

# CH1 Vector Space Textbook

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## Vector Space

Definition ( Vector Space ).

A vector space (or linear space)  $W$  over a *field*<sup>2</sup>  $F$  consists of a set on which two operations (called addition and scalar multiplication, respectively) are defined so that for each pair of elements  $x, y$ , in  $W$  there is a unique element  $x + y$  in  $W$ , and for each element  $a$  in  $F$  and each element  $x$  in  $W$  there is a unique element  $ax$  in  $W$ , such that the following conditions hold.

There are eight rules for calculating the vector space !

Please do not forget !

Definition ( Definition of Subspace ).

A subset  $W$  of a vector space  $W$  over a field  $F$  is called a subspace of  $W$  if  $W$  is a vector space over  $F$  with the operations of addition and scalar multiplication defined on  $W$ .

Theorem (1.3).

Let  $V$  be a vector space and  $W$  a subset of  $V$ . Then  $W$  is a subspace of  $V$  if and only if the following three conditions hold for the operations defined in  $V$ .

- (a)  $0 \in W$ .
- (b)  $x + y \in W$  whenever  $x \in W$  and  $y \in W$ .
- (c)  $cx \in W$  whenever  $c \in F$  and  $x \in W$ .

Theorem (1.4).

Any intersection of subspaces of a vector space  $W$  is a subspace of  $V$ .

Definition.

A vector space  $V$  is called direct sum of  $W_1$  and  $W_2$  if  $W_1$  and  $W_2$  are subspace of  $V$

such that  $W_1 \cap W_2 = \{0\}$  and  $W_1 + W_2 = V$ . We denote that  $V$  is the direct sum of  $W_1$  and  $W_2$  by writing  $V = W_1 \oplus W_2$ .

### Example

Question ( Exercise 19 ) :

Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$ . Prove that  $W_1 \cup W_2$  is a subspace of  $W$  if and only if  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ .

Solution.

1. Let  $W_1$  and  $W_2$  be subspace of  $W$ ,  $W_1 \cup W_2$  is a subspace
2. Suppose  $W_1 \cup W_2$  is a subspace of  $W$  while  $W_1 \not\subseteq W_2$  and  $W_2 \not\subseteq W_1 \rightarrow \exists x \in W_1, x \notin W_2, \exists y \in W_2, y \notin W_1$   
If  $W_1 \cup W_2$  is a subspace,  $x, y \in W_1 \cup W_2 \rightarrow x + y \in W_1 \cup W_2$
3.  $x + y \in W_1 \cup W_2 \therefore x + y \in W_1$  or  $x + y \in W_2$   
Since  $W_1$  and  $W_2$  are subspaces of  $V \therefore x \in W_1, y \in W_2$  or  $x \in W_2, y \in W_1$   
 $\rightarrow \leftarrow$  with (1)

Definition ( Definition of Linear Combination ).

Let  $W$  be a vector space and  $S$  a nonempty subset of  $V$ . A vector  $v \in V$  is called a linear combination of vectors of  $S$  if there exist a finite number of vectors  $u_1, u_2, \dots, u_n$  in  $S$  and scalars  $a_1, a_2, \dots, a_n$  in  $F$  such that  $v = a_1 u_1 + a_2 u_2 + \dots + a_n u_n$ . In this case we also say that  $W$  is a linear combination of  $u_1, u_2, \dots, u_n$  and call  $a_1, a_2, \dots, a_n$  the coefficients of the linear combination.

Definition ( Definition of Span ).

Let  $S$  be a nonempty subset of a vector space  $V$ . The span of  $S$ , denoted  $\text{span}(S)$ , is the set consisting of all linear combinations of the vectors in  $S$ . For convenience, we define  $\text{span}(\emptyset) = \{0\}$ .

### Example

$$S = \{(0, 0, 1), (1, 0, 0)\}$$

$$\text{Span}(S) = \{a_1(0, 0, 1) + a_2(1, 0, 0) \mid a_1, a_2 \in F\}$$

Theorem (1.5).

The span of any subset  $S$  of a vector space  $W$  is a subspace of  $V$ . Moreover, any subspace of  $W$  that contains  $S$  must also contain the span of  $S$ .

Definition ( Linearly Dependent ).

A subset  $S$  of a vector space  $W$  is called linearly dependent if there exist a finite number of distinct vectors  $u_1, u_2, \dots, u_n$  in  $S$  and scalars  $a_1, a_2, \dots, a_n$ , not all zero, such that

$$a_1u_1 + a_2u_2 + \dots + a_nu_n = 0.$$

In this case we also say that the vectors of  $S$  are linearly dependent.

Definition ( Definition of Linearly Independent ).

A subset  $S$  of a vector space that is not linearly dependent is called linearly independent. As before, we also say that the vectors of  $S$  are linearly independent.

Theorem (1.6).

Let  $W$  be a vector space, and let  $S_1 \subseteq S_2 \subseteq V$ . If  $S_1$  is linearly dependent, then  $S_2$  is linearly dependent.

Corollary. Let  $W$  be a vector space, and let  $S_1 \subseteq S_2 \subseteq V$ . If  $S_2$  is linearly independent, then  $S_1$  is linearly independent.

Theorem (1.7).

Let  $S$  be a linearly independent subset of a vector space  $V$ , and let  $v$  be a vector in  $V$  that is not in  $S$ . Then  $S \cup \{v\}$  is linearly dependent if and only if  $v \in \text{span}(S)$ .

Definition ( Definition of Bases ).

A basis  $\beta$  for a vector space  $V$  is a linearly independent subset of  $V$  that generates  $V$ . If  $\beta$  is a basis for  $V$ , we also say that the vectors of  $\beta$  form a basis for  $V$ .

Theorem (1.8).

Let  $V$  be a vector space and  $\beta = \{u_1, u_2, \dots, u_n\}$  be a subset of  $V$ . Then  $\beta$  is a basis for  $V$  if and only if each  $v \in V$  can be uniquely expressed as a linear combination of vectors of  $\beta$ , that is, can be expressed in the form

$$v = a_1u_1 + a_2u_2 + \dots + a_nu_n$$

for unique scalars  $a_1, a_2, \dots, a_n$ .

Theorem (1.9).

If a vector space  $V$  is generated by a finite set  $S$ , then some subset of  $S$  is a basis for  $V$ . Hence  $V$  has a finite basis.

Theorem (1.10 Replacement Theorem).

Let  $V$  be a vector space that is generated by a set  $G$  containing exactly  $n$  vectors, and let  $L$  be a linearly independent subset of  $V$  containing exactly  $m$  vectors. Then  $m \leq n$  and there exists a subset  $H$  of  $G$  containing exactly  $n - m$  vectors such that  $L \cup H$  generates  $V$ .

Theorem (Corollary 1.).

Let  $V$  be a vector space having a finite basis. Then every basis for  $V$  contains the same number of vectors.

Definition ( Definition of Dimension ).

A vector space is called finite-dimensional if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for  $V$  is called the dimension of  $V$  and is denoted by  $\dim(V)$ . A vector space that is not finite-dimensional is called infinite-dimensional .

## Chapter 2

### § 2-1 Linear transformation

Definition (Definition of Linear Transformation).

Let  $V$  and  $W$  be vector spaces (over  $F$ ). We call a function  $T : V \rightarrow W$  a linear transformation from  $V$  to  $W$  if, for all  $x, y \in V$  and  $c \in F$ , we have

$$(a) \quad T(x + y) = T(x) + T(y)$$

$$(b) \quad T(cx) = cT(x)$$

Definition (Definition of nullity and rank).

Let  $V$  and  $W$  be vector spaces and let  $T : V \rightarrow W$  be linear. If  $N(T)$  and  $R(T)$  are finite-dimensional, then we define the nullity of  $T$ , denoted  $\text{nullity}(T)$ , and the rank of  $T$ , denoted  $\text{rank}(T)$ , to be the dimensions of  $N(T)$  and  $R(T)$ , respectively.

Theorem.

Let  $V$  and  $W$  be vector spaces and  $T : V \rightarrow W$  be linear. Then  $N(T)$  and  $R(T)$  are subspaces of  $V$  and  $W$ , respectively.

Theorem.

Let  $V$  and  $W$  be vector spaces, and let  $T : V \rightarrow W$  be linear. If  $\beta = \{v_1, v_2, \dots, v_n\}$  is a basis for  $V$ , then  $R(T) = \text{span}(T(\beta)) = \text{span}(\{T(v_1), T(v_2), \dots, T(v_n)\})$ .

#### Example 11

Question :

Let  $T : P_2(R) \rightarrow P_3(R)$  be the linear transformation defined by

$$T(f(x)) = 2f'(x) + \int_0^x 3f(t)dt$$

Solution.

Now  $R(T) = \text{span}(\{T(1), T(x), T(x^2)\}) = \text{span}(\{3x, 2 + \frac{3}{2}x^2, 4x + x^3\})$ .

Since  $\{3x, 2 + \frac{3}{2}x^2, 4x + x^3\}$  is linearly independent,  $\text{rank}(T) = 3$ . Since  $\dim(P_3(R)) = 4$ ,  $T$  is not onto. From the dimension theorem (Thm 2.3),  $\text{nullity}(T) + 3 = 4$ . So  $\text{nullity}(T) = 1$ , and therefore,  $N(T) \neq 0$ . We conclude from Theorem 2.4 that  $T$  is not one-to-one.

Example 12

Question :

Let  $T : F^2 \rightarrow F^2$  be the linear transformation defined by

$$T(a_1, a_2) = (a_1 + a_2, a_1)$$

Solution.

It is easy to see that  $N(T) = \{ 0 \}$  ; so  $T$  is one-to-one. Hence Theorem 2.5 tells us that  $T$  must be onto.

Theorem (2.3 Dimension Theorem).

Let  $V$  and  $W$  be vector spaces, and let  $T : V \rightarrow W$  be linear.

If  $V$  is finite-dimensional, then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V).$$

Theorem (2.4).

Let  $V$  and  $W$  be vector spaces, and let  $T : V \rightarrow W$  be linear. Then  $T$  is one-to-one if and only if  $N(T) = \{ 0 \}$ .

Theorem (2.5).

Let  $V$  and  $W$  be vector spaces of equal (finite) dimension, and let  $T : V \rightarrow W$  be linear. Then the following are equivalent.

- (a)  $T$  is one-to-one.
- (b)  $T$  is onto.
- (c)  $\text{rank}(T) = \dim(V)$ .

### Example

Let  $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ ,  $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$ , determine that  $T$  is linear, 1-1, onto or not.

Solution.

- Calim :  $T$  is linear.

Let  $x = (a_1, a_2)$ ,  $y = (b_1, b_2)$

$$\begin{aligned} T(cx + y) &= T(c(a_1, a_2) + (b_1, b_2)) = T(ca_1 + b_1, ca_2 + b_2) \\ &= (ca_1 + b_1 + ca_2 + b_2, 0, 2ca_1 + 2b_1 - ca_2 - b_2) \\ &= (c(a_1 + a_2) + (b_1 + b_2), 0, c(2a_1 - a_2) + (2b_1 - b_2)) \\ &= c(a_1 + a_2, 0, 2a_1 - a_2) + (b_1 + b_2, 0, 2b_1 - b_2) \\ &= cT(x) + T(y) \end{aligned}$$

$\therefore T$  is linear

- 1-1 and onto

By Thm 2.2 in text book choose a basis for  $\mathbb{R}^2$ ,  $\beta = \{(1, 0), (0, 1)\}$

$$R(T) = \text{span}(T(\beta)) = \text{span}(\{T(1, 0), T(0, 1)\}) = \text{span}((1, 0, -1), (1, 0, -2))$$

Clearly it is L.I.  $\implies \text{rank}(T) = 2$ , and apply the Dimension Theorem  $\text{rank}(T) = 2 \neq 3 = \dim(\mathbb{R}^3)$ , and nullity( $T$ ) = 1, then it is not onto.

By Thm 2.4 it is not one to one.

### Example

Let  $V$  and  $W$  be vector spaces, let  $T : V \rightarrow W$  be linear, and let  $\{w_1, w_2, \dots, w_k\}$  be a linearly independent subset of  $R(T)$ . Prove that if  $S = \{v_1, v_2, \dots, v_k\}$  is chosen so that  $T(v_i) = w_i$  for  $i = 1, 2, \dots, k$ , then  $S$  is linearly independent.

Solution. Calim :  $\sum_{i=1}^n a_i v_i = 0 \implies a_1 = a_2 = a_3 = \dots = a_n = 0$

$$\begin{aligned} \text{Let } \sum_{i=1}^n a_i v_i = 0 \text{ then } T\left(\sum_{i=1}^n a_i v_i\right) &= 0 \text{ Since } T \text{ is linear, } T\left(\sum_{i=1}^n a_i v_i\right) = \\ \sum_{i=1}^n a_i T(v_i) &= \sum_{i=1}^n a_i w_i = 0 \text{ Since } S \text{ is L.I. } \implies a_1 = a_2 = \dots = a_n = 0 \end{aligned}$$

Theorem (2.6).

Let  $V$  and  $W$  be vector spaces over  $F$ , and suppose that  $\{v_1, v_2, \dots, v_n\}$  is a basis for  $V$ .

For  $w_1, w_2, \dots, w_n$  in  $W$ , there exists exactly one linear transformation  $T : V \rightarrow W$  such that  $T(v_i) = w_i$  for  $i = 1, 2, \dots, n$ .