1. Suppose that $V = R(T) \bigoplus W$ and W is T-invariant.
   (a) Prove that $W \subseteq N(T)$.
   (b) Show that if V is finite-dimensional, then $W = N(T)$.
   (c) Show by example that the conclusion of (b) is not necessarily true if v is not finite-dimensional.

Solution:

(a) $T(W) \subseteq W$ (T-invariant)
$R(T) \cap T(W) \subseteq R(T) \cap W$
$\Rightarrow R(T) \cap T(W) \subseteq \{0\}$
$\because$ W is a subspace
$\therefore W \subseteq V$
$\Rightarrow T(W) \subset T(V)$
$R(T) \cap T(W) = T(W)$
$\because R(T) \cap T(W) \subseteq \{0\} \ \Rightarrow T(W) \subseteq \{0\}$
for each $w \in W$,$T(W) = 0$
$\Rightarrow w \in N(T)$ for all $x \in V$
$\Longrightarrow W \subseteq N(T)$
(b) We assume that V is a finite-dimensional vector space.
By dimension theorem, we can obtain $dim(V) = nullity(T) + rank(T)$
$\because V = W + R(T)$ (by direct sum)
$dim(V) = dim(W + R(T)) = dim(W) + dim(R(T)) = rank(T) + nullity(T)$
$\Rightarrow dim(W) = nullity(T) = dim(N(T))$
$\Rightarrow W = N(T)$
(c) Let assume V be non finite-dimensional vector space
let's define V be a set of all polynomial over the field of $\mathbb{R}$
Let's define T(g(x)) = $\dfrac{df(x)}{dx}$ for all g(x) $\in$ V
$N(T) = \{g(x) \in V : T(g(x)) = 0\}$
$= \left\{ g(x) \in V : \dfrac{df(x)}{dx} = 0 \right\}$
$= \{g(x) \in V : g(x) = c\}$ where c is orditrary constant
$= \mathbb{R}$
$\Rightarrow$ N(T) = $\mathbb{R}$
If W = $\{0\} \Rightarrow$ R(T) $\cap$ W = $\{0\}$
Here V = R(T) $\bigoplus$ W, we got R = N(T), W = $\{0\}$
but the definition of direct sum and dimension theorem is still valid if W $\neq$ N(T)

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2. Assume that W is a subspace of a vector space V and that $T : V \to V$ is linear.
   Prove that the subspaces $\{0\}$,V,$R(T)$,$N(T)$ are all $T - invarient$.

Solution:

(i)
Let v be a vector $\in \{0\}$

T(v)=T(0)=0 $\in \{0\}$
$\because v \in \{0\}$ and $T(v) \in \{0\}$
$\therefore$ $\{0\}$ is $T-invarient$
(ii)
Let v be a vector $\in V$
$\because T : V \to V$
$\therefore T(v) \in V$
$\therefore V$ is $T-invarient$
(iii)
Let v be a vector $\in R(T)$
$\exists w \in V s.t. T(w) = v$
$T(T(w)) = T(v)$
$\because T : V \to V$ and $v \in R(T)$
$\therefore T(v) \in R(T)$
$\therefore R(T)$ is $T-invarient$
(iiii)
Let v be a vector $\in N(T)$
$T(v) = 0$
$T(T(v)) = T(0) = 0$
$\because T : V \to V$ and $v \in N(T)$
$\therefore t(v) \in N(T)$
$N(T)$ is $T-invarient$

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3. Let V be the vector space
Define the functions $T, U : V \to V$ by

$$T(a_1, a_2, \ldots) = (a_2, a_3, \ldots) \; and \; U(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots).$$

T and U are called the left shift right shift operators on V, respectively
(a) Prove that T and U are linear.
(b) Prove that T is onto, but not one-to-one.
(c)Prove that U is one-to-one, but not onto.

Solution:
(a)
$T((a_1, a_2, \ldots) + (b_1, b_2 \ldots))$
$= T(a_1 + b_1, a_2 + b_2, \ldots)$
$= (a_2 + b_2, a_3 + b_3, \ldots)$
$= (a_2, a_3, \ldots) + (b_2, b_3, \ldots)$
$= T(a_1, a_2, \ldots) + T(b_1, b_2, \ldots)$

$T(c(a_1, a_2, \ldots))$
$= T(ca_1, ca_2, \ldots)$
$= (ca_2, ca_3, \ldots)$
$= c(a_2, a_3, \ldots)$
$= cT(a_1, a_2, \ldots)$
$\therefore T$ is linear

$U((a_1, a_2, \ldots) + (b_1, b_2 \ldots))$

$= U(a_1 + b_1, a_2 + b_2, \ldots)$
$= (0, a_1 + b_1, a_2 + b_2, \ldots)$
$= (0, a_1, a_2, \ldots) + (0, b_1, b_2, \ldots)$
$= U(a_1, a_2, \ldots) + U(b_1, b_2, \ldots)$

$U(c(a_1, a_2, \ldots))$
$= U(ca_1, ca_2, \ldots)$
$= (0, ca_1, ca_2, \ldots)$
$= c(o, a_1, a_2, \ldots)$
$= cU(a_1, a_2, \ldots)$
$\therefore U$ is linear

(b)
$\because T(1, a_2, \ldots) = T(2, a_2, \ldots) = (a_2, a_3 \ldots)$
$\therefore T$ is not one-to one
For any vector $x = (x_1, x_2, \ldots)$ in V, we can find $y = (1, x_1, x_2, \ldots)$ in V which satisfies $T(y) = x$
$\Rightarrow x \in R(T)$
$\therefore V \subseteq R(T)$
$\because R(T) \subseteq V$
$\therefore R(T) = V$
$\Rightarrow T$ is onto
(c)
$U(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots)$
Clearly, if $U(a_1, a_2, \ldots) = (0, 0, 0, \ldots)$,then $a_1 = a_2 = \ldots = 0$
$\therefore N(U) = 0$
$\Rightarrow U is one to one$
$\because (a_1, a_2, \ldots) = (0, a_1, a_2, \ldots)$
$\Rightarrow$ There are not solution for function $U(a_1, a_2, \ldots) = (2, a_1, a_2, \ldots)$
$\therefore U$ is not onto.

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4. 2-1 14.
Let V and W be vector space and $T : V \to W$ be linear.
(a)Prove that T is 1-1 if and only if T carries L.I. subsets of V onto L.I. subsets of W.
(b)Suppose that T is 1-1 and that S is a subset of V. Prove that S is L.I. if and only if $T(S)$ is L.I.
(c)Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and T is 1-1 and onto. Prove that T(β)=$\{T(v_1), \ldots, T(v_n)\}$ is a basis for W.

Solution:
(a)
($\Rightarrow$)Given $S \subseteq V$,$U \subseteq W$ are L.I. subset.
Claim : $U = \{w_1, \ldots\} \subseteq W$ is a L.I. subset.
Given $S = \{v_1, \ldots\} \subseteq V$
$\sum a_i w_i = \sum a_i T(v_i) = \mathrm{T}(\sum a_i v_i) = 0$
$\because$ T is 1-1 $\Leftrightarrow$ N(T)=$\{0\}$
$\therefore \sum a_i v_i = 0$
$\because$ S is L.I. $\Rightarrow a_i$ are all zero $\therefore$ U is L.I.

($\Leftarrow$) Suppose T is not 1-1, then $\exists$ distinct vectors v_1, v_2 such that
T(v_1) = T(v_2) $\Rightarrow w_1 = w_2$ 矛盾 $\therefore$ T is 1-1
(b)
($\Rightarrow$) Claim : T(S) is L.I.
$\sum a_i T(v_i)$ = T($\sum a_i v_i$) = 0
$\because$ T is 1-1 $\Leftrightarrow$ N(T)={0}
$\therefore \sum a_i v_i = 0$
$\because$ S is L.I. $\Rightarrow a_i$ are all zero $\therefore$ T(β) is L.I.
($\Leftarrow$)Claim : S is L.I.
$\because$ T(β) is L.I. $\therefore \sum a_i T(v_i) = 0$, a_i are all zero
$0 = \sum a_i T(v_i)$ = T($\sum a_i v_i$)
$\because$ T is 1-1 $\Leftrightarrow$ N(T)={0}
$\therefore \sum a_i v_i = 0$, a_i are all zero
$\therefore$ S is L.I.
(c)
<1> Claim : T(β)=$\{T(v_1), \ldots, T(v_n)\}$ is L.I. $\Rightarrow \sum_{i=1}^{n} a_i T(v_i)$ = T($\sum_{i=1}^{n} a_i v_i$) = 0
$\because$ T is 1-1 $\Leftrightarrow$ N(T)={0}
$\therefore \sum_{i=1}^{n} a_i v_i = 0$, a_i are all zero
$\therefore$ T(β) is L.I.
<2>$\because$ T is 1-1 $\Leftrightarrow$ N(T)={0} $\Leftrightarrow$ nulity(T) = 0
By Dimension Theorem dim(v) = rank(T) = dim(R(T))
By Thm2.2 R(T) = Span(T(β))
V = Span(T(β)) $\therefore$ T(β) can generates V
By <1>,<2> T(β) is a basis of V

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5. T:$P_2(R) \to P_3(R)$defined by T(f(x)) = xf(x)+f'(x)
prove that T is a linear transformation,and find bases for both N(T) and R(T). Then compute the nullity and rank of T, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether T is one-to-one or onto.

Solution:
1.prove T is linear transformation.
Let f(x),g(x)$\in P_2 R$
T(f(x)+g(x))=x(f(x)+g(x))+(f(x)+g(x))'
=xf(x)+xg(x)+f'(x)+g'(x)
=xf(x)+f'(x)+xg(x)+g'(x)
=T(f(x))+T(g(x))
T(cf(x))=xcf(x)+cf'(x)
=c(xf(x)+f'(x))
=cT(f(x))

2.
(1)find basis for N(T)
suppose f(x)$\in P_2 R$
let f(x)=ax^2+bx+c=0
$\because$T(f(x))=x(ax^2+bx+c)+2ax+b=ax^3+bx^2+c=0
$\Rightarrow$ a=0,b=0,c=0
$\therefore$N(T)={0}
$\Rightarrow \emptyset$ is a basis of N(T)

(2) find basis for R(T)
By theorem 2.2
We know that R(T)=span(T(1),T(x),T(x^2))=span(x,x^2+1,x^3+2x)
suppose (x,x^2+1,x^3+2x)is L.I.
let ax+b(x^2+1)+c(x^3+2x)=cx^3+bx^2+(a+2)x+b=0
Then we only have a=b=c=0
⇒(x,x^2+1,x^3+2x)is L.I.
∵(x,x^2+1,x^3+2x) is L.I. and R(T)=span(x,x^2+1,x^3+2x)
∴(x,x^2+1,x^3+2x)is a basis of R(T)

3.compute the nullity and rank of T, and verify the dimension theorem
state the dimension theorem: if T:V→W,and V is finite dimension
then nullity(T)+rank(T)=dim(V)
By2. we know that nullity(T)=0
rank(T)=3
dim($P_2(R)$)=3
then 0+3=3

4. whether T is one-to-one or onto
∵N(T)={0} and rank(T)≠dim($P_3(R)$)
∴ T is one-to-one,but not onto

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6. For Exercises 2 through 6, prove that T is a linear transformation, and find bases for both N(T) and R(T). Then compute the nullity and rank of T, and verity the dimension theorem. Finally, use the appropriate theorems in this section to determine whether T is one-to-one or onto.

4. $T : M_{2\times2}(F) \to M_{2\times2}(F)$ defined by

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}$$

Solution:

$< 1 > proof:$

1. T is linear
2. bases for N(T), R(T)
3. nullity(T), rank(T)
4. dimension theorem
5. T is one-to-one or onto

1.

(1) $T(0) = T\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(2) Given $x = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}, y = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$

$T(cx + y) = T\left(\begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}\right)$

$= T\begin{pmatrix} ca_{11} + b_{11} & ca_{12} + b_{12} & ca_{13} + b_{13} \\ ca_{21} + b_{21} & ca_{22} + b_{22} & ca_{23} + b_{23} \end{pmatrix}$

$= \begin{pmatrix} 2(ca_{11} + b_{11}) - (ca_{12} + b_{12}) & ca_{13} + b_{13}) + 2(ca_{12} + b_{12}) \\ 0 & 0 \end{pmatrix}$

$= \left(\begin{pmatrix} c(2a_{11} - a_{12}) & c(a_{13} + 2a_{12}) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2b_{11} - b_{12}) & b_{13} + 2b_{12} \\ 0 & 0 \end{pmatrix}\right)$

$= cT(x) + T(y)$

2.

$(1) T(x) = 0 \Rightarrow \begin{cases} 2a_{11} - a_{12} = 0 \\ a_{13} + 2a_{12} = 0 \end{cases} \Rightarrow \begin{cases} 2a_{11} = a_{12} \\ a_{13} = -2a_{12} = -4a_{11} \end{cases}$

$\Rightarrow N(T) = \begin{pmatrix} a_{11} & 2a_{11} & -4a_{11} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

$\Rightarrow$ basis for N(T) :$\left\{\begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$

(2) basis for R(T) : $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$

3.

$nullity(T) = 4$

$rank(T) = 2$

4.

dimension theroem

dim(V)=rank(T)+nullity(T)

$\therefore dim(M_{2\times 3)} = nullity(T) + rank(T)$

$\therefore 6 = 4 + 2$ 5.

T is not 1-1 $\because N(T) = \begin{pmatrix} a_{11} & 2a_{11} & -4a_{11} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

T is not onto $\because R(T) = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}, x, y \in F$

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7. Verify $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ where $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$ is linear transformation, find bases for both N(T) and R(T). Th use the appropriate theorems in section 2.1 to determine whether T is one to one or onto.

Solution:

$< 1 > T(c(a_1, a_2) + (a_3, a_4))$

$= T(ca_1 + a_3, ca_2 + a_4)$

$= (ca_1 + a_3 + ca_2 + a_4, 0, 2ca_1 + 2a_3 - ca_2 - a_4)$

$= ((ca_1 + ca_2) + (a_3 + a_4), 0, (2ca_1 - ca_2) + (2a_3 - a_4))$

$= ((ca_1 + ca_2), 0, (2ca_1 - ca_2)) + ((a_3 + a_4), 0, (2a_3 - a_4))$

$= (c((a_1 + a_2), 0, (2a_1 - a_2)) + ((a_3 + a_4), 0, (2a_3 - a_4))$

$= cT(a_1, a_2) + T(a_3, a_4)$

T is a Linear transformation

$< 2 > Claim : N(T) = \{0\}$

$Let\ T(a_1, a_2) = 0$

$(a_1 + a_2, 0, 2a_1 - a_2) = 0$

$$\begin{cases} a_1 + a_2 = 0 \\ 2a_1 - a_2 = 0 \end{cases}$$

$$\longrightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \end{cases}$$

$\therefore (0, 0)$is the base for N(T)

The bases of the $\mathbb{R}^2$ is $\beta\ = \{(1, 0), (0, 1)\}$ (by thm)

$R(T) = span(T(\beta)) = span(T(1, 0), (0, 1)) = span(\{(1, 0, 2), (1, 0, -1)\})$

$\therefore \{(1, 0, 2), (1, 0, -1)\}$are the bases for R(T)

$\because dim(N(T)) = dim((0, 0)) = 0$

$\therefore nullity(T) = 0$

$\because dim(R(T)) = dim(\{(1, 0, 2), (1, 0, -1)\}) = 2$

$\therefore rank(T) = 2$

$\because dim(R^2) = 2$

$\therefore dim(V) = nullity(T) + rank(T)$

$\because nullity(T) = 0(bythm)$

$\Longrightarrow$ T is 1-1

$\because dim(W) = dim(R^3) = 3 \neq rank(T) = 2$

$\Longrightarrow$ T is not onto (by thm)

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