MATH 127 midterm review

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Derivative

Definition Let $y = f(x)$ be a function. The $\boxed{\textbf{derivative of } y}$ is the function

$$\frac{dy}{dx} := f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Remark If you want to talk about the derivative at a input a, you use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} = f'(a).$$

Remark

- Geometrically, the derivative represents the slope of the tangent line!
- Physically, the derivative represents rate of change (e.g. speed is the rate of change of distance, and acceleration is the rate of change of speed, etc.)

Notations for derivatives

Notation

The n-th derivative is $\boxed{f^{(n)}(x) = \frac{d^n y}{dx^n}}$.

The n-th derivative at a is $\boxed{f^{(n)}(a) = \left.\frac{d^n y}{dx^n}\right|_{x=a}}$.

Example The second derivative of $y = x^2$ is $\frac{d^2 y}{dx^2} = 2$.

Remark You don't have to always use the letters x, y.

The second derivative of $z = u^2$ is $\frac{d^2 z}{du^2} = 2$.

Examples of non-differentiable functions

The following are some (not all) of the cases where a function is not differentiable.

Example If f has a **sharp corner** at $x = a$, e.g. $f(x) = |x|$.

Example If f has a **vertical tangent** at $x = a$, e.g. $f(x) = x^{1/3}$.

Example If f is **not continuous** at $x = a$, e.g. $f(x) = \begin{cases} 1, & \text{if } x \leq 1, \\ 0.5, & \text{if } x > 1. \end{cases}$

Common derivatives

You should know the following from high school.

1. If $f(x) = c$ for c a constant, then $f'(x) = 0$.
2. If $f(x) = x^n$ for an integer $n \neq 0$, then $f'(x) = nx^{n-1}$.
3. If $f(x) = e^x$, then $f'(x) = e^x$.
4. If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$.
5. If $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$.

Suppose c is a number, and f, g, h are functions.

1. If $h(x) = cf(x)$, then $h'(x) = cf'(x)$. (Constant rule)
2. If $h(x) = f(x) + g(x)$, then $h'(x) = f'(x) + g'(x)$. (Sum rule)
3. If $h(x) = f(x) \cdot g(x)$, then $h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$. (Product rule)
4. If $h(x) = f \circ g(x) = f(g(x))$, then $h'(x) = f'(g(x))g'(x)$. (Chain rule)
5. I hate the quotient rule, do you remember why? (Quotient rule)

Example

Question Assume the mass M (in kg) of an object of radius r (in cm) is

$$M(r) = \frac{e^r + 3r}{r^3}$$

Find the rate of change of the mass with respect to the radius.

1. We are looking for rate of change of M with respect to r. Should I use the notation
$$\frac{dM}{dr} \text{ or } \frac{dy}{dx}$$
2. We want to find the derivative. What rule should I begin with?
   - Product rule or quotient rule
   - Sum rule
   - Chain rule
3. What is the unit for the derivative?
   - Answer: The unit of derivative is the unit of the output divided by unit of input, so kg/cm.

Point-slope form

Definition The $\boxed{\textbf{point–slope form}}$ of a line with slope m through the point (x_0, y_0) is

$$\boxed{y = \text{slope} \cdot (x - x\text{-value}) + y\text{-value}}$$

i.e.

$$y = m(x - x_0) + y_0.$$

Definition The $\boxed{\textbf{tangent line}}$ of a function $y = f(x)$ at $x = a$ is a line, with slope $\boxed{f'(a)}$ through the point $(a, f(a))$. Therefore its point-slope form is

$$y = \text{slope} \cdot (x - x\text{-value}) + y\text{-value}$$

i.e.

$$y = f'(a)(x - a) + f(a)$$

Example

Question What is the equation of the tangent line of $f(x) = x^2$ at $x = 2$?

1. You use the point-slope form.
2. Which point should the line go through?
   - It should be $(x_0, y_0) = (2, f(2))$.
3. What is the slope of the line?
   - Slope is represented by derivative.

Tangent line approximation

Definition Given $y = f(x)$, if b is close to a, then you can approximate $f(b)$ by computing the tangent line at $x = a$ and subbing $x = b$ into that:

$$\text{true value at } b \approx \text{tangent line's value at } b$$

$$f(b) \approx f'(a)(b-a) + f(a).$$

Example Use tangent line approximation of $y = x^3$ at $x = 2$ to approximate $(2.01)^3$.

1. What is the tangent line at $x = 2$?
    - Use point-slope form.
2. What is a and what is b?
3. Draw the graph of the function to help visualize.

Differential approximation

We have

$$f(b) \approx f'(a)(b-a) + f(a).$$

Rearranging, we can say

$$f(b) - f(a) \approx f'(a)(b-a).$$

Here $(b-a)$ represents change in x, so we rewrite as Δx. Also $f(b) - f(a)$ represents change in y, so we rewrite as Δy.

$$\boxed{\Delta y \approx f'(a)\Delta x.}$$

This is the **differential approximation** of $y = f(x)$ near $x = a$.

Example Use differential approximation of $y = x^3$ at $x = 2$ to approximate $(2.01)^3 - 2^3$.

1. What is the derivative at $x = 2$?
2. What is a?
3. What is Δx and what is Δy?

Differentials

The approximation

$$\Delta y \approx f'(a)\Delta x$$

is not perfect, but when Δx is super small, we know Δy has high precision. This **infinitesimal behavior** is recorded by the notation

$$\boxed{dy = f'(x)dx}$$

Remark For us, dx, dy are just symbols, they are not numbers nor functions. Saying $dx = 0$ or $dx = 1$, or $dx = x^2$ does not make any sense.

Differential equations

A **differential equation** is an equation involving an unknown function, its variable, and its derivatives.

Example The equation

$$f'(x) = f(x)$$

is a differential equation, because it involves the unknown function $f(x)$, the variable x, and the derivative $f'(x)$.

Example The equation

$$\frac{d^2y}{dx^2} = x$$

is a differential equation, because it involves the unknown function y, the variable x, and the second derivative y.

Order of a differential equation

Definition The $\boxed{\textbf{order}}$ of a differential equation is the order of the highest derivative involved in the equation.

Example The equation

$$y'' + (y')^{999} = y$$

has order 2, because the highest derivative it involves is the *second* derivative $f''(x)$. Note that the exponent on $(y')^{999}$ is **irrelevant**.

Solution of a differential equation

We know

$$y = e^x, y = 2e^x$$

are both solutions to the differential equation

$$y' = y.$$

To check this, you just substitute the candidate into the equation and check whether equality indeed hold.

Sometimes you have multiple possibilities of solutions. In this case, the solutions are

$$f(x) = Ce^x$$

for some constant C. We say C is a **parameter** of the **family of solutions**.

Remark For more difficult differential equations, you might even have many parameters!

Exercise

Question Come up with a differential equation and propose a solution. Of course you can come up with a simple example like $y' = 1$. Try to come up with something more complicated and check whether your proposed solution is indeed a solution.

Left Riemann Sum Approximation

Suppose we want

- to approximate the total change in $F(x)$ when x goes from a to b.

We know

- the differential approximation is good enough for our purpose when the change in x is small.

We cut the interval $[a, b]$ into n pieces $[x_0, x_1], \dots, [x_{n-1}, x_n]$ evenly so that

- $x_0 = a$, $x_n = b$;
- The change is just $\Delta x_{i\text{-th piece}} = \dfrac{b-a}{n}$.

For n really big, the change $\Delta x_{i\text{-th piece}}$ will be very small.

$$\boxed{\Delta F_{\text{total}} = \sum_{i=0}^{n-1} \Delta F_{i\text{-th piece}} \approx \sum_{i=0}^{n-1} F'(x_i)\Delta x_{i\text{-th piece}} = \sum_{i=0}^{n-1} \frac{F'(x_i)(b-a)}{n}.}$$

This is called the **left Riemann sum approximation**.

Question Geometrically, we are approximating area under $F'(x)$ using rectangles. Can you draw the picture to visualize this?

Definite Integral

Recall the Riemann sum approximation becomes more and more accurate when Δx is smaller.

Definition The **definite integral** of $f(x)$ from a to b is

$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{f(x_i)(b-a)}{n}}$$

Remark

- Geometrically, this represents the area under $f(x)$ from a to b.
- Physically, this represents the accumulation of something from time a to time b if its rate of change if $f(x)$, e.g. speed is the accumulation of acceleration, so speed is the integral of acceleration.

The Fundamental Theorem of Calculus

Theorem (The Fundamental Theorem of Calculus (FTC)) *If* $F'(x) = f(x)$, *then*

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a).$$

So if we know $f(x)$ and want to calculate an integral, all we need is to find $F(x)$ and apply the above theorem. This brings us to the next topic.

Anti-derivatives

Definition Suppose $F'(x) = f(x)$, then we say $F(x)$ is an **anti-derivative** of $f(x)$.

There can be many anti-derivatives. If we know $F(x)$ is the anti-derivative of $f(x)$, then adding a constant C to it will still make it an anti-derivative.

Definition Suppose $F(x)$ is the anti-derivative of $f(x)$, then the family of anti-derivatives of $f(x)$ is denoted by

$$\int f(x)dx = F(x) + C$$

where C is called the **constant of integration**.

Common anti-derivatives

Keep the following in mind.

1. $\int 0 dx = C$;
2. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$;
3. $\int e^x dx = e^x + C$;
4. $\int \cos x dx = \sin x + C$;
5. $\int \sin x dx = -\cos x + C$;

If k is a number and f, g are two functions with anti-derivatives F, G respectively, then

$$\int \Big(k \cdot f(x) + g(x)\Big) dx = k \cdot F(x) + G(x) + C$$

because the derivative of $kF(x) + G(x)$ is $\boxed{k \cdot f(x) + g(x)}$.

Example

Question If my speed is $(6x^2 + 2e^x)$ m/s, what could my distance function be at time x?

1. Distance is the integral of speed, so this question is asking you to solve an integral.
2. What is the unit of the final answer?

Example Find the solutions to the first order differential equation

$$y' = \cos(u)$$

You should get a family of solution with a parameter C.

Example Solve the initial value problem

$$\frac{dy}{dx} = 24x^5, \quad y(1) = 3$$

You should get a single solution. The condition $y(1) = 3$ should help you get determine an exact value for the parameter C.

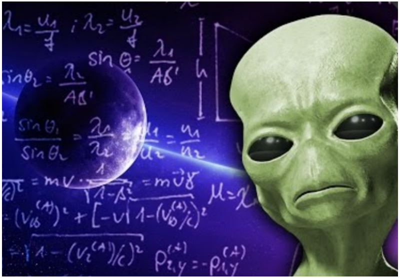
Piecewise defined functions

The most basic example of piecewise defined function is.

$$f(x) = \begin{cases} +x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$$

Question If $x < -2$, what is $|x|$?

Question Come up with a real life situation where you need piecewise defined functions, say with 3 or more conditions.



Expressing functions with absolute values

Question Express
$$f(x) = x|x^2 - 1| + 5|2x - 3|$$
without the use of the absolute value symbol.

1. Where did the function change behavior?
   These points are called **transition point**.
2. Split the x-axis using the transition points.
3. Write down the expression of the function on each piece, and combine everything into a piecewise defined function.

Expressing errors with absolute values

Remark If $|x - y| \leq z$ for some number z, that means the difference between x and y is at most z. This idea is used to express a range for error.

Example Suppose I measured my height h to be 175cm, but my ruler has an error of at most 5cm. So

$$|h - 175| \leq 5.$$

Example Solve the inequality $|h - 175| < 5$.

1. We don't like the absolute value sign, so first rewrite the left hand side into a piecewise defined function like before.
2. Then solve the inequalities on each piece. This you know how to do.

Alternatively, you can say that my height must be between 170 and 180 by the interpretation of the error. So the answer is $170 < h < 180$.

Derivative of absolute value

Recall that

$$\boxed{|x| = \sqrt{x^2}}$$

so we have

$$\begin{aligned}
(|x|)' = \left(\sqrt{x^2}\right)' &= \frac{1}{2\sqrt{x^2}} \cdot (x^2)', \text{ by chain rule} \\
&= \frac{2x}{\sqrt{2x^2}} \\
&= \frac{x}{\sqrt{x^2}} \\
&= \frac{x}{|x|}
\end{aligned}$$

Integral functions

Definition For a number a, and a function $f(x)$,

$$g(t) = \int_a^{t} f(x)dx$$

is a function with variable t. Such a function is called an **integral function**.

Question

1. We write $g(t)$ instead of $g(x)$, why?
2. What are the geometric and physical interpretations of $g(t)$?

Theorem (FTC - Part 1) *The derivative of* $g(t) = \int_a^{t} f(x)dx$ *is*

$$g'(t) = f(t).$$

Example What is the derivative of

$$g(t) = \int_0^{\sin(t)} \cos(x)dx$$

Don't forget to use the chain rule.

Inverse functions

Definition Suppose f is a function. The inverse f^{-1} is the function such that

$$f^{-1}(f(x)) = x \text{ and } f(f^{-1}(x)) = x \qquad \text{(cancellation rule)}$$

The domain and range of f^{-1} are the range and domain of f, respectively.

Remark Always write down the domain and range when you deal with inverse functions.
Also, you should check whether an inverse even exists using horizontal line test.

Example On which interval does $\sin(x)$ admit an inverse? What are the domain and range of the inverse if it exists?

(A) $[0, \pi/2)$;

(B) $[0, \pi)$;

(C) $[-\pi/2, \pi/2)$.

Example

Example We know $\tan(x)$ is invertible on the interval $(-\pi/2, \pi/2)$. Let us call this inverse $\tan^{-1}$.

1. What is $\tan^{-1}(0)$?
2. What is $\tan^{-1}(\tan(0))$?
3. What is $\tan(\tan^{-1}(123))$?
4. What is $\tan^{-1}(\tan(2\pi))$?

Think of it like this

$\tan^{-1}(123)$ **means the** tan **of this thing is** 123**.**

3. tan of $\tan^{-1}(123)$ is 123, so

$$\tan(\tan^{-1}(123)) = 123.$$

4. $\tan^{-1}(\tan(2\pi))$ is something such that the tan of it is $\tan(2\pi)$, but does that mean that thing has to be 2π?

Exponential function

Definition For $\boxed{a > 0}$, the function

$$\boxed{a^x}$$

is called an **exponential function** (in the variable x).

Assuming $a, b > 0$, we have

1. $a^x \cdot a^y = a^{x+y}$;
2. $\dfrac{1}{a^x} = a^{-x}$;
3. $(a^x)^y = a^{xy}$;
4. $(ab)^x = a^x b^x$;
5. $a^0 = 1$.

Question Do you remember the derivative and integral of e^x? What about 2^x?

Logarithmic function

Definition For $\boxed{a > 0 \text{ and } a \neq 1}$, the inverse function of a^x is

$$\log_a x$$

such a function is called a **logarithmic function** (in the variable x).

Question What are the domain and range for log functions? What about their derivatives?

Assuming $a > 0$ and $a \neq 1$, we have

1. $a^{\log_a x} = \boxed{x}$;
2. $\log_a(a^x) = \boxed{x}$;
3. $\log_a(xy) = \boxed{\log_a x + \log_a y}$.
4. $\log_a(x^y) = \boxed{y \log_a x}$.
5. $\log_a x = \dfrac{\ln x}{\ln a}$.

Approximating $\ln x$

We know $\int \frac{1}{x} = \ln|x|$. We also have a formula

$$\boxed{\ln t = \int_1^t \frac{1}{x} dx.}$$

Example Approximate $\ln 4$ using right Riemann approximation by dividing a interval into 6 pieces.

1. Under which function is the area you are approximating?
2. From where to where are should this area go?
3. How long is the base of each rectangle?

Logarithmic scale

Definition We say a function is a **logarithmic scale of base** a if

adding b to the output means the input has to be multiplied by a^b.

Example Some aliens measure earthquake on a base-9 logarithmic scale. The input is badness and the output is the level.

1. If one earthquake is level 3 and another has level 6.5, then the badness of the second one is

$$\boxed{9^{3.5} \text{ times more bad}}$$

than the first one.

2. We can compute this value using exp rules

$$9^{3.5} = 9^3 \cdot 9^{0.5} = 729 \cdot 3 = 2187$$

Trig functions

You should know the following (from high school or from a quick review of course notes):

1. How to convert between radian and degree.
2. How to find $\sin\theta, \cos\theta, \tan\theta$ of special angles. (What are the special angles?)
3. Given one of $\sin\theta, \cos\theta, \tan\theta$, draw a triangle for the angle and find the other two values for that angle.
4. Trig identities. (You need to memorize these)
5. Given one of $\sin\theta, \cos\theta, \tan\theta$, find all possible values of θ within a specific interval, e.g. $[0, 4\pi]$.
6. Graph a trig function.

Example Suppose a thing is moving according to a sin wave. It has period a. An average height of b, an amplitude of c. The function that models this thing is

$$\boxed{f(x) = c\sin\left(\frac{2\pi}{a}x\right) + b}$$

1. In practice, you might be given other data about the shape of the wave, and deduce the period/height/amplitude.
2. Conversely, you should be able to retrieve data about the wave by looking at its formula.

Inverse of $\sin$ and all that

Definition The function

$$\boxed{\arcsin(x)}$$

is the inverse of $\sin(x)$ with respect to the "special domain" $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Definition The function

$$\boxed{\arccos(x)}$$

is the inverse of $\cos(x)$ with respect to the "special domain" $[0, \pi]$.

Definition The function

$$\boxed{\arctan(x)}$$

is the inverse of $\tan(x)$ with respect to the "special domain" $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Inverse of $\sin$ and all that

Example You know how to find things like $\tan(\arctan(\frac{3}{4}))$, but can you find

$$\sin\left(\arctan\left(\frac{3}{4}\right)\right)?$$

1. To begin, think of $\arctan\left(\frac{3}{4}\right)$ as the angle θ such that when you apply tan to it, you get $\frac{3}{4}$.
2. Then draw the triangle for θ, and deduce sin of θ from the triangle.

(Anti-)derivatives of trigs

Just memorize them.

Function	Derivative
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\cot(x)$	$-\csc(x)^2$
$\arcsin(x)$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\dfrac{1}{1+x^2}$

Integral	Result
$\int \tan(x)dx$	$\ln\lvert\sec(x)\rvert + C$
$\int \sec(x)dx$	$\ln\lvert\sec(x)+\tan(x)\rvert + C$
$\int \csc(x)dx$	$-\ln\lvert\csc(x)+\cot(x)\rvert + C$

Factorial

Definition For any integer $n \geq 1$,

$$n! = 1 \cdot 2 \cdots n$$

read as "n factorial". We also define specially

$$0! = 1$$

Example Simplify $\dfrac{n!}{(n+1)!}$.

Definition The terms $\binom{n}{m}$ are called binomial coefficients, given by

$$\binom{n}{m} = \frac{n!}{(n-m)!m!}$$

The integers n, m must satisfy $\boxed{}$.

Implicitly defined function

Remember the formula of a circle? it is

$$x^2 + y^2 = 1$$

The radius of this circle is $\boxed{}$.
This is an example of an implicit function; an explicit expression for it is

$$y = \pm\sqrt{1 - x^2}$$

But a function can only have one output given a input, so this is not a function.

Function given by an equation, which does not have an explicit form, are called **implicitly defined function**.

Implicitly defined function

Suppose we are given an implicit function

$$\boxed{\text{(something about } x, y) \quad = \quad \text{(something else about } x, y)}$$

We can take derivative on both sides with respect to x and get

$$\boxed{\text{(thing about } x, y, y') \quad = \quad \text{(another thing about } x, y, y')}$$

Then we isolate y' and get the derivative

$$\boxed{y' = \frac{dy}{dx} = \quad \text{(whatever you get in terms of } x, y)}$$

Now if you want the tangent line at $(x, y) = (a, b)$, you use the point-slope formula:

$$\boxed{y = \text{slope} \cdot (x - a) + b}$$

The slope is just the derivative at $(x, y) = (a, b)$, i.e.

$$\boxed{\text{slope} = \left.\frac{dy}{dx}\right|_{x=a, y=b}.}$$

Log diff of product of a bunch of things

Example Consider $y = \dfrac{(x+1)^2(x-2)^3}{x^5\sqrt{x-1}}$ where $x > 2$.

1. What is the domain of this function if I don't say $x > 2$?
2. Take log of both sides and get an implicit function.
3. Make the function easier using log rules.
4. Take derivative and find y'. This should be much faster than doing product rules with the original function.
5. Why do we need $x > 2$?