Contents

I Chapter 1

I.1 Lecture 1

I.1.1 conics

(a) Since $y-x^2$ is irreducible, $I(Y)=(y-x^2)$. Hence $A(Y)=k[x,y]/(y-x^2)=k[x,x^2]=k[x]$.

(b) Similarly, $A(Z)=k[x,y]/(xy-1)=k[x,\frac{1}{x}]=k[x]_x$ is a local ring, not isomorphic to a polynomial ring.

*(c) By diagonalizability of symmetric bilinear forms, there exists change of coordinates such that $V(f)$ is isomorphic to one of the following:

$$V(xy-1),V(y-x^2),V(x^2-y^2),V(y^2)$$

Since f is irreducible, it must be the first 2 case.

I.1.2 twisted cubic curve

The maps $t\mapsto(t,t^2,t^3)$ and $(x,y,z)\mapsto x$ gives isomorphism $Y\cong\mathbb{A}^1$. So Y is affine variety. $I(Y)$ generated by $y-x^2,z-x^3$ because given $f(x,y,z)\in I(Y)$, write $f=(y-x^2)g(x,y,z)+(z-x^3)h(x,z)+F(x)$, then $F(t)=0$ for all t and so $F=0$.

$A(Y)=A(\mathbb{A}^1)=k[t]$ is given by parametric representation $x=t,y=t^2,z=t^3$.

I.1.3 x

$$\begin{aligned}Y=&V(x^2-yz,xz-x)\\=&V(x^2-yz,x)\cup V(x^2-yz,z-1)\\=&V(yz,x)\cup V(x^2-y,z-1)\\=&V(x,y)\cup V(x,z)\cup V(x^2-y,z-1):=X\cup Y\cup Z\end{aligned}$$

Remains to show each part is irreducible. $I(X)=(x,y)$ is prime, and so is $I(Y)$, so they are irreducible. $I(Z)=(z-1,y-x^2)$ because for any $f\in I(Z)$, write $f=(z-1)g(x,y,z)+(y-x^2)h(x,y)+F(x)$, then $F(x)=0$ for all x because $(x,x^2,1)\in Z$, so $F=0$. Now $A(Y)=k[x,x^2,1]=k[x]$ integral domain, so Z irreducible.

I.1.4 x

Need to show there is an open set in $\mathbb{A}^2$ that is not a union of products of open sets in $\mathbb{A}^1\times\mathbb{A}^1$. Let $f=x-y$. Then $X=\mathbb{A}^2-V(f)$ does not contain any element of form (t,t). Let Y,Z be non-empty open sets in $\mathbb{A}^1$, which are cofinite. Then $Y\cap Z$ is non-empty and there is $(t,t)\in Y\times Z$. Hence X is not open in the product topology.

I.1.5 x

Suppose $B=A(X)=k[x_1,...,x_n]/I(X)$, then since $I(X)$ is radical, there is no nilpotent element in B.

Conversely, suppose B is finitely generated, say $B\cong k[x_1,...,x_n]/I$ where I is a radical ideal. Then by correspondence between radical ideals and algebraic sets, $I=I(X)$ for some X.

I.1.6 x

Suppose X is irreducible and Y is open in X. Say $Y = Y_1 \cup Y_2$ where Y_1, Y_2 are closed in Y. This means there are X_1, X_2 closed in X where $Y_1 = X_1 \cap X$, $Y_2 = X_2 \cap X$. Now $X = X_1 \cup X_2 \cup (X - Y)$. If Y is a proper subset, then $X_1 = X$ or $X_2 = X$. So Y is irreducible. Note that Y intersects all open subsets of X because otherwise it would be contained in a closed proper subset $Z \subsetneq X$, and we would have $X = Z \cup (X - Y)$, contradicting irreducibility of X. Hence Y is irreducible and dense.

Suppose $\overline{Y} = X \cup Z$ for some closed sets X, Z. Then $Y = (X \cap Y) \cup (Z \cap Y)$. Since Y is irreducible, either $X \supseteq Y$ or $Z \supseteq Y$. Since X, Z are closed, one of them must be $\overline{Y}$.

I.1.7 x

(a) (i) $\iff$ (ii) If there is a set without minimal element, pick smaller and smaller elements and get a infinite descending chain; given a descending chain, minimal element is where it stabilizes.

(i) $\iff$ (iii) complement of descending chain condition for closed sets is the same as ascending for open sets.

(ii) $\iff$ (iv) minimal closed set is the same as maximal open set.

(b) Given open cover, pick elements from it to get an ascending chain. The chain stabilizes to the whole set since noetherian, which gives a finite subcover.

(c) Any nonempty family of open subsets correspond to a family of open sets in the bigger space, which is noetherian so has a maximal element, which correspond to maximal element in the smaller space, so the smaller space is noetherian.

(d) Let X be noetherian, then X is a union of finitely many irreducible subsets by prop 1.5. Take Y irreducible in X with at least 2 elements. Then by (c) Y is noetherian. However, pick two elements in Y, and separate them with open sets of $Y_1, Y_2 \subseteq X$, then $Y = (Y - Y_1) \cup (Y - Y_2)$, contradicting irreducibility. Hence all irreducible components of X have one element. So X must be finite with discrete topology.

I.1.8 x

Let $H = V(f)$ where f is irreducible. The irreducible components of $H \cap Y$ correspond to minimal prime ideals $\mathfrak{p}$ containing both (f) and $I(Y)$. These prime ideals are minimal prime ideals $\mathfrak{q}$ in $A(Y)$ that contain (f). Since $A(Y)$ is noetherian and f is not a zero devisor ($Y \not\subseteq H$) nor a unit, by 1.11A they have height 1. By 1.8A,

$$\dim A(Y)/\mathfrak{q} = \dim A(Y) - \operatorname{height} \mathfrak{q} = r - 1$$

and we have $A(Y)/\mathfrak{q} \cong k[x_1, ..., x_n]/\mathfrak{p}$ so the dimension of the irreducible components of $H \cap Y$ is $r - 1$.

I.1.9 x

Let Y, Z, H be varieties. Note that irreducible component of $(Y \cup Z) \cap H$ are irreducible components of $Y \cap H$ and $Z \cap H$.

Now apply previous question repeatedly for each generator. The dimension decreases by 1 if we have $Y \not\subseteq H$ and does not decrease if it is contained in the hypersurface. Thus all irreducible components of $Z(\mathfrak{a})$ have dimension at least $n - r$.

I.1.10 x

(a) A chain of irreducible closed subsets of Y correspond to a chain in X, so $\dim Y \leq \dim X$.

(b) Let $Z_0 \subset ... \subset Z_n$ be a chain of distinct irreducible closed subsets of X. Say Z_0 intersects U. We claim this chain intersects U at a chain of length n. Suppose for a contradiction $\phi \neq U \cap Z_0 \neq U \cap Z_i = U \cap Z_j$ for some $i < j$. By 1.1.6 U is dense in Z_i, Z_j. Thus

$$\overline{U \cap Z_i} = Z_i \subsetneq Z_j = \overline{U \cap Z_j}$$

a contradiction. Hence $\dim U$ is at least the length of n, and so $\dim X = \sup \dim U_i$.

(c) $X = \{1, 2, 3\}$, open sets are subsets that do not contain 1. Then $\{2, 3\}$ is dense but has dimension 2, whereas X has dimension 3.

(d) If $Y \neq X$, then attach X to any chain in Y to get a bigger chain.

(e) $\mathbb{N}$ where the closed sets are all ordinals.

I.1.11 x*

Consider map $k[x, y, z] \to k[t]$ by $x \mapsto t^3, y \mapsto t^4, z \mapsto t^5$. Then kernel is $I(Y)$ (using fact k is infinite). Since $k[t]$ is an integral domain and $A(Y) = k[x, y, z]/I(Y)$ is injected into it, $A(Y)$ is an integral domain and $I(Y)$ is prime. By 1.8A(a), $\dim A(Y) = \operatorname{tran}_k k(t^3, t^4, t^5) = 1$, and by 1.8A(b), height $I(Y) = 2$.

Observe $I(Y)$ contains $y^3 - x^4, z^3 - x^5$, and $z^4 - y^5$. Need to show not generated by 2 elements.

I.1.12 x

$(x^2 - 1)^2 + y^2$. If it were reducible in $\mathbb{R}[x, y]$, it must be a product of two linear polynomials in y, but $f = ((x^2 - 1) + iy)((x^2 - 1) - iy)$, the factors are not real.

I.2 Lecture 2

I.2.1 x

Consider the set of points $Z(\mathfrak{a}) \subseteq \mathbb{A}^{n+1}$. Note that $P \neq 0 \in Z(\mathfrak{a})$ iff the equivalent class of P in $\mathbb{P}^n$ is inside $Z(\mathfrak{a}) \subseteq \mathbb{P}^n$. Therefore $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ by affine Nullstellensatz, and if f is homogeneous with $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in $\mathbb{P}^n$, then $f(P) = 0$ for all $P \in Z(\mathfrak{a}) - \{0\}$ in $\mathbb{A}^n$. If $\deg f > 0$ then $f(0) = 0$, so $f \in \sqrt{\mathfrak{a}}$ and $f^q \in \mathfrak{a}$ for some $q > 0$.

I.2.2 x

(i)⇒(ii) obvious.

(i)⇒(ii) Since S is a domain, $\sqrt{\alpha}$ is a homogeneous ideal. By 1.2.1 it contains $(x_0, \dots, x_n) = S_+$.

(iii)⇒(ii) obvious.

(ii)⇒(iii) For each x_i there is some q_i such that $x_i^{q_i} \in \alpha$. Take $Q = \max q_i$ then $\alpha \supseteq S_Q$.

I.2.3 x

(a) Suppose $f(P) = 0$ for all $f \in T_2$, then true for T_1, so $Z(T_2) \subseteq Z(T_1)$.

(b) Suppose $f(P) = 0$ for all $P \in Y_2$, then true for all Y_1 and $I(Y_2) \subseteq I(Y_1)$.

(c) If $f(P) = 0$ for all $P \in Y_1 \cup Y_2$, then $f \in I(Y_1)$ and $I(Y_2)$, and true conversely.

(d) $\mathfrak{a}$ does not contain any unit so by 1.2.1 $I(Z(\mathfrak{a})) = \sqrt{(\mathfrak{a})}$.

(e) $Z(I(Y))$ is closed and contains Y. So $Z(I(Y)) \supseteq \overline{Y}$. Also, $I(\overline{Y}) \subseteq I(Y)$ by (b), so $Z(I(Y)) \subseteq Z(I(\overline{Y}))$ by (a). By (d), $Z(I(\overline{Y})) = \overline{Y}$.

I.2.4 x

(a) Follows from 1.2.2 and 1.2.3.

(b) Follows from 1.2.3(c). condition should be *homogeneous* prime ideal?

(c) S itself is prime.

I.2.5 x

(a) Follows from 1.2.3 and that S is noetherian.

(b) Follows from Prop 1.5.

I.2.6 x

Let $\varphi_i : U_i \to \mathbb{A}^n$ be the homeomorphism from 2.2. Le Y_i be $\varphi(Y \cap U_i)$. Then $Y_i = Z(\alpha(I(Y)))$ and $A(Y_i) = k[..., \hat{x_i}, ...]/I(Y_i)$. Given an element f in $A(Y_i)$, note that $f(x_0/x_i, ..., x_n/x_i)$ is a degree 0 element in $S(Y)_{x_i}$. Conversely, given an element of degree 0, it must be of form f/x_i^d where $d = \deg f$, so $\alpha(f) \in A(Y_i)$. Also note that elements in $I(Y_i) = \sqrt{\alpha(I(Y))} = \alpha I(Y)$ correspond to elements in $I(Y)_{x_i}$ so this correspondence is well defined.

Thus $A(Y_i)$ is identified with the subring of degree 0 elements in $S(Y)_{x_i}$. Now attaching the new variable x_i and x_i^{-1} gives all elements in $S(Y)_{x_i}$. Hence $A(Y_i)[x_i, x_i^{-1}] = S(Y)_{x_i}$. Now the transcendence degree of fraction field of $A(Y_i)[x_i, x_i^{-1}]$ is $\dim A(Y_i) + 1$. Whereas the fraction field of $S(Y)_{x_i}$ and $S(Y)$ are the same, so the transcendence degree, which correspond to dimension of the ring, is

$$\dim S(Y) = \dim Y_i + 1$$

By proof of 1.1.10(b), $\dim Y = \dim Y_i$ for some i, but we have just shown that $\dim Y_i$ are all equal for Y_i non-empty.

I.2.7 x

(a) Follows from 1.2.6.

(b) By 1.1.10 Y has same dimension as some Y_i. By Prop 1.10 Y_i has same dimension as $\overline{Y}_i$. By previous question $\dim \overline{Y}_i = \dim \overline{Y}$. Hence $\dim Y = \dim \overline{Y}$.

I.2.8 hypersurface

If Y is hypersurface, then $Y_i = Z(\alpha(I(Y))$ are hypersurfaces. So $\dim Y = n - 1$.

Suppose $\dim Y = n - 1$, then $\dim Y_i = n - 1$ for all Y_i non-empty, so $Y_i = Z(f_i)$ are hypersurfaces. Now $Y = Z(x_i^d \beta(f))$ for some d. Since Y irreducible, the polynomial must also be irreducible.

(can also use the fact if $\dim Y = n - 1$ then $\dim S(Y) = n$ correspond to prime ideal of height 1).

I.2.9 projective closure

(a) Assume Y non-empty. Let $\beta I(Y)$ also denote the ideal generated by $\beta I(Y)$. For $P \in Y$, $(1, P) \in Z(\beta(I(Y))$, so $Z(\beta(I(Y))$ contains $\overline{Y}$, which means $\beta I(Y) \subseteq \sqrt{\beta I(Y)} = IZ(\beta(I(Y)) \subseteq I(\overline{Y})$.

Let f be homogeneous such that $f(1, P) = 0$ for all $P \in Y$. This means $\alpha(f) \in I(Y)$ and $f \in \beta I(Y)$. Hence $I(\overline{Y}) \subseteq I((1, Y)) \subseteq \beta I(Y)$.

(b) Recall $I(Y)$ is generated by $y - x^2, z - x^3$. $I(\overline{Y})$ must contain $\beta(xy - z) = xy - zu$, which is not generated by $uy - x^2$, $u^2 z - x^3$ for degree reason. find precise generators?

I.2.10 cone over a projective variety

(a) $C(Y) = Z(I(Y))$ for if $P \in C(Y)$ then $f(P) = 0$ for all $f \in I(Y)$. If $f(P) = 0$ for f homogeneous in $I(Y)$, then $f(kP) = 0$ for all k so $P \in Y$ as a projective point, which means it is in $C(Y)$. (special case when $P = 0$).

(b) $C(Y)$ is irreducible iff $I(Y)$ is prime.

(c) $\dim C(Y) = \dim A(C(Y)) = \dim S(Y) = \dim Y + 1$

I.2.11 linear projective variety

(a) $Z(f_1, ..., f_n) = Z(f_1) \cap ... \cap Z(f_n)$.

(b) Let r be the size of minimal linear generators of $I(Y)$. Change coordinates and get $S(Y) = k[x_r, ..., x_n]$ has dimension $n - r + 1$, so $\dim Y = n - r$.

(c) By linear algebra, $I(Y \cap Z)$ is generated by at most $(n-r)+(n-s) \leq n$ generators, so $\dim S(Y \cap Z) \geq r + s - n + 1 \geq 1$, $Y \cap Z$ non-empty. If $Y \cap Z$ non-empty then $\dim S(Y \cap Z) \geq r + s - n + 1$ impleis $\dim Y \cap Z \geq r + s - n$.

I.2.12 d-Uple embedding

(a) Note that Y vanishes on $\mathfrak{a}$. So $\mathfrak{a} \subseteq I(Y)$. Conversely if f vanishes on Y, then $f(M_0(a), ..., M_N(a)) = 0$ for all a, so $f(M_0, ..., M_N) = 0$ as a polynomial, and f is in $\ker\theta$. Hence $\mathfrak{a} = I(Y)$ is homogeneous. Since image of θ is integral domain, $\mathfrak{a}$ is prime.

(b) Suppose $P' = (b_0, ..., b_N)$ vanishes on $\mathfrak{a}$. Take $P = (a_0, ..., a_n)$ where $a_i = \sqrt[d]{b_{j,i}}$, $M_{j,i} = x_i^d$. Then we must have $P' = \rho_d(P)$ because each M_ℓ^d can be written as product of $M_{j,i}$

(c) For $f \in k[x_0, ..., x_n]$ homogeneous, let $g \in k[x_1, ..., x_N]$ be such that $\theta(g) = f^d$. Then $f(P) = 0$ iff $g(\rho_d(P)) = f^d(P) = 0$ and closed sets are mapped to closed sets. For $g \in k[x_1, ..., x_N]$ homogeneous, we have $\theta(g)(P) = 0$ iff $g(\rho_d(P)) = 0$, so inverse of closed sets are closed. Hence homeomorphism.

(d) Let Y be twisted cubic curve in $\mathbb{P}^3$. The twisted cubic curve has ideal generated by kernel of the map $(x, y, z) \mapsto (t, t^2, t^3)$. By 1.2.9 $I(Y)$ is then generated by homogeneous elements inside $\ker\varphi$ where $\varphi(u, x, y, z) = (1, t, t^2, t^3)$. Note that take $(M_0, ..., M_3) = (x_0^3, x_0^2x_1, x_0x_1^2, x_1^3)$ and the kernel of θ is also generated by those elements. Hence the twisted cubic curve is the 3-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$.

I.2.13 veronese surface

Y has dimension 2 by the homeomorphism. A curve in Y is a hypersurface defined by irreducible f, by previous part, $V = Z(g)$ for some $\theta(g) = f^d$ gives required hypersurface. The intersection is exactly Z because dimension are both 1. V can be taken to be irreducible because Z is contained in one of its irreducible component (Y, Z are irreducible).

I.2.14 segre embedding

Let $\mathfrak{a}$ be the kernel of $k[\{z_{ij}\}] \to k[..., x_i, ..., y_j, ...]$ which sends z_{ij} to x_iy_j. Then $\mathfrak{a}$ prime and homogeneous ideal of the image of ψ by same argument as in 1.2.12.

To show image is closed, note if $P' = (b_{ij})$ vanishes and some $b_{\ell m}$ is non-zero, then $P' = \psi(P)$, $P = (a_1, ..., c_1, ...)$ where $a_i = b_{im}/b_{\ell m}$ and $c_j = b_{\ell j}/b_{\ell m}$. explicit description for $\mathfrak{a}$ is: $(z_{ij}z_{kl} - z_{il}z_{kj})_{0\le i,k\le n, 0\le j,l\le m}$

I.2.15 the quadric surface

(a) Let embedding be $(x_0, x_1), (y_0, y_1) \mapsto (x_0y_0, x_0y_1, x_1y_0, x_1y_1) = (x, z, w, y)$. Suppose $xy = zw$, let $P = (x/w, 1), (x/z, 1)$ or $(z/y, 1), (w/y, 1)$. If $w = y = 0$ let $P = (1, 0), (x/z, 1)$. If $z = y = 0$, let $P = (1, 0), (1, 0)$. This shows the image is equal to $Z(xy - zw) = Q$.

(b) Let $L_t = Z(x - tw, ty - z)$, $M_t = Z(x - tz, ty - w)$. Then $L_t \cap M_u$ has 1 solution in projective plane because the four linear equations have rank 3.

(c) Consider $Q \cap Z(x - y)$. Is a curve because isomorphic to conic $Z(y^2 - zw)$ in $\mathbb{P}^2$.

I.2.16 x

(a) Suppose $x^2 = yw$ and $xy = zw$. When $w = 1$, the curve is parametrized by $\mathbb{A}^1$ and is twisted cubic curve (take projective closure gives desired projective curve). When $w = 0$, we have $x = 0$, and the curve is $Z(x, w)$ a line.

(b) $C \cap L$ contains $(0, 0, 1)$ only. But $I(C) = (y)$, $I(L) = (x^2 - yz)$, and $I(P) = (x, y)$. Now $I(C) + I(L) \neq I(P)$ because we can not get x.

I.2.17 complete intersection *

(a) $Z(f_1, ..., f_q) = Z(f_1) \cap ... \cap Z(f_q)$. So by 1.1.8, $\dim Y \geq n - q$.or we can prove $\mathfrak{a}$ has height at most q?

(b) follows from (a).

(c) Let H_1 be $Z(uy - x^2)$, H_2 be $Z(z^2u - y^3)$. Then when $u = 1$ we have the affine twisted cubic curve and when $u = 0$, we have $x = 0, y = 0$, the only point is $(0, 0, 0, 1)$, giving us exactly the projective twisted cubic curve (by the homeomorphism from 1.2.12).Need to show not generated by 2 elements.

I.3 Lecture 3

I.3.1 conics part 2

(a) By 1.1.1 any conic has coordinate ring isomorphic to $k[x]$ or $k[x]_x$. Now $k[x]$ is isomorphic to the coordinate ring of $\mathbb{A}^1$. On the other hand, $k[x]_x$ is the regular functions on $\mathbb{A}^1 - \{0\}$, which is affine by the projection of $V(1-xy)$ to the x-axis.

(b) If they were isomorphic, they must have the same regular functions, where constant functions correspond to constant functions. But non-constant functions on $\mathbb{A}^1$ are not invertible, whereas a proper open subset (infinite because it has dimension 1), say does not contain a, has regular function $1/(x-a)$.

(c) By diagonalizability of symmetric bilinear forms, any conic is isomorphic to $Z(x^2+y^2+z^2)$ or $Z(x^2+y^2)$ or $Z(x^2)$. Since irreducible, must be $Z(x^2+y^2+z^2) \cong Z(xy-z^2)$. Consider the isomorphism

$$(x,y,z) \mapsto (x,z), \text{ on } x \neq 0$$

$$(x,y,z) \mapsto (z,y), \text{ on } y \neq 0$$

(well defined regular function) with inverse $(s,t) \mapsto (s^2,t^2,st)$.

(d) Two lines in $\mathbb{P}^2$ must intersect by 2.11(c).

(e) If affine isomorphic to projective, then $A(X) = \mathcal{O}(Y) = k$, so $I(X)$ is maximal, X a point.

I.3.2 x

(a) Inverse $(x,y) \mapsto y/x$. Bijections on dimension 1 varieties are homeomorphic because topology is cofinite topology. (or observe $f(t)=0$ iff $f(y/x)x^d = 0$ at (t^2,t^3), and $f(x,y)=0$ iff $f(t^2,t^3) = 0 \in k[t]$). φ is not isomorphism because $A(\mathbb{A}^1) = k[t]$, $A(X) = k[t^2,t^3]$ (by 1.3.3(c)) is not integrally closed (t is integral over $k[t^2,t^3]$).

(b) Injective because $(a+b)^p = a^p + b^p$ (homomorphisms on field always injective). Surjective because algebraically closed. Not isomorphism because $f(t^p) = t$ for all $t \in \mathbb{A}^1$ implies $f(t^p) = t \in k[t]$.

I.3.3 pullback

Isomorphisms perserve local rings

(a) For each $f \in \mathcal{O}_{\varphi(P),Y}$, define $\varphi_P^*(f) = f(\varphi) \in \mathcal{O}_{P,X}$. Since φ morphism, f regular function on a neighbourhood around P. (basically $\alpha(\varphi)$ in prop 1.3.5 in the affine case)

(b) Need to show $f(\varphi^{-1})$ is regular for all $f \in \mathcal{O}_{P,X}$. By surjectivity of φ_P^*, we have $f(\varphi^{-1}) = g\varphi(\varphi^{-1}) = g$ is regular as required.

(c) Suppose $(f/g)(\varphi) = \varphi_P^*(f/g) = 0$ on some open set around P. Then $f/g = 0$ on $\varphi(U) \cap V$ where U is open dense in X. Since $\varphi(X)$ is dense in Y, $\varphi(U) \cap V$ is dense in V. Therefore $f \in I(\varphi(U) \cap V)$ and $Z(I(\varphi(U) \cap V)) = Y$ vanishes on f, so $f/g = 0$.

I.3.4 d-Uple embedding part 2

By 1.3.3(b), need to show $\varphi : a \mapsto (M_0(a), ..., M_N(a))$ regular (because it is a polynomial function) and $\varphi^* : f \mapsto f(M_0, ..., M_N)$ surjective on local rings. Given $f \in \mathcal{O}_P$, write $f = F/G$ where F, G homogeneous. Then $f = F^N/G^N$ is of form $h(M_0, ..., M_N)/\ell(M_0, ..., M_N)$ for some h/ℓ in $\mathcal{O}_{\varphi(P)}$.

I.3.5 x

H is isomorphic to $\varphi(H)$ where φ is d-Uple embedding, and d is degree of H. Now defining polynomial of H correspond to a linear homogeneous polynomial H', $\varphi(\mathbb{P}^n - H) = (\mathbb{P}^N - H') \cap \varphi(\mathbb{P}^n)$. Now $\mathbb{P}^N - H'$ is affine (by change of coordinate it is the same as $\mathbb{P}^n - \{x_0 = 0\} = U_0 \cong \mathbb{A}^n$) and $\varphi(\mathbb{P}^n)$ is a closed subset of it.

I.3.6 x

$\mathcal{O}(X)$ consists functions f/g where g can only vanish on $(0,0)$. However $Z(g)$ is infinite, so g must not vanish at all and g is a unit (or g divides f). Thus $\mathcal{O}(X) = k[x,y]$. If it were affine then by Theorem 1.3.5, the identity map on $k[x,y]$ correspond to the identity map from $\mathbb{A}^n$ to Y, which is not possible.

I.3.7 x

(a) By 1.2.8, curves in $\mathbb{P}^2$ are of form $Z(f)$. Now $Z(f) \cap Z(g) = Z(f,g)$. Inside $\mathbb{A}^3$, (f,g) has infinitely many zeros because it has dimension 1, therefore it has at least one zero in $\mathbb{P}^2$.

(b) If Y does not intersect a hyperspace, then it is closed set in $\mathbb{P}^n - H$ which is affine by 1.3.5. Thus Y is affine. However by 1.3.2(e) Y must contain only one point, but $\dim Y \geq 1$.

I.3.8 x

Suppose we have regular function f/g, then g must not vanish on $P^n - (H_0 \cap H_1)$. However, by 1.3.7 the hypersurface $Z(g)$ must intersect $Z(x_2, ..., x_n)$ at some point $P = (a, b, 0, ..., 0)$. Now a, b can not both be 0, so $P \in P^n - (H_0 \cap H_1)$, a contradiction. Hence g is a unit and f/g is a constant (since regular functions on projective have degree 0).

I.3.9 x

The embedding is $(x,y) \mapsto (x^2, xy, y^2)$. So $I(Y)$ is the kernel of $k[x,y,z] \to k[x,y], x \mapsto x^2, y \mapsto xy, z \mapsto y^2$. Now $S(Y)$ is isomorphic to the image $k[x^2, xy, y^2]$ which is not isomorphic to $S(X) = k[x,y]$ because it is not UFD $(x^2y^2 = (xy)^2)$.

Note: Also note UDF because x^2 can not be factored as a product of primes, as x^2 itself is not prime for the same reason.

Find rings where there are no prime elements? given curve, $\dim \mathcal{O}_P = 1$, so it has unique prime ideal, which will not be principal if P is singular, so $\mathcal{O}_P$ noetherian domain with no prime.

I.3.10 subvarieties

Let f be a regular function on Y' Then for any open set $V \subseteq Y'$, we can consider it as open set in Y, so $f \circ \varphi : \varphi^{-1}V \to k$ is regular by definition of morphism. This means $f \circ \varphi : \varphi^{-1}V \cap X' \to k$ is also regular, so $\varphi|_{X'}$ is a morphism.

I.3.11 x

Suppose X' contains P, then $I(X')$ corresponds to a prime ideal containing $I(X)$, and contained inside $\mathfrak{m}_P$. This means they are in (injective) correspondence with primes in $\mathcal{O}_P = A(X)_{\mathfrak{m}_P}$. Conversely, given a prime ideal $\mathfrak{p}$ in $\mathcal{O}_P$, it must be contained in $\mathfrak{m}_P$, so $\mathfrak{p} \cap A(X)$ is a prime ideal subvariety in X containing P.

I.3.12 x

If X quasi-affine, it has same dimension as closure and same local ring at P (by taking intersections of domains of regular functions).

If X projective variety, P is in some non-empty Y_i, which has same dimension as X by 1.2.6 and same local ring by proof of 1.3.4(b).

If X is quasi-projective, then same dimension as its closure and same local ring at P for the same reason as above.

I.3.13 local ring of subvariety

Suppose $f/g \in \mathcal{O}_{Y,X}$ and $f/g \neq 0$ on some point P in Y. Then the set where $f \neq 0$ on X is $X - Z(f)$ is open, so f is invertible with inverse having domain $U - Z(f)$ which intersects Y at P. On the other hand, if f vanishes on Y, then it has no inverse in $\mathcal{O}_{Y,X}$ because we can not find domain V, as $U \cap Y$ is open (so dense in $\overline{Y}$) and $V \cap Y$ is open, so $U \cap V \cap Y$ is non-empty. Now observe the maximal ideal is exactly made of non-units, so the ring is local.

Note that if two functions in $\mathcal{O}_{Y,X}$ agree on Y, then their difference is 0 in the residue field. Therefore the residue field is identified with $K(Y)$. The prime ideals corresponds to closed subvarieties of X that contain Y by a similar argument as 1.3.11. Therefore by Thm 1.1.8A (any prime ideal can be extended to a chain, so any variety can be extended to a chain filling dimension), $\dim \mathcal{O}_{Y,X} = \dim X - \dim \overline{Y} = \dim X - \dim Y$.

When Y is not a point, $K(Y)$ is not algerbaically closed by Theorem I.4.9 (it is isomorphic to $k(x_1, ..., x_r)(y)$ where x_i transcendental y algerbaic).

I.3.14 projection from a point

(a) By change of coordinates, assume $\mathbb{P}^n = H = Z(x_0)$, $P = (1, 0, ..., 0)$. Let $Q = (a_0, a_1, ..., a_{n+1})$, then the line containing P, Q is $Z(a_j x_i = a_i x_j : i, j = 1, ..., n+1)$. By linear algebra, the set of homogeneous equations have rank n, so it is indeed a line and note it intersects $\mathbb{P}^n$ at $(0, a_1, ..., a_{n+1})$. Hence the map φ is the projection map onto the last $n+1$ coordinates, which is a morphism because it is a polynomial map.

(b) Note that $\varphi(Y)$ is parametrized by (t^3, t^2u, u^3). Indeed, this parametrization is invertible by $(t/u, 1) = (x/y, 1)$ When $y \neq 0$ or $(1, 0)$ when $z \neq 0$, $(1, y/x)$ when $x \neq 0$. Thus it is closed of dimension 1 in $\mathbb{P}^2$ and therefore must be defined by the irreducible polynomial $y^3 = zx^2$, a cuspidal cubic.

I.3.15 products of affine varieties

(a) Suppose $X \times Y = Z_1 \cup Z_2$. Since $x \times Y \cong Y$ irreducible, $x \times Y$ is either inside Z_1 or Z_2. Let $X_i = \{x \in X : x \times Y \subseteq Z_i\}$. Then

$$x \in X_i \iff x \times Y \subseteq Z_i \iff f(x, Y) = 0 \text{ for all } f \in I(Z_i)$$
$$\iff x \in Z(f(x, y) \in k[x] : f \in I(Z_i), y \in Y)$$

Hence X_i is closed. and $X = X_1$ or $X = X_2$, giving $Z_1 = X \times Y$ or $Z_2 = X \times Y$.

(b) Consider the k-algebra homomorphism $k[x, y] \to A(X) \otimes A(Y)$. Suppose F is in the kernel, then by the equational characterization of when a tensor product element is 0, we know F can be written as

$$F = \sum_i f_i(x) g_i(y) = \sum_i \sum_j (a_{ij} h_j(x)) g_i(y)$$

such that $\sum_i a_{ij} g_i(y) = 0$ in $A(Y)$, this means $F = \sum_j h_j(x) g_j(y)$ where $g_j \in I(Y)$. As a result, $F \in I(X \times Y)$. On the other hand, suppose $F = \sum f_i(x) g_i(y) \in I(X \times Y)$, we can add in elements of $I(X)k[x, y]$ without affecting its image. Do so until we get an expression for F with minimal number of summands. If $g_i(y) \in I(Y)$ for all i, then it is 0 in $A(X) \otimes A(Y)$. Otherwise, $F(x, P) = \sum g_i(P) f_i(x) \in I(X)$ for some P and $g_1(P) \neq 0$. We have $f_1(x) = (F(x, P) - \sum_{i>1} g_i(P) f_i(x))/g_1(P)$. Substitute back $f_1(x)$ into the expression, then subtract $F(x, P) g_1(y)/g_1(P) \in I(X)k[x, y]$ and contradict minimality. Hence kernel is equal to $I(X \times Y)$.

(c) Projections are morphisms because they are polynomial maps. Given Z with morphisms $f : Z \to X, g : Z \to Y$, define $h(P) = (f(P), g(P))$ is required morphism (by Lemma 1.3.6, coordinatewise regular).

(d) Given chains of closed subvarieties of X, Y, say $X_0, ..., X_n$, $Y_0, ..., Y_m$, we have $X_0 \times Y_0, ..., X_n \times Y_0, ..., X_n \times Y_n$ gives $\dim X \times Y \geq \dim X + \dim Y$.

Suppose $\dim K(X \times Y) > m + n$, say with transcendence basis $\{x_i, y_j\}$, then we know there are at least $n+1$ x_i's in that basis, or $m+1$ y_j's. In either case, we get a polynomial in $I(X)$ or $I(Y)$ that gives a dependent relation between those elements.

I.3.16 products of projective varieties*

(a) Take $U_i = \mathbb{P}^n - Z(x_i)$, $V_i = \mathbb{P}^m - Z(x_i)$. Then $X = \cup(U_i \cap X)$ and $Y = \cup(V_i \cap Y)$ and $X \times Y = \cup(U_i \cap X \times V_j \cap Y)$. It suffices to prove for each part, so assume $x_0, y_0 \neq 0$.

$$(P,Q) \in X \times Y \iff P \in \mathbb{P}^n - Z(I), Q \in \mathbb{P}^n - Z(J)$$

$$\iff \psi(P,Q) \notin Z(I_0), \psi(P,Q) \notin Z(J_0)$$

where I_0 is I but in the coordinates corresponding to $x_0y_0, ..., x_ny_0$, and J_0 correspond to $x_0y_0, ..., x_0y_m$

Hence the image is open.

(b)

$$(P,Q) \in X \times Y \iff P \in Z(I), Q \in Z(J)$$

$$\iff \psi(P,Q) \in Z(I_0, I_1, ..., I_n, J_0, ..., J_m)$$

(c)* Try a similar proof as affine case. On each $\mathbb{P}^n - Z(x_{rs})$, define projection maps $\pi_X(b_0, ..., b_N) = (a_0, ..., a_n)$ where $a_i = b_{is}/b_{rs}$ and $\pi_X(b_0, ..., b_N) = (c_0, ..., c_m)$ where $c_j = b_{rj}/b_{rs}$. Is morphism because rational functions on each open sets.

I.3.17 normal varieties

(a) Conics in $\mathbb{P}^2$ are equivalent by change of coordinates. By 1.3.9 we can assume $S(Y) = k[x^2, xy, y^2]$. Suppose

$$t^n + f_{n-1}(x^2, xy, y^2)t^{n-1} + ... + f_0(x^2, xy, y^2) = 0$$

The elements in the quotient field all have even degree, which means degree of t is even, but then it can not have denominator and thus its terms are divisible by one of x^2, xy, y^2. Hence $S(Y)$ is integrally closed.

By 1.3.4 $S(Y)_{(\mathfrak{m}_P)} = \mathcal{O}_P$. Suppose

$$x^n + \frac{a_{n-1}}{s_{n-1}}x^{n-1} + ... + \frac{a_0}{s_0} = 0$$

let $s = s_{n-1}...s_0$, then multiply s^n we see sx is integral over $S(Y)$, so $x \in S(Y)_{\mathfrak{m}_P}$, also x has degree 0 by observing the equation. Hence $\mathcal{O}_P$ is integrally closed. (integral closure is local property)

or just use the fact they are isomorphic to $\mathbb{P}^1$

(b) By the proof above, it suffices to show $S(Y)$ is integrally closed. Note that Q_1 is the image of serge embedding $\mathbb{P}^1 \times \mathbb{P}^1$, so $S(Y)$ is isomorphic to $k[x_0y_0, x_0y_1, x_1y_0, x_1y_1]$. The elements in quotient field have degree of x equal to degree of y, as a result the integral elements are made of terms divisible by $x_0y_0, x_0y_1, x_1y_0, x_1y_1$. Hence integrally closed.

(c) Let $P = (0,0)$, then $\mathcal{O}_P = A(X)_{\mathfrak{m}_P} = k[x,y]/(x^3 - y^2)_{(x,y)}$. Since (x,y) generate all non-unit elements, localizing does nothing, so $\mathcal{O}_P = A(X) = k[t^2, t^3]$, which is not integrally closed.

(d) If Y normal, then $\mathcal{O}_P = A(X)_{\mathfrak{m}_P}$ integrally closed for all P. Now $A(X) = \cap_P A(X)_{\mathfrak{m}_P}$. An integral element in $A(X)$ would by definition be an integral element in $A(X)_{\mathfrak{m}_P}$ for all P. Therefore $A(X)$ is integrally closed. Conversely, if $A(X)$ is integrally closed, then localization is closed by proof in part (a).

(e) Consider the integral closure of $A(Y)$ in $K(Y)$. By 1.3.9A, it is a finitely generated A-module and finitely generated k-algebra. By 1.1.5 since $K(Y)$ is an integral domain, it is $A(\tilde{Y})$ for some affine variety $\tilde{Y}$. if $\varphi : Z \to Y$ is dominant, then $\varphi^* : A(Y) \to A(Z)$ is injective, where $A(Z)$ is integrally closed.

Let $\pi : \tilde{Y} \to Y$ be the map corresponding to the inclusion $A(Y) \to A(\tilde{Y})$.

Extend φ^* to map $K(Y) \to K(Z)$ since injective, then this gives us unique map on the integral closures, namely $F^* : A(\tilde{Y}) \to A(Z)$. Now define $\theta : Z \to \tilde{Y}$ corresponding to F^* and we have $\pi \circ \theta = \varphi$ (because the diagram for coordinate rings commute).

I.3.18 projective normal varieties

(a) As shown in proof of 1.3.17(a).

(b) $S(Y) = k[t^4, t^3u, tu^3, u^4]$. In the quotient field we have $tu^3/u^4 = t/u$, so t^2/u^2, so t^2u^2, but then $(t^2u^2)^2 = t^4u^4$, so not integrally closed. The curve is image of $\mathbb{P}^1$ by isomorphism (perserves local rings by 1.3.3(a)), which is normal, so it is normal.

(c) Inverse is $(x/y, 1)$ when $w \neq 0$ and $(1, y/x)$ when $x \neq 0$.

I.3.19 automorphisms of $\mathbb{A}^n$**

(a) Follows from inverse function theorem the jacobian of automorphism is invertible at each point. It is a invertible polynomial because inverse is also a polynomial function. Hence it must be non-zero constant.

(b) **

I.3.20 x(*)

(a) Suppose f is regular on $Y - P$, then $f \in \cap_{Q \neq P} A(Y)_{\mathfrak{m}_Q}$ f might not have unique representation, we only know it has polynomial representation in $A(Y)$ if it is in all $\mathcal{O}_P$. Need f integral over $A(Y)_{\mathfrak{m}_P}$??

(b) $1/x$ with $Y = Z(y) \subseteq \mathbb{A}^2$, $P = (0, 0)$.

I.3.21 group varieties

(a) Polynomial function where inverse map is $a \mapsto -a$.

(b) Inverse given by $x \mapsto 1/x$ rational function well defined on $\mathbb{A}^1 - \{0\}$.

(c) Given $f, g \in \mathrm{Hom}(X, G)$, we have $fg \in \mathrm{Hom}(X, G)$ with $fg(x) = \mu(f(x), g(x))$ is a morphism because $X \times Y$ is product in category of varieties (so $x \mapsto (f(x), g(x))$ is morphism) What about quasi-affine?? does lemma 1.3.6 work with quasi-affine?. $f^{-1}(x) = f(x)^{-1}$ is composition of morphisms. Identity is constant function onto the identity of G.

(d) Morphisms from X to $\mathbb{A}^1$ is the same as regular function from X to k (compose with identity gives regular function). Since $\mathcal{O}(X)$ is additive group, $\mathrm{Hom}(X, G_a) \cong \mathcal{O}(X)$.

(e) Morphisms from X to $\mathbb{A}^1 - \{0\}$ with multiplicative inverse are the same as regular functions with multiplicative inverse, so units in $\mathcal{O}(X)$.

I.4 Lecture 4

I.4.1 x

Function that is f on U and g on V is regular on $U \cup V$ by definition.

I.4.2 x

Suppose φ is morphism from U to Y, ψ is morphism from V to Y and $\varphi = \psi$ on $U \cap V$. Combine the functions to get a function F on $U \cup V$, continuous since inverse of open is open. Given regular $f : W \to k$ function on Y. Then $F^{-1}(W) = \varphi^{-1}W \cup \psi^{-1}W$, which is regular on each part, so $F \circ f$ is regular. Hence F regular map.

I.4.3 x

(a) Suppose $g/h = x_1/x_0$ on $\mathbb{P}^2 - Z(h) - Z(x_0)$, where $Z(h)$ misses a point in $Z(x_0)$. Now $Z(x_0)$ is infinite, so $Z(x_0) - Z(h)$, an open dense set, is infinite, so $Z(x_0) - Z(h)$ contains some point other than $(0, 0, 1)$. Consider the open set $h \neq 0$, $x_0 g \neq x_1 h, x_0 \neq 0$, which are all non-empty by our construction, and as a result $g/h \neq x_1/x_0$ at that point, a contradiction.

(b) When considered as a map from $\mathbb{P}^2 \to \mathbb{P}^1$, with $\varphi(x_0, x_1, x_2) = (1, x_1/x_0)$, it is equal to $(x_0, x_1, x_2) \mapsto (x_0, x_1)$, defined on all points other than $(0, 0, 1)$.

If it extends to morphism on $\mathbb{P}^2$, then it restricts to U_0 to $\varphi(0, x_1, x_2) = (0, 1)$ by continuity. But it must also restirct to U_1 to $\varphi(x_0, 0, x_2) = (1, 0)$, and as a result $\varphi(0, 0, 1) = (1, 0) = (0, 1)$, a contradiction. Hence the open set it is defined on is $\mathbb{P}^2 - (0, 0, 1)$.

I.4.4 x

(a) Any conic in $\mathbb{P}^2$ isomorphic to $\mathbb{P}^1$.
(b) Use map $(x, y) = (t^2, t^3) \mapsto (1, y/x)$ with inverse $(x_0, x_1) \mapsto ((\frac{x_1}{x_0})^2, (\frac{x_1}{x_0})^3)$.
(c) The map is $(x, y, z) \mapsto (x, y)$ with inverse $(x, y) \mapsto ((y^2 - x^2)x, (y^2 - x^2)y, x^3)$ because

$$y^2 z = x^2(x + z)$$

$$y^2/x^2 = x/z + 1$$

$$z = \frac{x^3}{y^2 - x^2}$$

I.4.5 x

By 1.2.15 it is the image of the serge embedding $(a_0, a_1), (b_0, b_1) \mapsto (a_0b_0, a_1b_0, a_0b_1, a_1b_1)$. Define rational map $(x, y, z, w) \mapsto (x/y, x/w, 1) = (a_0/a_1, b_0/b_1, 1)$ with inverse $(x, y, z) \mapsto (xy/z^2, y/z, x/z, 1)$ we see Q is birational to $\mathbb{P}^2$, but it is not isomorphic to $\mathbb{P}^2$ because from 1.2.15 it has dimension 1 subvarieties that do not intersect.

I.4.6 plane cremona transformations

(a) Note that if the inverse is defined on an open set, it is automatically dominant so birational. The inverse, if we check, is exactly φ itself.
(b) Take U, V to be the set of elements with no zero coordinate.
(c) Claim that it is defined on the set of points where no two coordinates are 0, namely $U = \mathbb{P}^2 - (0, 0, 1) - (0, 1, 0) - (1, 0, 0)$. Suppose for a contradiction that it were defined on $(0, 0, 1)$, then if we restrict it to the closed subset $Z(x)$ it would still be a rational map. We must have $\varphi(0, k, 1) = (k, 0, 0) = (1, 0, 0)$ for all $k \neq 0$, which by continuity means $\varphi(0, 0, 1) = (1, 0, 0)$. On the other hand, if we restrict to $Z(y)$, we would obtain $\varphi(0, 0, 1) = (0, 1, 0)$, a contradiction.

I.4.7 x

Since there is an open affine cover, assume X, Y affine. Let $\varphi^* : \mathcal{O}_{P,X} \to \mathcal{O}_{Q,Y}$ be an isomorphism. Then define $\varphi : X \to Y$ by $\varphi(P) = (\varphi^*(y_1)(P), ..., \varphi^*(y_n)(P))$, and similar for its inverse. Then this map is defined on the intersection of defining sets of $\varphi^*(y_i)$. We need $\varphi(P) = Q$. This is because φ^* is an isomorphism of k-algebras $A(X)_{\mathfrak{m}_P}$ and $A(Y)_{\mathfrak{m}_Q}$, so it sends elements in $\mathfrak{m}_P$ to elements in $\mathfrak{m}_Q$ and for all $f \in \mathfrak{m}_Q$, $0 = \varphi^*(f)(P) = f(\varphi^*(y_1)(P), ..., \varphi^*(y_n)(P)) = f(\varphi(P))$, thus $\varphi(P) = Q$.

I.4.8 x

(a) True for $\mathbb{A}^n$ and $\mathbb{P}^n$. By I.4.9, assume it contains hypersurface in $\mathbb{P}^{r+1}$. Projection of $H \to \mathbb{P}^r$ from a point not on H is a rational map. Inverse of points are lines in $\mathbb{P}^{r+1}$ through P, which intersect H at finitely many points (at least 1), and therefore the map is finite-to-one, and surjective.
(b) Closed sets in curves are finite, so any bijection is a homeomorphism.

I.4.9 x

Since X has open affine cover and rational maps only need to be on an open set, take X affine inside $U_0 = \mathbb{P} - Z(x_0)$. $x_1, ..., x_n$ generate $K(X)$, so it has a subset which is transendence basis by 1.4.7A. Write $K(X) = k(x_1, ..., x_r, y)$ where $y = c_{r+1}x_{r+1} + ... + c_n x_n$ by 1.4.6A. With change of coordinates, assume $y = x^{r+1}$. Now the projection from $(0, ..., 0, 1)$ to the plane $x_n = 0$ gives a map from $K(\mathbb{P}^{n-1} - U_0) = k(x_1, ..., x_{n-1})$ to $K(\mathbb{P}^n - U_0) = k(x_1, ..., x_n)$ where $x_i \mapsto x_i$. Now the inverse image of $K(X)$ would be the function field of the image of X by the projection. This is the isomorphism $k(x_1, ..., x_{r+1}) \to k(x_1, ..., x_r, y)$, so the projection is birational of X onto its image. need check

Repeat until dimension is $n - 1$ to get a birational map between X and a hypersurface in $\mathbb{P}^n$.

I.4.10 x

Blow-up of $\mathbb{A}^2$ in $\mathbb{A}^2 \times \mathbb{P}^1$ at O is defined by $xu = ty$. The total inverse image of Y is defined by $y^3 = x^3$ and $xu = ty$. When $t = 0, u = 1$, we have $x = 0 = y$ which is inside E. When $t \neq 0$, set $t = 1$, then

$$y^2 = x^3, y = xu$$

$$x^2u^2 - x^3 = x^2(u^2 - x) = 0$$

Now $x = 0$, $y = 0$ corresponds to the curve E, and the other part is given by homogeneous equations $u^2 = xt^2, yt^3 = u^3$. When $u \neq 0$, we have

$$y^2 = x^3, ty = x$$

$$y^2(1 - t^3y) = 0$$

the other part is defined by $u^3 = t^3y, xt^2 = u^2$. Hence the curve $u^3 = t^3y, xt^2 = u^2$ contains $\varphi^{-1}(Y - O)$, which intersects E at only $(0, 0, 0, 1)$. Therefore $\varphi^{-1}(Y - O)$ must be dense in it and it is exactly $\tilde{Y}$. Are there steps to follow to find $\tilde{Y}$ in general?

The map $\varphi : \tilde{Y} \to Y$ is bijective, therefore homeomorphism by 1.4.8, but not isomorphism by 1.3.2.

I.5 Lecture 5

I.5.1 x

All curves singular at $(0, 0)$.

Observe lowerst degree term. (a) is tacnode. (b) is node. (c) is cusp. (d) is triple point.

I.5.2 x

(a) Jacobian matrix is $(y^2, 2xy, -2z)$ which is 0 when $y = z = 0$. So pinch point.

(b) Jacobian is 0 when $x = y = z = 0$, so conical double point.

(c) Singular at $x = y = 0$, same as cusp curve, double line.

I.5.3 Multiplicities

(a) Suppose $\mu_P(Y) = 1$, then $f = ax + by + g(x, y)$ where g has terms of degree > 1. Then Jacobian matrix is $(a + g_x, b + g_y)$ which does not vanish at $(0, 0)$, so not singular. Conversely, if f has terms of degree > 2, then Jacobian matrix has entries with terms degree > 1, which vanish at $(0, 0)$, so singular.

(b) (a) has 2, (b) has 2, (c) has 2, (d) has 3.

I.5.4 intersection multiplicity (*)

(a) By Eisenbud 3.7, there is a filtration

$$0 = M_0 \subseteq ... \subseteq M_n = \mathcal{O}_P/(f,g) = M$$

where $M_i/M_{i-1} = A/P_i$ for some P_i prime ideal in $A = \mathcal{O}_P$. (M is finitely generated because $\mathcal{O}_P$ is Noetherian). Note that the primes here in $\mathcal{O}_P$ are maximal because the they are in fact associated prime, which means they must contain (f,g).

Now $A/P_i = k$ (mentioned on page 16), so

$$\operatorname{length}_A M = \sum \operatorname{length}_A M_i/M_{i-1} = \sum \operatorname{length}_A k = n = \dim_k M$$

is finite.

Note that k-subspaces of $k[x,y]/(f,g)$ are subspaces of M, so

$$\dim k[x,y]/(f,g) \leq \dim_k M$$

Say $f = \sum_{i\geq \mu_P(Y)} f_i(x-a, y-b)$, $g = \sum_{i\geq \mu_P(g)} g_i(x-a,y-b)$, we have $\{x^iy^j : i < \mu_P(Y), j < \mu_P(Z)\}$ in A is linearly independent over k, so

$$\operatorname{length}_{\mathcal{O}_P} A \geq \dim_k A \geq \mu_P(Y)\mu_P(Z)$$

(b) Let $f = x - yt$ where $x - yt$ does not divide lowest degree terms of g. M is generated by $1, ..., (x - uy)^{\mu_P(g)-1}$. And $\dim_k M = \mu_P(g)$.

There are infinitely many t to choose.

(c) By linear transformation assume L is $z = 0$. Then since $L \neq Y$, assume $Y = Z(f)$ where f is not a multiple of x, y, nor z. Say $f(x = 1, y, z) = g(y) + zh(y,z)$ where g has degree n. Let $g(y) = \prod(y - a_i)^{d_i}$. By subtracting off multiples of z from f and observing that $\{1, ..., (y-a_i)^{d_i-1}\}$ is a basis, we see the sum of multiplicities is n. Now in the affine chart $y \neq 0$, we consider the intersections where $x = 0$, we have $f(y = 1) = h(x) + zH(x,z)$ where $h(x)$ have degree $d - n$ (complement to g), so in total the multiplicities sum to $d - n$, and combine all intersections we get multiplicity d.

I.5.5 x

$x^d + y^d + z^d$ irreducible by Eisenstein's criterion. Jacobian is $(dz^{d-1}, dz^{d-1}, dz^{d-1})$ which is non-zero in $\mathbb{P}^2$, unless $d = p$. If $d = p$, then use $zx^{d-1} + xy^{d-1} + yz^{d-1}$ (irreducible by dividing z then apply Eisenstein on prime ideal (y)). Giving Jacobian $((d-1)zx^{d-2} + y^{d-1}, (d-1)xy^{d-2} + z^{d-1}, (d-1)yz^{d-2} + x^{d-1})$. Since $d - 1 = p - 1 = -1$, we have singularity at $zx^{d-2} = y^{d-1}, yz^{d-2} = x^{d-1}, xy^{d-2} = z^{d-1}$. Multiply by x, z, y respectively and get $zx^{d-1} = xy^{d-1} = yz^{d-1}$. Substitute back to the polynomial and get they all need to be 0 unless $p = 3$. So one of x, y, z is 0, and as a result by the above equations $x = y = z = 0$, not possible. If $p = 3$, use $x^d + zx^{d-1} + xy^{d-1} + yz^{d-1}$.

I.5.6 blowing up curve singularities

(a) Assume k characteristic 0. Let Y be $xy = x^6 + y^6$. Blow up at O with $xu = ty$ in $\mathbb{A}^2 \times \mathbb{P}^1 = \{(x, y; t : u)\}$. Consider singularities in each affine cover. When $u \neq 0$, set $u = 1$, then

$$ty^2 = t^6y^6 + y^6, ty = x$$

$$y^2(t - t^6y^4 - y^4) = 0$$

If $y = 0$, then $x = 0$ and corresponds to E. The other part of the curve, when viewed as in $\mathbb{A}^3$ is

$$(t - y^4 - t^6y^4, x - ty)$$

$$A(X) = k[ty, t, y]/(t - y^4 - t^6y^4) = k[t, y]/(t - y^4 - t^6y^4)$$

is integral domain, so irreducible. Jacobian is

$$\begin{bmatrix} 0 & -4y^3 - 4t^6y^3 & 1 - 6t^5y^4 \\ 1 & -t & -y \end{bmatrix}$$

which is not full rank when $-4y^3 - 4t^6y^3 = 1 - 6t^5y^4 = 0$. So we need

$$t - y^4 - t^6y^4, -4y^3 - 4t^6y^3, 1 - 6t^5y^4$$

If $y = 0$, then t must also be 0 by the first equation, so no solution. Suppose $t^6 = -1$, then first equation becomes $t = 0$, a contradiction. Hence no solution in the curve.

The case when $u \neq 0$ is by symmetry.

Now let Y be $x^3 = y^2 + x^4 + y^4$. Then when $t \neq 0$, we have

$$t^3y^3 = y^2 + t^4y^4 + y^4, ty = x$$

$$y^2(t^3y - 1 - t^4y^2 - y^2) = 0, ty = x$$

Curve

$$t^3y - 1 - t^4y^2 - y^2 = 0, ty = x$$

Has Jacobian

$$\begin{bmatrix} 0 & t^3 - 2y(t^4 + 1) & 3t^2y - 4t^3y^2 \\ 1 & -t & -y \end{bmatrix}$$

is rank 1 when $t^3 - 2y(t^4+1) = 3t^2y - 4t^3y^2 = 0$, and also $t^3y - 1 - t^4y^2 - y^2 = 0$. If $y = 0$, then $t = 0$ by first equation, and no solution for third equation. If $y \neq 0$, then we are inside $\varphi^{-1}(Y - O)$ which is isomorphic to $Y - O$, non-singular. Hence non-singular.

When $u \neq 0$,

$$x^3 = u^2x^2 + u^4x^4 + x^4, y = ux$$

$$x^2(x - u^2 - u^4x^2 - x^2) = 0, y = ux$$

$$x - u^2 - u^4x^2 - x^2 = 0, y = ux$$

Jacobian

$$\begin{bmatrix} 1 - 2x(u^4 + 1) & 0 & -2u - 4u^3x^2 \\ -u & 1 & -x \end{bmatrix}$$

When $x = 0$, full rank, so nonsingular by previous arguments.

(b) By linear change of coordinates, assume Y is defined by $xy + f$ with node at O. Blowup and set $u = 1$, then

$$ty^2 + f(ty, y) = 0, ty = x$$

$$y^2(t + g(t, y)) = 0, ty = x$$

where $g(t, y)$ has no constant in y. Consider curve $t + g(t, y), ty - x$. Jacobian is

$$\begin{bmatrix} 0 & g_y & 1 - g_t \\ 1 & -t & -y \end{bmatrix}$$

If $y = 0$, then $g_t = 0$ since g has no constant in y. Thus rank 2. If $y \neq 0$, then we are in $\phi^{-1}(Y - O)$ which is non-singular. Hence the curve is nonsingular on $u \neq 0$. On $t \neq 0$, use symmetry.

(c) Let Y be $x^2 = x^4 + y^4$. THen blow-up and focus on the affine plane $t \neq 0$, we have

$$x^2 = x^4 + u^4x^4, y = xu$$

$$x^2(1 - x^2 - u^4x^2) = 0, y = xu$$

Consider the curve in $\mathbb{A}^3$, then by I.2.9, Thus $\tilde{Y}$ is

$$t^4(1-x^2) = u^4x^2, ty = xu$$

which intersects E at $t=0, u=1$. So we suspect singularity at $u=1$, where

$$t^2y^2 = t^4y^4 + y^4, ty = x$$

$$y^2(t^2 - t^4y^2 - y^2), ty = x$$

Now the curve $t^2 - t^4y^2 - y^2$ is indeed a node with tangents $y-t, y+t$.

(d) O is triple point because multiplicty 3. Blow up and get on $t \neq 0$,

$$x^3(u^3 - x^2), y = xu$$

Now $u^3 - x^2$ is a double point (cusp) at $u = x = 0$.

Note the closure is $u^3 = x^2t, ty = xu$, when $t = 0$, there is no solution, so by (a) one more blow-up resolves (all) singularity.

I.5.7 x

(a) Since defined by 1 equation, singularity when $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$. By I.5.8, this can only happen when $x=y=z=0$ because Y is non-singular. Since f degree >1, the partials vanish at P, so P is the only singular point.

(b) $\mathbb{A}^3 \times \mathbb{P}^2 = \{(x,y,z;t:u:v)\}$, blowup defined by $f, xu = yt, yv = zu, zt = xv$. When $t \neq 0$, write $t = 1$ and

$$f, xu = y, z = xv$$

Substitute into f to get $f(x, xu, xv)$, which is

$$x^d f(1,u,v)$$

Now $x = 0$ gives the curve E, so the blow-up is defined by $f(1,u,v), xu = y, xv = z$. View as a variety in $\mathbb{A}^5$, has Jacobian

$$\begin{bmatrix} 0 & u & v \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ \frac{\partial f(1,u,v)}{\partial u} & x & 0 \\ \frac{\partial f(1,u,v)}{\partial v} & 0 & x \end{bmatrix}$$

Note that $\frac{\partial f(1,u,v)}{\partial u}$ and $\frac{\partial f(1,u,v)}{\partial v}$ can not both be 0 at some u, v because Y (defined by $f(x,y,z)$) is its projective closure in $\mathbb{P}^2$ and is non-singular. Hence $\tilde{X}$ is non-singular in the open affine set $t \neq 0$. The other cases are by symmetry.

(c) X defined by $f(1,u,v), xu = y, z = xv$, so $\tilde{X}$ defined by the closure

$$f(t,u,v), xu = yt, zt = xv$$

Inside E, $x = y = z = 0$, so $\varphi^{-1}(P)$ is defined by $f(t,u,v)$ which is exactly Y.

I.5.8 x

(a) Since partials of homogeneous polynomials is homogeneous, the rows of Jacobian change by a (non-zero) scalar multiple for each projective coordinates for P, so the rank does not change.

(b) Now assume x_0, set $x_0 = 1$, then the affine Jacobian matrix is

$$\|\frac{\partial f_i(1, x_1, ..., x_n)}{\partial x_j}(a_1, ..., a_n)\| = \|\frac{\partial f_i}{\partial x_j}(1, a_1, ..., a_n)\|$$

has rank $n - r$ iff non-singular at P.

(c) don't know what this is for. Note that Euler's lemma implies that if $\frac{\partial f}{\partial x_i}(a_0, ..., a_n) = 0$ for all i, then $(a_0, ..., a_n)$ is automatically inside $V(f) \subseteq \mathbb{P}^n$ and is singular.

I.5.9 x

Suppose f not irreuducible. Say $f = gh\ell$ where g, h irreducible. By I.3.7, take $P \in V(g,h)$. Now $\frac{\partial f}{\partial x} = \frac{\partial g\ell}{\partial x}h + \frac{\partial h}{\partial x}g\ell$ vanishes at P, and similarly for $\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$.

I.5.10 x

(a) Follows from $\dim T_P(X) = \dim \mathfrak{m}/\mathfrak{m}^2$.

(b) $\varphi^* : \mathcal{O}_{\varphi(P)} \to \mathcal{O}_P$ corresponds to φ. Now φ maps things in $\mathfrak{m}_{\varphi(P)}$ to things inside $\mathfrak{m}_P$, so it induces a map from $\mathfrak{m}_{\varphi(P)}/\mathfrak{m}^2_{\varphi(P)}$ to $\mathfrak{m}_P/\mathfrak{m}_P^2$. Now take the dual map (transpose).

(c) φ is the map $(x,y) \mapsto x$. So $\varphi^*(f) = f \circ \varphi$ for $f \in \mathcal{O}_0 = k[x]_{(x)}$. Now for $f \in \mathfrak{m} = (x)$, we have $\varphi^*(f)(x,y) = xg(x) = y^2g(y^2) \in (y^2)$. On the other hand, $\mathcal{O}_O = k[x,y]/(x-y^2)_{(x,y)} = k[y]_{(y)}$ with maximal ideal (y), so $\varphi^*(f)$ is 0 in $(x)/(x^2)$, and as a result, $T_0(\varphi) = 0$.

I.5.11 the elliptic quartic curve in $\mathbb{P}^3$

If $y \neq 0$, define $\varphi^{-1}(x,y,z) = (x,y,z,(x^2-xz)/y)$. Then the inverse image is indeed inside Y. When $x+z \neq 0$, define $w = yz/(x+z)$. Suppose $y = 0 = x+z$, then in the inverse image, we must have $x^2 - xz = 0$, so $x = z = 0$, and we must have $w = 1$. Since we are exclusing $(0,0,0,1)$, this gives us an isomorphism from $Y - (0,0,0,1)$ to the curve $y^2 - x^3 + xz^2$ minus $(1,0,-1)$.

Since $y^2z - x^3 + xz^2$ irreducible nonsingular (by I.5.9), the original curve is irreducible nonsingular.

I.5.12 quadric hypersurfaces

(a) Follows from diagonalizability of symmetric bilinear forms.

(b) By Eisenstein criterion.

(c) $\operatorname{Sing} Q = Z(2x_0, ..., 2x_r)$ is linear of dimension $n-r-1$.

(d) Say $Q = Z(f) = Z(x_0^2 + ... + x_r^2)$. Let $Y \subseteq \mathbb{P}^r$ be defined by the same polynomial. Then Y is nonsingular. Let $Z = (x_0, ..., x_r)$ be the axis and now every point $(x_0, ..., x_r, ..., x_n)$ on Q satisfies $(x_0, ..., x_r, 0, ..., 0) \in Y + 0$, $(0, ..., 0, x_{r+1}, ..., x_n) \in Z + 0$, so it is on a line between a point on Y and a point on Z.

I.5.13 x

Suppose there is a cover $\{U_i\}$ of X such that Y is closed in U_i for each i, then $X - Z = (X-Z) \cap (\cup U_i) = (\cup U_i - Z)$ is open. Hence we can assume affine. Let the integral closure of $A(Y)$ be A' by theorem I.3.9A, it is generated $A(Y)$-module by finitely many $f_i \notin A(Y)$. Consider the closed set Z that is complement to the intersection of the defining sets of f_i. If $P \notin Z$, then P is defined on all f_i and therefore $A' \subseteq \mathcal{O}_P$, so $A'_{\mathfrak{m}_P} \subseteq \mathcal{O}_P$ and $\mathcal{O}_P$ is integrally closed, so P normal. If $P \in Z$, then it is not defined on some f_i, so $f_i \notin \mathcal{O}_P$ and P not normal.

I.5.14 analytically isomorphic singularities* (*)

Example of completion: The completion of $k[x_1, ..., x_n]$ with respect to maximal ideal $(z_1, ..., z_n)$ is $k[[x_1, ..., x_n]]$. Consider the sequence $(f_1 + \mathfrak{m}, f_2 + \mathfrak{m}^2, ...)$, this gives a power series $g = f_1 + (f_2 - f_1) + ...$ where each $f_i - f_{i-1}$ has degree $i-1$.

(a) Assume affine, let $Y = Z(f), Z = Z(g)$. Assume $P = Q = 0$, so $\hat{\mathcal{O}_P} = k[[x,y]]/(f)$, $\hat{\mathcal{O}_P} = k[[x,y]]/(g)$. Since k-algebra isomorphism, $\varphi(f(x,y)) = f(\varphi(x), \varphi(y))$. Note $\hat{\mathfrak{m}}$ is mapped to $\hat{\mathfrak{m}}$, so $\varphi(x), \varphi(y) \in \hat{\mathfrak{m}}$. Now $f(\varphi(x), \varphi(y)) = \varphi(f(x,y)) \in (g)$ so $\varphi((f)) \subseteq (g)$ and conversely $\varphi^{-1}((g)) \subseteq (f)$, so $\varphi((f)) = (g)$, and we must have $\varphi(f) = g$ by multiplying φ by some unit. Note that $\varphi(f) \in \hat{\mathfrak{m}}^{\mu(f)}$ so $\mu_P(Y) \leq \mu_Q(Z)$, and apply the argument to φ^{-1} to get $\mu_P = \mu_Q$.

(b) By I.5.12, all polynomials in two variables are reducible to linear factors. So g_s, h_t have no common factors, their intersection must be empty (only $(0,0)$ vanish on both), so $\sqrt{(g_s, h_t)} = S$ or S_+, if it is S

then we are done, otherwise, we want to show it contains S_{r+1}. it does not generate max ideal, for example, $g_s = x^2, h_t = y^3$

(c) By (b) and argument from I.5.6.3, any ordinary double point have completion of local ring isomorphic to $k[[x,y]]/(xy)$. For triple point, apply (b) twice, to get $f = gh(ag+bh)$. Send it to $xy(ax+by)$, then map it to $\frac{1}{ab}xy(x+y)$ to get isomorphic to $k[[x,y]]/(xy(x+y))$.

When there are four directions of lines intersecting at the origin, we can not change them linearly to get them isomorphic to each other, from linear algebra, we know there is one parameter left. what about non-linear k-algebra isomorphisms? not sure since we are dealing with formal power series.

(d) linear transformation to get y^2+x^2 lowest term or y^2+f_r lowest terms for $r>2$. If we have y^2+x^2y, send y to $y-\frac{1}{2}x^2$, then

$$y^2 + x^2y \mapsto y^2 - \frac{1}{4}x^4$$

now map x to $(1/4)^{1/4}x$ and get $y^2 - x^4$, which gives tacnode.

In general, how do we cancel y in f_r using isomorphisms of complete local rings?

I.5.15 families of plane curves

(a) Since principal ideal of f has elements with coefficients corresponding to projective coordinates, this gives a correspondence.

(b) The partial derivatvies of f are polynomials with coefficients that are integer multiples of a_j (cofficients of f) where $j = 0, ..., N$. Apply Theorem I.5.7A and get that they are 0 at the same time at some point (which corresponds to singularity by Euler's lemma from I.5.8) iff the a_j's are in a closed set in $\mathbb{P}^N$. The complement of this set, by I.5.9, the corresponding polynomials are irreducible, so nonsingular curves.

I.6 Lecture 6

I.6.1 x

(a) Y is birationally equivalent to $\mathbb{P}^1$. By I.6.7, Y is isomorphic to an open subset of $C_K = \mathbb{P}^1$. Since Y is not isomorphic to $\mathbb{P}^1$, Y is a proper open subset, which is isomorphic to an open subset of $\mathbb{A}^1$.

(b) Say Y is isomorphic to $\mathbb{A}^1 - \{a_1, ..., a_n\}$, then it is isomorphic to $V((x-a_1)...(x-a_n)y-1)$ via projection map.

(c) It suffices to show $K(Y)$ is unique factorization domain, but $K(Y) = K(\mathbb{P}^1) = S(\mathbb{P}^1)_{((0))}$ is a subring of the fraction field of $S(\mathbb{P}^1)$, which is $k(x,y)$.

I.6.2 an elliptic curve

(a) Jacobian is $(3x^2-1, 2y)$, so singular at $(\sqrt{1/3}, 0)$, but this point is not contained in Y, so nonsingular. Hence $A(Y) = k[x,y]/(y^2-x^3-x)$ is integrally closed.

(b) Since x is not constant on Y, $x \notin k$. Since k is algebraically closed, x is transcedental over k, so $k[x]$ is polynomial ring. To show A is integral closure of $k[x]$, we only need to show y is integral over $k[x]$. Now y is a root of $t^2 - x^3 + x = 0$.

(c) The automorphism is the pullback of the isomorphism $(x,y) \mapsto (x,-y)$ on Y. Given $a = f(x,y)$, replace y^2 by x^3-x and get $a = g(x)y + h(x)$, so $N(a) = h(x)^2 - g(x)^2y^2 = h(x)^2 - g(x)^2(x^3-x) \in k[x]$. Also, $N(ab) = ab\sigma a\sigma b = N(a)N(b)$.

(d) Suppose a is a unit, then $N(a)$ is a unit in $k[x]$, so $N(a) \in k$. Write $N(a) = h(x)^2 - g(x)^2(x^3-x) = \alpha \in k$, then

$$(h+\sqrt{\alpha})(h-\sqrt{\alpha}) = g^2x(x+1)(x-1)$$

which is not possible unless $g = 0$ and $\deg h = 0$, so $a \in k$. Now $N(x) = x^2$, so if $x = ab$ is reducible, then $N(a) = N(b) = x$, but we have seen $N(a)$ has degree at least 2. Similar for $N(y) = x^3 - x$. Hence $x \neq y$ are irreducible, but now $y^2 = x(x^2-1)$, so not unique factorization domain.

(e) By I.6.1, since $A(Y)$ is not unique factorization domain, Y is not rational.

I.6.3 x

(a) Consider $\mathbb{A}^2 - 0 \mapsto \mathbb{P}^1$ by $(x, y) \mapsto (x : y)$. Note that this map is constant on each line, so if extended, it must still be constant on each line, which is not possible.

(b) Consider $\mathbb{A}^1 - 0 \mapsto \mathbb{A}^1$ by $x \mapsto 1/x$. Since open sets around 0 contain open real interval around 0, and rational polynomials are continuous functions with repsect to Euclidean topology, this can not be extended.

I.6.4 x

Let f be defined on U. Let $P \in Y$ with $P \in V$ where $f = g/h$ on V, then define $\varphi(P) = (g(P) : h(P)) \in \mathbb{P}^1$, otherwise $\varphi(P) = (1 : 0)$ (since Y projective, $\deg g = \deg h$). Note this is well-defined morphism. For each $P = (a, 1) \in \mathbb{P}^1$, $\varphi^{-1}(P) = Y \cap Z(g - ah)$. Since g/h is not constant on Y, this intersection is proper and thus finite. Similarly for $Y \cap Z(ag - h)$. The case when $a = 0$ follows from that g, h are not 0 on Y.

I.6.5 x

Take closure and assume Y is projective variety. It suffices to show X is closed in its closure, so assume Y is its closure. So Y is a projective curve. Consider $X \hookrightarrow X$. Extend by prop I.6.8 and get $\varphi : Y \to X$. But since Y is the closure of X, by continuity it must send the extra points in Y to the closure of the original image, so closure of X is contained in X. Hence X closed.

I.6.6 automorphisms of $\mathbb{P}^1$

(a) Bijective by construction. This is morphism because on $y \neq 0$, send $(x, 1)$ to $(ax + b, cx + d)$. On $x \neq 0$, send $(1, y)$ to $(a + by, c + dy)$. So automorphism.

(b) By Cor I.6.12, automorphisms on nonsingular projective $\mathbb{P}^1$ correspond to k-automorphisms on $K(\mathbb{P}^1) = k(x)$.

(c) Let $\varphi : k(x) \to k(x)$ automorphism. Say $x \mapsto \alpha = f(x)/g(x)$. Then $\alpha g(y) - f(y)$ has x as a root in $k(\alpha)[y]$. Note that α has transcendence degree 1 by automorphism. By Gauss's lemma (can apply since UFD), this is irreducible over $k[\alpha][y]$, so irreducible over $k(\alpha)[y]$. Since x is degree 1 over $k(x)$ and $k(\alpha) = k(x)$, the minimal polynomial is degree 1. So $1 = \max(\deg(f), \deg(g))$ and $x \mapsto (ax + b)/(cx + d)$. Since not a unit, $ad - bc \neq 0$. Hence $\mathrm{PGL}(1) = \operatorname{Aut} \mathbb{P}^1$

I.6.7 x

Say $\varphi : \mathbb{A}^1 - \{P\} \to \mathbb{A}^1 - \{Q\}$ isomorphism. View them as subvarieties of $\mathbb{P}^1$, and extend using Prop I.6.8. By uniqueness we know we must arrive at $\mathbb{P}^1 \to \mathbb{P}^1$ automorphism, therefore the number of missing points are equal.

Consider $\mathbb{A}^1 - \{1, 2, 3, 4, 5\}$ and $\mathbb{A}^1 - \{1, 2, 4, 7, 9\}$. Again extending must give us an automorphism $x \mapsto (ax+b)/(cx+d)$ (fractional linear transformation from last question), and try to solve it gives a system of linear equations about a, b, c, d, so suitable choices of the finite sets give us no solution.

I.7 Lecture 7

I.7.1 x

(a) Recall that the image of $\mathbb{P}^n$ is $Z(\mathfrak{a})$ where $\mathfrak{a}$ is the kernel of the map $\theta : k[y_0, ..., y_N] \to k[x_0, ..., x_n]$. So $S(\mathbb{P}^n) = k[y_0, ..., y_N]/\ker\theta = \operatorname{im}\theta$. (note that the map sends the degree ℓ components to degree ℓd components.) The image is the subalgebra generated by monomials of degree d. Therefore

$$\varphi_{S(\mathbb{P}^n)}(\ell) = \varphi_{\operatorname{im}\theta}(\ell d) = \text{number of degree } \ell d \text{ monomials} = \binom{\ell d + n}{n}$$

$$P_{S(\mathbb{P}^n)}(z) = \binom{zd + n}{n}$$

leading coefficient is $d^n/n!$. Since image has dimension n, the degree is d^n.

(b) Using a similar argument, $\varphi_{S(\mathbb{P}^r\times\mathbb{P}^s)}(\ell)$ is equal to number of degree 2ℓ monomials in $k[...,x_i,...,y_j,...]$ whose degree in x is ℓ, and degree in y is ℓ, which is

$$\binom{\ell+r}{r}\binom{\ell+s}{s}$$

So leading coefficient is $1/r!s!$ and since dimension is $(r+s)!$, the degree is $(r+s)!/r!s! = \binom{r+s}{r}$.

I.7.2 x

(a) $P_S = \binom{z+n}{n}$ has constant term 1, so $P_S(0) = 1$ and $p_a(\mathbb{P}^1) = 0$.

(b) A plane curve in $\mathbb{P}^2$ is a hypersurface, so by I.7.6(d) $p_H(z) = \binom{z+2}{2} - \binom{z-d+2}{2}$, evaluate at 0 and get $1 - \binom{2-d}{2} = 1 - \frac{1}{2}(2-d)(1-d)$. Therefore $p_a(Y) = \frac{1}{2}(d-1)(d-2)$.

(c) By I.7.6(d) $p_H(z) = \binom{z+n}{n} - \binom{z-d+n}{n}$, so

$$p_a(Y) = (-1)^n\binom{-d+n}{n} = \frac{(-1)^n(n-d)(n-d-1)...(-d+1)}{n!} = \frac{(d-n)(d-n+1)...(d-1)}{n!} = \binom{d-1}{n}$$

(d) Since complete intersection, say $Y = Z(f)\cap Z(g)$ with $I(Y) = (f,g)$, then

$$0 \to S/(fg) \to S/(f)\oplus S/(g) \to S/(f,g) \to 0$$

is exact. Also,

$$0 \to S(-a-b) \xrightarrow{\cdot(f,g)} S(-b)\oplus S(-a) \xrightarrow{gh-fr} S \to S/(f,g) \to 0$$

is exact Thus

$$P_{S(f,g)} = P_S - P_{S(-a)} - P_{S(-b)} + P_{S(-a-b)}$$

Set $z = 0$ and get

$$P_{S(f,g)}(0) = 1 - \binom{3-a}{3} - \binom{3-b}{3} + \binom{3-a-b}{3} = \frac{1}{2}ab(4-a-b)$$

So $p_a(Y) = \frac{1}{2}ab(a+b-4)+1$.

(e) Note that serge embedding image has ideal that is the kernel of $k[\{z_{ij}\}] \to k[x_i]\otimes k[y_j]$ where $z_{ij}\mapsto x_i\otimes y_j$. Similarly the ideal for $Y\times Z$ would be the kernel of the map $k[\{z_{ij}\}] \to k[x_i]/I(Y)\otimes k[y_j]/I(Z)$. Note that the image of this map is the subring where the degree of x component equals that of y component. Thus $I(Y\times Z) = \oplus_i (k[x_i]/I(Y))_i \otimes (k[y_j]/I(Z))_i$. Now tensor product multiplies dimension, so the hilbert polynomial is the product of the Hilbert polynomials for Y and Z. So $P_YP_Z = P_{Y\times Z}$.

Now

$$(-1)^{r+s}(P_X(0)-1) = (-1)^{r+s}(P_Y(0)-1)(P_Z(0)-1) + (-1)^{r+s}(P_Y(0)-1) + (-1)^{r+s}(P_Z(0)-1)$$

$$P_X(0)-1 = (P_Y(0)-1)(P_Z(0)-1) + (P_Y(0)-1) + (P_Z(0)-1)$$

$$P_X(0) = P_Y(0)P_Z(0)$$

I.7.3 the dual curve

Given nonsingular $P = (a,b,c) \in Y$, apply linear transformation and assume $P = (1,0,0)$. Write $f(x,y,z) = yg(x,y)+zh(x,y,z)$. Consider the line defined by jacobian of Y, $T_P(Y) = (\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P), \frac{\partial f}{\partial z}(P)) = (0,\alpha,\beta) \neq 0$. We shall show this line $\alpha y + \beta z$ intersects Y on the affine chart $x \neq 0$ with multiplicity > 1. On the affine chart, we have $F(y,z) := f(1,y,z)$. Its lowest degree component is not a constant because $F(0,0) = 0$. Since $(\frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}) = (\alpha,\beta) \neq 0$, the lowest degree component must be the line $\alpha y + \beta z$. Hence by proof of I.5.4 the intersection multiplicity is > 1. Also the line is unique since it has to agree with tangent direction of F (otherwise intersection multiplicity $= 1$).

This is a morphism because linear transformation does not affect derivative, so the map is just the polynomial map

$$(\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P), \frac{\partial f}{\partial z}(P))$$

which is indeed defined on nonsingular points of Y.

I.7.4 x

By Bezout, lines meeting with multiplicity 1 would meet Y at exactly d points. By I.7.3, the lines that do not meet Y at d points are those that are tangent lines, or pass through a singular point of Y (if singularity, then by I.5.3 multiplicity > 1, and by I.5.4(a), intersection multiplicity > 1). Now a point (a, b, c) in $(\mathbb{P}^2)^*$ crosses a singularity (x, y, z) in Y iff $ax + by + cz = 0$. Since there are finitely many singularities, these are in a closed set. On the other hand, the set of tangent lines are inside the dual curve Y^* which has dimension 1. Thus we have a proper closed subset, and its complement is the required nonempty open set.

I.7.5 x

(a) Since $d > 1$, Y is not a line. Pick any two points on Y and consider the line connecting them, their intersection multiplicities must sum up to at most d by Bezout, and at least 1. By I.5.4(a), this means multiplicity for each of them is at most $d - 1$.

(b) only defined multiplicities for points on plane curves, so assume Y is a plane curve

Assume the point with multiplicity d is $(0, 0, 1)$ by change of coordinates. On the affine chart $z \neq 0$, we have the curve is defined by $f(x, y) + g(x, y) = 0$ where f is degree $d-1$, g degree d homogeneous. Consider the map $(x, y) \mapsto x/y$. The inverse is $t \mapsto (ut, u)$ where $u := -\frac{f(t,1)}{g(t,1)}$. This is indeed inverse because

$$f(x, y) = -g(x, y)$$

$$y^{d-1} f(x/y, 1) = -y^d g(x/y, 1)$$

$$y = -\frac{f(x/y, 1)}{g(x/y, 1)}$$

Thus an open subset is isomorphic to an open subset of $\mathbb{A}^1 \subseteq \mathbb{P}^1$, so rational.

I.7.6 linear varieties

If linear variety then degree 1, then it is an intersection of hyperplanes. By Theorem I.7.7, intersection of hyperplanes have degree 1 and are irreducible.

Suppose Y has degree 1 and pure dimension r. By Prop I.7.6(b), Y is irreducible. If $Y \subseteq H$ for any hyperplane H (which is isomorphic to $\mathbb{P}^{n-1}$ by a linear isomorphism), then $Y \subseteq \mathbb{P}^{n-1}$ and by induction on n is linear variety. (base case dimension 0)

Suppose now Y is not inside any hyperplane, then $Y \cap H$ has degree 1 by Theorem I.7.7, so by induction is linear variety. Consider a line connecting any two points P, Q on Y. The line is inside some hyperplane H, so P, Q are in the linear variety $H \cap Y$. So the line is also in $H \cap Y$. Therefore Y contains any line connecting any two points so linear (because its cone is a linear subspace in the vector space $\mathbb{A}^{n+1}$).

I.7.7 x(*)

(a) Assume $P = (1, 0, ..., 0)$. Consider dominant affine map from cone of $Y \subseteq \mathbb{A}^{n+1}$ to $X' = X - Z(x_0)$ by $(x_0, ..., x_n) \mapsto (1, x_1, ..., x_n)$. Which gives injection from $A(X')$ to integral domain $A(Y)$, so X irreducible and has dimension at most $r + 1$. (the projective closure of X' would contain all lines PQ so must be X).

Note that by cutting with hyperplanes containing P, we can arrive at an intersection which has finitely many points plus P, or a degree d curve (by Bezout). In the latter case, since $d > 1$, d is not a line, so there is some Q where the line PQ is not contained in Y. In all cases, we see X strictly contains Y.

Now since X is irreducible, strictly contains Y, it must have dimension $r + 1$.

(b) Note that if $Y = Y_1 \cup Y_2$ then $X = \overline{X_1 \cup X_2} = X_1 \cup X_2$. Cut Y by some hyperplane containing P to reduce its dimension, then by Bezout the degree of the irreducible components of the intersection sum to at most d, so by induction the corresponding cones have degree sum to at most $d - 1$.

but why are all irreducible components of $X \cap H$ cones over irreducible components of $Y \cap H$? and why is generaic H intersect at X with multiplicity 1?

I.7.8 x

By I.7.7(a), Y is contained in X of dimension $r + 1$ with degree 1, which by I.7.6 is a linear variety. By Prop I.7.6(d), Y is defined by a qudratic in $\mathbb{P}^{r+1}$.

II Chapter 2

II.1 Lecture 1

II.1.1 x

Use the construction from definition 1.2. Let $\mathscr{F}^+(U)$ defined by the set of functions from U to $\cup \mathscr{F}_P = A$, where for each $P \in U$, there is a neighbourhood $V_P \subseteq U$ of P and some $t \in \mathscr{F}(V) = A$, such that $t = s(Q)$ for all $Q \in V$. Such functions are just a continuous functions on the discrete topology on A because for any point $a \in A$, $s^{-1}(a) = \cup_{P \in s^{-1}(a)} V_P$ is open, and conversely, if continous, then take $V_P = s^{-1}(s(P))$. Thus this is exactly the constant sheaf $\mathscr{A}$ defined in example 1.0.3.

II.1.2 x

(a) $(\ker \varphi)_P$ contains pairs $\langle U, s\rangle$ where U open around P, $s \in \ker \varphi(U)$. Such an element would be mapped by φ_P to $\langle U, 0\rangle$, so it is in $\ker \varphi_P$.

φ_P is the map that sends $\langle U, s\rangle \in \mathscr{F}_P$ to $\langle U, \varphi(s)\rangle \in \mathscr{G}_P$. If $\langle U, s\rangle \in \ker \varphi_P$, then $\varphi(s)|_V = 0$ for some $V \subseteq U$. Thus $\langle U, s\rangle = \langle V, s|_V\rangle$ is of form of an element in $(\ker \varphi)_P$ ($\varphi(s|_V) = \varphi(s)|_V$ by definition).

Note that for any point P, $\mathscr{F}_P = \mathscr{F}_P^+$ because an element $\langle U, s\rangle \in \mathscr{F}_P$ corresponds to function $\langle U, s'\rangle \in \mathscr{F}_P^+$ where $s'(Q) = \langle U, s\rangle \in \mathscr{F}_Q$ for any $Q \in U$. We have $s(Q)$ is locally given by the section $s \in \mathscr{F}(V)$. Conversely, given $\langle U, s'\rangle \in \mathscr{F}_P^+$, let $s'(P) = s$. There is a neighbourhood V of P such that there is $t \in \mathscr{F}(V)$, $s'(Q) = \langle V, t\rangle \in \mathscr{F}_Q$ for all $Q \in V$. In particular, $s'(P) = \langle V, t\rangle = \langle U, s\rangle$, so $t = s|_V$, and $\langle U, s'\rangle = \langle V, s'|_V\rangle$ now corresponds to $\langle U, s\rangle$.

The above paragraph shows $(\operatorname{im} \varphi)_P$ can be considered where $\operatorname{im} \varphi$ is the presheaf, consisting of element $\langle U, \varphi(s)\rangle$ where $s \in \mathscr{F}(U)$. Now is the image of $\langle U, s\rangle$ by φ_P so inside $\operatorname{im} \varphi_P$.

Conversely, the images of φ_P are of form $\langle U, \varphi(s)\rangle$, which is inside $(\operatorname{im} \varphi)_P$.

(b) φ injective iff $\ker \varphi = 0$ iff $(\ker \varphi)_P = 0$ for all P (by axiom (3) of definition of sheaf) iff $\ker \varphi_P = 0$ (by part (a)) iff φ_P injective.

φ surjective iff $\operatorname{im} \varphi = \mathscr{G}$ iff inclusion $\operatorname{im} \varphi \to \mathscr{G}$ is isomorphism iff $\operatorname{im} \varphi_P \to \mathscr{G}_P$ isomorphism for all P (by prop 1.1) iff φ_P surjective.

(c) follows from (a).

II.1.3 x

(a) Let $P \in X$. Given $s \in \mathscr{G}(U)$ where U neighbourhood of P, there is cover $\{U_i\}$ and $t_i \in \mathscr{F}(U_i)$ such that $\varphi(t_i) = s|_{U_i}$. Thus

$$\varphi_P(\langle U_i, t_i\rangle) = \langle U_i, s|_{U_i}\rangle = \langle U, s\rangle$$

for any i such that $P \in U_i$. Therefore φ_P is surjective for all P and φ is surjective.

(b) Consider the constant sheaf on $\{0,1\}$ with discrete topology and abelian group A. Then $\mathscr{A}(\{0\}) = \mathscr{A}(\{0\}) = A$, $\mathscr{A}(\{0,1\}) = A \oplus A$. Now consider the morphism $\varphi : \mathscr{A} \to \mathscr{A}$ with $\varphi(\{0,1\})$ projecting onto first coordinate. Then image of φ is the constant presheaf associated to A. By 1.1 $\operatorname{im}\varphi$ is isomorphic to $\mathscr{A}$, and thus by the uniqueness of the morphism from universal property, we must have $\operatorname{im}\varphi = \mathscr{A}$ as a subsheaf. Hence the map is surjective but $\varphi(X)$ is not. can also prove this by using part (a) with $U = \{0,1\}$, $s = (0,1)$.

II.1.4 x

(a) Since $\varphi(U)$ injective for all U, φ injective, so φ_P injective for all P, so φ_P^+ injective for all P because $\mathscr{F}_P^+ = \mathscr{F}_P$, so φ^+ injectvie.

(b) the inclusion map i : image presheaf $\to \mathscr{G}$ induces injection $i^+ : \operatorname{im}\varphi \to \mathscr{G}^+ = \mathscr{G}$.

II.1.5 x

Isomorphism iff map on stalks isomorphism iff map on stalks injective and surjective iff injective and surjective.

II.1.6 x

(a) The map on stalk is exact sequence so use 1.2(c). Need to show $(\mathscr{F}/\mathscr{F}')_P = \mathscr{F}_P/\mathscr{F}'_P$. Given $\langle U, s + \mathscr{F}'(U)\rangle$ where $s \in \mathscr{F}/\mathscr{F}'(U)$, it corresponds to $\langle U, s\rangle + \mathscr{F}'_P$ because $s \in \mathscr{F}'(U)$ iff $\langle U, s\rangle \in \mathscr{F}'_P$. Observe the kernel is indeed $\mathscr{F}'$.

(b) Since $\mathscr{F}'$ injects into $\mathscr{F}$, it is isomorphic to the subsheaf that is its image. We have induced maps on stalks

$$(\mathscr{F}/\mathscr{F}')_P = \mathscr{F}_P/\mathscr{F}'_P \cong \mathscr{F}''_P$$

for all P, so $\mathscr{F}/\mathscr{F}' \cong \mathscr{F}''$

II.1.7 x

Follows from 1.6.

II.1.8 x

If φ injective, then $\varphi(U)$ injective. Suppose ψ has kernel $\operatorname{im}\varphi$. Then $(\operatorname{im}\varphi)(U) = (\ker\psi)(U) = \ker(\psi(U))$. Since $\operatorname{im}\varphi$ is a sheaf because φ is injective, $(\operatorname{im}\varphi)(U) = \operatorname{im}(\varphi(U))$. Hence $\ker\psi(U) = \operatorname{im}\varphi(U)$, so left exact.

II.1.9 direct sum

Suppose U_i is a cover of U and $(s,t)|_{U_i} = 0$, then since restriction map is defined coordinatewise $s|_{U_i} = t|_{U_i} = 0$ so $s = t = 0$. Similar for the other axiom in the definition.

This is a direct sum of category because the morphisms correspond to pairs of morphisms.

II.1.10 direct limit

Let $\mathscr{F}$ be the presheaf, so $\varinjlim \mathscr{F}_i = \mathscr{F}^+$. Given a collection of morphisms $\mathscr{F}_i \to \mathscr{G}$, define $\mathscr{F} \to \mathscr{G}$ by $\langle i, s\rangle \mapsto \varphi(s)$. Well-defined if we assume the morphisms compatible with the direct system. Unique because all values are determined by some $\mathscr{F}_i \to \mathscr{G}$. Now by prop 1.2 there is a unique morphism from $\mathscr{F}^+$ to $\mathscr{G}$ with the universal property.

II.1.11 x

Suppose U_j is a cover and for each j, there is some $\langle i_j, s_j\rangle \in \varinjlim \mathscr{F}_i(U_i)$, such that $\langle i_j, s_j\rangle|_{U_j\cap U_k} = \langle i_k, s_k\rangle|_{U_j\cap U_k}$ for any j,k. This means we can replace j,k by something bigger in the directed system, since X is noetherian, there is a finite subcover, so repeat this finitely many times and find a common i_J. Now apply condition (4) for sheaves to $\mathscr{F}_{i_J}$ we see there is some $\langle i_J, s\rangle \in \varinjlim \mathscr{F}_i(U)$ that restricts to the s_j.

II.1.12 inverse limit

Suppose a sequence $(s_i) \in \varprojlim \mathscr{F}(U)$ is 0 in restriction to U_j, then $s_i|_{U_j} = 0$ for each i, so $s_i = 0$ in $\mathscr{F}(U)$ and $(s_i) = 0$.

Suppose $(s_i)_j$ inside each $\varprojlim \mathscr{F}(U_j)$ where $(s_{ij})|_{U_j\cap U_k} = (s_{ik})|_{U_j\cap U_k}$. Pick s_i coordinatewise using axiom (4) and form (s_i), which restricts to $(s_i)_j$ as required. Suppose s_I is sent to t_i for some $I \geq i$. Then (then $t_i = s_i$ by uniqueness of gluing) $s_i|_{U_j} = \varphi_{Ii}(s_I|_{U_j}) = t_i|_{U_j}$ for all j, so by axiom (3), $s_i = t_i$. Thus the choice is compatible with the inverse system.

II.1.13 espace étalé of a presheaf

Suppose $s \in \mathscr{F}^+(U)$, need to show it is a continuous section of the map π. For $P \in U$, since $s(P) \in \mathscr{F}_P$, we can write $s(P) = t_P$ for some $t \in \mathscr{F}(U)$, so π sends it back to P, hence this is a section of π. By definition of $\mathscr{F}^+$, there is some V, for each $Q \in V$, we have $s(Q) = s_Q$ for some fixed $s \in \mathscr{F}(U)$, thus on V it agrees with the map $\bar{t}$, which is continuous. Since U can be covered by such V, s is continous on U with respect to the topology on $\text{Spé}(\mathscr{F})$.

Conversely, given continuous section $s : U \to \text{Spé}(\mathscr{F})$, we see it must map P inside $\mathscr{F}_P$. For each $P \in U$, let $t_P = s(P)$. Consider $V' := \{t_Q : Q \in U\}$ which is the image of U by $\bar{t}$ and by definition of topology on $\text{Spé}(\mathscr{F})$, it is open. Thus $V := s^{-1}(V')$ is open and contains P, and inside which $s(Q) = t_Q$, so by definition $s \in \mathscr{F}^+$.

Now $\mathscr{F}$ is a sheaf iff θ in prop 1.2 is isomorphism iff $\mathscr{F}(U) = \mathscr{F}^+(U) = \text{Spé}(\mathscr{F})$.

II.1.14 support

Need to show $\{P \in U | s_P = 0\}$ open. For each P where $s_P = 0$, we can write it as $\langle U, s\rangle = \langle V, s|_V\rangle = 0$ where $s|_V = 0$ and V is open around P. Now any $Q \in V$ satisfies $s_Q = 0$ inside $\mathscr{F}_Q$. Hence eahc point has a neighbourhood that is also contained in $\{P \in U | s_P = 0\}$, so it is open.

1.19(b) gives open support. (note that $\mathcal{O}_P$ is non-zero stalk for algebraic varieties because it contains k).

II.1.15 sheaf $\mathscr{H}om$

For $\varphi, \psi \in \text{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$, we can define $\varphi + \psi$ by $\varphi + \psi(V) = \varphi(V) + \psi(V)$ is an abelian group.

Need to show presheaf $U \mapsto \text{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is sheaf. Suppose $\varphi \in \mathscr{H}om(\mathscr{F}, \mathscr{G})(U)$ restricts to 0 on cover U_i. Then $\varphi(U)(s)|_{U_i} = \varphi(U_i)(s|_{U_i}) = \varphi|_{U_i}(U_i)(s|_{U_i}) = 0$, so $\varphi = 0$.

Suppose a morphism defined on each $\mathscr{F}|_{U_i}$ consistently. Then again $\varphi(U)(s)|_{U_i} = \varphi|_{U_i}(U_i)(s|_{U_i})$, and we can find some $\varphi(U)(s) \in \mathscr{G}_U$ so that φ is a morphism of sheaves. We need to show $\varphi(U)$ is a morphism of abelian groups. This is because restriction map is homomorphism of groups, so $\varphi(st)$ and $\varphi(s)\varphi(t)$ restrict to the same elements $\varphi|_{U_i}(U_i)(s|_{U_i})$ in $\mathscr{G}|_{U_i}$, so they are the same by axiom (3).

II.1.16 flasque sheaves

(a) Map is surjection unless V has more connected components than U. If $U = V \cup W$ where V, W open, then $\overline{V}, \overline{W}$ are closed proper subsets and U is not irreducible. Since all open subsets of irreducible space are irreducible, the restriction maps are all surjective.

(b) by 1.8, only need to show $\mathscr{F}(U) \to \mathscr{F}''(U)$ is surjective. Suffices to show image is sheaf (then inclusion from image at U is the same as $\mathscr{G}(U)$, i.e. surjective). This is the same as showing $U \mapsto \mathscr{F}(U)/\ker\varphi(U) = \mathscr{F}(U)/\operatorname{im}\psi(U) = \mathscr{F}(U)/\mathscr{F}'(U)$ is a sheaf, where $\psi : \mathscr{F}' \to \mathscr{F}$ is injectvie.

Locality: Suppose $s|_{U_i}$ restricts to an element inside $\mathscr{F}'(U_i)$, then by gluing together the restrictions in the sheaf $\mathscr{F}'$, we see s is the unique element with such feature and lies inside $\mathscr{F}'(U)$. So $s = 0$ in the quotient.

Gluing: Suppose we want to glue together $s_i + \mathscr{F}'(U_i)$, where $s_i|_{U_i \cap U_j} + t_{ij} = s_j|_{U_i \cap U_j}$ for some $t_{ij} \in \mathscr{F}'(U_i \cap U_j)$. Since flasque, there is some $T_{ij} \in \mathscr{F}'(U_i)$ such that $T_{ij}|_{U_i \cap U_j} = t_{ij}$. Fix j, and replace s_i by $s_i + T_{ij}$ for all $i \neq j$. Now we can glue together s_i, s_j inside $U_i \cup U_j$.

Consider the collection of sets of indices $i \in I$ paired with $\{t_i\} \subseteq \mathscr{F}(U)$ such that $t_i = s_i \mod \mathscr{F}'(U_i)$ and $t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$. Order them by inclusion on both parts. Given a chain, take the union, and we see it is again an element of this collection that is upper bound. Thus there is a maximal element $(M, \{t_i : i \in M\})$. Glue them together inside $\mathscr{F}(\cup_M U_i)$ to get t. If this maximal element does not cover the whole index set, then we can find some U_k that has not been covered and apply the last paragraph to glue together t and s_k, and contradict maximality. Hence M is the whole index set and $\bar{t} \in \mathscr{F}(U)/\mathscr{F}'(U)$ is the requied lift.

(c) By part (b), we get two exact sequences with open sets U and V. Use diagram chasing from something like five lemma (also use that morphism commutes with restrictions).

(d) $f_*\mathscr{F}(U) \to f_*\mathscr{F}(V)$ is the same map as $\mathscr{F}(f^{-1}U) \to f_*\mathscr{F}(f^{-1}V)$ which is surjectvie since $\mathscr{F}$ is flasque (inclusion perserved).

(e) $\mathscr{F}$ is identified with $\mathscr{F}^+$ which is a subsheaf of $\mathscr{G}$. It is flasque because any map on V can be extended to U by sending P to s_P (fix arbitrary $s \in \mathscr{F}(U)$).

II.1.17 skyscraper sheaves

A point Q is in $\{P\}^-$ iff any open set around Q intersects P iff $i_P(A)_Q = A$.

$i_*(A)$ is defined by $A(i^{-1}(V))$ for any V open in X. Note that since V open, V contains P iff V intersects $\{P\}^-$. So $i_*(A)(V) = 0$ for V not containing P. Note that an open subset of $\{P\}^-$ must be connected (otherwise we can make the closure smaller), so $i^{-1}(V)$ is connected for any V containing P and the constant sheaf gives us $i_*(A)(V) = A$.

II.1.18 adjoint property of f^{-1}

Let F be the presheaf $U \mapsto \lim_{V \supseteq f(U)} f_*\mathscr{F}(V) = \lim_{V \supseteq f(U)} \mathscr{F}(f^{-1}V)$. Define $\langle s, f^{-1}(V)\rangle \mapsto s|_U$ (well defined because $f^{-1}(V)$ contains U). By universal property we can extend it to a natural map from $f^{-1}f_*\mathscr{F}$ to $\mathscr{F}$.

Let G be the presheaf on X that $W \mapsto \lim_{V \supseteq f(W)} \mathscr{G}(V)$. Let $s \in \mathscr{G}(U)$. The elements of $f_*f^{-1}\mathscr{G}(U) = f^{-1}\mathscr{G}(f^{-1}U)$ are functions s from $f^{-1}(U)$ to $\cup_{P \in f^{-1}(U)} G_P$ satisfying construction in definition 1.2. Elements of G_P are of form $\langle t, W\rangle$ where $W \subseteq X$ contains P and $t = \langle u, V\rangle$ where $V \subseteq Y$ contains $f(W)$ and $u \in \mathscr{G}(V)$. Send s to $s(P) = \langle t, f^{-1}(U)\rangle$ where $t = \langle s, U\rangle = s_P$ (ok because U contains $ff^{-1}U$). And we have natural map $\mathscr{G} \to f_*f^{-1}\mathscr{G}$.

Given $\varphi : f^{-1}\mathscr{G} \to \mathscr{F}$, then since f_* is a functor, we have $f_*\varphi : f_*f^{-1}\mathscr{G} \to f_*\mathscr{F}$, which gives us a map

$$\mathscr{G} \to f_*f^{-1}\mathscr{G} \to f_*\mathscr{F}$$

On the other hand, given $\mathscr{G} \to f_*\mathscr{F}$, we get $f^{-1}\mathscr{G} \to f^{-1}f_*\mathscr{F} \to \mathscr{F}$.

These maps are inverses of each other by construction. need check!

II.1.19 extending a sheaf by zero

(a) If $P \in Z$, then stalk $i_*\mathscr{F}_P = \{\langle s, V\rangle : V \subseteq X\} = \{\langle s, V \cap Z\rangle\} = \{\langle s, V\rangle : V \subseteq Z\} = \mathscr{F}_P$. If $P \notin Z$, then $\langle s, V\rangle = \langle s, U \cap V\rangle = 0$ because $i_*\mathscr{F}(U \cap V) = \mathscr{F}(\phi) = 0$.

(b) Note stalk of presheaf is the same as stalk of sheaf associated to it. Now if $P \in U$, then elements of $\mathscr{F}_P$ correspond to the ones in $(j_!\mathscr{F})_P$ because $\langle s, V\rangle = \langle 0, U\rangle$ for any V not inside U. If $P \notin U$, then any open set containing V is not inside U, so $(j_!\mathscr{F})_P = 0$.

Suppose a sheaf $\mathscr{G}$ satisfies this property and restriction to U is $\mathscr{F}$. Let $j_!(F)$ be th presheaf for $j_!(\mathscr{F})$. Define $\varphi : j_!(F) \to \mathscr{G}$ by $\varphi(V)$ is the 0 map for V not inside U, $\varphi(U)$ is the identity map. Extend this to

$j_!(\mathscr{F}) \to \mathscr{G}$ using universal property. Then the induced maps on stalks are identity maps by construction. Thus by prop 1.1 $\mathscr{G}$ and $j_!(\mathscr{F})$ are isomorphic. Thus unique.

(c) Let $P \in Z$, then the corresponding sequence for stalks at P becomes

$$0 \to 0 \to \mathscr{F}_P \to \mathscr{F}_P \to 0$$

which is exact by our identification in (a). (also using the fact $(\mathscr{F}|_Z)_P = \mathscr{F}_P$ for $P \in Z$, 0 for $P \notin Z$). Similar for $P \in U$.

II.1.20 subsheaf with supports

(a) Locality: suppose $s|_{U_i} = 0$ in all $\Gamma_{Z\cap U_i}(U_i, \mathscr{F}|_{U_i})$. observe from definition that restriction to an open set is simply retriction of the sheaf (no worry about limits). Then since restriction maps are induced from the original $\mathscr{F}$ and $\Gamma_{Z\cap V}(V, \mathscr{F}|_V)$ are subgroups of $\mathscr{F}(V)$, by locality of $\mathscr{F}$, $s = 0$.

Gluing: Suppose s_i are elements of $\mathscr{F}(U_i)$ whose support contained in $Z \cap U_i$, that is for all $P \notin Z$, $\langle s_i, U_i\rangle = \langle 0, W_i\rangle$ for some W_i containing P. Glue them together using $\mathscr{F}$ to get s. Now suppose $P \in U - Z$, then $P \in U_i - Z$ for some i, so $s_P = \langle s, U\rangle = \langle s_i, U_i\rangle = \langle 0, W_i\rangle = 0$, and $s \in \Gamma_{Z\cap U}(U, \mathscr{F}|_U)$ as required.

(b) Since $\Gamma_Z(X, \mathscr{F})$ is subgroup of $\Gamma(X, \mathscr{F})$, there is injection morphism $\varphi : \mathscr{H}^0_Z(\mathscr{F}) \to \mathscr{F}$. By definition, if $P \notin U$, then $\mathscr{H}^0_Z(\mathscr{F})_P = 0$.

Let $\varphi : \mathscr{F} \to j_*(\mathscr{F}_U) = j_*j^{-1}\mathscr{F}$ be the map from 1.18. Let $s \in \mathscr{F}(V) \in \ker\varphi(V)$, then by our construction in 1.18, $\langle s_P, j^{-1}(V)\rangle = 0$ in $(\mathscr{F}|_U)_P = \mathscr{F}_P$ for all $P \in j^{-1}(V) = V \cap U$. This means the support of S is contained in $V - U = V \cap Z$, namely $s \in \mathscr{H}^0_Z(\mathscr{F})$. Thus the sequence is exact.

Alternative proof: Use the restriction map $\mathscr{F}(V) \to \jmath_*\mathscr{F}|_U(V) = \mathscr{F}(V \cap U)$. If s restricts to 0, then that means $s_P = 0$ for any $P \in V \cap U$, so s is supported by elements in $V \cap Z$.

Suppose $\mathscr{F}$ is flasque. Then since restriction map is surjective, the map $\mathscr{F} \to \jmath_*\mathscr{F}|_U$ is too.

Observe $j_*(\mathscr{F}_U)$ has stalk $\mathscr{F}_P$ at $P \subseteq U$.

II.1.21 some examples of sheaves on varieties

(a) If regular function vanishes on cover, then it vanishes on whole set (holds true for functions in general). By I.4.1, we can glue regular functions to get a regular function that also vanish on Y.

(b) Consider the sequence

$$0 \to \mathscr{I}_Y \to \mathcal{O}_X \to i_*(\mathcal{O}_Y) \to 0$$

By 1.19(a), $i_*(\mathcal{O}_Y)_P = 0$ for $P \notin Y$ and $(\mathcal{O}_Y)_P$ if $P \in Y$. If $P \notin Y$, observe that local functions at P do not care about Y, so $(\mathscr{I}_Y)_P = \mathcal{O}_X$ and the sequence is exact. If $P \in Y$, then since any regular function on Y at P is defined on an open set of Y containing P, it is defined on the corresponding open set in X, so $(\mathcal{O}_X)_P \to (\mathcal{O}_Y)_P$ is surjection. On the other hand, the kernel of this map would be maps that vanish on Y, namely $\mathscr{I}_Y$.

(c) Similar to before (part of the argument only used that Y is closed). Consider $(i_*(\mathcal{O}_P) \oplus i_*(\mathcal{O}_Q))_P$, since we can find open sets around P that do not contain P, this is just $(\mathcal{O}_P)_P = k$, so $(\mathcal{O}_X)_P$ is onto it. Similar for Q. Hence the map is surjective and sequence is exact.

Now note the induced map $k = \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathscr{F}) = \Gamma(\{P\}, \mathcal{O}_{\mathscr{P}}) \oplus \Gamma(\{Q\}, \mathcal{O}_{\mathscr{Q}}) = k \oplus k$ is not surjective.

(d) Since regular functions on open sets are rational functions, they inject into K. Since X irreducible, it is connected. So $\mathscr{K}$ maps everything to K, so the stalks are all K. Consider sequence

$$0 \to \mathcal{O} \to \mathscr{K} \to \sum i_P(I_P) \to 0$$

Since single point P closed, stalk at $Q \neq P$ for $i_P(I_P)$ is 0 and $I_P = K/\mathcal{O}_P$ at P. Thus the induced seuqence is

$$0 \to \mathcal{O}_P \to K \to K/\mathcal{O}_P \to 0$$

which is exact.

(e) Take global section we have

$$0 \to k = \Gamma(X, \mathcal{O}) \to K = \Gamma(X, \mathscr{K}) \to \sum K/\mathcal{O}_P = \sum i_*(I_P)(X) \to 0$$

$$0 \to k \to K \to \sum K/\mathcal{O}_P \to 0$$

Surjective since rational function defined on open set, which is $\mathbb{P}^1$ minus finitely many points, so the elements in $\sum K/\mathcal{O}_P$ have 0 coordinates everywhere but finitely many, take their sum to get pre-image. The kernel would be regular on X, so just k. Thus sequence is exact.

II.1.22 glueing sheaves

Define $\mathscr{F}(V)$ the disjoint union of $\mathscr{F}_i(V_i)$ mod the equivalence relation that $s|_{U_i \cap U_j} \sim \varphi_{ij}(s|_{U_i \cap U_j})$. Then $\mathscr{F}|_{U_i}$ is just $\mathscr{F}_i$ because all other sheaves are modded out. The maps ψ_i agree on $U_i \cap U_j$ by construction.

Locality: Let V_j be a cover of V and $s|_{V_j} = 0$. Note that $V_{ij} = V_j \cap U_i$ is also a cover and also restrict to 0, so $s|_{V \cap U_i} = 0$ for all i by locality of the sheaf $\mathscr{F}_i$. Hence by our definition s can only be 0.

Gluing: Similar, if we want to glue in V_j, first glue V_{ij} to get s_i in $V \cap U_i$. Then we must have $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ by locality because they agree on all the V_{ij}. Now take equivalent class $\langle s_i, U_i \rangle$.

Alternative: define $\mathscr{F}(U) = 0$ for U not in any U_i, and $\mathscr{F}(U) = \mathscr{F}_i(U)$ if in some U_i. Then take sheaf associated with this presheaf. check this

Note: Let $\mathscr{F}$ presheaf on X, $\mathscr{G}$ sheaf on Y open in X. Suppose $\mathscr{F}|_Y \to \mathscr{G}$ morphism, which is isomorphism at all $U \subseteq Y$, then induced map gives isomorphism $\mathscr{F}^+|_Y \to \mathscr{G}$ because the stalks all give isomorphisms for $P \in Y$. Where $\mathscr{F}|_U$ is defined as the presheaf on Y equal to $\mathscr{F}$

Uniqueness: By locality sheafs are determined by restrictions on U_i, and the ψ_i would give the uniquess up to isomorphism.

II.2 Lecture 2

II.2.1 x

Note that prime ideals in A_f are those in A that do not contain f. Thus we have a homeomorphism φ between $D(f)$ and $\operatorname{Spec} A_f$ by $\varphi(\mathfrak{p}) = \mathfrak{p}A_f$.

Need to define an isomorphism between sheefs $\mathcal{O}_X|_{D(f)}(\varphi^{-1}U)$ and $\mathcal{O}_{\operatorname{Spec} A_f}(U)$. Given a section $s \in \mathcal{O}_{\operatorname{Spec} A_f}(U)$, define $\varphi^{\#}(s)(\mathfrak{p})$ to be the image of s in the stalk (namely $s(\mathfrak{p})$ by prop 2.2(a))

$$(\mathcal{O}_{\operatorname{Spec} A_f})_{\mathfrak{p}} = (A_f)_{\varphi(\mathfrak{p})} = A_{\mathfrak{p}} = (\mathcal{O}_X)_{\mathfrak{p}} = (\mathcal{O}_X|_{D(f)})_{\mathfrak{p}}$$

for $\mathfrak{p} \in \varphi^{-1}(U)$. The above chain of equations shows $\varphi^{\#}$ is isomorphism (locality condition from definition is perserved). Hence $(D(f), \mathcal{O}_X|_{D(f)}) \cong (\operatorname{Spec} A_f, \mathcal{O}_{\operatorname{Spec} A_f})$.

II.2.2 x

Since X is covered by affine schemes, we can assume $X = \operatorname{Spec} A$ affine. Now U is covered by open sets $D(f)$, which by 2.1 is affine.

II.2.3 reduced scheme

(a) Let U be open in X. Suppose $(U, \mathcal{O}_X|_U)$ is reduced. Let s be nilpotent in $\mathcal{O}_X(X)$, then $s|_U$ would be nilpotent in $\mathcal{O}_X(U) = \mathcal{O}_X|_U(U)$, namely $s|_U = 0$. If this is true for a cover, then $s = 0$. Hence it suffices to show a cover of X is reduced. Also note that the stalk for restriction is the same as the stalk for orginal scheme. Thus we can assume $X = \operatorname{Spec} A$ is affine. Since any open set is covered by $D(f)$, we only need to show is for open sets $D(f)$.

Observe that nilpotent element in $S^{-1}A$, say $(a/b)^n = 0$ iff $sa^n = 0$ for some $s \in S$. Suppose $A_{\mathfrak{p}}$ has no nilpotent element for any $\mathfrak{p}$. We need to show $\mathcal{O}(D(f)) = A_f$ has no nilpotent element for all $f \in A$.

Suppose the conrtary, then for some f, $f^m a^n = 0$ for some $a \neq 0$. Take $\mathfrak{p} \in D(f)$, then this means a is nilpotent in $A_{\mathfrak{p}}$, a contradiction.

Conversely, suppose A_f has no nilpotent element but $sa^n = 0$ for some $s \notin \mathfrak{p}$, then A_s would have nilpotent element, a contradiction.

Alternative: Suppose $\langle U, s\rangle$ nilpotent in stalk, then $\langle U, s^n\rangle = \langle V, 0\rangle$ for some V, which means $(s|_V)^n = 0$, but $\mathcal{O}_X(V)$ has no nilpotent, so $s|_V = 0$ and $\langle U, s\rangle = \langle V, 0\rangle = 0$. Conversely suppose $s^n = 0$ in $\mathcal{O}_X(U)$, then $\langle U, s^n\rangle = 0$, since stalks have no nilpotent, $s|_V = 0$ for some $P \subseteq V \subseteq U$. Since P arbitrary, this gives a cover and so $s = 0$ by locality.

(b) Since homomorphisms map nilpotents to nilpotents, the restirction maps are well-defined, so it is a presheaf. Note that $(\mathcal{O}_X)_{\text{red}}|_U = (\mathcal{O}_X|_U)_{\text{red}}$, so we can assume $X = \operatorname{Spec} A$ affine to show $(X, (\mathcal{O}_X)_{\text{red}})$ is scheme. Note that since all primes contain the nilradical, the quotient does not change spectrum, so $X = \operatorname{Spec} A_{\text{red}}$. Thus $f(\mathfrak{p}) = \mathfrak{p}A_{\text{red}}$ homeomorphism from $\operatorname{Spec} A$ to $\operatorname{Spec} A_{\text{red}}$.

Consider the projection map $(\mathcal{O}_X)_{\mathfrak{p}} \to ((\mathcal{O}_X)_{\text{red}})_{\mathfrak{p}}$ that sends $\langle s, U\rangle$ to $\langle s, U\rangle$, well-defined by the definition of reduced sheaf. Now $\langle s, U\rangle$ is in the kernel iff $\langle s|_V, V\rangle$ is nilpotent for some V iff $\langle s, U\rangle$ is nilpotent in the stalk $(\mathcal{O}_X)_{\mathfrak{p}}$. Hence $((\mathcal{O}_X)_{\text{red}})_{\mathfrak{p}} = ((\mathcal{O}_X)_{\mathfrak{p}})_{\text{red}} = (A_{\mathfrak{p}})_{\text{red}} = (A_{\text{red}})_{\mathfrak{p}}$ (because localization pass through quotient).

Therefore elements of $(\mathcal{O}_X)_{\text{red}}(U)$ are maps $s : U \to (A_{\text{red}})_{\mathfrak{p}}$ and check to see this gives $(\mathcal{O}_X)_{\text{red}} = \mathcal{O}_{\operatorname{Spec} A_{\text{red}}}$ and $(X, (\mathcal{O}_X)_{\text{red}})$ is a scheme. (locally ringed space because stalks local).

Finally, back to when X is covered by affine schemes, for each of the affine scheme, the homomorphism $\varphi : A \to A_{\text{red}}$, induces morphism $(g, g^\#) : \operatorname{Spec} A_{\text{red}} \to \operatorname{Spec} A$ where g is f^{-1} homeomprhism defined above. Now glue them together via setting $g^\#(s)$ to be the unique lift of $g^\#(s|_{U_i})$, which will be local homomorphism because the stalks of X are stalks of U_i.

(c) Let $\varphi : Y_{\text{red}} \to Y$ be the homeomorphism on the space, then set $g = \phi^{-1} \circ f$. For the function on sheaf, by universal property, it suffices to show $f^\#$ is determined by a homomorphism $g : (\mathcal{O}_Y)'_{\text{red}} \to f_*\mathcal{O}_X$ where $(\mathcal{O}_Y)'_{\text{red}}$ is the presheaf in part (b). We see g must be $g(s) = f^\#(s)$, it is well-defined because if s is nilpotent, then since $\mathcal{O}_X$ is reduced, $f^\#(s)$ must be 0, and g extends to some $g^\# : (\mathcal{O}_Y)_{\text{red}} \to f_*\mathcal{O}_X$. Observe that on the stalks $g^\#_P = g_P$, so since $f^\#$ is local (maps non-units to non-units), $g^\#$ is also local. Hence g is a morphism as required.

II.2.4 x(*).

Given a ring homomorphism $\varphi : A \to \Gamma(X, \mathcal{O}_X)$, we need to define a morphism $f : X \to \operatorname{Spec} A$. Let X be covered by $\operatorname{Spec} B_i$, then φ restricts to

$$\varphi_i : A \to \Gamma(X, \mathcal{O}_X) \xrightarrow{\rho_{X\,\operatorname{Spec} B_i}} B_i$$

and induces maps $f_i : \operatorname{Spec} B_i \to \operatorname{Spec} A$. Cover $\operatorname{Spec} B_i \cap \operatorname{Spec} B_j$ by affine $U_{ijk} = \operatorname{Spec} C_{ijk}$, then we have

$$A \to \Gamma(X, \mathcal{O}_X) \xrightarrow{\rho_{X\,\operatorname{Spec} B_i}} B_i \xrightarrow{\rho_{\operatorname{Spec} B_i, U_{ijk}}} C_{ijk}$$

where $\rho_{\operatorname{Spec} B_i, U_{ijk}} : \Gamma(\operatorname{Spec} B_i, \mathcal{O}_{B_i}) \to \Gamma(U_{ijk}, \mathcal{O}_{B_i})$, its inverse of any prime ideal is prime, which corresponds to elements of $\operatorname{Spec} C_{ijk}$. Therefore the induced map is in fact $f_i|_{\operatorname{Spec} C_{ijk}}$. On the other hand, we also have

$$A \to \Gamma(X, \mathcal{O}_X) \xrightarrow{\rho_{X\,\operatorname{Spec} B_j}} B_j \xrightarrow{\rho_{\operatorname{Spec} B_j, U_{ijk}}} C_{ijk}$$

which induces $f_j|_{\operatorname{Spec} C_{ijk}}$. But these two maps are the same because

$$\rho_{\operatorname{Spec} B_j, U_{ijk}} \circ \rho_{X, \operatorname{Spec} B_j} = \rho_{\operatorname{Spec} B_i, U_{ijk}} \circ \rho_{X, \operatorname{Spec} B_i} = \rho_{X, U_{ijk}}$$

Hence the f_i agree on intersections and we can glue them together.

From φ_i we also get $f_i^\# : \mathcal{O}_{\operatorname{Spec} A}(V) \to \Gamma(f_i^{-1}(V), \mathcal{O}_X|_{B_i})$, defined by

$$f_i^\#(s)(\mathfrak{p}) = (\varphi_i)_{\mathfrak{p}} \circ s \circ f_i(\mathfrak{p})$$

Now $f^{-1}(V) = \cup f_i^{-1}(A)$ and we want to glue the maps together using glueing from sheafs (axiom (4)). Need to check they are compatible, namely

$$\rho^{f_i^{-1}(V)}_{f_i^{-1}(V)\cap f_j^{-1}(V)}(f_i^{\#}(s)) = \rho^{f_j^{-1}(V)}_{f_i^{-1}(V)\cap f_j^{-1}(V)}(f_j^{\#}(s))$$

it suffices to show

$$(\varphi_i)_{\mathfrak{p}} \circ s \circ f_i = (\varphi_j)_{\mathfrak{p}} \circ s \circ f_j$$

agree on the subset $f_i^{-1}(V) \cap f_j^{-1}(V)$ of $\operatorname{Spec} B_i \cap \operatorname{Spec} B_j$, which follows from a similar argument using C_{ijk} as before.

Observe if we take the global section in $f^{\#}$, then by 2.3, it is exactly what we get from glueing φ_i, which is just φ (by our way of glueing maps of sheafs), so the constructions are mutually inverse of each other, therefore bijection.

II.2.5 x

Space of $\operatorname{Spec} \mathbb{Z}$ is the set of prime numbers $\{\mathbb{Z}/p\mathbb{Z} : p \text{ prime or } 0\}$. Closed sets are finite sets not containing 0 (corresponding to prime divisors of natural numbers), or everything.

Let $(f, f^{\#})$ be a morphism from X to $\operatorname{Spec} \mathbb{Z}$. This corresponds to a ring homomorphism $\mathbb{Z} \to \Gamma(X, \mathcal{O}_X)$ by 2.4, which is unique. Hence $\operatorname{Spec} \mathbb{Z}$ is final object of category of schemes.

II.2.6 x

Is empty. A morphism from $\operatorname{Spec} 0$ would have associated map $f^{\#} : \mathcal{O}_X \to f_*(\mathcal{O}_{\operatorname{Spec} 0})$, $\varphi^{\#}(V) : \mathcal{O}_X(V) \to 0$ is the 0 map, unique. Map on space is $\phi = f : \phi \to X$ unique.

II.2.7 x

$\operatorname{Spec} K$ has one point $\{(0)\}$, so the space map is determined by $x = f((0))$. And we just need to give the associated map $f^{\#} : \mathcal{O}_X(V) \to \mathcal{O}_{\operatorname{Spec} K}(f^{-1}(V))$. It requires $f^{\#}_{(0)} : \mathcal{O}_x \to K$ to be local, which means $\mathfrak{m}_x$ is mapped to 0, and corresponds to a inclusion map $k(x) \to K$.

Conversely, given $k(x) \to X$, we can define $\mathcal{O}_x \to K$. Want to define $f^{\#}$. When V does not contain x, the map is just 0. When V contains x, simply set

$$f^{\#} : \mathcal{O}_X(V) \hookrightarrow \mathcal{O}_x \to \Gamma(\operatorname{Spec} K, \mathcal{O}_{\operatorname{Spec} K}) = K$$

II.2.8 x

Since X is over k, there is a morphism $X \to \operatorname{Spec} k$ given by some $g(x) = (0)$, with $g_x^{\#} : k \to \mathcal{O}_x$ local. The ring of dual numbers is over k via the induced map from injection $h^{\#}_{(\epsilon)} : k \to k[\epsilon]/\epsilon^2$, which is $(\epsilon) \mapsto (0)$.

Prime ideals of $k[\epsilon]/\epsilon^2$ corespond to those of $k[\epsilon]$ containing (ϵ^2), which are just (ϵ). Thus a k-morphism from $\operatorname{Spec} k[\epsilon]/\epsilon^2$ to X has space map determined by picking $x \in X$. The associated map requires

$$f^{\#}_{(\epsilon)} : \mathcal{O}_x \to k[\epsilon]/\epsilon^2$$

local, namely elements of $\mathfrak{m}_x$ are mapped inside (e). This means m_x^2 is mapped to 0 and we have a map

$$\mathcal{O}_x/\mathfrak{m}_x^2 \to k[\epsilon]/\epsilon^2$$

Since units are mapped to units, we have k-vector space homomorphism

$$\mathcal{O}_x/\mathfrak{m}_x \oplus \mathfrak{m}_x/\mathfrak{m}_x^2 \to k \oplus (e) \cong k \oplus k$$

The second part corresponds to an element in T_x. Now since this is a k-morphism, we need

$$h^{\#}_{(\epsilon)} = f^{\#}_{(\epsilon)} \circ g^{\#}_x : k \to \mathcal{O}_x \to k[\epsilon]/\epsilon^2$$

but the image of $h^{\#}_{(\epsilon)}$ is all of k. So k injects into $k(x)$, then injects back into k and giving an isomorphism, meaning $k(x) = k$ and x is rational.

Conversely, given x rational via $\psi : k(x) \to k$ and an element of $T \in T_x$, define $f((\epsilon)) = x$ and

$$f^{\#}(V) : \mathcal{O}_X(V) \hookrightarrow \mathcal{O}_x \to \Gamma(\operatorname{Spec} k[\epsilon]/\epsilon^2, \mathcal{O}_{\operatorname{Spec} k[\epsilon]/\epsilon^2}) = k[\epsilon]/\epsilon^2$$

by $f^{\#}(a) = \psi(m_0) + T(m_1)\epsilon$ where m_0 is image of a in $k(x)$, m_1 is image in $\mathfrak{m}_x/\mathfrak{m}_x^2$, check well-defined.

II.2.9 x

By 2.2 X has open cover of affine schemes $\operatorname{Spec} A_i$. Let Z be irreducible closed, say $Z = \cup U_i = \cup V(\mathfrak{a}_i)$, where $U_i = Z \cap \operatorname{Spec} A_i$ also irreducible closed in $\operatorname{Spec} A_i$. Suppose $\mathfrak{a}_i$ has more than 1 minimal primes over it, say $\mathfrak{p}_j$. Then $V(\mathfrak{a}_i) = \cup V(\mathfrak{p}_j)$ is reducible. So $\mathfrak{a}_i$ has only one minimal prime $\mathfrak{p}_i$ over it, and $\mathfrak{p}_i$ is the unique generic point in U_i. Since U_i dense, $\mathfrak{p}_i$ generic point of Z. For $i \neq j$, suppose $\mathfrak{p}_i$ is not in $\operatorname{Spec} A_j$, then its closure is contained in $X - \operatorname{Spec} A_i$. Therefore if Z were to intersect $\operatorname{Spec} A_i$, then $\mathfrak{p}_i$ must all be the same point in X, so the generic point is unique.

II.2.10 x

Prime ideals are irreducible polynomials, which are linear or qudratic. Closed sets are closed points corresponding to these prime ideals, which correspond to elements in $\mathbb{C}$ modulo their conjugate ($\{x + yi : y \geq 0\}$).

II.2.11 x

Elements of $\operatorname{Spec} k[x]$ are irreducible polynomials f. $\mathcal{O}_{(f)} = k[x]_{(f)}$. Note that (f) is maximal ideal. So $k[x]_{(f)}/(f) = (k[x]/(f))_{(f)} = k[x]/(f)$. Since $k[x]/(f)$ is a vector space of dimension $n = \deg f$ over k, it has to be $\mathbb{F}_{p^n}$. Thus given a residue field, number of points with this residue field is equal to number of irreducible polynomials (monic) of degree n. to find the exact number, use the fact that $x^{p^n} - x$ is the product of all irreducible polynomials of degree dividing n, because its roots are in $\mathbb{F}_{p^n}$ and every polynomial of degree d have root in $\mathbb{F}_{p^d}$.

II.2.12 glueing lemma

Space of X would be the disjoint union of X_i quotient the equivalence relation $x_i \sim \varphi_{ij}(x_i)$ (check this is equivalence relation using assumptions). Define open sets to be open iff the restriction to X_i are open. Define sheaf $\mathcal{O}_X$ by glueing $\mathcal{O}_{X_i}$ using 1.22.

II.2.13 x

(a) $\Rightarrow$ by I.1.7 (subspace of noetherian is noetherian by I.1.7(c)). $\Leftarrow$ since X itself is open.

(b) Since X affine, every open cover can be written in form of cover by $D((f_i))$ where $f_i \in A$. Thus $\cap V(f_i) = \phi$. This means no prime ideal contains $(f_i)_{i \in I}$, in particular not even maximal ideals. Therefore there exists some combination $\sum_{i=1}^n a_i f_i = 1$, and $D(f_1), ..., D(f_n)$ is finite subcover.

Consider $A = k[x_1, x_2, ...]$ and open set $\cup_{i=1}^{\infty} D(x_i)$, which contains prime ideals of form $(x_1, ..., x_{i-1}, x_{i+1}, ...)$. Thus this cover has no finite subcover.

(c) Consider a cover $\cup D(f_i)$, then $\cap V(f_i) = V(\alpha)$. So the prime ideals that contain α are those that contain all the f_i. Since A noetherian, $(f_i)_{i \in I}$ is finitely genrated by $(f_1, ..., f_n)$. Thus containing $f_1, ..., f_n$ iff contain all f_i, and $\cap_{i=1}^n V(f_i) = V(\alpha)$, and $D(f_1), ..., D(f_n)$ is finite subcover.

(d) Consider $A = k[x_1, x_2, ...]/(x_1^2, x_2^2, ...)$, note the reduced ring is k, so $\operatorname{Spec} A = \operatorname{Spec} A_{\text{red}}$ contains only one element, and must be noetherian. A is not noetheiran because $(x_1, x_2, ...)$ not finitely generated.

II.2.14 x

(a) If everything in S_+ nilpotetnt, then they are in all the prime ideals, so not included in $\operatorname{Proj} S$. Conversely, if all prime homogeneous ideal contain S_+, then S_+ is inside the nilradical.

(b) Consider the complement $Z = \{\mathfrak{p} \in \operatorname{Proj} T : \varphi(S_+) \subseteq \mathfrak{p}\} = V(\varphi(S_+))$. Since S_+ is a homogeneous ideal, so is $\varphi(S_+)$. Thus Z is closed.

Define $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ is a map from U to $\operatorname{Proj} S$ because $\varphi^{-1}(\mathfrak{p})$ is homogeneous prime does not contain S_+. Define $f^\#$ similar to proof of prop 2.3(b), where we localize φ to $\varphi_{(\mathfrak{p})} : S_{(\varphi^{-1}(\mathfrak{p}))} \to T_{(\mathfrak{p})}$.

(c) Observe that if a prime ideal contains all of T_d, then it contains all of T_+ (take the d-th power). Since the image of S_+ contains $\varphi(S_d) = T_d$, the set Z is empty and $U = \operatorname{Proj} T$. Suppose $\mathfrak{p}, \mathfrak{q}$ are two homogeneous prime ideals of T, say $t \in \mathfrak{q} - \mathfrak{p}$, then $t^d \in \mathfrak{q} - \mathfrak{p}$, so $\varphi^{-1}(t^d) \in \varphi^{-1}(\mathfrak{q}) - \varphi^{-1}(\mathfrak{p})$ and f is injective. To show surjective, we need to show the image of any homogeneous prime is homogeneous prime. Suppose $xy \in \varphi(\mathfrak{p})$, then $x^d y^d \in \varphi(\mathfrak{p})$, so $\varphi^{-1}(x)^d \varphi^{-1}(y)^d \in \mathfrak{p}$ and $\varphi^{-1}(x) \in \mathfrak{p}$ or $\varphi^{-1}(y) \in \mathfrak{p}$, thus x or y in $\varphi(\mathfrak{p})$ and the image is indeed prime; homogeneous because the map is graded. Hence f is a homeomorphism. On the other hand, $\varphi_{(\mathfrak{p})}$ is isomorphism because we have $\varphi_{(\mathfrak{p})}(a/b) = \varphi_{(\mathfrak{p})}(ac^{d_0}/bc^{d_0})$ where $c \in S_+ - \mathfrak{p}$, and $\varphi_{\geq d_0}$ is isomorphism.

(d) Observe $\operatorname{Proj} S$ is covered by $D_+(x_i)$ by prop 2.5. By uniquess of glueing sheafs (1.22), we know $\operatorname{Proj} S$ is the glueing of

$$(D_+(x_i)|\mathcal{O}|_{D_+(x_i)}) \cong \operatorname{Spec} S_{(x_i)} = \operatorname{Spec} k[x_1/x_i, ..., x_n/x_i]/I = \operatorname{Spec} A(V_i)$$

where V_i is the natural affine cover of V. But from proof of prop 2.6, we know $t(V)$ is the glueing (using method from 2.12) of $t(V_i) = \operatorname{Spec} A(V_i)$. Hence $t(V) = \operatorname{Proj} S$.

II.2.15 x(*).

(a) Since $(V, \mathcal{O})$ covered by open affine sets, we can assume V is affine. Let $A = k[x_1, ..., x_n]/I$ be coordinate ring of V, then $t(V) \cong (\operatorname{Spec} A, \mathcal{O}_A)$. $\mathfrak{p}$ is a closed point implies $\mathfrak{p} = \mathfrak{m}_P$ is maximal ideal, so its stalk is $A_\mathfrak{p} = \mathcal{O}_P$ which has residue field k.

Conversely, if $\mathfrak{p}$ is not maximal, so $\dim A/\mathfrak{p} > 0$, but then by I.1.8A(a), $K(A/\mathfrak{p})$ has transcendence degree > 0 over k. Now as an A module, $K(A/\mathfrak{p})$ is the same as $A/\mathfrak{p}$ localized at $\mathfrak{p}$, which is $A_\mathfrak{p}/\mathfrak{p}$ the residue field. Hence the residue field is not k.

(b) We have local homomorphism $f_P^\# : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ where the maximal ideal is mapped inside the maximal ideal. So we get

$$\mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P} \to \mathcal{O}_{X,P}/\mathfrak{m}_P = k$$

giving injection

$$\mathcal{O}_{Y,f(P)}/m_{f(P)} \to k$$

On the other hand since Y is k-scheme, we have map $Y \to \operatorname{Spec} k$ giving injection $k \to \mathcal{O}_{Y,f(P)}/m_{f(P)}$. This means $\mathcal{O}_{Y,f(P)}/m_{f(P)}$ is a field extension of degree 0 over k so is k is this valid?.

(c) Given $\varphi : V \to W$ morphism. φ carries regular functions on W to regular functions on V, giving a map on sheafs

$$\varphi^\# : \mathcal{O}_W(U) \to \mathcal{O}_V(\varphi^{-1}(U))$$

Define corresponding map of schemes $f : t(V) \to t(W)$ by $f(Y) = \overline{\varphi(Y)}$. $f^\# : \alpha_*(\mathcal{O}_W)(U) \to f_*\alpha_*(\mathcal{O}_V)(U)$ can be defined to be $\varphi^\#$ (which is local since we are composing maps) because the two sheafs are just

$$\mathcal{O}_W(\alpha^{-1}(U)) \to \mathcal{O}_V(\alpha^{-1} f^{-1}(U))$$

$$\mathcal{O}_W(U) \to \mathcal{O}_V(\varphi^{-1}U)$$

where U is corresponding open set in W (recall α is bijection between open sets).

Since closed poitns of $t(V)$ and $t(W)$ are just points in V and W, we see f restricted to those closed points must be φ, so the natural map is injective.

Given a morphism $f : t(V) \to t(W)$, by part (b), it must send closed points to closed points, which induces a map $\varphi : V \to W$. φ is continous because f is and open sets of $t(V)$ correspond to open sets in V by α, i.e. $\varphi^{-1}(U) = \alpha^{-1} f^{-1}(\alpha U)$.

Consider $f^{\#} : \mathcal{O}_W(U) \to \mathcal{O}_V(\varphi^{-1}U)$, induce $f^{\#}_P : \mathcal{O}_{W,\varphi(P)} \to \mathcal{O}_{V,P}$. Since local, it further gives us a map on the residue field $f^{\#}_P : k \to k$. Now since f is a k-morphism, by argument from part (b) we know this has to be identity (since it perserves k-structure). Therefore $f^{\#}$ maps a regular function on W to a regular function on V whose value at $\varphi(P)$ is the original value at P, which means the pullback of φ is exactly $f^{\#}$. I.e. φ carries regular function to regular function, and is a morphism of varieties.

Finally, we need to show $f(Y) = \overline{\varphi(Y)}$. But if this is not the case, then even the domain/range of $f^{\#}$ would not agree with $\varphi^{\#}$ on some open set. i'm tired of this question so i wont write the details, which means this sentense may be total bullshit

II.2.16 x

(a) Recall stalk of restriction is the same as orginal stalk. So we can assume $X = \operatorname{Spec} B$ affine. Now $f \in \Gamma(X, \mathcal{O}_X) = B$ is not contained in maximal ideal $\mathfrak{m}_\mathfrak{p} \subseteq \mathcal{O}_{X,\mathfrak{p}}$ iff f is not inside $\mathfrak{p}$, so $X_f = D(f)$.

(b) Exists finite affine cover $U_i = \operatorname{Spec} B_i$ of X, and a restircts to 0 in all the X_f. It suffices to show $f^n a|_{U_i} = 0$. Now a restricts to 0 in $D(\overline{f})$ means a is 0 in the localized ring $B_{i,\overline{f}}$, so $a = 0/1 \in A_{\overline{f}}$, which means $\overline{f}^n a = 0$ for some n. Since finite cover, we can find n large enough for all covers. Hence $f^n a = 0$.

(c) Let $U_i = \operatorname{Spec} B_i$. let $b_i = b|_{U_i \cap X_f}$, then we can find $f^{n_i} b_i \in B_i$. Take maximum by finiteness and $f^N b_i \in B_i$. We want to lift this to A, which requries compatibility in overlaps. Observe that $(b_i - b_j)|_{U_i \cap U_j \cap X_f} = 0$, so $f^{n_{ij}}(b_i - b_j)|_{U_i \cap U_j} = 0$ for some n_{ij} by part (b). Take maximum again and now $f^N b_i - f^N b_j = 0$ in overlaps, hence we can lift to some a that restricts to $f^N b$.

(d) Consider map $b \mapsto (\overline{f^n b})/f^n$ where $\overline{f^n b}$ denotes the element in A that restricts to $f^n b$, this is unique up to a power of f since glueing in sheafs is unique. Well-defined because multiplying varying n does not change the image in A_f. Injective because if $b = 0$, then $\overline{f^n b}$ restricts to 0, but since 0 restricts to 0, by uniquess, we must have the image is 0.

For any $a \in A$, let $b = a|_{X_f}$. For each n, consider the lift of $\beta := b/f^n \in B_{i,f}$ from the $U_i \cap X_f$ up to X_f, then $a = \overline{f^n \beta}$ and $\beta \mapsto a/f^n$, hence surjective.

II.2.17 a criterion for affiness

(a) Since X covered by $f^{-1}(U_i)$, the map is homeomorphism on space. Now $f^{\#}$ is isomorphism on any open set contained in U_i. Since morphism of sheafs compatible with restrictions, $f^{\#}$ on bigger open sets would be lifts, nad by uniqueness of lift, this is well-defined isomorphism of sheafs.

(b) Suppose there is finite set $f_1, ..., f_r$ that generate A. Suppose X_{f_i} are affine, they are isomorphic to $\operatorname{Spec} A_{f_i}$ by 2.16(d) (intersection of X_{f_i} are quasi-compact because they are $\operatorname{Spec}(A_{f_i})_{f_j}$ by 2.16(a)). Suppose $x \notin X_{f_i}$ for all i, and $\sum a_i f_i = 1$. Then $(f_i)_x = \langle f_i|_U, U\rangle$ is in the maximal ideal $\mathfrak{m}_x$. This means $(\sum a_i f_i)_x = \langle 1|_U, U\rangle = 1$ is in $\mathfrak{m}_x$, a contradiction. Hence X_{f_i} cover X.

Since $A \to A_{f_i}$, by prop 2.3(b), we have a morphism $f : X_{f_i} = \operatorname{Spec} A_{f_i} \to \operatorname{Spec} A$. Note that the prime ideals of A_{f_i} correspond to those in A which do not contain f_i. In general, Theorem: if φ is a locatization morphism $A \to A_S$, then $\operatorname{Spec} \varphi$ is a homeomorphism from $\operatorname{Spec} A_S$ onto the subspace $\{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \cap S = \phi\}$.

Since $f_1, ..., f_r$ generate unit ideal, we see the above give a covering $\operatorname{Spec} A_{f_i}$ of $\operatorname{Spec} A$. Now by glueing we get a map $X \to \operatorname{Spec} A$ that restricts to the identity $X_{f_i} \to \operatorname{Spec} A_{f_i}$, hence $X = \operatorname{Spec} A$.

Conversely, suppose X is affine. Then take $f = 1$.

II.2.18 x

(a) f is nilpotent iff in nilradical which is intersection of all prime iff $D(f)$ empty.

(b) Suppose φ injective. Consider localized map at P. If $\varphi_P(a/b) = \varphi(a)/\varphi(b) = 0$, then $q\varphi(a) = 0$ for some $q \notin \mathfrak{p}$, and $qa = 0$, implying $a/b = 0$ in $A_{\varphi^{-1}(\mathfrak{p})}$. Hence $\varphi_\mathfrak{p}$ injective. Recall $f^{\#}_\mathfrak{p} = \varphi_\mathfrak{p}$ (Indeed, given

$s \in (\mathcal{O}_{\operatorname{Spec} A})_{f(\mathfrak{p})}$ which corresponds to the evaluation at $f(\mathfrak{p})$, namely $s(f(\mathfrak{p})) \in A_{\varphi^{-1}\mathfrak{p}}$. On the other hand we have by construction from prop 2.3 $f^{\#}(s) = \varphi_{\mathfrak{p}} \circ s \circ f \in (\mathcal{O}_{\operatorname{Spec} B})_{\mathfrak{p}}$, which corresponds to the evaluation at $\mathfrak{p}$, namely $\varphi_{\mathfrak{p}}(s(f(\mathfrak{p})) \in B_{\mathfrak{p}})$. So $f^{\#}$ injective by 1.2(b) (which states induced map on stalks injective implies injective). Conversely, since φ is just global section of $f^{\#}$, injectivity follows.

In this case, the closure of $f(Y)$ would be the set of primes that contain some element in the image. Observe that the conplement of the closure of a point $\mathfrak{p}$ is the union of $D(f)$ through all $f \in \mathfrak{p}$. Therefore the complement of the closure of $f(Y)$ is

$$\bigcap_{\mathfrak{p} \subseteq B} \bigcup_{f \in \varphi^{-1}\mathfrak{p}} D(f) = \bigcup_{f \in \varphi^{-1}(\cap \mathfrak{p})} D(f)$$

Since injective function sends nilradical to nilradical, by part (a), the union is empty, hence the closure is the whole space and f is dominant.

(c) Need to show f injective, image closed, and sends closed sets to closed sets. $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ injective because φ is surjective (if $\mathfrak{p} \neq \mathfrak{q}$, then $f(\mathfrak{p}) - f(\mathfrak{q})$ contains $\varphi^{-1}(\mathfrak{p} - \mathfrak{q})$). It sends closed sets to closed sets of its image because

$$f(V(\mathfrak{a})) = \{f(\mathfrak{p}) : \mathfrak{p} \supseteq \mathfrak{a}\} = \{\mathfrak{q} \in f(Y) : \mathfrak{q} \supseteq \varphi^{-1}(\mathfrak{a})\} = V(f(\mathfrak{a})) \cap f(Y)$$

(note that the second equality follows from $\varphi^{-1}(\mathfrak{p}) \supseteq \varphi^{-1}(\mathfrak{a}) \Rightarrow \mathfrak{p} \supseteq \mathfrak{a}$, which uses that φ is surjective). To show $f(Y)$ is closed, suppose $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}') \in f(Y)$ and $\mathfrak{p} \subseteq \mathfrak{q} \in X$. Suppose $\varphi(a)\varphi(b) = \varphi(q) \in \varphi(\mathfrak{q})$, then $ab - q \in \ker\varphi$ in A. Now since $\mathfrak{q} \supseteq \mathfrak{p} = \varphi^{-1}(\mathfrak{p}') \supseteq \ker\varphi$, we know $ab - q \in \mathfrak{q}$ and so $a \in \mathfrak{q}$ or $b \in \mathfrak{q}$. Hence $\varphi(\mathfrak{q})$ is prime and $\mathfrak{q} = \mathfrak{q} + \ker\varphi = \varphi^{-1}(\varphi(\mathfrak{q})) \in f(Y)$ and $f(Y)$ is closed.

Similar as before, the localized maps $\varphi_{\mathfrak{p}}$ are surjective. Hence $f^{\#}$ surjective by 1.2(b).

(d) Note that $\psi : A/\ker\varphi \to B$ is an injection. So we have $F : Y \to X' = \operatorname{Spec}(A/\ker\varphi)$ dominant and $F^{\#}$ injective.

We also have $\pi : A \to A/\ker\varphi$ surjective, so $G : X' \to X$ homeomorphism to a closed subset of X, and $G^{\#}$ surjective.

Now observe $\varphi = \psi \circ \pi$, and $\varphi^{-1}\mathfrak{p}$ is always a prime ideal containing $\ker\varphi$, so we see from contruction in proof of prop 2.3, that $f = G \circ F$, $f^{\#} = F^{\#} \circ G^{\#}$.

Since $f^{\#}$ is surjective, we must have $F^{\#}$ surjective. Since f homeomrphism onto closed set, F must be homeomorphism to a closed subset, but since it is dominant, it is just homeomorphism (this sentence not necessary). Therefore ψ induces an isomorphism, and ψ itself must be an isomorphism (because it is global section of $f^{\#}$). So φ surjective.

II.2.19 x

(i)⇒(ii) Suppose $\operatorname{Spec} A$ disconnected, namely $V(\mathfrak{a}) \cap V(\mathfrak{b}) = \phi$ and $V(\mathfrak{a}) \cup V(\mathfrak{b}) = \operatorname{Spec} A$. Note that $\mathfrak{a} + \beta = A$ because otherwise there would be some maximal ideal containg $\mathfrak{a} + \mathfrak{b}$. Thus we can pick f, g such that $f + g = 1$ and $V(f) \cup V(g) = \operatorname{Spec} A$. This means each prime ideal contains either f or g, which means they all contain fg, and fg is nilpotent, say $f^n g^n = 0$. On the other hand $f^n \neq 0, g^n \neq 0$. Note that $V(f) = V(f^n)$, which means $V(f^n) \cap V(g^n) = \phi$. So similar as before we can pick e_1, e_2 in $(f^n), (g^n)$ such that$e_1 + e_2 = 1$, and now $e_1 e_2$ is a multiple of $f^n g^n = 0$. Also, $e_1(e_1 + e_2) = e_1, e_1^2 = e_1$, and similarly $e_2^2 = e_2$, as required.

(ii)⇒(i) Disconnected by the closed sets $V((e_1)), V((e_2))$ because any prime ideal contains $e_1 e_2 = 0$, which means they must contain e_1 or e_2, but $e_1 + e_2 = 1$, so they contain exactly one.

(iii)⇒(ii) Take $(1, 0), (0, 1)$.

(ii)⇒(iii) Consider $\varphi : A \to A/(e_2) \times A/(e_1)$. Surjective because $\varphi(ae_1 + be_2) = (a, b)$. Injective because $\varphi(r) = 0$ implies $\varphi(re_1 + re_2) = 0$ implies $e_2 | r$, $e_1 | r$, which means $r = re_1 + re_2 = 0$.

II.3 Lecture 3

II.3.1 x

The if direction is by definition. Suppose f is locally of finite time, meaning there is a cover $V_i = \operatorname{Spec} B_i$ of Y such that $f^{-1}(V_i)$ covered by $U_{ij} \operatorname{Spec} A_{ij}$ with A_{ij} finitely generated B_i algebra.

Note that $f|_{U_{ij}} : U_{ij} \to V_i$ are induced by homomorphisms $\varphi_{ij} : B_i \to A_{ij}$ where $f|_{U_{ij}}(\mathfrak{p}) = \varphi_{ij}^{-1}(\mathfrak{p})$. Let $g \in V_i$, then $D(g)$ are prime ideals in B_i not containing g, and $f^{-1}(D(g))$ are prime ideals in A_{ij} whose inverse image by φ_{ij} do not contain g, which are those that do not contain $\varphi_{ij}(g)$, namely $D(\varphi_{ij}(g))$. Therefore $f^{-1}(D(g)) = \cup_j D(\varphi_{ij}(g)) = \cup \operatorname{Spec}(A_{ij})_{\varphi_{ij}(g)}$. Observe that if $a_1, ..., a_n$ is a set of generator of A_{ij} over B_i, then add in $1/\varphi_{ij}(g)$ and we get a set of generator of $(A_{ij})_{\varphi_{ij}(g)}$. Thus they are finitely generated B_i algebras. In particular, finitely generated over $(B_i)_g$.

Let $V = \operatorname{Spec} B$ open in Y. Then V is covered by sets of form $D(g) = \operatorname{Spec}(B_i)_g$ (we are denoting image of g in B_i as g too by abuse of notation), whose inverse image are covered by spectra of finitely generated $(B_i)_g$ algebras. Observe by 2.16 that $B_g \cong (B_i)_g$. Now observe that if A is a finitely generated B_g algebra with generators $a_1, ..., a_n$, then $a_1, ..., a_n, 1/g$ generates A over B. Hence we can conclude that $f^{-1}(V)$ is covered by $\operatorname{Spec} A_j$ where A_j are finitely generated B-algebra.

II.3.2 x

Suppose f is quasi-compact with cover $V_i = \operatorname{Spec} B_i$. In particular, $f^{-1}(V_i)$ is a finite union of open affine schemes $U_{ij} = \operatorname{Spec} A_{ij}$. We showed in 3.1 that for $g \in B_i$, $f^{-1}(D(g)) = \cup \operatorname{Spec}(A_{ij})_{\overline{g}}$. Recall from 2.13(b) each $\operatorname{Spec}(A_{ij})_{\overline{g}}$ is quasi-compact.

Note that a finite union of quasi-compact space is quasi-compact: given cover of the union, they induce a cover for each element which has a finite cover; take the collection of those finite cover and we have a finite cover for the union. Hence $D(g)$ is quasi-compact.

Let $V = \operatorname{Spec} B$ open in Y. Then $V \cap V_i$ is covered by subsets of form $D(g)$ where $g \in B_i$. Since V is quasi-compact (2.13(b)), it is a finite union of $D(g)$. Now $f^{-1}(V) = \cup f^{-1}D(g)$ is a finite union of quasi-compact space, which means it is quasi-compact.

II.3.3 x

(a) Suppose locally finite type and quasi-compact, then by definition it is finite type.

Suppose finite type, then by definition it is locally finite type. Each $f^{-1}(V_i)$ is a finite union of open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$. Since U_{ij} are quasi-compact (2.13(b)), $f^{-1}(V_i)$ is too, as shown in 3.2. Hence f is quasi-compact.

(b) Follows from 3.1 and 3.2.

(c) We know $f^{-1}(V)$ is covered by finitely many open affines $U_j = \operatorname{Spec} A_j$, where A_j finitely generated B-algebra. Let $U = \operatorname{Spec} A \subseteq f^{-1}(V)$. Then U is covered by $D(g) \subseteq U_j$ where $(A_j)_g = A_g$. Since A_j are finitely generated, $(A_j)_g$ are (by adding $1/g$ to basis). Hence we can assume A_{g_i} finitely generated for some $g_1, ..., g_n$ such that $(g_1, ..., g_n)$ is the unit ideal.

Let $a_{ij}/g_i^{m_j}$ be the generators of A_{g_i}. Then for any $a \in A$, we can write

$$a = F_i(a_{i1}/g_i^{m_1}, ..., a_{in_i}/g_i^{m_{n_i}}) = F_i(g, a_{i1}, ..., a_{in_i})/g_i^{M_i}$$

$$g_i^M a = F_i(g_i, a_{i1}, ..., a_{in_i})$$

for some polynomial F_i over B in n_i variables.

Say $a_1 g_1 + ... + a_n g_n = 1$, then $(a_1 g_1 + ... + a_n g_n)^{nM} a = a$, but the terms on the left hand side are all multiples of $g_i^M a$, and therefore arbitrary a is generated by $a_1, ..., a_n, g_1, ..., g_n, a_{11}, ..., a_{nn_n}$, and A finitely generated.

II.3.4 x

Suppose f is finite, with Y covered by $V_i = \operatorname{Spec} B_i$, $f^{-1}(V_i) = U_i = \operatorname{Spec} A_i$. We have maps $\varphi_i : B_i \to A_i$. Let $V = \operatorname{Spec} B \subseteq Y$. Note that for each $g \in B_i$, $f^{-1}(D(g)) = D(\varphi(g)) = \operatorname{Spec}(A_i)_{\varphi(g_j)}$ is also affine, and observe that since A_i is finite B_i module, $(A_i)_{\varphi(g_j)}$ is finite $(B_i)_{g_j}$ module.

So we can assume that a finite subset of V_i form a cover for V, and then we would have $D(g) = D(\overline{g})$ for any $g \in B_i$ where $\overline{g}$ is image of g in B. To summarize, V is covered by finitely many $D(\overline{g_j})$ where $\overline{g_j} \in B$ and $f^{-1}(D(\overline{g_j})) = D(\varphi_i(g_j)) =: \operatorname{Spec} A_j$. and $\overline{g_j}$ generate the unit ideal of B.

Let $A = \Gamma(U, \mathcal{O}_X|_U)$. Using 2.4 we get map $\varphi : B \to A$ that is compatible with φ_i via restriction map of sheafs. Note that $U_{\varphi(\overline{g_j})}$ (notation from 2.16), which corresponds to elements of U where the stalk of $\overline{g_j}$ is not in the maximal ideal. Using the fact that $f^\#$ is local homomorphism, this means $U_{\varphi(\overline{g_j})} = \operatorname{Spec} A_j$. Finally note that $\varphi(\overline{g_j})$ generates unit ideal of A, so by 2.17, U is affine.

Now $A_{\varphi(g_j)}$ is a finite B_{g_j} module. A similar argument as 3.3(c) shows A is a finite B-module.

II.3.5 x

(a) Let y be contained in $V = \operatorname{Spec} B$, $f^{-1}(V) = U = \operatorname{Spec} A$, then restrict f and assume $f : U \to V$. f is induced by $\varphi : B \to A$ where A is finite B-module.

By 3.10, $f^{-1}(y)$ is homeomorphic to the fibre $X_y = X \times_Y \operatorname{Spec} k(y) = \operatorname{Spec}(A \otimes_B k(y))$ (by construction in thm 3.3), where $k(y) = B_\mathfrak{p}/\mathfrak{p}$ is viewed as a B-algebra. We see $A \otimes_B k(y)$ is a finite dimensional $k(y)$-module (vector space) because A is finite over B. Write $R = A \otimes_B k(y)$, then $R/\mathfrak{p}$ is integral domain. Now for each $r \in R$, $\{1, ..., r^n\}$ is linearly dependent and gives minimal polynomial $f(x) \in k(y)[x]$, and giving a way to define r^{-1}. Hence $R/\mathfrak{p}$ is a field, so all prime ideals of R are maximal. Since ideals are subspaces, the ring R is artinian. Recall that artinian rings have finitely many maximal ideals, so $\operatorname{Spec} R$ is finite.

Proof: suppose $\mathfrak{m}_1, \mathfrak{m}_2, ...$ are maximal ideals, then $\mathfrak{m}_1 ... \mathfrak{m}_{k-1} = \mathfrak{m}_1 ... \mathfrak{m}_k \subseteq \mathfrak{m}_k$ implies $\mathfrak{m}_j \subseteq \mathfrak{m}_k$ for some $j \leq k-1$, a contradiction, so $\mathfrak{m}_1, \mathfrak{m}_1\mathfrak{m}_2, ...$ is a strict descending chain.

(b) First show $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is closed. Consider

$$\varphi : B \to B/\ker\varphi \to A$$

The map $B \to B/\ker\varphi$ is surjective and induces $\operatorname{Spec}(B/\ker\varphi) \to \operatorname{Spec} B$ whose image is closed by 2.18(c). We need to show $\operatorname{Spec} A \to \operatorname{Spec} B/\ker\varphi$ is closed. So we may assume $B \to A$ is injective.

Since A is finite over B, A is integral extension of B important! Let $V(\mathfrak{a}) \subseteq \operatorname{Spec} A$, $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}') \in f(V(\alpha))$, and $\mathfrak{q} \supseteq \mathfrak{p}$. By going up theorem, there is some $\mathfrak{q}'$ over $\mathfrak{q}$, that also contains $\mathfrak{p}'$, so $\mathfrak{q}' \in V(\mathfrak{a})$ and $\mathfrak{q} \in f(V(\mathfrak{a}))$. Hence $f(V(\mathfrak{a}))$ is closed under specialization. We shall show it is exactly $V(\varphi^{-1}(\mathfrak{a}))$, by showing all minimal primes $\mathfrak{p}$ over $\varphi^{-1}(\mathfrak{a})$ is contained in it. Consider the injection $B' = B/\varphi^{-1}(\mathfrak{a}) \to A' = A/\mathfrak{a}$, then $B'_\mathfrak{p} \to A'_\mathfrak{p}$ is injection since localization is exact, which means $A'_\mathfrak{p} \neq 0$ and its maximal ideal must contract to $\mathfrak{p}$ in $B'_\mathfrak{p}$ as it is the unique prime ideal, which gives us a prime ideal $\mathfrak{q}$ in A containing $\mathfrak{a}$ that pulls back to $\mathfrak{p}$, as required.

Now given $f : X \to Y$ finite, say $Y = \cup V_i$, $f^{-1}(V_i) = U_i$ where U_i, V_i affine. Observe a subset is closed iff it is closed in each element of the cover, so we are done.

(c) Consider $\operatorname{Spec} k[x,y]/(xy-1)$ glued together with $\operatorname{Spec} k = \{0\}$, then project onto the x-axis $\operatorname{Spec} k[x]$ with map induced by $k[x] \to k[x,y]/(xy-1)$ and $k[x] \to k, x \mapsto 0$. Then $\operatorname{Spec} k[x,y]/(xy-1)$ is closed but is mapped to $\mathbb{A}^1_k - \{0\}$ not closed, so by (b), f not finite.

II.3.6 function field

The generic point is contained in some open affine subset, and since stalk on restriction does not change, need to show $\mathcal{O}_\mathfrak{p}$ of $\operatorname{Spec} A$ is field. Since generic, $\mathfrak{p}$ is contained in every prime ideal, namely $\mathfrak{p} = \operatorname{nil} A$. Since integral scheme, it is reduced so $\operatorname{nil} A = 0$. Hence $K(X) = A_\mathfrak{p} = A_0$ is the fraction field of A.

Now if $U = \operatorname{Spec} B$ is open affine subset of X but ξ is not in U, then the closure of ξ does not intersect U, a contradiction. Hence $\xi \in U$ and $K(X)$ is the fraction field of B.

II.3.7 x

Take $V = \operatorname{Spec} B \subseteq Y$ open, since irreducible, it is dense. We showed in 3.6 V must contain generic point. Since finite type, $f^{-1}(V)$ is covered by finitely many $U_i = \operatorname{Spec} A_i$. So $f^{-1}(V) \to V$ is dominant, generically finite, and finite type. So we can assume $Y = \operatorname{Spec} B$, $X = f^{-1}(V)$. Pick one of U_i and call it $\operatorname{Spec} A$.

Now A is finitely generated B-algebra, so $K(B) \otimes_B A = B^{-1}A$ it is finitely generated $K(B)$-algebra. By Noether normalization lemma, $B^{-1}A$ is a finitely generated $K(B)[y_1, ..., y_d]$-module where y_i are algebraically independent. So $B^{-1}A$ is integral over $K(B)[y_1, ..., y_d]$ and by going up, the induced morphism $\operatorname{Spec} B^{-1}A \to \operatorname{Spec} K(B)[y_1, ..., y_d]$ is surjective.

Let $g = f|_{\operatorname{Spec} A}$. By 3.10 $g^{-1}(\eta) = \operatorname{Spec} A_i \times_Y \operatorname{Spec} k(\eta) = \operatorname{Spec}(A_i \otimes_B K(Y))$, which only has finitely many points. Note that if $d > 0$, then $(f(y_1))$ would be prime ideal for any f irreducible over $K(B)$ (infinite because irreducible polynomials correspond to the algebraic closure of $K(B)$ which is infinite). Hence $d = 0$ and $B^{-1}A$ is finite $K(B)$-module. Thus $K(A) = K(B^{-1}A)$ is finite field extension of $K(B)$. WARNING: it is only a vector space. For it to be a field extension, need to show $K(B)$ injects into $K(A)$, this is where we use dominant: f factors as

$$f : \operatorname{Spec} A \to \operatorname{Spec}(B/\ker\varphi) \to \operatorname{Spec} B$$

where $\operatorname{Spec}(B/\ker\varphi) \to \operatorname{Spec} B$ homeomorphism onto a closed set of $\operatorname{Spec} B$. Since f dominant, it must be surjective and all prime ideals containing $\ker\varphi$ correspond to prime ideals in B. Since Y integral, the intersection of primes is 0, and therefore $\ker\varphi$ is 0 and $B \to A$ is injective; therefore $K(B) \to K(A)$ injective.

Now take $\operatorname{Spec} A$ to be an open set of form $D(f)$ in $\cap \operatorname{Spec} A_i$ (finite intersection, so open, non-empty because they all contain generic point). Then A is finitely generated by a_i as a B-algebra. Since A finite extension of $K(B)$, $f_i(a_i) = 0$ for some f_i with coefficients in $K(B)$. Clear denominator to get $g_i(a_i) = 0$ with g_i coefficients in B. Let c be the product of the leading coefficients of g_i. Then a_i are all integral over B_c, thus $B_c \otimes_B A = A_c$ is finitely generated B_c-module where c refers to image of c in A too. Now observe that $D(c) = \operatorname{Spec} B_c$ inside Y is open and its inverse is $\cup \operatorname{Spec}(A_i)_c = \operatorname{Spec} A_c$ (the union can be ignored because we picked $\operatorname{Spec} A$ to be in the intersection of all $\operatorname{Spec} A_i$, and $D(c)$ inside all the $\operatorname{Spec} A_i$ is just the same set as $D(c)$ in $\operatorname{Spec} A$).

In conclusion, $\operatorname{Spec} B_c$ is the required open dense subset whose inverse image is $\operatorname{Spec} A_c$, finite as required.

II.3.8 normalization

Let U, V be open affine in X, $\tilde{U} = \operatorname{Spec} \tilde{A}, \tilde{V} = \operatorname{Spec} \tilde{B}$. To glue, need isomorphism $U' \to V'$, where $U' = i^{-1}(U \cap V)$, and $i : \tilde{U} \to U$ is induced map by $A \to \tilde{A}$. Cover $U \cap V$ by $D(f) = \operatorname{Spec} A_f = \operatorname{Spec} B_f$. Back in I.3.17 we showed localization commutes with integral closure. So $\operatorname{Spec}(\tilde{B})_f = \operatorname{Spec} \tilde{B_f} = \operatorname{Spec} \tilde{A_f} = \operatorname{Spec}(\tilde{A})_f$. Since all the f generate the unit ideal, they generate the unit ideal in the integral closure, so U', V' are indeed covered by $\operatorname{Spec} \tilde{A}_f, \operatorname{Spec} \tilde{B}_f$. Now we can glue.

Since we have maps $\tilde{U} \to U$, they glue to map $\tilde{X} \to X$. We saw in 3.7 that a dominant map on integral shcemes is induced by injection, so given $f : Z = \cup \operatorname{Spec} A_i \to X = \cup \operatorname{Spec} B_j$, where A_i integrally closed, they are induced by $\varphi : B_j \to A_i$. Since φ injective, this extends to $\varphi : K(B_j) \to K(A_i)$, which restricts to unique maps on $\tilde{B}_j \to \tilde{A}_i = A_i$ (image of integral element is again integral by factoring through polynomial). Thus $B_j \to \tilde{B}_j \to A_i$. Glue to see that f factors unique through $Z \to \tilde{X} \to X$.

If X is finite type over k, say X covered by $\operatorname{Spec} A$ where A finitely generated k-algebra. By Thm I.3.9A, $\tilde{A}$ is finitely generated A-module, so $\operatorname{Spec} \tilde{A} \to \operatorname{Spec} A$ finite.

II.3.9 the topological space of a product

(a) From construction in theorem 3.3, we know $\mathbb{A}^1_k \times_{\operatorname{Spec} k} \mathbb{A}^1_k = \operatorname{Spec}(k[x] \otimes_k k[x]) = \operatorname{Spec} k[x, y] = \mathbb{A}^2_k$. Given prime ideal $(f(x)), (g(y))$ in $\mathbb{A}^1_k \times \mathbb{A}^1_k$, we get prime ideal $(f(x), g(y))$ in $\mathbb{A}^2_k$, but in the other direction, given $(x - y) \in \mathbb{A}^2_k$, its projection is (0) onto $\mathbb{A}^k_1$. So the space $\mathbb{A}^2_k$ is not the product set.

Open sets are different: All open sets of $\mathbb{A}^1_k$ are of form $D(f)$ since $k[x]$ is PID. Given $D(f) \times D(g)$, where f, g are polynomials in $k[x]$, we can find irreducible polynomial h that is not divisible by fg (because the algebraic closure of k is infinite). Therefore $(h, h) \in D(f) \times D(g)$. Now consider open set $D(y - x) = D(x \otimes 1 - 1 \otimes x) \subseteq \mathbb{A}_k^2$ which can not contain (h, h) for any h, thus the topology of $\mathbb{A}_k^2$ is different from product topology.

(b) The product is $\operatorname{Spec}(k(s) \otimes_k k(t))$. Note that $k(s) \otimes_k k(t)$ are sums of rational polynomials $f(s) \otimes g(t)$, which corresponds to the set of $f(s,t)/g(s)h(t)$ for arbitrary $f, g, h \in k[x]$. Thus $k(s) \otimes_k k(t) = (k[s]k[t])^{-1}k[s,t]$. The spectrum would then correspond to the set of prime ideals in $k[s,t]$ that do not intersect $k[s]k[t]$, so we exlude all the maximal ideals (points) and verticle and horizontal lines, giving us irreducible curves of degree at least 2, or lines of form $as + bt + c$ where $a, b \neq 0$. As a result, every point is closed in this scheme. generalize this for k not algebraic

II.3.10 fibres of a morphism

(a) By glueing, we can assume $X = \operatorname{Spec} A$. Since $k(y)$ does not change when we restrict, we can assume $Y = \operatorname{Spec} B$. Thus f induces $\varphi : B \to A$ with $f^{-1}(y)$ is the set of primes $\mathfrak{q} \subseteq A$ such that $\varphi^{-1}(\mathfrak{q}) = y$.

On the other hand, $X_y = \operatorname{Spec} A \times_{\operatorname{Spec} B} \operatorname{Spec} k(y) = \operatorname{Spec}(A \otimes_B k(y))$, with the following commutative diagram

$$\begin{tikzcd} & A \otimes_B k(y) & \\ A \arrow[ur] & & k(y) \arrow[ul] \\ & B \arrow[ul, "\varphi"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

where $\psi : B \to B_y/y = k(y)$. Observe a prime ideal in $A \otimes_B k(y)$ corresponds to a prime ideal $\mathfrak{q}$ in A and a prime ideal in $k(y)$. A prime ideal in $k(y)$ is just 0, so $\psi^{-1}(0) = y$. By the commutative diagram we see $\varphi^{-1}(\mathfrak{q}) = y$. This means the image of $p_1 : \operatorname{Spec}(A \otimes_B k(y)) \to \operatorname{Spec} A$ is $f^{-1}(y)$. It remains to show p_1 is a homeomorphism onto the image. Now

$$p_1(V(\mathfrak{a})) = \{p_1(\mathfrak{p}) : \mathfrak{p} \supseteq \mathfrak{a}\} = \{\mathfrak{q} \in p_1(\operatorname{Spec}(A \otimes_B k(y))) : \mathfrak{q} \supseteq \varphi_1^{-1}(\mathfrak{a})\} = V(\varphi_1^{-1}(\mathfrak{a})) \cap p_1(\operatorname{Spec}(A \otimes_B k(y)))$$

is closed. The second equality requires $\varphi_1^{-1}(\mathfrak{p}) \supseteq \varphi_1^{-1}(\mathfrak{a}) \Rightarrow \mathfrak{p} \supseteq \mathfrak{a}$. Let $\alpha \in \mathfrak{a}$. Observe that elements of $A \otimes_B k(y) = A \otimes_B (B/y)_y$ can be written as

$$\sum x_i \otimes a_i/b_i = \sum (1 \otimes \frac{1}{b_i})(\frac{1}{a_i} x_i \otimes 1) = \sum (1 \otimes \frac{1}{\prod b_i})(\frac{1}{\prod_{j \neq i} b_j a_i} x_i \otimes 1) = (1 \otimes 1/b)\varphi_1(x)$$

where b_i, a_i are not inside y because otherwise we would have $(x_i \otimes a_i/b_i) = 0$ and we can omit it. Now we can write $\alpha = (1 \otimes 1/b)\varphi_1(x)$ and therefore $\varphi_1(x) \in \mathfrak{a}$ (because $(1 \otimes 1/b)$ is a unit) and $x \in \varphi_1^{-1}(\mathfrak{a}) \subseteq \varphi_1^{-1}(\mathfrak{p})$. Now we see $\alpha \in \mathfrak{p}$. Hence $\alpha \subseteq \mathfrak{p}$ as required.

(b) Let y be the idel $(s - a)$. X_y is homeomorphic to $f^{-1}(y)$, and $f^{-1}(s - a)$ is the set of prime ideals $\mathfrak{p}$ in $k[s,t]/(s - t^2)$ that has inverse image $(s - a)$. This means $s - a \in \mathfrak{p}$ and the only such prime ideals (irreducible subvarieties) are $P = (s - a, t - \sqrt{a}), Q = (s - a, t + \sqrt{a})$. The residue field of then would then be the residue field of $\mathcal{O}_P$ and $\mathcal{O}_Q$ (as in varieties) which we know is k.

When $a = 0$, we have $f^{-1}(y) = \{(s,t)\}$ has one point. In fact,

$$\begin{aligned} X_y &= \operatorname{Spec}(k[s,t]/(s - t^2) \otimes_{k[s]} k(y)) = \operatorname{Spec}(k[s,t]/(s - t^2) \otimes_{k[s]} k[s]/(s - a)) \\ &= \operatorname{Spec} k[s,t]/(s - t^2, s - a) = \operatorname{Spec} k[t]/(t^2 - a) \end{aligned}$$

When $a = 0$, $t^2 = 0$ in $k[t]/(t^2)$ so it is not reduced.

When η is generic, namely $\eta = (0)$, we have $k(\eta) = k(s)$. X_η can only contain the zero ideal, namely $(s - t^2)$ in $k[s,t]/(s - t^2)$.

$$X_\eta = \operatorname{Spec}(k[s,t]/(s - t^2) \otimes_{k[s]} k(s)) = \operatorname{Spec}(k[s])^{-1}k[s,t]/(s - t^2)$$

the quotient field is then $K(k[s,t]/(s - t^2)) = k(t)$ over $k(s)$ where $k(s)$ is embedded in $k(t)$ with $s \mapsto t^2$. This means it is degree 2 field extension over $k(s)$ (the minimal polynomial of t over $k(s)$ is $t^2 - s$).

II.3.11 closed subschemes

(a) We can assume X', X, Y affine and then glue. Say $f : \operatorname{Spec} A \to \operatorname{Spec} B$, then since closed immersion, by 2.18, $\varphi : B \to A$ is surjective and $A = B/\ker\varphi$. Now we have $B' \to B' \otimes_B A = B'/\ker\varphi B'$ is surjective.

(b)* Embed Y into X by the closed immersion. Cover Y by $\operatorname{Spec} B_j$. Since X quasi-compact, cover $\operatorname{Spec} B_i$ by $D(f_{ij})$. Add in a cover $D(f_k)$ for $X - Y$ to get a cover for X, then take a finite subcover. Now we have Y is covered by finitely many open affine $D(f_i)$, where $f_1, ..., f_r$ generate unit ideal of A. Since restirction does not change stalk, we know $D(f_i)$ are of form Y_{f_i} where f_i are viewed as image in $B = \Gamma(Y, \mathcal{O}_Y)$ (using 2.4). By 2.17 we know Y is affine. By 2.18(d) we know $B = A/\ker\varphi$.

(c) Let U_i be affine cover of X and let $Y_i = f^{-1}(U_i)$. We need to show $Y_i \to U_i$ factors through Y_i'. Assume $X = \operatorname{Spec} A$ affine. Then $Y = V(I) = Y' = V(J)$ where $J \subseteq I$. By previous part we know $Y = \operatorname{Spec} A/I$, $Y' = \operatorname{Spec} A/J$. Now $A \to A/I$ factors through A/J because $J \subseteq I$, as required.

(d) By glueing, assume $Z = \operatorname{Spec} A$, $X = \operatorname{Spec} B$. f is induced by $\varphi : B \to A$, which factors as $B \to \varphi(B) \to A$. Now $\operatorname{Spec}\varphi(B) \to X$ is a closed immersion and $Z \to \operatorname{Spec}\varphi(B)$ factors through. Assign it the reduced induced subscheme structure.

Suppose A is reduced. Then Y is closed containing the image of Z. Suppose $D(f)$ does not intersect image of Z, which means $\varphi^{-1}(\mathfrak{p})$ always contain f for any prime $\mathfrak{p} \subseteq A$. This means $f \in \cap f^{-1}(\mathfrak{p}) = f^{-1}(\cap\mathfrak{p}) = f^{-1}(\operatorname{nil} A) = f^{-1}(0)$, so $f \in \ker\varphi$ and as a result, $f^{-1}(\mathfrak{p})$ contains f for any prime ideal of the ring $\varphi(B)$. Hence $D(f)$ does not intersect Y, and Y is the closure of $f(Z)$.

II.3.12 closed subschemes of $\operatorname{Proj} S$

(a) Since φ surjective and perserve degree, $\varphi(S_+) = T_+$, so $U = \{\text{primes that do not contain all of } T_+\} = \operatorname{Proj} T$ by definition.

Similar proof as 2.18(c) (localized maps $\varphi_{(\mathfrak{p})}$ is surjective because degree perserving).

(b) Let $J = I' \cup I$, then we have maps $S/I \to S/J$ and $S/I' \to S/J$ which are isomorphisms on degree $d \geq d_0$. Thus by 2.14(c), $\operatorname{Proj}(S/I) = \operatorname{Proj}(S/J) = \operatorname{Proj}(S/I')$. Since $S \to S/J$ factors through S/I and S/I', we see they all determine the same subscheme.

II.3.13 properties of morphisms of finite type

(a) Let $f : X \to Y$ closed immersion. Observe restriction to open subset also closed immersion Restriction of sheaf map is surjective because sheafification of restriction is the same as restriction of sheafification: Consider sheafification $\mathscr{F}|_U \to \mathscr{F}|_U^+$ and map $\mathscr{F}|_U \to \mathscr{F}^+|_U$ extends to $\mathscr{F}|_U^+ \to \mathscr{F}^+|_U$, since on stalks, it is composition of isomorphisms, so the whole map is isomorphism. So by 3.11(b) inverse of open affine is open affine. In particular, $f^{-1}(\operatorname{Spec} A) = \operatorname{Spec} A/\mathfrak{a}$ and $A/\mathfrak{a}$ is a finitely generated A-module. Hence finite.

(b) By 3.3(a) suffices to show it is locally finite type. Let $V = \operatorname{Spec} B$ open in Y. Since open immersion, $f^{-1}(V)$ is isomorphic to $V \cap f(X)$ open, which is covered by $D(f) = \operatorname{Spec} B_f$. Now B_f is finitely generated B-algebra (by $1/f$).

(c) Follows from 3.3(b) and if A is finitely generated B-algebra and B is finitely generated C-algerba, then A is finitely generated C-algebra (take union of generators).

(d) Assume X, S, S' affine then glue. Say A is finitely generated B-algebra and $\operatorname{Spec} A \to \operatorname{Spec} B$ is extended to $\operatorname{Spec}(A \otimes_B B') \to \operatorname{Spec} B'$. Since we have $B \to B'$, $A \otimes_B B'$ is a finitely generated B'-algebra (tensor generators with 1).

(e) Suffices to show affine case. Say A, B are finitely generated C-algebra with generators $\{a_i\}, \{b_j\}$. Consider $A \otimes_C B$ (generated by pure tensor) is generated by $\{a_i \otimes 1, 1 \otimes b_j\}$

(f) Let $V_i = \operatorname{Spec} B_i$ open affine cover of Y. Then $f^{-1}(V_i)$ is quasi-compact and is finite union of $\operatorname{Spec} A_{ij}$. Given any $\operatorname{Spec} C \subseteq Z$, $g^{-1}(\operatorname{Spec} C)$ is covered by open sets of form $\operatorname{Spec}(B_i)_h$, and observe $f^{-1}(\operatorname{Spec}(B_i)_h)$ is covered by $\operatorname{Spec}(A_{ij})_h$ where h is viewed as image of $h \in B_i$ in A. By 3.3(c) and $g \circ f$ is finite type, we know $(A_{ij})_h$ is finitely generated C-algebra. Since C embeds into $(B_i)_h$, $(A_{ij})_h$ is a finitely generated $(B_i)_h$-algebra. This means $(A_{ij})_h$ is finitely generated B_i-algebra. Again since $f^{-1}(V_i)$ is quasi-finite, and since we can go through $\operatorname{Spec} C$ (and cover all of Z) to get a collection of $\operatorname{Spec}(A_{ij})_g$ that covers all of $\operatorname{Spec} A_{ij}$. Now pick a finite subcover and we see f is finite type.

(g) If Y noetherian, it is covered by finitely many noetherian $\operatorname{Spec} B_i$ and each pulls back to finitely many $\operatorname{Spec} A_{ij}$ such that A_{ij} finitely generated B_i-algebra. Write $A_{ij} = B_i[y_1, ..., y_r]/\ker\varphi$ where φ is the natural surjection, then by Hilbert basis theorem and that quotient of noetherian is noetherian, A_{ij} are and thus X is.

II.3.14 x

Since being closed in cover implies closed, we know closed points are all the closed points in affine cover. Thus can assume $X = \operatorname{Spec} A$. Now since X finite type over k, A is finitely generated k-algebra by 3.3(c). The closed points are maximal ideals. Therefore we need to show the intersection of all maximal ideals is contained in the intersection of all prime ideals, namely the jacobson radical is equal to nilradical. Note that finitely generated algebra over jacobson ring is jacobson ring, use generalized nullstellensatz on $k[x_1, ..., x_n]$ then take quotient? so we are done.

Embed $k[x]_{(x)}$ with maximal ideal (x) and nilradical (0). Not what we want.

II.3.15 geometrically irreducible/reduced/integral

(a) If X were reducible, then it has open subsets U, V with $U \cap V = \phi$. Let $p_1 : X \times_S Y \to X$, we have $p_1^{-1}(U), p_1^{-1}(V)$ open and disjoint, so $X \times_S Y$ is reducible.

Observe a space is irreducible iff it is covered by irreducible open subspaces that are not disjoint. Note that by our hypothesis X must be irreducible, so $X \otimes_S Y$ is covered by $\operatorname{Spec} A \otimes_S Y$, not disjoint. Thus we can assume $X = \operatorname{Spec} A$ is affine, and A is finitely generated k-algebra.

(i)$\Rightarrow$(ii) Suppose $X \times_k \overline{k}$ irreducible. Note $X \times_k \overline{k} = X \times_k k_s \times_k \overline{k}$ is a base extension of $X \times_k k_s$. So we have projection map $X \times_k \overline{k} \to X \times_k k_s$. Now the image of irreducible space is irreducible, and since $k_s \to \overline{k}$ is injective, $A \otimes_k k_s \to A \otimes_k \overline{k}$ injective and projection is dominant. Now we conclude the whole space is irreducible (if two disjoint open sets, then intersect with the dense image and get the image is reducible, contraidction). Hence $X \times_k k_s$ is irreducible.

(i)$\Rightarrow$(iii) Since every two field extensions has a common extension (take $K \otimes_k L$ mod a maximal ideal), we only need to show $X \times_k K$ irreducible for K extension of $\overline{k}$. Suppose $X \times_k \overline{k}$ is irreducible, observe $A \otimes_k K = (A \otimes_k \overline{k}) \otimes_{\overline{k}} K$. So we can assume k algebraically closed and $\operatorname{Spec} A$ irreducible. Note that for any maximal ideal $\mathfrak{m} \subseteq A$, $A/\mathfrak{m}$ is a finite extension over k, so $A/\mathfrak{m} = k$. Suppose $(\sum a_i \otimes b_i)(\sum a_i' \otimes b_i')$ is nilpotent where we assume the expressions are minimal, i.e. b_i are linearly independent, b_i' linearly independent over k. Observe $A/\mathfrak{m} \otimes_k K = k \otimes_k K = K$, so take quotient in $\mathfrak{m}$, we have $(\sum \overline{a_i} b_i)(\sum \overline{a_i'} b_i')$ nilpotent and since K a field, $(\sum \overline{a_i} b_i) = 0$ or $(\sum \overline{a_i'} b_i') = 0$, so by linear independence $a_i \in \mathfrak{m}$ or $a_i' \in \mathfrak{m}$ for all $\mathfrak{m}$. This splits the set of maximal ideals into two sets, and by 3.14 the closed points is dense, so since irreducible, one of the sets must contain all the maximal ideals, namely $a_i \in \mathfrak{m}$ for all $\mathfrak{m}$ (or α_i'). Since intersection of all $\mathfrak{m}$ is jacobson radical which is equal to nilradical (again by 3.14). As a result $\sum a_i \otimes b_i$ is nilpotent (or a_i'). Hence $A \otimes_k K$ has prime nilradical and spectrum is irreducible.

(ii)$\Rightarrow$(i) If $\operatorname{char} k = 0$, then $\overline{k} = k_s$. Suppose $\operatorname{char} k = p$. Suppose X_{k_s} irreducible. Let $(\sum \overline{a_i} b_i)(\sum \overline{a_i'} b_i')$ be nilpotent in $A \otimes_k \overline{k}$. Since $\overline{k}$ is a purely inseparable extension of k_s, $b_i^{p^{n_i}} \in k_s$ for some n_i. Thus we can take power p^N for N large enough and see $(\sum A_i \otimes B_i)(\sum A_i' \otimes B_i')$ is nilpotent in $A \otimes_k k_s$. Since nilradical of $A \otimes_k k_s$ is prime, we know $\sum A_i \otimes B_i$ is nilpotent (or A_i'), and thus $\sum a_i \otimes b_i$ is nilpotent. Hence the nilradical of $A \otimes_k \overline{k}$ is prime and $X_{\overline{k}}$ irreducible. this argument looks weird, did I miss something?

(b) (iii)$\Rightarrow$(i)$\Rightarrow$(ii) follows from injection $A \otimes k \to A \otimes k'$ for field extension k' over k (nilradical is 0). (i)$\Rightarrow$(iii) follows similar argument as above.

We shall show (ii)$\Rightarrow$(i). Since $X \otimes_k k_p$ is finite type over k_p (finite type stable under base extension), it suffcies to show if A reduced, finitely generated k-algebra, then $A \otimes K$ reduced for $K = \overline{k}$ separable over k. Let $\sum a_i \otimes b_i$ be nilpotent, then b_i have finite degree, so replace K by $k(b_1, ..., b_n)$ and assume K is finite over k. By primitive element theorem, $K = k(\alpha) = k[x]/f$ where f is separable. Observe that since f is separable over any extension over k, $A/\mathfrak{m} \otimes_k K = A/\mathfrak{m}[x]/(f)$ is also reduced. Observe A injects into $\prod A/\mathfrak{m}$ because the jacobson radical is equal to nilradical which is 0 in this case. Since K is flat (Eisenbud prop

6.1), $A \otimes K$ injects into $\prod A/\mathfrak{m} \otimes K$. Note that K is finitely presented k-module (noetherian),

$$R^m \to R^n \to K \to 0$$

since tensoring is right exact,

$$\begin{array}{ccccccc} \prod A/\mathfrak{m} \otimes R^m & \longrightarrow & \prod A/\mathfrak{m} \otimes R^n & \longrightarrow & \prod A/\mathfrak{m} \otimes K & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (\prod A/\mathfrak{m}) \otimes R^m & \longrightarrow & (\prod A/\mathfrak{m}) \otimes R^n & \longrightarrow & (\prod A/\mathfrak{m}) \otimes K & \longrightarrow & 0 \end{array}$$

The left two verticle rows are isomorphism because $(\prod A/\mathfrak{m}) \otimes R^m = (\prod A/\mathfrak{m})^m = \prod (A/\mathfrak{m})^m = \prod A/\mathfrak{m} \otimes R^m$, so by five lemma we have

$$\prod A/\mathfrak{m} \otimes K = \prod (A/\mathfrak{m} \otimes K) = \prod (A/\mathfrak{m}[x]/(f))$$

is reduced (multiplication is componentwise).

(c) Consider $X = \operatorname{Spec} \mathbb{F}_p(t)[x]/(x^p - t)$. Since $x^p - t$ irreducible by Eisenstein criterion, it is a prime ideal and therefore maximal in the PID $\mathbb{F}_p(t)[x]$, so the ring is in fact a field and thus integral. However, extend to $\overline{k} = \overline{\mathbb{F}_p}$ we see $x^p - t = (x - \sqrt[p]{t})^p$, so $X_{\overline{k}}$ not irreducible nor reduced.

II.3.16 noetherian induction

Suppose $\mathscr{P}$ does not hold for X over some closed subset Y_1. If it were to hold for all proper closed subsets of Y_1, it would hold for Y_1, so there is some Y_2 on which it is not true. Continue and we find an infinite descending chain which contradicts noetherian.

II.3.17 zariski spaces

(a) Follows from 2.9.

(b) Let Y be a minimal nonempty closed subset, then it is irreducible and has a unique generic point. However the closure of any point of Y would be Y. Thus by uniqueness it can only contain one point.

(c) Given two points x, y, if the closure of x does not contain y, then the complement is open containing only y. If the closure of x contains y, then note that x is the unique generic point of $\{x\}^-$ (irreducible), so $\{y\}^-$ must not contain x and thus complement is open and only contains x.

(d) Since closure of generic point x is the whole space, there is no non-empty open set that does not contain x.

(e) Closed points iff minimal in partial ordering $x_1 > x_0 \Leftrightarrow x_1 \rightsquigarrow x_0$, simply by definition. Irreducible components of X are maximal irreducible closed subsets of X, whose generic point corresponds to maximal element of the partial ordering by uniqueness.

Closed subset of course contains closure of all its points, so it contains all specialization of its points. Given open set U containing x_0, if $x_1 > x_0$, then $U \cap \{x_1\}^-$ is open nonempty in $\{x_1\}^-$, and therefore by (d) contains x_1, so stable under generalization.

(f) Note that a chain of closed subsets in $t(X)$ corresponds to a chain of closed subsets of X, so $t(X)$ noetherian. Let $t(Y)$ be a irreducible closed subspace of $t(X)$. If $Y = Z_1 \cup Z_2$ is reducible, then $t(Y) = t(Z_1) \cup t(Z_2)$ reducible. Thus Y is irreducible, which means Y is the unique generic point of $t(Y)$. Hence $t(X)$ is Zariski space.

Suppose X is a Zariski space. Then the closure of two points are not the same by uniqueness of generic point, so α injective. Elements of $t(X)$ are irreducible subspaces of X, which are image of their respective generic points so α surjective. Inverse of $t(Y)$ is just Y. Image of closed set Y would be all of $t(Y)$ (because existence of generic point). Hence homeomorphism.

II.3.18 constructible sets

(a) if direction follows from definition. Suppose Y is constructible. Note suppose we have $(U_1 \cap Z_1) \cup ... \cup (U_n \cap Z_n)$, then intersecting with open/closed sets or unioning open/closed sets give a new set of the same form, taking complement also give the same form by distributing the unions and intersections. So by induction on number of operations we see Y is of such form. Observe $(U_1 \cap Z_1) - (U_2 \cap Z_2) = (U_1 \cap (Z_1 - U_2)) \cup ((U_1 - Z_2) \cap Z_1)$ so we can replace with a finer union and repeat until we get a disjoint union of locally open subsets (process stops since X noetherian).

(b) Suppose $Y = (U_1 \cap Z_1) \cup ... \cup (U_n \cap Z_n)$. Suppose dense, then closed set $Z_1 \cup ... \cup Z_n$ is dense and must contain generic point. Since all open sets U_i contain generic point by 3.17(d), Y does too. If Z_i contains the generic point, then we see $Z_i = X$ and Y contains open set U_i.

(c) Suppose S constrictible and stable under specialization, say $S = (U_1 \cap Z_1) \cup ... \cup (U_n \cap Z_n)$. Given a point in Z_i its closure is contained in Z_i, so suffices to show $U \cap Z$ is closed if it is stable under specialization. Since U stable under generalization and contains some point in Z, it contains the generic point of Z. Since $U \cap Z$ stable under specialization, it contains the closure of the generic point of Z, which is Z. Hence intersection is just Z, closed. Observe a set is stable under specialization iff complement is stable under generalization (if $x > y$, y in U, then x not in U implies x in U^C but then specialize and get $y \in U^C$, contradiction). Thus constructible and stable under generalization iff open follows by De Morgan's law plus the closed case.

(d) Follows from inverse image commutes with intersection, union, and complement.

II.3.19 x

(a) By taking finite union, we have $Y = \cup \operatorname{Spec} B_i$. Now cover $f^{-1}(Y) = \cup \operatorname{Spec} A_j$ with finitely many affine open and we see we can assume X, Y affine and only show $f(X)$ constructible. We can reduce X, Y (does not change topological space) and then take Y to be the closure of image (which is scheme theoretic closure by 3.11) and assume f dominant. View closed irreducible subsets of X as closed subscheme with closed immersion $X' \to X$ which is homeomorphism onto image, so we can assume X irreducible, which means closure of image, which is Y, is irreducible. Thus A, B are integral noetherian. In 3.7 we showed dominant maps on integral domains are induced by injections. (after this question, we can just say by 3.18(b) $f(X)$ (constructible) must contain generic point (0) which means $\varphi^{-1}(0) = 0$ and $\ker \varphi = (0)$). Thus B injects into A.

(b)* With induction, can assume B has only one generator b over A. So $B = A[x]/\mathfrak{p}$. If $\mathfrak{p} \neq 0$, b satisfies some degree n $f(x)$ (not necessarily monic) in $A[x]$. Now extend φ by sending $\varphi(x)$ to $1/\sqrt[n]{\varphi(a)}$ times a root of $f(x/\sqrt[n]{a})$ in K, where a is the leading coefficient of f? If $\mathfrak{p} = 0$, send x to ?

Apply this with $b = 1$. Now $\mathfrak{p} \in D(a)$ becomes images of some $\mathfrak{q} \in D(1) = X$, (embedd A into quotient field of $A/\mathfrak{p}$, then image of a is non-zero, but how does this give us $\mathfrak{q}$?) so $f(X)$ contains an open set of Y. i give up. this question is starred so it's ok?

(c) Suppose F is closed in Y and for all G closed proper in F, $f(X) \cap G$ is constructible. If F is reducible, then apply to irreducible components and combine together to get constructible set. So let F be irreducible with generic point $\mathfrak{p}$, so $F = V(\mathfrak{p})$. Now if $f(X) \cap F$ is not dense, say it is contained in some $G \subseteq F$, then $f(X) \cap F = f(X) \cap G$ is constructible. Otherwise, say $F = V(\mathfrak{p}) = \operatorname{Spec} A/\mathfrak{p}$ closed immersion into Y. Note that $f^{-1}(F)$ are those primes in B whose imgage in A contains $\mathfrak{p}$, which is $V(\mathfrak{p}B) = \operatorname{Spec} B/\mathfrak{p}B$. Now we have map $A/\mathfrak{p} \to B/\mathfrak{p}B$ that induces $f^{-1}(F) \to F$ dominant. If $B = A[x]$, then $\mathfrak{p}B$ is prime because $B/\mathfrak{p}B = A/\mathfrak{p}[x]$ integral domain. If $B = A[x]/\mathfrak{q}$, then since A embeds into B, $A \cap \mathfrak{q} = 0$ and the generator of B satisfies some polynomial, which means B is finitely generated A-module, so integral over A, and $\mathfrak{p}$ lifts to some prime in B by going up theorem. Therefore $\operatorname{Spec} B/\mathfrak{p}B$ in any case is homeomorphic to some $B/\mathfrak{q}$ integral domain. Since dominant between integral schemes, $A/\mathfrak{p} \to B/\mathfrak{q}$ is injective, and we can apply part (b) to get a nonempty open set of F contained in $F \cap f(X)$. Take this open set union its complement in F, which is constructible by hypothesis, we get $F \cap f(X)$ is constructible.

(d) Projection of $y - x^2$ onto y-axis.

II.3.20 dimension

(a) By I.1.10(b), $\dim X = \sup \dim \operatorname{Spec} A_i$. On the other hand, if $P \in X \subseteq \operatorname{Spec} A_i$ is closed point (maximal), then

$$\dim \operatorname{Spec} A_i = \dim A_i = \operatorname{height} \mathfrak{m} + \dim A_i/\mathfrak{m} = \operatorname{height} \mathfrak{m} = \dim A_{\mathfrak{m}} = \dim \mathcal{O}_P$$

Since P is arbitrary, and X integral means irreducible means $\operatorname{Spec} A_i$ all intersect, we see $\dim \operatorname{Spec} A_i$ are all equal and $\dim X = \dim \mathcal{O}_P$ for all P.

(b) Follows from 3.6.

(c) Let $Y = V(\mathfrak{a})$

$$\operatorname{codim}(Y, X) = \operatorname{codim}(\operatorname{Spec} A/\mathfrak{a}, \operatorname{Spec} A) = \inf_{\mathfrak{p} \supseteq \alpha} \operatorname{codim}(\operatorname{Spec} A/\mathfrak{p}, \operatorname{Spec} A)$$

$$= \inf_{\mathfrak{p} \supseteq \alpha} \operatorname{height} \mathfrak{p} = \inf_{\mathfrak{p} \in Y} \dim A_{\mathfrak{p}} = \inf_{\mathfrak{p} \in Y} \dim \mathcal{O}_{\mathfrak{p}, X}$$

(d)

$$\dim Y + \operatorname{codim} Y = \dim A/\mathfrak{a} + \inf_{\mathfrak{p} \in Y} \operatorname{height} \mathfrak{p} = \max_{\mathfrak{p} \in Y} \dim A/\mathfrak{p} + \inf_{\mathfrak{p} \in Y} \operatorname{height} \mathfrak{p}$$

which is achieved when $\mathfrak{p}$ is minimal over Y, and we always get $\dim A = \dim X$ as the sum.

(e) By (a) we can give a chain of length $\dim X$ starting at Z_0 such that $Z_0 \cap U \neq \phi$. Then U is open, thus dense in each irreducible Z_i, which means $U \cap Z_i = U \cap Z_j$ implies $Z_i = Z_j$ and $i = j$. Thus $Z \cap U$ is a chian of length n too and $\dim U = \dim X$.

(f) Assume $X = \operatorname{Spec} A$, then $X' = \operatorname{Spec} A \otimes_k k'$ induced by the following

$$\begin{tikzcd}
& A \otimes_k k' & \\
A \arrow[ur] & & k' \arrow[ul] \\
& k \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Now a prime ideal in $A \otimes_k k'$ restricts to 0 in k' and a prime in A, with a similar argument from 3.10(a) shows the prime ideals in $A \otimes_k k'$ are in one to one correspondence with prime ideals of A.

II.3.21 x

(a) R has maximal ideal generated by $u \in R$, which has height 1. $\dim R[t] = 2$ because we have $(0), (u), (u, t)$.

However, $(ut - 1)$ is a maximal ideal because $R[t]/(ut - 1) = R(u^{-1})$ is a field, but its height is 1.

(d) Take $Y = V(ut - 1)$.

(e) Take $U = D(u)$, then $R[t]_u = R_u[t]$ has dimension 1 because R_u is fraction field of R.

II.3.22 dimension of the fibres of a morphism *

Skipped

II.3.23 x

In the affine case, since $t(V) = \operatorname{Spec} A$ where A is the coordinate ring of V, this follows from I.3.15(b). use some sort of universal property (I.3.16(b) and definition of fibred product) for projective case? what is coordinate ring of $X \times Y$ projective (serge embedding)?

II.4 Lecture 4

II.4.1 x

Let $f : X \to Y$ be finite. By Cor 4.6(f), f is separated since Y has affine cover with inverse image of each affine set affine, and $\operatorname{Spec} A \to \operatorname{Spec} B$ has diagonal map $\operatorname{Spec} A \to \operatorname{Spec}(A \otimes_B A)$ induced by $A \otimes_B A \to A$, $a \otimes c \mapsto ac$, which is surjective so closed immersion. By Cor 4.8(f), we need to show $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is universally closed assuming it is finite. Write $A = Bx_1 \oplus ... \oplus Bx_n$. Let $Y' \to \operatorname{Spec} B$ be any morphism and $g : \operatorname{Spec} A \times_{\operatorname{Spec} B} Y' \to Y'$ base extension. Cover Y' with $\operatorname{Spec} C \subseteq Y'$. Then

$$g^{-1}(\operatorname{Spec} C) = \operatorname{Spec} A \otimes_B C = \operatorname{Spec} Cx_1 \oplus ... \oplus Cx_n$$

so we see g is finite, and therefore closed by 3.5(b).

II.4.2 x

Consider $h : X \to Y \times_S Y$ where $p_1 \circ h = f$ and $p_2 \circ h = g$. Since $p_1 \circ h|_U = p_2 \circ h|_U$, h factors through $\Delta : Y \to Y \times_S Y$ (by universal property of $Y \times_S Y$). Thus $h(U)$ is a subset of $\Delta(Y)$. Since Y is separated, $\Delta(Y)$ is closed. Since U dense, $h(X)$ is in $\Delta(Y)$. Take p_i and get $f = g$ on all of X.

Need to show $f^{\#}$ and $g^{\#}$ also agree. It suffices to check on the stalks, which means we can assume $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. Let $\varphi, \psi : B \to A$ be induced by f, g. Let $a = \varphi(b) - \psi(b)$, then $a|_U = 0$ (because $f = g$ on U as morphisms) and $a_{\mathfrak{p}} = 0 \in A_{\mathfrak{p}}$ for any $\mathfrak{p} \in U$, implying $a \in \mathfrak{p}$. Hence $U \subseteq V(a)$ and therefore $V(a) = X$, and a is nilpotent, so $a = 0$ since X reduced.

When X not reduced, say $X = \operatorname{Spec} k[x]/(x^2), Y = \operatorname{Spec} k[x], S = \operatorname{Spec} k$. Then consider the maps induced by $\varphi, \psi : k[x] \to k[x]/(x^2)$ with $x \mapsto 0$, $x \mapsto x$. Then the space map $f = g$ because Y is a single point, but they do not agree as morphisms.

When Y is not separated, say Y is the affine line with double origin. Let f, g map the affine line into Y using each origin. They agree on open dense set $U = Y -$ origin.

II.4.3 x

Observe $U \cap V = \Delta^{-1}(U \times_S V)$. Since X separated, Δ is closed immersion, say $U = \operatorname{Spec} A, V = \operatorname{Spec} B, S = \operatorname{Spec} C$, then $U \cap V = \operatorname{Spec}(A \otimes_C B)$.

II.4.4 x

Consider the graph morphism $\Gamma_f : X \to X \times_S Y$ induced by identity on X and f. We want to show it is closed immersion. Suffice to do it for an affine cover for $X \times_S Y$.

By glueing we can assume $S = \operatorname{Spec} R$ affine (still separated by argument from 4.6(f)). Since S noetherian, and X, Y finite type, they are all noetherian. Let $V_i = \operatorname{Spec} B_i$ cover Y,$U_{ij} = \operatorname{Spec} A_{ij}$ cover V_i. Then $X \times_S Y$ is covered by $U_{ij} \times_S V_j$, and

$$\Gamma_f^{-1}(U_{ij} \times_S V_j) = \Gamma_f^{-1}(U_{ij} \times_S Y) \cap \Gamma_f^{-1}(X \times_S V_j) = U_{ij} \cap f^{-1}(V_j) = \operatorname{Spec} A_{ij}$$

Now we have

$$\Gamma_f : \operatorname{Spec} A_{ij} \to \operatorname{Spec}(A_{ij} \otimes_R B_j)$$

Say f restricted on $\operatorname{Spec} A_{ij}$ is φ, then we see Γ_f is induced by

$$A_{ij} \otimes_R B_j \to A_{ij}$$

$$(x, y) \mapsto x\varphi(y)$$

which is surjective (take $y = 1$). Hence Γ_f is a closed immersion.

Now $f(Z)$ is the image of p_2 of the image of $\Gamma_f(Z)$, which is a closed subset of the subscheme (check) $Z \times_S Y$. Since Z is proper, $Z \to S$ is universally closed, so p_2 is closed and the image is closed.

We need to show $f(Z)$ with reduced induced structure is proper. Suppose $W \to S$ is any morphism. Consider $f(Z) \otimes_S W \to W$. Since $Z \otimes_S Y \to f(Z)$ is a surjection, we only need to show $Z \otimes_S Y \otimes_S W \to f(Z) \otimes_S W \to W$ is closed. This follows from that $Z \otimes_S Y$ is proper (properness stable under base extension) and the map is the projection $(Z \otimes_S Y) \otimes_S W \to W$ (consider commutative diagram).

II.4.5 x

(a) By 2.9 we can take unique generic point of X, say ξ with $\mathcal{O}_\xi = K$. Let R be a valuation ring of K/k, say with center x. So R dominates $\mathcal{O}_{x,X} = \mathcal{O}_{x,\overline{\{\xi\}}}$ and by Lemma 4.4 gives us a morphism T to X, sending $\mathfrak{m}_R$ to x. Since separated, by uniqueness from Theorem 4.3, the center X is unique.

(b) Existence follows from Theorem 4.7.

(c) *

(d) Can assume X affine, and we want to find two centerse in $X = \operatorname{Spec} A$. Since X finite type over K assume $A = k[x_1, ..., x_n]/I$, in particular, A is not a field. Let $a \notin k$ be a non-unit of A. Since k is algebraically closed, a is transcendental, and $k[a^{-1}]_{a^{-1}}$ is a local ring contained in K. It is in fact a valuation ring because the maximal ideal is principal. Now R has no center because if it were to dominate $A_\mathfrak{p}$, then $\mathfrak{p}$ must contain a^{-1} or $\mathfrak{p} = 0$, but then we would have $K \subseteq R \subsetneq K$. This contradicts the existence from (b).

II.4.6 x

Write $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. Let $K = K(X)$, let R be any DVR on K/k dominating $\varphi(B)$, $T = \operatorname{Spec} R$. By Theorem 4.7, there is a map $T \to X$ which induces $A \to R$. It is injective because the image $T \to X$ contains a generic point (so dominant). Therefore A is a subring of all DVR over $\varphi(B)$. By Theorem 4.11A, A is a subring of the integral closure of $\varphi(B)$. Since A is finitely generated algebra over B, it is a finite B-module. because if x integral then $B[x]$ finite, and if M finite over N, N finite over B, then M finite over B

II.4.7 schemes over $\mathbb{R}$ (*)

(a) Uniqueness follows from that

$$\mathbb{R}[x_1, ..., x_n]/I \otimes_\mathbb{R} \mathbb{C} \cong \mathbb{R}[x_1, ..., x_m]/J \otimes_\mathbb{R} \mathbb{C}$$

implies $\mathbb{R}[x_1, ..., x_n]/I \cong \mathbb{R}[x_1, ..., x_n]/J$ (and glueing).

Given $x \in X$, pick affine $\operatorname{Spec} A = \operatorname{Spec} \mathbb{C}[x_1, ..., x_n]/I$ that contains both x and $\sigma(x)$. Note that σ semilinear, induced by some involution $\varphi : A \to A$ with φ acting as conjugation on $\mathbb{C}$. Now let A_0 be the subring fixed by φ. It suffices to identify $\operatorname{Spec} A_0 \otimes_\mathbb{R} \mathbb{C}$ with $\operatorname{Spec} A$ then glue. This requires an isomorphism $A_0 \otimes_\mathbb{R} \mathbb{C} \to A$. Observe that A_0 is generated by $x_1 + \sigma(x_1), ..., x_n + \sigma(x_n)$ not enough? and as a result $A_0 \otimes_\mathbb{R} \mathbb{C}$ is isomorphic to A (via identification involving α, check).

Separateness follows from Theorem 4.3 (at most one extension):

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X_0 \arrow[r, hook] \arrow[d] & X \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \mathbb{R} \arrow[r] & \operatorname{Spec} \mathbb{C} \end{tikzcd}$$

(b) If X_0 is, then we have $X = \operatorname{Spec} A_0 \otimes_\mathbb{R} \mathbb{C}$ affine. If $X = \operatorname{Spec} A$, then by previous part we have $X_0 = \operatorname{Spec} A_0$.

(c) Use universal property, f is obtained by tensoring $f_0 p_1$ and αp_2 (check commutative diagram).

(d) By construction from above $X_0 = \operatorname{Spec} A_0$ where $A_0 = \mathbb{R}[x + \sigma(x)] \cong \mathbb{R}[x]$.

(e) When σ is the conjugate function, then just take $X_0 = \mathbb{P}^1_\mathbb{R}$. Otherwise it swaps the two affine planes, so by our above construction, we need to glue together the real schemes obtained from the affine cover $\operatorname{Spec} \mathbb{C}[x, y]_{(x+y)}$, $\operatorname{Spec} \mathbb{C}[x, y]_{(x-y)}$. Note that

$$\mathbb{C}[x, y]_{(x+y)} = \mathbb{C}[x, y, z]/(z(x + y) - 1)$$

corresponds to

$$\mathbb{R}[x+y, y+x, z]/(z(x+y)-1) \cong \mathbb{R}[x,y]/(xy-1)$$

and

$$\mathbb{C}[x,y]_{(x-y)} = \mathbb{C}[x,y,z]/(z(x-y)-1)$$

corresponds to

$$\mathbb{R}[x+y, y+x, 1/(x-y)+1/(y-x)] = \mathbb{R}[x]$$

How do they glue to a conic in $\mathbb{P}^2_{\mathbb{R}}$?

II.4.8 x

(d) Base change and get $X \times_S X' \to X$. Compose and get $X \times_S X' \to S$. Base change and get $X \times_S \times X' \times_S (Y \times_S Y') = X \times_S X' \to Y \times_S Y'$. (check compatibility of last equality).

(e) $\Delta : Y \to Y \times_Z Y$ is closed immersion since g separate. Base extension and get $X \times_Y Y = X \to X \times_Y Y \times_Z Y = X \times_Z Y$. Since $X \to Z$ has $\mathscr{P}$, base change and get $X \times_Z Y \to Y$ (projection) has $\mathscr{P}$. Compose and we are done.

(f) $f : X \to Y$ factors through $X \to Y_{\text{red}} \to Y$. Since $Y_{\text{red}} \to Y$ closed immersion (because induced morphism bundle-wise surjective ($A \to A_{\text{red}}$)), it is separate and since $X \to Y$ has $\mathscr{P}$, so does $X \to Y_{\text{red}}$ by (e). Since $X_{\text{red}} \to X$ closed immersion, it also has $\mathscr{P}$.

II.4.9 x

Consider serge embedding $\mathbb{Z}[z_{ij}] \to \mathbb{Z}[..., x_i, ..., y_j, ...]$ by $z_{ij} \mapsto x_i y_j$. This is surjective on each affine cover (allowing z_{ij}^{-1}, we get x_k/x_i, y_k/y_j from z_{kj}/z_{ij} and z_{ik}/z_{ij}) so closed immersion from $\mathbb{P}^r_{\mathbb{Z}} \times \mathbb{P}^s_{\mathbb{Z}}$ to $\mathbb{P}^{rs+r+s}_{\mathbb{Z}}$.

Let $f : X \xrightarrow{i} \mathbb{P}^r_Y \to Y$, $g : Y \xrightarrow{j} \mathbb{P}^s_Z \to Z$ projective. Let $id \times j : \mathbb{P}^r_Y = \mathbb{P}^r_Z \times Y \to \mathbb{P}^r_Z \times_Z \mathbb{P}^s_Z \Longrightarrow \mathbb{P}^{rs+r+s}_Z$ closed immersion because j is closed immersion. Now $X \to Z$ factors through $X \to \mathbb{P}^{rs+r+s}_Z \to Z$ where first is a composition of closed immersions. Hence projective.

II.4.10 x

(a) Say L is finite over $K(x_1, ..., x_n)$. Localize $\mathcal{O}[x_1, ..., x_n]$ at $\mathfrak{m}\mathcal{O}[x_1, ..., x_n]$ and get a new noetherian local domain with quotient field $K(x_1, ..., x_n)$ which dominates $\mathcal{O}$. Thus we can assume L is finite over K. Let $x_1, ..., x_n$ independent generators of $\mathfrak{m}$. Then $\alpha = (x_1)$ in $\mathcal{O}' = \mathcal{O}[x_2/x_1, ..., x_n/x_1]$ is not the unit ideal. Let $\mathfrak{p}$ be minimal prime inside α. Then $\mathfrak{p}$ contains some $(\sum_{i>1} r_i x_i) x_1^N$ while not containing $\sum_{i>1} r_i x_i/x_1$. Localize at $\mathfrak{p}$ and divide to get x_1^{N+1} inside $\mathfrak{p}$ and thus $x_1 \in \mathfrak{p}$ and $\mathfrak{m} \subseteq \mathfrak{p}$. Thus $\mathcal{O}'_{\mathfrak{p}}$ dominates $\mathcal{O}$. Now take integral closure, localize at maximal ideal as question suggests. (recall I.6.2A noetherian local domain of dimension 1 is DVR iff it is integrally closed.

(b) Follows from above and the proof of Theorem 4.3.

II.4.11 examples of valuation rings

(a) By I.6.1A and 4.11(a) suffices to show any valuation ring K/k is noetherian. why?

(b) (1) Since we are looking at local ring, consider an affine neighbourhood Spec A of x_1 where x_1 (generic in Y) on X corresponds to a the whole curve Y which is a height 1 ideal $\mathfrak{p}$ in A and $\mathcal{O}_{x_1,X} = A_{\mathfrak{p}}$. Now this is noetherian local domain of dimension 1. Since X nonsingular, by I.5.1 the local ring is regular definition of nonsingular scheme?, and by I.6.2A it is DVR. By 4.5, the center is exactly x_1.

(2) By (1), R is DVR. Since image of Y is $\{x_0\}$, this induces an injection $f^{\#}_{x_0}$ (because birational maps are dominant) that is also a local homomorphism by definition. Thus R dominates $\mathcal{O}_{x_0,X}$, so center at x_0. binational morphism?

(3) R dominates R_0 which dominates $\mathcal{O}_{x_0,X}$ (otherwise there would be come $\mathcal{O}_{x_i,X}$ that does not dominate $\mathcal{O}_{x_0,X}$ which is not possible by the chain of domination) so center X_0. definition of blowing up of scheme?

II.5 Lecture 5

II.5.1 x

recall that when tensoring with free module of finite rank, everything in $\mathscr{E} \otimes \mathscr{F}$ can be written uniquely of form $\sum e_i \otimes f_i$ where $\{e_i\}$ fixed basis. Note tensoring only gives presheaf, but we can find isomorphism of sheaves by first finding a morphism of presheaf to sheaf then check isomorphism on stalks.

(a) Isomorphism of sheaf can be checked locally, so assume free of finite rank. Dual generated by dual basis $\check{e}_i(e_j) = \delta_{ij}$, double dual given by $a \mapsto (\check{e}_i(a))_i$ isomorphic to $\mathscr{E}$.

(b) Again check locally. An element e of $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})(U)$ can be identified with $(f_1, ..., f_n)$ where $f_i = e(e_i) \in \mathscr{F}$. Map this to $\sum \check{e}_i \otimes f_i$. This is isomorphism (on stalks) because uniqueness mentioned above.

(c) Given a map $\varphi : \mathscr{E} \otimes \mathscr{F} \to \mathscr{G}$, define $\varphi' : \mathscr{F} \to \mathscr{H}om(\mathscr{E}, \mathscr{G})$ by sending f to $\psi_f : \mathscr{E} \to \mathscr{G}$ where $\psi_f(e_i) = \varphi(e_i \otimes f)$. Check details for homomorphism, etc.

(d) Check locally, assume $\mathscr{E} = \mathcal{O}_Y^n$. Then

$$f^*(\mathcal{O}_Y^n) = f^{-1}(\mathcal{O}_Y^n) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X = f^{-1}(\mathcal{O}_Y)^n \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X = \mathcal{O}_X^n$$

(passing through direct limit). So

$$f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{O}_Y^n)) = f_*(\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^n) = f_*(\mathscr{F}^n) = f_*(\mathscr{F})^n = f_*(\mathscr{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y^n = f_*(\mathscr{F}) \otimes_{\mathcal{O}_Y} \mathscr{E}$$

II.5.2 x

(a) There are two non-empty open sets, X and the generic point $\{(0)\}$, corresponding to rings R and K. So a $\mathcal{O}_X$-module involves an R-module M and a K-vector space L. We need $\varphi : M \to L$ to be compatible with restriction $\varphi' : R \to K$. Namely $\varphi(rm) = \varphi'(r)\varphi(m)$. This induces a homomorphism $\rho : M \otimes_R K \to L$ by $m \otimes \frac{r}{s} \mapsto \frac{\varphi'(r)}{\varphi'(s)}\varphi(m)$, where we view K as an R-module by r'. Conversely, given such a ρ, we have $\rho(rm \otimes 1) = \varphi'(r)\rho(m \otimes 1)$ automatically is a restriction map that defines the sheaf.

(b) Quasi-coherent on $X = \operatorname{Spec} R$ iff $\mathscr{F} = \tilde{M}$, and $L \cong M_{(0)} = M \otimes_R R_{(0)} = M \otimes_R K$ and restriction map is identified with the localization map. This happens iff ρ is an isomorphism.

II.5.3 x

Given a morphism $\varphi : M \to \Gamma(X, \mathscr{F})$, define sheaf morphism $\psi : \tilde{M} \to \mathscr{F}$ on open sets $D(f)$ then glue. Set $\psi : M_{D(f)} \to \mathscr{F}(D(f))$ where $\psi(m/f^n) = \varphi(m)/f^n$ (this is the only possible map because it has to be an $A_{(f)}$-module homomorphism). Need to check this is compatible with restriction maps. Given $D(g) \subseteq D(f)$ where $g = fh$, we have m/f^n restricts to $h^n m/g^n$, sent to $\varphi(h^n m)/g^n = h^n\varphi(m)/g^n$ by ψ. On the other hand, $\varphi(m)/f^n$ restricts to $h^n\varphi(m)/g^n$ as required. (also check details for how we lift using that it is compatible with restrictions).

Conversely, given an element of $\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathscr{F})$, take global section and get a map in $\operatorname{Hom}_A(M, \Gamma(X, \mathscr{F}))$.

II.5.4 x

Suppose $\mathscr{F}$ quasi-coherent, then it is locally $\tilde{M}$. Since localization goes through quotients, we have $(M/N)^\sim = \tilde{M}/\tilde{N}$. Thus take a free resolution of M, $\oplus_i \mathcal{O}_X \to \oplus_j \mathcal{O}_X \to M$ and we see it is the cokernel of free modules. Converse follows from Prop 5.7.

When X is noetherian, M finitely generated is the same as finitely presented, which means it is the cokernel of free modules of finite rank.

II.5.5 x

(a) Let $A = k, B = k[x]$, $M = k[x]$. Then $f : \operatorname{Spec} B \to \operatorname{Spec} A$ gives $f_*(\tilde{M}) = ({}_A M)^\sim$ by 5.2, but M is not finitely generated when viewed as an A-module.

(b) Same proof as 3.13. Let $f : X \to Y$ closed immersion. Let $V = \operatorname{Spec} A \subseteq Y$ open, $U = f^{-1}(V)$. Then $U \to V$ a closed immersion, say with immage $\operatorname{Spec} A/I$. So $U = \operatorname{Spec} A/I$. Now A/I is finitely generated A-module, therefore f is finite.

(c) By 3.5(b) finite morphisms are closed, so it sends $\operatorname{Spec} B$ to an open set of Y. For any $V = \operatorname{Spec} A \subseteq f(\operatorname{Spec} B)$, let $U = f^{-1}(\operatorname{Spec} A) = \operatorname{Spec} C$. By 5.4, $\mathscr{F}_U = \tilde{M}$ where M is finitely generated C-module. Since f is finite, C is finitely generated A-module. Hence M is finitely generated A-module and $f_*\mathscr{F}$ is coherent.

II.5.6 support

(a) Note that the image of $m \in M$ inside $M_\mathfrak{p}$ is zero when something outside $\mathfrak{p}$ kills m. Thus the support are those ideals that contain $\operatorname{Ann} M$, thus $\operatorname{Supp} m = V(\operatorname{Ann} M)$.

(b) Suppose $M_\mathfrak{p} = 0$, then there exists $\{a_i\} \subseteq M - \mathfrak{p}$ such that each $m \in M$ is killed by some a_i. Note the set of elements annihilated by a_0 form a submodule, and those killed by a_0a_1, etc, giving us an ascending chain. Since A noetherian and M finitely generated, the chain stops and gives us some $a_1...a_N$. This is inside $\operatorname{Ann} M$ because otherwise we can add some a_i to make the set killed by it larger. It is also not inside $\mathfrak{p}$ for $A - \mathfrak{p}$ is closed under multiplication. Therefore $M_\mathfrak{p} = 0$ iff $\mathfrak{p}$ contains $\operatorname{Ann} M$. Hence $\operatorname{Supp} \mathscr{F} = V(\operatorname{Ann} M)$.

(c) Since being closed is a local property, we can check on affine open sets. Then it follows directly from part (b).

(d) Let $U = X - Z$. By Prop 5.4, $\mathscr{F}|_U$ is quasi-coherent, so $j_*(\mathscr{F}|_U)$ is quasi-coherent by Prop 5.8 since X is noetherian. By 1.20(b) and Prop5.7, $\mathscr{H}^0_Z(\mathscr{F})$ is quasi-coherent. Thus it is the sheaf of modules associated to $\mathscr{H}^0_Z(\mathscr{F})(X) = \Gamma_Z(X, \mathscr{F})$. Now it suffices to show $\Gamma_\mathfrak{a}(M) \cong \Gamma_Z(X, \mathscr{F})$.

We need a map

$$\{m \in M : \mathfrak{a}^n m = 0\} = \Gamma_\mathfrak{a}(M) \to \Gamma_Z(X, \mathscr{F}) = \{m \in M : \operatorname{Supp} m \subseteq V(\mathfrak{a}))\} = \{m \in M : \mathfrak{a} \subseteq \sqrt{\operatorname{Ann} m})\}$$

Suppose $\mathfrak{a} \subseteq \sqrt{\operatorname{Ann} m}$. Since A is noetherian, it is finitely generated, so there is N large enough (take $\sum n_i$ for generators) so that $\mathfrak{a}^N \subseteq \operatorname{Ann} m$. Therefore the above two submodules are the same, giving us the isomorphism.

(e) Quasi-coherent case follows from previous argument. Coherent case is similar. From 1.20(b) we get

$$0 \to \mathscr{H}^0_Z(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U)$$

On open affine sets $V = \operatorname{Spec} A$, where A is noetherian since X is, we have $M = \Gamma(V, \mathscr{F}_V)$ finitely generated A-module so it is noetherian, and $\Gamma(V, \mathscr{H}^0_Z(\mathscr{F})|_V)$ as a submodule is therefore also finitely generated. Hence coherent.

II.5.7 x

(a) Say $\mathscr{F}_x$ has basis represented by pairs (U_i, m_i) (finite since coherent). Take intersection $U = \cap U_i$ and $\mathscr{F}_x$ is generated by pairs $(U, m_i|_U)$. Now $x_i|_U$ generates a free submodule of $\mathscr{F}(U)$. Say $\mathscr{F}(U)$ is generated by $m_1, ..., m_n, y_1, ..., y_n$, where $(y_j)_x$ are combinations of $(m_i)_x$ in the stalks. Then we can find V_j such that (V_j, y_j) is combination of (V_j, m_i). Take intersection $V = \cap V_i$ and we see V is the free module generated by $m_1, ..., m_n$, and for any open subsets of V, they must be too by the compatibility of restriction maps.

(b) If direction is by part (a). Suppose $\mathscr{F}$ is locally free. Then for each x there is a neibourhood U with $\mathscr{F}|_U$ free. Since coherent, there is a minimal rank of the free neighbouhoods of x inside U, giving a neighbourhood V where $\mathscr{F}|_V = \oplus \mathcal{O}_X|_V$.

(c) Suppose $\mathscr{F}$ is invertible, we claim $\check{\mathscr{F}} \otimes \mathscr{F} \cong \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F})$ (5.1) is isomorphic to $\mathcal{O}_X$. For each $a \in \mathcal{O}_X(X)$, it induces a map $f(U) : \mathscr{F}(U) \to \mathscr{F}(U)$ which multiplies by $a|_U$. Conversely, consider the following commutative diagram

$$\begin{tikzcd} \mathscr{F}(X) \arrow[r] \arrow[d] & \mathscr{F}(X) \arrow[d] \\ \prod \mathscr{F}(U_i) \arrow[r] & \prod \mathscr{F}(U_i) \end{tikzcd}$$

since the verticle arrows are injective, any map $f : \mathscr{F} \to \mathscr{F}$ is in fact determined by how it acts on the cover this argument also works for stalks and maps $\mathscr{F} \to \mathscr{G}$. Since $\mathscr{F}$ is locally free of rank 1, we can take an affine cover $U_i = \operatorname{Spec} A_i$ of rank 1 and the maps on each open set can only be multiplication by an element of A_i. Say $1 \in A_i$ is mapped to a_i, and a_i glue to a in $\mathcal{O}_X(X)$ (because restrictions compatible with these multiplication maps). Now observe that multiplication by a induces a homomorphism that agrees with f on the cover, so it must be f. Now we have identified $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{F})$ with $\mathcal{O}_X$, as required.

can also just show $\check{\mathscr{F}} \otimes \mathscr{F} \to \mathcal{O}_X$ by $\varphi \otimes m \mapsto \varphi(m)$ is isomorphism on stalks

Suppose $\mathscr{F} \otimes \mathscr{G} = \mathcal{O}_X$. By (b) we check stalks, so can assume $X = \operatorname{Spec} A$, $\mathscr{F} = \tilde{M}, \mathscr{G} = \tilde{N}$, and $M_\mathfrak{p} \otimes N_\mathfrak{p} = A_\mathfrak{p}$. Rewrite and assume $M \otimes N = R$ where R local and M, N finitely generated. This means

$$M/\mathfrak{m}M \otimes_{R/\mathfrak{m}} N/\mathfrak{m}N \cong R/\mathfrak{m}$$

is a one dimensional vector space over $R/\mathfrak{m}$. Since tensoring vector psace multiplies dimension, $M/\mathfrak{m}$ is dimension 1 over $R/\mathfrak{m}$. By Nakayama a minimal set of generator has 1 element m in it and generates M over R. Now $M = R/\operatorname{Ann} m$, but since $M \otimes N = R$, we must have $\operatorname{Ann}(m)R = \operatorname{Ann}(m)M \otimes N = 0$ meaning $\operatorname{Ann} m = 0$ and $M = R$ as required.

II.5.8 x

(a) Suffice to check locally. Assume X affine and $\mathscr{F} = \tilde{M}$ where M has a minimal set of generators of size g, say $\{m_1, ..., m_g\}$. By Nakayama, a minimal set of generator for $M_\mathfrak{p}$ form a basis for $M_\mathfrak{p}/\mathfrak{m}_p = M_\mathfrak{p} \otimes_{A_\mathfrak{p}} (A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}) = M_\mathfrak{p} \otimes_{A_\mathfrak{p}} k(\mathfrak{p})$, so the dimension $\varphi(\mathfrak{p})$ is at most g.

When $n = 0$, we have $\{x \in X : \varphi(x) \geq 0\} = X$ is closed. When $n = 1$, $\{x \in X : \varphi(x) \geq 0\}$ is the support of $\mathscr{F}$, so by 5.6(c) it is closed. Suppose for induction that $P_{n-1} = \{x \in X : \varphi(x) \geq n-1\}$ is closed. Suppose $\varphi(\mathfrak{p}) = n - 1 < g$ for some $\mathfrak{p} \in P_{n-1}$, then we know $\{m_1, ..., m_g\}$ is not a minimal set of generator for $M_\mathfrak{p}$, say $\{m_1, ..., m_{n-1}\}$ is a minimal subset. Now take $M' = M/(Am_1 + ... + Am_{n-1})$ and we have $M'_\mathfrak{p} = 0$. Conversely, if $M'_\mathfrak{p} = 0$, then we know $\{m_1, ..., m_{n-1}\}$ is a set of generators for $M_\mathfrak{p}$, but since $\mathfrak{p} \in P_{n-1}$, it must be minimal. Therefore $\{x \in X : \varphi(x) \geq n\}$ corresponds to

$$\{x \in X : \varphi(x) \geq n-1\} \cap \bigcup_{1 \leq i_1 < ... < i_{n-1} \leq g} \operatorname{Supp}(M/(Am_{i_1} + ... + Am_{i_{n-1}}))^\sim$$

which is a finite union of closed sets by the $n = 1$ case, so closed.

(b) Cover X by disjoint sets $\{x \in X : \varphi(x) = n\}$. It suffices to show that they are all open because then since X is connected, one of them must be all of X and $\varphi(x)$ is constant. Since X is locally free, we have shown in 5.7 that around each point x there is a free neighbourhood of rank equal to the rank of $\mathscr{F}_x$, so we are done.

(c) Let $\{m_1, ..., m_g\}$ be a set of generators for M, then as seen before, for each $\mathfrak{p}$, a subset $\{m_1, ..., m_n\}$ is a minimal set of generators for $M_\mathfrak{p}$. This means we can write $m_j - \sum_{i=1}^n c_i m_i = 0$ for $j > n$ and $c_i \in A_\mathfrak{p}$. For any prime ideal $\mathfrak{q} \subseteq \mathfrak{p}$, we see this relation still holds and $M_\mathfrak{q}$ also has minimal generators $\{m_1, ..., m_n\}$ too.

Suppose $\sum_{i=1}^n b_i m_i = 0$ for some $b_i \in A_\mathfrak{p}$, this relation also holds over $M_\mathfrak{q}$. Note that b_i are non-units because otherwise the set would not be minimal. Clear denominator and get a relation $\sum a_i m_i = 0$ where a_i now must lie in all prime ideals of $A_\mathfrak{p}$, so in its nilradical. Since X is reduced, $a_i = 0$ for all i and we see $M_\mathfrak{p}$ is free.

II.5.9 x

(a) By Prop 5.12(b), $\Gamma_*(\tilde{M}) = \oplus\Gamma(X, \tilde{M}(n)) = \oplus\Gamma(X, (M(n))\tilde{})$. For any homogeneous element m of degree n, map m to the function that sends $\mathfrak{p}$ to $m_\mathfrak{p} \in M(n)_{(\mathfrak{p})}$, which is a function in $\Gamma(X, \tilde{M}(n))$. This gives a homomorphism from M to $\Gamma_*(\tilde{M})$.

(b) Observe that S_0 and S_1 are noetherian, so by Hilbert basis theorem S is noetherian. M is finitely generated over S, so by Prop 5.11(c) $\tilde{M}$ is coherent.

Take a finite filtration

$$0 = M^0 \subseteq ... \subseteq M^r = M$$

where $M^i/M^{i-1} \cong (S/\mathfrak{p}_i)(n_i)$. This gives short exact sequences

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & M_n^{i-1} \arrow[r] \arrow[d] & M_n^i \arrow[r] \arrow[d] & (S/\mathfrak{p}_i)(n_i)_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(X, \tilde{M}^{i-1}(n)) \arrow[r] & \Gamma(X, \tilde{M}^i(n)) \arrow[r] & \Gamma(X, (S/\mathfrak{p}_i)\tilde{\ }(n_i+n)) &
\end{tikzcd}$$

By five lemma and an induction, we see it suffices to show $(S/\mathfrak{p})(n) \to \Gamma_*((S/\mathfrak{p})\tilde{\ }(n))$ is isomorphism on large degree. In particular, we can assume $n = 0$. Note that given a function from X to $\amalg(S/\mathfrak{p})_\mathfrak{q}$ must map any $\mathfrak{q} \notin V(\mathfrak{p})$ to something in $(S/\mathfrak{p})_\mathfrak{q} = 0$. Thus this function can be identified with a function from $V(\mathfrak{p})$ to $\amalg(S/\mathfrak{p})_\mathfrak{q}$, so we can replace $X = \operatorname{Proj} S$ by $\operatorname{Proj} S/\mathfrak{p}$. Thus we have reduced to the case when S is a domain, and we need to show $S(d) \cong \Gamma(X, \mathcal{O}_X(d))$ for large d.

Let $x_1, ..., x_r \in S_1$ be generators of S_1 as A-module. By proof of Theorem 5.19 we let $S' = \oplus_{n\geq 0}\Gamma(X, \mathcal{O}_X(n))$ which contains S and contained in $\cap S_{x_i}$. Then S' is a finitely generated S-module, say generated by $\{z_1, ..., z_t\}$, and we have $yz_i \in S_{\geq n}$ for some large n and all $y \in S_{\geq n}$. Say $r = \sum a_i z_i$ has degree $d > \max \deg z_i + n$, then we must have $a_i \in S_{\geq n}$, and so r is an element of degree d inside $S_{\geq n}$, so it is in fact inside S_d. Now we have $S_d \subseteq S'_d = \Gamma(X, \mathcal{O}_X(d)) \subseteq S_d$, and we are done.

(c) We shall show that $\tilde{\ }$ is faithfully full and each coherent $\mathcal{O}_X$-module is of form $\tilde{M}$ for some M quasi-finitely generated graded S-module. The latter statement follows from part (b) (that $\Gamma_*(\tilde{M}) \approx M$) and $\Gamma_*(\tilde{M})^\sim = \tilde{M}$.

To see faithfully full, note that $\operatorname{Hom}_S(M, N)$ natually gives $\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \tilde{N})$, well defined with respect to the equivalence relation because we can twist up to d, get map, and twist back. The inverse can be obtained by applying Γ and twisting to get a map in $\operatorname{Hom}_S(\Gamma_*(\tilde{M}), \Gamma_*(\tilde{N}))$ which is in the same equivalent class as $\operatorname{Hom}_S(M, N)$.

II.5.10 x

(a) Suppose $s = \sum a_i f_i$ is in the saturation where f_i have degree i. Then $x_i^n s \in I$ for some n and since I is homogeneous, $x_i^n a_i f_i \in I$, therefore $a_i f_i \in \overline{I}$, so $\overline{I}$ is homogeneous.

(b) Let I be the saturation and generated by $s_1, ..., s_n$. Pick N large enough so that $x_i^{N-\deg s_j} s_j \in I_1, I_2$ for all i, j. Now observe that $(I_1)_{\geq Nn}, (I_2)_{\geq Nn}$ must be generated by $\{x_i^{N-\deg s_j} s_j\}$, and by the argument from 3.12, we know they define the same closed subscheme.

Suppose they define the same subscheme, then they define the same sheaf of ideal $\mathscr{I}$ (on the affine cover $D_+(x_i)$ by 5.10). Suppose $x_i^n s = t \in I_1$, then $s/x_i^{\deg s} = t/x_i^{n+\deg s}$ is an element of $\mathscr{I}(D_+(x_i))$, which means there is some $t' \in I_2$ such that $t'/x_i^m = t/x_i^{n+\deg s}$, as a result $x_i^M s = t' x_i^{M+\deg s - \deg t'} \in I_2$ for some large M. Hence $s \in \overline{I_2}$ and $\overline{I_1} = \overline{I_2}$.

(c) Let $Y = \operatorname{Proj} S/I$ with sheaf of ideal $\mathscr{I} = \tilde{I}$. Suppose $x_i^n s \in I$ for some n for each i where s homogeneous, then $s/x_i^{\deg s} \in I_{(x_i)} = \mathscr{I}(D_+(x_i))$, and we can glue to get an element t in $\mathscr{I}(X) = I$. We would have $s/x_i^{\deg s} = t/x_i^{\deg t}$ (image of t in $D_+(x_i)$) for all i, so

$$t x_0^{\deg s - \deg t} = ... = t x_r^{\deg s - \deg t}$$

and we can conclude (cancel out t) $\deg s = \deg t$ and $s = t$. Hence $I = \overline{I}$ is saturated.

(d) Given two different saturated ideals, they must define two different subschemes by (b). Given subschemes, we can get saturated ideals back by (c), which gives us the 1-1 correspondence.

II.5.11 cartesian product

First we need to define projection map $p_1 : \operatorname{Proj}(S \times_A T) \to X = \operatorname{Proj} S$. Note that $S \times_A T$ is generated by $x_i \otimes y_j$ where $x_i \in S_d$, $y_j \in T_d$. Thus we need to define the map on open sets $D_+(x_i \otimes y_j) = \operatorname{Spec}(S \times_A T)_{(x_i \otimes y_j)}$. We shall maps its element to $\operatorname{Spec} S_{(x_i)}$ via the homomorphism

$$S_{(x_i)} \to (S \times_A T)_{(x_i \otimes y_j)}$$

$$s/x_i^f \mapsto (s \otimes y_j^f)/(x_i \otimes y_j)^f = s/x_i^f \otimes 1$$

(we can glue because compatibile on $S_{(x_i x_k)} \to (S \times_A T)_{(x_i x_k \otimes y_j y_\ell)}$). Define projection $p_2 \operatorname{Proj}(S \times_A T) \to Y$ similarly.

Also note that locally we have

$$(S \times_A T)_{(x_i \otimes y_j)} \cong S_{(x_i)} \otimes_A T_{(y_j)}$$

$$(s \otimes t)/(x_i \otimes y_j)^f \mapsto (s/x_i^f) \otimes (t/y_j^f)$$

$$(s x_i^g \otimes t y_j^f)/(x_i \otimes y_j)^{f+g} \leftarrow (s/x_i^f) \otimes (t/y_j^g)$$

so

$$\operatorname{Spec}(S \times_A T)_{(x_i \otimes y_j)} \cong \operatorname{Spec} S_{(x_i)} \times_A \operatorname{Spec} T_{(y_j)}$$

and glueing (from Theorem 3.3?) gives

$$f : \operatorname{Proj}(S \times_A T) \cong X \times_A Y$$

Need $\mathcal{O}(1)$ isomorphic to the pullback f^* of, but since we defined p_1 to take things from $\operatorname{Proj}(S \times_A T)$, this is canceled, $p_1^*(\mathcal{O}_X(1)) \otimes p_2^*(\mathcal{O}_Y(1))$. Suffices to check on stalks. Now $\mathcal{O}(1)_\mathfrak{p} = (S \times_A T)(1)_{(\mathfrak{p})}$ which are the degree 1 elements of $(S \times_A T)_\mathfrak{p}$. Observe the stalks of inverse image are stalks at the image point, namely

$$p_1^*(\mathcal{O}_X(1))_\mathfrak{p} = \mathcal{O}_X(1)_{p_1(\mathfrak{p})} \otimes_{(\mathcal{O}_X)_{p_1(\mathfrak{p})}} (\mathcal{O}_{S \times_A T})_\mathfrak{p} = S(1)_{(p_1(\mathfrak{p}))} \otimes_{S_{(p_1(\mathfrak{p}))}} (S \times_A T)_{(\mathfrak{p})}$$

$$p_1^*(\mathcal{O}_X(1)) \otimes p_2^*(\mathcal{O}_Y(1)) = S(1)_{(p_1(\mathfrak{p}))} \otimes (S \times_A T)_{(\mathfrak{p})} \otimes T(1)_{(p_2(\mathfrak{p}))}$$

Pick affine chart that $\mathfrak{p}$ is in and view it as a prime ideal of $(S \times_A T)_{(x_i \otimes y_j)}$, and use above construction we see it suffices to show

$$(S \times_A T)(1)_{(x_i \otimes y_j),\mathfrak{p}} \cong S(1)_{(x_i),p_1(\mathfrak{p})} \otimes (S \times_A T)_{(x_i \otimes y_j),\mathfrak{p}} \otimes T(1)_{(y_j),p_2(\mathfrak{p})}$$

So we set

$$\frac{s \otimes t}{(x_i \otimes y_j)^f (a \otimes b)} \mapsto \frac{s}{x_i^f a} \otimes 1 \otimes \frac{t}{y_j^f b}$$

where $(a \otimes b) \in (S \times_A T)_{(x_i \otimes y_j)} - \mathfrak{p}$ and as a result, $a \otimes 1, 1 \otimes b \notin \mathfrak{p}$, so $a \notin p_1^{-1}(\mathfrak{p}), b \notin p_2^{-1}(\mathfrak{p})$. The inverse is defined similar as before (split the middle tensor to the sides first). Also note the degree match up so the above map is well-defined and we are done.

II.5.12 x

(a) By glueing assume $Y = \operatorname{Spec} A$ affine? Suppose $i : X \to \mathbb{P}_Y^r$ and $j : X \to \mathbb{P}_Y^s$ immersions with $i^*(\mathcal{O}(1)) = \mathscr{L}, j^*(\mathcal{O}(1)) = \mathscr{M}$. By 5.11, we can lift to $X \xrightarrow{f} \mathbb{P}_Y^r \times \mathbb{P}_Y^s \xrightarrow{g} \mathbb{P}_Y^{r+s+rs}$, with $p_1 : \mathbb{P}_Y^r \times \mathbb{P}_Y^s \to \mathbb{P}_Y^r$, $p_2 : \mathbb{P}_Y^r \times \mathbb{P}_Y^s \to \mathbb{P}_Y^s$. Since Y affine, we can write $\mathbb{P}_Y^r \times \mathbb{P}_Y^s = \operatorname{Proj}(A[x_1, ..., x_r] \times A[y_1, ..., y_s])$ and apply 5.11 and remark 5.16.1 to get $g^*(\mathcal{O}(1)) = \mathcal{O}_{\mathbb{P}_Y^r \times \mathbb{P}_Y^s}(1) = p_1^*(\mathcal{O}(1)) \otimes p_2^*(\mathcal{O}(1))$.

pullback distributes over tensor product (at least for locally free sheafs) proof:

$$\begin{aligned}\mathrm{Hom}(f^*(\mathscr{F}\otimes\mathscr{G}),\mathscr{K}) &= \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}\otimes\mathscr{G}, f_*\mathscr{K}) \\ &= \mathrm{Hom}(\mathscr{G}, \mathscr{H}om(\mathscr{F}, f_*\mathscr{K}))(5.1(c)) \\ &= \mathrm{Hom}(\mathscr{G}, f_*\mathscr{H}om(f^*\mathscr{F}, \mathscr{K}))(\text{by construction from 1.18}) \\ &= \mathrm{Hom}(f^*\mathscr{F}\otimes f^*\mathscr{G}, \mathscr{K})\end{aligned}$$

So

$$f^*(\mathcal{O}(1)) = f^*p_1^*(\mathcal{O}(1))\otimes f^*p_2^*(\mathcal{O}(1))$$

composition of pullback is pullback of composition (clear for pushforward, so a similar argument as above shows it for pullback)

$$f^*(\mathcal{O}(1)) = (p_1\circ f)^*(\mathcal{O}(1))\otimes(p_2\circ f)^*(\mathcal{O}(1)) = i^*(\mathcal{O}(1))\otimes j^*(\mathcal{O}(1)) = \mathscr{L}\otimes\mathscr{M}$$

Hence $\mathscr{L}\otimes\mathscr{M}$ is very ample.

(b) Assume Z affine? Let $f : X \xrightarrow{i} \mathbb{P}^r_Y \to Y, g : Y \xrightarrow{j} \mathbb{P}^s_Z \to Z$ where factor through immersions i, j with $i^*(\mathcal{O}(1)) = \mathscr{L}$, $j^*(\mathcal{O}(1)) = \mathscr{M}$. Let

$$X \xrightarrow{i\times f} (\mathbb{P}^r_Z\times Y)\times Y \xrightarrow{p_1'\times j} \mathbb{P}^r_Z\times_Z\mathbb{P}^s_Z \xrightarrow{h} \mathbb{P}^{rs+r+s}_Z$$

Now $h^*(\mathcal{O}(1)) = \mathcal{O}(1) = p_1^*(\mathcal{O}(1))\otimes p_2^*(\mathcal{O}(1))$ by 5.11. Pull back second coordinate to get

$$f^*(p_2\circ j)^*(\mathcal{O}(1)) = f^*\mathscr{M}$$

Pull back first coordinate and get

$$i^*(p_1\circ p_1')^*(\mathcal{O}(1)) = i^*(\mathcal{O}(1)) = \mathscr{L}$$

where first equality is by remark 5.16.1. Combine together and we get $\mathscr{L}\otimes f^*\mathscr{M}$ is very ample.

II.5.13 d-uple embedding

Let $\{x_i\}$ generate S_0, then observe that the degree d monomials generate $S_1^{(d)}$, call them $\{M_i\}$. Now consider the open set $D_+(M_i)$. On X we have $\operatorname{Spec} S_{(M_i)}$, on $Y = \operatorname{Proj} S^{(d)}$ we have $\operatorname{Spec} S^{(d)}_{(M_i)}$. Observe that elements of $S_{(M_i)}$ are exactly s/M_i^f where $s\in S$ has degree df, which are just elements in $\operatorname{Spec} S^{(d)}_{(M_i)}$. Thus we found affine covers where X and Y are isomorphic on each affine set. These isomorphisms are compatible because on intersections $D_+(M_i)\cap D_+(M_j) = D_+(M_iM_j)$ we have a similar argument with degree $2d$. Thus we can glue using 1.22, and get isomorphism $f : X \cong \operatorname{Proj} S^{(d)}$.

We would like to show $f^*\mathcal{O}(1) = \mathcal{O}(d)$. Suffices to check on stalks. We have $\mathcal{O}(d)_{\mathfrak{p}} = S(d)_{(\mathfrak{p})}$ and

$$f^*\mathcal{O}(1)_{\mathfrak{p}} = S^{(d)}(1)_{(f(\mathfrak{p}))}\otimes_{S^{(d)}_{(f(\mathfrak{p}))}} S_{(\mathfrak{p})}$$

where if we pick an affine chart and view $\mathfrak{p}$ as a prime ideal of $S_{(M_i)}$, then $f(\mathfrak{p}) = \mathfrak{p}$ and on this chart, we have

$$f^*\mathcal{O}(1)_{\mathfrak{p}} = S^{(d)}(1)_{(M_i),\mathfrak{p}}\otimes_{S^{(d)}_{(M_i),\mathfrak{p}}} S_{(M_i),\mathfrak{p}} = S^{(d)}(1)_{(M_i),\mathfrak{p}}$$

because $S^{(d)}_{(M_i)} = S_{(M_i)}$. Then we have required isomorphism for a similar reason as before

$$S(d)_{(M_i)} \cong S^{(d)}(1)_{(M_i)}$$

$$S(d)_{(M_i),\mathfrak{p}} \cong S^{(d)}(1)_{(M_i),\mathfrak{p}}$$

II.5.14 x

(a) We are assuming the local rings are integrally closed domains and X is connected. We want to show X is integral. We know X is reduced since local rings are reduced. Suppose X (noetherian) has irreducible components $V(I)$ and $V(J)$. Since irreducible components are maximal closed subsets, I, J can be taken as minimal primes. Since X connected, the components must intersect at some $\mathfrak{p}$. Now minimal primes of $A_\mathfrak{p}$ correspond to minimal primes of $\mathfrak{p}$. Since $A_\mathfrak{p}$ integral domain, it only has one minimal prime, a contradiction. Hence X is integral.

Note that $X = \operatorname{Proj} S = \operatorname{Proj} k[x_1, ..., x_n]/I$. We want to show S is a domain. Since I is homoegeneous, suffices to show $fg = 0$ implies $f = 0$ or $g = 0$ for homogeneous f, g. Suppose $fg = 0$. We know $S_{(x_i)}$ is integral domain, and $(f/x_i^{\deg f})(g/x_i^{\deg g}) = 0$, so $x_i^n f = 0$ for some n for each i. So f is in the saturation of I, and therefore inside I. Hence $f = 0$ (or $g = 0$ respectively) and S is a domain.

Consider $\mathscr{S} = \oplus_{n\geq 0}\mathcal{O}_X(n)$. Then

$$\mathscr{S}_\mathfrak{p} = \oplus S(n)_{(\mathfrak{p})}$$

are elements of $S_\mathfrak{p}$ of non-negative degrees. Suppose

$$t^n + \sum_{i=0}^{n-1} \frac{s_i}{f_i} t^i = 0$$

for some t in fraction field of S and $s_i/f_i \in \mathscr{S}_\mathfrak{p}$. Split this relation apart by degrees and assume each term has degree equal to t^n, namely $\deg s_i - \deg f_i = (n - i) \deg t$. Divide by f^{-n} where x is not in $\mathfrak{p}$, $\deg x = \deg t$.

$$(t/x)^n + \sum_{i=0}^{n-1} \frac{s_i}{f_i x^{n-i}} (t/x)^i = 0$$

so t/x is integral over $S_{(\mathfrak{p})}$, which is integrally closed, so $t \in S(\deg t)_{(\mathfrak{p})} \subseteq \mathscr{S}_\mathfrak{p}$. Hence $\mathscr{S}_\mathfrak{p}$ is integrally closed.

Now if we have

$$t^n + \sum_{i=0}^{n-1} st^i = 0$$

for $s \in \mathcal{O}_X(U)$, then it holds for all stalk $\mathfrak{p} \in U$, and we have $t \in \mathscr{S}_\mathfrak{p}$ for all $\mathfrak{p} \in U$. Using the fact $A = \cap A_\mathfrak{p}$ for integral domain A, we know $t \in \mathcal{O}_X(U)$, and so $\mathscr{S}$ is a sheaf of integrally closed rings. In particular, S' is integrally closed and since it is integral over S by proof of Theorem 5.19, it is the integral closure of S.

(b) Consider the map $\alpha : S \to \Gamma_*(\tilde{S}) = \Gamma_*(\mathcal{O}_X)$ from 5.9 (a). By 5.9(b) this is isomorphism for large degree, so $S_d = S'_d = \Gamma(X, \mathcal{O}_X(d))$ for large positive d.

(c) For sufficiently large d, by 5.13 $S^{(d)} = S'^{(d)}$, which is integrally closed since S' is (compare degree). Hence the d-uple embedding $\operatorname{Proj} S^{(d)}$ is projectively normal (use the homomorphism $k[x_0, x_1, ..., x_N] \to S^{(d)}$ from 2.12).

(d) Consider the surjection

$$\Gamma(\mathbb{P}^r, A[x_0, ..., x_r](n)^\sim) = \Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to \Gamma(X, \mathcal{O}_X(n)) = \Gamma(X, S(n)^\sim)$$

induced by $A[x_0, ..., x_r] \to S = A[x_0, ..., x_r]/I$. Looking at the restriction to affine cover then glue together, we see the image of $A_n = \Gamma(\mathbb{P}^r, A[x_0, ..., x_r](n)^\sim)$ (by Prop 5.13) always lie in S_n. Therefore the image of the surjection

$$\bigoplus_{n\geq 0} \Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to \bigoplus_{n\geq 0} \Gamma(X, \mathcal{O}_X(n)) = S'$$

is S. So $S = S'$ is integrally closed domain (this part of the proof did not use connectedness).

II.5.15 x

(a) Follows from that every module is the union of its finitely generated submodules (for each $m \in M$, take $\langle m \rangle$).

(b) Let $i : U \to X$, then $i_*\mathscr{F}$ is quasi-coherent by Prop 5.8(c). By (a) $i_*\mathscr{F}$ is a union of coherent sheafs $\mathscr{F}_\alpha$. Let $m_1, ..., m_n$ generate $\mathscr{F}(U)$, they also generate $\mathscr{F}(V)$ as $\mathcal{O}_U(V)$-modules for any $V \subseteq U$ by definition. Say m_i is contained in $\mathscr{F}_{\alpha_i}$, then take $\mathscr{F}'$ to be the finite union $\mathscr{F}_{\alpha_i}$, and we see it is coherent with $\mathscr{F}'|_U = \mathscr{F}$.

(c) Let $\rho : \mathscr{G}(V) \to i_*(\mathscr{G}|_U)(V) = \mathscr{G}(U \cap V)$ restriction map. Consider $\rho^{-1}(i_*\mathscr{F})$ defined by

$$\rho^{-1}(i_*\mathscr{F})(V) = \rho^{-1}(i_*\mathscr{F}(V)) = \rho^{-1}(\mathscr{F}(U \cap V))$$

Note the stalks are perserved so $\rho^{-1}(i_*\mathscr{F})(V)_x = \mathscr{F}_x$. Apply the same argument as before we find a coherent subsheaf $\mathscr{F}'$ of $\rho^{-1}(i_*\mathscr{F}) \leq \mathscr{G}$ with $\mathscr{F}'|_U = \mathscr{F}$.

(d) Cover X with open affine X_i. First extend $\mathscr{F}$ on $U_1 = U \cap X_1$ to $\mathscr{F}_1$ on X_1 with $\mathscr{F}_1 \subseteq \mathscr{G}|_{X_1}$. Now take $U_2 = U \cap X_2 \cup X_1$ and extend. Repeat finitely many times since X is noetherian.

(e) Take finite affine cover X_i with $\mathscr{F}|_{X_i} = \tilde{M}_i$. For each element s of $\tilde{M}_i$, take the subsheaf of $\mathscr{F}|_{X_i}$ generated by s and extend to a coherent subsheaf of $\mathscr{F}$. Now take s to be generators and take union of the finitely many coherent subsheaves, and note that by glueing we indeed get all of $\mathscr{F}$.

II.5.16 tensor operation on sheaves

(a) Since stalk of sheaf associated to presheaf is just stalk of the presheaf, if $\mathscr{F}$ locally free, then the stalks of $T^r(\mathscr{F})$ are of form $T^r(\mathcal{O}_x^n)$ free module generated by r-tuples with rank n^r. For $S^r\mathscr{F}$, it is free module generated by r-tuples mod out permutation, so $\binom{n+r-1}{n-1}$ (stars and bars). Finally, $\wedge^r(\mathscr{F})$ generated by those with distinct coordinates up to permutation, so $\binom{n}{r}$.

(b) Consdier $(\bigwedge^{n-r}\mathscr{F})^\vee \otimes \bigwedge^n \mathscr{F} = \mathscr{H}om_{\mathcal{O}_X}(\bigwedge^{n-r}\mathscr{F}, \bigwedge^n \mathscr{F})$. Recall these homomorphisms are determined on stalks. Such a map on stalks looks like

$$\mathcal{O}_x^{\binom{n}{n-r}} = \mathcal{O}_x^{\binom{n}{r}} \to \mathcal{O}_x^{\binom{n}{n}} = \mathcal{O}_x$$

which is determined by the values of the standard basis, so identified with $\mathcal{O}_x^{\binom{n}{r}}$ itself. By compatibility with restriction maps, we know these stalks are exactly the stalks of those maps induced by elements of $\wedge^r\mathscr{F}$. (check)

(c) Define on (free) open sets then glue. Assume $\mathscr{F}, \mathscr{F}', \mathscr{F}''$ are free, then $\mathscr{F} = \mathscr{F}' \oplus \mathscr{F}''$ and

$$S^r(\mathscr{F}) \cong \bigoplus_{p=0}^{r} S^p(\mathscr{F}') \otimes S^{r-p}(\mathscr{F}'')$$

$$v_1 \otimes ... \otimes w_{r-p} \leftarrow\!\shortmid (v_1 \otimes ... \otimes v_r) \otimes (w_1 \otimes ... \otimes w_{r-p})$$

So take F^i to be

$$\bigoplus_{p=i}^{r} S^p(\mathscr{F}') \otimes S^{r-p}(\mathscr{F}'')$$

(d) Same argument works. Set $r = n$, then for any $p \neq n'$, either p is too large that $\bigwedge^p(\mathscr{F}') = 0$ or too small so $\bigwedge^{r-p}(\mathscr{F}'') = 0$.

(e) Use induction, suppose $T^{n-1}f^*\mathscr{F} = f^*T^{n-1}\mathscr{F}$, then

$$\begin{aligned}
T^n f^*\mathscr{F} =& f^*\mathscr{F} \otimes T^{n-1}f^*\mathscr{F} \\
=& f^{-1}\mathscr{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \otimes_{\mathcal{O}_X} f^*T^{n-1}\mathscr{F} \\
=& f^{-1}\mathscr{F} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}T^{n-1}\mathscr{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X
\end{aligned}$$

and $f^{-1}\mathscr{F} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}T^{n-1}\mathscr{F} = f^{-1}(\mathscr{F} \otimes T^{n-1}\mathscr{F}) = f^{-1}T^n\mathscr{F}$ (we should be able to define presheaf map between $\lim_V \mathscr{F}(V) \otimes \lim_V \mathscr{G}(V)$ and $\lim_V \mathscr{F} \otimes \mathscr{G}(V)$ then look at stalks for a sheaf isomorphism?) So $T^n f^*\mathscr{F} = f^*T^n\mathscr{F}$.

For symmetric and exterior case, consider the kernel $\mathscr{I}$ of $T^n f^*\mathscr{F} \to \wedge^n f^*\mathscr{F}$, and the kernel of $f^*T^n\mathscr{F} \to f^* \wedge^n \mathscr{F}$ which on stalks are of form $x \otimes x$, so the surjective maps have the same kernels on the stalks, which gives an isomorphism $f^* \wedge^n \mathscr{F} \to T^n f^*\mathscr{F}/\mathscr{I} = \wedge^n f^*\mathscr{F}$. (This argument is very sketchy and probably wrong).

II.5.17 affine morphisms

(a) See 2.14.

(b) Quasi-compact because affine schemes are quasi-compact. Separated by Cor 4.6(f), see 4.1.

(c) For each $V = \operatorname{Spec} A \subseteq Y$ open, define $f : \operatorname{Spec} \mathscr{A}(V) \to V$ by the homomorphism $f' : A \to \mathscr{A}(V)$, $1 \mapsto 1$, where we know $\mathscr{A}(V)$ is an $\mathcal{O}_V = A$-algebra.

On V, we have $\mathscr{A}|_V = \mathscr{A}(V)\tilde{}$. So for any $U = \operatorname{Spec} B$ in V with induced homomorphism $A \to B$, the map $f^{-1}(U) \hookrightarrow f^{-1}(V)$ is induced by $\varphi : \mathscr{A}(V) = \mathcal{O}_{\operatorname{Spec} \mathscr{A}(V)}\mathscr{A}(V) \to \mathcal{O}_{\operatorname{Spec} \mathscr{A}(V)}(f^{-1}(U)) = \mathscr{A}(U)$. We need to check this agrees with the restriction map from the sheaf of algebra, which we call ψ. Now consider the definition of φ, we view $\mathscr{A}(V)$ as the set of functions from $\operatorname{Spec} \mathscr{A}(V)$ to $\amalg \mathscr{A}(V)_\mathfrak{p}$ by sending $\mathfrak{p} \subseteq \mathscr{A}(V)$ to $a_\mathfrak{p}$, and restrict the domain to $\operatorname{Spec} \mathscr{A}(U)$. Say a is restricted to $\varphi(a)$ which maps $\mathfrak{q}$ to $\varphi(a)_\mathfrak{q}$ for all $\mathfrak{q} \subseteq \mathscr{A}(U)$, and similarly, restricted to $\psi(a)$ where $\mathfrak{q}$ is mapped to $\psi(a)_\mathfrak{q} \in \mathscr{A}(U) \otimes \operatorname{Spec} B_\mathfrak{q}$ for all $\mathfrak{q} \subseteq B$. Suppose $\psi(a) = b$, namely for each $\mathfrak{q}$, there is some $c \notin \mathfrak{q}$ such that $c \cdot (a - b) = f'(c)(a - b) = 0$. For each $\mathfrak{q} \subseteq \mathscr{A}(U)$, take $c \notin f'^{-1}(\mathfrak{q})$, then $f'(c) \notin \mathfrak{q}$ and $f'(c)(a - b) = 0$, so $a = b$ in the localization $\mathscr{A}(U)_\mathfrak{q}$ and we must have $\varphi(a) = b$. Hence $\varphi = \psi$ and the two restrictions are the same as requried.

Therefore f is compatible over $D(f)$ (restricting to $D(f)$ is the same as taking the map induced by $\mathcal{O}_Y(D(f)) \to \mathscr{A}(D(f))$), so we can glue and get $f : X \to Y$. Uniqueness follows from uniqueness of glueing (1.22, 2.12).

(d) This is because on each $V = \operatorname{Spec} A \subseteq Y$, $\mathcal{O}_X|_{f^{-1}(V)} = \mathcal{O}_{\operatorname{Spec} \mathscr{A}(V)} = \mathscr{A}(V)\tilde{}$, so by Prop 5.2(d),

$$f_*(\mathscr{A}(V)\tilde{}) = ({}_A\mathscr{A}(V))\tilde{} = \mathscr{A}(V)\tilde{} = \mathscr{A}|_V$$

Therefore glueing gvies $f_*(\mathcal{O}_X) = \mathscr{A}$.

Suppose $f : X \to Y$ affine. Then

$$f_*(\mathcal{O}_X|_{f^{-1}(V)}) = f_*(\mathcal{O}_X(V)\tilde{}) = ({}_A\mathcal{O}_X(V))\tilde{} = ({}_AB)\tilde{}$$

Define $\mathscr{A}(V) = B$ and check compatibility with restrictions using compatibility of f. Then by uniqueness from (c), $X \cong \mathbf{Spec}\mathscr{A}$.

(e) Given as quasi-coherent $\mathcal{O}_X$-module $\mathscr{M}$, as shown above f_* gives a quasi-coherent $\mathscr{A}$-module. Suppose we have a quasi-coherent $\mathscr{A}$-module $\mathscr{M}$. Define $\tilde{\mathscr{M}}$ on X by setting

$$\tilde{\mathscr{M}}|_{f^{-1}V} = \mathscr{M}|_V = (\mathscr{M}|_V)\tilde{}$$

on the space $\operatorname{Spec} \mathscr{A}(V)$. This is the equivalence requried. (basically corollary 5.5?)

II.5.18 vector bundles

(a) Suppose $V = \operatorname{Spec} A \subseteq U_i \cap U_j$. By 5.17(c), ther map $f^{-1}(V) \to f^{-1}(U_i)$ is induced by $S(\mathscr{E})|_U \to S(\mathscr{E})|_V$. Note that (note from 3.13) restriction pass through sheafification, so we have $S(\mathscr{E}|_U) = S(\mathscr{E})|_U$, so they are also free and the restriction map, by definition is just $\mathcal{O}(U_i)[x_1, ..., x_n] \to \mathcal{O}(V)[z_1, ..., z_n] \cong S(\mathscr{E}(V))$ with suitable basis chosen, $\mathcal{O}(U_i) \to \mathcal{O}(V)$ restriction map, and $x_\ell \mapsto z_\ell$ (these are generators for $\mathscr{E}|_{U_i}$ and $\mathscr{E}_V$). Similarly we have a map $\mathcal{O}(U_j)[y_1, ..., y_n] \to \mathcal{O}(V)[z'_1, ..., z'_n]$, where y_ℓ sent to z'_ℓ is a linear combination of z_ℓ's (again, because the basis are chosen from the generators for modules, i.e. corresponding to things in $S^1(...)$). Now $\psi_j \circ \psi_i^{-1}$ is induced by $\mathcal{O}_V[z_1, ..., z_n]$ to $S(\mathscr{E}(V))$, then sent to $\mathcal{O}_V[z'_1, ..., z'_n]$. Since z'_ℓ are linear combinations of z_ℓ's, this map is indeed a linear automorphism.

If different basis were chosen, then simply take g to be the change of basis map (indeed linear) (identity on the space of X, and change basis for the sheaf on free affine open sets, then glue), and since the open sets are still the same, we do not need to check the other conditions.

(b) Assume $Y = \operatorname{Spec} A$ affine, $X = \mathbb{A}^n_A$. A section $s : Y \to X$ comes from $\theta : A[x_1, ..., x_n] \to A$, which correspond to $(\theta(x_1), ..., \theta(x_n))$. This means $A \to A[x_1, ..., x_n]$ is a section for θ. On the other hand, anything that $A \to A[x_1, ..., x_n]$ is a section for induces a section for f. Since $A \to A[x_1, ..., x_n]$ is the natural map induced by projection $\mathbb{A}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} A \to \operatorname{Spec} A$, we have A is just sent to A, and the sections correspond exactly to $\mathcal{O}^n_Y$, is locally free with $\mathcal{O}_Y$-structure.

(c) Given $s \in \Gamma(V, \mathscr{E}^\vee)$, view s as an element of $\operatorname{Hom}(\mathscr{E}|_V, \mathcal{O}_V)$, and extends to $\mathcal{O}_V$-algebra homomorphism $\varphi : S(\mathscr{E}|_V) \to \mathcal{O}_V$. s induces a map $V = \mathbf{Spec}\mathcal{O}_V \to \mathbf{Spec}S(\mathscr{E}|_V) = f^{-1}(V)$. We need to show that this is a section. It suffices to show on each affine set $U = \operatorname{Spec} A$ on which $\mathscr{E}|_V$ is free and $f^{-1}(U)$ isomorphic to $\mathbb{A}^n_U$. By part (b) any map induced by any A-algebra homomorphism is a section, so we are done. Conversely given a section $V \to f^{-1}(V) = \mathbf{Spec}S(\mathscr{E}|_V)$, on affine sets we have induced maps $\mathcal{O}_{f^{-1}(V)}(f^{-1}(U)) \to \mathcal{O}_V(U)$ which gives a cover and glue to an $\mathcal{O}_V$-module homomorphism $f_*\mathbf{Spec}S(\mathscr{E}|_V) = S(\mathscr{E}|_V) \to \mathcal{O}_V$ by 5.17(d). Now take the degree 1 part and get a map in $\operatorname{Hom}(\mathscr{E}|_V, \mathcal{O}_V) = \Gamma(V, \mathscr{E}^\vee)$. Now this is an isomorphism ($\mathcal{O}_Y$-module sturcture is perserved when we used the $\mathcal{O}_V$-algebra map).

(d) By (a) we can construct a vector bundle using a locally free sheaf. Conversely, by (b) we can get a locally free sheaf, now by (c) and 5.1(a), taking the dual of this sheaf gives us back the original. It remains to show that all vector bundles are of form $\mathbb{V}(\mathscr{E})$. Suppose $f : X \to Y$ is a vector bundle, then for any affine $V = \operatorname{Spec} A \subseteq Y$, $f^{-1}(V)$ is isomorphic to $\mathbb{A}^n_V = \mathbf{Spec}S(\mathcal{O}^n_V)$. On $U_i \cap U_j$ we have glueing map $\psi_j \circ \psi_i^{-1}$ which induced a glueing map for $\mathcal{O}^n_{U_i}$ and $\mathcal{O}^n_{U_j}$. So we can glue and get a locall free $\mathscr{E}$. are the categories equivalent? (is this a functor that gives identification of morphisms)