

PMATH 464 Algebraic geometry

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This note is intended as additional material for selected lectures. You are recommended to take your own notes in lecture and do your own research. There may be concepts from other courses (Galois theory, algebraic topology, commutative algebra, etc) that you might have not seen. You are expected to learn these concepts yourself. This note will guide you through the research process if you are lost on how to begin, and you will hopefully be able to learn later materials by yourself. Feel free to contact me if you want advice on how to research for a certain topic.

1. WEEK 1

1.1. The graph of a function is a Riemannian surface. Suppose $f : \mathbb{C} \rightarrow \mathbb{C}$ is a complex single-valued function. The graph of f is $\text{graph}(f) = \{(x, f(x)) | x \in \mathbb{C}\}$. For example, if $f(x) = x^2$, then the graph is $\{(x, x^2) | x \in \mathbb{C}\} \subseteq \mathbb{C}^2$. Consider the map

$$\pi : \text{graph}(f) \rightarrow \mathbb{C}, \quad \pi(x, f(x)) = x$$

This is a continuous bijection, so topologically the graph is just a copy of \mathbb{C} . This is analogous to the real case, where the parabola $\{(x, x^2) | x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ can be projected down to the x -axis which is a continuous bijection, so topologically the parabola is just a line.

Exercise 1.1. The map π is a continuous bijection (homeomorphism), but it can have more structure. Which one of the following properties does π satisfy and which does it not?

π is a ...

- (1) diffeomorphism
- (2) biholomorphism
- (3) symplectomorphism
- (4) conformal map
- (5) isometry

Remark 1.2. The graph of a holomorphic function $f : \mathbb{C} \rightarrow \mathbb{C}$ is always a Riemann surface. More generally, the graph of a holomorphic function $f : X \rightarrow Y$ between complex manifolds is a complex manifold. Look up Riemannian surfaces if you are interested. \diamond

The graph of a multi-valued function is more complicated. We will see in the next section that the graph of the complex circle $x^2 + y^2 = 1$ is a (Riemann) surface. For the function $y = x^{1/2}$, the graph is easy: it is just \mathbb{C} because of the projection $(y^2, y) \mapsto y$ is a continuous bijection. Essentially this is due to the inverse function $x = y^2$ is single-valued. This is analogous to the fact that the graph of the real function $y = \pm\sqrt{x}$ is just the parabola rotated by 90 degrees, and can be projected to the y -axis.

Exercise 1.3. You may have seen in complex analysis the Riemann surface that is the graph of the complex multi-valued function $y = \log x$. Look up what this surface is and find a picture of it.

1.2. Gluing the branch cuts of the circle $x^2 + y^2 = 1$. Consider the “complex circle” $\{(x, y) \in \mathbb{C}^2 | x^2 + y^2 = 1\}$. Since this is a subspace defined by one equation in the 2-dimensional space \mathbb{C}^2 , we expect it to be of dimension 1. However, this is complex dimension 1, which is real dimension 2, so the “complex circle” is a surface in the real world, and we imagine it.

The set $\{(x, y) \in \mathbb{C}^2 | x^2 + y^2 = 1\}$ can be also interpreted as the graph of the multi-valued function $y = (1 - x^2)^{1/2}$. This is analogous to the example that the parabola $\{(x, y) \in \mathbb{R}^2 | y = x^2\}$ is the graph of the single valued function $f(x) = x^2$, where $\text{graph}(f) = \{(x, f(x)) | x \in \mathbb{R}\}$.

Recall from complex analysis that the function $y = f(x) = (1 - x^2)^{1/2}$ can be interpreted as a complex single-valued function with a branch-cut. For our purpose the branch cut we choose is by removing the interval $[-1, 1]$. Consequently, we have two ways of defining f

$$f_1 : \mathbb{C} \setminus [-1, 1] \rightarrow \mathbb{C}, f_2 : \mathbb{C} \setminus [-1, 1] \rightarrow \mathbb{C}$$

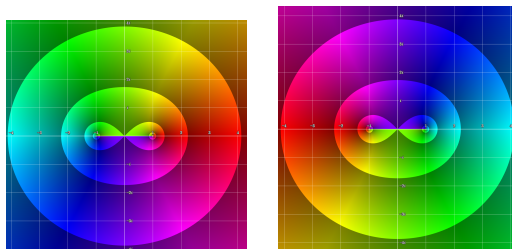
such that $f_1(x), f_2(x)$ are the two values of $(1 - x^2)^{1/2}$ for each x in their domains.

Consider the map

$$\pi_1 : \text{graph}(f_1) \rightarrow \mathbb{C} \setminus [-1, 1], \quad \pi_1(x, f_1(x)) = x.$$

This map is a continuous bijection, which means the graph of f_1 we are interested in is homeomorphic to $\mathbb{C} \setminus [-1, 1]$. You can think of this as cutting a small segment into the middle of a piece of paper with a scissor. We can write down at each point x , the value of $f_1(x)$ to keep track of the function f_1 , but topologically it is a flat piece of paper. Similarly, the graph of f_2 is also homeomorphic to $\mathbb{C} \setminus [-1, 1]$.

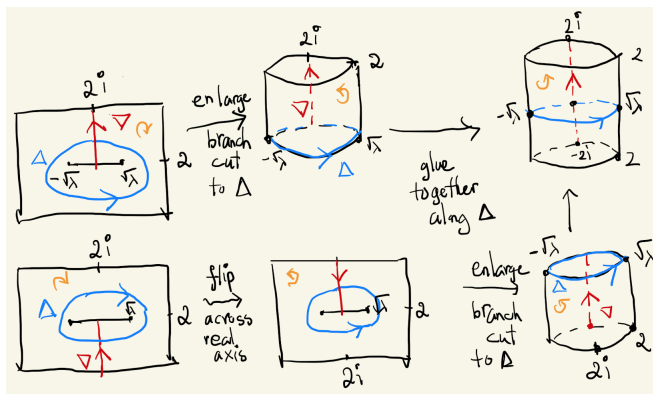
You have seen in class the following two images for a visualization of these graphs, which can be thought of as coloring the paper based on the value of f_1 and f_2 .



Let us refer to the pictures. In image 1, the point 1.1 is mapped to red, while in the second image the point -1.1 is mapped to red, which reflects the fact that $f_1(1.1) = f_2(-1.1) = \text{red}$.

Now we need to combine the two graphs into a single surface to get the (multi-valued) graph $\{(x, y) \in \mathbb{C}^2 | x^2 + y^2 = 1\}$. This is done by glueing the two pieces of papers together along the interval $[-1, 1]$. Note that when the paper is cut, the interval $[-1, 1]$ splits into two, one attached to the top half of the paper, and one to the bottom. In the first image, the top half is green and the bottom half blue, and vice-versa for the second image. When glueing them together, we match the colors and obtain a single surface which looks like a tube.

Exercise 1.4. I found the following image online for demonstration. Try researching yourself and see if you can find more detailed explanations of this process.



More precisely, let $\overline{\text{graph}(f_1)}$ be the topological space obtained by attaching the two points $\{\pm 1\}$ and two copies of $(-1, 1)$ (forming top and bottom copies of $[-1, 1]$), and similarly $\overline{\text{graph}(f_2)}$. The graph of the complex circle is

$$\left(\overline{\text{graph}(f_1)} \sqcup \overline{\text{graph}(f_2)} \right) / \sim$$

where \sqcup denotes disjoint union, and \sim is the equivalence relation that identifies the copies of $[-1, 1]$ by color.

Exercise 1.5. Make the above explanation completely rigorous (using algebra instead of pictures and colors). Recall the definition of quotient space and gluing from algebraic topology.

1.3. The monodromy group of the multi-valued function $y = x^{1/2}$.

There are many ways of observing the monodromy of a covering map, for which you may refer to your own notes from the lecture. Here we give some detail for the definition using “curve lifting”.

The most common definition for *monodromy* you will find in the context of covering maps. Let X, Y be connected topological spaces. Suppose $p : X \rightarrow Y$ is a covering map of degree d for some positive integer d . We know from definition the fiber $F = p^{-1}(y)$ consists of d points for any $y \in Y$. Also, as a property of covering maps, we have the *unique lifting property* which says: if $\gamma : [0, 1] \rightarrow Y$ is a path starting at $\gamma(0)$ and ending at $\gamma(1)$, then we may specify a starting point $x \in p^{-1}(\gamma(0))$, and there exists a unique lift $\tilde{\gamma} : [0, 1] \rightarrow X$ that starts at x and ends at some point in $p^{-1}(\gamma(1))$.

Exercise 1.6. Recall the definition of a covering space and the proof of unique lifting property from algebraic topology.

To define a monodromy, we also need the notion of *fundamental groups* $\pi_1(X)$. As explained in class, this can be naively understood as the group of loops on X , up to *homotopy*, i.e. two loops are considered equivalent if you can interpolate them from one to the other via a continuous family of other loops. The addition operator of the fundamental group is given by concatenating two loops.

For example, if X is $\mathbb{C} \setminus \{0\} \cong \mathbb{R}^2 \setminus \{0\}$, then the fundamental group $\pi_1(X)$ is \mathbb{Z} , where $1 \in \mathbb{Z}$ corresponds to a loop that wraps the origin once counterclockwise, and -1 corresponds to a loop that wraps around the origin once clockwise. If a loop does not wrap around the origin, then you can shrink it to a single point (the constant loop) and say this loop is homotopic to the constant loop, which corresponds to the identity $0 \in \mathbb{Z}$.

If γ is the loop $\gamma : [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$ given by the circle $\gamma(t) = e^{2\pi it}$, then the equivalence class of γ in the fundamental group, denoted $[\gamma]$, correspond to the number 1 in \mathbb{Z} . If σ is the double speed of γ , given by $\sigma(t) = e^{4\pi it}$, which wraps around the origin two times, then $[\sigma]$ corresponds to 2.

Exercise 1.7. Recall the definition of fundamental group from algebraic topology. What is the fundamental group of $\mathbb{R} \setminus \{\pm 1\}$? Also take note that I have simplified the notations by assuming X, Y are connected.

Definition 1.8. The *monodromy* of the covering map $p : X \rightarrow Y$ is the group homomorphism

$$\begin{aligned} \varphi : \pi_1(Y) &\rightarrow \text{Aut}(F) \cong S_d \\ [\gamma] &\mapsto (\rho : F \rightarrow F, \rho(x) = \tilde{\gamma}_x(1)) \end{aligned}$$

where $F = p^{-1}(y)$ for any $y \in Y$, S_d denotes the symmetric group of degree d , and $\tilde{\gamma}_x$ is the unique lift of γ starting at x .

The automorphism $\varphi([\gamma])$, which is a permutation on a set with d elements, is called the *monodromy action* associated to γ .

The *monodromy group* of the covering map $p : X \rightarrow Y$ is the image of φ , which is a subgroup of the symmetric group S_d .

Exercise 1.9. Convince yourself why the above definition is independent of the choice of y .

In complex analysis, the function $x^{1/2}$ only makes sense as a complex, single-valued function after we choose a branch cut on \mathbb{C} . However, we can also interpret it as “multi-valued function” $y = x^{1/2}$, sending each $x \in \mathbb{C}$ to its two square roots.

Exercise 1.10. Look up a rigorous definition for “multi-valued function”.

Now we want to make sense of the phrase “monodromy group of the multi-valued function $y = x^{1/2}$ ”. But we only know what a monodromy is for covering maps. Therefore we should try to construct a covering map out of $y = x^{1/2}$, then consider the monodromy of that covering map. The map we want is

$$f : \mathbb{C} \rightarrow \mathbb{C}, \quad x \mapsto x^2,$$

which is the inverse of the function $y = x^{1/2}$. Since $x^{1/2}$ sends one point to two values, the inverse sends two points to the same value, making f a covering map of degree 2. Of course there is the exceptional point 0, over which the function is one to one. Hence we need to take out the origin, and obtain a genuine covering map

$$p := f|_{\mathbb{C} \setminus \{0\}} : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\}.$$

Exercise 1.11. The circle $\gamma(t) = e^{2\pi it}$ lifts via the covering map p to a path $\tilde{\gamma}$ which starts at the point 1. What is the endpoint $\tilde{\gamma}(1)$ of this path? What about the lift whose starting point is -1 ?

Exercise 1.12. Based on the above exercise, what is the monodromy group of p ? Remember it should be a subgroup of S_2 . What is the monodromy action of the circle γ ?

Now it makes sense to talk about the monodromy group of $y = x^{1/2}$: it is simply the monodromy of p .

Finally, for your assignment, you will be computing the monodromy group of the multi-valued function $x^2 + y^2 = 1$ (or some other more complicated function). If you do not know how to proceed, try to first define the covering map p . The map p might not be directly analogous to the definition above, but that is what you need to figure out. Alternatively, you may use other examples such as braid groups to come up with your own definition of the monodromy. It will work as long as you have the correct understanding.