

CH1 Vector Space Textbook

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Vector Space

Definition (Vector Space).

A vector space (or linear space) W over a *field*² F consists of a set on which two operations (called addition and scalar multiplication, respectively) are defined so that for each pair of elements x, y , in W there is a unique element $x + y$ in W , and for each element a in F and each element x in W there is a unique element ax in W , such that the following conditions hold.

There are eight rules for calculating the vector space !

Please do not forget !

Definition (Definition of Subspace).

A subset W of a vector space W over a field F is called a subspace of W if W is a vector space over F with the operations of addition and scalar multiplication defined on W .

Theorem (1.3).

Let V be a vector space and W a subset of V . Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V .

- (a) $0 \in W$.
- (b) $x + y \in W$ whenever $x \in W$ and $y \in W$.
- (c) $cx \in W$ whenever $c \in F$ and $x \in W$.

Theorem (1.4).

Any intersection of subspaces of a vector space W is a subspace of V .

Definition.

A vector space V is called direct sum of W_1 and W_2 if W_1 and W_2 are subspace of V

such that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. We denote that V is the direct sum of W_1 and W_2 by writing $V = W_1 \oplus W_2$.

Example

Question (Exercise 19) :

Let W_1 and W_2 be subspaces of a vector space V . Prove that $W_1 \cup W_2$ is a subspace of W if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Solution.

1. Let W_1 and W_2 be subspace of W , $W_1 \cup W_2$ is a subspace
2. Suppose $W_1 \cup W_2$ is a subspace of W while $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1 \rightarrow$
 $\exists x \in W_1, x \notin W_2, \exists y \in W_2, y \notin W_1$
 If $W_1 \cup W_2$ is a subspace, $x, y \in W_1 \cup W_2 \rightarrow x + y \in W_1 \cup W_2$
3. $x + y \in W_1 \cup W_2 \therefore x + y \in W_1$ or $x + y \in W_2$
 Since W_1 and W_2 are subspaces of $V \therefore x \in W_1, y \in W_2$ or $x \in W_2, y \in W_1$
 $\rightarrow \leftarrow$ with (1)

Definition (Definition of Linear Combination).

Let W be a vector space and S a nonempty subset of V . A vector $v \in V$ is called a linear combination of vectors of S if there exist a finite number of vectors u_1, u_2, \dots, u_n in S and scalars a_1, a_2, \dots, a_n in F such that $v = a_1u_1 + a_2u_2 + \dots + a_nu_n$. In this case we also say that W is a linear combination of u_1, u_2, \dots, u_n and call a_1, a_2, \dots, a_n the coefficients of the linear combination.

Definition (Definition of Span).

Let S be a nonempty subset of a vector space V . The span of S , denoted $\text{span}(S)$, is the set consisting of all linear combinations of the vectors in S . For convenience, we define $\text{span}(\emptyset) = \{0\}$.

Example

$$S = \{(0, 0, 1), (1, 0, 0)\}$$

$$\text{Span}(S) = \{a_1(0, 0, 1) + a_2(1, 0, 0) \mid a_1, a_2 \in F\}$$

Theorem (1.5).

The span of any subset S of a vector space W is a subspace of V . Moreover, any subspace of W that contains S must also contain the span of S .

Definition (Linearly Dependent).

A subset S of a vector space W is called linearly dependent if there exist a finite number of distinct vectors u_1, u_2, \dots, u_n in S and scalars a_1, a_2, \dots, a_n , not all zero, such that

$$a_1u_1 + a_2u_2 + \dots + a_nu_n = 0.$$

In this case we also say that the vectors of S are linearly dependent.

Definition (Definition of Linearly Independent).

A subset S of a vector space that is not linearly dependent is called linearly independent. As before, we also say that the vectors of S are linearly independent.

Theorem (1.6).

Let W be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent.

Corollary. Let W be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent, then S_1 is linearly independent.

Theorem (1.7).

Let S be a linearly independent subset of a vector space V , and let v be a vector in V that is not in S . Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.

Definition (Definition of Bases).

A basis β for a vector space V is a linearly independent subset of V that generates V . If β is a basis for V , we also say that the vectors of β form a basis for V .

Theorem (1.8).

Let V be a vector space and $\beta = \{u_1, u_2, \dots, u_n\}$ be a subset of V . Then β is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β , that is, can be expressed in the form

$$v = a_1u_1 + a_2u_2 + \dots + a_nu_n$$

for unique scalars a_1, a_2, \dots, a_n .

Theorem (1.9).

If a vector space V is generated by a finite set S , then some subset of S is a basis for V . Hence V has a finite basis.

Theorem (1.10 Replacement Theorem).

Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V .

Theorem (Corollary 1.).

Let V be a vector space having a finite basis. Then every basis for V contains the same number of vectors.

Definition (Definition of Dimension).

A vector space is called finite-dimensional if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for V is called the dimension of V and is denoted by $\dim(V)$. A vector space that is not finite-dimensional is called infinite-dimensional .

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§ 2-1 Linear transformation

Definition (Definition of Linear Transformation).

Let V and W be vector spaces (over F). We call a function $T : V \rightarrow W$ a linear transformation from V to W if, for all $x, y \in V$ and $c \in F$, we have

$$(a) \quad T(x + y) = T(x) + T(y)$$

$$(b) \quad T(cx) = cT(x)$$

Definition (Definition of nullity and rank).

Let V and W be vector spaces and let $T : V \rightarrow W$ be linear. If $N(T)$ and $R(T)$ are finite-dimensional, then we define the nullity of T , denoted $\text{nullity}(T)$, and the rank of T , denoted $\text{rank}(T)$, to be the dimensions of $N(T)$ and $R(T)$, respectively.

Theorem.

Let V and W be vector spaces and $T : V \rightarrow W$ be linear. Then $N(T)$ and $R(T)$ are subspaces of V and W , respectively.

Theorem.

Let V and W be vector spaces, and let $T : V \rightarrow W$ be linear. If $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for V , then $R(T) = \text{span}(T(\beta)) = \text{span}(\{T(v_1), T(v_2), \dots, T(v_n)\})$.

Example 11

Question :

Let $T : P_2(R) \rightarrow P_3(R)$ be the linear transformation defined by

$$T(f(x)) = 2f'(x) + \int_0^x 3f(t)dt$$

Solution.

Now $R(T) = \text{span}(\{T(1), T(x), T(x^2)\}) = \text{span}(\{3x, 2 + \frac{3}{2}x^2, 4x + x^3\})$.

Since $\{3x, 2 + \frac{3}{2}x^2, 4x + x^3\}$ is linearly independent, $\text{rank}(T) = 3$. Since $\dim(P_3(R)) = 4$, T is not onto. From the dimension theorem (Thm 2.3), $\text{nullity}(T) + 3 = 4$. So $\text{nullity}(T) = 1$, and therefore, $N(T) \neq \{0\}$. We conclude from Theorem 2.4 that T is not one-to-one.

Example 12

Question :

Let $T : F^2 \rightarrow F^2$ be the linear transformation defined by

$$T(a_1, a_2) = (a_1 + a_2, a_1)$$

Solution.

It is easy to see that $N(T) = \{ 0 \}$; so T is one-to-one. Hence Theorem 2.5 tells us that T must be onto.

Theorem (2.3 Dimension Theorem).

Let V and W be vector spaces, and let $T : V \rightarrow W$ be linear.

If V is finite-dimensional, then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V).$$

Theorem (2.4).

Let V and W be vector spaces, and let $T : V \rightarrow W$ be linear. Then T is one-to-one if and only if $N(T) = \{ 0 \}$.

Theorem (2.5).

Let V and W be vector spaces of equal (finite) dimension, and let $T : V \rightarrow W$ be linear. Then the following are equivalent.

- (a) T is one-to-one.
- (b) T is onto.
- (c) $\text{rank}(T) = \dim(V)$.

Example

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$, determine that T is linear, 1-1, onto or not.

Solution.

- Calim : T is linear.

Let $x = (a_1, a_2)$, $y = (b_1, b_2)$

$$\begin{aligned} T(cx + y) &= T(c(a_1, a_2) + (b_1, b_2)) = T(ca_1 + b_1, ca_2 + b_2) \\ &= (ca_1 + b_1 + ca_2 + b_2, 0, 2ca_1 + 2b_1 - ca_2 - b_2) \\ &= (c(a_1 + a_2) + (b_1 + b_2), 0, c(2a_1 - a_2) + (2b_1 - b_2)) \\ &= c(a_1 + a_2, 0, 2a_1 - a_2) + (b_1 + b_2, 0, 2b_1 - b_2) \\ &= cT(x) + T(y) \end{aligned}$$

$\therefore T$ is linear

- 1-1 and onto

By Thm 2.2 in text book choose a basis for \mathbb{R}^2 , $\beta = \{(1, 0), (0, 1)\}$

$$R(T) = \text{span}(T(\beta)) = \text{span}(\{T(1, 0), T(0, 1)\}) = \text{span}((1, 0, -1), (1, 0, -2))$$

Clearly it is L.I. $\implies \text{rank}(T) = 2$, and apply the Dimension Theorem $\text{rank}(T) = 2 \neq 3 = \dim(\mathbb{R}^3)$, and nullity(T) = 1, then it is not onto.

By Thm 2.4 it is not one to one.

Example

Let V and W be vector spaces, let $T : V \rightarrow W$ be linear, and let $\{w_1, w_2, \dots, w_k\}$ be a linearly independent subset of $R(T)$. Prove that if $S = \{v_1, v_2, \dots, v_k\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, 2, \dots, k$, then S is linearly independent.

Solution. Calim : $\sum_{i=1}^n a_i v_i = 0 \implies a_1 = a_2 = a_3 = \dots = a_n = 0$

$$\begin{aligned} \text{Let } \sum_{i=1}^n a_i v_i = 0 \text{ then } T\left(\sum_{i=1}^n a_i v_i\right) &= 0 \text{ Since } T \text{ is linear, } T\left(\sum_{i=1}^n a_i v_i\right) = \\ \sum_{i=1}^n a_i T(v_i) &= \sum_{i=1}^n a_i w_i = 0 \text{ Since } S \text{ is L.I. } \implies a_1 = a_2 = \dots = a_n = 0 \end{aligned}$$

Theorem (2.6).

Let V and W be vector spaces over F , and suppose that $\{v_1, v_2, \dots, v_n\}$ is a basis for V .

For w_1, w_2, \dots, w_n in W , there exists exactly one linear transformation $T : V \rightarrow W$ such that $T(v_i) = w_i$ for $i = 1, 2, \dots, n$.