Ch1. Vector Space

<u>Definition</u> (Vector Space).
A vector space (or linear space) W over a $Field$ F consists of a set on which two operations (called addition and scalar multiplication, respectively) are defined so that for each pair of elements x, y, in W there is a unique element $x + y$ in W, and for each element a in F and each element x in W there is a unique element ax in W, such that the following conditions hold.

There are eight rules for calculating the vector space !
Please do not forget !

§1-3 Subspace

<u>Definition</u> (Definition of Subspace).
A subset W of a vector space W over a field F is called a subspace of W if W is a vector space over F with the operations of addition and scalar multiplication defined on W.

Theorem (1.3).
Let V be a vector space and W a subset of V. Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V.
(a) $0 \in$ W.
(b) $x + y \in W$ whenever $x \in$ W and $y \in$ W.
(c) $cx \in$ W whenever $c \in$ F and $x \in$ W.

Theorem (1.4).
Any intersection of subspaces of a vector space W is a subspace of V.

Example

Exercise 19 :
Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of W if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Solution.

1. Let W_1 and W_2 be subspace of W, $W_1 \cup W_2$ is a subspace

2. Suppose $W_1 \cup W_2$ is a subspace of W while $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$

   $\implies \exists\, x \in W_1\,,\, x \notin W_2\,,\, \exists\, y \in W_2\,,\, y \notin W_1$
   If $W_1 \cup W_2$ is a subspace , $x, y \in W_1 \cup W_2 \implies x + y \in W_1 \cup W_2$

3. $x + y \in W_1 \cup W_2 \quad \therefore x + y \in W_1$ or $x + y \in W_2$
   Since W_1 and W_2 are subspaces of V $\therefore x \in W_1, y \in W_2$ or $x \in W_2, y \in W_1$
   $\rightarrow\leftarrow$ wtih our suppose in 1.

Definition.
A vector space V is called direct sum of W_1 and W_2 if W_1 and W_2 are subspace of V such thay $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. We denote that V is the direct sum of W_1 and W_2 by Writting $V = W_1 \oplus W_2$.

§1-4 Linear Combination and System of Linear Equations

Definition (Definition of Linear Combination).
Let W be a vector space and S a nonempty subset of V. A vector $v \in V$ is called a linear combination of vectors of S if there exist a finite number of vectors $u_1, u_2, \cdots, u_n$ in S and scalars $a_1, a_2, \cdots, a_n$ in F such that $v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$. In this case we also say that W is a linear combination of $u_1, u_2, \cdots, u_n$ and call $a_1, a_2, \cdots, a_n$ the coefficients of the linear combination.

Definition (Definition of Span).
Let S be a nonempty subset of a vector space V. The span of S, denoted span(S), is the set consisting of all linear combinations of the vectors in S. For convenience, we define span(ϕ) = $\{0\}$.

Example

Question (Exercise 5 (a) (e)) :
In each part, determine whether the given vector is in the span of S.

a. (2,-1,1), S = {(1,0,2),(-1,1,1)}

e. $-x^3 + 2x^2 + 3x + 3$, S = $\left\{x^3 + x^2 + x + 1, x^2 + x + 1, x + 1\right\}$

Solution.

(a)

1. $\text{Span}(S) = \{a_1(1,0,2) + a_2(-1,1,1) \mid a_1, a_2 \in \mathrm{F}\}$

2. $\begin{cases} a_1 - a_2 = & 2 \\ 0a_1 + a_2. = & -1 \\ 2a_1 + a_2 = & 1 \end{cases} \implies a_1 = 1, a_2 = -1$

3. Since there exist scalars a_1, a_2 such that $(2,-1,1) = a_1(1,0,2) + a_2(-1,1,1)$, we conclude that $(2,-1,1)$ is in the span of S.

(e)

1. $\text{Span}(S) = \{a_1(x^3 + x^2 + x + 1) + a_2(x^2 + x + 1) + a_3(x + 1)\}$

2. $\begin{cases} a_1 + 0a_2 + 0a_3 = & -1 \\ a_1 + a_2 + 0a_3 = & 2 \\ a_1 + a_2 + a_3 = & 3 \\ a_1 + a_2 + a_3 = & 3 \end{cases} \implies a_1 = -1, a_2 = 3, a_3 = 1$

3. Since there exist scalars a_1, a_2, a_3 such that $-x^3 + 2x^2 + 3x + 3 = a_1(x^3 + x^2 + x + 1) + a_2(x^2 + x + 1) + a_3(x + 1)$, we conclude that $(2,-1,1)$ is in the span of S.

Example

Question (Exercise 6) :
Show that the vectors $(1,1,0),(1,0,1)$, and $(0,1,1)$ generate F^3.

Solution. Let $\mathrm{S}=\{(1,0,1),(1,0,1),(0,1,1)\}$.
Claim : $\mathrm{S}=\{(1,0,1),(1,0,1),(0,1,1)\}$ is linearly independent.

1. $a_1(1,0,1)+a_2(1,0,1)+a_3(0,1,1)=0 \implies a_1=a_2=a_3=0$
2. $b_1(1,0,1)+b_2(1,0,1)+b_3(0,1,1)=(x,y,z)$
$\implies b_1=\dfrac{x+y-z}{2}, b_2=\dfrac{x-y+z}{2}, b_3=\dfrac{-x+y+z}{2}$

Since $\mathrm{S}=\{(1,0,1),(1,0,1),(0,1,1)\}$ is linearly independent, Span(S) can generate F^3.

Example

$\mathrm{S}=\{(0,0,1),(1,0,0)\}$
$Span(S)=\{a_1(0,0,1)+a_2(1,0,0) \mid a_1,a_2\in \mathrm{F}\}$

Theorem (1.5).
The span of any subset S of a vector space W is a subspace of V. Moreover, any subspace of W that contains S must also contain the span of S.

§1-5 Linear dependence and linear independence

<u>Definition</u> (Linearly Dependent).
A subset S of a vector space W is called linearly dependent if there exist a finite number of distinct vectors $u_1,u_2,...,u_n$ in S and scalars $a_1,a_2,...,a_n$, not all zero, such that

$$a_1u_1+a_2u_2+\cdots+a_nu_n=0.$$

In this case we also say that the vectors of S are linearly dependent.

<u>Definition</u> (Definition of Linearly Independent).
A subset S of a vector space that is not linearly dependent is called linearly independent. As before, we also say that the vectors of S are linearly independent.

Theorem (1.6).
Let W be a vector space,and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent.

Corollary. Let W be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent, then S_1 is linearly independent.

Example

Question (Exercise 2 (a) (c) (e)) :
Determine whether the following sets are linearly dependent or linearly independent.

a. $\left\{ \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}, \begin{bmatrix} -2 & 6 \\ 4 & -8 \end{bmatrix} \right\}$

c. $\left\{ x^3 + 2x^2, -x^2 + 3x + 1, x^3 - x^2 + 2x + 1 \right\}$

e. $\{(1, -1, 2), (1, -2, 1), (1, 1, 4)\}$ in $\mathbb{R}^3$

Solution.

a. Since $2 \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 6 \\ 4 & 8 \end{bmatrix} = 0 \implies$ linearly dependent by Thm 1.5.

Theorem (1.7).
Let S be a linearly independent subset of a vector space V, and let v be a vector in V that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.

§1-6 Bases and Dimension

Definition (Definition of Bases).
A basis β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.

Theorem (1.8).
Let V be a vector space and $\beta = \{u_1, u_2, ..., u_n\}$ be a subset of V. Then β is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β, that is, can be expressed in the form

$$\text{v} = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$$

for unique scalars $a_1, a_2, \cdots, a_n$.

Theorem (1.9).
If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence V has a finite basis.

Theorem (1.10 Replacement Theorem).
Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V.

Theorem (Corollary 1.).
Let V be a vector space having a finite basis. Then every basis for V contains the same number of vectors.

<u>Definition</u> (Definition of Dimension).
A vector space is called finite-dimensional if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for V is called the dimension of V and is denoted by $dim(V)$. A vector space that is not finite-dimensional is called infinite-dimensional .