Chap.6 Inner Product Space

§6.1 Inner product and norms

Example

Let A, B $\in M_{m\times n}(F)$,

$$\langle \mathrm{A}, \mathrm{B}\rangle = \mathrm{Tr}\,(\mathrm{B}^{*}\mathrm{A})$$

Determine the $\langle\cdot\,,\,\cdot\rangle$ is an inner prduct.

solution:.

- Claim 1. $\langle \mathrm{A}, \mathrm{A}\rangle \geq 0$
- Claim 2. $\langle \mathrm{A}, \mathrm{B}\rangle = \overline{\langle \mathrm{B}, \mathrm{A}\rangle}$
- Claim 3. $\langle k\mathrm{A} + \mathrm{B}, \mathrm{C}\rangle = k\langle \mathrm{A}, \mathrm{C}\rangle + \langle \mathrm{B}, \mathrm{C}\rangle$

> Notice : A vector space V over F endowed with a specific inner product is called a inner product space. If $F = \mathbb{C}$ we called it "Complex inner product space", whereas if $F = \mathbb{R}$ we called it "real inner product space".

§6.2 Gram-Schmidt Orthogonalization Process

Definition. V be a inner product space. A subset of V is a orthonormal basis for V if it is an ordered basis that is orthonormal.

Theorem. V be a inner product space , $S = \{v_1, \cdots, v_k\}$ be a orthogonal subset of V consisting of nonzero vectors. If $y \in \mathrm{Span}(S)$ then

$$y = \sum_{i=1}^{k} \frac{\langle y, v_i\rangle}{\|v_i\|^2} \cdot v_i$$

Proof.

$\because y \in \mathrm{Span}(S) \implies y = \sum_{i=1}^{k} a_i v_i$ where $a_1, \cdots, v_k \in F$

For $1 \leq i \leq k$, $\langle y, v_i \rangle = \langle \sum_{i=1}^{k} a_i v_i, v_j \rangle = \sum_{i=1}^{K} a_i \langle v_i, v_j \rangle \ldots (1)$

$\because \beta$ is an orthogonal subset $\implies \langle v_i, v_j \rangle = 0$, $\forall i, j$, $i \neq j \therefore (1) = a_j \langle v_j, v_j \rangle = a_j \|v_j\|^2$

$\implies$ for $1 \leq j \leq k, \langle y, v_j \rangle = a_j \|v_j\|^2 \implies a_j = \dfrac{\langle y, v_j \rangle}{\|v_j\|^2}$

$y = \sum_{j=1}^{k} a_j v_j = y = \sum_{j=1}^{k} \dfrac{\langle y, v_j \rangle}{\|v_j\|^2} \cdot v_i$

$\square$