CH1 Vector Space Textbook

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Vector Space

Definition (Vector Space).

A vector space (or linear space) W over a $Field$ F consists of a set on which two operations (called addition and scalar multiplication, respectively) are defined so that for each pair of elements x, y, in W there is a unique element $x + y$ in W, and for each element a in F and each element x in W there is a unique element ax in W, such that the following conditions hold.

There are eight rules for calculating the vector space !

Please do not forget !

Definition (Definition of Subspace).

A subset W of a vector space W over a field F is called a subspace of W if W is a vector space over F with the operations of addition and scalar multiplication defined on W.

Theorem (1.3).

Let V be a vector space and W a subset of V. Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V.

(a) $0 \in$ W.

(b) $x + y \in W$ whenever $x \in$ W and $y \in$ W.

(c) $cx \in$ W whenever $c \in$ F and $x \in$ W.

Theorem (1.4).

Any intersection of subspaces of a vector space W is a subspace of V.

Definition.

A vector space V is called direct sum of W_1 and W_2 if W_1 and W_2 are subspace of V such thay $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. We denote that V is the direct sum of W_1 and W_2 by Writting $V = W_1 \oplus W_2$.

Example

Question (Exercise 19) :
Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of W if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Solution.

1. Let W_1 and W_2 be subspace of W, $W_1 \cup W_2$ is a subspace
2. Suppose $W_1 \cup W_2$ is a subspace of W while $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$

   $\implies \exists\, x \in W_1\,,\ x \notin W_2\,,\ \exists\, y \in W_2\,,\ y \notin W_1$
   If $W_1 \cup W_2$ is a subspace , $x, y \in W_1 \cup W_2 \implies x + y \in W_1 \cup W_2$
3. $x + y \in W_1 \cup W_2 \quad \therefore x + y \in W_1$ or $x + y \in W_2$
   Since W_1 and W_2 are subspaces of V $\therefore x \in W_1, y \in W_2$ or $x \in W_2, y \in W_1$
   $\rightarrow\leftarrow$ wtih our suppose in 1.

Definition (Definition of Linear Combination).
Let W be a vector space and S a nonempty subset of V. A vector $v \in V$ is called a linear combination of vectors of S if there exist a finite number of vectors $u_1, u_2, \cdots, u_n$ in S and scalars $a_1, a_2, \cdots, a_n$ in F such that $v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$. In this case we also say that W is a linear combination of $u_1, u_2, \cdots, u_n$ and call $a_1, a_2, \cdots, a_n$ the coefficients of the linear combination.

Definition (Definition of Span).
Let S be a nonempty subset of a vector space V. The span of S, denoted span(S), is the set consisting of all linear combinations of the vectors in S. For convenience, we define span(ϕ) = $\{0\}$.

Example

Question (Exercise 5 (a) (e)) :
In each part, determine whether the given vector is in the span of S.

a. (2,-1,1), S = {(1,0,2),(-1,1,1)}

e. $-x^3+2x^2+3x+3$, S = $\left\{x^3+x^2+x+1, x^2+x+1, x+1\right\}$

Solution.

(a)

1. $\text{Span}(S) = \{a_1(1,0,2)+a_2(-1,1,1) \mid a_1, a_2 \in \text{F}\}$

2. $\begin{cases} a_1 - a_2 = & 2 \\ 0a_1 + a_2. = & -1 \\ 2a_1 + a_2 = & 1 \end{cases} \implies a_1 = 1, a_2 = -1$

3. Since there exist scalars a_1, a_2 such that $(2,-1,1) = a_1(1,0,2)+a_2(-1,1,1)$, we conclude that $(2,-1,1)$ is in the span of S.

(e)

1. $\text{Span}(S) = \{a_1(x^3+x^2+x+1)+a_2(x^2+x+1)+a_3(x+1)\}$

2. $\begin{cases} a_1 + 0a_2 + 0a_3 = & -1 \\ a_1 + a_2 + 0a_3 = & 2 \\ a_1 + a_2 + a_3 = & 3 \\ a_1 + a_2 + a_3 = & 3 \end{cases} \implies a_1 = -1, a_2 = 3, a_3 = 1$

3. Since there exist scalars a_1, a_2, a_3 such that $-x^3+2x^2+3x+3 = a_1(x^3+x^2+x+1)+a_2(x^2+x+1)+a_3(x+1)$, we conclude that $(2,-1,1)$ is in the span of S.

Example

Question (Exercise 6) :
Show that the vectors $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$ generate F^3.

Solution. Let $\mathrm{S} = \{(1,0,1),(1,0,1),(0,1,1)\}$.

Claim : $\mathrm{S} = \{(1,0,1),(1,0,1),(0,1,1)\}$ is linearly independent.

1. $a_1(1,0,1) + a_2(1,0,1) + a_3(0,1,1) = 0 \implies a_1 = a_2 = a_3 = 0$
2. $b_1(1,0,1) + b_2(1,0,1) + b_3(0,1,1) = (x,y,z)$
   $\implies b_1 = \dfrac{x+y-z}{2}, b_2 = \dfrac{x-y+z}{2}, b_3 = \dfrac{-x+y+z}{2}$

Since $\mathrm{S} = \{(1,0,1),(1,0,1),(0,1,1)\}$ is linearly independent, Span(S) can generate F^3.

Example

$\mathrm{S} = \{(0,0,1),(1,0,0)\}$
$Span(S) = \{a_1(0,0,1) + a_2(1,0,0) \mid a_1, a_2 \in \mathrm{F}\}$

Theorem (1.5).
The span of any subset S of a vector space W is a subspace of V. Moreover, any subspace of W that contains S must also contain the span of S.

Definition (Linearly Dependent).
A subset S of a vector space W is called linearly dependent if there exist a finite number of distinct vectors $u_1, u_2, ..., u_n$ in S and scalars $a_1, a_2, ..., a_n$, not all zero, such that

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0.$$

In this case we also say that the vectors of S are linearly dependent.

Definition (Definition of Linearly Independent).
A subset S of a vector space that is not linearly dependent is called linearly independent. As before, we also say that the vectors of S are linearly independent.

Theorem (1.6).
Let W be a vector space,and let $\mathrm{S}_1 \subseteq \mathrm{S}_2 \subseteq \mathrm{V}$. If S_1 is linearly dependent, then S_2 is linearly dependent.
Corollary. Let W be a vector space, and let $\mathrm{S}_1 \subseteq \mathrm{S}_2 \subseteq \mathrm{V}$. If S_2 is linearly independent, then S_1 is linearly independent.

Example

Question (Exercise 2 (a) (c) (e)) :
Determine whether the following sets are linearly dependent or linearly independent.

a. $\left\{ \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}, \begin{bmatrix} -2 & 6 \\ 4 & -8 \end{bmatrix} \right\}$

c. $\left\{ x^3 + 2x^2, -x^2 + 3x + 1, x^3 - x^2 + 2x + 1 \right\}$

e. $\{(1, -1, 2), (1, -2, 1), (1, 1, 4)\}$ in $\mathbb{R}^3$

Solution.

a. Since $2 \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 6 \\ 4 & 8 \end{bmatrix} = 0 \implies$ linearly dependent by Thm 1.5.

Theorem (1.7).
Let S be a linearly independent subset of a vector space V, and let v be a vector in V that is not in S. Then $\mathrm{S} \cup \{v\}$ is linearly dependent if and only if v $\in$ span(S).

<u>Definition</u> (Definition of Bases).
A basis β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.

Theorem (1.8).
Let V be a vector space and $\beta = \{u_1, u_2, ..., u_n\}$ be a subset of V. Then β is a basis for V if and only if each v $\in$ V can be uniquely expressed as a linear combination of vectors of β, that is, can be expressed in the form

$$\mathrm{v} = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

for unique scalars $a_1, a_2, \cdots, a_n$.

Theorem (1.9).
If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence V has a finite basis.

Theorem (1.10 Replacement Theorem).
Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then m $\leq$ n and there exists a subset H of G containing exactly n $-$ m vectors such that L $\cup$ H generates V.

Theorem (Corollary 1.).
Let V be a vector space having a finite basis. Then every basis for V contains the same number of vectors.

<u>Definition</u> (Definition of Dimension).
A vector space is called finite-dimensional if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for V is called the dimension of V and is denoted by $dim(\mathrm{V})$. A vector space that is not finite-dimensional is called infinite-dimensional .

Chapter 2 Linear Transformation

§ 2-1 Linear transformation

Definition (Definition of Linear Transformation).
Let V and W be vector spaces (over F). We call a function T : V $\to$ W a linear transformation from V to W if, for all x, y $\in$ V and c $\in$ F, we have

(a) T(x + y) = T(x) + T(y)

(b) T(cx) = cT(x)

Definition (Definition of nullity and rank).
Let V and W be vector spaces and let T : V $\to$ W be linear. If N(T) and R(T) are finite-dimensional, then we define the nullity of T, denoted nullity(T), and the rank of T, denoted rank(T), to be the dimensions of N(T) and R(T), respectively.

Theorem.
Let V and W be vector spaces and T : V $\to$ W be linear. Then N(T) and R(T) are subspaces of V and W, respectively.

Theorem.
Let V and W be vector spaces, and let T : V $\to$ W be linear. If $\beta = \{v_1, v_2, ..., v_n\}$ is a basis for V , then R(T) = span(T(β)) = span($\{$T(v_1), T(v_2), ..., T(v_n)$\}$).

Example 11

Question :
Let T : $P_2(R) \to P_3(R)$ be the linear trans formation defined by

$$T(f(x)) = 2f'(x) + \int_0^x 3f(t)dt$$

Solution.
Now R(T) $= span(\{\mathrm{T}(1), \mathrm{T}(x), \mathrm{T}(x^2)) = span(\{3x, 2 + \frac{3}{2}x^2, 4x + x^3\})$.
Since { $3x$, $2 + \frac{3}{2}x^2$, $4x + x^3$ } is linearly independent, rank(T) = 3. Since 2 $dim(P3(R)) = 4$, T is not onto. From the dimension theorem (Thm 2.3) , nullity(T) + 3 = 3. So nullity(T) = 0, and therefore, N(T) = 0. We conclude from Theorem 2.4 that T is one-to-one.

Example 12

Question :

Let T : $F^2 \to F^2$ be the linear transformation defined by

$$T(a_1, a_2) = (a_1 + a_2, a_1)$$

Solution.

It is easy to see that N(T) = { 0 } ; so T is one-to-one. Hence Theorem 2.5 tells us that T must be onto.

Theorem (2.3 Dimension Theorem).

Let V and W be vector spaces, and let T : V $\to$ W be linear.

If V is finite-dimensional, then

$$\mathrm{nullity(T) + rank(T) = dim(V)}.$$

Theorem (2.4).

Let V and W be vector spaces, and let T : V $\to$ W be linear. Then T is one-to-one if and only if N(T) = 0 .

Theorem (2.5).

Let V and W be vector spaces of equal (finite) dimension, and let T : V $\to$ W be linear. Then the following are equivalent.

(a) T is one-to-one.

(b) T is onto.

(c) rank(T) = dim(V).

Example

Let $\mathrm{T}:\mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $\mathrm{T}(\mathrm{a}_1,\mathrm{a}_2)=(\mathrm{a}_1+\mathrm{a}_2,0,2\mathrm{a}_1-\mathrm{a}_2)$, determine that T is linear , 1-1 , onto or not.

Solution.

- Calim : T is linear.
  Let $x=(a_1,a_2)$, $y=(b_1,b_2)$
  $$\begin{aligned}\mathrm{T}(\mathrm{cx}+\mathrm{y}) &= \mathrm{T}\left(\mathrm{c}\left(\mathrm{a}_1,\mathrm{a}_2\right)+\left(\mathrm{b}_1,\mathrm{b}_2\right)\right)=\mathrm{T}\left(\mathrm{ca}_1+\mathrm{b}_1,\mathrm{ca}_2+\mathrm{b}_2\right)\\ &= \left(\mathrm{ca}_1+\mathrm{b}_1+\mathrm{ca}_2+\mathrm{b}_2,0,2\mathrm{ca}_1+2\mathrm{b}_1-\mathrm{ca}_2-\mathrm{b}_2\right)\\ &= \left(\mathrm{c}\left(\mathrm{a}_1+\mathrm{a}_2\right)+\left(\mathrm{b}_1+\mathrm{b}_2\right),0,\mathrm{c}\left(2\mathrm{a}_1-\mathrm{a}_2\right)+\left(2\mathrm{b}_1-\mathrm{b}_2\right)\right)\\ &= \mathrm{c}\left(\mathrm{a}_1+\mathrm{a}_2,0,2\mathrm{a}_1-\mathrm{a}_2\right)+\left(\mathrm{b}_1+\mathrm{b}_2,0,2\mathrm{b}_1-\mathrm{b}_2\right)\\ &= \mathrm{cT}(\mathrm{x})+\mathrm{T}(\mathrm{y})\end{aligned}$$
  $\therefore$ T is linear

- 1-1 and onto

  By Thm 2.2 in text book choose a basis for $\mathbb{R}^2, \beta=\{(1,0),(0,1)\}$

  $\mathrm{R}(\mathrm{T})=\mathrm{span}\left(\mathrm{T}(\beta)\right)=\mathrm{span}(\{\mathrm{T}(1,0),\mathrm{T}(0,1)\})=\mathrm{span}\left((1,0,-1),(1,0,-2)\right)$

  Clearly it is L.I. $\Longrightarrow$ $\mathrm{rank}(\mathrm{T})=2$, and apply the Dimension Theorem $\mathrm{rank}(\mathrm{T})=2\neq 3=\dim(\mathbb{R}^3)$, and $\mathrm{nullitily}(\mathrm{T})=1$, then it is not onto.

  By Thm 2.4 it is not one to one.

Example

Let V and W be vector spaces, let $\mathrm{T}:\mathrm{V}\to\mathrm{W}$ be linear, and let $\{w_1,w_2,\ldots,w_k\}$ be a linearly independent subset of R(T). Prove that if $S=\{v_1,v_2,\ldots,v_k\}$ is chosen so that $\mathrm{T}\left(v_i\right)=w_i$ for $i=1,2,\ldots,k$, then S is linearly independent.

Solution. Calim : $\displaystyle\sum_{i=1}^{n} a_i v_i=0 \Longrightarrow a_1=a_2=a_3=\cdots=a_n=0$

Let $\displaystyle\sum_{i=1}^{n} a_i v_i=0$ then $\mathrm{T}\left(\displaystyle\sum_{i=1}^{n} a_i v_i\right)=0$ Since T is linear , $\mathrm{T}\left(\displaystyle\sum_{i=1}^{n} a_i v_i\right)=\displaystyle\sum_{i=1}^{n} a_i \mathrm{T}\left(v_i\right)=\sum_{i=1}^{n} a_i w_i=0$ Since S is L.I. $\Longrightarrow a_1=a_2=\cdots=a_n=0$

Theorem (2.6).

Let V and W be vector spaces over F, and suppose that $\{\, v_1, v_2, ..., v_n \,\}$ is a basis for V.

For $w_1, w_2, ..., w_n$ in W, there exists exactly one linear transformation T : V $\rightarrow$ W such that T(v_i) = w_i for i = 1,2,...,n.