# Chapter 2 Linear Transformation

## § 2-1 Linear transformation

Definition (Definition of Linear Transformation).
Let V and W be vector spaces (over F). We call a function T : V → W a linear transformation from V to W if, for all x, y ∈ V and c ∈ F, we have

(a) T(x + y) = T(x) + T(y)

(b) T(cx) = cT(x)

Definition (Definition of nullity and rank).
Let V and W be vector spaces and let T : V → W be linear. If N(T) and R(T) are finite-dimensional, then we define the nullity of T, denoted nullity(T), and the rank of T, denoted rank(T), to be the dimensions of N(T) and R(T), respectively.

Theorem.
Let V and W be vector spaces and T : V → W be linear. Then N(T) and R(T) are subspaces of V and W, respectively.

Theorem.
Let V and W be vector spaces, and let T : V → W be linear. If $\beta = \{v_1, v_2, ..., v_n\}$ is a basis for V , then $\mathrm{R(T)} = \mathrm{span}(\mathrm{T}(\beta)) = \mathrm{span}(\{\mathrm{T}(v_1), \mathrm{T}(v_2), ..., \mathrm{T}(v_n)\})$.

Example 11

Question :
Let T : $\mathrm{P}_2(R) \to \mathrm{P}_3(R)$ be the linear trans formation defined by

$$T(f(x)) = 2f'(x) + \int_0^x 3f(t)dt$$

solution:.
Now $\mathrm{R(T)} = \mathrm{Span}(\{\mathrm{T}(1), \mathrm{T}(x), \mathrm{T}(x^2)) = \mathrm{Span}(\{3x, 2 + \frac{3}{2}x^2, 4x + x^3\})$.
Since { $3x$ , $2 + \frac{3}{2}x^2$, $4x + x^3$ } is linearly independent, rank(T) = 3. Since 2 dim$(P_3(\mathbb{R})) = 4$ , T is not onto. From the dimension theorem (Thm 2.3) , nullity(T) + 3 = 3. So nullity(T) = 0, and therefore, N(T) = 0. We conclude from Theorem 2.4 that T is one-to-one.

Example 12

Question :
Let T : $F^2 \to F^2$ be the linear transformation defined by

$$T(a_1, a_2) = (a_1 + a_2, a_1)$$

solution:.
It is easy to see that N(T) = { 0 } ; so T is one-to-one. Hence Theorem 2.5 tells us that T must be onto.

Theorem (2.3 Dimension Theorem).
Let V and W be vector spaces, and let T : V $\to$ W be linear.
If V is finite-dimensional, then

$$\mathrm{nullity(T) + rank(T) = dim(V)}.$$

Theorem (2.4).
Let V and W be vector spaces, and let T : V $\to$ W be linear. Then T is one-to-one if and only if N(T) = 0 .

Theorem (2.5).
Let V and W be vector spaces of equal (finite) dimension, and let T : V $\to$ W be linear. Then the following are equivalent.

(a) T is one-to-one.

(b) T is onto.

(c) rank(T) = dim(V).

Example

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ , $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$ , determine that T is linear , 1-1 , onto or not.

solution:.

- Calim : T is linear.
  Let $x = (a_1, a_2)$ , $y = (b_1, b_2)$
  $$\begin{aligned} T(cx + y) &= T\left(c\left(a_1, a_2\right) + \left(b_1, b_2\right)\right) = T\left(ca_1 + b_1, ca_2 + b_2\right) \\ &= \left(ca_1 + b_1 + ca_2 + b_2, 0, 2ca_1 + 2b_1 - ca_2 - b_2\right) \\ &= \left(c\left(a_1 + a_2\right) + \left(b_1 + b_2\right), 0, c\left(2a_1 - a_2\right) + \left(2b_1 - b_2\right)\right) \\ &= c\left(a_1 + a_2, 0, 2a_1 - a_2\right) + \left(b_1 + b_2, 0, 2b_1 - b_2\right) \\ &= cT(x) + T(y) \end{aligned}$$
  $\therefore$ T is linear
- 1-1 and onto
  By Thm 2.2 in text book choose a basis for $\mathbb{R}^2, \beta = \{(1,0),(0,1)\}$
  $R(T) = \text{span}\left(T(\beta)\right) = \text{span}(\{T(1,0), T(0,1)\}) = \text{span}\left((1,0,-1),(1,0,-2)\right)$
  Clearly it is L.I. $\implies$ rank(T) $= 2$ , and apply the Dimension Theorem rank(T) $= 2 \neq 3 = \dim(\mathbb{R}^3)$, and nullitily(T) $= 1$ , then it is not onto.
  By Thm 2.4 it is not one to one.

Example

Let V and W be vector spaces, let $T : V \to W$ be linear, and let $\{w_1, w_2, \ldots, w_k\}$ be a linearly independent subset of R(T). Prove that if $S = \{v_1, v_2, \ldots, v_k\}$ is chosen so that $T\left(v_i\right) = w_i$ for $i = 1, 2, \ldots, k$, then $S$ is linearly independent.

solution:. Calim : $\sum_{i=1}^{n} a_i v_i = 0 \implies a_1 = a_2 = a_3 = \cdots = a_n = 0$

Let $\sum_{i=1}^{n} a_i v_i = 0$ then $T\left(\sum_{i=1}^{n} a_i v_i\right) = 0$ Since T is linear , $T\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i T\left(v_i\right) = \sum_{i=1}^{n} a_i w_i = 0$ Since S is L.I. $\implies a_1 = a_2 = \cdots = a_n = 0$

Theorem (2.6).
Let V and W be vector spaces over F, and suppose that { $v_1, v_2, ..., v_n$ } is a basis for V.

For $w_1, w_2, ..., w_n$ in W, there exists exactly one linear transformation T : V $\to$ W such that T($v_i$) = $w_i$ for i = 1,2,...,n.