CH1 Vector Space Textbook

May 2, 2022

Vector Space

<u>Definition</u> (Vector Space).
A vector space (or linear space) W *over a* $field^2$ F *consists of a set on which two operations (called addition and scalar mul- tiplication, respectively) are defined so that for each pair of elements* x, y, *in* W *there is a unique element* $x + y$ *in* W, *and for each element a in F and each element* x *in* W *there is a unique element* ax *in* W, *such that the following conditions hold.*

There are eight rules for calculating the vector space !
Please do not forget !

<u>Definition</u> (Definition of Subspace).
A subset W *of a vector space* W *over a field* F *is called a subspace of* W *if* W *is a vector space over* F *with the operations of addition and scalar multiplication defined on* W.

Theorem (1.3).
Let V be a vector space and W a subset of V. Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V.
(a) $0 \in \mathrm{W}$.
(b) $x + y \in W$ *whenever* $x \in \mathrm{W}$ *and* $y \in \mathrm{W}$.
(c) $cx \in \mathrm{W}$ *whenever* $c \in \mathrm{F}$ *and* $x \in \mathrm{W}$.

Theorem (1.4).
Any intersection of subspaces of a vector space W *is a subspace of* V.

<u>Definition</u>. *A vector space* V *is called direct sum of* W_1 *and* W_2 *if* W_1 *and* W_2 *are subspace of* V *such thay* $W_1 \cap W_2 = \{0\}$ *and* $W_1 + W_2 = \mathrm{V}$. *We denote that* V *is the direct sum of* W_1 *and* W_2 *by Writting* $\mathrm{V} = W_1 \oplus W_2$.

Example

Question (Exercise 19) :
Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of W if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Solution.

1. *Let* W_1 *and* W_2 *be subspace of* W, $W_1 \cup W_2$ *is a subspace*
2. *Suppose* $W_1 \cup W_2$ *is a subspace of* W *while* $W_1 \nsubseteq W_2$ *and* $W_2 \nsubseteq W_1 \rightarrow \exists\ x \in W_1, x \notin W_2$, $\exists\ y \in W_2, y \notin W_1$
   If $W_1 \cup W_2$ *is a subspace ,* $x, y \in W_1 \cup W_2 \rightarrow x + y \in W_1 \cup W_2$
3. $x + y \in W_1 \cup W_2 \quad \therefore x + y \in W_1$ *or* $x + y \in W_2$
   Since W_1 *and* W_2 *are subspaces of* V $\quad \therefore x \in W_1, y \in W_2$ *or* $x \in W_2, y \in W_1$
   $\rightarrow\leftarrow$ *wtih (1)*

Definition (Definition of Linear Combination).
Let W *be a vector space and* S *a nonempty subset of* V. *A vector* $v \in V$ *is called a linear combination of vectors of* S *if there exist a finite number of vectors* $u_1, u_2, ..., u_n$ *in* S *and scalars* $a_1, a_2, ..., a_n$ *in* F *such that* $v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$. *In this case we also say that* W *is a linear combination of* $u_1, u_2, ..., u_n$ *and call* $a_1, a_2, ..., a_n$ *the coefficients of the linear combination.*

Definition (Definition of Span).
Let S *be a nonempty subset of a vector space* V. *The span of* S, *denoted span*(S)*, is the set consisting of all linear combinations of the vectors in* S. *For convenience, we define span*(ϕ) = $\{0\}$.

Example

$S = \{(0, 0, 1), (1, 0, 0)\}$
$Span(S) = \{a_1(0, 0, 1) + a_2(1, 0, 0) \mid a_1, a_2 \in F\}$

Theorem (1.5).
The span of any subset S *of a vector space* W *is a subspace of* V. *Moreover, any subspace of* W *that contains* S *must also contain the span of* S.

Definition (Linearly Dependent).
A subset S *of a vector space* W *is called linearly dependent if there exist a finite number of distinct vectors* $u_1, u_2, ..., u_n$ *in* S *and scalars* $a_1, a_2, ..., a_n$*, not all zero, such that*

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0.$$

In this case we also say that the vectors of S are linearly dependent.

Definition (Definition of Linearly Independent).
A subset S *of a vector space that is not linearly dependent is called linearly independent. As before, we also say that the vectors of* S *are linearly independent.*

Theorem (1.6).
Let W *be a vector space,and let* $S_1 \subseteq S_2 \subseteq V$*. If* S_1 *is linearly dependent, then* S_2 *is linearly dependent.*
Corollary. Let W *be a vector space, and let* $S_1 \subseteq S_2 \subseteq V$*. If* S_2 *is linearly independent, then* S_1 *is linearly independent.*

Theorem (1.7).
Let S *be a linearly independent subset of a vector space* V*, and let v be a vector in V that is not in S. Then* $S \cup \{v\}$ *is linearly dependent if and only if v* $\in$ *span(S).*

Definition (Definition of Bases).
A basis β *for a vector space V is a linearly independent subset of V that generates V. If* β *is a basis for V, we also say that the vectors of* β *form a basis for V.*

Theorem (1.8).
Let V *be a vector space and* $\beta = \{u_1, u_2, ..., u_n\}$ *be a subset of* V*. Then* β *is a basis for V if and only if each v* $\in$ *V can be uniquely expressed as a linear combination of vectors of* β*, that is, can be expressed in the form*

$$\mathrm{v} = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

for unique scalars $a_1, a_2, ..., a_n$*.*

Theorem (1.9).
If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence V has a finite basis.

Theorem (1.10 Replacement Theorem).
Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \le n$ *and there exists a subset H of G containing exactly* $n \mp m$ *vectors such that* $L \cup H$ *generates V.*

Theorem (Corollary 1.).
Let V be a vector space having a finite basis. Then every basis for V contains the same number of vectors.

Definition (Definition of Dimension).
A vector space is called finite-dimensional if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for V is called the dimension of V and is denoted by dim(V)*. A vector space that is not finite-dimensional is called infinite-dimensional .*

Chapter 2

§ 2-1 Linear transformation

Definition (Definition of Linear Transformation).
Let V *and* W *be vector spaces (over* F*). We call a function* $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ *a linear transformation from* V *to* W *if, for all* $x, y \in \mathrm{V}$ *and* $c \in \mathrm{F}$*, we have*

(a) $\mathrm{T}(x + y) = \mathrm{T}(x) + \mathrm{T}(y)$

(b) $\mathrm{T}(cx) = c\mathrm{T}(x)$

Definition (Definition of nullity and rank).
Let V *and* W *be vector spaces and let* $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ *be linear. If N(T) and R(T) are finite-dimensional, then we define the nullity of T, denoted nullity(T), and the rank of T, denoted rank(T), to be the dimensions of N(T) and R(T), respectively.*

Theorem.
Let V and W be vector spaces and $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ *be linear. Then N(T) and R(T) are subspaces of* V *and* W*, respectively.*

Theorem.
Let V and W be vector spaces, and let $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ *be linear. If* $\beta = \{v_1, v_2, ..., v_n\}$ *is a basis for* V*, then* $\mathrm{R}(\mathrm{T}) = \mathrm{span}(\mathrm{T}(\beta)) = \mathrm{span}(\{\mathrm{T}(v_1), \mathrm{T}(v_2), ..., \mathrm{T}(v_n)\})$.

Example 11

Question :
Let $\mathrm{T} : \mathrm{P}_2(R) \to \mathrm{P}_3(R)$ be the linear trans formation defined by

$$T(f(x)) = 2f'(x) + \int_0^x 3f(t)dt$$

Solution.
Now $\mathrm{R}(\mathrm{T}) = span(\{\mathrm{T}(1), \mathrm{T}(x), \mathrm{T}(x^2)) = span(\{3x, 2 + \frac{3}{2}x^2, 4x + x^3\})$.
Since { $3x$ *,* $2+\frac{3}{2}x^2$*,* $4x+x^3$ *} is linearly independent, rank(*T*) = 3. Since* $2\, dim(P3(R)) =$ 4*, T is not onto. From the dimension theorem (Thm 2.3), nullity(T) + 3 = 3. So nullity(T) = 0, and therefore, N(T) = 0. We conclude from Theorem 2.4 that T is one-to-one.*

Example 12

Question :
Let $\mathrm{T} : \mathrm{F}^2 \to \mathrm{F}^2$ be the linear transformation defined by

$$T(a_1, a_2) = (a_1 + a_2, a_1)$$

Solution.
It is easy to see that N(T) = { 0 } ; so T is one-to-one. Hence Theorem 2.5 tells us that T must be onto.

Theorem (2.3 Dimension Theorem)**.**
Let V and W be vector spaces, and let $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ *be linear.*
If V is finite-dimensional, then

$$\mathrm{nullity(T)} + \mathrm{rank(T)} = \mathrm{dim(V)}.$$

Theorem (2.4)**.**
Let V and W be vector spaces, and let $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ *be linear. Then T is one-to-one if and only if N(T) = 0 .*

Theorem (2.5)**.**
Let V and W be vector spaces of equal (finite) dimension, and let $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ *be linear. Then the following are equivalent.*

(a) T is one-to-one.

(b) T is onto.

(c) rank(T) = dim(V).

Example

Let $\mathrm{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $\mathrm{T}(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$, determine that T is linear , 1-1 , onto or not.

Solution.

- *Calim :* T *is linear.*
  Let $x = (a_1, a_2)$, $y = (b_1, b_2)$
  $$\begin{aligned} \mathrm{T}(cx + y) &= \mathrm{T}\left(c\left(a_1, a_2\right) + \left(b_1, b_2\right)\right) = \mathrm{T}\left(ca_1 + b_1, ca_2 + b_2\right) \\ &= \left(ca_1 + b_1 + ca_2 + b_2, 0, 2ca_1 + 2b_1 - ca_2 - b_2\right) \\ &= \left(c\left(a_1 + a_2\right) + \left(b_1 + b_2\right), 0, c\left(2a_1 - a_2\right) + \left(2b_1 - b_2\right)\right) \\ &= c\left(a_1 + a_2, 0, 2a_1 - a_2\right) + \left(b_1 + b_2, 0, 2b_1 - b_2\right) \\ &= c\mathrm{T}(x) + \mathrm{T}(y) \end{aligned}$$
  $\therefore$ T *is linear*

- *1-1 and onto*
  By Thm 2.2 in text book choose a basis for $\mathbb{R}^2, \beta = \{(1, 0), (0, 1)\}$
  $\mathrm{R}(\mathrm{T}) = \mathrm{span}\left(\mathrm{T}(\beta)\right) = \mathrm{span}(\{\mathrm{T}(1, 0), \mathrm{T}(0, 1)\}) = \mathrm{span}\left((1, 0, -1), (1, 0, -2)\right)$
  Clearly it is L.I. $\Longrightarrow$ $\mathrm{rank}(\mathrm{T}) = 2$ *, and apply the Dimension Theorem* $\mathrm{rank}(\mathrm{T}) = 2 \neq 3 = \dim(\mathbb{R}^3)$*, and* $\mathrm{nullitily}(\mathrm{T}) = 1$ *, then it is not onto.*
  By Thm 2.4 it is not one to one.

Example

Let V and W be vector spaces, let $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ be linear, and let $\{w_1, w_2, \ldots, w_k\}$ be a linearly independent subset of R(T). Prove that if $S = \{v_1, v_2, \ldots, v_k\}$ is chosen so that $\mathrm{T}\left(v_i\right) = w_i$ for $i = 1, 2, \ldots, k$, then S is linearly independent.

Solution. *Calim :* $\sum_{i=1}^{n} a_i v_i = 0 \implies a_1 = a_2 = a_3 = \cdots = a_n = 0$

Let $\sum_{i=1}^{n} a_i v_i = 0$ *then* $\mathrm{T}\left(\sum_{i=1}^{n} a_i v_i\right) = 0$ *Since* T *is linear ,* $\mathrm{T}\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i \mathrm{T}\left(v_i\right) = \sum_{i=1}^{n} a_i w_i = 0$ *Since* S *is L.I.* $\Longrightarrow$ $a_1 = a_2 = \cdots = a_n = 0$

Theorem (2.6)**.**
Let V and W be vector spaces over F, and suppose that { $v_1, v_2, \ldots, v_n$ *} is a basis for V. For*

$w_1, w_2, ..., w_n$ *in W, there exists exactly one linear transformation* $T : V \to W$ *such that* $T(v_i) = w_i$ *for* $i = 1,2,...,n$.