**Classification**

**order**

$$\frac{dy}{dx} = y^2$$

which is 1st order, $x$ independent variable, $y$ dependent variable

$$\frac{d^4y}{dt^4} + 5\frac{d^2x}{dt^2} + 3x\sin t$$

which is 4th order
because above equation only have 1 independent variable, they are ordinary differential equations(ODEs).

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = r$$

which is 1st order

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

which is 2nd order
above equation have more than one independent variable, they are partial differential equations(PDEs).

$n$th-order ODE: $F(x, y, y', \cdots, y^{(n)}) = 0$
In certain condition on $F$, it can be written as

$$y^{(n)} = f(x, y, y', \cdots, y^{(n)}) = 0(\star)$$

Example $(y')^2 + y' + xy = 0$

$$y' = \frac{-1 \pm \sqrt{1 - 4xy}}{2}$$

**Definition.** *a function $\phi(x)$ is called a solution of $(\star)$ on $a < x < b$ if $\phi^{(n)}$ exists on $a < x < b$ and*

$$\phi^{(n)}(x) = f(x, \phi(x), \phi'(x), \cdots, \phi^{(n-1)}(x)) \ \forall a < x < b$$

**Example.** Verify that $y = e^{2x}$ is a solution of $y'' + y' - 6y = 0$

*Proof.* $y'' + y' - 6y = 4e^{3x} + 2e^{2x} - 6e^{3x} = 0 \ \forall -\infty < x < \infty$
$\therefore y = e^{2x}$ is a solution on $-\infty < x < \infty$ ∎

**Note.** $y' = \dfrac{xy}{x+y+1}$ is derivative form $\Leftrightarrow$ $dy = \dfrac{xy}{x+y+1}dx$ or $xydx - (x+y+1)dy = 0$ is differential form.

**Definition.** *An ODE of order $n$ is called linear if it may be written in the form*

$$b_0(x)y^{(n)} + b_1(x)y^{(n-1)} + \cdots + b_{n-1}(x)y' + b_n(x)y = R(x)$$

*where $b_0 \neq 0$, An ODE that is not linear is called nonlinear ODE.*

**Example**

linear

$$\begin{aligned} y'(x) + 5y'(x) + 6y(x) &= 0 \\ y'''(x) + x^2y''(x) + x^3y'(x) &= xe^x \end{aligned}$$

non linear

$$\begin{aligned} y''(x) + 5y'(x) + 6y^2(x) &= 0 \\ y''(x) + 5(y'(x))^3 + 6y(x) &= 0 \end{aligned}$$

Initial-Value Problem(IVP): same point (and 1st order)

$$\begin{cases} \dfrac{d^2y}{dx^2} + y = 0 \\ y(1) = 3 \\ y(1) = 2 \end{cases}$$

Boundary-Value Problem(BVP): two or more different points

$$\begin{cases} \dfrac{d^2y}{dx^2} + y = 0 \\ y(1) = 3 \\ y(2) = 2 \end{cases}$$

**Theorem** (Existence and uniqueness)**.** *Consider*

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

*where* $x_0, y_0 \in \mathbb{R}$ *are given*
*Let* $T = \{(x,y) \mid |x - x_0| \le a, |y - y_0| \le b\}$, *where* $a, b > 0$. *Suppose that* $f$ *and* $fy$ *are continuous in* $T$. *Then (IVP) has a unique solution defined on* $[x_0 - h, x_0 + h]$ *for some* $h > 0$

§ **Separable equation** $A(x)dx = B(y)dy$

**Example.**
(1) $\dfrac{dy}{dx} = \dfrac{2y}{x}$
*Solution.*

$$\begin{aligned} \frac{1}{y}dy &= \frac{2}{x}dx \\ \implies \int \frac{1}{y}dy &= \int \frac{2}{x}dx \\ \implies \ln|y| &= 2\ln|x| + C \end{aligned}$$

■

(2) $\begin{cases} (1+y^2)dx + (1+x^2)dy = 0 \\ y(0) = -1 \end{cases}$
*Solution.*

$$\begin{aligned} (1+y^2)dx &= -(1+x^2)dy \\ \implies \frac{dx}{-(1+x^2)} &= \frac{dy}{(1+y^2)} \\ \implies \int \frac{1}{1+x^2}dx &= -\int \frac{1}{1+y^2}dy \end{aligned}$$

you can let $x = \tan\theta \implies dx = \sec^2\theta d\theta$
$\therefore \int \dfrac{1}{1+x^2}dx = \int \cos^2\theta \sec^2\theta d\theta = \theta + C = \tan^{-1}x + C$
$\implies \tan^{-1}x = -\tan^{-1}y + C$
$y(0) = -1 \implies 0 = \dfrac{\pi}{4} + C \implies C = \dfrac{\pi}{-4}$, $\therefore \tan^{-1}x = -\tan^{-1}y - \dfrac{\pi}{4}$

■

(3) $\begin{cases} 2x(y+1)dx - ydy = 0 \\ y(0) = -2 \end{cases}$

*Solution.* $\int 2xdx = \int \dfrac{y}{y+1}dy \implies x^2 = y - \ln|y+1| + C$

$y(0) = -2 \implies 0 = -2 + c \implies c = 2 \;\therefore x^2 + y - \ln|y+1| + 2$ ■

## § Homogeneous equations

**<u>Definition</u>.** *a function $f(x,y)$ is said to be homogeneous of degree $k$ in $x$ and $y$ if and only if*

$$f(\lambda x, \lambda y) = \lambda^k f(x,y)$$

**Example.** $f(x,y) = x^2 + y^2$

$$\begin{aligned} f(\lambda x, \lambda y) &= (\lambda x)^2 + (\lambda y)^2 \\ &= \lambda^2(x^2 + y^2) \\ &= \lambda^2 f(x,y) \end{aligned}$$

$\therefore f(x,y)$ is homogeneous , $k = 2$

**Theorem.** *If $M(x,y)$ and $N(x,y)$ are both homogeneous and of the same degree, then $\dfrac{M(x,y)}{N(x,y)}$ is homogeneous of degree zero.*

*Proof.* Set $f(x,y) = \dfrac{M(x,y)}{N(x,y)}$. By definition, we assume $M$ and $N$ are homogeneous of degree $k$, so

$$M(\lambda x, \lambda y) = \lambda^k M(x,y) \text{ and } N(\lambda x, \lambda y) = \lambda^k N(x,y)$$

$\therefore f(\lambda x, \lambda y) = \dfrac{M(\lambda x, \lambda y)}{N(\lambda x, \lambda y)} = \dfrac{\lambda^k}{\lambda^k} \cdot \dfrac{M(x,y)}{N(x,y)} = \lambda^0 \dfrac{M(x,y)}{N(x,y)}$ ■

**Theorem.** *If $f(x,y)$ is homogeneous of degree zero in $x$ and $y$, then $f(x,y) = g(\frac{y}{x})$ for some function $y$.*

*Proof.* By assumption,

$$f(\lambda x, \lambda y) = \lambda^0 f(x,y) = f(x,y)$$

Take $\lambda = \frac{1}{x}$. Then $f(x,y) = f(1, \frac{y}{x}) = g(\frac{y}{x})$, where $g(v) = f(1,v)$. ■

**Corollary.** *If $M(x,y)$ and $N(x,y)$ are both homogeneous and of the same degree, then $\dfrac{M(x,y)}{N(x,y)} = g(\frac{y}{x})$ for some function $g$.*

<u>**Definition.**</u> *$M(x,y)+N(x,y)dy=0$ is said to be homogeneous if it can be written as the form $\dfrac{dy}{dx}=g(\frac{y}{x})$ for some function $g$*

**Example.** $(x^2-3y^2)dx+2xydy=0(\star)$

$$\frac{dy}{dx}=-\frac{x^2-3y^2}{2xy}=-\frac{1-3(\frac{y}{x})^2}{2\cdot\frac{y}{x}}=g(\frac{y}{x})\text{ where }g(v)=\frac{1-3v^2}{-2v}$$

**Remark.** *If $M(x,y)$ and $N(x,y)$ are homogeneous of the same degree, then $M(x,y)dx+N(x,y)dy=0$ is homogeneous.*

*Proof.* By assumption and corollary, $\dfrac{M(x,y)}{N(x,y)}=g(\dfrac{y}{x})$ for some function $g$. $\therefore Mdx+Ndy=0 \implies \dfrac{dy}{dx}=-\dfrac{M(x,y)}{N(x,y)}=-g(\dfrac{y}{x})$
$\therefore Mdx+Ndy$ is homogeneous. ∎

**How to solve homogeneous equation**
Suppose$M(x,y)dx+N(x,y)dy=0(\star)$ is homogeneous.
Let $y=vx \implies \dfrac{dy}{dx}=\dfrac{dv}{dx}x+v(1)$
$\because(\star)$ is homogeneous $\therefore$ By definition, $(\star)\Leftrightarrow\dfrac{dy}{dx}=g(\dfrac{y}{x})(2)$
where $g$ is a function.

$$\begin{aligned}\text{put (1) to (2)}\quad &\implies \frac{dv}{dx}x+v=g(v)\\ &\implies \frac{dv}{dx}x=g(v)-v\\ &\implies \frac{1}{g(v)-v}dv=\frac{1}{x}dx,\ \text{which is separable}\end{aligned}$$

$\therefore$ The solution is $\int\dfrac{1}{g(v)-b}dv=\int\dfrac{1}{x}dx$
**Example.** $(x^2-xy+y^2)dx-xydy=0$—(1)

*Solution.*

$$\begin{aligned}M(\lambda x,\lambda y)&=(\lambda x)^2-(\lambda x)(\lambda y)+(\lambda y)^2\\ &=\lambda^2(x^2-xy+y^2)\\ &=\lambda^2M(x,y)\end{aligned}$$

$$\begin{aligned} N(\lambda x, \lambda y) &= -(\lambda x)(\lambda y) \\ &= -\lambda^2 xy \\ &= \lambda^2 N(x,y) \end{aligned}$$

■

so (1) is homogeneous.
Let $y = vx \implies \dfrac{dy}{dx} = \dfrac{dv}{dx}x + v$

$$\begin{aligned} (1) \quad &\implies \quad \frac{dy}{dx} = -\frac{x^2 - xy + y^2}{xy} = \frac{1 - \frac{y}{x} + (\frac{y}{x})^2}{\frac{y}{x}} \\ &\implies \quad \frac{dv}{dx}x + v = \frac{1 - v + v^2}{v} \\ &\implies \quad \frac{dv}{dx}x = \frac{1 - v + v^2}{v} - v = \frac{1-v}{v} \\ &\implies \quad \int \frac{v}{1-v} dv = \int \frac{1}{x} \\ &\implies \quad \frac{v - 1 + 1}{1 - v} = -1 - \frac{1}{v-1} \\ &\implies \quad -v - \ln|v - 1| = \ln|x| + c \end{aligned}$$

**Example.** $xydx + (x^2 + y^2)dy = 0$

*Solution.* $\dfrac{dy}{dx} = \dfrac{xy}{-(x^2+y^2)} = -\dfrac{\frac{y}{x}}{1 + (\frac{y}{x})^2} = g(\frac{y}{x})$ —(1),
where $g(v) = \dfrac{-v}{1+v^2}$, so the equation is homogeneous.
Let $y = vx \implies \dfrac{dy}{dx} = \dfrac{dv}{dx}x + v$

$$
\begin{aligned}
(1) \quad & \Longrightarrow \quad \frac{dv}{dx}x + v = -\frac{v}{1+v^2} \\
& \Longrightarrow \quad \frac{dv}{dx} \cdot x = -\frac{v}{1+v^2} - v = -\frac{2v+v^3}{1+v} \\
& \Longrightarrow \quad \int \frac{1+v^2}{2v+v^3} dv = -\int \frac{1}{x} dx \\
& \Longrightarrow \quad \int (\frac{0.5}{v} + \frac{0.5v+0}{2+v^2}) = -\int \frac{1}{x} dx \\
& \therefore \quad 0.5 \int \frac{1}{v} + \frac{v}{2+v^2} dv = -\ln|x| + c \\
& \Longrightarrow \quad 0.5 \ln|v| + 0.25 \ln|2+v^2| + -\ln|x| + c \\
& \Longrightarrow \quad 0.5 \ln|\frac{y}{x}| + 0.25 \ln|2 + \frac{y^2}{x^2}| = -\ln|x| + c
\end{aligned}
$$

■

## § **Exact equation**

**Definition.** *$M(x,y)dx + N(x,y)dy = 0$ is called an exact equation if there exists a function $F(x,y)$ such that $F_x = M$ and $F_y = N$*

**Example.**
$y^2dx + 2xydy = 0$ Set $F(x,y) = xy^2 \implies Fx = y^2$ and $Fy = 2xy$
$\therefore$ exact equation.
**How to solve homogeneous equation**
Suppose $M(x,y)dx + N(x,y) = 0$ —($\star$) is exact
$\implies \exists$ a function $F(x,y)$ such that $Fx = M$ and $Fy = N$

$$
\begin{aligned}
(\star) \quad & \Longrightarrow \quad Fxdx + Fydy = 0 \\
& \Longrightarrow \quad dF = 0 \\
& \Longrightarrow \quad F = C, \text{ where } C \text{ is an arbitrary constant}
\end{aligned}
$$

**Theorem.** *Suppose $M, N, My, Nx$ are continuous. Then $Mdx + Ndy = 0$ is an exact equation $\Leftrightarrow$ $My = Nx$*

*Proof.* ($\Leftarrow$) Suppose $My = Nx$. Claim ($\star$) is exact

$$
\begin{cases} Fx = M \text{—(1)} \\ Fy = N \text{—(2)} \end{cases}
$$

(1) $\Leftrightarrow F(x,y) = \int M(x,y)\partial x + \phi(y)$ for some function $\phi$

$$
\begin{aligned}
(1)(2) \quad &\Leftrightarrow \quad \frac{\partial}{\partial y}\int M(x,y)\partial x + \phi'(y) = N(x,y) \\
&\Leftrightarrow \quad \phi'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\partial x = N(x,y) - \int My(x,y)\partial x
\end{aligned}
$$

We compute

$$
\frac{\partial}{\partial}\left[N(x,y) - \int My(x,y)\partial x\right] = Nx(x,y) - My(x,y) = 0
$$

This implies $N(x,y) - \int My(x,y)\partial x$ is independent of $x$
$\therefore \phi(y) = \int \left[N(x,y) - \int My(x,y)\partial x\right] dy$
$\therefore F(x,y) = \int M(x,y)\partial x + \int \left[N(x,y) - \int My(x,y)\partial x\right] dy$ satisfy (1)(2)
$\therefore Mdx + Ndy = 0$ is exact.

∎

**Example.** $3x(xy-2)dx + (x^3+2y)dy = 0$

*Solution.* $\therefore My = 3x^2 = Nx \;\; \therefore$ exact

Find a function $F$ s.t. $\begin{cases} Fx = M = 3x^2y - 6x \text{— (1)} \\ Fy = N = x^3 + 2y \;\; \text{— (2)} \end{cases}$

$$
\begin{aligned}
(1) \implies F \quad &= \quad \int (2x^3 - xy^2 - 2y + 3)\partial x + \phi(y) \\
&= \quad \frac{1}{3}x^4 - \frac{1}{2}x^2y^2 - 2xy + 3x + \phi(y) \\
&\implies \quad -x^2y - 2x = Fy = -x^2y - 3x + \phi'(y) \implies \phi'(y) = c \\
&\implies \quad \phi(y) = 0 \implies F = x^4 - \frac{1}{2}x^2y^2 - 2xy + 3x
\end{aligned}
$$

$\therefore \frac{1}{2}x^4 - \frac{1}{2}x^2y^2 - 2xy + 3x = C$ is the general solution ∎

§ **Linear equations**: $A(x)y' + B(x)y = c(x)$, where $A(x) \neq 0$
Suppose $c(x) = 0$

$$
\begin{aligned}
&\implies \quad A(x)y' + B(x)y = 0 \\
&\implies \quad A(x)y' = -B(x)y \implies \frac{1}{y}dy = \frac{B(x)}{-A(x)}dx, \text{ which is separable}
\end{aligned}
$$

Suppose $B(x) = 0 \implies$ it's easy to solve
Suppose $c(x) \neq 0$ and $B(x) \neq 0$

$A(x)y' + B(x)y = c(x) \implies y' + p(x)y = Q(x)$, where $p(x) = \dfrac{B(x)}{A(x)}$ and $Q(x) = \dfrac{C(x)}{A(x)}$
$\Leftrightarrow dF = 0 \Leftrightarrow Fxdx + Fydy = 0$
$\Leftrightarrow (P(x)y - Q(x))dx + 1dy = 0$—(1) is not exact.
Let $v = v(x)$, (1) $\times v \implies v(P(x)y - Q(x))dx + v(x)dy$ — (2)

$$\begin{aligned}
\text{(2) is exact} \quad &\Leftrightarrow \quad \frac{\partial}{\partial y}\left[v(x)P(x)y - Q(x)\right] = v'(x) \\
&\Leftrightarrow \quad v(x)P(x) = v'(x) \Leftrightarrow P(x)dx = \frac{1}{v}dv \\
&\Leftrightarrow \quad \ln|v| = \int P(x)dx \Leftrightarrow |v| = e
\end{aligned}$$

Let $v(x) = e^{\int p(x)dx}$, which is called the integrating factor of (1)
$A(x)y' + B(x)y = C(x) \implies y' + \dfrac{B(x)}{A(x)}y = \dfrac{C(x)}{A(x)}$, and let $\dfrac{B(x)}{A(x)} = p(x)$

$$\mu(x) = e^{\int p(x)dx}$$

$$e^{\int p(x)dx}y' + p(x)e^{\int p(x)dx}y = Q(x)e^{\int p(x)dx}$$

**Example.** $2(y - 4x^2)dx + xdy = 0$

*Solution.* $x\dfrac{dy}{dx} + 2y = 8x^2 \implies \dfrac{dy}{dx} + \dfrac{2}{x}y = 8x - (1)$

$$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln x^2} = x^2$$

$$\begin{aligned}
(1) \times \mu(x) \quad &\implies \quad x^2\frac{dy}{dx} + 2xy + 8x^3 \\
&\implies \quad [x^2y]' = 8x^3 \\
&\implies \quad x^2y = \int 8x^3dx = 2x^4 + c
\end{aligned}$$

■

**Example** $ydx + (3x - xy + 2)dy = 0$

*Solution.* $(3x - xy + 2)\dfrac{dy}{dx} + y = 0$ which is not linear
$y\dfrac{dx}{dy} + (3 - y)x = -2$, which is linear

$\implies \dfrac{dx}{dy} + \dfrac{3-y}{y}x = -\dfrac{2}{y}$ – (1)

$\mu(y) = e^{\int \frac{3-y}{y}dy} = e^{3\ln|y|-y} = |y|^3e^{-y} = \pm y^3e^{-y} = y^3e^{-y}$

(1) $\times\mu(y) \implies y^3e^{-y}\dfrac{dx}{dy} + y^3e^{-y}\left(\dfrac{3-y}{y}\right)x = -2y^2e^{-y}$

$\implies \dfrac{d}{dy}\left[y^3e^{-y}x\right] = -2y^2e^{-y}$

$\implies y^3e^{-y}x = -2\int y^2e^{-y}dx$ ■

## 5. Additional topics on equations of order one

(A) Find an integrating factor
Consider

$$Mdx + Ndy = 0 \text{ – (1)}$$

Suppose (1) is not exact, let $\mu = \mu(x,y)$
(1) $\times\mu \implies \mu Mdx + \mu Ndy = 0$ – (2)
(2) is exact iff $\dfrac{\partial}{\partial y}(\mu, M) = \dfrac{\partial}{\partial x}(\mu N)$ By thm 2.3, iff $\mu_y M + \mu M_y = \mu_x N + \mu N_x$ – (3)
Note that (3) is a 1st linear PDE and connot be solved in general
Suppose $\mu = \mu(x)$ :

$$\begin{aligned}(3) \quad &\Leftrightarrow \quad \mu M_y = \mu_x N + \mu N_x \\ &\Leftrightarrow \quad N - \frac{du}{dx} = (My - Nx)\mu \\ &\Leftrightarrow \quad \frac{1}{\mu}\frac{du}{dx} = \frac{My - Nx}{N}\end{aligned}$$

Suppose $\dfrac{My - Nx}{N}$ depends only on $x$. Then

$\int \dfrac{1}{\mu}du = \int \dfrac{My - Nx}{N}dx \implies \ln|\mu| \implies e^{\int \frac{My-Nx}{N}dx} \implies$

$\mu = \pm e^{\int \frac{My-Nx}{N}dx} = e^{\int \frac{My-Nx}{N}dx}$ Take positibity

$\therefore \mu(x) = e^{\frac{My-Nx}{N}dx}$ is an integrating factor of (1)
Suppose $u = \mu(y)$,

$$\begin{aligned}
(2) \quad &\Leftrightarrow \quad u_y M + u M_y = u N_x \\
&\Leftrightarrow \quad \frac{du}{dy} M = (Nx - My)u \\
&\Leftrightarrow \quad \frac{1}{u}\frac{du}{dy} = \frac{Nx - My}{M}
\end{aligned}$$

and $\dfrac{1}{u}\dfrac{du}{dy}$ depending only on $y$.
Suppose $\dfrac{Nx - My}{M}$ depends only on $y$, then

$$\begin{aligned}
\int \frac{1}{u} du &= \int \frac{Nx - My}{M} dy \\
\implies |u| &= e^{\int \frac{Nx - My}{M} dy} \\
\implies u &= \pm e^{\int \frac{Nx - My}{M} dy}
\end{aligned}$$

Take positivity of $u \implies u(y) = e^{\int \frac{Nx - My}{M} dy}$ is an integrating factor of (1)
Conclusion

(1) Suppose $\dfrac{My - Nx}{N}$ depends only on $x$, then $u(x) = e^{\int \frac{My - Nx}{N} dx}$ is an integrating factor of (1)

(2) Suppose $\dfrac{Nx - My}{M}$ depends only on $y$, then $u(y) = e^{\int \frac{Nx - My}{M} dy}$ is an integrating factor of (1)

**Example** $(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0$

*Solution.* $My = 4x + 6y, \; Nx = 2x + 3y$
$\because My \neq Nx \therefore$ not exact
$\dfrac{My - Nx}{N} = \dfrac{2x + 4y}{x(x + 2y)} = \dfrac{2}{x}$, which depends only on $x$
$\mu(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln |x|} = x^2$ is an integrating factor
$\mu \times (1) \implies (vx^3 y + 3x^2 y^2 - x^3)dx + (x^4 + 3x^3 y)dy = 0$, which is exact

$$\begin{cases} Fx = 4x^3 y + 3x^2 y^2 - x^3 - (2) \\ Fy = x^4 + 2x^3 y - (3) \end{cases}$$

■

(2) $\implies F = x^4y + x^3y^2 - \frac{1}{4}x^4 + \phi(y) \implies Fy = x^4 + 2x^3y + \phi'(y) \implies$ $\phi(y) = 0 \implies \phi(y) = 0 \implies F = x^4y + x^3y^2 - \frac{1}{4}x^4$

$\therefore x^4y + x^2y^2 - \frac{1}{4}x^4 = C$ is the general solution.

**Example.** $y(x + y + 1)dx + x(x + 3y + 2)dy = 0$

*Solution.* $My = x + 2y + 1,\ Nx = 2x + 3y + 2$

$\because My \neq Nx \therefore$ not exact $\dfrac{My - Nx}{N} = \dfrac{-x - y - 1}{x(x + 3y + 2)}$, which depends on both $x$ and $y$. $\dfrac{Nx - My}{M} = \dfrac{x + y + 1}{y(x + y + 1)} = \dfrac{1}{-y}$, which depends only on $y$, $\therefore \mu(y) = e^{\int \frac{1}{y}dy} = e^{\ln|y|} = |y| = y$ taking positive

(1) $\times\mu(y) \implies (xy^2 + y^3 + y^2)dx + (x^2y + 3xy^2 + 2xy)dy = 0$, which is exact.

$$\begin{cases} Fx = xy^2 + y^3 + y^2 \\ Fy = x^2y + 2xy^2 + 2xy \text{ −(3)} \end{cases}$$

$\implies F = \frac{1}{2}x^2y^2 + xy^3 + xy^2 + \phi(y)$

$\implies Fy = x^2y + 3xy^2 + 2xy + \phi'(y)$ − (2)

(2)(3) $\implies \phi'(y) = 0 \implies \phi(y) = 0$

$\therefore F = \frac{1}{2}x^2y^2 + xy^3 + xy^2$

$\therefore \frac{1}{2}x^2y^2 + xy^3 + xy^2 = 0$ is the general solution ■

(B) substitution

**Example.** $(x + 2y - 1)dx + 3(x + 2y)dy = 0$

*Solution.* Let $v = x + 2y \implies dv = dx + 2dy$

$\implies (v - 1)(dv - 2dy) + 3vdy = 0 \implies (v - 1)dv - 2(v - 1)dy + 3vdy$

$\implies (v - 1)dv + (v + 2)dy = 0 \implies (v - 1)dv = -(v + 2)dy$

$\implies \int \frac{v - 1}{v + 2}dv = -\int 1dy \implies v - 3\ln|v + 2| = -y + c$

$\implies x + 2y - 3\ln|x + 2y + 2| = -y + c$ ■

**Example.** $(1 + 3x\sin y)dx - x^2\cos ydy = 0$

*Solution.* Let $v = \sin y \implies dv = \cos ydy$

$\implies (1 + 3xv)dx - x^2dv = 0 \implies -x^2\frac{dv}{dx} + 3xv = -1$ is linear

$\implies \frac{dv}{dx} - \frac{3}{x}v = \frac{1}{x^2}$ −(1)

$\mu(x) = e^{-\int \frac{3}{x} dx} = e^{-3\ln|x|} = |x|^{-3} = x^{-3}$ taking positivity

$\implies x^{--3}\dfrac{dv}{dx} - 3x^{-4}v = x^{-5} \implies x^{-3}v = \int x^{-5}dx = \dfrac{1}{-4}x^{-4} + C$

$\implies x^{-3}\sin y = \dfrac{1}{-4}x^{-4} + C$ ■

(C) Benoull's equation's: $y' + p(x)y = Q(x)y^n$ – (1)
Suppose $n = 1$ (1) is linear and separable,
Suppose $n \neq 1$, (1) $\implies y^n y^1 + p(x)y^{1-n} = Q(x)$ – (2)
$\implies$ Set $z = y^{1-n} \implies z' = (1-n)y^{-n}y'$
(2) $\implies \dfrac{1}{1-n}z' + p(x)z = Q(x)$, which is linear.

**Example.** $y(6y^2 + x - 1)dx + 2xdy = 0$

*Solution.* $2x\dfrac{dy}{dx} - (x+1)y = -6y^3 \implies 2xy^{-3}y' - (x+1)y^{-2} = -6$
Set $z = y^{-2} \implies z' = -2y^{-3}y' \therefore -xz' - (x+1)z = -6$ is linear

$\implies z' + \dfrac{x+1}{x}z = \dfrac{6}{x},\ \mu(x) = e^{\int \frac{x+1}{x}dx} = e^{x+\ln|x|} = |x|e^x = xe^x$

(1) $\times\mu \implies (xe^x z)' = 6e^x \implies xe^x z = 6x^x + c$ ■

**Example** $6y^2dx - x(2x^3 + y)dy = 0$

*Solution.* $-x(2x^3 + y)\dfrac{dy}{dx} + 6y^2 = 0$ is not Bonuliy

$6y^2\dfrac{dx}{dy} - xy = 2x^4$, which is a Berrnoulli's equation

$\implies 6y^2x^{-4}\dfrac{dx}{dy} - yx^{-3} = 2$

Set $z = x^{-3} \implies z' = -3x^{-4}x' \implies -2y^2z' - yz = 2$, is linear

$\implies z' + \dfrac{1}{2y}z = -y^{-2}$

$\mu(y) = e^{\int \frac{1}{2y}dy} = e^{\frac{1}{2}\ln|y|} = \sqrt{|y|} = \sqrt{y}$ (taking positive)

$\implies y^{\frac{1}{2}}z' + \frac{1}{2}y^{\frac{1}{-2}}z = -y^{\frac{3}{-2}} \implies y^{\frac{1}{2}}z = 2y^{-\frac{1}{2}} + c$

$\implies y^{\frac{1}{2}}x^{-3} = 2y^{-\frac{1}{2}} + c$ ■

(D) $(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0 \mid c_1^2 + c_2^2 \neq 0$
Case I: $\frac{a_2}{a_1} \neq \frac{b_2}{b_1}$
Let $x = u + h$ and $y = v + k$, $h, k$ are constants to be determined later.
$\implies dx = du$ and $dy = dv$
$\implies [a_1u + b_1v + (a_1h + b_1k + c_1)]du + [a_2(u+h) + b_2(v+k) + c_2]dv = 0$
$\implies [a_1u + b_1v + (a_1h + b_1 + c_1)]du + [a_2u + b_2v + (a_2h + b_2k + c_2)]dv = 0$

Take $h, k$ such that $\begin{cases} a_1h + b_1k + c_1 = 0 \\ a_2h + b_22k + c_2 = 0 \end{cases}$
$\implies (a_1u + b_1v)du + (a_2u + b_2v)dv = 0$,which is homogeneous.
Case II $\frac{a_2}{a_1} = \frac{b_2}{b_2} = l \implies a_2 = la_1$ and $b_2 = lb_1$
$\implies (a_1x + b_1y + c_1)dx + (la_1x + b_1y + c_2)\mathrm{dy} = 0$
Let $v = a_1x + b_1y \implies dv = a_1dx + b_1dy$
$\implies \frac{1}{a_1}(v + c_1)(dv - b_1dy) + (lv + c_2)dy = 0$
$\implies \frac{1}{a_1}(v + c_1)dv - \frac{b_1}{a_1}(v + c_1)dy + (lv + c_2)dy = 0$
$\implies \frac{1}{a_1}(v + c_1)dv + [lv + c_2 - \frac{b_1}{a_1}(v + c_1)]dy = 0$ which is separable.

**Example.** $(x + 2y - 4)dx - (2x + y - 5)dy = 0$

*Solution.* $\begin{cases} h + 2k - 4 = 0\text{–}(1) \\ 2h + k - 5 = 0\text{–}(2) \end{cases}$
$(1) \times (2) - (2) \implies 3k - 3 = 0 \implies k = 1, h = 2$
Let $x = u + 2,\ y = v + 1 \implies dx = du$ and $dy = dv$
$\therefore [(u + 2) + 2(r + x) - 4]du - [2(u + 2) + (v + x) - 5]dv = 0$
$\implies (u + 2v)du - (2u + v)du = 0 \implies \dfrac{dv}{du} = \dfrac{u + 2v}{2u + v} = \dfrac{1 + 2\frac{v}{u}}{2 + \frac{v}{u}}$
Let $v = uw \implies \dfrac{dv}{du} = w + u\dfrac{dw}{wu}\ \therefore w + u\dfrac{dw}{du} = \dfrac{1 + 2w}{2 + w}$
$\implies u\dfrac{dw}{du} = \dfrac{1 + 2w - 2w - w^2}{2 + w} = \dfrac{1 - w^2}{2 + 2} \implies \int \dfrac{2 + w}{1 - w^2} = \int \dfrac{1}{u}du$
$\implies \dfrac{3}{-2}\ln|w - 1| + \dfrac{1}{2}\ln|w + 1| = \ln|u| + c$
$\implies \dfrac{3}{-2}\ln|\dfrac{y - 1}{x - 2} - 1| + \dfrac{1}{2}\ln|\dfrac{y - 1}{x - 2} + 1| = \ln|x - 2| + c$ ■

**Example.** $(2x + 3y - 1)dx + (2x + 3y + 2)dy = 0$
Let $v = 2x + 3y \implies dv = 2dx + 3dy$
$\therefore \dfrac{1}{2}(v - 1)(dv - 3dy) + (v + 2)dy = 0$
$\implies \dfrac{1}{2}(v - 1)dv - \dfrac{3}{2}(v - 1)dy + (v + 2)dy = 0$
$\implies \dfrac{1}{2}(v - 1)dv = [\dfrac{3}{2}(v - 1) - 2(v + 2)]dy \implies \int \dfrac{v - 1}{v - 7}dv = \int 1dy$
$\implies v + 6\ln|v - 7| = y + c \implies 2x + 3y + 6\ln|2x + 3y - 7| = y + c$

## 6. Linear Differential

Form ($n$th-order)
($\star$) $b_0(x)y^n(x) + b_1(x)y^{n-1}(x) + \cdots + b_{n-1}(x)y'(x) + b_n(x)y(x) = R(x)$
where $b_0(x) \neq 0$

**Definition.** *If $R(x) = 0$, then ($\star$) is said to be homogeneous. Otherwise, it is said to be nonhomogeneous*

**Definition.** *Let $I$ be an interval. If $b_0(x) \neq 0$, $\forall x \in I$ and $b_0, b_1, \cdots, b_n, R \in C(I)$, then ($\star$) is said to be normal on $I$.*

**Example.** $(x-1)y' + y = \sin x$
1st order, linear, nonhomogeneous equation. normal on any interval $I$ where $1 \notin I$
**Example.** $3y'' + xy = 0$
2nd order, linear, homogeneous equation. normal on any interval $I$

**Theorem.** *Let $y_1, \cdots, y_k$ be solution of*

$$(\star) b_0(x)y^{(n)} + b_1(x)y^{(n-1)} + \cdots + b_n(x)y$$

*in $I$. Then, $\forall c_1, \cdots, c_n \in \mathbb{R}$, $y = c_1y_1 + \cdots + c_ny + n$ is also a solution of ($\star$) on $I$*

*Solution.* $b_0(x)y^{(n)} + b_1(x)y^{(n-1)} + \cdots + b_{n-1}(x)y' + b_n(x)y$
$= b_0(x)[c_1y_1^{(n)} + \cdots + c_ky_k^{(n)}] + \cdots + b_1(x)[c_1y_1 + \cdots + c_ky_k]$
$= c_1[b_0y_1^{(n)} + b_1(x)y_1^{(n-1)} + \cdots + b_{n-1}(x)y_1' + b_n(x)y_1] + \cdots + c_k[b_0(x)y_k^{(n)} + b_1(x)y_k^{n-1} + \cdots + b_{n-1}(x)y_k + b_n(x)y_k] = 0$
$\therefore y = c_1y_1 + \cdots + c_ky_k$ is also a solution of ($\star$) on $I$ ■

**Definition.** *Let $y_1, \cdots, y_k$ be functions. Then $\forall c_1, \cdots, c_k \in \mathbb{R}$, $c_1y_1 + \cdots + c_ky_k$ is called a linear combination of $y_1, \cdots, y_k$*

**Remark.** *We may restate Thm 6.1 as follows:*
*Any linear combination of solutions of ($\star$) is also a solution of ($\star$)*

**Example.** $y'' + y = 0$
$y = \sin x$ and $y = \cos x$ are solutions on $\mathbb{R}$. By Thm 6.1, $\forall c_1, c_2 \in \mathbb{R}$, $y = c_1 \sin x + c_2 \cos x$ is also a solution on $\mathbb{R}$

**Theorem.** *Consider ($\star$) is normal in $I$ and $x_0 \in I$. For any given $y_0, \cdots, y_{n-1} \in \mathbb{R}$, ($\star$) has a unique solution $y = y(x)$ in $I$ satisfying*

$$y(x_0 = y_0),\ y'(x_0) = y_1, \cdots, y^{n-1}(x_0) = y_{n-1}$$

,

**Example** $\begin{cases} y'' + y = 0 \text{ -- } (1) \\ y(0) = 0,\ y'(0) = 1 \text{ -- } (2) \end{cases}$
$\because$ (1) is normal on $\mathbb{R}$ and $0 \in \mathbb{R}$
$\therefore$ By Thm 6.2, (1) has a unique solution on $\mathbb{R}$ satisfying (2). Indeed,
$y = c_1 \sin x + c_2 \cos x$ is a solution of (1) on $\mathbb{R}$, $\forall c_1, c_2 \in \mathbb{R}$
$y(0) = 0 \implies c_2 = 0 \implies y = c_2 \sin x \implies y' = c_1 \cos x$
$y'(0) = 1 \implies c_1 = 1 \therefore y = \sin x$ is the unique solution.
**Example.**
$\begin{cases} x^2 y'' + 2xy' - 12y = 0 \text{ ---}(1) \\ y(1) = 4,\ y'(1) = 5 \text{ ---}(2) \end{cases}$
$\because$ (1) is normal on $(-\infty, 0)$ or $(0, \infty)$, and $1 \in (0, \infty)$
$\therefore$ By Thm 6.2, (1) has a unique solution in $(0, \infty)$ satisfying (2)

**Definition**. *Let $f_1, \cdots, f_n$ be functions in $[a, b]$. If there exists $c_1, \cdots, c_n \in \mathbb{R}$, not all zero, such that*

$$c_1 f_1(x) + \cdots + c_n f_n(x) = 0, \forall x \in [a, b],$$

*then $f_1, \cdots, f_n$ are said to be linearly dependent on $[a, b]$. Otherwise, they are said to be linearly independent on $[a, b]$.*

**Remark.** *If $f_1, \cdots, f_n$ are linear dependent on $[a, b]$, then*

$$c_1 f_1(x) + \cdots + c_n f_n(x) = 0,\ \forall [a, b]$$

*implies $c_1 = \cdots = c_n = 0$*

**Example.** $x, 2x$ are linear dependent on $\mathbb{R}$

**Definition**. *let $f_1, \cdots, f_n$ be $n$ functions*
$\begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{n-1}(x) & f_2^{n-1}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$ *is called the Wronskian of $f_1, \cdots, f_n$*

**Theorem.** *Suppose $(\star)$ is normal on $[a, b]$, and suppose $y_1, \cdots, y_n$ are solutions of $(\star)$ on $[a, b]$. Then $y_1, \cdots, y_n$ are linear independent on $[a, b]$ iff $w[y_1, \cdots, y_n](x_0) \neq 0$ for some $x_0 \in [a, b]$*

*Proof.* ($\Rightarrow$) Suppose $w[y_1, \cdots, y_n](x) = 0,\ \forall x \in [a, b]$. Pick any point $x_0 \in [a, b]$. Then $w[y_1, \cdots, y_n](x_0) = 0$, i.e.

$$\begin{vmatrix} y_1(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & \cdots & y_n'(x_0) \\ \vdots & & \vdots \\ y_1^{n-1}(x_0) & \cdots & y_n^{n-1}(x_0) \end{vmatrix} = 0$$

Thus, $\exists c_1, \cdots, c_n \in \mathbb{R}$, not all are zero such that

$$\begin{cases} y_1(x_0)c_1 + y_2(x_0)c_2 + \cdots + y_n(x_0)c_n = 0 \\ Y_1'(x_0)c_1 + y_2'(x_0)c_2 + \cdots + y_n'(x_0)c_n = 0 \\ \vdots \\ y_1^{n-1}(x_0) + y_2^{(n-1)}(x_0)c_2 + \cdots + y_n^{n-1}(x_0)c_n = 0 \end{cases}$$

Let $y(x) = c_1y_1(x) + \cdots + c_ny_n(x)$

$\because$ $y_1, \cdots, y_n$ are solutions of $(\star)$ on $[a, b]$ $\therefore$ By Thm 6.1, $y$ is also a solution of $\star$ on $[a, b]$

In addition, by (1), $y(x_0) = y'(x_0) = \cdots = y^{n-1}(x_0) = 0$. By Thm 6.2, $y = 0$ in $[a, b]$ $\therefore$ $c_1, \cdots, c_n$ are not all zero such that (2) holds.

$\therefore$ $y_1, \cdots, y_n$ are linear dependent on $[a, b]$

$(\Leftarrow)$ Suppose $w[y_1, \cdots, y_n](x_0) \neq 0$ for some $x_0 \in [a, b]$

$$\implies \begin{vmatrix} y_1(x_0) & \cdots & y_n(x_0) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x_0) & \cdots & y_n^{(n-1)} \end{vmatrix} \neq 0 \text{ — (3)}$$

Let $c_1, \cdots, c_n \in \mathbb{R}$ such that $c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) = 0,\ \forall x \in [a, b]$,

$\implies c_1y_1'(x) + c_2y_2'(x) + \cdots + c_ny_n'(x) = 0,\ \forall x \in [a, b]$

$\implies c_1y_1''(x) + c_2y_2''(x) + \cdots + c_ny_n''(x) = 0,\ \forall x \in [a, b]$

$\vdots$

$\implies c_1y_1^{(n-1)}(x) + c_2y_2^{(n-1)}(x) + \cdots + c_ny_n^{(n-1)} = 0,\ \forall x \in [a, b]$

$\because$ $x_0 \in [a, b]$

$$\begin{cases} c_1y_1(x_0) + c_2y_2(x_0) + \cdots + c_ny_n(x_0) = 0 \\ c_1y_1'(x_0) + c_2y_2'(x_0) + \cdots + c_ny_n'(x_0) = 0 \\ \vdots \\ c_1y^{n-1}(x_0) + c_2y^{(n-1)}(x_0) + \cdots + c_ny_n^{(n-1)}(x_0) = 0 \end{cases}$$

By (3), $c_1 = \cdots = c_n = 0$. So $y_1, \cdots, y_n$ are linear independent on $[a, b]$ ∎

**Example.** $y'' + y = 0$ has two solutions $\sin x$ and $\cos x$ on $\mathbb{R}$

$$w[\sin x, \cos x] = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1 \neq 0,\ \forall x \in \mathbb{R}$$

$\therefore$ By Thm 6.3, $\sin x$ and $\cos x$ are linear independent on $\mathbb{R}$

**Example.** $y''' - 2y'' - y' + 2y = 0$ has solution: $e^x,\ e^{-x}, e^{2x}$ on $\mathbb{R}$

$$w[e^x, e^{-x}] = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} = e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = -6e^{2x} \neq 0$$

$\therefore e^{-x}, e^{-x}, e^{2x}$ are linear independent on $\mathbb{R}$

**Theorem.** *Suppose $(\star)$ is normal on $[a, b]$, and $y_1, \cdots, y_n$ are linear independent solution of $(\star)$ on $[a, b]$. For any solution $\phi$ of $(\star)$ on $[a, b]$, $\exists \overline{c_1}, \cdots, \overline{c_n} \in \mathbb{R}$, such that $\phi(x) = \overline{c_1}y_1(x) + \cdots + \overline{c_n}y_n(x), \forall x \in [a, b]$*

*Proof.* $\because y_1, \cdots, y_n$ are linear independent solution of $(\star)$ on $[a, b]$. By Thm 6.3, $\exists x_0 \in [a, b]$ such that $w[y_1, \cdots, y_n](x_0) \neq 0$,

i.e. $\begin{vmatrix} y_1(x_0) & \cdots & y_n(x_0) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x_0) & \cdots & y_n^{(n-1)} \end{vmatrix} \neq 0$

$\implies \exists \overline{c_1}, \cdots, \overline{c_n} \in \mathbb{R}$ such that

$$\begin{cases} \overline{c_1}y_1(x_0) + \overline{c_2}y_2(x_0) + \cdots + \overline{c_n}y_n(x_0) = 0 \\ \overline{c_1}y_1'(x_0) + \overline{c_2}y_2'(x_0) + \cdots + \overline{c_n}y_n'(x_0) = 0 \\ \vdots \\ \overline{c_1}y^{n-1}(x_0) + \overline{c_2}y^{(n-1)}(x_0) + \cdots + \overline{c_n}y_n^{(n-1)}(x_0) = 0 \end{cases}$$

let $y(x) = \overline{c_1}y_1(x) + \cdots + \overline{c_n}y_n(x)$

$\therefore y_1, \cdots, y_n$ are solutions of $(\star)$ on $[a, b]$

$\because$ By Thm 6.1, $y$ is also a solution of $(\star)$ on $[a, b]$.

By (1), $y(x_0) = \phi(x_0),\ y'(x_0) = \phi'(x_0), \cdots, y^{(n-1)}(x_0) = \phi^{(n-1)}(x_0)$.

By Thm 6.2, $y(x) = \phi(x),\ \forall x \in [a, b]$,

i.e. $\phi(x) = c_1y_1(x) + \cdots + c_ny_n(x) \forall x \in [a, b]$ ■

**<u>Definition</u>.** *Let $(\star)$ be normal in an interval $I$. Suppose $y_1, \cdots, y_n$ are linear independent solutions of $(\star)$ on $I$. Then $y = c_1y_1 + \cdots + c_ny_n$ is called the general solution of $(\star)$ on $I$, where $c_1, \cdots, c_n$ are arbitrary constants.*

**Example.** $y'' + y = 0$ has solutions linear independent $\sin x$ and $\cos x$

$\therefore$ The general solutions is $y = c_1 \sin x + c_2 \cos x$, where $c_1$ and $c_2$ are arbitrary constants. Consider the nonhomogeneous equation

(NH) $b_0(x)y^n(x) + b_1(x)y^{(n-1)}(x) + \cdots + b_{n-1}(x)y'(x) + b_n(x)y(x) = R(x)$

and its corresponding homogeneous equation.

(H) $b_0(x)y^{(n)}(x) + b_1(x)y^{n-1}(x) + \cdots + b_{n-1}(x)y'(x) + b_n(x)y(x) = 0$

**Theorem.** *Let $v$ be any solution of (NH) and let $u$ be any solution of (H). Then $u + v$ is also a solution of (NH).*

*Proof.*
$b_0(x)[u+v]^{(n)} + b_1(x)[u+v]^{(n-1)} + \cdots + b_{n-1}(x)[u+v]' + b_n(x)[u+v]$
$= b(x)[u^{(n)} + v^{(n)}] + b_1(x)[u^{(n-1)} + v^{(n-1)}] + \cdots + b_{n-1}(x)[u' + v'] + b_n(x)[u+v]$
$= [b_0(x)u^{(n)} + b_1(x)u^{n-1} + \cdots + b_{n-1}(x)u' + b_n(x)u] + [b_0(x)v^{(n)} + b_1(x)u^{(n-1)} + \cdots + b_{n-1}(x)v' + b_n(x)v]$
$= 0 + R(x)$ ($\because$ $u$ is an root of (H) and $v$ is an root of (NH))
$\therefore u+v$ is a solution of (NH) ■

**Example.** $y'' + y = x$ has a solution $x$, $y'' + y = 0$ has a solution $\sin x$. By Thm 6.8 $x + \sin x$ is a solution of $y'' + y = x$

**Remark.** *Let $y_p$ be a particular solution of (NH) and $y_c = c_1y_1 + \cdots + c_ny_n$ be the general solution of (H). Then, $\forall c_1, \cdots, c_n \in \mathbb{R}$, $y_c + y_p$ is a solution of (NH).*

**Theorem.** *Let $y_p$ be a particular solution of (NH) and $y_c = c_1y_1 + \cdots + c_ny_n$ be the general solution of (H). Then every solution $y$ of (NH) can be expressed in the form $y = y_c + y_p$ for suitable choice of $c_1, \cdots, c_n$*

*Proof.* $\because$ $y$ and $y_p$ are solution of (NH).
$\therefore b_0(x)y^{(n)} + b_1(x)y^{(n-1)} + \cdots + b_{n-1}(x)y' + b_n(x)y = R(x)$
$b_0(x)y_p^{(n)} + b_1(x)y_p^{(n-1)} + \cdots + b_{n-1}(x)y_p' + b_n(x)y_p = R(x)$
(1)-(2) $\implies b_0(x)[y-y_p]^{(n)} + b_1(x)[y-y_p]^{(n-1)} + \cdots + b_{n-1}(x)[y-y_p] + b_n(x)[y-y_p] = 0$
$\implies y - y_p$ is a solution of (H).
By Thm 6.4, $\exists c_1, \cdots, c_n \in \mathbb{R} \ni [y - y_p] = c_1y_1 + \cdots + c_ny_n$
$y = c_1y_1 + \cdots + c_ny_n + y_p = y_c + y_p$ ■

**Definition.**

(1) *The general solution of (H) is called the complementary function of (NH). We denote it by $y_c$*
(2) *The general solution of (NH) is $y = y_c + y_p$, where $y_p$ is any particular solution of (NH).*

**Example.** $y'' = 4$ and $y'' = 0 \implies y' = c_1 \implies y = y_c = c_1x + c_2$, where $c_1, c_2$ are arbitrary constant
$y'' = 4 \implies y' = 4x = y = y_p = 2x^2$
$\therefore$ The general solution of $y'' = 4$ is $y = y_c + y_p = c_1x + c_2 + 2x^2$

**Definition.**
*Let* $A = a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n$
$B = b_0D^n + b_1D^{n-1} + \cdots + b_{n-1}D + b_n$
*we define* $A + B = (a_0 + b_0)D^n + \cdots + (a_{n-1} + b_{n-1})D + (a_n + b_n)$

**Example.** $A = 3D^2 - D + x - 2,\ B = x^2D^2 + 4D = 7$
$\implies A + B = (3 + x^2)D^2 + 3D + x + 5$

**Remark.** *Let $A$ be a $n$th order linear differential operator, $c_1, c_2$ be constant, $f_1, f_2$ be two functions with $f_1^{(n)}$ and $f_2^{(n)}$ exists.*
*Then $A(c_1f_1 + c_2f_2) = c_1Af_1 + c_2Af_2$(i.e. $A$ is linear)*

*Proof.* Write $A = a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n$
$\implies A(c_1f_1 + c_2f_2) = a_0(c_1f_1 + c_2f_2)^{(n)} + a_1(c_1f_1 + c_2f_2)$
$= a_0(c_1f_1 + c_2f_2)^{(n)} + a_1(c_1f_1 + c_2f_2)^{(n-1)} + \cdots + a_{n-1}(c_1f_1 + c_2f_2)' + a_n(c_1f_1 + c_2f_2)$
$= a_0(c_1f_1^{(n)} + c_2f_2^{n-1}) + a_1(c_1f_1^{n-1} + c_2f_2^{n-1}) + \cdots + a_{n-1}(c_1f_1' + c_2f_2') + a_n(c_1f_1 + c_2f_2)$ ■

The fundamental low:
Let $A, B, C$ be linear differential operators. Then

(1) $A + B = B + A$
(2) $(A + B) + C = A + (B + C)$
(3) $(AB)C = A(BC)$
(4) $A(B + C) = AB + AC$
(5) $AB = BA$ if $A, B$ are with constant cofficients

Let $a_0, a_1, \cdots, a_n$ be constants
$a_ny^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0 - (1)$
$\Leftrightarrow (a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = 0$
Let $y = e^{nx}$
Put $y = e^{nx}$ into (1) $\implies a_0(m^ne^{mx}) + a_1(m^{n-1}e^{mx}) + \cdots + a_{n-1}(me^{mx}) + a_ne^{mx} = 0$