MATH 127 Calculus for the Sciences

Lecture 10

Today's lecture

Last time
Inverse functions

- How to find
- Domain and range
- Use horizontal line test
- Cancellation rule

This time
Course note coverage Section 2.4.1–2.4.3
Exponential and Logarithmic functions

1. Definition
2. Rules
3. their derivatives

Quiz is today! Don't forget to go.

Exponential function

Definition For $\boxed{a > 0}$, the function

$$\boxed{a^x}$$

is called an **exponential function** (in the variable x).

Question Why do we ask for $a > 0$?

1. What happens if $a = 0$?

   $f(x) = 0^x$. $f(-1) = 0^{-1} = \cancel{\frac{1}{0}}$ $f(0) = \cancel{0^0}$

2. What happens if $a < 0$?

   $f(x) = (-1)^x$ $f(\frac{1}{2}) = (-1)^{\frac{1}{2}} = \cancel{\sqrt{-1}}$

Exponential rules

Recall the following exponent rules:

Assuming $a, b > 0$, we have

1. $a^x \cdot a^y = \boxed{a^{x+y}}$;
1. $\dfrac{1}{a^x} = \boxed{a^{-x}}$;
1. $(a^x)^y = \boxed{a^{xy}}$;
1. $(ab)^x = \boxed{a^x b^x}$;
1. $a^0 = \boxed{1}$.

Crazy example

Example Where did the following argument go wrong?

1. We know $(-1)^2 = 1$, so $\sqrt{(-1)^2} = 1$.
2. Rewrite $\sqrt{(-1)^2} = ((-1)^2)^{\frac{1}{2}}$.
3. By exponent rule,

$$((-1)^2)^{\frac{1}{2}} \underset{(A)}{=} (-1)^{2\cdot\frac{1}{2}} \underset{(B)}{=} (-1)^1 \underset{(C)}{=} -1.$$

4. Hence $1 = -1$.

Step (A) fails because the base of exponent is (-1) but $-1<0$.

Natural exponential

Remark When $a = e$, we get **the *natural* exponential function** is

$$e^x \qquad (e^x)' = e^x$$

Sometimes we write

$$\exp(x) = e^x$$

because expressions like

$$\exp\left(\sum_{i=1}^{n} \frac{q^i}{i}\right) \quad \text{are easier to read than} \quad e^{\sum_{i=1}^{n} \frac{q^i}{i}}.$$

Derivative of exponential

We have seen

$$(e^x)' = e^x$$

But what about

$$(a^x)'$$

For your eyes only.

I hate a^x. They should ban it. Make a law that make them ban it. Sign one of those executive orders. If you ever see a^x again, spit on it. Always write

$$a^x = \boxed{e^{x \cdot \ln a}}$$

By the way, things like 2^3 ares okay. Just don't let me see you writing $\boxed{2.5^x}$. I don't know how to take derivatives of those, but I do know

$$(e^{x \cdot \ln a})' = (\ln a) \cdot \boxed{e^{x \cdot \ln a}}.$$

The same for x^x. Get rid of that. If you see someone writing x^x, they're just pretending to be smart because that looks like a super difficult expression. So you call them stupid and just rewrite

$$x^x = e^{x \ln x}. = \exp(x \cdot \ln x)$$

And guess what, they don't teach you this in the textbook. This is secret knowledge from me. Eventually you'll learn about taking limit of certain functions, and if they ever ask something like the limit of $x^{\sin x}$, those kids in the other sections will be so confused. But not you. You know what to do.

Simplify by changing base to e

By exponent rules, we have

$$e^{x \cdot \ln a} = (e^{\ln a})^x = a^x$$

so we can always just deal with the left-hand side, whose derivative we know is

$$\underline{(e^{x \cdot \ln a})'} = \boxed{\ln a} \cdot e^{x \ln a}.$$

Question So what is

$$\int a^x dx?$$

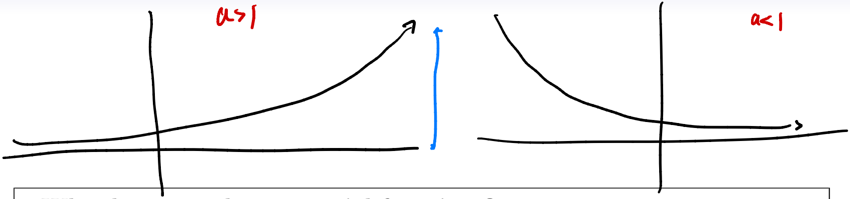
Wrong question. You should do the following

$$\int e^{x \ln a} dx = \boxed{\frac{e^{x \ln a}}{\ln a} + C}$$

$$\left(\frac{e^{x \ln a}}{\ln a}\right)' = \frac{e^{x \ln a} \cdot \ln a}{\ln a}$$

$$= e^{x \ln a}$$

Motivation



Why do we need exponential functions?
Many things in real life are modeled by exponential functions

1. Population growth – of germ, humans, animals, aliens.
2. Radioactive decay – half life (if you base is < 1, then the function decreases)
3. Other examples?

Logarithmic function

Remember inverse functions from last time? The $\ln x$ is the inverse of e^x.

Definition For $\boxed{a > 0 \text{ and } a \neq 1}$, the inverse function of a^x is

$$\underline{\log_a x}$$

such a function is called a **logarithmic function** (in the variable x).

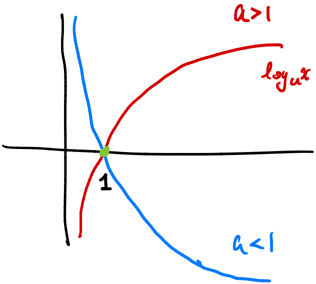
Don't forget what I told you on Monday. When dealing with inverse functions, always write down the domain and range.

The range of a^x is $(0, \infty)$. So the domain of $\log_a x$ is

$$\boxed{(0, \infty)}$$

The domain of a^x is $(-\infty, \infty)$. So the range of $\log_a x$ is

$$\boxed{(-\infty, \infty)}$$



STOP IT.

Did you forget what I just told you? Instead of $f(x) = a^x$, you should be looking at the inverse function of

$$f(x) = e^{x \ln a}$$

Then we can try to find the inverse as usual. Write $y = e^{x \ln a}$, then try to write x in terms of y:

$$\ln y = x \ln a$$

$\ln y = \ln e^{x \ln a}$
$= x \cdot \ln a$

so

$$x = \frac{\ln y}{\ln a}.$$

So the inverse function is

$$f^{-1}(y) = \frac{\ln y}{\ln a}.$$

This explains why $a \neq 1$, because if $a = 1$, then the denominator becomes 0.

Logarithmic rules

If you absolutely have to work with $\log_a x$, then recall the following log rules:

Assuming $a > 0$ and $a \neq 1$, we have

1. $a^{\log_a x} = \boxed{x}$; by cancellation rule
2. $\log_a(a^x) = \boxed{x}$;
3. $\log_a(xy) = \boxed{\log_a x + \log_a y}$.
4. $\log_a(x^y) = \boxed{y \log_a x}$.

0. $\boxed{\log_a x = \dfrac{\ln x}{\ln a}}$.

$$a^{\log_a x} = e^{\log_a x \cdot \ln a} = e^{\frac{\ln x}{\ln a} \cdot \ln a} = e^{\ln x} = x$$

What is the anti-derivative of $\frac{1}{x}$?

$$(x^n)' = n x^{n-1} \qquad \left(\frac{x^8}{8}\right)' = \frac{8x^7}{8}$$

$$\int x^7 dx = \frac{x^8}{8} + C \qquad (x^0)' = (1)' = 0$$

What is the anti-derivative of $\frac{1}{x} = x^{-1}$? Usually we would say the anti-derivative of x^n is

$$\boxed{\frac{x^{n+1}}{n+1} + C}$$

but we can't take $n = -1$.

What is the anti-derivative of $\frac{1}{x}$?

Consider the following

$$\boxed{e^{\ln x} = x}$$

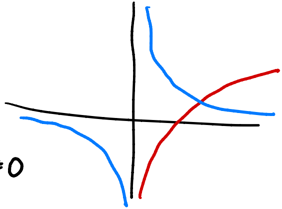
Take derivative on both sides, using chain rule on the left, we get

$$\boxed{e^{\ln x}} \cdot (\ln x)' = \boxed{1}$$

$$\boxed{x} \cdot (\ln x)' = \boxed{1}$$

Therefore

$$(\ln x)' = \boxed{\frac{1}{x}} \qquad x \neq 0$$



But the domain is only $\boxed{(0,\infty)}$. So this is not a complete solution to the anti-derivative of $\frac{1}{x}$. We will get to it next time.

$\underline{\text{Example}}$ Find the derivative of $\sin(x)^{\exp(2x^2)}$

$$\left(e^{\exp(2x^2)\cdot \ln \sin(x)}\right)' = \left(\underbrace{\exp(2x^2)\cdot \ln \sin x}\right)' \cdot e^{\exp(2x^2)\cdot \ln \sin x}$$

$$(\ln u)' = \frac{1}{u}$$

$$= \left(\underbrace{\exp(2x^2)}' \cdot \ln \sin x + \exp(2x^2)\cdot (\ln \sin x)'\right) \cdot \text{---}$$

$$= \left(4x \cdot e^{2x^2} \cdot \ln \sin x + \exp(2x^2)\cdot \frac{1}{\sin x}\cdot (\cos x)\right) \cdot \text{----}$$