MATH 127 Calculus for the Sciences

Lecture 8

Today's lecture

Last time

Functions involving absolute values:

- How to write them as piecewise defined functions?
- What are transition points? ✓
- What are their derivatives? ✓

Today

Course note coverage Section 2.2

Integral functions:

- Quick review of theory (you should have read about it in course note before today)
- Examples

Quiz 3 is next Wednesday!

Coverage:

1. Topics that were on the coverage of Quiz 2 might show up in Quiz 3.
2. Left/ right Riemann sum approximation, e.g. do the left Riemann sum approximation by splitting interval into 8 pieces.
3. Definite integral, e.g. evaluate this definite integral/what does the definite integral of speed represent?
4. Indefinite integral, e.g. evaluate this indefinite integral (don't forget $+C$).
5. Initial value problem, e.g. Given $y' = ...$, and initial condition $y(0) = ...$, solve for y.
6. Piecewise defined function, e.g. express a function by defining it piecewise.
7. Absolute value function, e.g. find the derivative of this function that has absolute value signs in it.

Check this out

https://chatgpt.com/share/68cc4e49-c3ac-8006-8f78-be51d3735a7d

I asked AI to generate a powerpoint that illustrates left Riemann sum vs right Riemann sum.

Integral functions

First, regarding integrals, we know

1. $y = f(x)$ is a $\boxed{\text{function}}$ with variable x.
2. $\int f(x)dx$ is a family of functions with variable x, with a constant of integration $\boxed{C}$
3. $\int_0^1 f(x)dx$ is a number, it has nothing to do with x or C!!!
4. $\int_0^t f(x)dx$ is a function with variable t.

Definition For a number a, and a function $f(x)$, we have

$$g(t) = \int_a^t f(x)dx$$

is a function with variable t. Such a function is called an **integral function**.

Example

Example

$$g(t) = \int_2^t 3x^2 dx = \boxed{x^3}\Big|_2^t = \boxed{t^3 - 2^3 =} t^3 - 8$$

Why do we need integral functions?
Because sometimes we have crazy functions like

$$f(t) = \int_0^t \frac{\sqrt{\sin(x)}}{x} dx$$

$$g(t) = \int_0^t e^{-x^2} dx$$

that we just don't know how to integrate. But the notion of integral functions still allow us to define and study them.

Derivative of integral functions

Let $\boxed{F(x)}$ be an anti-derivative of $f(x)$
(we might not know how to compute it, but let us denote it abstractly as $F(x)$).
Then

$$g(t) = \int_a^{t} f(x)dx = F(x)\Big|_a^t = \underline{F(t)} - \underline{F(a)}.$$

Hence

$$g'(t) = F'(t) = \boxed{f(t)}.$$

Theorem (FTC - ~~Part 1~~) *Suppose $f(x)$ is continuous and $g(t) = \int_a^{t} f(x)dx$.*
Then

$$g'(t) = f(t).$$

$$g'(x) = f(x)$$

Example 1

Let $r(t)$ be the rate in mmol/hr at which insulin is produced at time t in hours in a patient.

1. What does the integral function

$$S(x) = \int_0^x r(t)dt$$

   represent?

2. What is the unit of the function $S(x)$?
3. If $r(t) = e^{-t}$, find the values of $S(1), S(3), S(5)$.
4. We want the insulin level of the patient to increase to 2 mmol. How long does it take for this to happen?

Example 1

Let $r(t)$ be the rate in mmol/hr at which insulin is produced at time t in hours in a patient.

1. What does the integral function

$$S(x) = \int_0^x r(t)dt \qquad \text{represent?}$$

   (A) The insulin level of the patient at time x.
   (B) The insulin level of the patient at time t.
   (C) The change of insulin level of the patient between time 0 and time x.
   (D) The change of insulin level of the patient between time 0 and time t.

2. What is the unit of the function $S(x)$?

   (A) mmol/hr
   (B) mmol^2/hr
   (C) mmol
   (D) mmol^2
   (E) something else?

Example 1

Let $r(t)$ be the rate in mmol/hr at which insulin is produced at time t in hours in a patient.

3. If $r(t) = e^{-t}$, find the values of $S(1), S(3), S(5)$.
4. We want the insulin level of the patient to increase to 2 mmol. How long does it take for this to happen?

3. $S(x) = \int_0^x r(t)\,dt$

$= \int_0^x e^{-t}\,dt$

$= \left(\boxed{-e^{-t}}\right)\Big|_0^x$

$= (-e^{-x}) - (-e^{0})$

$= 1 - e^{-x}$

$\left(\boxed{-e^{-t}}\right)' = +e^{-t}$

so $S(1) = 1 - e^{-1}$

$S(3) = 1 - e^{-3}$

$S(5) = 1 - e^{-5}$

4. Since $e^{-x} > 0$, $1 - e^{-x} < 1$ so $S(x)$ will never reach 2

Example 2

Consider the function

$$g(u) = \int_0^u h(v)dv,$$

such that $h(0) = 1, h(1) = 3, h(2) = 1, h(3) = 0.5$.

1. What is $g'(u)$?
2. Suppose $g(1) = 0.5$. What is the tangent line of g at $u = 1$?
3. Express $g(4)$ as an definite integral of $h(v)$.
4. Approximate the integral you wrote down in the previous part, using left Riemann sum approximation with $n = 4$ subdivisions.

Example 2

Consider the function

$$g(u) = \int_0^u h(v)dv,$$

such that $h(0) = 1, h(1) = 3, h(2) = 1, h(3) = 0.5$.

1. What is $g'(u)$?

   Answer: $g'(u) = \boxed{h(u)}$ by $\boxed{\text{FTC.}}$.

2. Suppose $g(1) = 0.5$. What is the tangent line of g at $u = 1$?

   Answer: The tangent line passes through g at point $\boxed{(1, 0.5)}$.

   The slope of the tangent line is $\boxed{g'(1) = h(1) = 3}$.

   The point-slope form of the tangent line is

$$y = \boxed{3}(x - \boxed{1}) + \boxed{0.5}.$$

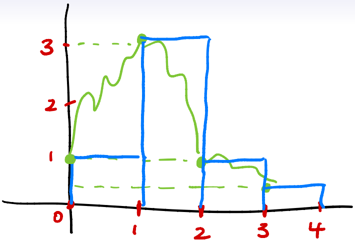
Extra exercise Approximate $g(1.01)$ using the tangent line of $g(u)$ at $u = 1$.

Example 2

Consider the function

$$g(u) = \int_0^u h(v)dv,$$

such that $h(0) = 1, h(1) = 3, h(2) = 1, h(3) = 0.5.$



3. Express $g(4)$ as an definite integral of $h(v)$.

   Answer: $g(4) = \boxed{\int_0^4 h(v)dv}$

4. Approximate the integral you wrote down in the previous part, using left Riemann sum approximation with $n = 4$ subdivisions.

$$\Delta h \approx \text{area of rectangles}$$
$$= 1 \times 1 + 1 \times 3 + 1 \times 1 + 1 \times 0.5$$
$$= 1 + 3 + 1 + 0.5$$
$$= 5.5$$