Advanced Calculus

Exercise 1 (Chapter 1 and 2)

1. (a) If $r, s \in \mathbb{Q}$ then $r + s$ and rs are rational.
   (b) If $r \in \mathbb{Q}$ with $r \neq 0$ and $x \in \mathbb{R} \setminus \mathbb{Q}$, then $r + x$ and rx are irrational.
2. (a) Show that $\mathbb{N}$ is unbounded above.
   (b) Show that for any real number x, there exists a positive integer n such that $n > x$.
   (c) Using (b) to prove the following **Archimedean property**:
   If $x > 0$ and $y \in \mathbb{R}$, then there exists a positive integer n such that $nx > y$.
   (d) Using (c) to prove the denseness of $\mathbb{Q}$ in $\mathbb{R}$:
   Let $a < b \in \mathbb{R}$ be distinct real numbers, then there exists a rational number $q \in \mathbb{Q}$ such that $a < q < b$.
3. Let A, B be two nonempty sets of $\mathbb{R}$.
   (a) Show that $\inf A \leq \sup A$.
   (b) Show that $\inf(-A) = -\sup A$ and $\sup(-A) = -\inf(A)$, where $-A = \{-a \mid a \in A\}$.
   (c) If A, B be two sets of positive numbers which is bounded above.
   Let $a = \sup A$, $b = \sup B$ and $C = \{ab \mid a \in A, b \in B\}$. Prove that $\sup C = ab$.
4. Prove or disprove the following statement by given a counterexample:
   (a) $\sup(A \cap B) \leq \inf\{\sup A, \sup B\}$.
   (b) $\sup(A \cap B) = \inf\{\sup A, \sup B\}$.
   (c) $\sup(A \cap B) \geq \sup\{\sup A, \sup B\}$.
   (d) $\sup(A \cap B) = \sup\{\sup A, \sup B\}$.
5. Let $A, B \subseteq \mathbb{R}$ such that $\sup A = \sup B$ and $\inf A = \inf B$. Does $A = B$?
6. Prove the following three important inequalities:
   (a) **(Young)** Let $a, b \geq 0$ and $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then
   $$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$
   (b) **(Hölder)** Let $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$, and $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then
   $$\sum_{j=1}^{n} |x_j y_j| \leq \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} \left(\sum_{j=1}^{n} |y_j|^q\right)^{1/q}$$
   (c) **(Minkowski)** Let $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$, and $p \geq 1$. Then
   $$\left(\sum_{j=1}^{n} |x_j + y_j|^p\right)^{1/p} \leq \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{1/p}$$
7. Prove the following statements of **De Morgan's Laws**: Let $A_1, A_2, \ldots, A_n$ be a collection of sets. Then
   (a) $\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c$ (b) $\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c$
8. Let $f : X \to Y$ be a function. If $B \subseteq Y$, we denote by $f^{-1}(B)$ the largest subset of X which f maps into B. That is,
   $$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$
   The set $f^{-1}(B)$ is called the **inverse image** of B under f. Prove the following for arbitrary $A, A_1, A_2 \subseteq X$ and $B, B_1, B_2 \subseteq Y$.

(a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

(b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. Given an example such that the inclusion is strict.

(c) $A \subseteq f^{-1}[f(A)]$ and $f[f^{-1}(B)] \subseteq B$. Given an example such that the inclusion is strict.

(d) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$ and $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

9. Let S be a relation and let $\mathscr{D}(S)$ be its domain. The relation S is said to be

   i) *reflexive* if $a \in \mathscr{D}(S)$ implies $(a, a) \in S$

   ii) *symmetric* if $(a, b) \in S$ implies $(b, a) \in S$,

   iii) *transitive* if $(a, b) \in S$ and $(b, c) \in S$ implies $(a, c) \in S$.

   A relation which is symmetric, reflexive, and transitive is called an equivalence relation.

   Determine which of these properties is possessed by S, if S is the set of all pairs of real numbers (x, y) such that

   a) $x \leq y$,
   b) $x < y$,
   c) $x < |y|$,
   d) $x^2 + y^2 = 1$,
   e) $x^2 + y^2 < 0$,
   f) $x^2 + x = y^2 + y$.

10. Let S be the collection of all sequences whose terms are the integers 0 and 1. Show that S is uncountable.

11. Let S denote the collection of all subsets of a given set T. Let $f : S \to \mathbb{R}$ be a real-valued function defined on S. The function f is called *additive* if $f(A \cup B) = f(A) + f(B)$ whenever A and B are disjoint subsets of T. If f is additive, prove that for any two subsets A and B we have:

    (a) $f(A \cup B) = f(A) + f(B - A)$

    (b) $f(A \cup B) = f(A) + f(B) - f(A \cap B)$