X^{n+1} \end{bmatrix} \cdot \begin{bmatrix} y \\ x \end{bmatrix}.$$
∎

Definition 1.6. Let D be a triangulated category. Let $D^{\leq 0}, D^{\geq 0}$ be two of its full subcategories. Write $D^{\leq n} := D^{\leq 0}[-n]$ and $D^{\geq n} := D^{\geq 0}[-n]$. The pair $(D^{\leq 0}, D^{\geq 0})$ is called a *t-structure* if

(1) $D^{\leq -1} \subsetneq D^{\leq 0}$ and $D^{\geq 1} \subsetneq D^{\geq 0}$.
(2) If $X \in D^{\leq 0}, Y \in D^{\geq 1}$, then $\mathrm{Hom}_D(X, Y) = 0$.
(3) For all $X \in D$, there exists exact triangle $X_0 \to X \to X_1 \to X_0[1]$ with $X_0 \in D^{\leq 0}$, $X_1 \in D^{\geq 1}$.

Our motivation for defining triangulated categories is so that we may generalize the derived category and can consider different t-structures on triangulated categories. For example, we have the standard t-structure on $D(\mathcal{A})$ where $D^{\geq 0}(\mathcal{A})$ are those complexes with $H^{<0}(A^\bullet) = 0$ for $i < 0$, and $D^{\leq 0}(\mathcal{A})$ for those with $H^{>0}(A^\bullet) = 0$. We also have the perverse t-structure defined in the next section.

Theorem 1.7. *If $(D^{\leq 0}, D^{\geq 0})$ is a t-structure of a triangulated category D, then $D^{\leq 0} \cap D^{\geq 0}$ is an abelian category, which we call the* heart *of this t-structure.*

The motivation for perverse sheaves was to recover Poincaré (Verdier) duality for "singular manifolds". The use of intersection cohomology was later developed into perverse sheaves.

Definition 1.8. Let X be an algebraic variety or analytic space over $\mathbb{C}$. A locally finite partition $X = \sqcup_{\alpha \in I} X_\alpha$ by locally closed subspace is a *stratification* of X if for every α,

$$\overline{X_\alpha} = \bigsqcup_{\beta \in J} X_\beta$$

for some subset $J \subseteq I$.

In general we write the derived category of coherent sheaves on X as $D(X) = D(\mathrm{coh}(X))$. We shall focus on constructable sheaves defined as follows.

Definition 1.9. Let $\underline{\mathbb{C}}_X$ be the constant sheaf on X. A $\underline{\mathbb{C}}_X$-module F is a *constructable sheaf* if there exists a stratification $X = \sqcup X_\alpha$ such that F_{X_α} is locally constant for all α. Denote $D^b_c(X)$ the bounded derived category of constructable sheaves.

Definition 1.10. The *perverse t-structure*, denoted by $({}^pD_c^{\leq 0}(X), {}^pD_c^{\geq 0}(X))$ on the triangulated category $D^b_c(X)$ is defined by

$$F^\bullet \in {}^pD_c^{\leq 0}(X) \text{ if } \dim \operatorname{supp} H^j(F^\bullet) \leq j \text{ for all } j,$$
$$F^\bullet \in {}^pD_c^{\geq 0}(X) \text{ if } \dim \operatorname{supp} H^j(\mathbb{D}F^\bullet) \leq j \text{ for all } j$$

where $\mathbb{D}F^\bullet$ denotes the dualizing complex of $F^\bullet$. A sheaf is a *perverse sheaf* if it is in the heart of the perverse t-structure.

Example 1.11. Let X be proper and smooth over $\mathbb{C}$ of dimension d. We ask when $L[n]$ is perverse for a locally constant sheaf L. Here from smoothness we have $\mathbb{D}(\cdot) = R\mathscr{H}om(\cdot, \underline{\mathbb{C}}_X[2d])$. Now $\dim \operatorname{supp} L = d$, and

$$H^j(L[n]) = \begin{cases} L, j = -n, \\ 0, j \neq -n. \end{cases}$$

So $L[n] \in {}^pD_c^{\leq 0}(X)$ if $d \leq n$, and $L[n] \in {}^pD_c^{\geq 0}(X)$ if $d \geq n$. Therefore $L[n]$ is a perverse sheaf if and only if $n = \dim X$.

Example 1.12. A similar argument shows that a skyscraper sheaf $\underline{\mathbb{C}}_p$ on a point is a perverse sheaf (viewed as a complex), and if L is supported on a curve C, then the complex $L[1]$ is perverse.

2. $P = W$ conjecture

Let X be a topological space. We have the following three cohomology theories: singular cohomology $H^*_{\mathrm{sing}}(X, \underline{\mathbb{Q}}_X)$, de Rham cohomology $H^*_{\mathrm{dR}}(X, \underline{\mathbb{R}}_X)$, and sheaf cohomology $H^*(X, \underline{\mathbb{Q}}_X)$.

The singular cohomology is computed by taking a flasque resolution of pre-sheaves C^i of i-simplices

$$0 \to \underline{\mathbb{Q}}_X \to C^0 \to C^1 \to \dots.$$

One can show that $H^i(X, \underline{\mathbb{Q}}_X) = R^i\Gamma(C^\bullet) = H^i_{\mathrm{sing}}(X, \underline{\mathbb{Q}}_X)$.

The de Rham cohomology is computed by taking a resolution of fine sheaves $A^i(X)$, which are the sheaves of i-forms on X

$$0 \to \underline{\mathbb{R}}_X \to A^0 \to A^1 \to \dots.$$

We have $H^i(X, \underline{\mathbb{R}}_X) = R^i\Gamma(A^\bullet) = H^i_{\mathrm{dR}}(X, \underline{\mathbb{Q}}_X)$.

2.1. **Mixed Hodge structures.** Let X be a compact Kähler manifold. We can forget the $\mathbb{C}$-structure and get the de Rham cohomology from sheaf cohomology

$$H^i_{\mathrm{dR}}(X, \mathbb{C}) = H^i(X, \underline{\mathbb{C}}_X).$$

Consider the Kähler differential Ω on X. The cohomology of $\Omega^p = \wedge^p\Omega$ gives us the Dolbeault cohomology

$$H^q(X, \Omega^p).$$

This can be computed using $A^{p,q}$, the sheaf of differential forms of type p, q, locally given by

$$\sum f_{i_1,\dots,i_p,j_1,\dots,j_q} dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\overline{z_{j_1}} \wedge \dots \wedge d\overline{z_{j_q}}$$

for smooth functions f. We have maps $A^{p,q} \to A^{p,q+1}$ by taking partial derivatives on $\overline{z_{j_1}}, \dots, \overline{z_{j_q}}$, giving us a fine resolution

$$0 \to \Omega^p \to A^{p,0} \to A^{p,1} \to \dots.$$

Taking its cohomology, we obtain

$$H^{p,q}(X) := H^q(X, \Omega^p).$$

Theorem 2.1 (Hodge Theorem). *We have the decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

with symmetry

$$H^{p,q}(X) = \overline{H^{q,p}(X)}.$$

We call the decomposition from above a *weight k Hodge decomposition*. For differential k-forms, we can locally decompose them by $dz_{i_1} \dots dz_{i_p} d\overline{z_{j_1}} \dots d\overline{z_{j_q}}$ for $p + q = k$. The above theorem says that such a decomposition is possible even on the level of cohomology.

Remark 2.2. If k is odd, the dimension of H^k must be even by the symmetry.

Recall that we were under the assumption of X a compact Kähler manifold. Let us now see what could happen when X is smooth but non-compact.

Example 2.3. Let $X = \mathbb{C}^* = \mathbb{A}^1_{\mathbb{C}} - \{0\}$. We have

$$\dim_{\mathbb{C}} H^0(\mathbb{C}^*, \mathbb{C}) = 1 \text{ with generator } 1,$$

$$\dim_{\mathbb{C}} H^1(\mathbb{C}^*, \mathbb{C}) = 1 \text{ with generator } dz/z,$$

so it is impossible for $\mathbb{C}^*$ to have a weight 1 Hodge decomposition as H^1 has odd dimension.

Let us complete $\mathbb{C}^*$ to $\mathbb{P}^1$ by adding in the points 0 and ∞. We have $\mathbb{P}^1 = \mathbb{A}^1_0 \cup \mathbb{A}^1_\infty$ with $\mathbb{A}^1_0 \cap \mathbb{A}^1_\infty = \mathbb{C}^*$. Using the Mayer-Vietoris sequence

$$\dots \to H^1(\mathbb{P}^1)| -> H^1(\mathbb{A}^1) \oplus H^1(\mathbb{A}_1) \to H^1(\mathbb{C}^*) \to H^2(\mathbb{P}^1) \to \dots$$

where $H(\mathbb{A}^1) = 0$, we get $H^1(\mathbb{C}^*) \cong H^2(\mathbb{P}^1)$, so $H^1(\mathbb{C}^*)$ admits a weight 2 Hodge structure.

We would like to obtain a Hodge theory on smooth non-compact algebraic variety U. Take a compactification $U \hookrightarrow X$ with X smooth such that the boundary divisor $D = X - U$ is a simple normal crossing. This is possible by results from minimal model program. We need to obtain the cohomology of X and $D = \sum D_i$ in order to get $H^*(U)$. This will give us a "mixed Hodge structure".

Let V be a finite dimensional real vector space. A $\mathbb{R}$-*Hodge decomposition of weight* k is a decomposition

$$V \otimes_{\mathbb{R}} \mathbb{C} = \oplus_{p+q=k} V^{pq}$$

such that $V^{pq} = \overline{V^{qp}}$. Such a decomposition is equivalent to a $\mathbb{R}$-*Hodge filtration*

$$V \otimes_{\mathbb{R}} \mathbb{C} = F^0 \supsetneq F^1 \supsetneq \dots$$

such that $F_p \cap \overline{F_q} = 0$ for any $p + q = k + 1$. The correspondence is given by $F^p = \oplus_{s \geq p} V^{st}, V^{pq} = F^p \cap \overline{F^q}$.

Definition 2.4. A $\mathbb{Z}$-*Mixed Hodge structure* on a finitely generated $\mathbb{Z}$-module V consists of two filtrations:

- an increasing filtration $W_\bullet$ on $V \otimes_{\mathbb{Z}} \mathbb{Q}$,
- a decreasing filtration $F^\bullet$ on $V \otimes_{\mathbb{Z}} \mathbb{C}$

such that for each k, the gradient piece

$$\mathrm{Gr}_k^W(V \otimes \mathbb{Q}) := W_k / W_{k-1}$$

has a Hodge structure of weight k.

Recall for $f : X \to Y$ a continuous map on topological spaces, the pushforward functor $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is left exact. We have right derived functor

$$Rf_* : D^+(\mathrm{Sh}(X)) \to D^+(\mathrm{Sh}(Y))$$

with

$$R^i f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$$
$$F \mapsto H^i(Rf_*(F)).$$

The sheaf $R^i f_*(F)$ is exactly the associated sheaf of the pre-sheaf given by $U \mapsto H^i(F|_{f^{-1}(U)})$. In the case f is the map from X to a point, we have $f_* F = \Gamma(X, F)$, so $Rf_* = R\Gamma, R^i f_* = H^i$.

Given a map $f : X \to Y$, we manually add on a map $g : Y \to \mathrm{pt}$, which gives us a commutative diagram

$$\begin{tikzcd} D^+(\mathrm{Sh}(X)) \arrow[r, "Rf_*"] \arrow[rr, bend right, "R(g\circ f)_*"'] & D^+(\mathrm{Sh}(Y)) \arrow[r, "Rg_*"] & D^+(\mathrm{Sh}(\mathrm{pt})) \end{tikzcd}.$$

Note that this diagram is in general not commutative for composition of maps $f \circ g$. From this we obtain

$$\mathbb{H}^*(Y, Rf_* F) = H^*(X, F)$$

so we can forget the map g and use this identity in general. Here $\mathbb{H}$ denotes hypercohomology of a complex, computed by taking a quasi-isomorphism to an injective resolution, then taking cohomology of global sections. We can also write

$$H^p(Y, R^q f_*(F)) \Rightarrow H^{p+q}(X, F)$$

as a spectral sequence.

Suppose $U \overset{f}{\hookrightarrow} X$ with $X - U = D_1 \cup \cdots \cup D_k$ simple normal crossing. Then from above we have

$$H^*(U, \underline{\mathbb{C}}_U) = H^*(X, Rf_* \underline{\mathbb{C}}_U).$$

It is known that the resolution

$$\underline{\mathbb{C}}_U \to \Omega_U^0 \to \Omega_U^1 \to \dots$$

is enough to compute $Rf_*\underline{\mathbb{C}}_U$ (though not enough for $H^*(U, \underline{\mathbb{C}}_U)$. Thus we would like to compute

$$H^*(X, [f_*\Omega_U^0 \to f_*\Omega_U^1 \to \dots])$$

Define the logarithmic poles $\Omega_X^p(\log D)$ on X as a sheaf with stocks

$$\left(\Omega_X^1(\log D)\right)_x = O_{X,x}\frac{dz_1}{z_1} \oplus \dots \oplus O_{X,x}\frac{dz_k}{z_k} \oplus O_{X,x}dz_{k+1} \oplus \dots \oplus O_{X,x}dz_n,$$

$$\Omega_X^p(\log D) = \bigwedge^p \Omega_X^1(\log D)$$

where $n = \dim X$ and x is a closed point in D locally given by $\{z_1 \dots z_k = 0\}$.

Proposition 2.5. *There exists a quasi-isomorphism*

$$\Omega_X^\bullet(\log D) \to f_*\Omega_U^\bullet.$$

Now we have $H^*(U, \mathbb{C}) = H^*(X, \Omega_X^\bullet(\log D))$. Consider the following definition

$$W_m(\Omega_X^p(\log D)) := \begin{cases} 0, m < 0, \\ \Omega_X^p(\log D), m \geq p, \\ \Omega_X^{p-m} \wedge \Omega_X^m(\log D), 0 \leq m \leq p. \end{cases}$$

We obtain an inclusion $W_m(\Omega_X^\bullet(\log D)) \hookrightarrow \Omega_X^\bullet(\log D)$ which induces a map on their cohomology. This gives us a mixed Hodge decomposition for $H^*(U, \mathbb{C})$ as follows

$$W_m H^k(U, \mathbb{C}) := \operatorname{im}\{\mathbb{H}^k(X, W_{m-k}\Omega_X^\bullet(\log D)) \to H^k(U, \mathbb{C})\}$$

$$F^p H^k(U, \mathbb{C}) := \operatorname{im}\{\mathbb{H}^k(X, F^p\Omega_X^\bullet(\log D)) \to H^k(U, \mathbb{C})\}$$

2.2. **Moduli of Higgs bundles.** Consider the real manifold $(S^1)^2 \times \mathbb{R}^2$, i.e. on each point of $\mathbb{R}^2$ we attach a torus $(S^1)^2$. The cohomology

$$H^*((S^1)^2 \times \mathbb{R}^2) \cong \mathbb{Q}^4$$

is decomposed into

$$H^0 \oplus H^1 \oplus H^2 \cong \mathbb{Q} \oplus \mathbb{Q}^2 \oplus \mathbb{Q}.$$

We discuss two ways of realizing this space as an algebraic variety:

(i) $\mathbb{C}^* \times \mathbb{C}^*$,
(ii) $E \times \mathbb{C}$ for some elliptic curve E.

Recall the weight filtration from the mixed Hodge structure is of form

$$W_\bullet \subseteq H^*(X)$$

with each $\operatorname{Gr}_k^W H^*(X)$ having a pure Hodge structure of weight k.

For (i), we have

$$H^*(\mathbb{C}^*) = W_2 \supseteq W_1 = W_0,$$

where Gr_2 is generated by dx/x, W_0 is generated by the fundamental class. This gives us

$$H^*(\mathbb{C}^* \times \mathbb{C}^*) = W_4 \supseteq W_2 \supseteq W_0$$

where Gr_4 is generated by $\frac{dx}{x} \wedge \frac{dy}{y}$, Gr_2 by $\frac{dx}{x}, \frac{dy}{y}$ and W_0 by the fundamental class.

For *(ii)*, consider the projection

$$h : E \times \mathbb{C} \to \mathbb{C}$$

giving us

$$H^*(E \times \mathbb{C}) = H^*(\mathbb{C}, Rh_*\underline{\mathbb{Q}}_{E\times\mathbb{C}})$$

where $Rh_*\underline{\mathbb{Q}}_{E\times\mathbb{C}} \in D^b_c(\mathbb{C})$. One can compute

$$Rh_*\underline{\mathbb{Q}}_{E\times\mathbb{C}} \cong \underline{\mathbb{Q}}_{\mathbb{C}} \oplus \underline{\mathbb{Q}}^2_{\mathbb{C}}[-1] \oplus \underline{\mathbb{Q}}_{\mathbb{C}}$$

inducing a *Leray filtration* $L_0 \subseteq L_1 \subseteq L_2$ by including term by term from this decomposition. Observe that $L_k(E\times\mathbb{C}) = W_{2k}(\mathbb{C}^*\times\mathbb{C}^*) = H^{\leq k}((S^1)^2\times\mathbb{R}^2)$.

Note that the cases (i) and (ii) can be viewed as moduli spaces associated with E and $\mathrm{GL}_1(\mathbb{C})$. For (i), we can consider $\mathbb{C}^*\times\mathbb{C}^*$ as the set of $\mathrm{GL}_1(\mathbb{C})$-representations of the fundamental group of a torus E, i.e.

$$\mathbb{C}^*\times\mathbb{C}^* = \{\pi_1(E)\to \mathrm{GL}_1(\mathbb{C})\} = \{\text{rank 1 local systems on } E\}.$$

For (ii), we view $E\times\mathbb{C}$ as the set of all *Higgs bundles* on E, that is pairs (L,Θ) such that L is a line bundle on E with fixed degree d, and $\Theta : L \to L\otimes\omega_E$

$$E\times\mathbb{C} = \{(L,\Theta) \text{ Higgs bundle}\}.$$

The map $h : E\times\mathbb{C}\to\mathbb{C}$ can be viewed as $\operatorname{tr}\Theta$ mapping to $H^0(E,\omega_E)=\mathbb{C}$, which we call the *Hitchin fibration.*

Now we may generalize this discussion to higher ranks. Let C be a curve with $g\geq 2$, and consider $\mathrm{GL}_n(\mathbb{C})$ for $n\geq 2$. Fix a degree d coprime to n (so we will obtain a smooth moduli space).

(i) (Betti moduli) Let

$$R = \left\{ A_1, B_1, \ldots, A_g, B_g \in \mathrm{GL}_n(\mathbb{C}) \,\middle|\, \prod_{i=1}^{g}[A_i,B_i] = e^{\frac{2\pi i d}{n}}\mathrm{Id} \right\}.$$

Consider the following GIT quotient

$$\begin{aligned}\mathcal{M}_{\mathrm{B}} &= R /\!\!/ (\mathrm{GL}_n(\mathbb{C}) \text{ action by conjugation})\\ &= \operatorname{Spec}(R^{\mathrm{GL}_n(\mathbb{C})})\end{aligned}$$

which is a smooth affine variety.

(ii) (Dolbeault moduli) Define

$$\mathcal{M}_{\mathrm{Dol}} = \text{moduli space of (slope) stable Higgs bundles of rank } n, \text{ degree } d \text{ over } C.$$

By (semi)-stability of a Higgs bundle $(E.\Theta)$, we mean that for all subbundle F such that

$$F \subsetneq E, \quad \operatorname{im}\Theta|_F \subseteq F\otimes\omega_C,$$

one has $\mu(F)(\leq)\mu(E)$. Note that n,d coprime means that semi-stability is equivalent to stability. The Hitchin fibration in this case, mapping to an affine space, is given by

$$h : \mathcal{M}_{\mathrm{Dol}} \to A = \bigoplus_{i=1}^{n} H^0(C,\omega_C^i)$$

$$(E,\Theta) \mapsto (\operatorname{tr}\Theta, \operatorname{tr}\wedge^2\Theta, \ldots, \operatorname{tr}\det\Theta)$$

i.e. this map records the characteristic polynomial of Θ.

Proposition 2.6. *$\mathcal{M}_{Dol}$ is smooth quasi-projective, and is projective relative to A via h.*

Analytically we have $\mathcal{M}_{\mathrm{Dol}} \xrightarrow{C^\infty} \mathcal{M}_{\mathrm{B}}$, giving us $H^*(\mathcal{M}_{\mathrm{Dol}}) \cong H^*(\mathcal{M}_{\mathrm{B}})$. One might ask in algebraic geometry, whether the equality we obtained before,

$$L_k(h : \mathcal{M}_{\mathrm{Dol}} \to A) = W_{2k}(\mathcal{M}_{\mathrm{B}}) = H^{\leq k}(-)$$

still holds. The answer is negative for all above equalities. However, we could modify L_k to using perverse sheaf. The new filtration we obtain, which we call P_k, will give us $P_k = W_{2k}$. This is the $P=W$ conjecture by de Cataldo-Hausel-Migliorini which has been proven by Davesh Maulik and Junliang Shen.

Denote $\mathrm{Perv}(X)$ the category of perverse sheaves on X, i.e. the heart of the perverse t-structure. Let ${}^p\tau_{\leq k}, {}^p\tau_{\geq k}$ be the *perverse truncation functors* associated to the t-structure, which are given by adjoints of the inclusions ${}^pD^{\leq k}, {}^pD^{\geq k} \hookrightarrow {}^pD$. We shall define the perverse filtration.

Let $f : X \to Y$ be a proper map of smooth varieties. Set $r := \dim X \times_Y X - \dim X$, so if f is equidimensional, then r is the relative dimension. We have

$$\begin{aligned} H^m(X, \mathbb{Q}) =& H^m(Y, Rf_*\underline{\mathbb{Q}}_X)) \\ =& H^{m+d_X-r}(Y, Rf_*\underline{\mathbb{Q}}_X)[r - dX]). \end{aligned}$$

where $d_X = \dim X$. Set

$$P_K := H^*(Y, {}^p\tau_{\leq k} Rf_*\underline{\mathbb{Q}}_X)[r - dX]),$$

giving us a filtration $P_0 \subseteq P_1 \subseteq \ldots \subseteq H^*(X)$.

Example 2.7. Let $(X, L = \mathcal{O}(1))$ be a smooth projective variety. We get maps

$$c_1(L)^i : H^{d_X-i}(X, \mathbb{Q}) \xrightarrow{\sim} H^{d_X+i}(X, \mathbb{Q}).$$

Let $f : X \to Y$ with L relatively ample, then we have maps

$$c_1(L)^i : \mathrm{Gr}^P_{r-i} H^*(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{Gr}^P_{r+i} H^*(X, \mathbb{Q}).$$

2.3. **Cohomology on moduli spaces.** A common method to study the cohomology of a moduli space is with its tautological classes. In our case we can use them to under stand the filtrations P and W.

Example 2.8. Let V be an $n + 1$ dimensional complex vector space. We have

$$\mathbb{P}(V) = \{L \subseteq V \text{ lines through origin}\}.$$

Above this space we have a line bundle $\mathcal{O}(1) \to \mathbb{P}(V)$ where the fibre over $[L]$ is L. One then observes that $H^*(\mathbb{P}(V))$ is generated by $c_1(\mathcal{O}(1))$, which we call the tautological class, as an algebra.

Example 2.9. Let E be an elliptic curve with a fixed point 0. Consider the Poincaré line bundle

$$P \to E \times \mathrm{Pic}^0(E)$$

where the fibre over (x, L) is L_x. We know $E \cong \mathrm{Pic}^0(E)$ by the map $x \mapsto \mathcal{O}_E(x - 0)$. Under this identification, we have $P = \mathcal{O}_{E\times E}(0 \times E + E \times 0 - \Delta)$. For $k \in \mathbb{N}$, $\gamma \in H^*(E)$, we set

$$c_k(\gamma) := \int_\gamma \mathrm{ch}_k(P) = p_*(q^*\gamma \cdot \mathrm{ch}_k(P)) \in H^*(\mathrm{Pic}^0(E))$$

to be the tautological classes, where p, q are projections from $E\times\mathrm{Pic}^0(E)$. One can show $H^*(\mathrm{Pic}^0(E))$ is generated by classes of form $c_k(\gamma)$.

Let C be a curve and fix $(n, d) = 1$. Recall $\mathcal{M}_{\mathrm{Dol}}$ is the moduli of stable Higgs bundles of rank n and degree d. There exists (non-unique) a universal bundle

$$U \to C \times \mathcal{M}_{\mathrm{Dol}}$$

whose fibre at $x, (E, \Theta)$ is E_x, and we may define tautological classes

$$c_k(\gamma) = \int_\gamma \mathrm{ch}_k(U)$$

for $\gamma \in H^*(C)$ and $k \in \mathbb{N}$.

Theorem 2.10. *$H^*(\mathcal{M}_{Dol})$ is generated by $c_k(\gamma)$ as an algebra.*

We recall the proof of a simpler case of the above theorem from the theory of stable bundles on curves.

Theorem 2.11 (Atiyah-Bott)**.** *Let n, d coprime and N the moduli of stable vector bundles on C of rank n and degree d. Then $H^*(N)$ is generated by tautological classes.*

Proof. Consider $N \times C \times N$ with maps p_{12}, p_{23}, p_{13} onto the corresponding terms. Let

$$H := R\mathscr{H}om(p_{12}^* U, p_{23}^* U),$$

then

$$Rp_{13*}H \in D^b \operatorname{coh}(N \times N).$$

For $(E, F) \in N \times N$, we have

$$K := Rp_{13*}H|_{(E,F)} = \operatorname{Ext}_C^\bullet(E, F).$$

Recall when $E \neq F$ are stable bundles, we have $\operatorname{Hom}(E, F) = 0$, so $K|_{N-\Delta}$ is a vector bundle of rank $d_N = \dim N$.

Now over $p = (E, E) \in \Delta$, we have $K = [K_0 \to K_1]$ where $H^0(K) = \operatorname{Hom}(E, F)$ and $H^1(K) = \operatorname{Ext}^1(E, F)$, i.e. we have the following exact sequence

$$0 \to \mathbb{C} = \operatorname{Hom}(E, E) \to (K_0)_p \to (K_1)_p \to \operatorname{Ext}^1(E, E) \to 0.$$

Now by Porteous formula (Fulton 14.4) and Grothendieck-Riemann-Roch, we obtain

$$[\Delta] = c_{d_N}(Rp_{13*}H)$$

as an expression in terms of $p_{12}^* \operatorname{ch}(U), p_{23}^* \operatorname{ch}(U), \operatorname{td}_C$. From the commutative diagram

$$\begin{tikzcd} H^*(N) \arrow[rr, "{[\Delta]=\mathrm{id}}"] \arrow[dr] & & H^*(N) \\ & H^*(C) \arrow[ur, "\Phi"] & \end{tikzcd}$$

we see $\operatorname{im} \Phi = H^*(N)$. ☐

Exercise 2.12. Consider C with $g = 2, n = 2, d = 1$. Let M be the moduli of stable Higgs bundles of rank n and degree d, and N be the moduli of stable vector bundles of rank n and degree d.

(1) Show that T^*N embeds into M by sending (E, ξ) to some (E, Θ).

(2) After attaching the Hitchin fibration, is

$$T^*N \to M \to A$$

surjective?

(3) Try computing $e(M) - e(N)$ equivariantly using the $\mathbb{C}^*$ action on M by

$$\lambda(E, \Theta) = (E, \lambda\Theta),$$

where the fixed locus is $M^{\mathbb{C}^*} = N \sqcup T$ with

$$T = \left\{ \text{stable Higgs bundles of form } \left(E = L_1 \oplus L_2, \Theta = \begin{bmatrix} 0 & \phi \\ 0 & 0 \end{bmatrix} \right) \right\}.$$

Exercise 2.13. Let n, d be coprime to each other.

(1) Suppose $F \in \operatorname{coh}(T^*C)$ with proper support over C. Say $p : T^*C \to C$ is the projection. Show F can be recovered from some Higgs sheaf $(E = p_*F, \Theta : E \to E \otimes \omega C)$.

(2) Suppose $C_\alpha \hookrightarrow T^*C$ is some integral curve with $p_\alpha : C_\alpha \to C$ having finite degree n. Show that if F is torsion free, generically rank 1 on C_α, then $p_{\alpha*}$ gives a stable Higgs bundle on C of rank n.

(3) Denote $\overline{J_{C_\alpha}}$ the compactified Jacobian, which parametrizes torsion free sheaves on C_α generically of rank 1 for some fixed degree. Note that if C_α is smooth, then $\overline{J} = J$ is the usual Jacobian. Show that

$$\begin{tikzcd} \overline{J_{C_\alpha}} \arrow[r, hook] \arrow[d] & M \arrow[d] \\ \{\alpha\} \arrow[r, hook] & A \end{tikzcd}$$

is a Cartisian diagram, where $\alpha \in A$ is a point determined by the curve C_α, which we call the *spectral data*.

2.4. **Support theorems.** In the study of the $P = W$ conjecture, the support of a morphism became an important topic. To define support, we first look at the widely known Beilinson–Bernstein–Deligne–Gabber decomposition theorem.

Let $f : X \to Y$ be a proper morphism with X, Y smooth. The BBDG decomposition calculates $Rf_*\underline{\mathbb{Q}}_X$:

$$Rf_*\underline{\mathbb{Q}}_X \cong \oplus_i {}^pH^i(Rf_*\underline{\mathbb{Q}}_X)[-i]$$

where ${}^pH^i = {}^p\tau_{\leq i} \circ {}^p\tau_{\geq i}$ is the cohomology induced by the perverse t-structure. Every summand

$${}^pH^i(Rf_*\underline{\mathbb{Q}}_X)[-i] \in \mathrm{Perv}(Y)$$

is semisimple in the sense that it can be written as a finite sum of form

$$\oplus_{\alpha\in I}\mathrm{IC}_{Z_\alpha}(L_\alpha).$$

For a closed subvariety $Z_\alpha \overset{i}{\hookrightarrow} Y$, say $U_\alpha \overset{j}{\hookrightarrow} Z_\alpha$ is open in Z_α and L_α is an irreducible local system on U_α, we define

$$\mathrm{IC}_{Z_\alpha}(L_\alpha) := i_* {}^pj_{!*}(L_\alpha[\dim Z_\alpha])$$

where ${}^pj_{!*}$ is certain abstract map from $\mathrm{Perv}(U_\alpha)$ to $\mathrm{Perv}(Z_\alpha)$. A fact from BBDG is that every perverse sheaf arrises in this way. We define the *support* of f to be

$$\mathrm{supp}(f) := \{Z_\alpha | \alpha \in I\}.$$

Note that an ideal situation would be if $\mathrm{supp}(f) = \{Y\}$, but in general, it is very difficult to compute the support. The Goresky-MacPherson inequality gives some restrictions to the support.

Theorem 2.14 (Goresky-MacPherson)**.** *Suppose $f : X \to Y$ has equi-dimensional fibre of dimension d.*

(1) $Z \in \mathrm{supp}(f)$ only if $\operatorname{codim} Z \leq d$.
(2) Equality holds only if Z can be detected by $R^{2d}f_*\underline{\mathbb{Q}}_X$, i.e. there exists some $V \subseteq Y$ open with

$$R^{2d}f_*\underline{\mathbb{Q}}_X|_V \cong i_{Z*}L_Z \oplus \textit{something else}$$

for some local system L_Z and inclusion $i_Z : Z \cap V \hookrightarrow V$.

Example 2.15. Suppose $f : S \to B$ is a map from a surface to a curve with elliptic fibration. Say the general fibre is a torus. Say at a point P, the fibre is a curve with two components; at a point R, the fibre is a cuspidal curve; and at Q a nodal curve. Then by the above theorem, Q, R must not be in the support of f because

$$R^2f_*\underline{\mathbb{Q}}_X = \underline{\mathbb{Q}}_B \oplus \underline{\mathbb{Q}}_P$$

as the H^2 of the fibre here counts the number of irreducible components. One can show we have $\mathrm{supp}(f) = \{B, P\}$.

Example 2.16. Suppose now B is a surface, where on a certain curve of in B the fibre is nodal, and at a point P on that curve the fibre has 2 components. We again have $R^2 f_* \underline{\mathbb{Q}}_X = \underline{\mathbb{Q}}_B \oplus \underline{\mathbb{Q}}_P$, so the only possible supports are B and P. By the theorem, P can not be in $\operatorname{supp}(f)$ as it has codimension 2. Hence $\operatorname{supp}(f) = \{B\}$ in this case.

Note that the GM inequaltiy is not as useful in the case $d = d_Y = \dim Y$, which is always greater than or equal $\operatorname{codim} Z$, for example in a Symplectic setting.

Proof. (sSketch) Suppose $Z \in \operatorname{supp} f$. Define

$$\operatorname{Ind}(Z) := \{i \in \mathbb{Z} | {}^p H^i(Rf_* \underline{\mathbb{Q}}_X) \text{ has } Z \text{ as one of its support}\}.$$

By Verdier duality, $\operatorname{Ind}(Z)$ is symmetric with respect to $d_X = \dim X$. Thus we can take some $m \in \operatorname{Ind}(Z)$ with $m \geq d_X$. Now ${}^p H^i(Rf_* \underline{\mathbb{Q}}_X)$ having a summand supported on Z means that there is some U open with local system L such that

$$L_U[\dim Z][-m]$$

is a summand of $Rf_* \underline{\mathbb{Q}}_X|_U$. Also, we know $Rf_* \underline{\mathbb{Q}}_X \in D^{[0,2d]}$ with respect to the standard t-structure, so

$$m - \dim Z \leq 2d.$$

This gives the desired result. $\square$

Next we consider the Ngo Support Theorem regarding the Dolbeault moduli. Denote $M = \mathcal{M}_{\text{Dol}}$ with Hitchin fibration $h : M \to A$. Define $A^0 := \{\alpha | C_\alpha \subseteq T^* C \text{ integral}\}$ to be the set of points which are spectral data of integral curves from Exercise 2.13, and let A^{sm} be those points corresponding to smooth curves. Let $M^0 = h^{-1}(A^0)$, and $M^{\text{sm}} = h^{-1}(A^{\text{sm}})$. We see $h^{\text{sm}} : M^{\text{sm}} \to A^{\text{sm}}$ has support A^{sm}. Denote $\mathcal{C} \to A^0$ the spectral curve where the fibre over α is C_α.

Theorem 2.17. *The support of $h^0 : M^0 \to A^0$ is $\operatorname{supp} h^0 = A^0$.*

Proof. (Sketch)

(1) Symmetries: $P^0 := \operatorname{Pic}^0(\mathcal{C}/A^0)$ is a commutative group scheme over A^0, which acts on M^0 by the fibrewise action $P_\alpha \curvearrowright \overline{J_{C_\alpha}}$ where

$$L_\alpha \cdot F_\alpha = L_\alpha \otimes F_\alpha.$$

(2) δ-function: For all $\alpha \in A^0$, the abelian group scheme P_α decomposes into a maximal affine part and abelian variety part

$$1 \to R_\alpha \to P_\alpha \to \text{Ab}_\alpha \to 1$$

where $\text{Ab}_\alpha = \operatorname{Pic}^0(\tilde{C_\alpha})$ for a normalization $\tilde{C_\alpha} \to C_\alpha$. Consider the function

$$\delta : A^0 \to \mathbb{N}$$

$$\alpha \mapsto \dim R_\alpha.$$

For any $Z \subseteq A^0$ closed and irreducible, we set $\delta(Z) := \delta(\beta)$ for a general $\beta \in \mathbb{Z}$.

(3) Serveri inequality: This is a classical result for curves. Suppose $\mathcal{C} \to A^0$ is a family of integral locally planar curves. Assume $\overline{J_{\mathcal{C}/\mathcal{A}'}}$ is smooth, then for all closed irreducible subvariety $Z \subseteq A^0$, we have $\delta(Z) \leq \operatorname{codim} Z$.

(4) Enhanced GM inequality: The original GM inequality says $Z \in \operatorname{supp} h^0$ implies $\operatorname{codim}(Z) \leq d$. The result of Ngo improved this bound to $\operatorname{codim}(Z) \leq \delta(Z)$, with equality holding if and only if Z is detected by $R^{2d} f_* \underline{\mathbb{Q}}$.

Combine (3) and (4), and a fact that $\mathcal{C}$ is integral and locally planar, we see $Z \in \operatorname{supp} h^0$ means it must be detected by $R^{2d} f_* \underline{\mathbb{Q}}$. But $R^{2d} f_* \underline{\mathbb{Q}}$ is the top degree cohomology of the fibres. Using the fact that $\overline{J_C}$ is irreducible, we get $R^{2d} f_* \underline{\mathbb{Q}}$ is a rank 1 local system on A^0, which means the only element of the support is A^0. ☐

2.5. **Betti moduli space.** Recall that a mixed Hodge structure is given by an increasing filtration $W_\bullet$ on $H^*(X, \mathbb{Q})$ and a decreasing filtration $F^\bullet$ on $H^*(X, \mathbb{C})$. Define

$$^k\mathrm{Hdg}^d(X) := (W_{2k} \cap F^k) \cap H^d(X).$$

Theorem 2.18 (Shende)**.** *Let $\mathcal{M}_B$ be the Betti moduli space. We have*

(1) $H^*(\mathcal{M}_B) = \oplus\, ^k Hdg^d(\mathcal{M}_B)$.
(2) $c_k(\gamma) \in {^k Hdg^*}(\mathcal{M}_B)$ for any tautological class.

Exercise 2.19. Prove (1) for torus $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$.

Proof. (Key ingredients) Using that the filtrations $W_\bullet, F^\bullet$ are multiplicative, and that $c_k(\gamma)$ generate H^*, it suffices to prove (2).

(1) We use the fact that $H^*(BGL_n) = \oplus\, ^k\mathrm{Hdg}^{2k}(BGL_n)$, where we have $H^*(BGL_n) = \mathbb{Q}[c_1, \ldots, c_n]$ for $c_k \in {^k\mathrm{Hdg}^{2k}}(BGL_n)$. When $n = 1$, we have $H^*(B\mathbb{G}_m) = H^*(\mathbb{CP}^\infty) = \mathbb{Q}[c_1]$, so this is similar to a pure Hodge structure.
(2) Obtain functoriality of $W_\bullet, F^\bullet, c_k(\gamma)$ from the c_k of the previous step.
(3)

Theorem 2.20 (Mellit)**.** *There exists $w \in W_4 H^2(\mathcal{M}_B)$ satisfying the Lefschetz symmetry, i.e. the map*

$$w^i : H^\bullet(\mathcal{M}_B) \to H^{\bullet+2i}(\mathcal{M}_B)$$

induces a map

$$w^i : \mathrm{Gr}^W_{d_{\mathcal{M}_B}-2i} H^\bullet(\mathcal{M}_B) \to \mathrm{Gr}^W_{d_{\mathcal{M}_B}+2i} H^{\bullet+2i}(\mathcal{M}_B)$$

The idea of the proof of this theorem is to first show for $\mathbb{C}^* \times \mathbb{C}^*$, then show $\mathcal{M}_{\mathrm{B}}$ admits a "cell decomposition" into $(\mathbb{C}^*)^\ell$.

☐

Proposition 2.21. *The $P = W$ conjecture is true if and only if for any set of tautological classes $\{c_{k_i}(\gamma_i)\}$, we have*

$$\prod c_{k_i}(\gamma_i) \in P_{\sum k_i} H^*(\mathcal{M}_{Dol})$$

Proof. For two filtrations with Lefschetz symmetry, if one is contained in the other, then they are equal. ☐

To end the discussion on $P = W$ conjecture, we give the steps of showing

$$\prod c_{k_i}(\gamma_i) \in P_{\sum k_i} H^*(\mathcal{M}_{\mathrm{Dol}}).$$

Step 1: Let $h : M \to A$ be the Hitchin fibration. For any element $\alpha \in H^\ell(M, \mathbb{Q}) = \mathrm{Hom}(\underline{\mathbb{Q}}_M, \underline{\mathbb{Q}}_M[\ell])$, we have an induced map

$$\alpha : Rh_*(\underline{\mathbb{Q}}_M) \to Rh_* \underline{\mathbb{Q}}_M[\ell].$$

Consider the universal family $U \to C \times M$, we have $c_k(U) \in H^{2k}(C \times M)$, giving us

$$\mathrm{ch}_k(U) : Rh_*(\underline{\mathbb{Q}}_M) \to Rh_* \underline{\mathbb{Q}}_M[2k].$$

Under this map, elements of form $^p\tau_{\leq i}(\ldots)$ are sent to elements of form $^p\tau_{\leq i-2k}(\ldots)$.

We claim that if we show that such images are of form ${}^p\tau_{\leq i-k}$, i.e. bring the bound from $i-2k$ to $i-k$, then we are done. Because if this holds, then we have

$$\cdot \cup c_k(\gamma) : P_i \to P_{i+k}$$

(where as the trivial bound is P_{i+2k}). Now apply this to the fundamental class $1 \in P_0 H^0(\mathcal{M}_{\mathrm{Dol}})$ and we find

$$\prod c_{k_i}(\gamma_i) = \prod c_{k_i}(\gamma_i) \cdot 1 \in P_{\sum k_i} H^*(\mathcal{M}_{\mathrm{Dol}}).$$

Step 2: We consider an ideal situation. Say $f : X \to Y$ is smooth and $\alpha \in H^\ell(X)$ inducing $\alpha : Rf_*\underline{\mathbb{Q}}_X \to Rf_*\underline{\mathbb{Q}}_X[\ell]$. For now let us make the assumptions (which we shall see is too good to be true in practice) that $\mathrm{supp}(f) = \{Y\}$ and that we only want to push the bound by 1 instead of by k. So we have

$$\alpha : {}^p\tau_{\leq i}(\dots) \mapsto {}^p\tau_{\leq i-1}(\dots)$$

if and only if the induced map on the i-th degree

$$\alpha : {}^pH^i(Rf_*\underline{\mathbb{Q}}_X) \to {}^pH^i(Rf_*\underline{\mathbb{Q}}_X[\ell]) = {}^pH^{i+\ell}(Rf_*\underline{\mathbb{Q}}_X)$$

is 0. Since we have full support, we have a map

$$\alpha : \mathrm{IC}_Y(L_i) \to \mathrm{IC}_Y(L_{i+\ell})$$

so the above map vanishes if and only if for some $U \subseteq Y$ open, $\alpha|_U : L_i \to L_{i+\ell}$ is 0. Now this can be verified by hand.

The problem is that in our case for the Hitchin fibration, $\mathrm{supp}(h)$ is really large. Though we know that $\mathrm{supp}(h^0) = A^0$, there can still be many supports in $A - A^0$. Also, the goal is to push the bound to k, not 1. For the latter problem, the "Global Springer theory" of Zhiwei Yun was used.

Step 3: It is possible to reduce to the ideal situation above using a stronger version of support theorem by Chaudouard-Laumon, and a critical chart construction by Maulic-Shen.

Consider a new moduli. For a point $p \in C$, let M^p be the moduli of slope stable bundles (E, Θ) where $\Theta : E \to E \otimes \omega_C(p)$. This gives us

$$h^p : M^p \to A^p := \oplus h^0(C, (\omega_C(p))^{\otimes i})$$

which we call the twisted Hitchin fibration.

Theorem 2.22 (Chaudouard-Laumon)**.** *We have*

$$\mathrm{supp}(h^p) = \{A^p\}.$$

Example 2.23. The basic idea when $g = 2, n = 2, d = 1$ is that a subscheme of form $Z = \{\text{non-reduced spectural curve}\}$ is might be a support for h, but never for h^p. In this case, we have $\dim M = 10, \dim A = 5, \dim Z = 2$, and $\delta(Z) = 3 = \mathrm{codim}\, Z$, so Z might be a support. On the other hand, with a point p, we have $\dim M^p = 13, \dim A^p = 7, \dim Z^p = 2$, and $\delta(Z^p) = 4 < \mathrm{codim}\, Z$, so it is not in the support by Ngo's support theorem.

3. Bridgeland stability

The references for this talk are *Fourier-Mukai Transforms in Algebraic Geometry* by Huybrechts, *Lectures on Bridgeland Stability* by Macrì-Schmidt, *Lectures on non-commutative K3 surfaces, Bridgeland stability, and moduli spaces* by Macrì-Stellari.

Definition 3.1. A *K3 surface* is a smooth projective surface $S/\mathbb{C}$ such that $\omega_S = \mathcal{O}_S$ and $\pi_1(S) = \{1\}$.

Basic examples of K3 surfaces are hypersurfaces of $\mathbb{P}^3$ of degree 4, and double covers of $\mathbb{P}^2$ ramified over a degree 6 curve.

Definition 3.2. A *hyperKähler (HK) variety* is a smooth projective variety X with $\pi_1(X) = 1$ and $H^0(X, \Omega_X^2) = \mathbb{C} \cdot \omega$ for some holomorphic symplectic form ω.

Remark 3.3. HyperKähler implies $\dim X$ is even with $\omega_X = \mathcal{O}_X$.

Example 3.4.
- K3 surfaces are HK.
- Hilbert schemes $X = S^{[n]}$ of a K3 surface S are HK.
- $M_H(S, \nu)$ the moduli of Geiseker-stable (with respect to some $H \in |\mathcal{O}(1)|$) sheaves on K3 surfaces of a fixed class ν (where ν is primitive and H is ν-generic) is HK. We know
$$T_{[E]}M \cong \mathrm{Ext}^1(E, E),$$
and the map
$$\mathrm{Ext}^1(E, E) \times \mathrm{Ext}^1(E, E) \to \mathrm{Ext}^2(E, E) \to \mathrm{Ext}^0(E, E)^\vee = \mathbb{C},$$
where the first arrow is the Yoneda map, and the second by Serre duality, gives us the 2-form for the HK structure.
- Let Y be a cubic 4-fold in $\mathbb{P}^5$. The Fano variety
$$F(Y) = \{[l] \in \mathrm{Grass}(2, 6) | l \subseteq Y\}$$
is a HK 4-fold which is deformation equivalent to $S^{[2]}$ for some K3 surface S.

In this section we will construct a moduli of stable objects for a cubic 4-fold, which will be a HK variety.

3.1. **Derived category of coherent sheaves.** From now on we denote $D(X) = D^b(\mathrm{coh}(X))$. Let X, Y be smooth projective varieties over $\mathbb{C}$. For $f : X \to Y$, recall derived functors
$$Rf_* : (F \xrightarrow{q.i.} I^\bullet) \mapsto f_*(I^\bullet),$$
$$Lf^* : (E^\bullet \to F) \mapsto f^*(E^\bullet),$$
where $E^\bullet$ is a resolution by locally free sheaves (we do not have enough projective objects). For $F, G \in D(X)$, we also have $F \otimes_L G \in D(X)$. For simplicity, we will omit the letters L and R from now on.

Definition 3.5. For $K \in D(X \times Y)$, the *Fourier-Mukai functor* associated to K is
$$\Phi_K : D(X) \to D(Y)$$
$$F \mapsto pr_*(pr^* F \otimes K).$$

Example 3.6. For $f : X \to Y$, if $\Gamma_f \subseteq X \times Y$ is the graph of f, then $\Phi_{\mathcal{O}_\Gamma} = f_*$.

Theorem 3.7. *If $F : D(X) \to D(Y)$ is an equivalence, then there exists a unique K such that $F = \Phi_K$.*

Theorem 3.8. *The following are preserved under derived equivalences:*

(1) $\dim X$,
(2) $\oplus_{m \geq 0} H^0(X, \pm m K_X)$,
(3) $\oplus_{p+q=k} H^{p,q}(X)$.

We consider some non-trivial examples of derived equivalence.

- Let A be an abelian variety, and $\hat{A}$ be its dual. Consider the Poincaré line bundle $P \to A \times \hat{A}$. A result of Mukai states $D(A) \xrightarrow{\Phi_P} D(\hat{A})$ is an equivalence.

- Let S be a K3 surface. The *Mukai cohomology*
$$\tilde{H}(S,\mathbb{Z}) := H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})$$
has a weight 2 Hodge structure given by
$$\tilde{H}^{2,0} = H^{2,0}, \tilde{H}^{0,2} = H^{0,2}, \tilde{H}^{1,1} = H^0 \oplus H^{1,1} \oplus H^4.$$
We define a pairing on $\tilde{H}$
$$\langle (a,b,c),(a',b',c')\rangle := bb' - ac' - a'c.$$
Define Mukai vector
$$v : K_0(S) \to \tilde{H}(S,\mathbb{Z})$$
$$[E] \mapsto \mathrm{ch}(E)\sqrt{\mathrm{td}}(S)$$
where K_0 is the Grothendieck group. Then one has $\langle v(E), v(F)\rangle = -\chi(E,F)$.

Theorem 3.9. *For K3 surfaces S, S', we have $D(S) \cong D(S')$ if and only if $\tilde{H}(S,\mathbb{Z}) \cong \tilde{H}(S',\mathbb{Z})$. The second $\cong$ refers to a Hodge isometry with respect to the decomposition and pairing..*

Proof. (Sketch, assuming the $\Rightarrow$ direction is proven) Suppose we have $\varphi : D(S) \xrightarrow{\sim} D(S')$. Consider $(0,0,1), (-1,0,0) \in \tilde{H}(S)$, which are mapped to v, v' by ϕ. The intersection matrix of these two vectors is then
$$U = \begin{bmatrix} v \cdot v & v \cdot v' \\ v' \cdot v & v' \cdot v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$
A result of Mukai states that there is a non-empty moduli of stable sheaves on S' of class v, and M is a K3 surface. Since we have found a v' such that $v \cdot v' = 1$, one can show M has a universal family $\mathcal{E} \to M \times S'$. Using some results from homological algebra, we can show
$$D(S') \xrightarrow{\Phi_{\mathcal{E}}} D(M)$$
is an equivalence. This induces a map $\Phi_M : \tilde{H}(S') \to \tilde{H}(M)$ where $\Phi_M(v) = (0,0,1)$. Thus we conclude $H^2(S) \cong H^2(M)$ is a Hodge isometry, and a general statement about K3 surfaces of Torelli implies $S \cong M$. □

3.2. **Semi-orthogonal decomposition.**

Definition 3.10. A *semi-orthogonal decomposition* of $D(X)$ is a sequence of triangulated subcategories $\langle A_1, \dots, A_n\rangle$ such that

1. $\mathrm{Hom}(F,G) = 0$ if $F \in A_i, G \in A_j$ for $i > j$,
2. for all $F \in D(X)$, there exists "filtration"
$$0 = F_n \to F_{n-1} \to \dots \to F_0 = F$$
with $\mathrm{Cone}(F_i \to F_{i-1}) \in A_i$.

Observe that the category $D(\mathbb{C}) = D(\operatorname{Spec}\mathbb{C})$ is equivalent to the category of finite dimensional graded vector spaces over $\mathbb{C}$ by
$$V^\bullet \mapsto \oplus H^i(V^\bullet).$$
For $E \in D(X)$, define
$$\phi_E : D(\mathbb{C}) \to D(X)$$
$$V \mapsto V \otimes E = \oplus V_i \times E[-i].$$
One can check that ϕ_E is fully faithful if and only if $\mathrm{Hom}(E,E) = \mathbb{C}$ and $\mathrm{Ext}^{>0}(E,E) = 0$. We call such an E an *exceptional* object. Define $\langle E\rangle = \mathrm{im}(\phi_E)$.

Example 3.11. The sheaf $\mathcal{O}(i)$ on $\mathbb{P}^n$ viewed as a complex is exceptional.

Definition 3.12. A triangulated subcategory $A \subseteq D(X)$ is *right admissible* if $\alpha : A \hookrightarrow D(X)$ admits a right adjoint $\alpha^! : D(X) \to A$.

Example 3.13. If E is exceptional, $\langle E \rangle$ is always right admissible. We also have

$$\phi_E^!(F) = R\operatorname{Hom}(E, F) \in D(\mathbb{C}).$$

Lemma 3.14. *If $A \subseteq D(X)$ is right admissible, then we have a semi-orthogonal decomposition*

$$D(X) = \langle A^\perp, A \rangle$$

where

$$A^\perp = \{F \in D(X) | \operatorname{Hom}(E, F) = 0 \textit{ for all } E \in A\}.$$

Proof. Say $\alpha : A \hookrightarrow D(X)$ is the inclusion. Let $F \in D(X)$. We have an element

$$\mathrm{Id} \in \operatorname{Hom}(\alpha^!(F), \alpha^!(F)) \cong \operatorname{Hom}(\alpha\alpha^!(F), F).$$

We can complete $\alpha\alpha^!(F) \to F$ to an exact triangle

$$\alpha\alpha^!(F) \to F \to B$$

where B is the cone, so it suffices to prove $B \in A^\perp$. From this triangle, we get a long exact sequence for $E \in A$ as follows

$$\ldots \to \operatorname{Hom}(\alpha(E), \alpha\alpha^!(F)) \to \operatorname{Hom}(\alpha(E), F) \to \operatorname{Hom}(\alpha(E), B)) \to \ldots$$

Note that $\operatorname{Hom}(\alpha(E), \alpha\alpha^!(F)) \cong \operatorname{Hom}_A(E, \alpha^!(F)) \cong \operatorname{Hom}(\alpha(E), F)$. One can also show that $\operatorname{Ext}^1(\alpha(E), \alpha\alpha^!(F)) \cong \operatorname{Ext}^1(\alpha(E), F)$. This gives us $\operatorname{Hom}(\alpha(E), B) = 0$. Since $E \in A$ is arbitrary, we conclude $B \in A^\perp$. □

Corollary 3.15. *Given exceptional objects $E_1, \ldots, E_n \in D(X)$ with $\operatorname{Ext}^\cdot(E_i, E_j) = 0$ for $i > j$, we call this an* exceptional collection*, and obtain semi-orthogonal decomposition*

$$D(X) = \langle Ku_X, E_1, \ldots, E_n \rangle$$

where $Ku_X = \langle E_1, \ldots, E_n \rangle^\perp$ is the Kuznetsov component.

Example 3.16. Let X be a Fano variety with $-K_X = rH$ for some $r > 0$. Then

$$\mathcal{O}_X, \mathcal{O}_X(H), \ldots, \mathcal{O}_X((r-1)H)$$

is an exceptional collection. So

$$D(X) = \langle \mathrm{Ku}_X, \mathcal{O}, \ldots, \mathcal{O}((r-1)H) \rangle.$$

Let $X = \mathbb{P}^n$, we know $K_{\mathbb{P}^n} = (-n-1)H$.

Theorem 3.17. *We have SOD $D(\mathbb{P}^n) = \langle \mathcal{O}, \ldots, \mathcal{O}(n) \rangle$.*

Lemma 3.18. *There exists a resolution on $\mathbb{P}^n \times \mathbb{P}^n$*

$$0 \to \Omega^n(n) \boxtimes \mathcal{O}(-n) \to \ldots \to \Omega^1(1) \boxtimes \mathcal{O}(-1) \to \mathcal{O}(\mathbb{P}^n \times \mathbb{P}^n) \to \mathcal{O}_\Delta \to 0.$$

Proof. We need to find a section $s \in H^0(\mathbb{P}^n \times \mathbb{P}^n, T_{\mathbb{P}^n}(-1) \boxtimes \mathcal{O}(1)$ that vanishes on Δ, then take Koszul resolution.

Consider $\mathbb{P}^n = \mathbb{P}(V)$ where $x \in \mathbb{P}(V)$ corresponds to a one dimensional subspace $\langle x \rangle \subseteq V$. Consider

$$\mathcal{O} \boxtimes \mathcal{O}(-1) \to V \boxtimes \mathcal{O} \to T_{\mathbb{P}^n}(-1) \boxtimes \mathcal{O}$$

where on the fibre over (x, y) we have $\langle y \rangle \to V \to V/\langle x \rangle$, which means the composition vanishes exactly on $(x, y) \in \Delta$.

□

Proof of theorem. Write $\mathcal{O}_\Delta \cong K$ in $D(\mathbb{P}^n \times \mathbb{P}^n)$. For $F \in D(\mathbb{P}^n)$, we have
$$F \cong \Phi_{\mathcal{O}_\Delta}(F) \cong \Phi_K(F) \in \langle \Phi_{K^0}(F), \dots \Phi_{K^{-n}}(F)\rangle$$
since each K^{-i} is a vector space.

Manually computing gives us
$$\Phi_{K^{-i}}(F) = R\Gamma(F \otimes \Omega^i(i)) \otimes \mathcal{O}(-i) \in \mathcal{O}(-i).$$
□

3.3. **Calabi-Yau categories.**

Definition 3.19. A *Serre functor* for a triangulated category D is an auto-equivalence S_D such that
$$\mathrm{Hom}(E, F)^\vee \cong \mathrm{Hom}(F, S_D(E))$$
where the isomorphism is functorial in E and F.

Remark 3.20. If S_D exists, it is unique.

Example 3.21. Let X be a smooth projective variety of dimension n, then
$$S_{D(X)} = - \otimes \omega_X[n].$$
Note that if E is a vector bundle on X, the
$$\begin{aligned} H^i(X, E) =& \mathrm{Hom}_{D(X)}(\mathcal{O}_X, E[i]) \\ =& \mathrm{Hom}_{D(X)}(E[i], \omega[n])^\vee \\ =& \mathrm{Hom}_{D(X)}(\mathcal{O}_X, E^\vee \otimes \omega[n-i])^\vee \\ =& H^{n-i}(X, E^\vee \otimes \omega)^\vee. \end{aligned}$$
So $S_{D(X)}$ is indeed given by the usual Serre duality.

Definition 3.22. A triangulated category D is *Calabi-Yau of dimension* n if $S_D = [n]$. It is *fractional Calabi-Yau of dimension* p/q if $S_D^q = [p]$.

Remark 3.23. Note that being fractional CY of dimension 4/2 is not necessarily CY of dimension 2.

Theorem 3.24 (Kuznetsov)**.** *Let $X \subseteq \mathbb{P}^n$ be a smooth Fano hypersurface of degree $d \leq n$. We have*
$$D(X) = \langle Ku_X, \mathcal{O}_X, \dots, \mathcal{O}_X(n-d)\rangle.$$
Then Ku_X has Serre functor with
$$S^{d/c} = \left[\frac{(n+1)(d-2)}{c}\right]$$
where $c = \gcd(d, n+1)$.

Example 3.25. For a cubic 3-fold, we have $S^3_{\mathrm{Ku}_X} = [5]$. For a cubic 4-fold, we have $S_{\mathrm{Ku}_X} = [2]$.

We shall prove the above theorem for cubic 4-folds in the remainder of this section. First we remark a theorem related to the previous example, as [2] is the Serre functor for a K3 surface.

Theorem 3.26 (Kuznetsov)**.** *Let X be some cubic 4-folds in $\mathbb{P}^5$. For a K3 surface S, we can find such X that $Ku_X = D(S)$. For a very general choice of X (away from countably many divisors) we have $Ku_X \ncong D(S)$ for any surface S.*

Theorem 3.27 (Bondal)**.** *Let X be a smooth projective variety with semi-orthogonal decomposition $D(X) = \langle A, B\rangle$. Then A, B are (left and right) admissible subcategories.*

Let $\alpha : A \hookrightarrow D$ be admissible, recall that $D = \langle A^\perp, A\rangle = \langle A, {}^\perp A\rangle$, where

$$ {}^\perp A = \{F | \operatorname{Hom}(F, E) = 0 \text{ for all } E \in A\}. $$

Definition 3.28. The *left, right mutation functors* through A are

$$ \mathbb{L}_A := ii^* : D \to D \text{ where } i : A^\perp \hookrightarrow D, i^* \dashv i, $$

$$ \mathbb{R}_A := jj^! : D \to D \text{ where } i : {}^\perp A \hookrightarrow D, j \dashv j^!. $$

For all $F \in D$, we hav ethe following exact triangles

$$ \alpha\alpha^!(F) \to F \to \mathbb{L}_A(F), $$

$$ \mathbb{R}_A(F) \to F \to \alpha\alpha^*(F). $$

Example 3.29. If $A = \langle E\rangle$, then the above exact triangles become

$$ R\operatorname{Hom}(E, F) \otimes E \xrightarrow{\text{eval}} F \to \mathbb{L}_A(F), $$

$$ \mathbb{R}_A(F) \to F \xrightarrow{\text{coeval}} R\operatorname{Hom}(F, E)^\vee \otimes E. $$

Example 3.30. We have $D(\mathbb{P}^1) = \langle \mathcal{O}, \mathcal{O}(1)\rangle$, and $\mathbb{L}_{\mathcal{O}}(\mathcal{O}(1)) = \mathcal{O}(-1)[1]$.

Proposition 3.31. *The mutation functors satisfy the following properties:*

(1) $\mathbb{L}_A(A) = \mathbb{R}_A(A) = 0$,

(2) The two maps ${}^\perp A \underset{\mathbb{L}_A}{\overset{\mathbb{R}_A}{\longleftrightarrow}} A^\perp$ *are inverse equivalences.*

(3) If there exists a Serre functor S_D*, then*

$$ S_{A^\perp} = S_D \circ \mathbb{R}_A|_{A^\perp}, \quad S_{A^\perp}^{-1} = \mathbb{L}_A \circ S_D^{-1}|_{A^\perp} $$

Proof of (3). By definition, the maps

$$ {}^\perp A \underset{S_D}{\overset{S_D^{-1}}{\longleftrightarrow}} A^\perp $$

are inverse equivalences. So by (2), $S_D \circ \mathbb{R}_A|_{A^\perp}$ and $\mathbb{L}_A \circ S_D^{-1}|_{A^\perp}$ are auto-equivalences on $A^\perp$. For $F, G \in A^\perp$, we can check

$$ \operatorname{Hom}_{A^\perp}(F, G) = \operatorname{Hom}_D(F, G) = \operatorname{Hom}_D(S_D^{-1}G, F)^\vee = \operatorname{Hom}_{A^\perp}(\mathbb{L}_A \circ S_D^{-1}G, F)^\vee. $$

□

Now we go back to $X \subseteq \mathbb{P}^5$ a cubic 4-fold. Let $A = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$. Then $S_{\mathrm{Ku}_X}^{-1} = \mathbb{L}_A \cdot \mathcal{O}(3)[-4]$ by the above proposition. We shall use the following general fact about Cartier divisors: for $F \in \mathrm{Ku}_X$, $i : X \hookrightarrow \mathbb{P}^5$, there exists an exact triangle

$$ i^* i_* F \to F \to F \otimes \mathcal{O}(-3)[2]. $$

Apply $S_{\mathrm{Ku}_X}^{-1}$, we have

$$ \mathbb{L}_A \mathcal{O}(3) \otimes i^* i_* F[-4] \to S_{\mathrm{Ku}_X}^{-1} F \to \mathbb{L}_A F[-2] = F[-2]. $$

Another fact we use is that $i^* i_* F \otimes \mathcal{O}(3)$ is inside A. Thus $\mathbb{L}_A$ of it is 0, and we obtain $S_{\mathrm{Ku}_X} = [2]$ as required.

3.4. **Bridgeland stability conditions.** Let D be a triangulated category. We will be considering $D = D(X)$ or $D = \mathrm{Ku}_X$. Denote $K_0(D)$ its Grothendieck group. Fix Λ a finite rank free abelian group, and $v : K_0(D) \to \Lambda$ a group homomorphism.

Example 3.32.
- For C a smooth projective curve, we may take $D = D(C)$, $\Lambda = H^0(C,\mathbb{Z}) \oplus H^2(C,\mathbb{Z})$, with $v = (\mathrm{rank}, \deg)$. This will give us the usual slope stability in the following construction.
- For S a K3 surface, let $\Lambda = \tilde{H}_{\mathrm{dg}}(S,\mathbb{Z}) := \tilde{H}^{1,1} \cap \tilde{H}(S,\mathbb{Z})$. We may take v to be the Mukai vector.

Definition 3.33. A pair $\sigma = (A, Z)$ on D is a *stability condition* if

- $Z : \Lambda \to \mathbb{C}$ is a group homomorphism, which we call the *central charge*,
- A is the heart of some bounded t-structure on D (so it is an abelian category),

such that

(1) for all $0 \neq E \in A$, $Z(E) := Z(vE) \in \{z \in \mathbb{C} | \operatorname{Im} z > 0 \text{ or } \operatorname{Im} z = 0, \operatorname{Re} z < 0\}$, i.e. the upper-half plane with the negative half of the x axis. Denote $\mu_\sigma = -\operatorname{Re} Z / \operatorname{Im} Z \in (-\infty, \infty]$ the slope. Say E is σ-semi-stable if for all $0 \neq F \subsetneq E$, $\mu(F) \leq \mu(E)$.

(2) For all $E \in A$, there exists a *Harder-Narasimhan filtration*

$$0 = E_0 \to E_1 \to \ldots \to E_n = E$$

where $F_j = E_j / E_{j-1}$ is σ-s.s. and $\phi^+ := \mu(F_1) > \cdots > \mu(F_n) =: \phi^-$.

(3) There exists a quadratic form Q on $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$ such that $Q|_{\ker Z}$ is negative definite and

$$Q(E) \geq 0$$

for all σ-s.s E. This corresponds to the discriminant and Bogomolov inequality for the usual slope stability.

Remark 3.34. The set of stability conditions on D with respect to some fixed Λ and v, denoted $\mathrm{stab}_{(\Lambda,v)} D$, can be considered as a topological space with the coarsest topology such that $\phi^+, \phi^-, \|Z\|$ are continuous maps for any E.

Theorem 3.35. *The map $\mathcal{Z} : \mathrm{stab}\, D \to \mathrm{Hom}(\Lambda, \mathbb{C})$ that sends (A, Z) to Z is a local homeomorphism.*

The local injectivity part can be proven using the HN-filtration from definition.

Example 3.36.
- Let $D = D(C)$, then $(A, Z) = (\mathrm{coh}(C), -\deg + i \,\mathrm{rank})$ is a stability condition. When $g > 1$, one can show all stability conditions are usual slope stability conditions.
- For surfaces S, we can not take $\mathrm{coh}(S)$ as A because a skyscraper sheaf satisfies $\deg = \mathrm{rank} = 0$, so part (1) of the definition does not hold. To fix this, we need to include data of higher cohomology into Z.

Definition 3.37. Let $A \subseteq D$ be the heart of some bounded t-structure. A pair of subcategories (T, F) is a *torsion pair* if

- $\mathrm{Hom}(T, F) = 0$,
- for all $E \in A - \{0\}$, there exists a unique $T_E \in T$, $F_E \in F$ such that

$$0 \to T_E \to E \to F_E \to 0$$

is a short exact sequence in A.

Example 3.38. When $A = \mathrm{coh}(X)$, we can take T be the torsion sheaves and F be the torsion free sheaves.

Lemma 3.39. *The* tilted heart

$$A^{\#} := \left\{ M \in D : H^j_A(M) = \begin{cases} 0, j \neq 0, -1, \\ \in T, j = 0 \\ \in F, j = -1 \end{cases} \right\}$$

is the heart of some bounded t-structure.

Note that if the pair (T, F) decomposes A into the torsion part and torsion free part. For $A^{\#}$, the definition says it decomposes into T and $F[1]$.

Example 3.40. The complex $[E^{-1} \xrightarrow{f} E^0]$ is in $A^{\#}$ if $\operatorname{coker} f \in T, \ker f \in F$.

Example 3.41. Let X be a smooth projective variety, H an ample line bundle. $\beta \in \mathbb{R}$. Set

$$T^{\beta}_H := \{E \in \operatorname{coh}(X) | \text{ all HN factors of } E \text{ satisfy } \mu_H > \beta\},$$
$$F^{\beta}_H := \{E \in \operatorname{coh}(X) | \text{ all HN factors of } E \text{ satisfy } \mu_H \leq \beta\}$$

where μ_H is the Gieseker slope with respect to H. Then (T^{β}, F^{β}) is a torsion pair. Denote the tilting heart of this pair by $\operatorname{coh}^{\beta}$.

Now let X be a surface. We have the following setups

- $\operatorname{ch}^{\beta}(E) := e^{-\beta H} \operatorname{ch}(E)$ the twisted Chern character,
- take λ to be the Lattice generated by $(H^2 \operatorname{rank}(E), H \operatorname{ch}(E), \operatorname{ch}_2(E)) \in \mathbb{Q}^3$ for all possible choices of E.

Theorem 3.42. *For $\alpha > 0, \beta \in \mathbb{R}$, $\sigma_{\alpha,\beta} = (\operatorname{coh}^b(X), Z_{\alpha,\beta})$ where*

$$Z_{\alpha,\beta} := \frac{1}{2}\alpha H^2 \operatorname{ch}^{\beta}_0 - \operatorname{ch}^{\beta}_2 + iH \operatorname{ch}^{\beta}_1$$

is a stability condition.

Proof for part (1) of definition. For simplicity assume $\beta = 0$. For T^0_H, by definition, we always have $H\chi_1 \geq 0$ with equality only for some 0-dimensional sheaf A. For F^0_H, we have $H \operatorname{ch}_1 \leq 0$, so for $F^0_H[1]$, we have $H \operatorname{ch}_1 \geq 0$ with equality on for Gieseker semi-stable torsion free sheaf B. Now in the $H \operatorname{ch}_1 = 0$ case, we have

$$Z_{\alpha,0}(A) = -1 < 0$$

and

$$\Delta_H(B) = -2(H^2 \operatorname{ch}_0 B) \operatorname{ch}_2 B + (H \operatorname{ch}_1 B)^2 \geq 0$$

by Bogomolov inequality. Since $H^2 \operatorname{ch}_0(E) > 0$, we conclude $\operatorname{ch}_2 E \leq 0$ and $Z_{\alpha,0}(B[1]) < 0$. $\square$

Remark 3.43. On 3-folds, $(\operatorname{coh}^{\beta}(X), Z_{\alpha,\beta})$ is no longer a stability condition, because the skyscraper sheaves have no $\operatorname{ch}_0, \operatorname{ch}_1, \operatorname{ch}_2$, so $Z(E)$ might be 0 for $E \neq 0$. We call such a stability condition *weak*. Similar as before, we need to somehow involve ch_3 in Z. It is conjectured that "tilt stable" objects in $(\operatorname{coh}^{\beta}, Z_{\alpha,\beta})$ satisfy some inequalities regarding ch_3 in the 3-fold case.

Now we define a stability condition on Kuznetsov components.

Proposition 3.44. *Suppose $D\langle D', E_1, \ldots, E_n\rangle$ where $D' = \langle E_1, \ldots, E_n\rangle^{\perp}$. Let $\sigma = (A, Z)$ be a weak stability condition, $A' = A \cap D', Z' = Z|_{K_0(D')}$. If the following are satisfied*

(1) $E_i \in A$,
(2) $S_D(E_i) \in A[1]$,
(3) $Z(E_i) \neq 0$,
(4) $0 \neq F \in A' \Rightarrow Z(F) \neq 0$,

then $\sigma' = (A', Z')$ is a stability condition on D'.

Proof. (Sketch) The key is to show that A' is the heart of some bounded t-structure on D'. For $F \in D'$ we need $H^i_A(F) \in A'$. To show this, let q be minimal such that $H^q_A(F) \neq 0$. Prove for this q and proceed by induction. Thus we need

$$\operatorname{Ext}^\bullet(E_i, H^q_A(F)) = 0.$$

Using the exact triangle

$$H^q(F)[-q] \to F \to \tau_A^{>q}(F)$$

we get

$$\operatorname{Ext}(E_i, H^q(F)) = \operatorname{Ext}(E_i, \tau^{>q}(F)[q-1]).$$

On one side, we have

$$\begin{aligned}\operatorname{Ext}^p(E_i, H^q(F)) =& \operatorname{Hom}(E_i, H^q(F)[p]) \\ =& \operatorname{Hom}(H^q(F)[p], S_D(E_i))^\vee \\ =& 0\end{aligned}$$

for $p > 1$ because $H^q \in A$ and $S_D(E_i) \in A[1]$. On the other hand,

$$\operatorname{Hom}(E_i, \tau_A^{>q}(F)[q-1+p]) = 0$$

if and only if $-p+1 \geq 0$, i.e $p \leq 1$ because $E_i \in A$ and $\tau^{>q}[q-1+p] \in D^{>-p+1}$. □

Example 3.45. Let $X \subseteq \mathbb{P}^4$ be a cubic 3-fold, $K_X = -2H$, $D(X) = \langle \mathrm{Ku}_X, \mathcal{O}, \mathcal{O}(1)\rangle$. We would like heart A such that $\mathcal{O}, \mathcal{O}(1) \in A$, $\mathcal{O}(-2)[3], \mathcal{O}(-1)[3] \in A[1]$. Start with the weak stability condition $(\mathrm{coh}^{-\frac{1}{2}}(X), Z_{\alpha,-\frac{1}{2}})$. For all $\alpha > 0$, we have $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(-2)[1], \mathcal{O}(-1)[1] \in \mathrm{coh}^{-\frac{1}{2}}(X)$, and for a close to 0, we have

$$\mu_{\sigma,\alpha,-\frac{1}{2}}(\mathcal{O}(-2)[1]) < \mu_{\sigma,\alpha,-\frac{1}{2}}(\mathcal{O}(-1)[1]) < 0 < \mu_{\sigma,\alpha,-\frac{1}{2}}(\mathcal{O}) < \mu_{\sigma,\alpha,-\frac{1}{2}}(\mathcal{O}(1)).$$

Tilt $\mathrm{coh}^{-\frac{1}{2}}$ again with respect to $\mu = 0$ for α closed to 0, we get a new condition that satisfy the requirements of the above proposition

$$\mathrm{coh}^0_{\alpha,-\frac{1}{2}}(X), Z^0_{\alpha,-\frac{1}{2}} := -iZ_{\alpha,-\frac{1}{2}}.$$

Hence applying the proposition, we get a stability condition on Ku_X.

Remark 3.46. For cubic 4-folds, take a line $\ell \subseteq X$, we get a conic fibration $X \to \mathbb{P}^3$. This gives us a sheaf of Clifford algebra B_0. Now Ku_X embeds into $D(\mathbb{P}^3, B_0)$ and we can replace $D(X)$ by $D(\mathbb{P}^3, B_0)$ to obtain a stability condition.

3.5. **Moduli of objects in derived category.**

3.5.1. *Moduli on a variety.* Let X be a scheme of finite type over $\mathbb{C}$. An object $E \in D(X)$ is a *perfect complex* if locally on X, E is isomorphic to a bounded complex of locally free sheaves. Write $D_{\mathrm{perf}}(X)$ the subcategory of perfect complexes.

Remark 3.47. We can perform derived pullbacks on perfect objects. Note that X is smooth if and only if $D_{\mathrm{perf}} = D(X)$.

Definition 3.48. Given a scheme T of finite type, $E \in D_{\mathrm{perf}}(X \times T)$, say E is universally gluable over T if for all $t \in T$, $\operatorname{Ext}^i(E_t, E_t) = 0$ for all $i < 0$.

Example 3.49. A vector bundle on $X \times T$ is universally gluable.

Definition 3.50. Let X be smooth projective, the moduli stack of perfect universally gluable (pug) complexes is the functor of groupoids

$$\mathcal{M}_{\mathrm{pug}}(X) : \mathrm{Sch}^{\text{fin. type}}/\mathbb{C} \to \mathrm{Grp}$$
$$T \mapsto \{\text{pug complexes on } T \times X\}.$$

Theorem 3.51. *The stack $\mathcal{M}_{pug}$ is an algebraic stack, locally of finite type over $\mathbb{C}$.*

3.5.2. *Moduli of σ-semi-stable objects.* Let X be smooth projective. Fix a stability condition $\sigma = (A, Z)$ with respect to Λ, v. Fix $w \in \Lambda$. Define

$$\mathcal{M}_\sigma(w) : T \mapsto \{E \in D_{\mathrm{perf}}(X \times T) | E_t \in A \text{ is } \sigma\text{-s.s.}, v(E_t) = w\},$$
$$\mathcal{M}^s_\sigma(w) : T \mapsto \{E \in D_{\mathrm{perf}}(X \times T) | E_t \in A \text{ is } \sigma\text{-s.}, v(E_t) = w\}.$$

One might as whether

$$\mathcal{M}^s_\sigma(w) \subseteq \mathcal{M}_\sigma(w) \subseteq \mathcal{M}_{\mathrm{pug}}(X)$$

are open or of finite type. This is known to be positive for X is a surface, and σ is constructed via tilting.

Another question is are there moduli spaces $M_\sigma(w)$ and $M^s_\sigma(w)$ for the moduli stacks that parameterize S-equivalence classes and isomorphism classes respectively. General results by Alper–Halpern-Leistner–Leistner gives good moduli spaces as algebraic spaces.

Say S is a proper algebraic space of finite type, $E \in D_{\mathrm{perf}}(X \times S)$ is a family of σ-semi-stable objects of class w. Define the Bayer-Macri divisor $D_{\sigma,E}$ on S by setting its intersection number with projective integral curves to be

$$D_{\sigma,E} \cdot C = \mathrm{Im}\left(-\frac{Z(p_{X*}(E|_{X\times C}))}{Z(w)}\right).$$

Lemma 3.52 (Positivity lemma)**.** *(1) $D_{\sigma,E}$ is nef on S.*

(2) $D_{\sigma,E} \cdot C = 0$ if and only if on two general points $c, c' \in C$, we have $E_c \sim E_{c'}$ where $\sim$ denotes S-equivalence.

On $M^s_\sigma(w)$, the Bayer-Macri divisor is strictly nef. A general result by Villalobos-Paz states that a proper algebraic space with "mild singularities" and a strictly nef divisor class is in fact a projective scheme.

3.5.3. *Moduli of families of Kuznetsov components.*

Definition 3.53. Given $f : X \to S$, a subcategory $D \subseteq D_{\mathrm{perf}}(X)$ is *S-linear* if $D \otimes f^*(D_{\mathrm{perf}}(X)) \subseteq D$. A SOD

$$D_{\mathrm{perf}}(X) = \langle D_1, \ldots, D_n \rangle$$

is S-linear if each D_i is.

Example 3.54. Let $X \subseteq \mathbb{P}^5 \times S$ be a family of cubic 4-folds. Then

$$D_{\mathrm{perf}}(X) = \langle \mathrm{Ku}_X, f^*D_{\mathrm{perf}}(S), f^*D_{\mathrm{perf}}(S) \otimes \mathcal{O}(1), f^*D_{\mathrm{perf}}(S) \otimes \mathcal{O}(2) \rangle$$

is a S-linear SOD.

Theorem 3.55 (Base change)**.** *For $s \in S$, we have*

$$D_{perf}(X_s) = \langle D_1|_s, \ldots, D_n|_s \rangle$$

for any S-linear SOD.

Definition 3.56. Given $f : X \to S$ smooth proper and $D \subseteq D_{\mathrm{perf}}(X)$ admissible and S-linear. Set

$$\mathcal{M}_{\mathrm{pug}}(D/S) : T/S \mapsto \{E \in D_T \text{ such that } E \text{ is universally gluable over } T\}.$$

Theorem 3.57. *$\mathcal{M}_{pug}(D/S)$ is an algebraic stack, locally of finite type over S.*

3.5.4. *Mukai's theorem for K3 categories.*

Proposition 3.58. *Let X be a cubic 4-fold, there exists Mukai cohomology $\tilde{H}(Ku_X;\mathbb{Z})$ with*

- *weight 2 Hodge structure,*
- *a Mukai vector $v : K_0(Ku_X) \to \tilde{H}$,*
- *a pairing compatible with $-\chi$ on K_0,*
- *that if $Ku_X \cong D(S)$, then $\tilde{H}(Ku_X) = \tilde{H}(S)$.*

Theorem 3.59 (BLMNPS)**.** *Let $0 \neq \nu \in \tilde{H}_{alg}(Ku_X;\mathbb{Z})$ (alg here means integral of type $(1,1)$) be a primitive vector. Take σ generic to ν, then $\mathcal{M}_\sigma(\nu)$ has a good moduli space $M_\sigma(\nu)$ which is smooth projective hyperKähler of dimension ν^2+2, deformation equivalent to $S^{[n]}$ for some K3 surface S.*

The first application is for any cubic 4-fold X, take $\lambda_1, \lambda_2 \in \tilde{H}_{\text{alg}}(\text{Ku}_X;\mathbb{Z})$ where $\lambda_i = v(i^*(\mathcal{O}_\ell(i)))$, then $\mathcal{M}_\sigma(\lambda_i)$ is a locally complete 20-dimensional family of polarized hyperKähler manifold. Fano varieties of lines correspond to $\mathcal{M}_\sigma(\lambda_1)$. LLSvS 8-folds from twisted cubics correspond to $\mathcal{M}_\sigma(\lambda_1 + 2\lambda_2)$. OG10 from elliptic quintics correspond to $\widetilde{\mathcal{M}_\sigma}(2\lambda_1 + 2\lambda_2)$.

Another application is the following theorem from Addington-Thomas and BLMNPS.

Theorem 3.60. *There exists K3 surface S such that $Ku_X \cong D(S)$ if and only if we can find vectors in $\tilde{H}_{alg}(Ku_X;\mathbb{Z})$ such that the intersection matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.*