MATH 127 Calculus for the Sciences

Lecture 16

Today's lecture

Last time

Calculus of trig functions

1. Derivatives and anti-derivatives of trig functions;
2. those things about inverses and reciprocals of trig functions.

This time

Course note coverage Section 2.6, 2.7.1

Finally done with trigs

Factorial Function, Implicitly Defined Functions

Quiz is today

Factorial function

From today, you will never be allowed to use exclamation mark in math! Sorry.

Definition For any integer $n \geq 1$,

$$n! = 1 \cdot 2 \cdots n$$

read as "n factorial".
We also define specially

$$0! = 1$$

Example We know

$$3! = \boxed{1 \cdot 2 \cdot 3} = \boxed{6}$$

Factorial function

Example What is 2.5!?

Example What is the derivative of $y = x!$?

(1) 2.5 is not an integer. so 2.5! does not make sense.

(2) Derivative only make sense for functions with real input.

but $x!$ takes input in integers.

Factorial function

Example Simplify $\dfrac{n!}{(n+1)!}$.

n is an integer

$\underline{n \geq 0}$ and $\boxed{\cancel{\begin{array}{c} n+1 \geq 0 \\ \Downarrow \\ n \geq -1 \end{array}}}$

$$\frac{n!}{(n+1)!} = \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdots\cancel{n}}{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdots\cancel{n}(n+1)}$$

$$= \frac{1}{n+1}$$

Binomial theorem

The main reason we care about factorial functions is be cause they show up when you try to expand $(x+y)^N$.

Theorem *For any integer $n \geq 1$, we have*

$$(x+y)^n = \binom{n}{n} x^n y^0 + \binom{n}{n-1} x^{n-1} y^1 + \cdots + \binom{n}{1} x^1 y^{n-1} + \binom{n}{0} x^0 y^n.$$

Definition The terms $\binom{n}{m}$ are called binomial coefficients, given by

$$\binom{n}{m} = \frac{n!}{(n-m)!m!} = \frac{\boxed{1\cdot 2\cdots(n-m)}(n-m+1)\cdots n}{\boxed{1\cdot 2\cdots(n-m)}\cdot 1\cdot 2\cdots m} = \frac{n(n-1)\cdots(n-m+1)}{1\cdot 2\cdots m}$$

count down m from n

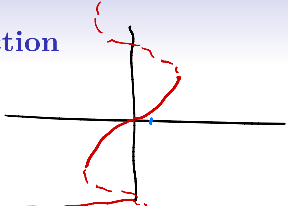
count up m fr 1

The integers n, m must satisfy $\boxed{n \geq 0,\ m \geq 0}$. $n \geq m$

Example

$$\binom{5}{3} = \frac{5!}{(5-3)!\cdot 3!} = \frac{\cancel{1\cdot 2\cdot 3}\cdot 4\cdot 5}{(1\cdot 2)(\cancel{1\cdot 2\cdot 3})} = 10$$

Implicitly defined function



Remember the formula of a circle? it is

$$\boxed{x^2 + y^2 = 1}$$

The radius of this circle is $\boxed{1}$.
This is an example of an implicit function, an explicit expression for it is

$$\boxed{y = \pm\sqrt{1 - x^2}}$$

But a function can only have one output given a input, so this is not a function.

Function given by an equation, which does not have an explicit form, are called **implicitly defined function**.

normal ✗ ordinary ✓ regular ✗ simple ✗ function	Implicitly defined function	parametrized function
$y=x$, $y=\arcsin(x)$	$x^2+y^2=2$	?
$y=x^2$,	$y^3=x^2$	
$y=\sqrt{x}$.	$y=x$	

Implicitly defined function

Example Some implicitly defined functions do not have an explicit $y = ...$ form, like

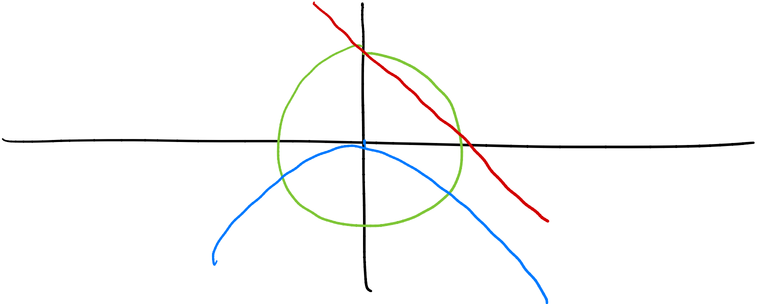
$$y^2 = 1 - x^2 \qquad y = \pm\sqrt{1-x^2}$$

But some do, like

$$y = 1 - x$$

and

$$x^2 + y = 0 \qquad y = -x^2$$



Implicitly defined function

You can take derivative of an implicitly defined function, by taking derivative of both sides of equations.

> **Example** Find y' (with respect to the input variable x) of the implicitly defined function
> $$x^2 + y^2 = 3$$

Derivative of LHS is

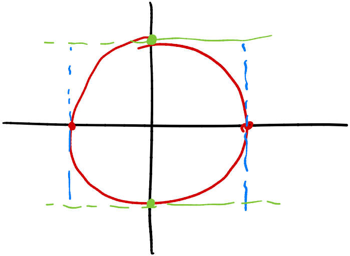
$2x + 2y \cdot y'$ by chain rule

Derivative of RHS is

0

So $2x+2y\cdot y' = 0$

$$y' = \frac{-2x}{2y} = \frac{-x}{y}$$



Implicitly defined function

You can take derivative of an implicitly defined function, by taking derivative of both sides of equations.

Example Find x' (with respect to the input variable z) of the implicitly defined function

$$\ln(x) + \sin(z)^2 = 3x + z$$

$$\frac{1}{x} \cdot x' + 2\sin(z) \cdot \cos(z) = 3x' + 1$$

$$\left(\frac{1}{x} - 3\right) x' = 1 - 2\sin(z)\cos(z)$$

$$x' = \frac{1 - 2\sin(z)\cos(z)}{\frac{1}{x} - 3}$$

Tangent line of implicitly defined functions

You can express the tangent line of an implicitly defined function

$$\text{(some function involving } x, y) = 0$$

as usual using point-slope form: say at a point (a, b)

$$\boxed{y = \left.\frac{dy}{dx}\right|_{x=a, y=b} \cdot (x - a) + b}$$

Example Find the tangent line at $(1, \sqrt{3})$ of

$$x^2 + y^2 = 4$$

(1) Find y'

$$2x + 2yy' = 0$$

$$y' = \frac{-2x}{2y} = \frac{-x}{y}$$

(2) $\left.\frac{dy}{dx}\right|_{x=1,\, y=\sqrt{3}} = -\frac{1}{\sqrt{3}}$

(3) The tangent line is

$$y = -\frac{1}{\sqrt{3}}(x - 1) + \sqrt{3}.$$

Example

Remark
Elliptic curves are implicitly defined functions used in encryption (so they are useful in real life I guess).

Example Find the tangent line at $(1, \overset{1}{\cancel{-2}})$ of the elliptic curve

$$y^2 = x^3 - x + 1$$

Find y'

$$2yy' = 3x^2 - 1$$

$$y' = \frac{3x^2-1}{2y}$$

$$\left.\frac{dy}{dx}\right|_{x=1,y=1} = \frac{3\cdot 1^2 - 1}{2\cdot 1} = 1$$

The tangent line through $(1,1)$ is

$$y = 1(x-1)+1$$

$$= x$$