MATH 127 Calculus for the Sciences

Lecture 2

September 5, 2025

Information

Instructor Jiahui (Kent) Huang; Email:j346huan@uwaterloo.ca

Learn Use the website called Learn to access course material, including

- Course outline; course schedule; course notes; practice exams.

Odyssey Use the website called Odyssey to find quiz/midterm/exam seating.

Piazza Use the forum called Piazza to ask questions, answer other people's questions, and discuss.

Quiz 1 is on Sept 10. The first quiz includes questions about high school algebra and calculus which we will not review in lecture.

Today's lecture

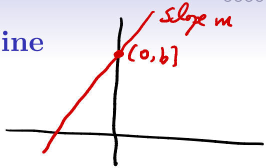
Last time
Differentiation: definitions and review of derivative rules.

This time
Course note coverage Section 1.1.3 - 1.1.4

Tangent lines
Point-slope form of a line
Point-slope form of the tangent line

Differentials
Differential approximation
Differentials

Point-slope form of a line



You have seen that a line is expressed as

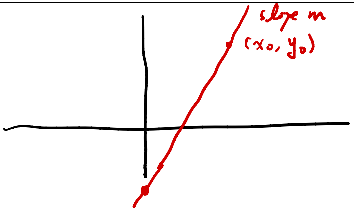
$$y = mx + b$$

where m is the slope, and b is the y-value of the line when it intersects the y-axis. This means the point $(0, b)$ is on the line.

Question How do we express a line with slope m, such that the point (x_0, y_0) is on the line?

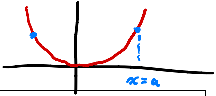
Definition The **point–slope form** of a line with slope m through the point (x_0, y_0) is

$$y = m(x - x_0) + y_0$$

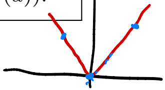


Tangent lines

Next we talk about the **tangent line**. For the notion of a tangent line to make sense, we should say "consider the **tangent line** of ..."



1. What? $\boxed{\text{A function } y = f(x).}$
2. At where? $\boxed{\begin{cases} \text{A point } (x_0, y_0) \text{ where } y_0 = f(x_0), \text{ or } \ldots \\ \text{The value } x = a, \text{ corresponding to the point } (a, f(a)). \end{cases}}$
3. When? $\boxed{\text{Assuming } y = f(x) \text{ is differentiable at } x = a.}$



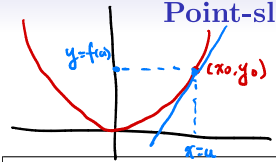
Example Let $y = x^2$. Then the point $(1, 1)$ is on this function. The following sentences makes sense:

The **tangent line** of $y = x^2$ at the point $(1, 1)$ is a line!

The **tangent line** of $y = x^2$ at $x = 1$ is a line!

Now a line has a point-slope form, so we can talk about point-slope forms of tangent lines.

Point-slope form of a tangent line



Recall The **point-slope form** of a line with slope m through point (x_0, y_0) is

$$y = m(x - x_0) + y_0$$

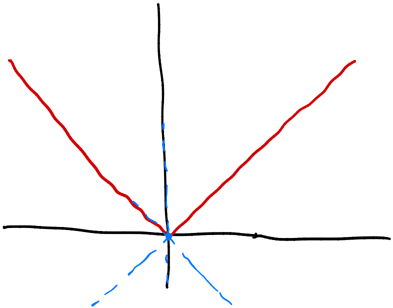
Definition The **tangent line** of a function $y = f(x)$ at $x = a$ is a line, with slope $\boxed{f'(a)}$ through the point $(a, f(a))$. Therefore its point-slope form is

$$y = \underbrace{f'(a)}_{\text{slope}} (x - \underbrace{a}_{x\text{-val}}) + \underbrace{f(a)}_{y\text{-val}}$$

We call this the **equation of the tangent line** of $y = f(x)$ at $x = a$.

Example

Question What is the equation of the tangent line of $f(x) = |x|$ at $x = 0$?

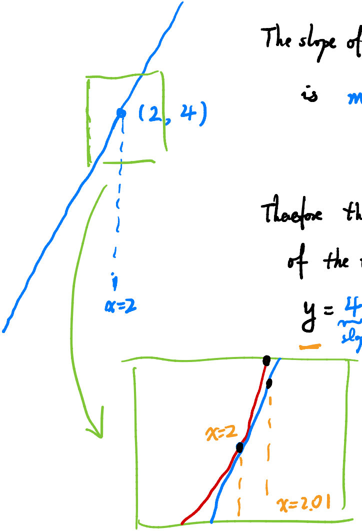


1. Show $f(x) = |x|$ is not differentiable at $x=0$

the function $f(x)$ is not differentiable at $x=0$. therefore the tangent line of $f(x)$ at $x=0$ does not exist.

Example

Question What is the equation of the tangent line of $f(x) = x^2$ at $x = 2$?



The slope of the tangent line

is $m = f'(2)$

$= 4$

Therefore the point-slope form

of the tangent line is

$$y = \underbrace{4}_{\text{slope}}(x - \underbrace{2}_{x\text{-val}}) + \underbrace{4}_{y\text{-val}}$$

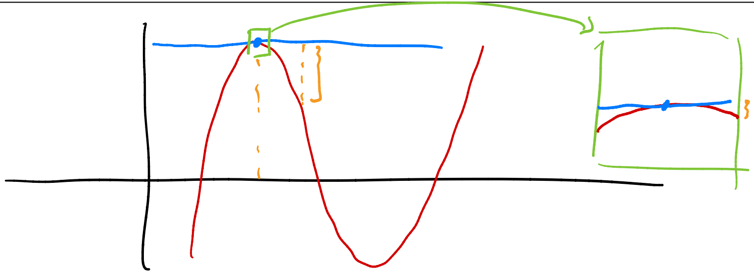
Tangent line approximation

The tangent line of a function at a point looks pretty close to the function itself, at least near that point. In real life, this can be used to approximate the original function.

Definition The **tangent line approximation** to $y = f(x)$ for values x near a

$$f(x) \approx f'(a)(x-a) + f(a).$$

Remark If a function behaves radically, the approximation might be off by a lot. So we need to go really close to the value $x = a$ to get a satisfying approximation, but how close? You will learn this in the future.



Examples

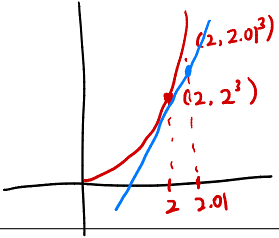
Example We want to approximate $y = f(x)$ at $x = b$ using tangent line at $x = a$. So we compute

$$f(b) \approx f'(a)(b-a) + f(a)$$

and say the right-hand side is an approximation of $f(b)$.

Example Use the equation of the tangent line of the function $y = x^3$ at $x = 2$ to approximate $(2.01)^3$.

$y' = 3x^2$



$y = 12(x-2) + 8$ is the tangent line

$$\begin{aligned}(2.01)^3 &\approx 12(2.01-2)+8\\ &= 12 \cdot 0.01 + 8\\ &= 8.12\end{aligned}$$

$(2.01)^3 = 8.1206\ldots$

Differential approximation

We want to approximate $y = f(x)$ at $x = b$ with tangent line at $x = a$,

$$f(b) \approx f'(a)(b-a) + f(a).$$

Alternatively, we can say

We want to approximate how much $y = f(x)$ changes when we go from $x = a$ to $x = b$ using tangent line at $x = a$, and get

$$f(b) - f(a) \approx f'(a)(b-a)$$

Alternatively, we can say

Definition We want to approximate the change in $y = f(x)$, denoted Δy, knowing the change in x from $x = a$ is $\Delta x = b - a$, using tangent line at $x = a$, and get

$$\Delta y \approx f'(a)\, \Delta x$$

This is the **differential approximation** of $y = f(x)$ near $x = a$.

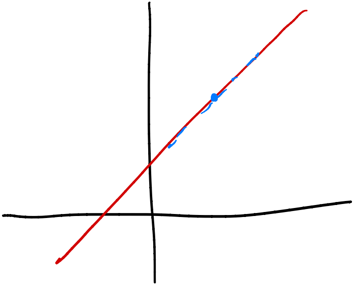
Differential approximation

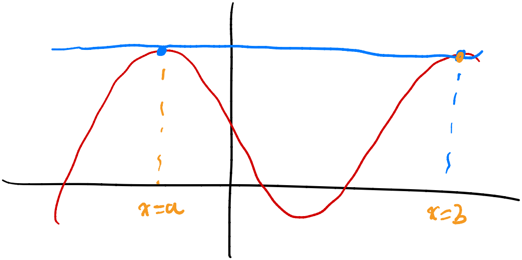
We have an *approxiamtion*

$$\Delta y \approx f'(a)\Delta x.$$

When will the equality hold genuinely?

- When $y = f(x)$ is itself a line, then approximating using tangent line is always perfectly precise.
- When $y = f(x)$ curves around and x is picked at just the right value, we could get lucky and get an exact equality of the approximation.
- When $y = f(x)$ is general, we can never guarantee the approximation to be perfect, except when $\Delta x = 0$, because if we don't change x, then y also does not change, so $\Delta y = 0$.





Differentials

When $y = f(x)$ is general, the approximation

$$\Delta y \approx f'(a)\Delta x$$

$0 = 0$

$\frac{\Delta y}{\Delta x} \approx f'(a)$

is not perfect unless Δx is exactly 0.

However, we still want to capture the notion that when Δx is super small, we know Δy has high precision. This is expressed by saying

$$\boxed{dy = f'(x)dx}$$

in the sense that when $\Delta x \to 0$ is smaller and smaller, we consider it as the **differential** dx, and when $\Delta y \to 0$, we consider it as the **differential** dy. This is compatible with the notation

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$$

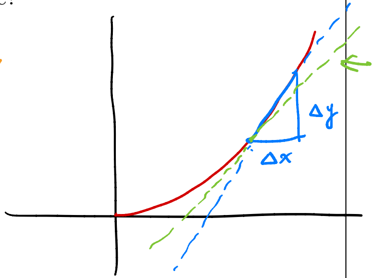
$\lim_{\substack{\varepsilon \to 0 \\ \Delta x}} \frac{f(x+\varepsilon) - f(x) \quad \Delta y}{\varepsilon \quad \Delta x}$

Remark For us, dx, dy are just symbols, they are not numbers nor functions. Saying $dx = 0$ or $dx = 1$, or $dx = x^2$ does not make any sense. In more advanced mathematics, there are more precise definitions of what these symbols should represent.

Examples of differentials

Question Let $y = f(x)$ be a differentiable function and say we are computing the approximation using the tangent line at $x = a$.
Which of the following sentences makes sense?

- When $x = 1$, we have $y = 2$. ✓
- When $\Delta x = 0.1$, we have $\Delta y = 0.05$. ✓
- When $dx = 0.1$, we have $dy = \frac{1}{2} dx$. ✗
- $\dfrac{\Delta y}{\Delta x} = f'(a)$ when $\Delta x \neq 0$. ✓
- $\dfrac{\Delta y}{\Delta x} \approx f'(a)$ when $\Delta x \neq 0$. ✓
- $\dfrac{dy}{dx} = f'(a)$ when $\cancel{dx \neq 0}$.
- $\dfrac{dy}{dx} = 2$. ✓



Units

Let $y = f(x)$ be a function with unit kg. Let x be the input with unit m.

Example

- The unit of $y = f(x)$ is $\boxed{kg}$.
- The unit of $\frac{dy}{dx}$ is $\boxed{kg/m}$.
- The unit of $f'(x)$ is $\boxed{kg/m}$.
- The unit of $\frac{\Delta y}{\Delta x}$ is $\boxed{kg/m}$.

Question The unit of $f''(x)$ is $\boxed{kg/m^2}$, because...

$f'(x)$ is a function with unit $\boxed{kg/m}$ whose input has unit $\boxed{m}$.

So the unit of $f''(x)$ is the quotient $\boxed{(kg/m)/m = kg/m^2}$