高等微積分

Exercise 1 (chapter 1 to 3)

1. (In Apostol Thm 1.1)
   Given two real number a and b such that $a \leq b + \epsilon$ for any $\epsilon > 0$, then $a \leq b$.
2. (a) If $r, s \in \mathbb{Q}$ then $r + s$ and rs are rational.
   (b) If $r \in \mathbb{Q}$ with $r \neq 0$ and $x \in \mathbb{R} \setminus \mathbb{Q}$, then $r + x$ and rx are irrational.
3. Let $f : X \to Y$ be a function. If $B \subseteq Y$, we denote by $f^{-1}(B)$ the largest subset of X which f maps into B. That is,
   $$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$
   The set $f^{-1}(B)$ is called the **inverse image** of B under f. Prove the following for arbitrary $A, A_1, A_2 \subseteq X$ and $B, B_1, B_2 \subseteq Y$.
   (a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
   (b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. Given an example such that the inclusion is strict.
   (c) $A \subseteq f^{-1}[f(A)]$ and $f[f^{-1}(B)] \subseteq B$. Given an example such that the inclusion is strict.
   (d) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$ and $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.
4. (a) Show that $\mathbb{N}$ is unbounded above.
   (b) Show that for any real number x, there exists a positive integer n such that $n > x$.
   (c) Using (b) to prove the following **Archimedean property**:
   If $x > 0$ and $y \in \mathbb{R}$, then there exists a positive integer n such that $nx > y$.
   (d) Using (c) to prove the denseness of $\mathbb{Q}$ in $\mathbb{R}$:
   Let $a < b \in \mathbb{R}$ be distinct real numbers, then there exists a rational number $q \in \mathbb{Q}$ such that $a < q < b$.
5. Let A, B be two nonempty sets of $\mathbb{R}$.
   (a) If $A \subseteq B$,then $\sup A \leq \sup B$ and $\inf A \geq \inf B$.
   (b) How to define $\sup \phi$ and $\inf \phi$ in $\mathbb{R}$?
   (c) Show that $\inf A \leq \sup A$.
   (d) Show that $\inf(-A) = -\sup A$ and $\sup(-A) = -\inf(A)$, where $-A = \{-a \ |a \in A\}$.
   (e) Show that $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$,
   where $A + B = \{a + b \ |a \in A, b \in B\}$, the **Minkowski sum** of A and B.
   (f) If A, B be two sets of positive numbers which is bounded above.
   Let $a = \sup A$, $b = \sup B$ and $C = \{ab \ |a \in A, b \in B\}$. Prove that $\sup C = ab$.
6. Prove or disprove the following statement by given a counterexample:
   (a) $\sup(A \cap B) \leq \inf\{\sup A, \sup B\}$.
   (b) $\sup(A \cap B) = \inf\{\sup A, \sup B\}$.
   (c) $\sup(A \cap B) \geq \sup\{\sup A, \sup B\}$.
   (d) $\sup(A \cap B) = \sup\{\sup A, \sup B\}$.
7. Let $A, B \subseteq \mathbb{R}$ such that $\sup A = \sup B$ and $\inf A = \inf B$. Does $A = B$?