MATH 127 Calculus for the Sciences

Lecture 4

September 10, 2025

Today's lecture

Office hours/Question session
Monday: 1430-1520 MC 5331
Wednesday: 1630-1710 (right before your quizzes)

Tutorial center
3rd floor MC

Last time
Differentials, Differential equations

This time
Course note coverage Section 1.2.1
Integration
Fundamental theorem of calculus

Motivation

Question Suppose an object has constant speed 1m/s, stating at position 0m. What is its position after 0.1s?

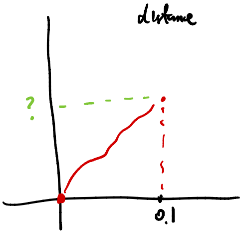
$$\text{distance} = \text{speed} \times \text{time}$$
$$= 1 \times 0.1$$
$$= 0.1\,\text{m}$$

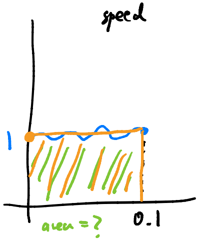
Question Suppose this object has speed 1m/s at the current moment, but in the future it might change its speed. What is its position after 0.1s?

Assuming speed varies "mildly"

then $\text{distance} \approx 1 \times 0.1$

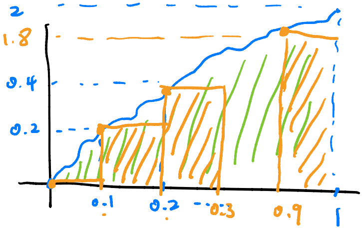
$= 0.1\text{m}$





Approximation by small intervals

Question Suppose this object has speed 0m/s at time $t = 0$, speed 0.2 at time 0.1, speed 0.4 at time 0.2, etc. What is its position after 1s?



distance = area under speed

$\approx$ area of rectangles.

$= 0.1\times 0 + 0.1\times 0.2 + 0.1\times 0.4 + \cdots + 0.1\times 1.8$

$= 0.9\,m$

Left Riemann Sum Approximation

Suppose we want

- to approximate the total change in $F(x)$ when x goes from a to b.

Suppose we know

- $F'(x) = f(x)$

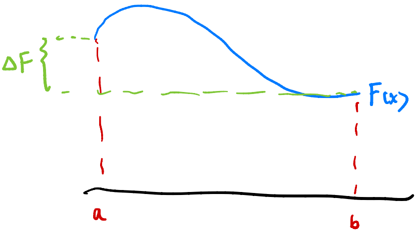
$\Delta F \approx f(x)\Delta x = f(a)(b-a)$

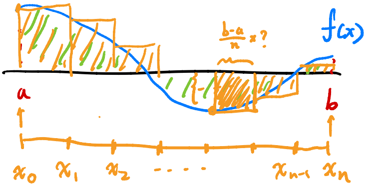
- the differential approximation is good enough for our purpose when the change in x is small.

We cut the interval $[a, b]$ into n pieces $[x_0, x_1], \ldots, [x_{n-1}, x_n]$ so that

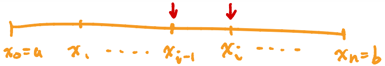
- $x_0 = a$, $x_n = b$;
- $x_i = x_{i-1} + \Delta x$ for each $i = 1, \ldots n$, where

$$\boxed{\Delta x = \frac{b-a}{n}}$$





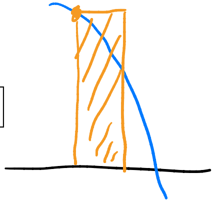
Left Riemann Sum Approximation



For n really big, the pieces $[x_{i-1}, x_i]$ will be very small, which gives relatively good approximations.

We can compute the change in $F(x)$ in these small intervals, then add them together, to get a somewhat accurate approximation for the total change. On the i-th interval, the change is approximately

$$\overbrace{\Delta F_{i\text{-th piece}}}^{\text{area of (i+1)-th}} \cancel{\approx} \underbrace{f'(x_i)}_{\text{height}}\underbrace{\overbrace{\Delta x}^{\frac{b-a}{n}}}_{\text{base}},$$



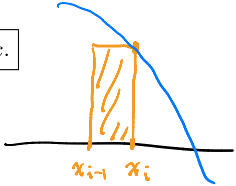
$$\boxed{\overbrace{\Delta F_{\text{total}}}^{\text{total area}} \approx \underbrace{f'(x_0)\Delta x}_{\text{area of 1st}} + \underbrace{f'(x_1)\Delta x}_{\text{area of 2nd}} + \cdots + \underbrace{f'(x_{n-1})\Delta x}_{\text{area of nth}}.}$$

This is called the **left Riemann sum approximation**.

If we use the right-hand side of the interval to approximamte, then

$$\boxed{\Delta F_{\text{total}} \approx f'(x_1)\Delta x + f'(x_2)\Delta x + \cdots + f'(x_n)\Delta x.}$$

is called the **right Riemann sum approximation**.



Definite Integral

Recall that the approximation

$$\boxed{\Delta F_{i\text{-th piece}} \approx f'(x_i)\Delta x.}$$

becomes more and more accurate when Δx is smaller and smaller, and will be totally accurate when Δx is 0, and this infinitesimal behavior is captured using **differentials**.

This means if we cut the interval into smaller and smaller pieces, our approximation of ΔF is more and more accurate.

Definition For each n, we cut the interval $[a, b]$ into n intervals

$$[x_0^{(n)}, x_1^{(n)}], \dots, [x_{n-1}^{(n)}, x_n^{(n)}]$$

with $\Delta x^{(n)} = \frac{b-a}{n}$. The **definite integral** of $f(x)$ from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \left(\begin{array}{l} \text{left or right Rieman sum approximation} \\ \text{with respect to } [x_0^{(n)}, x_1^{(n)}] \dots [x_{n-1}^{(n)}, x_n^{(n)}] \end{array} \right)$$

The Fundamental Theorem of Calculus

$n \to \infty$

As mentioned above, taking the limit, we should get a totally accurate approximation of ΔF. Indeed, this is a mathematically proven theorem.

Theorem (The Fundamental Theorem of Calculus (FTC)) *If $F(x)$ is a differentiable function with $F'(x) = f(x)$, then*

$$F(b) - F(a) = \int_a^b f(x)dx.$$

Notation We sometimes write

$F(b) - F(a)$

$$F(x)\Big|_a^b = \cancel{f(b) - f(a)}$$

for simplicity.

Example

Question Evaluate

$$\int_1^{10} \boxed{(x^2 - 1)} dx$$

The **integrand** is $f(x) = \boxed{x^2 - 1}$. By a lucky guess, we see if we take

$$\boxed{F(x) = \frac{x^3}{3} - x}.$$

then

$$F'(x) = f(x).$$

So by $\boxed{\text{fundamental theorem of Calculus}}$, we have

$$\int_1^{10} \underbrace{(x^2-1)}_{f(x)} dx = F(x)\Big|_1^{10} = \left(\frac{10^3}{3} - 10\right) - \left(\frac{1^3}{3} - 1\right) = \dots$$