§ Linear Transformations and Matrices

2-1 Linear Transformations, Null spaces, and Ranges.

13 Let V and W be vector spaces, let T: V $\to$ W be linear, and let $\{ w_1, \cdots, w_k \}$ be a linearly independent subset of R(T). Prove that if $S = \{ v_1, \cdots, v_k \}$ is chosen so that $\mathrm{T}(v_i) = w_i$ for $i = 1, 2, \cdots, k$, then S is linearly independent.

Solution. **Claim.** S is linearly independent
there exist $a_1, \cdots, a_k \in \mathrm{F} \implies$

$$\sum_{i=1}^{k} a_i v_i = 0 \implies \mathrm{T}(\sum_{i=1}^{k} a_i v_i) = 0 \implies \sum_{i=1}^{k} a_i \mathrm{T}(v_i) = 0 \implies \sum_{i=1}^{k} a_i w_i = 0$$

$\because \{ w_1, \cdots, w_k \}$ is linearly independent $\therefore \sum_{i=1}^{k} a_i v_i = 0$ only $a_i = 0, i = 1, \cdots, k$

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14 Let V and W be vector spaces and T:V $\rightarrow$ W be linear.

(a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

(b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if T(S) is linearly independent.

(c) Suppose $\beta = \{ v_1, v_2, \cdots, v_n \}$ is a basis for V and T is one-to-one and onto. Prove that T(β) = {T(v_1),T(v_2),$\cdots$,T(v_n)} is a basis for W.

Solution. (a) ($\Rightarrow$) let $\{ s_1, \cdots, s_n \}$ be a linearly independent subset of S

Claim. $\{ T(s_1), \cdots, T(s_n) \}$ is a linearly independent subset of W

$$\sum_{i=1}^{n} a_i T(s_i) = 0 \implies \sum_{i=1}^{n} T(a_i s_i) = 0 \implies T(\sum_{i=1}^{n} a_i s_i) = 0$$

$\because$ T is one-to-one $\therefore \sum_{i=1}^{n} a_i s_i = 0$ only scalars are 0

$\{ T(s_1), \cdots, T(s_n) \}$ is a linearly independent subset of W

($\Leftarrow$) let $x, y \in$V, β is a basis of V, $\beta = \{ v_1, v_2, \cdots, v_n \}$

$x = a_1 v_1 + \cdots + a_n v_n, y = b_1 v_1 + \cdots + b_n v_n$

T(x) = T(y) $\implies$ T(x)-T(y) = 0 $\implies$ T($x - y$) = 0 $\implies$ T($(a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n$) = 0

$\implies (a_1 - b_1)T(v_1) + \cdots + (a_n - b_n)T(v_n) = 0 \because \{ T(v_1), \cdots, T(v_n) \}$ is linearly independent

$\therefore (a_1 - b_1)T(v_1) + \cdots + (a_n - b_n)T(v_n) = 0$ only $a_1 - b_1 = \cdots = a_n - b_n = 0$

$\implies a_1 = b_1, \cdots, a_n = b_n \implies T(x) = T(y)$ only $x = y$

T is one-to-one

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(b) let S = $\{ s_1, s_2, \cdots, s_n \}$

($\Rightarrow$) **Claim.** $\{ T(s_1), \cdots, T(s_n) \}$ is linearly independent

$a_1T(s_1) + \cdots + a_nT(s_n) = 0 \implies T(a_1s_1) + \cdots + T(a_ns_n) = 0 \implies T(a_1s_1 + \cdots + a_ns_n) = 0$

$\because$ S is linearly independent, T is one-to-one $\therefore T(a_1s_1 + \cdots + a_ns_n) = 0$ only scalars are 0

$\{ T(s_1), \cdots, T(s_n) \}$ is linearly independent

($\Leftarrow$) **Claim.** $\{ s_1, \cdots, s_n \}$ is linearly independent

$a_1s_1 + \cdots + a_ns_n = 0 \implies T(a_1s_1 + \cdots + a_ns_n = 0) \implies a_1T(s_1) + \cdots + a_nT(s_n) = 0$

$\because \{ T(s_1), \cdots, T(s_n) \}$ is linearly independent, $\therefore a_1 = \cdots = a_n = 0$

$\{ s_1, \cdots, s_n \}$ is linearly independent

(c) **Claim.** T(β) can generate W

$\because$ T is onto, by Thm 2.2, T : V $\to$ W be linear, β is a basis of V, then R(T) = span(T(β)) = W

T(β) can generate W

Claim T(β) is a linearly independent set of W

by (b), T is one-to-one,S is a linearly independent subset of V, T(S) is linearly independent

T(S) is a basis of W

17 Let V and W be finite-dimensional vector spaces and T:V $\to$ W be linear.

(a) Prove that if dim(V) ¡ dim(W), then T cannot be onto.

(b) Prove that if dim(V) ¿ dim(W), then T cannot be one-to-one.

Solution. let dim(V)= m,dim(W)=n, β_v is a basis of V, β_w is a basis of W, $\beta_v = \{\, v_1, \cdots, v_m \,\}$, $\beta_w = \{\, w_1, \cdots, w_n \,\}$

(a) by dimension theorem

$\because$ rank(T) $\leq$ dim(V) ¡ dim(W) $\therefore$ R(T) $\neq$ W

(b) **Claim.** T is one-to-one

by Theorem 2.4, T is one-to-one then N(T)= $\{\, 0 \,\}$, nullity(T) = 0

$\therefore$ R(T) is a subspace of W

dim(V) = rank(T) + nullity(T) > dim(W) $\Longrightarrow$ rank(T) > dim(W)$\rightarrow\leftarrow$

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21 Let V be the vector space of sequences described in Example 5 of Section 1.2. Define the functions T,U:V $\to$ V by

$$\mathrm{T}(a_1, a_2, \cdots) = (a_2, a_3, \cdots) \text{ and } \mathrm{U}(a_1, a_2, \cdots) = (0, a_1, a_2, \cdots).$$

T and U are called the **left shift** and **right shift** operators on V, respectively.

(a) Prove that T and U are linear.

(b) Prove that T is onto, but not one-to-one.

(c) Prove that U is one-to-one, but not onto.

Solution. (a) let $v_1, v_2 \in$V $c \in$ F, $v_1 = (a_1, a_2, \cdots), v_2 = (b_1, b_2, \cdots)$

Prove T is linear

$\mathrm{T}(cv_1 + v_2) = \mathrm{T}((ca_1, ca_2, \cdots) + (b_1, b_2, \cdots)) = \mathrm{T}(ca_1 + b_1, ca_2 + b_2, \cdots) = (ca_2 + b_2, ca_3 + b_3, \cdots) = c(a_2, a_3, \cdots) + (b_2, b_3, \cdots) = c\mathrm{T}(v_1) + \mathrm{T}(v_2)$

$\mathrm{T}((0, 0, \cdots)) = (0, 0, \cdots)$

$\therefore$ T is a linear function

Prove U is linear

$\mathrm{U}(cv_1 + v_2) = \mathrm{U}((ca_1, ca_2, \cdots)+(b_1, b_2, \cdots)) = \mathrm{U}(ca_1 + b_1, \cdots) = (0, ca_1 + b_1, \cdots) = c(0, a_1, \cdots) + (0, b_1, \cdots) = c\mathrm{U}(v_1) + \mathrm{U}(v_2)$

$\mathrm{U}(0, 0, \cdots) = (0, 0, \cdots)$

$\therefore$ U is a linear function

(b) let $v_1 = (a_1, a_2, \cdots) \in V, a_1, a_2, \cdots \in F$

$\exists(a_2, \cdots) = T(a_2, a_3, \cdots) \in V$ $\therefore$ T is onto

$\mathrm{T}(1, 0, \cdots) = \mathrm{T}(2, 0, \cdots) = (0, 0, \cdots)$

T is not one-to-one

(c) $\mathrm{U}(a_1, a_2, \cdots) = (0, 0, \cdots)$ only $a_1 = a_2 = \cdots = 0 \implies N(U) = \{0\}$

by Theorem 2.4, U is linear, if N(V) = $\{0\}$, then U is one-to-one

$(1, 2, 3, \cdots) \in V \notin R(U)$, U is not onto

$\blacksquare$

26 Using the notation in the definition above, assume that T: V $\rightarrow$ V is the projection on W_1 along W_2.

(a) Prove that T is linear and $W_1 = \{ x \in V \mid T(x) = x \}$.

(b) Prove that $W_1 = R(T)$ and $W_2 = N(T)$.

Solution. (a) let $x, y \in V, x = w_1 + w_2, y = w_1' + w_2', w_1, w_1' \in W_1, w_2, w_2' \in W_2, c \in F$

Claim. T is linear

$T(cx + y) = T((cw_1 + w_1') + (cw_2 + w_2')) = cw_1 + w_1' = cT(x)+T(y)$

$T(0) = T(0 + 0) = 0$ T is linear

(b) let $w_1 \in W_1, w_2 \in W_2$

$R(T) = \{ T(x) \mid x \in V \} = \{ T(w_1 + w_2) | w_1 \in W_1, w_2 \in W_2 \} = \{ T(w_1) + T(w_2) \mid w_1 \in W_1, w_2 \in W_2 \} = \{ T(w_1) \mid w_1 \in W_1 \} = W_1$

let $x \in N(T) \implies T(x) = 0 \implies x \in W_2 \implies N(T) \subseteq W_2$

let $x \in W_2 \implies T(x) = 0 \implies x \in N(T) \implies W_2 \subseteq N(T)$

$\therefore N(T) = W_2$

$\because T(w_1) = 0$ only $w_1 = 0, T(w_2) = 0$ when $w_2 \in W_2$

$\therefore N(T) = \{ w_2 \mid w_2 \in W_2 \} = W_2$

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35 Let V be a finite-dimensional vector space and T: V $\to$ V be linear.

(a) Suppose that V = R(T) + N(T). Prove that V = R(T) $\oplus$ N(T).

(b) Suppose that R(T) $\cap$ N(T) = $\{0\}$. Prove that V = R(T) $\oplus$ N(T).

Solution. (a) **Claim** R(T) $\cap$ N(T) = $\{0\}$

by Exercise 1.6.29 and dimension Theorem, W_1, W_2 is a subspace of V, dim($W_1 + W_2$) = dim(W_1)+dim(W_2)-dim($W_1 \cap W_2$)

dim(R(T) $\cap$ N(T)) = dim(R(T)) + dim(N(T)) - dim(V) = 0

$\because$ R(T),N(T) is a subspace, $\therefore$ R(T) $\cap$ N(T) = $\{0\}$

$\because$ R(T), N(T) is a subspace of V, R(T) $\cap$ N(T) = 0

$\therefore$ R(T) $\oplus$ N(T) = V

(b) by dimension theorem and Exercise 1.6.29

W_1, W_2 is a subspace of V, dim($W_1 + W_2$) = dim(W_1)+dim(W_2)-dim($W_1 \cap W_2$)

dim(R(T)+N(T)) = dim(R(T)) + dim(N(T)) - dim(R(T) $\cap$ N(T)) = dim(R(T)) + dim(N(T)) = dim(V)

$\because$ R(T) + N(T) is a subspace of V, dim(R(T)+N(T)) = dim(V)

$\therefore$ R(T)+N(T) = V

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2-2 The Matrix Representation of a Linear Transformation.

11 Let V be an n-dimensional vector space, and let $T : V \to V$ be a linear transformation. Suppose that W is a T-invariant subspace of V having dimension k. Show that there is a basis β for V such that $[T]_\beta$ has the form

$$\begin{pmatrix} A & B \\ O & C, \end{pmatrix}$$

where A is a $k \times k$ matrix and O is the $(n-k) \times k$ zero matrix.

let β_w is a basis of W, $\beta_w = \{ v_1, v_2, \cdots, v_k \}, v_1, \cdots, v_k \in V$
we can extended β_w to β which is a basis of V, $\beta = \{ v_1, v_2, \cdots, v_n \}, v_{k+1}, \cdots, v_n \in V$
let $x \in V, \exists$ unique $a_ij \in F(1 \leq i, j \leq n), T(v_j) = \sum_{i=1}^{n} a_{ij} v_i, 1 \leq j \leq n$
$\because W$ is a T-invariant subspace of $V \implies T(w) \in W, w \in W$
$\therefore T(v_j) = \sum_{i=1}^{n} a_{ij} v_i (i \leq j \leq k) = \sum_{i=1}^{k} a_{ij} v_i \implies ([T]_\beta)_{ij} = 0 (k+1 \leq j \leq n)$

12 Let V be a finite-dimensional vector space and T be the projection on W along W', where W and W' are subspaces of V. Find and ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

let $\beta_w = \{ w_1, w_2, \cdots, w_n \}, \beta_{w'} = \{ w'_1, w'_2, \cdots, w'_m \}$, by Exercise 1.6.33, W_1 and W_2 is a subspace of V, $W_1 \oplus W_2 = V$, β_1, β_2 is a basis of W_1, W_2, then $\beta_1 \cup \beta_2$ is a basis of $V \implies \beta_w \cup \beta_{w'}$ is a basis of V

$$
\begin{array}{rcl}
T(w_1) & = & 1 \cdot w_1 + 0 \cdot w_2 + \cdots + 0 \cdot w_n + 0 \cdot w'_1 + \cdots + 0 \cdot w'_m \\
T(w_2) & = & 0 \cdot w_1 + 1 \cdot w_2 + \cdots + 0 \cdot w_n + 0 \cdot w'_1 + \cdots + 0 \cdot w'_m \\
\vdots & & \\
T(w_n) & = & 0 \cdot w_1 + 0 \cdot w_2 + \cdots + 1 \cdot w_n + 0 \cdot w'_1 + \cdots + 0 \cdot w'_m \\
T(w'_1) & = & 0 \cdot w_1 + 0 \cdot w_2 + \cdots + 0 \cdot w_n + 0 \cdot w'_1 + \cdots + 0 \cdot w'_m \\
\cdots & & \\
T(w'_1) & = & 0 \cdot w_1 + 0 \cdot w_2 + \cdots + 0 \cdot w_n + 0 \cdot w'_1 + \cdots + 0 \cdot w'_m
\end{array}
$$

$\implies \left([T]_{\beta_w \cup \beta_{w'}}\right)_{ij} = 0$ when $i \neq j$, $\therefore \beta_w \cup \beta_{w'}$ is a basis of V
$[T]_{\beta_w \cup \beta_{w'}}$ is a diagonal matrix

13 Let V and W be vector spaces, and let T and U be nonzero linear transformations from V into W. If $R(T) \cap R(U) = \{ 0 \}$, prove that $\{ T, U \}$ is a linearly independent subset of $L(V, W)$.

Claim $\{T, U\}$ is linearly independent, let $x \in V$

$$\begin{aligned} a_1T(x) + a_2U(x) &= 0 \ (a_1, a_2 \in \mathrm{F}) \\ a_1T(x) &= -a_2U(x) \in R(T) \cap R(u) \\ \implies a_1T(x) &= -a_2U(x) = 0 \\ \implies a_1T(x) + a_2U(x) &= 0, \text{ only } a_1 = a_2 = 0 \end{aligned}$$

$\therefore \{T, U\}$ is linearly independent.

14 Let $V = P(R)$, and for $j \geq 1$ define $T_j(f(x)) = f^{(j)}(x)$, where $f^{(j)}(x)$ is the jth derivation of $f(x)$. Prove that the set $\{T_1, T_2, \cdots, T_n\}$ is a linearly independent subset of $L(V)$ for any positive integer n.

let $c_1, c_2, \cdots, c_n \in F, f(x) \in P(R)$
$f(x) = a_0 + a_1x + \cdots + a_nx^n + \cdots, \ a_0, a_1, \cdots \in \mathrm{F}$
$c_1T_1(f(x)) + c_2T_2(f(x)) + \cdots + c_nT_n(f(x)) = 0$
$\Rightarrow c_1(1 \cdot a_1 + \cdots + n \cdot a_nx^{n-1} + \cdots) + \cdots + c_n(n!a_n + \dfrac{(n+1)!}{(n+1-n)!}a_{n+1}x + \cdots) = 0$
$\Rightarrow (c_1a_1 + \cdots + n!c_na_n) + \cdots + (c_1\dfrac{(n+1)!}{n!} + c_2\dfrac{(n+2)!}{n!} + \cdots)x^n + \cdots = 0$
only $c_1 = c_2 = \cdots = c_n = 0 \quad \therefore \{T_1, T_2, \cdots, T_n\}$ is linearly independent

16 Let V and W be vector spaces such that $\dim(V) = \dim(W)$, and let $T : V \to W$ be linear. Show that there exist ordered bases β and γ for V and W, respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix

Suppose $\dim(V) = n$, Claim nullity$(T) = k$
let $\{v_1, v_2, \cdots, v_k\}$ is a basis of $N(T)$
$\because N(T)$ is a subspace of V, we can extend $\{v_1, v_2, \cdots, v_k\}$ to $\{v_1, v_2, \cdots, v_n\}$ be a basis of V
by dimension theorem, $\dim(V) = \text{nullity}(T) + \text{rank}(T) \implies \mathrm{R}(T) = n - k$
pick $\{u_{k+1}, \cdots, u_n\}$ where $u_{k+1} = T(v_{k+1}), \cdots, u_n = T(v_n)$ be a basis of $R(T)$
$[T]_\beta^\gamma = \begin{pmatrix} 0 & 0 \\ 0 & I_{(n-k)} \end{pmatrix}$

2-3 Composition of Linear Transformations and Matrix Multiplication.

11 Let V be a vector space, and let $T : V \to V$ be linear. Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$.

($\Rightarrow$) let $v \in V \implies T(T(v)) = 0 \in N(T) \implies \{\, T(v) \mid v \in V \,\} \subseteq N(T)$
($\Leftarrow$)$R(T) \subseteq N(T)$, let $v \in V,\ T(T(v)) = T(v')\ v \in N(T) = 0$

12 Let V, W, and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear.

(1) Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?
(2) Prove that if UT is onto, then U is onto. Must T also be onto?
(3) Prove that if U and T are one-to-one and onto, then UT is also

(a) let $v_1, v_2 \in V$,
$UT(v_1) = UT(v_2)$ only $v_1 = v_2 \implies U(T(v_1)) = U(T(v_2))$ only $v_1 = v_2$
Claim T is not one-to-one
$\exists v_1, v_2 \in V, T(v_1) = T(v_2)$ but $v_1 \neq v_2$
$U(T(v_1)) = U(T(v_2))$ when $v_1 \neq v_2 \rightarrow\leftarrow$

(b) let $v_1 \in V$, $R(UT) = Z$, Claim U is not onto $\implies \exists z \in Z, z \notin R(U)$
$R(UT) = \{\, UT(v) \mid v \in V \,\} = \{\, U(w) \mid w \in R(T) \,\}$
$\subseteq \{\, U(w) \mid w \in W \,\}\, (R(T) \subseteq W) = R(U)$
$\implies z \notin R(UT) \implies R(UT) \neq Z (\rightarrow\leftarrow)$

(c) Claim UT is one-to-one, let $v_1, v_2 \in V$, pick $w_1, w_2 \in W,\ z_1, z_2 \in Z$
$T(v_1) = w_1,\ T(v_2) = w_2,\ U(w_1) = z_1,\ U(w_2) = z_2$
$UT(v_1) = UT(v_2) \Rightarrow U(T(v_1)) = U(T(v_2)) \Rightarrow U(w_1) = U(w_2) \Rightarrow z_1 = z_2$
$\because U, T$ is one-to-one, $z_1 = z_2$ only $w_1 = w_2,\ w_1 = w_2$ only $v_1 = v_2$
$\therefore UT$ is one-to-one
Claim UT is onto, let $z \in Z$
$\because U, T$ is onto, $R(U) = Z,\ R(T) = W$
$R(UT) = \{\, UT(v) \mid v \in V \,\} = \{\, U(T(v)) \mid v \in V \,\} = \{\, U(w) \mid w \in W \,\}$
$= R(U) = Z$

16 Let V be a finite-dimensional vector space, and let $T : V \to V$ be linear.

(a) If $\operatorname{rank}(T) = \operatorname{rank}(T^2)$, prove that $R(T) \cap N(T) = \{\, 0 \,\}$. Deduce that $V = R(T) \oplus N(T)$
(b) Prove that $V = R(T^k) \oplus N(T^k)$ for some positive integer k.

(a) $R(T) = \{ T(x) \mid x \in V \}$ $R(T^2) = \{ T(x) \mid x \in R(T) \} \implies R(T^2) \subseteq R(T)$

let $\beta = \{ v_1, v_2, \cdots, v_k \}$ be a basis of $R(T^2)$

$\because \operatorname{rank}(T^2) = \operatorname{rank}(T) \therefore \beta$ also be a basis of $R(T) \implies R(T^2) = R(T)$

Suppose $x \in R(T) \cap N(T)$ but $x \neq 0 \in \mathrm{F}$ i.e. $\exists\, y \in V, T(y) = x$

by Dimension Theorem, $\operatorname{nullity}(T^2) = \dim(V) - \operatorname{rank}(T^2) = \operatorname{nullity}(T)$

$\because N(T) \subseteq N(T^2), \operatorname{nullity}(T) = \operatorname{nullity}(T^2)$ $\therefore N(T) = N(T^2)$

$\because x \in N(T) \therefore T(T(y)) = T(x) = 0 \implies y \in N(T) \implies x = 0 (\rightarrow\leftarrow)$

by Exercise 2.1.35,$R(T) \cap N(T) = \{ 0 \} \implies R(T) \oplus N(T) = V$

(b) not today

17 Let V be a vector space. Determine all linear transformations $T : V \to V$ such that $T = T^2$, and show that $V = \{ y \mid T(y) = y \} \oplus N(T)$.

let $W = \{ y \mid T(y) = y \}, x \in V$, Claim W is a subspace of V
$T(ay_1 + y_2) = aT(y_1) + T(y_2) = ay_1 + y_2 \implies ay_1 + y_2 \in W$,$T(0) = 0$
$\therefore W$ is a subspace of V
Claim $x - T(x) \in N(T)$
$T(x - T(x)) = T(x) - T(T(x)) = T(x) - T(x) = 0 \therefore x - T(x) \in N(T)$
Claim $R(T) \subseteq W(T)$, let $T(x) \in R(T), x \in V$
Case1. $T(x) = 0 \in W(T)$
Case2. $T(x) = y, y \in V,\ y \neq 0,\ T(x) = T^2(x) = y \implies T(y) = y \in W$
$\therefore R(T) \subseteq W(T)$
Claim $V \subseteq W + N(T),\ x = T(x) + (x - T(x)) \in (W + N(T))$
$\because W,\ N(T)$ is a subspace of V, $W + N(T) \in V$
$\therefore W + N(T) = V$
Claim $W \cap N(T) = \{ 0 \}$
let $x \in W \cap N(T),\ T(x) = x = 0$ only $x = 0$

2-4 Invertibility & Isomorphisms.

15 Let V and W be finite-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Suppose that β is a basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W.

($\Rightarrow$) T is onto $\implies R(T) = W = \text{span}(T(\beta))$, let $\beta = \{ v_1, v_2, \cdots, v_n \}$
Claim $T(\beta)$ is linearly independent $\Rightarrow a_1T(v_1)+\cdots+a_nT(v_n) = 0, a_1, \cdots, a_n \in \mathrm{F}$
$\implies T(a_1v_1 + \cdots + a_nv_n) = 0$
$\because T$ is one-to-one, $N(T) = \{ 0 \}$
$a_1T(v_1) + \cdots + a_nT(v_n) = 0$ only $a_1 = \cdots = a_n = 0$
$\therefore T(\beta)$ is a basis of W

($\Leftarrow$) let $\beta = \{ v_1, \cdots, v_n \},\ x \in V,\ x = a_1v_1 + \cdots + a_nv_n,\ a_1, \cdots, a_n \in \mathrm{F}$
$R(T) = \text{span}(T(\beta)) = W \implies T$ is onto
$T(x) = T(a_1v_1+\cdots+a_nv_n) = a_1T(v_1)+\cdots+a_nT(v_n) = 0$ only $a_1 = \cdots = a_n = 0$
$\implies T(x) = 0$ only $x = 0 \implies N(T) = \{ 0 \} \implies T$ is one-to-one
$\therefore T$ is an isomorphism

17 Let V and W be finite-dimensional vector spaces and $T : V \to W$ be an isomorphism. Let V_0 be a subspace of V.

(a) Prove that $T(V_0)$ is a subspace of W.

(b) Prove that $\dim(V_0) = \dim(T(V_0))$.

let $\beta_0 = \{ v_1, \cdots, v_k \}$ be a basis of V_0, i.e. $\text{span}(\beta_0) = V_0$

(a) $T(V_0) = \{ T(a_1v_1 + \cdots + a_kv_k) \mid a_1, \cdots, a_k \in \mathrm{F} \}$
$= \{ a_1T(v_1) + \cdots + a_kT(v_k) \mid a_1, \cdots, a_k \in \mathrm{F} \}$
let $v'_1, v'_2 \in T(V_0),\ c \in \mathrm{F}$
$v'_1 = a_1T(v_1) + \cdots + a_kT(v_k),\ a_1, \cdots, a_k \in \mathrm{F}$
$v'_2 = b_1T(v_1) + \cdots + b_kT(v_k),\ b_1, \cdots, b_k \in \mathrm{F}$
(1) $cv'1 + v'2 = ca_1T(v_1) + \cdots + ca_kT(v_k) + b_1T(v_1) + \cdots + b_kT(v)k)$
$= (ca_1 + b_1)T(v_1) + \cdots + (ca_k + b_k)T(v_k) \in T(V_0)$
(2) $0 \in V_0 \implies 0 \in T(V_0)$
$T(V_0)$ is a subspace of W

(b) Claim $T(\beta)$ is a basis of $T(V_0)$, $\text{span}(\beta) = V_0 \implies \text{span}(T(\beta_0)) = V_0$
Claim $T(\beta)$ is linearly independent
$a_1T(v_1) + \cdots + a_kT(v_k) = 0 \implies T(a_1v_1 + \cdots + a_kv_k) = 0$
$\because T$ is one-to-one $\implies T(a_1v_1 + \cdots + a_kv_k) = 0$ only $a_1v_1 + \cdots + a_kv_k = 0$
$\implies a_1 + \cdots + a_k = 0 \implies T(\beta)$ is linearly independent
$\dim(V_0) = \dim(T(V_0)) = k$

21 Let V and W be finite-dimensional vector spaces with ordered bases $\beta = \{ v_1, v_2, \cdots, v_n \}$ and $\gamma = \{ w_1, w_2, \cdots, w_m \}$, respectively. By Theorem 2.6, there exist linear transformations $T_{ij} : V \to W$ such that

$$T_{ij}(v_k) = \begin{cases} w_i \text{if} k = j \\ 0 \text{if} k \neq j. \end{cases}$$

First prove that $\{ T_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n \}$ is a basis for $L(V, W)$. Then let M^{ij} be the $m \times n$ matrix with 1 in the ith row and jth column and 0 elsewhere, and prove that $[T_{ij}]_\beta^\gamma = M^{ij}$. Again by Theorem 2.6, there exists a linear transformation $\Phi : L(V, W) \to M_{m\times n}(F)$ such that $\Phi(T_{ij}) = M^{ij}$. Prove that Φ is an isomorphism.

Claim $\{ T_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n \}$ is linearly independent, let $x \in V$

$$\sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}T_{ij}(x) = 0 \implies \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}T_{ij}(b_1v_1 + b_2v_2 + \cdots + b_nv_n) = 0$$

$$\implies \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}b_jw_i = 0 \implies \sum_{j=1}^{n} b_j \sum_{i=1}^{m} a_{ij}w_i = 0$$

$$\implies \sum_{j=1}^{n} b_j(a_{ij}w_1 + \cdots + a_{mj}w_m) = 0$$

$$\implies (\sum_{j=1}^{n} b_ja_{1j})w_1 + \cdots + (\sum_{j=1}^{n} b_ja_{mj})w_m = 0 \text{ only } a_{ij} = 0\ (1 \leq i \leq m,\ 1 \leq j \leq n)$$

$\therefore \{ T_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n \}$ is linearly independent

Claim $\text{span}(\{ T_{ij} \mid 1 \leq j \leq m, 1 \leq j \leq n \}) = L(V, M)$

$\text{span}(\{ T_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n \}) \subseteq L(V, W)$ **trivial**

$L(V, W) \subseteq \text{span}(\{ T_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n \})$

let $T \in L(V, W)$ $x \in V,\ x = a_1v_1 + \cdots + a_nv_n$

$$T(x) = a_1T(v_1) + \cdots + a_nT(v_n) = a_1(\sum_{i=1}^{m} b_{i1}T_{i1}(v_1)) + \cdots + a_n(\sum_{i=1}^{m} b_{in}T_{in}(v_n))$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij}T_{ij}(v_j) \implies T = \sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij}T_{ij} \in \text{span}(T_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n)$$

$\{ T_{ij \ \mid \ 1\leq i\leq m,\ 1\leq j\leq n} \}$ is a basis of $L(V, W)$

$$\text{let } x \in V,\ x = a_1v_1 + \cdots + a_nv_n, T_{ij}(x) = a_jw_i \implies \begin{cases} T_{ij}(v_1) = 0 \\ T_{ij}(v_2) = 0 \\ \vdots \\ T_{ij}(v_j) = 1 \times w_i \\ \vdots \\ T_{ij}(v_n) = 0 \end{cases}$$

$$\implies [T_{ij}]_\beta^\gamma = M^{ij}$$

Claim Φ is one-to-one, let $T \in L(V,W),\ T = \sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij}T_{ij}$

$\Phi(T) = 0 \implies \Phi(\sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij}T_{ij}) = 0 \implies \sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij}\Phi(T_{ij}) = 0$

$\implies \sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij}M_{ij} = 0$ only $c_{ij} = 0(1 \leq i \leq m,\ 1 \leq j \leq n) \implies N(T) = \{\, T_0 \,\} \implies \Phi$ is one-to-one

Claim $R(\Phi) = M_{m\times n}(\mathrm{F})$

by Dimension Theorem, V is finite vector space, T is linear transformation,

$\dim(V) = \mathrm{rank}(T) + \mathrm{nullity}(T) \implies \dim(L(V,W)) = \mathrm{rank}(\Phi)$

$\implies \mathrm{rank}(\Phi) = m \times n$

$\therefore R(\Phi)$ is a subspace of $M_{m\times n}(\mathrm{F})$, $\dim(R(\Phi)) = M_{m\times n}(\mathrm{F})$

$\implies R(\Phi) = M_{m\times n}((F)) \ \therefore \Phi$ is onto

$\because L(V,W)$ is a finite Vector Space, Φ is one-to-one & onto

$\implies L(V,W)$ is a isomorphism