Chap.5 Diagonalization

§5-1 Eigenvalue and Eigenvectors

This chapter begins the "second half" of the linear algebra. The first half was about "A$x = b$". The new problem "A$x = \lambda x$" will still be solved by simplifying a matrix - making ait diagonal if possible.

1. Does there exist an ordered basis β for V such that $[T]_\beta$ is diagonal matrix ?
2. If such a basis exists how can it be found ?

<u>Definition</u>. $T \in L(\mathrm{V})$, V is finite dimensional vector space. T is "diagonalizable" if there exist $\beta \ni [T]_\beta$ is a diagonal matrix. A square matrix A is called diagonalizable if L_A is diagonalizable.

Note that if $D = [T]_\beta$ is a diagonal matrix, for $\beta = \{v_1, \cdots, v_n\}$, we have

$$\begin{cases} T(v_1) = & D_{11}v_1 + D_{21}v_2 + \cdots + D_{1n}v_n \\ T(v_2) = & D_{12}v_1 + D_{22}v_2 + \cdots + D_{n2}v_n \\ \quad\vdots & \\ T(v_n) = & D_{1n}v_1 + D_{2n}v_2 + \cdots + D_{nn}v_n \end{cases}$$

$\because [T]_\beta$ is diagonal matrix $\Longrightarrow \forall\, i, j$, $i \neq j$, $D_{ij} = 0$ and $D_{ji} = 0$.
$\therefore$ for $1 \leq j \leq n$, $T(v_j) = D_{jj}v_j$ take $D_{jj} = \lambda_j$ we obtain the formulation

$$\mathrm{A}x = \lambda x$$

Notice : To diagonalize the matrix or a linear operator is to find a basis of eigenvalue and the corresponding eigenvectors.

Example

Definition. Let A $\in$ $M_{n\times n}(F)$. The polynomial $f(t) = \det(\mathrm{A} - t\mathrm{I})$ is called a characteristic polynomial of A.

Definition. Let $T \in L(V)$, V is finite dimensional vector space , β is a ordered basis for V. We define the characteristic polynomial of T to be the characteristic polynomial of $A = [T]_\beta$ i.e. $f(t) = \det([T]_{beta} - t\mathrm{I})$.

Remark. The eigenvalue of $[T]_\beta$ is also an eigenvalue of T.

• Some properties between T and $[T]_\beta$:
V : n-dimensional vector space , $T \in L(V)$, define determinant of T denoted $\det(T)$ is choosing any ordered basis β for V , define $\det(T) = \det([T]_\beta)$ then the followings are true :

1. If β and γ are two ordered basis for $V \implies \det([T]_\beta) = \det([T]_\gamma)$
2. T is invertible $\Leftrightarrow \det(T) \neq 0$
3. T is invertible $\implies \det(T^{-1}) = \det(T)^{-1}$
4. If $U \in L(V)$, then $\det([TU]_\beta) = \det([T]_\beta)\det([U]_\beta)$
5. $\det(T - \lambda\mathrm{I}) = \det([T]_\beta - \lambda\mathrm{I})$