Chapter 2 Linear Transformations and Matrices

§ 2-1 Linear transformation, Null Spaces, and Ranges

Definition (Definition of Linear Transformation).

Let V and W be vector spaces (over F). We call a function $T : V \to W$ a linear transformation from V to W if, for all $x, y \in V$ and $c \in F$, we have

(a) $T(x + y) = T(x) + T(y)$

(b) $T(cx) = cT(x)$

Definition (Definition of nullity and rank).

Let V and W be vector spaces and let $T : V \to W$ be linear. If N(T) and R(T) are finite-dimensional, then we define the nullity of T, denoted nullity(T), and the rank of T, denoted rank(T), to be the dimensions of N(T) and R(T), respectively.

Theorem.

Let V and W be vector spaces and $T : V \to W$ be linear. Then N(T) and R(T) are subspaces of V and W, respectively.

Theorem.

Let V and W be vector spaces, and let $T : V \to W$ be linear. If $\beta = \{v_1, v_2, \cdots, v_n\}$ is a basis for V , then $R(T) = \text{span}(T(\beta)) = \text{span}(\{T(v_1), T(v_2), ..., T(v_n)\})$.

Example 11

Question :

Let T : $P_2(R) \to P_3(R)$ be the linear trans formation defined by

$$T(f(x)) = 2f'(x) + \int_0^x 3f(t)dt$$

solution:.

Now $\mathrm{R(T)} = \mathrm{Span}(\{\mathrm{T}(1), \mathrm{T}(x), \mathrm{T}(x^2)) = \mathrm{Span}(\{3x, 2 + \frac{3}{2}x^2, 4x + x^3\})$.

Since { $3x$, $2 + \frac{3}{2}x^2$, $4x + x^3$ } is linearly independent, rank(T) = 3. Since 2 $\dim(P_3(\mathbb{R})) = 4$, T is not onto. From the dimension theorem (Thm 2.3) , nullity(T) + 3 = 3. So nullity(T) = 0, and therefore, N(T) = 0. We conclude from Theorem 2.4 that T is one-to-one.

Example 12

Question :

Let T : $F^2 \to F^2$ be the linear transformation defined by

$$T(a_1, a_2) = (a_1 + a_2, a_1)$$

solution:.

It is easy to see that N(T) = { 0 } ; so T is one-to-one. Hence Theorem 2.5 tells us that T must be onto.

Theorem (2.3 Dimension Theorem).

Let V and W be vector spaces, and let T : V $\to$ W be linear.

If V is finite-dimensional, then

$$\mathrm{nullity(T)} + \mathrm{rank(T)} = \dim(\mathrm{V}).$$

Theorem (2.4).

Let V and W be vector spaces, and let $T : V \to W$ be linear. Then T is one-to-one if and only if $N(T) = 0$.

Theorem (2.5).

Let V and W be vector spaces of equal (finite) dimension, and let $T : V \to W$ be linear. Then the following are equivalent.

(a) T is one-to-one.

(b) T is onto.

(c) $\mathrm{rank}(T) = \dim(V)$.

Example

Let $\mathrm{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $\mathrm{T}(\mathrm{a}_1, \mathrm{a}_2) = (\mathrm{a}_1 + \mathrm{a}_2, 0, 2\mathrm{a}_1 - \mathrm{a}_2)$, determine that T is linear , 1-1 , onto or not.

solution:.

- Calim : T is linear.

  Let $x = (a_1, a_2)$, $y = (b_1, b_2)$

  $$\begin{aligned}
  \mathrm{T}(\mathrm{cx} + \mathrm{y}) = \mathrm{T}\left(\mathrm{c}\left(\mathrm{a}_1, \mathrm{a}_2\right) + (\mathrm{b}_1, \mathrm{b}_2)\right) &= \mathrm{T}\left(\mathrm{ca}_1 + \mathrm{b}_1, \mathrm{ca}_2 + \mathrm{b}_2\right) \\
  &= (\mathrm{ca}_1 + \mathrm{b}_1 + \mathrm{ca}_2 + \mathrm{b}_2, 0, 2\mathrm{ca}_1 + 2\mathrm{b}_1 - \mathrm{ca}_2 - \mathrm{b}_2) \\
  &= \left(\mathrm{c}\left(\mathrm{a}_1 + \mathrm{a}_2\right) + \left(\mathrm{b}_1 + \mathrm{b}_2\right), 0, \mathrm{c}\left(2\mathrm{a}_1 - \mathrm{a}_2\right) + \left(2\mathrm{b}_1 - \mathrm{b}_2\right)\right) \\
  &= \mathrm{c}\left(\mathrm{a}_1 + \mathrm{a}_2, 0, 2\mathrm{a}_1 - \mathrm{a}_2\right) + \left(\mathrm{b}_1 + \mathrm{b}_2, 0, 2\mathrm{b}_1 - \mathrm{b}_2\right) \\
  &= \mathrm{cT}(\mathrm{x}) + \mathrm{T}(\mathrm{y})
  \end{aligned}$$

  $\therefore$ T is linear

- 1-1 and onto

  By Thm 2.2 in text book choose a basis for $\mathbb{R}^2, \beta = \{(1,0), (0,1)\}$

  $\mathrm{R}(\mathrm{T}) = \mathrm{span}\left(\mathrm{T}(\beta)\right) = \mathrm{span}(\{\mathrm{T}(1,0), \mathrm{T}(0,1)\}) = \mathrm{span}\left((1,0,-1), (1,0,-2)\right)$

  Clearly it is L.I. $\implies$ $\mathrm{rank}(\mathrm{T}) = 2$, and apply the Dimension Theorem $\mathrm{rank}(\mathrm{T}) = 2 \neq 3 = \dim(\mathbb{R}^3)$, and $\mathrm{nullitily}(\mathrm{T}) = 1$, then it is not onto.

  By Thm 2.4 it is not one to one.

Example

Let V and W be vector spaces, let T : V $\to$ W be linear, and let $\{w_1, w_2, \ldots, w_k\}$ be a linearly independent subset of R(T). Prove that if $S = \{v_1, v_2, \ldots, v_k\}$ is chosen so that $\mathrm{T}(v_i) = w_i$ for $i = 1, 2, \ldots, k$, then S is linearly independent.

solution:.

Calim : $\sum\limits_{i=1}^{n} a_i v_i = 0 \implies a_1 = a_2 = a_3 = \cdots = a_n = 0$

Let $\sum\limits_{i=1}^{n} a_i v_i = 0$ then $\mathrm{T}\left(\sum\limits_{i=1}^{n} a_i v_i\right) = 0$

Since T is linear , $\mathrm{T}\left(\sum\limits_{i=1}^{n} a_i v_i\right) = \sum\limits_{i=1}^{n} a_i \mathrm{T}(v_i) = \sum\limits_{i=1}^{n} a_i w_i = 0$

Since S is L.I. $\implies a_1 = a_2 = \cdots = a_n = 0$

Theorem (2.6).

Let V and W be vector spaces over F, and suppose that $\{ v_1, v_2, \cdots, v_n \}$ is a basis for V. For $w_1, w_2, \cdots, w_n$ in W, there exists exactly one linear transformation T : V $\to$ W such that T(v_i) = w_i for i = 1,2,$\cdots$,n.

§ 2-2 The Matrix Representation of a Linear transformation

<u>Definition</u> (Definition of an Ordered Basis).

Let V be a finite-dimensional vector space. An ordered basis for V is a basis for V endowed with a specific order; that is, an ordered basis for V is a finite sequence of linearly independent vectors in V that generates V.

Definition (Definition for the symbol $[x]_\beta$).

Let $\beta = \{u_1, u_2, \cdots, u_n\}$ be an ordered basis for a finite- dimensional vector space V. For $x \in$ V, let $a_1, a_2, \cdot$, an be the unique scalars such that $x = \sum_{i=1}^{\mathrm{n}} a_i u_i$ We define the coordinate vector of x relative to β, denoted $[x]_\beta$, by

$$[x]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Notice that $[u_i]_\beta = e_i$ in the preceding definition. It is left as an exercise to show that the correspondence $x \to [x]_\beta$ provides us with a linear transformation from V to F^n. We study this transformation in Section 2.4 in more detail.

Definition (Definition for the symbol $[\mathrm{T}]_\beta^\gamma$).

Using the notation above,we call the m $\times$ n matrix A defined by $\mathrm{A}_{ij} = \mathrm{a}_{ij}$ the matrix representation of T in the ordered bases β and γ and write $\mathrm{A} = [\mathrm{T}]_\beta^\gamma$. If $\mathrm{V} = \mathrm{W}$ and $\beta = \gamma$, then we write $\mathrm{A} = [\mathrm{T}]_\beta$. Notice that the jth column of A is simply $[\mathrm{T}(v_j)]_\gamma$. Also observe that if $\mathrm{U} : \mathrm{V} \to \mathrm{W}$ is a linear transformation such that $[\mathrm{U}]_\beta^\gamma = [\mathrm{T}]_\beta^\gamma$, then $\mathrm{U} = \mathrm{T}$ by the corollary to Theorem 2.6 (p. 73).

Definition (The addition and multiplication for Linear Transformation).

Let $\mathrm{T}, \mathrm{U} : \mathrm{V} \to \mathrm{W}$ be arbitrary functions, where V and W are vector spaces over F, and let $a \in$ F. We define $\mathrm{T} + \mathrm{U} : \mathrm{V} \to \mathrm{W}$ by $(\mathrm{T} + \mathrm{U})(x) = \mathrm{T}(x) + \mathrm{U}(x)$ for all $x \in$ V, and $a\mathrm{T} : \mathrm{V} \to \mathrm{W}$ by $(a\mathrm{T})(x) = a\mathrm{T}(x)$ for all $x \in$ V.

Of course, these are just the usual definitions of addition and scalar multiplication of functions. We are fortunate, however, to have the result that both sums and scalar multiples of linear transformations are also linear.

Theorem (2.7).

Let V and W be vector spaces over a field F, and let $\mathrm{T}, \mathrm{U} : \mathrm{V} \to \mathrm{W}$ be linear.

(a) For all $a \in F$, $a\mathrm{T} + \mathrm{U}$ is linear.

(b) Using the operations of addition and scalar multiplication in the preceding definition, the collection of all linear transformations from V to W is a vector space over F.

<u>Definition</u> (Definition of the Linear Operator).

Let V and W be vector spaces over F. We denote the vector space of all linear transformations from V into W by L(V, W). In the case that V = W, we write L(V) instead of L(V, W).

Theorem (2.8).

Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively, and let $\mathrm{T}, \mathrm{U} : \mathrm{V} \to \mathrm{W}$ be linear transformations. Then

(a) $[\mathrm{T} + \mathrm{U}]_\beta^\gamma = [\mathrm{T}]_\beta^\gamma + [\mathrm{U}]_\beta^\gamma$ and

(b) $[a\mathrm{T}]_\beta^\gamma = a[\mathrm{T}]_\beta^\gamma$ for all scalars a.

$\mathcal{Q}snake$: Thm 2.8 means that $[\mathrm{x}]_\beta^\gamma$ has the characteristic of " linear ", too.

§ 2-3 Composition of Linear Transform