MATH 127 Calculus for the Sciences

Lecture 17

Today's lecture

Last time

Calculus of trig functions

1. Factorial Function
2. Implicitly Defined Functions

This time

Course note coverage Section 2.7.2

Logarithmic differentiation

- Derivative of x^x: I taught you how to do this, but there is an alternative way to do this called...
- Logarithmic differentiation
- my way is better, but they still want you to know this. rip

Recap

Suppose we are given an implicit function

$$\boxed{(\text{something about } x, y) \quad = \quad (\text{something else about } x, y)}$$

We can take derivative on both sides with respect to x and get

$$\boxed{(\text{thing about } x, y, y') \quad = \quad (\text{another thing about } x, y, y')}$$

Then we isolate y' and get the derivative

$$\boxed{y' = \frac{dy}{dx} = \quad (\text{whatever you get in terms of } x, y)}$$

Now if you want the tangent line at $(x, y) = (a, b)$, you use the point-slope formula:

$$\boxed{y = \text{slope} \cdot (x - a) + b}$$

The slope is just the derivative at $(x, y) = (a, b)$, i.e.

$$\boxed{\text{slope} = \left.\frac{dy}{dx}\right|_{x=a, y=b}.}$$

Motivation

Example

1. What is the derivative of x^2?
2. What is the derivative of 2^x?
3. What is the derivative of x^x? What is the domain of this function? $x>0$

(1) $(x^2)' = \frac{d(x^2)}{dx} = 2x$

(2) $2^x = e^{x \cdot \ln 2}$

$(e^{x\ln 2})' = \ln 2 \cdot e^{x \ln 2}$

$= (\ln 2)\, 2^x$

(3) $x^x = e^{x \ln x}$

$(e^{x\ln x})' = e^{x\ln x}(x\ln x)'$

$= e^{x\ln x}(\ln x + x\frac{1}{x})$

$= e^{x \ln x}(\ln x + 1) = x^x(\ln x + 1)$

Logarithmic differentiation

Example Here is another way of taking derivative of $y = x^x$ for $x > 0$.

1. Take ln of both sides and get

$$\ln y = \boxed{\ln x^x} = \boxed{x \ln x}$$

2. This is an implicit function now, remember how to take derivative of this?

   2.1 The derivative of LHS with respect to x is

$$(\ln y)' = \frac{1}{y} \cdot \boxed{y'} = \boxed{\frac{1}{x^x} \cdot \boxed{y'}}$$

   by $\boxed{\text{chain}}$ rule.

   2.2 The derivative of RHS with respect to x is

$$(\boxed{x \ln x})' = (\boxed{x})'\boxed{\ln x} + (\boxed{\ln x})'\boxed{x} = \boxed{\boxed{\ln x + 1}}$$

   by $\boxed{\text{product}}$ rule.

Logarithmic differentiation

Example Here is another way of taking derivative of $y = x^x$ for $x > 0$.

3. Combining the previous two steps, we get

$$\frac{y'}{x^x} = \ln x + 1 ,$$

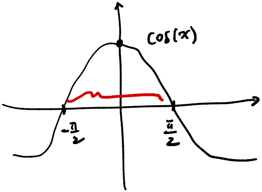
this makes sense because $x^x \neq 0$ by $x > 0$.

4. Hence

$$y' = x^x(\ln x + 1)$$

Example

Example Determine the derivative of $y = \cos(x)^{\sin(x)}$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$



$$\ln y = \ln \cos(x)^{\sin(x)} = \sin(x) \ln \cos x$$

$$\boxed{\frac{1}{y} \cdot \frac{dy}{dx}} = \cos(x) \ln\cos x + \sin(x) \left(\ln \cos x\right)'$$

$$= \cos(x) \ln \cos x + \sin(x) \left(\frac{-\sin(x)}{\cos x}\right)$$

$$= \boxed{\cos(x) \ln \cos(x) - \frac{\sin^2(x)}{\cos(x)}}$$

$$\frac{dy}{dx} = y \left(\cos x \ln \cos x - \frac{\sin^2 x}{\cos x}\right)$$

$$= \cos(x)^{\sin x} \left(\cos x \ln \cos x - \frac{\sin^2 x}{\cos x}\right).$$

Example

Example Consider $y = \dfrac{(x+1)^2(x-2)^3}{x^5\sqrt{x-1}}$ where $x > 2$.

1. What is the domain of this function if I don't say $x > 2$?
2. Take its derivative directly using quotient and product rule.

(1) $x > 1$

(2) Too much work.

Example

Example Consider $y = \dfrac{(x+1)^2(x-2)^3}{x^5\sqrt{x-1}}$ where $x > 2$.

3. Take ~~log~~ ln of both sides and see what happens.
4. Why did I say $x > 2$?

(3)

$$\ln y = \ln \frac{(x+1)^2(x-2)^3}{x^5 (x-1)^{\frac{1}{2}}}$$

$$= \ln\left[(x+1)^2(x-2)^3 x^{-5}(x-1)^{-\frac{1}{2}}\right]$$

$$= \ln(x+1)^2 + \ln(x-2)^3 + \ln x^{-5} + \ln(x-1)^{-\frac{1}{2}}$$

$$= 2\ln(x+1) + 3\ln(x-2) - 5\ln x - \tfrac{1}{2}\ln(x-1)$$

$$\frac{1}{y}\frac{dy}{dx} = 2\cdot\frac{1}{x+1} + 3\cdot\frac{1}{x-2} - 5\frac{1}{x} - \frac{1}{2}\cdot\frac{1}{x-1}$$

$$\frac{dy}{dx} = y\left(\frac{2}{x+1} + \frac{3}{x-2} - \frac{5}{x} - \frac{1}{2(x-1)}\right) ✓$$