1. Error Analysis

Definition:

let x is a value, $\tilde{x}$ is a estimated value

(1) absolute error, $E_a = |x - \tilde{x}|$

(2) relation error, $E_r = |\frac{x - \tilde{x}}{x}|$

(3) percentage error, $E_p = 100 \times |\frac{x - \tilde{x}}{x}|$

$\exists \epsilon > 0, |x - \tilde{x}| < \epsilon$, Then ϵ is upper limit of the absolute error measures the absolute accuracy.

1.1. Error in Implementation of Numerical Methods.

(1) Round-off Error

(2) Overflow & Underflow

(3) Floating Point Arithmetic and Error Propagation

(4) Truncation Error

(5) Machine eps (Epsilon)

(3)Floating Point Arithmetic and Error Propagation.

Let x_1, x_2 are values, E_1, E_2 are error of x_1, x_2,We want to check the change of error in $" + "," - "," * ","/"$

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Let $x = x_1 + x_2$, error of x is E
Then $x + E = x_1 + x_2 + E_1 + E_2 \implies E = E_1 + E_2$
by triangle inequality
Absolute Error $= |E| \leq |E_1| + |E_2|$
Relative Error $= \frac{|E|}{|x|} \leq \frac{|E_1|}{|x|} + \frac{|E_2|}{|x|}$

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Let $x = x_1 * x_2$
Then $x + E = (x_1 + E_1)(x_2 + E_2) = x_1x_2 + E_2x_1 + E_1x_2 + E_1E_2$
Absolute Error $= |E| \leq |x_2E_1| + |x_1E_2|$
Relative Error $= \dfrac{|E_1|}{|x|} \leq \dfrac{|E_1|}{|x_1|} + \dfrac{|E_2|}{|x_2|}$

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Let $x = x_1/x_2$
$x + E_x = \dfrac{x_1 + E_1}{x_2 + E_2}\left(\dfrac{x_2 - E_2}{x_2 - E_2}\right) = \dfrac{x_1x_2 + E_1x_2 - x_1E_2}{x_2^2 - E_2^2} + E_1E_2$
Absolute Error $= |E_x| = |\dfrac{E_1x_2 - x_1E_2}{x_2^2}| \leq \dfrac{|E_1|}{|x_2|} + \dfrac{|x_1E_2|}{x_2^2}$
Relative Error $= \dfrac{|E_x|}{|x|} \leq \dfrac{|E_1|}{|x_1|} + \dfrac{|E_2|}{|x_2|}$

(4)Truncation Error. Cause by approximation infinite with its finite terms.
Use Taylor series ($f(x) \in P(C)$) as example
Let $x = a, f(x) = f(a) + f'(a)(x-a) + f''(a)\dfrac{(x-a)^2}{2!} + \cdots + \dfrac{(x-a)^n}{n!}f^n(a) + \cdots + Rn$
$Rn = \displaystyle\int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt$

Thm 1(First Mean Value Theorem)

If g is continuous on $[a, x]$, then $\exists\ \xi$ between a and x s.t.

$$\int_a^x g(t)\ dt = g(\xi)(x-a)$$

Thm 2(Second Mean Value Theorem)

If g, h is differentiable and integrable on $[a, x]$, h does not change sign on $[a, x]$ then $\exists\ \xi$ that $a \leq \xi \leq x$ s.t.

$$\int_a^x g(t)h(t)\ dt = g(\xi)\int_a^x h(t)\ dt$$

since $t \in [a, x], h(t) = (x-t)^n\dfrac{1}{n!},\ f^{(n+1)}(t)$ is continuous
$\exists\ \xi \in [a, x],\ R_n = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}f^{(x+1)}(\xi),\ \xi \in [a, a+h]$
(Ref. Violin page:799)
since power series convergent, $R_n(x) \to 0,\ as_n \to \infty$

Definition

Given $\{\, a_n \,\}\{\, b_n \,\}$, $b_n \geq 0,\ \forall n \geq 1$
$a_n = O(b_n)$ if $\exists\ M > 0 \to |a_w| \leq M b_n\ \forall\ n \geq 1$
$R_n(x) = O(h^{n+1})$

1.2. Condition & Stability.

Condition number is sensitivit of the function
Stability is used to describe the sensitity of the process
Condition number of the $f(n)$

$$\text{CN} = \frac{|\dfrac{f(x) - f(\tilde{x})}{x - \tilde{x}}|}{|\dfrac{x - \tilde{x}}{x}|} = |\frac{f(x) - f(\tilde{x})}{x - \tilde{x}}| \cdot |\frac{x}{f(x)}| = |\frac{x}{f(x)} \cdot f'(x)|$$

by Mean Value Theorem,

$$\frac{f(x) - f(\tilde{x})}{x - \tilde{x}} \approx f'(x)$$

when CN ≤ 1 is **well condition**, other is **ill condition**
when the function is more sensitive to change, the condition number will be more big.

2. Methods for solutions of the Equation $f(x) = 0$

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