Chap.4 Determinants

§4-1 Determinants of order 2

Definition. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2matrix with entries from a field F, then we define the determinant of A, denoted $\det(A)$, to be the scalar $ad - bc$.

Example

§4-2 Determinants of order n

Definition. Let $A \in M_{n\times n}(F)$. If $n = 1$, so that $A = (A_{11})$, we define $\det(A) = A_{11}$.For $n \geq 2$, we define $\det(A)$ recursively as

$$\mathrm{det(A)} = \sum_{\mathrm{j=1}}^{\mathrm{n}} (-1)^{1+\mathrm{j}} \mathrm{A_{1j}} \cdot \mathrm{det}(\widetilde{\mathrm{A}}_{\mathrm{1j}}).$$

The scalar

$$(-1)^{1+\mathrm{j}}\mathrm{det}(\widetilde{\mathrm{A}}_{\mathrm{1j}}).$$

is called the cofactor of the entry of A in row i,column j. Letting

$$\mathrm{c_{ij}} = (-1)^{\mathrm{i+j}}\mathrm{det}(\widetilde{\mathrm{A}}_{\mathrm{ij}}).$$

denote the cofactor of the row i, column j entry of A, we can express the formula for the determinant of A as

$$\mathrm{det(A)} = \mathrm{A_{11}c_{11}} + \mathrm{A_{12}c_{12}} + \cdots + \mathrm{A_{1n}c_{1n}}.$$

Theorem.

The determinant of a square matrix can be evaluated by cofactor expansion along any row. That is, if $A \in M_{n\times n}(F)$, then for any integer $i(1 \leq i \leq n)$,

$$\mathrm{det(A)} = \sum_{\mathrm{j=1}}^{\mathrm{n}} (-1)^{\mathrm{i+j}} \mathrm{A_{ij}} \cdot \mathrm{det}(\widetilde{\mathrm{A}}_{\mathrm{ij}}).$$

Example

Theorem.

The determinant of an $n \times n$ matrix is a linear function of each row when the remaining rows are held fixed. That is, for $1 \leq r \leq n$, we have

$$\det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u + kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + \mathrm{k}\det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}.$$

whenever k is a scalar and u, v, and each a_i are row vectors in F^n

Theorem.

Let $A \in M_{n\times n}(F)$, and let B be a matrix obtained by adding a multiple of one row of A to another row of A. Then $det(B) = det(A)$.

The following rules summarize the effect of an elementary row operation on the determinant of a matrix $A \in M_{n\times n}(F)$.

(a) If B is a matrix obtained by interchanging any two rows of A, then $det(B) =$ -$det(A)$.

(b) If B is a matrix obtained by multiplying a row of A by a nonzero scalar k, then $det(B) = kdet(A)$.

(c) If B is a matrix obtained by adding a multiple of one row of A to another row of A, then $det(B) = det(A)$.

Example

§4-3 Properties of determinants

Theorem.

For any $A, B \in M_{n\times n}(F), det(AB) = det(A) \cdot det(B)$.

Corollary. A matrix $A \in M_{n\times n}(F)$ is invertible if and only if $det(A) \neq 0$. Furthermore, if A is invertible, then $det(A^{-1}) = \dfrac{1}{det(A)}$

Theorem.

For any $A \in M_{n\times n}(F), det(A^t) = det(A)$