

**0.0.1. In calculus.** iPad test

- (1) Extreme Value Theorem: Every continuous function  $f : [a, b] \rightarrow \mathbb{R}$  admit both max and min value  $\Rightarrow$  Compact set
- (2) Intermediate value Theorem: Given continuous function  $f : [a, b] \rightarrow \mathbb{R}$  for all  $f(a) \leq \lambda \leq f(b) \exists c \in [a, b] \ni f(c) = \lambda \Rightarrow$  connected set

How to prove a statement:  $HP$ , then  $Q, P \Rightarrow Q$

$$\left\{ \begin{array}{l} \text{Direct Proof} \\ \text{Indirect Proof} \left\{ \begin{array}{l} \text{contrapositive } \sim Q \Rightarrow \sim P \\ \text{by contradiction} \end{array} \right. \\ \text{Mathematical Induction} \end{array} \right.$$

## 1. Some preliminary

### 1.1. Set Theory.

Set is a collection which has two presentation

- (1) List  $\{a, b, c, \dots\}$
- (2)  $\{x \mid x \in \text{alphabet}\}$

We will assume that you are familiar with some basic set theory e.g. union, intersection, difference

### 1.2. The Number System.

$\mathbb{N} = \{1, 2, 3, \dots\}$  the set of all positive integers  $n$  natural numbers

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$  the set of all integers called the ring of integers

$\mathbb{Q} = \left\{ \frac{m}{n} : n, m \in \mathbb{Z}, n \neq 0 \right\}$  the set of all rational numbers

$\mathbb{R}$  the set all of real numbers on the real number field on real line

$\mathbb{C} = \{z = a + ib \mid a, b \in \mathbb{R}\}$  the set of all complex numbers or the complex number field on complex plane, where  $i = \sqrt{-1}$

and  $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

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### Remark.

- (1)  $x + 2 = 0$  no root in  $\mathbb{N}$   
 $3x - 5 = 0$  no root in  $\mathbb{Z}$   
 $x^2 + 1 = 0$  no root in  $\mathbb{R}$
- (2) One can construct  $\mathbb{Q}$  from  $\mathbb{Z}$  in algebraic way, called the fraction field of  $\mathbb{Z}$
- (3) One can construct  $\mathbb{R}$  from  $\mathbb{Q}$  in two ways:
  - Using Dedekind cut which is given in the appendix of Rudin p17-21
  - Using completion of matrix space
- (4) One can construct  $\mathbb{C}$  from in complex analysis

### Example.

- (1) Between any two rational numbers, there is another one

*Proof.* Let  $r, s \in \mathbb{Q}$  with  $r < s$ , then  $\frac{r+s}{2} \in \mathbb{Q}$  and  $r < \frac{r+s}{2} < s$

$$r = \frac{m_1}{n_1}, s = \frac{m_2}{n_2}, \frac{r+s}{2} = \frac{\frac{m_1}{n_1} + \frac{m_2}{n_2}}{2} = \frac{m_1 n_2 + n_1 m_2}{2 n_1 n_2} \in \mathbb{Q}$$
$$s = \frac{s+s}{2} > \frac{r+s}{2} > \frac{r+r}{2} = r$$

■

- (2)  $x^2 = \frac{4}{9}$  has exactly two rational solutions, namely,  $\pm \frac{2}{3}$
- (3)  $x^2 = 2$  has exactly two real root, namely,  $\pm \sqrt{2}$
- (4) Is there any rational roots of  $x^2 = 2$ ? i.e., is  $\sqrt{2}$  rational?

Suppose  $r = \frac{m}{n} \in \mathbb{Q}$ , is a root of  $x^2 = 2$ , where  $(m, n) = 1$

Then  $\frac{m^2}{n^2} = 2 \implies m^2 = 2n^2 \implies 2 \mid m^2 \implies 2 \mid m \implies 4 \mid m^2 \implies 4 \mid 2n^2 \implies 2 \mid n^2 \implies 2 \mid n \implies (n, m) \neq 1$

- (5) Let  $A = \{r \in \mathbb{Q} \mid r > 0 \text{ \& } r^2 < 2\}$ ,  $B = \{r \in \mathbb{Q} \mid r > 0 \text{ \& } r^2 > 2\}$

Then  $A$  contains no largest numbers, i.e. max element &  $B$  contains no smallest numbers, i.e. min element

*Proof.*  $A$  contains no largest numbers  $\Leftrightarrow$  given  $r \in A$ ,

$\exists s \in A \ni s > r$

Now, given  $r \in A$ , Let  $s = r - \frac{r^2 - 2}{r + 2} = \frac{2r + 2}{r + 2}$  ( $\star_1$ )

$\Rightarrow s^2 - 2 = \frac{2(r^2 - 2)}{(r + 2)^2}$  ( $\star_2$ )

Now,  $r \in A, r^2 < 2 \Rightarrow r^2 - 2 < 0 \therefore$

( $\star_1$ )&( $\star_2$ )  $\Rightarrow s > r$  &  $s^2 < 2 \Rightarrow s \in A$  ■

(6) As you know, in calculus, the sequence  $\{1, 1.4, 1.41, 1.414, 1.4142, \dots\}$  does not converge in  $\mathbb{Q}$ , but it converges to  $\sqrt{2}$  in  $\mathbb{R}$

### 1.3. Order Sets.

**Definition** (Relation).

Let  $X$  be a nonempty set  $A$ , relation on  $X$  is a subset  $R$  of  $X \times X = \{(x, y) \mid x, y \in X\}$

Let  $R$  be a relation on  $X$ , if  $(x, y) \in R$ , then we say that  $x$  is related to  $y$ , and is written as  $xRy$  ( $x \sim y$ )

**Definition** (Order Set). An ordered set on  $S$ , is a relation denoted by " $<$ " on  $S$ , satisfy:

(i) The law of trichotomy

Given  $x, y \in S$ , one and only one of the following holds:

$x < y, x = y, y < x$

(ii) Transitivity: if  $x < y$  &  $y < z$ , then  $x < z$

#### **Notation**

(1)  $x < y$  means " $x$  is less than  $y$ " or " $x$  is smaller than  $y$ "

(2)  $y > x$  means  $x < y$

(3)  $x \leq y$  means  $x < y$  or  $x = y$ , i.e. the negative of  $x > y$

**Definition** (**bdd**). Let  $S$  is an ordered set &  $E \subseteq S$  ( $E \neq \emptyset$ )

- $E$  is bounded above if  $\exists \alpha \in S \Rightarrow x \leq \alpha \forall x \in E$   
such  $\alpha$  is called an upper bound of  $E$

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- *$E$  is bounded below if  $\exists \beta \in S \ni \beta \leq x, \forall x \in E$ , such  $\beta$  is called a lower bdd of  $E$*
- *$E$  is bdd is  $E$  is both bdd above and below.*

**Definition** (least upper bound). Let  $S$  be an ordered set and  $E \subseteq S$  ( $E \neq \emptyset$ ) bdd above. An element  $\alpha \in S$  is called the last upper bound or supremum of  $E$  if

- (i)  $\alpha$  is an upper bound of  $E$
- (ii)  $\alpha$  is the smallest such one.

Equivalently,

- (i')  $x \leq \alpha, \forall x \in E$
- (ii') if  $\beta < \alpha$ , then  $\beta$  is not an upper bdd of  $E$ , i.e.  $\exists x \in E \ni x > \beta$

Such  $\alpha$  (if exists) is denoted by

$$\alpha = \sup(E)$$

similarly, one can defined the greatest lower bdd of infimum of  $E$

**Remark.** if  $\sup(E)$  exists then it is unique

suppose  $\alpha \neq \alpha'$  both lub of  $E$

$\therefore$  by trichotomy,  $\alpha > \alpha'$  or  $\alpha = \alpha'$  or  $\alpha < \alpha'$  ( $\rightarrow \leftarrow$ )

**Definition** (least upper bdd property). A ordered set  $S$  is said to have the least upper bdd property if  $E \subseteq S$ ,  $E \neq \emptyset$  and  $E$  is bdd above, then  $\sup(E)$  exists in  $S$

**Example.**

- (1) In  $\mathbb{Q}$  with the normal ordining

$$A = \{r \in \mathbb{Q} \mid r > 0, r^2 < 2\} \text{ \& } B = \{r \in \mathbb{Q} \mid r > 0, r^2 > 2\}$$

Then  $A$  is bdd above, in fact, bdd by every element in  $B$ , but

$\sup(A)$  does not exist in  $\mathbb{Q}$  ( $\therefore$  by Ex1.5)

- (2)  $B$  is bdd below by every element of  $A$  and  $\inf B$  does not exists
- (3) Note that  $\sup(E) \& \inf(E)$  may not in  $E$  even if exist

**Remark.**

- (1) By the Example above,  $\mathbb{Q}$  with the usual ordering has no l.u.b property
- (2) In 1.5 we will explain that  $\mathbb{R}$  with usual ordering has the l.u.b. property. However, we usually adopt the follwing

**The Axiom of Completeness or Least upper bdd property:**

Every nonempty subset  $E$  of  $\mathbb{R}$  which is bdd above has l.u.b

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**Theorem** (l.u.b.p.  $\rightarrow$  g.l.b.p.). *Let  $S$  is an ordered set if  $S$  has the l.u.b. property, then  $S$  has the g.l.b. property, i.e. if  $\emptyset \neq B \subseteq S$  is bdd below, then  $\inf(B)$  exists in  $S$*

*Proof.* (★)

Given  $B(\neq \emptyset) \subseteq S$  which is bdd below

Let  $L = \{a \in S \mid a \text{ is a lower bdd of } B\}$

- $L \neq \emptyset$  ( $\because B$  is bdd below)
- $L$  is bdd above (in fact, every element in  $B$  is an upper bound of  $L$ )  
 $\implies \forall a \in L \implies a \leq x, \forall x \in B \implies x \text{ is an upper bound of } L$
- $\sup(L) = \alpha$  exists by assumption

**Claim**  $\alpha = \inf B$

- (i)  $\alpha$  is a lower bdd of  $B$ , i.e.  $\alpha \leq x, \forall x \in B$

By  $\alpha = \sup L$ , if  $r < \alpha$ , then  $r$  is not an upper bdd of  $L$  ( $\because \alpha$  is the smallest one). Hence,  $r \notin B$  ( $\because$  every element of  $B$  is an upper bdd of  $L$ ), so  $\alpha \leq x, \forall x \in B$

We have proved  $(r < \alpha \implies r \notin B) \implies (r \in B \implies r \geq \alpha)$

- (ii)  $\alpha$  is the greatest one

if  $\alpha < \beta$  and  $\beta$  is a lower bdd of  $B$ , then  $\beta \in L$ , i.e.  $\beta$  is not a lower bdd of  $B$ , so  $\alpha$  is the greatest one. Therefore,  $\alpha = \inf(B)$

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**Remark.** Let  $E(\neq \emptyset) \subseteq \mathbb{R}$  be bdd below, then  $\inf(E)$  exists and  $\inf(E) = -\sup(-E)$ , where  $-E = \{-x \mid x \in E\}$

### 1.4. Field.

Recall the addition & multiplication in  $\mathbb{R}$

$$+ : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}((a, b) \mapsto a + b)$$

$$\times : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}((a, b) \mapsto a \cdot b = ab)$$

**Definition.** Let  $X$  is a nonempty set, binary operation on  $X$  is a function,  $\circ : X \times X \rightarrow X$

**Definition.** Let  $F$  be a nonempty set, we say that  $F$  is a field  $((F, +, \cdot)$  is a field) if there are two binary operator called addition " $+$ " and multiplication " $\cdot$ " on  $F$  property

**Axioms for " $+$ "**

(A1) Commutative:  $\forall x, y \in F, x + y = y + x$

(A2) Associative:  $\forall x, y, z \in F, (x + y) + z = x + (y + z)$

(A3) Additive identity or zero element:  $\exists 0 \in F \implies x + 0 = 0 + x = x, \forall x \in F$

(A4) Additive inverse on negative: For each  $x \in X, \exists -x \in F \implies x + (-x) = (-x) + x = 0$

i.e.  $(F, +)$  is an abelian group **Axioms for multiplication**

(M1) Commutative:  $\forall x, y \in F, xy = yx$

(M2) Associative:  $\forall x, y, z \in F, (xy)z = x(yz)$

(M3) Muti identity:  $\exists 1 \neq 0$  in  $F \ni x1 = 1x = x$

(M4) Multiplicative inverse: For each  $x \neq 0, \exists x^{-1} \in F \implies xx^{-1} = x^{-1}x = 1$

i.e.  $(F = F \cdot \{0\}, \cdot)$  is an abelian group

**Distributive Law**

(D1)  $\forall x, y, z \in F, (x, y)z = xz + yz \ \& \ x(y + z) = xy + xz$

**Induction from Axioms**

let  $(F, +, \cdot)$  be a field, we list a series of basic identity as you learn in high school in the real number system

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- (a) Cancellation law for " + " :  $x + y = x + z \implies y = z$   
 $\because x + y = x + z \implies (-x) + (x + y) = (-x) + (x + z) \implies$   
 $((-x) + x) + y = ((-x) + x) + z$   
 $\implies 0 + y = 0 + z \implies y = z$
- (b) 0 is "1"  
 suppose  $0' \in F$  is another element satisfy  $A_3$ , then  $0 = 0 + 0' = 0'$
- (c)  $x + y = x \implies y = 0$  by (a)  $\because x + y = x + 0 \implies y = 0$
- (d) negative  $-x$  of  $x$  is "1"  
 if  $x' \in F$ , is another negative of  $x$ , then  $x + x' = x' + x = 0$   
 From  $x + x' = 0 \implies (-x) + (x + x') = -x + 0 = -x$
- (e)  $x + y = 0 \implies y = -x$   
 $x + y = 0 \implies (-x) + (x + y) = (-x) + 0 \implies ((-x) + x) + y =$   
 $-x$   
 $\implies 0 + y = -x \implies y = -x$
- (f)  $-(-x) = x$   
 $-(-x) + (-x) = 0$ , By (d)  $x = -(-x)$
- (a') cancellation law  
 if  $x \neq 0$ , then  $xy = xz \implies y = z$ ,  $\because (x^{-1})(xy) = (x^{-1})(xz)$   
 $\implies (x^{-1})(xy) = (x^{-1}x)z \implies 1y = 1z \implies y = z$
- (b') 1 is "1"  
 if  $1'$  is another identity, then  $1 = 11' = 1'$
- (c')  $x \neq 0$  &  $xy = x \implies y = 1$   
 $xy = x1 \implies y = 1$
- (d') For  $x \neq 0$  in  $F$ ,  $x^{-1}$  is "1"  
 if  $x$  is another one, i.e.  $x'x = xx' = 1 \implies (x^{-1})(xx') =$   
 $(x^{-1})1 = x^{-1}$
- (f')  $x \neq 0 \implies (x^{-1})^{-1} = x$   
 $(x^{-1})^{-1}(x^{-1}) = 1 \implies x = (x^{-1})^{-1}$
- (g')  $0x = x0 = 0$   
 $(0 + 0)x = 0x + 0x \implies 0x = 0$
- (h')  $x \neq 0$  &  $y \neq 0 \implies xy \neq 0$ , equivalently  $xy = 0 \implies x = 0$  or  
 $y = 0$   
 $\because xy = 0$  then  $(x^{-1})(xy) = ((x^{-1}x)y = 1y = y(\rightarrow \leftarrow)$
- (i')  $(-x)y = -(xy) = x(-y)$



$$\begin{aligned}
& \because [(-x) + x]y = 0y = 0 = (-x)y = -(xy) \implies (-x)y = -(xy) \\
& \text{(j')} \quad (-x)(-y) = xy \\
& \quad \because (-x)(-y) = -(x(-y)) \text{ by (i)} \\
& \quad = -(-(xy)) = xy
\end{aligned}$$

$$\begin{aligned}
& \text{(k)} \quad -x = (-1)x \\
& \quad \because (1-1)x = 0x = 0 = 1x + (-1)x = x + (-1)x \implies (-1)x = -x
\end{aligned}$$

**Definition** (**Order Field**). Let  $F$  is a field, we say that  $F$  is an order field if there is an ordering " $<$ " satisfying

- (1) if  $x < y$ , then  $x + z < y + z$ ,  $\forall z \in F$
- (2) if  $x > y$  and  $y > 0$ , then  $xy > 0$

**Example.**  $\mathbb{Q}$  and  $\mathbb{R}$  are order field under the usual ordering  
Some basic properties of ordered field, let  $F$  be an ordered field with ordering " $<$ "

$$\begin{aligned}
& \text{(a)} \quad x > 0 \implies -x < 0 \\
& \quad \because x > 0 \implies x + (-x) > 0 + (-x) \implies 0 > -x \\
& \text{(b)} \quad x > y \Leftrightarrow x - y > 0 \\
& \quad \because x > y \implies x + (-y) > y + (-y) \implies x - y > 0 \\
& \quad x - y > 0 \implies x - y + y > y \implies x + 0 > y \implies x > y \\
& \text{(c)} \quad x > 0 \text{ and } y < z \implies xy < xz \\
& \quad \because x > 0 \text{ and } y < z \implies x > 0 \text{ and } z - y < 0 \implies x(z - y) > 0 \\
& \quad \implies xz + x(-y) > 0 \\
& \quad \implies xz - xy > 0 \implies xz > xy \\
& \text{(d)} \quad x < 0 \text{ and } y < z \implies xy > xz \\
& \quad \because x < 0 \text{ and } y < z \implies -x > 0 \text{ and } z - y > 0 \implies \\
& \quad (-x)(z - y) > 0 \implies -xz + xy > 0 \\
& \quad \implies xy > xz \\
& \text{(e)} \quad \forall x \neq 0 \text{ in } F, x^2 > 0 \\
& \quad \because x > 0 \implies x \cdot x > x0 \text{ by (c) or} \\
& \quad x < 0 \implies -x > 0 \text{ by (a)} \implies -x > 0 \text{ by (a)} \implies (-x)^2 > 0 \\
& \quad \implies x^2 > 0 \\
& \text{(f)} \quad 1 > 0, -1 < 0
\end{aligned}$$

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$$\begin{aligned}
& \because 1 \neq 0 \implies 1^2 > 0 \text{ by (e)} \implies 1 > 0 \\
(g) \quad & 0 < x < y \implies 0 < \frac{1}{y} < \frac{1}{x} \\
& \because \text{Note that } \forall u \in \mathbb{F}, u > 0 \implies \frac{1}{u} = u^{-1} > 0 \\
& \because \text{if } \frac{1}{u} < 0, \text{ then } u \cdot \frac{1}{u} < 0 \text{ by (e)} \implies 1 < 0 (\rightarrow \leftarrow) \therefore \frac{1}{u} > 0 \\
& \text{Now, } \frac{1}{x}, \frac{1}{y} > 0 \text{ from } x < y \text{ we get } (\frac{1}{x} \cdot \frac{1}{y})x < (\frac{1}{x} \cdot \frac{1}{y})y \implies 0 < \\
& \frac{1}{y} < \frac{1}{x}
\end{aligned}$$

**Remark.** By (e)(f), we conclude that  $\mathbb{C}$  is not an ordered field  
 $\because \mathbb{C}$  were an ordered field, then by (e),  $i^2 > 0 \implies -1 > 0 (\rightarrow \leftarrow)$   
 $\therefore \mathbb{C}$  is not an order field

### 1.5. The Real Number Field $\mathbb{R}$ .

**Theorem.** There exists an ordered field  $\mathbb{R}$  containing  $\mathbb{Q}$  which has the l.u.b. property. Moreover, such  $\mathbb{R}$  is unique up to order-isomorphism  
Such  $\mathbb{R}$  is called the real number field or real number system or real line

**Note.** if " $<$ " and " $<'$ " are two orders on  $\mathbb{R}$ , then

$$\exists f(\mathbb{R}, <) \rightarrow (\mathbb{R}, <') \text{ such that}$$

- (i)  $f$  is a field isomorphism,  
i.e.  $\forall a, b \in \mathbb{R}, f(a + b) = f(a) + f(b),$   
 $f(ab) = f(a)f(b), f(1) = 1$
- (ii)  $f$  preserves ordering,  $a < b \implies f(a) < f(b)$

Note. In most book, above theorem is taken as axiom, called the least upper bound property of  $\mathbb{R}$  or the completeness axiom of  $\mathbb{R}$ . However, it can be constructed from  $\mathbb{Q}$ . There are two ways to construct it

- (1) Using Dedekind cut as in the Appendix of Chapter 1.
- (2) Using Cauchy sequence to get the completion of  $\mathbb{Q}$ .

**Theorem.**

- (a) The Archimedean property of  $\mathbb{R}$  : Given  $x, y \in \mathbb{R}$  with  $x > 0$ ,  $\exists n \in \mathbb{N} \implies nx > y$
- (b)  $\mathbb{Q}$  is dense in  $\mathbb{R}$  :  $\forall x, y \in \mathbb{R}$  with  $x < y$ ,  $\exists r \in \mathbb{Q} \implies x < r < y$

*Proof.*

- (a) Let  $A = \{ nx \mid n \in \mathbb{N} \} \subseteq \mathbb{R}$   
if (a) were false, then  $A$  is bdd above by  $y$ , since  $\mathbb{R}$  has the l.u.b property  
 $\alpha = \sup A$  exists in  $\mathbb{R}$ , since  $x > 0$ ,  $\alpha - x < \alpha \implies \alpha - x$  is not an upper bdd of  $A$   
 $\implies \exists m \in \mathbb{N} \ni mx > \alpha - x \implies (m + 1)x > \alpha (\rightarrow \leftarrow)$
- (b) Since  $x < y$ ,  $y - x > 0$ , by (a),  $\exists n \in \mathbb{N} \implies n(y - x) > 1$

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By (a) again,  $\exists m_1, m_2 \in \mathbb{N} \implies m_1 = m_1 1 > n_x \text{ \& } m_2 = m_2 \cdot 1 > -n_x$

we have  $-m_2 < n_x < m_1$ , choose  $m \in \mathbb{Z} \implies -m_2 \leq m \leq m_1 \text{ \& } m - 1 \leq n_x < m$

(in fact,  $m = [n_x] + 1$ , where  $[z]$  is the greatest integer of  $z$ )

we have  $n_x < m < 1 + n_x < ny (\because n(y - x) > 1) \implies x < \frac{m}{n} < y$

Let  $r = \frac{m}{n} \in \mathbb{Q}$ , then  $x < r < y$

■

An application of the density property of  $\mathbb{Q}$  in  $\mathbb{R}$  :

Given  $x \in \mathbb{R} - \mathbb{Q}$  i.e.  $x$  is an irrational numbers, i.e.  $\forall \epsilon > 0, \exists r \in \mathbb{Q} \implies |x - r| < \epsilon$

equivalently,  $\exists$  a sequence  $\{r_n\}$  in  $\mathbb{Q} \implies r_n \rightarrow x$

In fact, one may choose  $\{r_n\}$  to  $\uparrow$  or  $\downarrow$

$\because \forall n \geq 1, \exists r_n \in \mathbb{Q} \implies x < r_n < \frac{1}{n} + x$  by Thm.1.3(b) By squeezing lemma,  $r_n \rightarrow x$  on  $n \rightarrow \infty$

**Theorem** (existence of  $n$ th root). Given  $x \in \mathbb{T}, x > 0 \text{ \& } n \in \mathbb{N}, \exists "1" y > 0 \implies y^n = x$

Such  $y$  is called the  $n$ th root of  $x$  & denoted by  $y = \sqrt[n]{x} = x^{\frac{1}{n}}$

*Proof.* **not important**

"1". Suppose  $y_1, y_2 > 0 \implies y_1^n = x \text{ \& } y_2^n = x$

Bt trichotomy, we have

(i)  $0 < y_1 < y_2 \implies y_1^n < y_2^n (\rightarrow \leftarrow)$

(ii)  $0 < y_2 < y_1 \implies y_2^n < y_1^n (\rightarrow \leftarrow)$

(iii)  $y_1 = y_2$

" $\exists$ ". Let  $E = \{t \in \mathbb{R} \mid t^n < x\}$

Claim:

- $E \neq \emptyset$ , Let  $t = \frac{x}{1+x}$ , then  $0 < t < 1$ , hence  $t^n < t < x$ ,  $\therefore t \in E \text{ \& } E \neq \emptyset$
- $E$  is bdd above, in fact  $E$  is bdd above by  $1+x$  if  $t > 1+x > 1$ , then  $t^n > t > x$ , so  $E$  is bdd above by  $1+x$

Therefore  $y = \sup E$  exists & is finite

- Claim  $y > 0 \text{ \& } y^n = x$ , clearly,  $y > 0 (\because \frac{x}{1+x} \in E \text{ \& } \frac{x}{1+x} > 0)$

by trichotomy, we have  $y^n < x$ ,  $y^n > x$ ,  $y^n = x$

Now, to show that (i) & (ii) are impossible, do (iii) holds  $y^n = x$

By the identity,  $b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \dots + a^{n-1})$

(i)  $y^n < x$  choose  $0 < h < 1 = \alpha$  &  $0 < \frac{x - y^n}{n(y + 1)^{n-1}}$ ,  $0 < h < \min \{ \alpha, \beta \}$

put  $a = y$ ,  $b = y + h$  in  $(\star)$ , we obtain

$$\begin{aligned} (y + h)^n - y^n &< hn(y + h)^{n-1} < hn(y + 1)^{n-1} < x - y^n \\ \implies (y + h)^n &< x \implies y + h \in E \text{ \& } y + h > y(\rightarrow \leftarrow) \therefore \text{(i) fails} \end{aligned}$$

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(ii)  $y^n > x$ , Let  $k = \frac{y^n - x}{ny^{n-1}}$ , Then  $0 < k < y$ ,  $k = \frac{y^n - x}{ny^{n-1}} < \frac{y^n}{ny^{n-1}} = \frac{y}{n} < y$   
 if  $t > y - k > 0$ , then  $y^n - t^n \leq y^n - (y - k)^n < kny^{n-1}$  by  $(\star) = y^n - x$   
 $\implies t^n > x \implies t \in E \implies E$  is bdd above by  $y - k \implies \sup E \leq y - k (\rightarrow \leftarrow)$   
 $\therefore$  (ii) fails ■

**Corollary.** Let  $a, b \in \mathbb{R}$  with  $a, b > 0$ ,  $n \in \mathbb{N}$  Then  $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$   
 $\because a^{\frac{1}{n}}, b^{\frac{1}{n}} > 0$  &  $(a^{\frac{1}{n}} \cdot b^{\frac{1}{n}})^n = ab$ , By (1) in Thm 1.4  $(a, b)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$

### infinite in $\mathbb{R}$

After discuss the real number  $\mathbb{R}$ , sometimes, we have to work with the extended real number system  $\mathbb{R}^* = [-\infty, \infty] = \mathbb{R} \cup \{+\infty, -\infty\}$  with observe,  $x \in \mathbb{R}$

$$\lim_{n \rightarrow \infty} (-n) = -\infty, \lim_{n \rightarrow \infty} n = \infty, \lim_{n \rightarrow \infty} \left(\frac{1}{n} + n\right) = \infty, \lim_{n \rightarrow \infty} (n^2 - n) = \infty$$

$$x \pm \infty = \pm\infty, 0 \cdot (\pm\infty) = 0, \infty - \infty \text{ is not define}$$

Element in  $\mathbb{R} \subseteq \mathbb{R}^*$  are called finite. Now, given any nonempty subset  $E \subseteq \mathbb{R}$ ,

$$\sup E = \begin{cases} +\infty & \text{if } E \text{ is not bdd above} \\ \text{finite} & \text{if } E \text{ is bdd above} \end{cases} \quad \& \quad \inf E = \begin{cases} -\infty & \text{if } E \text{ is not bdd below} \\ \text{finite} & \text{if } E \text{ is bdd below} \end{cases}$$

Note that if  $A \subseteq B$ , then  $\sup A \leq \sup B$  &  $\inf A \geq \inf B$   
 $\therefore \emptyset \subseteq B, \forall B \subseteq \mathbb{R}$ , One may define  $\sup \emptyset = -\infty, \inf \emptyset = +\infty$

### 1.6. The Complex Number Field $\mathbb{C}$ .

Consider the contention product  $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a, b) \mid a, b \in \mathbb{R}\}$

Note that  $(a, b) = (c, d) \Leftrightarrow a = c$  &  $b = d$ , From now, we can write  $\mathbb{C} = \mathbb{R}^2$

**Operation on  $\mathbb{C}$**  Given  $(a, b), (c, d) \in \mathbb{C}$

(1)  $(a, b) + (c, d) = (a + c, b + d)$

(2)  $(a, b)(c, d) = (ac - bd, ad + bc)$

It is easy to see that, with these operations,  $\mathbb{C}$  is a field.

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**Note that**

- the zero element is  $(0, 0)$
- the negative of  $(a, b)$  is  $-(a, b) = (-a, -b)$
- the identity is  $(1, 0)$
- if  $(a, b) \neq (0, 0)$ , then  $(a, b)^{-1} = \left( \frac{1}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right)$

$\mathbb{R}$  is a subset of  $\mathbb{C}$  (not very important)  
consider that map

$$f : \mathbb{R} \rightarrow \mathbb{C} \text{ define by } f(a) = (a, 0), \quad a \in \mathbb{R}$$

we have (1)  $f$  is injective (2)  $f(1) = (1, 0) \quad \because \forall a, b \in \mathbb{R}$

$$f(a + b) = (a + b, 0) = (a, 0) + (b, 0) = f(a) + f(b), \quad f(a \cdot b) = (ab, 0) = (a, 0) \cdot (b, 0)$$

$f$  is a field homomorphism

$\therefore f : \mathbb{R} \rightarrow \mathbb{C}$  is an injective and isomorphism

Therefore, we identify  $\mathbb{R}$  with  $f(\mathbb{R})$  through the injective  $f$

i.e.  $a \in \mathbb{R}$  is identified with  $f(a, 0)$  in  $\mathbb{C}$

$$ab = (a, 0) \cdot (b, 0), \quad a + b = (a, 0) + (b, 0) \quad \forall a, b \in \mathbb{R}$$

**Change  $(a, b)$  to  $a + bi$**

Now, we can transform an element  $(a, b) \in \mathbb{C}$  into the normal form:

$$(a, b) = (a, 0) + (0, b) = (a, 0)(1, 0) + (b, 0)(0, 1) = a1 + bi = a + ib, \\ \text{where } i = (0, 1)$$

Therefore, from now on, we write  $\mathbb{C} = \{ a + ib \mid a, b \in \mathbb{C} \}$

An element  $z = a + ib \in \mathbb{C}$  is called a complex number

Hence, under this notation,  $z = a + ib, w = c + id \in \mathbb{C}$

$$(1) \quad z + w = (a + c) + i(b + d)$$

$$(2) \quad zw = (ac - bd) + i(ad + bc)$$

and the  $a$  is called the real part of  $z$ ,  $a = \operatorname{Re}(z)$ ,  $b$  is called imaginary part of  $z$ ,  $b = \operatorname{Im}z$



**Some basic properties of complex numbers whose proofs are easy**

$\forall z, w \in \mathbb{C}$

$$\begin{aligned}
 & \cdot \quad \overline{z+w} = \bar{z} + \bar{w} \quad \cdot \quad \overline{zw} = \bar{z} \cdot \bar{w} \quad \cdot \quad \operatorname{Re} z = \frac{z + \bar{z}}{2} \\
 & \cdot \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i} \quad \cdot \quad |z| = 0 \Leftrightarrow z = 0 \quad \cdot \quad \text{Triangle inequality} \\
 & \quad \quad \quad |z+w| \leq |z| + |w| \\
 & \cdot \quad ||z| - |w|| \leq |z - w| \quad \cdot \quad \mathbb{C} \text{ is not an ordered field} \quad \cdot \quad |z|^2 = z\bar{z} \\
 & \cdot \quad |\bar{z}| = |z| \quad \cdot \quad |\operatorname{Re} z| \leq |z|, |\operatorname{Im} z| \leq |z| \quad \cdot \quad |zw| = |z||w|
 \end{aligned}$$

*Proof.*  $|z+w| \leq |z| + |w|$   
 $|z+w|^2 = (z+w)(\overline{z+w}) = (z+w)(\bar{z} + \bar{w}) = z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w}$   
 $= |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2 \leq |z|^2 + 2|z\bar{w}| + |w|^2 = |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2$   
 $\therefore |z+w| \leq |z| + |w|$  ■

**Theorem** (basic algebraic theorem).

- (a)  $x^2 + 1$  has no root in  $\mathbb{R}$   
 (b)  $x^2 + 1$  has two distinct roots in  $\mathbb{C}$

*Proof.*

- (a)  $1 > 0, x^2 > 0, \forall x \in \mathbb{R} - \{0\} \implies x^2 + 1 > 0 \forall x \neq 0$   
 $0^2 + 1 = 1 > 0, \therefore x^2 + 1 > 0, \forall x \in \mathbb{R}$ . Hence,  $x^2 + 1 = 0$   
 has no root in  $\mathbb{R}$   
 (b)  $i^2 = (0, 1)(0, 1) = (0 - 1, 0) = (-1, 0) = -1$   
 $(-i)^2 = (-(0, 1))^2 = (0, -1)^2 = (0, -1)(0, -1) = -1, \therefore \pm i$   
 are root of  $\mathbb{C}$  ■

**Conclusion:** Every non const polynomial  $f(x) \in \mathbb{R}[x]$  has  $n$  roots where  $n = \deg f(x)$

**The complex root is even**

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### no important proof

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ ,  $a_n \neq 0$ ,  $n \geq 1$

if  $\alpha = a + ib \in \mathbb{C}$  is a root of  $f(x)$ , then

$$0 = f(\alpha) = a_n \alpha^n + a_{n-1} \alpha^{n-1} + \cdots + a_1 \alpha + a_0$$

$$0 = f(\bar{\alpha}) = a_n \bar{\alpha}^n + a_{n-1} \bar{\alpha}^{n-1} + \cdots + a_1 \bar{\alpha} + a_0$$

$$\therefore (x - \alpha) | f(x), (x - \bar{\alpha}) | f(x) \implies (x - \alpha)(x - \bar{\alpha}) | f(x) \implies$$

$$(x^2 - (\alpha + \bar{\alpha})x + |\alpha|^2) | f(x)$$

$$\implies (x^2 - 2ax + (a^2 + b^2)) | f(x)$$

$\therefore$  quadratic function must have two roots in  $\mathbb{C}$

### The fundamental Theorem of Algebra

Every non zero polynomial  $f(x) \in \mathbb{C}[x]$  has at least one root in  $\mathbb{C}$

Therefore, if  $\deg f(x) = n$ , then  $f(x)$  has  $n$  roots in  $\mathbb{C}(C, M)$

$\therefore f(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_t)^{e_t} (a_1 x^2 + b_1 x + c_1)^{l_1} \cdots (a_s x^2 + b_s x + c_s)^{l_s}$ , where  $\lambda_1, \dots, \lambda_t \in \mathbb{R}$ ,  $a_i, b_i, c_i \in \mathbb{R}$  &  $e_1 + \cdots + e_t + 2l_1 + \cdots + 2l_s = \deg f(x)$  which shows that all roots of  $f(x)$  are in  $\mathbb{C}$

In fact, we have the famous theorem: The fundamental theorem of algebra

Every non zero polynomial  $f(x) \in \mathbb{C}[x]$  has at least one root in  $\mathbb{C}$

$\therefore$  if  $\deg f(x) = n$ , then  $f(x)$  has  $n$  roots in  $\mathbb{C}(C, M)$

**Theorem** (Cauchy-Schering Inequal). *Given  $z_1, \dots, z_n, w_1, \dots, w_n \in \mathbb{C}$ , we have*

$$\left| \sum_{j=1}^n z_j \bar{w}_j \right| \leq \left( \sum_{j=1}^n |z_j|^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^n |w_j|^2 \right)^{\frac{1}{2}}$$

and " $=$ " holds  $\Leftrightarrow \exists \lambda \in \mathbb{C} \ni w_j = \lambda z_j$ ,  $1 \leq j \leq n$ ,

In patricial, if  $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$ , then

$$\left| \sum_{j=1}^n x_j y_j \right| \leq \left( \sum_{j=1}^n x_j^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^n y_j^2 \right)^{\frac{1}{2}}$$

and " $=$ " holds  $\Leftrightarrow \exists t \in \mathbb{R} \ni y_j = t x_j$ ,  $1 \leq j \leq n$

### Real Version

*Proof.*  $\forall t \in \mathbb{R}, 0 \leq \sum_{i=1}^n (x_i + ty_i)^2 = \sum_{i=1}^n (x_i^2 + 2tx_iy_i + t^2y_i^2) = (\sum_{i=1}^n y_i^2)t^2 + t(\sum_{i=1}^n 2x_iy_i) + \sum_{i=1}^n x_i^2$  Not all  $x_i = 0$  and  $y_i = 0$   
we use  $b^2 - 4ac \implies 0 \geq (\sum_{i=1}^n 2x_iy_i)^2 - 4(\sum_{i=1}^n x_i^2)(\sum_{i=1}^n y_i^2) \implies (\sum_{i=1}^n x_iy_i)^2 \leq (\sum_{i=1}^n x_i^2)(\sum_{i=1}^n y_i^2)$  ■

### Complex Version

*Proof.* we may assume that not all  $z_i = 0$  and not all  $w_j = 0$  (otherwise it's trivial)

$$\begin{aligned} \forall \lambda \in \mathbb{C}, 0 &\leq \sum_{j=1}^n |z_j - \lambda w_j|^2 \\ &= \sum_{j=1}^n (z_j - \lambda w_j)(\overline{z_j} - \overline{\lambda w_j}) \\ &= \sum_{j=1}^n [|z_j|^2 - \lambda z_j \overline{w_j} - \lambda \overline{z_j} w_j + |\lambda|^2 |w_j|^2] \end{aligned}$$

$$\begin{aligned} \text{Taking } \lambda &= \frac{\sum_{j=1}^n z_j \overline{w_j}}{\sum_{j=1}^n |w_j|^2} \\ &= \sum_{j=1}^n |z_j|^2 - \frac{2|\sum_{j=1}^n z_j \overline{w_j}|^2}{\sum_{j=1}^n |w_j|^2} \\ &= \sum_{j=1}^n |z_j|^2 - 2 \frac{|\sum_{j=1}^n z_j \overline{w_j}|^2}{\sum_{j=1}^n |w_j|^2} + \left( \frac{|\sum_{j=1}^n z_j \overline{w_j}|}{\sum_{j=1}^n |w_j|^2} \right)^2 \sum_{j=1}^n |w_j|^2 \\ &= \sum_{j=1}^n |z_j|^2 - \frac{|\sum_{j=1}^n z_j \overline{w_j}|^2}{\sum_{j=1}^n |w_j|^2} \\ &\implies \left| \sum_{j=1}^n z_j \overline{w_j} \right|^2 \leq \left( \sum_{j=1}^n |z_j|^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^n |w_j|^2 \right)^{\frac{1}{2}} \end{aligned}$$

**Remark.** From the proof, the " $=$ " holds in CS inequality  $\Leftrightarrow \exists \lambda \in \mathbb{C} \ni z_j = \lambda w_j \forall j = 1, 2, \dots, n$ , i.e.  $(z_1, \dots, z_n) = \lambda(w_1, \dots, w_n)$

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### 1.7. Euclidean Spaces $\mathbb{R}^n$ .

**Definition.** the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$

$$= \{ x = (x_1, \dots, x_n) \mid x_i \in \mathbb{R}, 1 \leq i \leq n \} = \mathbb{R} \times \dots \times \mathbb{R}$$

Note that

$$(x_1, \dots, x_n) = (y_1, \dots, y_n) \Leftrightarrow x_i = y_i \quad \forall 1 \leq i \leq n$$

We are going to introduce the structure of  $\mathbb{R}^n$

- vector space
- inner product space
- normed linear space
- matrix space

**Definition.** Two operation on  $\mathbb{R}^n$  as follows:

- Addition  $+: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $(x, y) \mapsto x + y = (x_1 + y_1, \dots, x_n + y_n)$
- Scalar multiplication  $\cdot: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $(a, x) \mapsto ax = (ax_1, \dots, ax_n)$

**we skip space example here.**

**Observe.** The real version of the CS inequality can be write:

$$| \langle x, y \rangle | = \left| \sum_{i=1}^n x_i y_i \right| \leq \left( \sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^n y_i^2 \right)^{\frac{1}{2}} = \|x\| \|y\|$$

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle = \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle \\ &= \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

### 1.8. Countability of Sets.

Given two nonempty set  $A, B$  and a function  $f: A \rightarrow B$ ,  $f(A) = \{ f(a) \mid a \in A \}$  is called the image of  $A$  under  $f$

**Some basic things**

$E \subseteq A$ ,  $f(E) = \{ f(a) \mid a \in E \}$  the image of  $E$  under  $f$

•  $f$  is injective(one-to-one)

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2) \Leftrightarrow f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$$

•  $f$  is surjective(onto) if  $f(A) = B$

•  $f$  is bijective if  $f$  is one-to-one and onto

Given  $F \subseteq B$ ,  $f^{-1}(F) = \{ x \in X \mid f(x) \in F \}$  called the inverse image of  $f$  under  $F$

**Example**

$$f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2, x \in \mathbb{R}$$

$$f^{-1}([0, 1]) = \{ x \in \mathbb{R} \mid f(x) \in [0, 1] \} = \{ x \in \mathbb{R} \mid x^2 \in [0, 1] \} = [-1, 1]$$

$$f^{-1}([-1, 1]) = [-1, 1]$$

**Properties of inverse image**

$$\bullet F_1 \subseteq F_2 \subseteq B \implies f^{-1}(F_1) \subseteq f^{-1}(F_2)$$

• Inverse image preserves set operation

$$\forall F_\alpha \subseteq B, \alpha \in I, F \subseteq B$$

$$(i) f^{-1}(\cup_{\alpha \in I} F_\alpha) = \cup_{\alpha \in I} f^{-1}(F_\alpha)$$

$$(ii) f^{-1}(\cap_{\alpha \in I} F_\alpha) = \cap_{\alpha \in I} f^{-1}(F_\alpha)$$

$$(iii) f^{-1}(B - F) = f^{-1}(B) - f^{-1}(F)$$

• Given  $S \subseteq A$ ,  $f'(f'(S)) \supseteq S$ , " = "  $\Leftrightarrow$  one-to-one, **example:**

$$f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2, S = [0, 1], f(S) = [0, 1], f^{-1}(f(S)) = f^{-1}([0, 1]) = [-1, 1]$$

• Given  $F \subseteq B$ ,  $f(f^{-1}(F)) \subseteq F$ , " = "  $\Leftrightarrow$  "onto", **example**

$$f(x) = x^2, x \in \mathbb{R}, F = [-1, 1], f(f^{-1}([-1, 1])) = f([-1, 1]) = [0, 1]$$

• For  $y \in B$ ,  $f^{-1}(\{y\}) = f^{-1}(y) = \{x \in A \mid f(x) = y\}$  the inverse image of  $y$ , **example**

$$f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2, f^{-1}(1) = \{1, -1\}, f^{-1}(2) = \emptyset$$

**Definition** (cardinality). Let  $A, B$  are two set ew say that  $A$  and  $B$  have the same cardinality if  $\exists$  a bijective map  $f : A \rightarrow B$ , which is denoted by  $A \sim B$ . or  $R = \{(A, B) \mid \exists \text{ a bij map } f : A \rightarrow B\} \subseteq \mathcal{P}(A \times B)$

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From now on, we write  $|A|$  as the cardinality of  $A$

**Claim** " $\sim$ " is an  $\equiv$  relation among all sets

- (i) Reflexion:  $\forall$  set  $A$ ,  $A \sim^{1_A} A$ , which  $1_A$  is identity mapping
- (ii) Symmetry:  $A \sim^f B \implies B \sim^{f^{-1}} A$
- (iii) Transitive:  $A \sim^f B \ \& \ B \sim^g C \implies A \sim^{g \circ f} C$

Properties of " $\equiv$ " classes on a set  $X$ . " $\sim$ " is an " $\equiv$ " relative on  $X$   
 Given  $x \in X$ ,  $[x] = \bar{x} = \{y \in X \mid y \sim x\}$  the " $\equiv$ " class determine by  $x$   
 So we gave some property:

- Any two " $\equiv$ " class  $[x]$  and  $[y]$  are either disjoint or identical, i.e.  $[x] = [y]$  or  $[x] \cap [y] = \emptyset$
- $\overline{X}$  is a disjoint union of " $\equiv$ " classes  
 $[A] = \{B \in \overline{X} \mid B \sim A\}$  the " $\equiv$ " class set by  $A$

Ant two element in an " $\equiv$ " class have the same cardinality

Notation For  $n \in \mathbb{N}$ ,  $\mathbb{N}_n = \{1, 2, \dots, n\}$

**Definition.** Let  $A$  be a set

- (a)  $A$  is a finite set if  $A = \emptyset$  or  $A \sim \mathbb{N}_n$  for some  $n \in \mathbb{N}$
- (b)  $A$  is a infinite set if  $A$  is not a finite set
- (c)  $A$  is countable if  $A \sim \mathbb{N}$
- (d)  $A$  is uncountable if  $A$  is not countable.
- (e)  $A$  is at most countable if  $A$  is finite or countable

**Remark.**

- (1) when  $A, B$  are finite sets,  $A \sim B \Leftrightarrow |A| = |B|$ , i.e.  $A, B$  have same number.
- (2) where  $A, B$  are infinite and  $A \sim B$ , i.e.  $|A| = |B|$ , the concept is abstract.
- (3)  $\{a, b, c\} \cup \mathbb{N} \sim \mathbb{N}$ ,  $f : \mathbb{N} \rightarrow \{a, b, c\} \cup \mathbb{N}$ ,  $f(1) = a$ ,  $f(2) = b$ ,  $f(3) = c, \dots$
- (4) Any finite set can not equivalent to a proper subset, i.e.  $A$  is finite,  $B \subseteq A$   
 Then  $A \sim B$ , In fact  $|B| < |A|$ , but infinite different
- (5) Any finite set  $A$  can be listed an  $A = \{a_1, \dots, a_n\}$  where  $n = |A|$

Now, we consider the case of countable set

Recall, in calculus, a real sequence  $\{a_n\}$ , e.g.

$$a_n = \frac{1}{n} \left\{ \frac{1}{n} \right\}, \quad a_n = 1 - \frac{1}{n} \left\{ 1 - \frac{1}{n} \right\}, \quad a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

**Definition.** Let  $X$  be a nonempty set, a sequence in  $X$  is a function  $a : \mathbb{N} \rightarrow X$ . Given a sequence  $=^a$  in  $X$ ,  $a$  is "1" determine by  $a(n)$ ,  $n \in \mathbb{N}$ . We write

$$a = \{a(1), a(2), \dots, a(n), \dots\} = \{a_1, a_2, \dots, a_n, \dots\} = \{a_n\} = \{a_n\}_{n=1}^{\infty}$$

**Remark.**

- (1) For a sequence  $\{a_n\}$  in  $X$ ,  $a_n$  may not be distinct.  
If all  $a_n$  are distinct, then we say that  $\{a_n\}$  is a distinct sequence in  $X$ .
- (2) We usually use  $\{a_n\}, \{b_n\}$  to denote sequence
- (3) A sequence  $\{a_n\}$  in  $X$  in fact is a function from  $\mathbb{N} \rightarrow X$ , So  $\{a_n \mid n \in \mathbb{N}\}$  is the image of the sequence.
- (4)  $\{a_n\}$  is a sequence,  $a_n$  is called the  $n^{\text{th}}$  term of the sequence.
- (5) A sequence in  $X$  may begin at 0, i.e.  $\{a_n\}_{n=0}$   
By a changing index, we can make it from  $\{b_n\}_{n=1}^{\infty}$ ,  $b_n = a_{n+1}$ ,  $n = 1, 2, \dots$

**Definition** (increasing).

A function  $a : \mathbb{N} \rightarrow \mathbb{N}$  is increasing,  $a$  is  $\uparrow$ , if  $a(n) \leq a(n+1) \forall n \geq 1$   
 $a$  is strictly increasing,  $a$  is st.  $\uparrow$ , if  $a(n) < a(n+1) \forall n \geq 1$

Now, given a st.  $\uparrow$  function  $n : \mathbb{N} \rightarrow \mathbb{N}$ , i.e.  $n(k) < n(k+1)$ ,  $k \geq 1$   
i.e.  $n_k < n_{k+1}$ ,  $k \geq 1$ , i.e.  $n_1 < n_2 < \dots < n_k < \dots$ , i.e.  $\{n_k\}_{k=1}^{\infty}$  is a st. sequence in  $\mathbb{N}$

**Definition.** Let  $\{a_n\}$  be a sequence in  $X$  and  $\{n_k\}$  be a st.  $\uparrow$  sequence in  $\mathbb{N}$ , then the sequence  $\{a_{n_k}\}$  is called a subsequence of  $\{a_n\}$   
In fact

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$$\mathbb{N} \xrightarrow{st.} \mathbb{N} \xrightarrow{seq} X \Rightarrow a \circ n : \mathbb{N} \rightarrow X \text{ is a function,}$$

hence, it also a sequence in  $X$

$$a \circ n = \{ a \circ n(k) \} = \{ a(n(k)) \} = \{ a_{n(k)} \} = \{ a_{n_k} \}$$

**Remark.** if  $\{ a_{n_k} \}$  is st.  $\uparrow$  in  $\mathbb{N}$ , then  $k \leq n_k \forall k \geq 1$

$\therefore$  By mathematical Induction

- $1 \leq n_1$
- Assume it's true for  $k \geq 2$ , i.e.  $k \leq n_k$
- Consider  $k + 1$ ,  $k + 1 \leq n_k + 1 \leq n_{k+1}$

### Example

Let  $\{ a_n \}$  be a sequence in  $X$ , then  $\{ a_{2k} \}$  and  $\{ a_{2k-1} \}$  are subsequence of  $\{ a_n \}$

Finally, we will assume that you are familiar with the following property of the countability of sets:

(1). Every subset of a countable set is at most countable. The proof needs the well ordering of  $\mathbb{N}$ : Every nonempty subset of  $\mathbb{N}$  has the smallest element.

$\therefore$  Let  $X$  be a countable set and  $A \subseteq X$  While  $X = \{x_1, x_2, \dots, x_n, \dots\}$

Let  $n_1 = \min k \mid x_k \in A \implies x_{n_1} \in A$

Let  $n_2 = \min\{k \mid x_k \in A - \{x_{n_1}\}\} \implies x_{n_2} \in A - \{x_{n_1}\}$

$\vdots$

Let  $n_k = \min\{k \mid x_k \in A - \{x_{n_1}, \dots, x_{n_{k-1}}\}\} \implies$

$x_{n_k} \in A - \{x_{n_1}, \dots, x_{n_{k-1}}\}$

$\vdots$  (The process will not stop if it is countable and finite)

$\therefore A = \{x_{n_1}, x_{n_2}, \dots\}$  is countable

(2).  $\mathbb{N} \times \mathbb{N}$  is countable. Hence if  $X$  and  $Y$  are countably then so is  $X \times Y$ .y induction, if  $x_1, \dots, x_n$  are countable, then so is  $X_1 \times \dots \times X_n$

$\therefore$  you can list  $\mathbb{N} \times \mathbb{N}$  like below:

$$\begin{array}{cccc} (1,1) & (1,2) & (1,3) & \dots \\ (2,1) & (2,2) & (2,3) & \dots \\ \vdots & \vdots & \vdots & \end{array}$$



(3). If  $Y$  is countable and  $f : X \rightarrow Y$  is injective, then  $X$  is countable

$$\begin{aligned} \because f(X) \subseteq Y &\implies f(X) \text{ is countable by (1)} \\ &\implies f : X \rightarrow f(X) \text{ is a bij function} \\ &\implies X \text{ is countable} \end{aligned}$$

(4). If  $f : X \rightarrow Y$  is surjective and  $X$  is countable, then  $Y$  is countable  
 $\forall y \in Y, f^{-1}(y) \neq \emptyset$ . Choose one  $x_y \in X \ni f(x_y) = y$  and fix it.

$$A = \{x_y \in X \mid y \in Y\} \subseteq X \implies A \text{ is countable } f : A \rightarrow Y$$

$f(x_y) = y$  is bijective  $\therefore Y$  is countable.

(5). Another way to see that  $\mathbb{N} \times \mathbb{N}$  is countable. Consider the function  
 $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  defined by  $f(m, n) = 2^m 3^n, (m, n) \in \mathbb{N} \times \mathbb{N}$ .

Claim:  $f$  is injective  $\because f(m, n) = f(k, l) \implies 2^m 3^n = 2^k 3^l \implies 2^{m-k} = 3^{l-n} \implies m-k = l-n = 0 \implies m = k, n = l \implies (m, n) = (k, l)$ . By (3)  $\mathbb{N} \times \mathbb{N}$  is countable.

(6). Countable union of countable sets is countable i.e.  $X_n : \text{countable}$   
 $n \in \mathbb{N} \implies \bigcup_{n=1}^{\infty} X_n$  is countable

We may assume that  $\{X_n\}$  are pairwise disjoint, i.e.  $X_n \cap X_m = \emptyset$  if  $n \neq m$ . Otherwise consider

$$\begin{aligned} \overline{X_n} &= X_n \times \{n\} \quad n \in \mathbb{N} \\ &= \{(x, n) \mid x \in X_n\} \end{aligned}$$

$$X_n \sim \overline{X_n}$$

and  $\overline{X_n}$  is countable

(7).  $\mathbb{Z}$  is countable,  $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup -\mathbb{N}$

(8).  $\mathbb{Q}$  is countable,  $\mathbb{Q} = \{\frac{m}{n}, n, m \in \mathbb{Z}\} \rightarrow \mathbb{Z} \times \mathbb{Z}$

or  $\mathbb{Q} = \bigcup_{n=-\infty}^{\infty} \{\frac{m}{n} \mid m \in \mathbb{Z}\}$

(9). The set  $\{a_n \mid a_n = 0 \text{ or } 1\}, n \in \mathbb{N}$  is uncountable

This can be proved by Cantor diagonal process

$\because$  if it is countable, then we can list it,  $a_0 A = \{a_1^{(1)}, a_2^{(2)}, \dots\}$  where

$$a^{(1)} = \{a_n^{(1)}\} = a_1^{(1)}, a_2^{(1)}, \dots; \quad a^{(2)} = \{a_n^{(2)}\} = a_1^{(2)}, a_2^{(2)}, \dots$$

Now, construct a sequence  $\{a_n\}$  in  $A \ni \{a_n\} \neq a^{(k)} \forall k \geq 1 (\rightarrow \leftarrow)$

## 1. SOME PRELIMINARY

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(10). Countable union of countable sets is countable

Recall, intervals in  $\mathbb{R}$ ,  $-\infty < a \leq b < \infty$ , following are finite bdd interval

$$\begin{aligned} (a, b) &= \{x \in \mathbb{R} \mid a < x < b\} \text{ open interval} \\ [a, b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\} \text{ closed interval} \\ (a, b] &= \{x \in \mathbb{R} \mid a < x \leq b\} \text{ open-closed} \\ [a, b) &= \{x \in \mathbb{R} \mid a \leq x < b\} \text{ closed-open} \end{aligned}$$

An interval  $I$  in  $\mathbb{R}$  is said to be non-degenerate if the endpoint of  $I$  are distinct i.e. length  $> 0$ . Otherwise, it is degenerate.

**Note.**

$$\begin{aligned} (0, 1) &\text{ is uncountable,} \\ \because (0, 1) &= \left\{ \sum_{n=1}^{\infty} \frac{a_n}{2^n} \mid a_n = 0 \text{ or } 1, n \in \mathbb{N} \right\} \\ x \in (0, 1) &\text{ has a unique binary representation, so } (0, 1) \sim A, \\ \text{where } A &\text{ is } \{ \{a_n\} \mid a_n = 0 \text{ or } 1 \} \text{ which is uncountable} \end{aligned}$$

If  $a = b$  then it's degenerate

All non-degenerate intervals (the length is greater than 0) in  $\mathbb{R}$  are uncountable.

$\because$  It sufficient to consider bdd non-degenerate interval in  $\mathbb{R}$ , given  $-\infty < a < b < \infty$

$(a, b)$  is uncountable( $\because (0, 1) \sim (a, b)$ )

Note that  $(0, 1) \sim \mathbb{R}(\because (0, 1) \rightarrow (\frac{\pi}{-2}, \frac{\pi}{2}) \rightarrow \mathbb{R})$

## 2. Basic Point Set Topology

To know the "closeness", "limit" and "continue"

**Notation.** Let  $X$  be a nonempty set. The power set of  $X$  is denoted by  $p(X)$  or  $2^X$ , i.e.  $\mathcal{P}(X) = 2^X$  which is the collect of all subset, if  $|X| = n$ , then  $|\mathcal{P}(X)| = 2^n$

### 2.1. Topological Spaces.

**Definition.** Let  $X$  be a nonempty set and  $\mathcal{T} \subseteq \mathcal{P}(X)$ , we say that  $\mathcal{T}$  is a topology on  $X$  if it satisfies

- (1)  $\emptyset, X \in \mathcal{T}$
- (2)  $\mathcal{T}$  is closed under arbitrary union,  
i.e.  $U_\alpha \in \mathcal{T}, \alpha \in I \implies \bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$
- (3)  $\mathcal{T}$  is closed under finite intersection  
i.e.  $U_1, \dots, U_n \in \mathcal{T} \implies U_1 \cap \dots \cap U_n \in \mathcal{T}$

In this chapter, the pair  $(X, J)$  or simply  $J$  is called a topological space and members in  $T$  are called open set in  $X$  or open subsets of  $X$

**Remark.**

- (1)  $X$ : a nonempty set, there is at least two trivial topology on  $X$ 
  - $\mathcal{P}(x)$  is the largest topology on  $X$  w.r.t inclusion  $X$  with this topology is called a discrete topological space
  - $\mathcal{T}_0 = \{\emptyset, X\}$  is the smallest topology on  $X$  w.r.t inclusion  $X$  with this topology is called an indiscrete topological space
- (2) How many topology can be define on  $\{a\}$ ,  $\{a, b\}$ ?

In the following  $X$  is a topology space

**Definition** (neighborhood). Let  $p \in X$ , a neighborhood of  $P$  is an open set  $U$  containing  $p$

**Definition** (Hausdorff space).  $X$  is a Hausdorff space if any two distinct points can be separated by open set, i.e.  $\forall p \neq q$  in  $X$ ,  $\exists$  neighborhood  $U$  of  $p$  and  $V$  of  $q \in U \cap V = \emptyset$

**Definition** (closed set). A subset  $F \subseteq X$  is said to be closed if  $F^C = X - F$  is open in  $X$

## 2. BASIC POINT SET TOPOLOGY

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**Theorem 2.1.** *The collection of all closed subsets of  $X$  satisfied*

- (a)  $\emptyset, X$  are closed
- (b) Arbitrary intersection of closed set is closed
- (c) Finite union of closed sets is closed

*Proof.*

- (a)  $X - \emptyset = X$  is open  $\therefore \emptyset$  is closed  
 $X - X = \emptyset$  is open  $\therefore X$  is closed
- (b) Given closed sets  $F_\alpha, \alpha \in I$ ,  $X - \bigcap_{\alpha \in I} F_\alpha = \bigcup_{\alpha \in I} (X - F_\alpha)$  is open,  $\therefore \bigcap_{\alpha \in I} F_\alpha$  is closed.
- (c) Given closed set  $F_1, \dots, F_n$ ,  $X - \bigcup_{i=1}^n F_i = \bigcap_{i=1}^n (X - F_i)$  is open,  $\therefore \bigcup_{i=1}^n F_i$  is closed. ■

**Definition.** *Let  $Y \subseteq X$  and*

$$\mathcal{T}_Y = \{U \cap Y \mid U \text{ is open in } X\}$$

**Theorem 2.2.**  *$\mathcal{T}_Y$  is also a topology space*

*Proof.* To prove  $\mathcal{T}_Y$  is a topology space, we take the topology's definition

- (a)  $\emptyset, Y \in \mathcal{T}_Y$  ( $\because \emptyset = \emptyset \cap Y, Y = X \cap Y$ )
- (b) Given  $U_\alpha \cap Y \in \mathcal{T}_Y, \alpha \in I$ , where  $U_\alpha$  is open in  $X$   
 $\bigcup_{\alpha \in I} (U_\alpha \cap Y) = (\bigcup_{\alpha \in I} U_\alpha) \cap Y \implies \bigcup_{\alpha \in I} (U_\alpha \cap Y) \in \mathcal{T}_Y$
- (c) Given  $U_1 \cap Y, \dots, U_n \cap Y$ , where  $U_i$  is open in  $X, 1 \leq i \leq n$   
 $\bigcap_{i=1}^n (U_i \cap Y) = (\bigcap_{i=1}^n U_i) \cap Y \implies \bigcap_{i=1}^n (U_i \cap Y) \in \mathcal{T}_Y$   
 $\therefore \mathcal{T}_Y$  is a topology on  $Y$  ■

**Definition.** *In Theorem 2.2, with the topology  $\mathcal{T}_Y$  on  $Y$ , is called a topological subspace of  $X$  and  $\mathcal{T}_Y$  is called the relative topology of  $Y$  in  $X$ . Members in  $\mathcal{T}_Y$  are called open set in  $Y$  or relative open sets in  $Y$ .*

### 2.2. Metric Spaces & Subspace.

In this chapter, we will introduce a class of topology space whose topology is induced by a metric.

**Definition.** Let  $X$  be a nonempty set. A metric or distance function is a function

$$d : X \times X \rightarrow \mathbb{R}, (a, b) \mapsto d(a, b)$$

satisfying:

- (a)  $\forall a, b \in X, d(a, b) \geq 0$  and  $d(a, b) = 0 \Leftrightarrow a = b$
- (b)  $\forall a, b \in X, d(a, b) = d(b, a)$  **symmetry**
- (c)  $\forall a, b, c \in X, d(a, b) \leq d(a, c) + d(c, b)$  **triangle inequality**

if  $d$  is a metric on  $X$ , then the pair  $(X, d)$  or simply  $X$  is called a metric space and  $\forall a, b \in X, d(a, b)$  is called the distance between  $a$  &  $b$

#### Examples

- (1) Let  $X$  be a nonempty set define by

$$d(a, b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b \end{cases}$$

Then  $d$  is a metric on  $X$ , called the discrete metric and with this metric  $X$  is called a discrete metric space. In particular, any set admits a metric.

- (2) The most important metric spaces are the Euclidean space  $\mathbb{R}^k$ , the metric  $d$  is called the Euclidean or standard or usual metric on  $\mathbb{R}^k$ . There are other metrics on  $\mathbb{R}^k$  induced the same metric topology on  $\mathbb{R}^k$ , in fact, they are all equivalent, e.g.  $\forall 1 \leq p \leq \infty$ , We can define a metric  $d_p$  on  $\mathbb{R}^k$  as follows

- $1 \leq p < \infty, d_p(x, y) = \|x - y\|_p = \left( \sum_{i=1}^k |x_i - y_i|^p \right)^{\frac{1}{p}}$
- $p = \infty, d_\infty(x, y) = \max_{1 \leq i \leq k} |x_i - y_i|$

Note that  $d_2 = d$  is the Euclidean metric on  $\mathbb{R}^k$

**Remark.** In fact, every normed linear space  $(V, \|\cdot\|)$  is a metric space whose metric is induced by its norm

## 2. BASIC POINT SET TOPOLOGY

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- (3) Let  $(X, d)$  be a metric space and  $Y \subseteq X$ ,  $Y \neq \emptyset$ . Then the restriction of  $d$  to  $Y \times Y$  is also a metric on  $Y$ , with this metric,  $Y$  is called a metric subspace of  $X$

**Definition** (ball). Given  $p \in X$  &  $r > 0$

$B(p, r) = \{x \in X \mid d(x, p) < r\}$  : open ball with center  $p$  and radius  $r$

$\overline{B}(p, r) = \{x \in X \mid d(x, p) \leq r\}$  : closed ball with center  $p$  and radius  $r$

**Example**

- (1) The discrete metric space  $X$  :  $p \in X$ ,  $r > 0$

$$B(p, r) = \begin{cases} \{p\} & \text{if } 0 \leq r \leq 1 \\ X & \text{if } r > 1 \end{cases}$$

$$\overline{B}(p, r) = \begin{cases} \{p\} & \text{if } 0 \leq r < 1 \\ X & \text{if } r \geq 1 \end{cases}$$

- (2) In the Euclidean space  $\mathbb{R}^k$ ,  $p \in \mathbb{R}^k$ ,  $r > 0$

$B(p, r) = \{x \in \mathbb{R}^k \mid \|x - p\| < r\}$  is a "true" open

$\overline{B}(p, r) = \{x \in \mathbb{R}^k \mid \|x - p\| \leq r\}$  is a "true" closed ball

In particular, for  $k = 1$  in  $\mathbb{R}$

$B(p, r) = (p - r, p + r)$  : a symmetric open interval

$\overline{B}(p, r) = [p - r, p + r]$  : a symmetric closed interval

However, w.r.t  $d_1$  &  $d_\infty$ , we have, e.g. in  $\mathbb{R}^2$

$B_1(0, 1) = \{(x, y) \mid |x - 0| + |y - 0| < 1\}$

$B_\infty(0, 1) = \{(x, y) \mid \max\{|x|, |y|\} \leq 1\}$

- (3) In  $\mathbb{C}$ ,  $z \in \mathbb{C}$ ,  $r > 0$

$$D(z, r) = \{w \in \mathbb{C} \mid |w - z| < r\}$$

is called an open disk with center  $z$  and radius  $r$

- (4) What is the open balls in  $S = [0, 1] \subseteq \mathbb{R}$ ?

$$B_S(0, \frac{1}{2}) = \{x \in S \mid |x - 0| < \frac{1}{2}\} = [0, \frac{1}{2}] = B(0, \frac{1}{2}) \cap [0, 1]$$

$$B_S(0, 3) = [0, 1] = B(0, 3) \cap [0, 1]$$

**Prop 2.2** Let  $S$  be a metric subspace of a metric space  $X$ , then  $\forall p \in S$  &  $r > 0$ ,  $B_S(p, r) = B(p, r) \cap S$

*Proof.*  $B_S(p, r) = \{x \in S \mid d(x, p) < r\} = \{x \in X \mid d(x, p) < r\} \cap S$   
 $= B(p, r) \cap S$  ■

### 2.3. Open Sets in Metric Spaces.

We will see that every metric on a set induce a topology on  $X$

**Definition** (interior point). Let  $S \subseteq X$  be a set, and  $p \in S$ , we say that  $p$  is an interior point of  $S$  if  $\exists r > 0, \exists B(p, r) \subseteq S$   
Denote by  $S^\circ$  or  $\text{int}(S)$  by the set of all interior point of  $S$

**Definition** (open). Let  $S \subseteq X$ , we say that  $S$  is open if all points of  $S$  are interior points of  $S$

**Remark.**

(1) Every open set  $S$  is a union of a open balls in  $X$ .

$\because \forall x \in S, x$  is an interior point of  $S, \exists r_x > 0 \ni B(x, r_x) \subseteq S$

$\therefore S = \bigcup_{x \in S} B(x, r_x)$

(2)  $S^\circ \subseteq S$  by definition

(3)  $S$  is open  $\Leftrightarrow S = S^\circ$

#### Prop 2.3

(a)  $S \subseteq T \implies S^\circ \subseteq T^\circ$

$$\because p \in S^\circ \implies \exists r > 0 \ni B(p, r) \subseteq S \subseteq T \implies p \in T^\circ$$

(b) Every open ball  $B(p, r)$  in  $X$  is open

$\because$  Give  $q \in B(p, r)$ , Let  $\delta = r - d(p, q)$ . Claim  $B(q, \delta) \subseteq B(p, r)$  which says  $q$  is an interior point of  $B(p, r)$ . Since  $q \in B(p, r)$  is arbitrary, so  $B(p, r)$  is open. Given  $x \in B(q, \delta)$

$$d(x, p) \leq d(x, q) + d(q, p) < \delta + d(q, p) = r - d(p, q) + d(q, p) = r$$

(c)  $\forall S \subseteq X, S^\circ$  is always open

$\because$  Given  $p \in S^\circ, \exists r > 0 \ni B(p, r) \subseteq S$

$$\implies B(p, r) \subseteq S^\circ \implies p \text{ is a interior point of } S^\circ$$

$\therefore S^\circ$  is open

(d)  $\forall S \subseteq X, S^{\circ\circ} = (S^\circ)^\circ = S^\circ$

$\because$  by definition of open set and (c)

## 2. BASIC POINT SET TOPOLOGY

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Now, let  $T = \{U \subseteq X \mid U \text{ is open in } X\}$

**Prop 2.4**  $T$  is a topology on  $X$ . In particular,  $X$  is a topology space.

*Proof.*

(i)  $\emptyset, X \in \mathcal{T}, \because \emptyset = \emptyset, X^o = X$

(ii)  $U_\alpha \in \mathcal{T}, \alpha \in I$  are open  $\implies \bigcup_{\alpha \in I} U_\alpha$  is open

Given an arbitrary point  $p \in \bigcup_{\alpha \in I} U_\alpha \implies \exists \alpha_0 \in I \ni p \in U_{\alpha_0}$

$U_{\alpha_0}$  is open,  $\exists r > 0 \ni B(p, r) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in I} U_\alpha$

$\because p$  is an interior point of  $\bigcup_{\alpha \in I} U_\alpha \therefore \bigcup_{\alpha \in I} U_\alpha$  is open, i.e.  $\bigcup_{\alpha \in I} U_\alpha \in T$

(iii)  $U_1, \dots, U_n \in T \implies U_1 \cap \dots \cap U_n \in T$

$\because$  Given  $p \in U_1 \cap \dots \cap U_n \implies p \in U_i \ 1 \leq i \leq n$ . Each  $U_i$  is open,

$\exists r_i > 0 \ B(p, r_i) \subseteq U_i, \ 1 \leq i \leq n \implies B(p, r) \subseteq U_1 \cap \dots \cap U_n$

$\implies p$  is an interior point of  $U_1 \cap \dots \cap U_n$

$p$  is arbitrary, so  $U_1 \cap \dots \cap U_n$  is open, i.e.  $U_1 \cap \dots \cap U_n \in T$

Therefore,  $T$  is a topology on  $X$

■

**Definition.** Let  $X$  be a metric space with metric  $d$ . The topology  $T$  in prop 2.4 is called the metric topology(include by  $d$ )

Let  $X$  be a metric space and  $Y \subseteq X$ , Then  $Y$  is a metric subspace of  $X$ , and  $\forall y \in Y, r > 0, B_Y(y, r) = B(y, r) \cap Y$ . In fact, we have more

**Prop 2.5** A subset  $A \subseteq Y$  is open in  $Y \iff A = U \cap Y$  for some open set  $U$  in  $X$ , in particular, the metric topology on  $T$  is just the relation topology of  $Y$  on  $X$

*Proof.*  $(\implies)$  suppose  $A \subseteq Y$  is open in  $Y$ . Then

$$A = \bigcup_{y \in A} B_Y(y, r_y) = \bigcup_{y \in A} (B(y, r_y) \cap Y) = \left( \bigcup_{y \in A} B(y, r_y) \right) \cap Y$$

Let  $U = \bigcup_{y \in Y} B(y, r_y)$ , then  $U$  is open in  $X$  and  $A = U \cap Y$

$(\Leftarrow)$  Suppose  $A = U \cap Y$  where  $U \subseteq X$  is open  $\forall y \in A, y \in U \cap Y \implies y \in U \implies \exists r > 0 \ni B(y, r) \subseteq U \implies B(y, r) \cap Y \subseteq U \cap Y = A \implies B_Y(y, r) \subseteq A, \therefore A$  is open in  $Y$ . ■

**Prop 2.6** Every metric space  $X$  is Hausdorff



*Proof.* Given  $p, q \in X$ ,  $p \neq q$ . Choose  $r = \frac{1}{2}d(p, q) > 0$ . Then  $B(p, r) \cap B(q, r) = \emptyset$ . So  $X$  is Hausdorff  
 $(\because x \in B(p, r) \cap B(q, r) \Rightarrow d(x, p) < r \text{ \& } d(x, q) < r \Rightarrow d(p, q) \leq d(p, x) + d(x, q) < r + r = 2r = d(p, q) \Rightarrow \text{contradiction})$  ■

**Remark.** Let  $S \subseteq X$ , where  $X$  is a metric space. Then  $S^\circ$  is the largest (w.r.t inclusion) open set contained in  $S$ .  $\because \forall$  open set  $U \subseteq S$ ,  $U^\circ \subseteq S^\circ \Rightarrow U \subseteq S^\circ \subseteq S$ . In fact,  $S^\circ = \bigcup_{U \subseteq S} U$  (which is the definition of intension of  $S$  in a topology space  $X$ )

#### 2.4. Closed Sets.

**Definition** (Closed set).  $F \subseteq X$  is closed  $\Leftrightarrow F^C = X - F$  is open in  $X$

By Theorem 2.1, the collection of all closed sets in  $X$  has the properties

- (i)  $\emptyset, X$  are closed in  $X$
- (ii)  $F_\alpha$  is closed in  $X$ ,  $\alpha \in I \Rightarrow \bigcap_{\alpha \in I} F_\alpha$  is closed in  $X$
- (iii)  $F_1, \dots, F_n$  are closed in  $X \Rightarrow \bigcup_{i=1}^n F_i$  is closed in  $X$

#### Example

Intersection of infinitely many open set may not be open

in  $\mathbb{R}$  with Euclidean topology,  $(-\frac{1}{n}, \frac{1}{n})$  is open in  $\mathbb{R} \forall n \geq 1 \Rightarrow \bigcap_{n=1}^\infty (-\frac{1}{n}, \frac{1}{n}) = \{0\}$  is not open

**Prop 2.8** Let  $X$  be a metric space and  $Y \subseteq X$  and  $B \subseteq Y$ , Then  $B$  is closed in  $Y \Leftrightarrow B = F \cap Y$  for some closed set  $F$  in  $X$

*Proof.*  $(\Rightarrow)$  Suppose  $B$  is close in  $Y \Rightarrow Y - B$  is open in  $Y \Rightarrow Y - B = U \cap Y$  (by prop 2.5) for some open set  $U$  in  $X \Rightarrow Y - (Y - B) = Y - (U \cap Y) \Rightarrow B = (X - U) \cap Y$ . where  $(X - U)$  is close.  
 $(\Leftarrow)$  Suppose  $B = F \cap Y$ , where  $F$  is closed in  $X \Rightarrow Y - B = Y - (F \cap Y) = (X - F) \cap Y \Rightarrow Y - B$  is open in  $Y \Rightarrow B$  is close in  $Y$ . ■

In metic space, one can use sequence to detect the closeness of a set

#### Example

1. We know that  $[a, b)$  is not closed in  $\mathbb{R}$ , however,  $\exists$  a sequence  $\{x_n\}$  in  $[a, b) \ni x_n \rightarrow b$  on  $n \rightarrow \infty$ , e.g.  $b - \frac{1}{n} \rightarrow b$

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2.  $A = \{\frac{1}{n} \mid n \geq 1\} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$  is not close in  $\mathbb{R}$   
 if  $\mathbb{R} - A$  will open, then  $\exists r > 0, B(0, r) \subseteq \mathbb{R} - A(\rightarrow \leftarrow)$   
 $A \cup \{0\}$  is closed in  $\mathbb{R}$

$$\mathbb{R} \setminus (A \cup \{0\}) = (-\infty, 0) \cup (1, \infty) \cup \left(\bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n}\right)\right) \text{ is open}$$

$\therefore A \cup \{0\}$  is closed

**Definition** (Adherent, clousure  $\dots$ ). Let  $X$  be a metric space with metric  $d$ ,  $T \subseteq X$  be a subset. (**important**)

(1.) A point  $p \in X$  is said to be an adherent point of  $T$  if  $\forall r > 0, B(p, r) \cap T \neq \emptyset$ , equivalent,  $\forall$  neighborhood  $U$  of  $p$ ,  $U \cap T \neq \emptyset$

(2.) Let  $\overline{T}$  or  $cl(T)$  denote the set of all adherent points of  $T$ , called the closure of  $T$ , i.e.  $\overline{T} = \{p \in X \mid p \text{ is an adherent point of } T\}$

(3.) A point  $p \in X$  is said to be a limit point or accumulation point of  $T$  if  $\forall r > 0, B(p, r) \cap T - \{p\} \neq \emptyset$ , equivalently,  $\forall$  neighborhood  $U$  of  $p$ ,  $U \cap T - \{p\} \neq \emptyset$

Denote by  $T'$  the set of all accumulation points of  $T$ , called the derived set of  $T$ .

(4.)  $p \in T$  and  $p \notin T'$ , then  $p$  is called an isolated point of  $T$ , i.e.  $\exists r > 0 \ni B(p, r) \cap T = \{p\}$

(5.) A subset  $T \subseteq X$  is said to be perfect if  $T$  is closed and every points of  $T$  is an accumulated point of  $T$ , i.e.  $T$  is closed &  $T' = T$

(6.) A subset  $T \subseteq X$  is said to be bounded if  $\exists R > 0$  and  $p \in X \ni T \subseteq B(p, R)$

(7.) A subset  $T \subseteq X$  is said to be dense if  $\overline{T} = X$ , e.g.  $\overline{\mathbb{Q}} = \mathbb{R}$

(8.) A point  $p \in X$  is said to be a boundary point of  $T$  if  $\forall r > 0, B(p, r) \cap T \neq \emptyset$  &  $B(p, r) \cap (X \setminus T) \neq \emptyset$ . Denote by  $\partial T$  or  $bd(T)$  the set of all boundary points of  $T$

**Prop 2.9** Let  $X$  be a metric space. All sets and point below are subset of  $X$

(1).  $S \subseteq T \implies \overline{S} \subseteq \overline{T}$  &  $S' \subseteq T'$

$\therefore p \in \overline{S} \implies \forall r > 0, B(p, r) \cap S \neq \emptyset \implies B(p, r) \cap T \neq \emptyset \implies p \in \overline{T}$   
 $p \in S' \implies \forall r > 0, B(p, r) \cap S - \{p\} \neq \emptyset \implies B(p, r) \cap T - \{p\} \neq \emptyset$

(2).  $\overline{T}$  is always closed in  $X$

We want to know  $\overline{T}$  is closed on  $X \rightarrow X - \overline{T}$  is open  $\rightarrow \forall p \in X - \overline{T}$  is an interior point  $\implies \exists r > 0, B(p, r) \subseteq X - \overline{T}$   
 $\because p \notin \overline{T} \implies \exists r' > 0 \ni B(p, r') \cap T = \emptyset$   
 But we want to get  $B(p, r') \cap \overline{T}$ , so we check every point in  $B(p, r')$  is not in  $\overline{T}$ , let  $q \in B(p, r')$ ,  $\exists \delta > 0, B(q, \delta) \subseteq B(p, r') \implies B(q, \delta) \cap T = \emptyset \implies q \notin \overline{T}$   
 because if  $q \in \overline{T}, \forall r > 0 \ni B(q, r) \cap T \neq \emptyset \implies B(p, r) \cap \overline{T} \neq \emptyset$

Let  $p \in X - \overline{T} \implies p \notin \overline{T} \implies \exists r > 0 \ni B(p, r) \cap T = \emptyset \implies B(p, r) \cap \overline{T} = \emptyset (\because \forall q \in B(p, r), \exists \delta > 0 \ni B(q, \delta) \subseteq B(p, r) \implies B(q, \delta) \cap T = \emptyset \implies q \notin \overline{T})$   
 $\therefore B(p, r) \subseteq X - \overline{T}$ ,  $\because p$  is an interior point of  $X - \overline{T}$ . Hence,  $X - \overline{T}$  is open, i.e.  $\overline{T}$  is closed.

(3).  $T \subseteq \overline{T} (\because \forall p \in T, \forall r > 0, B(p, r) \cap T \neq \emptyset)$

(4).  $p \in T' \implies \forall r > 0, B(p, r) \cap T - \{p\}$  is an infinite set, say  $x_1, \dots, x_n$ , Let  $\delta = \frac{1}{2} \min\{d(p, x_i) \mid 1 \leq i \leq n\}$ . Then  $B(p, \delta) \cap T - \{p\} = \emptyset (\rightarrow \leftarrow)$  to  $p \in T', x \in B(p, \delta) \cap T - \{p\} \implies d(x, p) < \delta \implies x = x_i$  for some  $1 \leq i \leq n$  & we get  $d(x_i, p) < \delta \leq \frac{1}{2}d(x_i, p)$   
 $\therefore$  no such  $x$  i.e.  $B(p, \delta) \cap T - \{p\} = \emptyset$

(5). Any finite subset of  $X$  has no accumulation points in  $X$  by (4). In particular, it is closed by (6)(c) below.

(6). TFAE

- (a)  $S$  is closed
- (b)  $S$  contains all it's adherent point, i.e.  $\overline{S} \subseteq S$
- (c)  $S$  contains all it's accumulation points, i.e.  $S' \subseteq S$
- (d)  $S = \overline{S}$

*Proof.* of (6)

(a)  $\implies$  (b) Suppose  $S$  is closed  $\implies X \setminus S$  is open  $\implies \forall p \in X - S \implies \exists r > 0 \ni B(p, r) \subseteq X \setminus S \implies B(p, r) \cap S = \emptyset \implies p \notin \overline{S}$   
 $\therefore \overline{S} \subseteq S$ , i.e. (b) holds

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(b)  $\Rightarrow$  (c)  $\because S' \subseteq \overline{S}$

(c)  $\Rightarrow$  (d) Suppose  $S' \subseteq S$ . To prove  $S = \overline{S}$  if not, then  $S \subsetneq \overline{S}$ , i.e.  
 $\exists p \in \overline{S} \ \& \ p \notin S \implies \forall r > 0, B(p, r) \cap S \neq \emptyset \ (\because p \in \overline{S})$

(d)  $\Rightarrow$  (a) by (2) ■

**(7).**  $\overline{S}$  is the smallest closed set in  $X$  containing  $S$

$\because$  We know that  $S \subseteq \overline{S}$ , if  $F$  is closed in  $X$  &  $F \subseteq S$ , then  $\overline{F} \subseteq \overline{S}$  by (1),  $F = \overline{F} \subseteq \overline{S}$  by (6),  $\therefore \overline{S}$  is the smallest such one.

**(8).** In fact,  $\overline{S} = \bigcap_{F \subseteq S} F$

**(9).**  $p \in S$  is an isolated point  $\Leftrightarrow \exists r > 0 \ni B(p, r) \cap S = \{p\}$

( $\Rightarrow$ ) Suppose  $p \in S$  is an isolated point of  $S$ . Then  $p \in S' \implies \exists r > 0 \ni B(p, r) \cap S - \{p\} = \emptyset \implies B(p, r) \cap S = \{p\}$

( $\Leftarrow$ ) Trivial

**(10).**  $S$  is dense in  $X \Leftrightarrow \forall p \in X \ \& \ r > 0, B(p, r) \cap S \neq \emptyset \Leftrightarrow \forall$   
open set  $U \neq \emptyset, U \cap S \neq \emptyset$

*Proof.* ( $\Rightarrow$ ) Suppose  $S$  is dense in  $X$ , i.e.  $\overline{S} = X$ , So  $\forall p \in X, p \in \overline{S} \implies \forall r > 0, B(p, r) \cap S \neq \emptyset$

( $\Leftarrow$ ) Suppose the condition holds,  $\forall p \in X \ \& \ r > 0, B(p, r) \cap S \neq \emptyset \implies p \in \overline{S} \implies X \subseteq \overline{S} \subseteq X, \therefore \overline{S} = X$  ■

**(11).**  $\partial S = \partial(X - S)$  In particular,  $\partial S = \overline{S} \cap \overline{(X - S)}$ , In particular,  $\partial S$  is closed in  $X$ ,  $\because$  It suffices to prove  $\partial S = \overline{S} \cap \overline{(X - S)}$ ,

$\because \partial(X - S) = \overline{X - S} \cap \overline{X - (X - S)} = \overline{X - S} \cap \overline{S} = \partial S$

$\forall p \in \partial S \implies \forall r > 0, B(p, r) \cap S \neq \emptyset \ \& \ p \in \overline{X - S} \implies p \in \overline{S} \cap \overline{X - S}$

$\therefore \partial S \subseteq \overline{S} \cap \overline{X - S},$

Conversely,  $p \in \overline{S} \cap \overline{X - S} \implies p \in \overline{S} \ \& \ p \in \overline{X - S} \implies \forall r > 0, B(p, r) \cap S \neq \emptyset \ \& \ B(p, r) \cap (X - S) \neq \emptyset \implies p \in \partial S$

$\therefore \overline{S} \cap \overline{(X - S)} \subseteq \partial S \therefore \partial S = \overline{S} \cap \overline{(X - S)}$

### 2.5. Examples.

We give some simple examples of open sets, closed sets, adherent, accumulation, isolated and boundary points.

1. In a discrete metric space  $X$ , every subset of  $X$  is both open and

$$\text{close, } \forall x \in X, B(p, r) \begin{cases} \{x\} & \text{if } 0 < r \leq 1 \\ X & \text{if } r > 1 \end{cases}$$

$\therefore$  Every singleton is open in  $X$ , so every subset of  $X$  is open.

2. In  $\mathbb{R}$ . Consider the set  $S = [0, 1) \cup \{3\}$ ,  $S^\circ = \emptyset$ ,  $S' = \{0\}$ ,  
 $\overline{S} = S \cup \{0\}$

3. In  $\mathbb{R}$ , consider the set  $S = \{\frac{1}{n} \mid n = 1, 2, \dots\}$ ,  $S^\circ = \emptyset$ ,  
 $S' = \{0\}$ ,  $\overline{S} = S \cup \{0\}$

4. In  $\mathbb{R}^2$ , consider  $S = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ ,  $S$  is open  
 $\overline{S} = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$

$$\partial S = \{(x, 0) \mid x \geq 0\} \cup \{(y, 0) \mid y \geq 0\}$$

5. Let  $B(0, 1)$  be the unit open ball in  $\mathbb{R}^k$ . Then  $\partial B(0, 1) = S^{k-1}$  is the unit  $(k-1)$ -sphere. In particular, for  $k = 2$ ,  $\partial B(0, 1) = S^1$  in the unit circle in the plane  $\mathbb{R}^2$ . Similarly, for the closed unit ball  $\overline{B}(0, 1)$  in  $\mathbb{R}^k$ . Now, we define some special sets in  $\mathbb{R}^n$

- Internals in  $\mathbb{R}$  :  $-\infty < a \leq b < \infty$   
 $[a, b]$  close interval which is closed in  $\mathbb{R}$   
 $(a, b)$  open interval which is closed in  $\mathbb{R}$   
 Infinite intervals:  
 $(-\infty, b]$  : close in  $\mathbb{R}$  ,  $(-\infty, b)$  open in  $\mathbb{R}$   
 whose length are  $\infty$  and boundary

$$\begin{cases} \partial(a, +\infty) = \partial(a, +\infty) = \{a\} \\ \partial(-\infty, b) = \partial(-\infty, b) = \{b\} \end{cases}$$

- $k$ -dimensional interval (rectangle or  $k$ -cell)  $I$

$$I = I_1 \times \dots \times I_k$$

where  $I_j$  is an interval in  $\mathbb{R}$ ,  $1 \leq j \leq k$

(i)  $I$  is bounded  $\Leftrightarrow$  each  $I_j$  is bounded

$I$  is unbounded  $\Leftrightarrow I_j \neq \emptyset$  & some  $I_j$  is unbounded

(ii)  $I = [a_1, b_1] \times \dots \times [a_k, b_k]$ ,  $-\infty < a_j \leq b_j < \infty$ ,  $1 \leq j \leq k$

$k$ -dimensional closed(compact) interval in  $\mathbb{R}^k$

6. Convex sets in  $\mathbb{R}^k$  (In vector space)

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**Definition.**  $S \subseteq \mathbb{R}^k$  is convex if  $\forall x, y \in S$ ,  $\overline{xy}$  is the line segment joining  $x$  &  $y$

Given  $p, q \in \mathbb{R}^k$ , The parametric equation of

$$\begin{aligned}\overleftrightarrow{pq} &= \{p + t(q - p) \mid t \in \mathbb{R}\}: \text{ the line in } \mathbb{R}^k \text{ throw } p \text{ and } q \\ \overrightarrow{pq} &= \{p + t(q - p) \mid t \geq 0\}: \text{ the ray in } \mathbb{R}^k \text{ throw } p \text{ and } q \\ \overline{pq} &= \{p + t(q - p) \mid 0 \leq t \leq 1\}: \text{ the line segment throw } p \text{ and } q\end{aligned}$$

$$\begin{aligned}x \in \overline{pq} \Leftrightarrow x &= p + t(q - p) \text{ for some } t \in [0, 1] \\ &= p + tq - tp \\ &= (1 - t)p + tq \\ &= \alpha p + \beta q, \text{ where } \alpha, \beta \geq 0 \text{ and } \alpha + \beta = 1\end{aligned}$$

and we called  $\alpha p + \beta q$  convex combination of  $p$  and  $q$ .

Note that all open balls, closed balls, intervals are convex in  $\mathbb{R}^k$

Given  $B(p, r)$ ,  $\forall x, y \in B(p, r)$  and  $\alpha, \beta \geq 0, \alpha + \beta = 1$

$$\begin{aligned}\| \alpha x + \beta y - p \| &= \| \alpha x + \beta y - (\alpha + \beta)p \| \\ &= \| \alpha x - \alpha p + \beta y - \beta p \| \\ &\leq \| \alpha x - \alpha p \| + \| \beta y - \beta p \| \\ &= \alpha \| x - p \| + \beta \| y - p \| \\ &< \alpha r + \beta r = (\alpha + \beta)r = r\end{aligned}$$

$\therefore \alpha x + \beta y \in B(p, r) \therefore B(p, r)$  is convex

**7.** Star-like sets in  $\mathbb{R}^k$  with w.r.t some point  $x_0$ ,  $S \subseteq \mathbb{R}^k$  is star-like w.r.t.  $x_0 \in S$  if  $\forall x \in S$ ,  $\overline{x x_0} \subseteq S$

**8.** We know that  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , hence  $\mathbb{Q}^k$  is dense in  $\mathbb{R}^k$ . Note that  $\mathbb{Q}^k$  is countable, hence  $\mathbb{R}^k$  has a countable dense subset  $\mathbb{Q}^k$ , i.e.  $\mathbb{R}^k$  is separable.

**9.**  $\partial \mathbb{Q} = \mathbb{R}$ ,  $\partial \mathbb{Q}^k = \mathbb{R}^k$

**10.**  $\mathbb{Z}$  is closed in  $\mathbb{R}$  and every point in  $\mathbb{Z}$  is isolated e.g.  $\{n\pi \mid n \in \mathbb{Z}\}$ ,  
 $\because \mathbb{R} - \mathbb{Z} = \bigcup_{n=-\infty}^{\infty} (n-1, n)$  is open  $\implies \mathbb{Z}$  is close. or  $\mathbb{Z}' = \emptyset \subseteq \mathbb{Z}$ ,  $\therefore \mathbb{Z}$   
is close.

**11.** Let  $S \subseteq \mathbb{R}$  be a nonempty set which is bounded above. Then  
 $\alpha = \sup S$  exists. Moreover,  $\alpha \in \overline{S}$ .  $\because \forall r > 0, \exists x_0 \in S \ni \alpha - r <$   
 $x_0 \leq \alpha < \alpha + r \implies (\alpha - r, \alpha + r) \cap S \neq \emptyset \implies \alpha \in \overline{S}$

### 2.6. Compact Set in Metric Space.

- Compact sets in metric space, which is closely related to the extreme value problem.
- Compact set  $\mathbb{R}^k$  will be discussed in next section.

**Definition.** Let  $X$  be a topology space and  $S \subseteq X$ . A collection  $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$  of open sets in  $X$  is called an open covering of  $S$  if

$$S \subseteq \bigcup_{\alpha \in I} U_\alpha$$

#### Example

- (1)  $X$  is discrete metric space. Then  $\{\{x\} \mid x \in X\}$  is an open covering of  $X$
- (2) In  $\mathbb{R}$ ,  $\{(0, 1 - \frac{1}{n}) \mid n \in \mathbb{N}\}$  is an open covering of  $(0, 1)$ . In fact,  
 $(0, 1) = \bigcup_{n=1}^{\infty} (0, 1 - \frac{1}{n})$
- (3)  $\{B(0, n) \mid n \in \mathbb{N}\}$  is an open covering of  $\mathbb{R}^k$

**Definition.** Let  $X$  be a topology space,  $S \subseteq X$  and  $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$  be an open covering of  $S$ . We say that  $\mathcal{U}$  has a countable(finite) sub covering of  $S$  if  $\exists$  a countable(finite) sub collection of  $\mathcal{U}$  which also covers  $S$ . i.e.  $\mathcal{U}$  has a countable(finite) subcovering in  $S$  if

$\exists$  a sequence  $\{\alpha_n\}$  in  $I \ni S \subseteq \bigcup_{n=1}^{\infty} U_{\alpha_n}$  (countable)

$\exists$  a sequence  $\{\alpha_n\}$  in  $I \ni S \subseteq U_{\alpha} \cup \dots \cup U_{\alpha_n}$  (finite)

**Definition** (compact). Let  $X$  be a topology space. A subset  $K \subseteq X$  is said to be compact if **every** open covering of  $K$  admit a finite subcovering

#### Examples

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- (1) Let  $X$  be a topology space and  $K \subseteq X$  be a finite set. Then  $K$  is compact.  $\because$  Given an arbitrary open covering  $\{U_\alpha \mid \alpha \in I\}$  of  $K$ , i.e.  $K \subseteq \bigcup_{\alpha \in I} U_\alpha \implies \exists \alpha_i \in I \ni x_i \in U_{\alpha_i}, 1 \leq i \leq n \implies K = \{x_1, \dots, x_n\} \subseteq \bigcup_{i=1}^n U_{\alpha_i} \therefore K$  is compact.
- (2) In a discrete metric space  $X$ , a subset  $K \subseteq X$  is compact  $\Leftrightarrow K$  is a finite set.
- (3)  $(0, 1)$  is not compact in  $\mathbb{R}(\{0, 1 - \frac{1}{n} \mid n \in \mathbb{N}\})$ , but  $[0, 1]$  is compact

**Theorem 2.10.** *Let  $X$  be a metric space and  $K \subseteq Y \subseteq X$ . Then  $K$  is compact in  $X \Leftrightarrow K$  is compact in  $Y$ .*

*Proof.*  $(\Rightarrow)$  Suppose  $K$  is compact in  $X$ . Given an open covering  $\{V_\alpha\}_{\alpha \in I}$  of open sets in  $Y$  which covers  $K$ . By Prop 2.5, each  $V_\alpha = U_\alpha \cap Y$ , where  $U_\alpha$  is open in  $X$ . Now,

$$K \subseteq \bigcup_{\alpha \in I} V_\alpha = \bigcup_{\alpha \in I} (U_\alpha \cap Y) = (\bigcup_{\alpha \in I} U_\alpha) \cap Y \implies K \subseteq \bigcup_{\alpha \in I} U_\alpha$$

By the compactness of  $K$  in  $X$ ,  $\exists \alpha_1, \dots, \alpha_n \in I \ni K \subseteq \bigcup_{i=1}^n U_{\alpha_i} \implies K \cap Y \subseteq (\bigcup_{i=1}^n U_{\alpha_i}) \cap Y \implies K \subseteq \bigcup_{i=1}^n (U_{\alpha_i} \cap Y) = \bigcup_{i=1}^n V_{\alpha_i}$   
 $\therefore K$  is compact in  $Y$

$(\Leftarrow)$  Suppose  $K$  is compact in  $Y$ . Given a open covering  $\{U_\alpha\}_{\alpha \in I}$  of  $K$  by open sets in  $X$ .

$$K \subseteq \bigcup_{\alpha \in I} U_\alpha \implies K \cap Y \subseteq (\bigcup_{\alpha \in I} U_\alpha) \cap Y \implies K \cap Y \subseteq \bigcup_{\alpha \in I} (U_\alpha \cap Y)$$

By Prop 2.5,  $\{U_\alpha \cap Y \mid \alpha \in I\}$  is an open covering of  $K$  by open set in  $Y$ . By assumption,  $K$  is compact in  $Y$ ,  $\exists \alpha_1, \dots, \alpha_n \in I \ni K \subseteq \bigcup_{i=1}^n (U_{\alpha_i} \cap Y) = (\bigcup_{i=1}^n U_{\alpha_i}) \cap Y \implies K \subseteq \bigcup_{i=1}^n U_{\alpha_i}$   
 $\therefore K$  is compact in  $X$  ■

**Definition.** *Let  $X$  be a metric space and  $S \subseteq X$  be a nonempty set. The diameter of  $S$  is defined to be  $\text{dia}(S) = \sup\{d(x, y) \mid x, y \in S\}$  which generated the diameter of a circle in  $\mathbb{R}^2$*

**Recall.** bounded set in a metric space, specially, in  $\mathbb{R}^k$   $X$  : metric space

$$\star S \subseteq X \text{ is bdd if } \exists p \in X \text{ and } r > 0 \ni S \subseteq B(p, r)$$



★ In  $\mathbb{R}^k$ ,  $S \subseteq \mathbb{R}^k$  is bounded of  $\exists M > 0 \ni \|x\| \leq M \forall x \in S$   
 $\because S \subseteq B(p, r) \implies \forall x_0 \in \mathbb{R}^k, \exists R_{x_0} > 0 \ni S \subseteq B(x_0, R_{x_0})$   
 for some  $p \in \mathbb{R}^k$  and  $r > 0$   
 In particular,  $S \subseteq B(0, M) = \{x \in \mathbb{R}^k \mid \|x\| < M\}$  for some  $M > 0$

**Theorem 2.11.** *Let  $X$  be a metric space and  $K \subseteq X$  be a compact set. Then  $K$  is closed and bounded*

*Proof.*  **$K$  is bounded**

Fix a point  $p \in K$ . Then  $K \subseteq \bigcup_{n=1}^{\infty} B(p, n)$ .  $\because K$  is compact  $\implies \exists N \in \mathbb{N} \ni K \subseteq B(p, 1) \cup \dots \cup B(p, N) \implies K \subseteq B(p, N) \therefore K$  is bounded

**$K$  is closed**, i.e.  $X - K$  is open

Fix  $p \in X - K$ . Then  $p \neq x, \forall x \in K$ . Hence,  $d(x, p) > 0, \forall x \in K$   
 Let  $r_x = \frac{1}{2}d(x, p) > 0, x \in K$ . Then  $\{B(x, r_x) \mid x \in K\}$  is an open covering of  $K$ .  $\because K$  is compact  $\implies \exists x_1, \dots, x_n \in K \ni B(x_1, r_{x_1}) \cup \dots \cup B(x_n, r_{x_n})$ . Let  $V = \bigcap_{i=1}^n B(p, r_{x_i}) = B(p, r)$ , where  $r = \min\{r_{x_1}, \dots, r_{x_n}\}$ . Then as we can see that  $V \subseteq X - K$ , all point in  $X - K$  are inner point. So  $X - K$  is open, i.e.  $K$  is close.

To show that  $V \subseteq X - K$ , i.e.  $V \cap K \neq \emptyset$ , it suffices to show

$$V \cap \left( \bigcup_{i=1}^n B(x_i, r_{x_i}) \right) = \emptyset$$

Now,

$$\begin{aligned} V \cap \left( \bigcup_{i=1}^n B(x_i, r_{x_i}) \right) &= \bigcup_{i=1}^n (V \cap B(x_i, r_{x_i})) \\ &\subseteq \bigcup_{i=1}^n (B(p, r_{x_i}) \cap B(x_i, r_{x_i})) = \emptyset \end{aligned}$$

■

**Remark.** *The converse of Thm 2.11 is false, i.e. closed & bounded may not be compact, e.g.  $X$  is an infinite set with discrete metric. Then  $X$  is not compact, but  $X$  is closed and bounded.*

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**Theorem 2.12.** *Let  $X$  be a metric space,  $K \subseteq X$  be compact &  $L \subseteq K$  be a closed set in  $X$ . Then  $L$  is compact.*

*Proof.* Let  $\{U_\alpha\}_{\alpha \in I}$  be an open covering of  $L$ . Then  $\{U_\alpha\}_{\alpha \in I} \cup \{X - L\}$  is an open covering of  $K$ . By the compactness of  $K$ ,  $\exists \alpha_1, \dots, \alpha_n \in I \ni K \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_n} \cup (X - L)$ . By  $L \subseteq K \therefore L$  is compact ■

**Corollary 2.13.**

(a) *Let  $X$  be a metric space,  $K \subseteq X$  be compact and  $F$  be a closed set in  $X$ . Then  $K \cap F$  is compact.*

(b) *If  $X$  is a compact metric space, then every closed subset  $F$  of  $X$  is compact.*

*Proof.*

(a)

$$\begin{aligned} K \text{ is compact} &\implies K \text{ is closed (Thm 2.11)} \\ &\implies K \cap F \text{ is closed in } X \\ &\implies K \cap F \text{ is compact} \end{aligned}$$

(b) follows (a)

■

**Remark.** *Let  $X$  be a metric space. If  $K$  is closed in  $X$  and  $F$  is closed in  $K$ , then  $F$  is closed in  $X$ .  $\therefore F$  is closed in  $F \implies F = L \cap F$ , where  $L$  is closed in  $K \implies F = L \cap K$ , where  $L$  is closed in  $X \implies F$  is closed in  $X$ .*

**Theorem 2.14.** *Let  $X$  be a metric space. Then  $X$  is compact **if and only if**  $\forall$  collection  $\{K_\alpha\}$  of closed subsets of  $X$  with FIP (finite intersection property, i.e.  $\forall \alpha_1, \dots, \alpha_n \in I, K_{\alpha_1} \cap \dots \cap K_{\alpha_n} \neq \emptyset$ ),  $\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$*

*Proof.*

( $\implies$ ) Suppose  $X$  is compact. Given a collection  $\{K_\alpha\}$  of closed subset of  $X$  with FIP. We must prove that  $\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$ . If  $\bigcap_{\alpha \in I} K_\alpha = \emptyset$ , then  $X = X - \bigcap_{\alpha \in I} K_\alpha = \bigcup_{\alpha \in I} (X - K_\alpha) \implies \exists \alpha_1, \dots, \alpha_m \in I \ni X = \bigcup_{i=1}^m (X - K_{\alpha_i}) = X - \bigcap_{i=1}^m K_{\alpha_i} \implies \bigcap_{i=1}^m K_{\alpha_i} = \emptyset (\rightarrow \leftarrow)$  to FIP.

( $\Leftarrow$ ) Suppose the condition holds. To prove that  $X$  is compact. If  $X$  were not compact, then  $\exists$  an open covering  $\{U_\alpha\}_{\alpha \in I}$  of  $X$  which has no finite subcovering. Let  $K_\alpha = X - U_\alpha, \alpha \in I$ . Then  $\{K_\alpha\}_{\alpha \in I}$  is a collection of closed subsets.

Claim it satisfies FIP. Given  $\alpha_1, \dots, \alpha_n \in I$ ,  $\bigcap_{i=1}^n K_i = \bigcap_{i=1}^n (X - U_{\alpha_i}) = X - \bigcup_{i=1}^n U_{\alpha_i}$

$$\begin{aligned} \therefore \bigcap_{i=1}^n K_i = \emptyset &\Leftrightarrow X - \bigcup_{i=1}^n U_{\alpha_i} = \emptyset \Leftrightarrow X = \bigcup_{i=1}^n U_{\alpha_i} (\rightarrow \leftarrow) \\ &\therefore \bigcap_{i=1}^n K_{\alpha_i} = \emptyset \end{aligned}$$

By assumption,  $\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$ , i.e.  $\bigcap_{\alpha \in I} (X - U_\alpha) \neq \emptyset \implies X - \bigcup_{\alpha \in I} U_\alpha \neq \emptyset \implies \bigcup_{\alpha \in I} U_\alpha \neq X (\rightarrow \leftarrow)$   
 $\therefore X$  must be compact. ■

**Corollary 2.15** (& 2.16). *Let  $X$  be a metric space and  $\{K_n\}_{n=1}^\infty$  be a decrease sequence of nonempty compact sets of  $X$ . Then  $\bigcap_{n=1}^\infty K_n \neq \emptyset$ . In addition, if  $\text{dia}_{n \rightarrow \infty} \rightarrow 0$ , then  $\bigcap_{n=1}^\infty K_n$  is a singleton.*

*Proof.*  $\forall j_1, \dots, j_k \in \mathbb{N}$ ,  $K_{j_1} \cap \dots \cap K_{j_k} \neq \emptyset$ ,  $K_{j_1} \cap \dots \cap K_{j_k} = K_t$ , where  $t = \max\{j_1, \dots, j_k\}$ . By Thm 2.14  $\bigcap_{n=1}^\infty K_n \neq \emptyset$ , if  $\lim_{n \rightarrow \infty} \text{dia}(K_n) = 0$  and  $p, q \in \bigcap_{n=1}^\infty K_n$  and  $p \neq q$ , then  $\text{dia}(K_n) \geq d(p, q) \forall n \geq 1 \implies \lim_{n \rightarrow \infty} \text{dia}(K_n) \geq d(p, q) > 0 (\rightarrow \leftarrow) \therefore \bigcap_{n=1}^\infty K_n = \{p\}$  is a singleton. ■

**Remark.** *The usual form of Cor 2.15,  $X$  is a metric space,  $\{K_n\}$  is a decrease sequence of nonempty closed sets in  $X$  with  $K_i$  is compact  $\implies \bigcap_{n=1}^\infty K_n \neq \emptyset$*

**Example** In  $\mathbb{R}$ ,  $\{(0, \frac{1}{n} \mid n \geq 1]\}$  is decrease and every finite subcollection of  $\{(0, \frac{1}{n} \mid n \geq 1]\}$  is nonempty, but  $\bigcap_{n=1}^\infty (0, \frac{1}{n}) = \emptyset$ ,  $\bigcap [0, \frac{1}{n}] = \emptyset$

**Theorem 2.17.** *Let  $X$  be a metric space and  $K \subseteq X$ , TFAE:*

- (i)  $K$  is compact
- (ii) Every infinite subset has an accumulation point in  $K$
- (iii)  $K$  is sequentially compact

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(iv)  $K$  is complete and totally bounded

**Definition** (Convergence).  $\{a_n\}$  converge if  $\exists a \in X \ni \forall \epsilon \geq 0 \exists N \ni \mathbb{N} \ni \forall n \geq N, d(a_n, a) < \epsilon$ . Such  $a$  is called the limit of  $\{a_n\}$ , which is denoted by  $\lim_{n \rightarrow \infty} a_n = a$  or  $a_n \rightarrow a$  on  $n \rightarrow \infty$ .

**Definition** (Cauchy). We say that  $\{a_n\}$  is Cauchy if  $\forall \epsilon > 0, \exists N \in \mathbb{N} \ni \forall n, m \geq N, d(a_n, a_m) < \epsilon$

**Definition.** A metric  $X$  is said to be sequence compact if every sequence has a convergent subsequence

**Definition.** A metric space  $X$  is said to be complete if every Cauchy sequence in  $X$  convergence.

**Definition.** Let  $X$  be a metric space &  $K \subseteq X$ . We say that  $K$  is totally bounded if  $\forall r > 0, \exists x_1, \dots, x_n \in K \ni K \subseteq B(x_1, r) \cup \dots \cup B(x_n, r)$

**Remark.** Totally bounded can implies bounded, but not converse.

$K$  is totally bounded, for  $r = 1, \exists x_1, \dots, x_n \in K \subseteq B(x_1, 1) \cup \dots \cup B(x_n, 1) \implies K \subseteq B(x_1, R)$  for some large  $R$

- Take an "**infinite**" set  $X$  with discrete metric. Then  $X$  is bounded (e.g.  $X \subseteq B(x_0, 2)$ , where  $x_0 \in X$ ) but for  $r = \frac{1}{2}$ ,  $X \not\subseteq B(x_1, \frac{1}{2}) \cup \dots \cup B(x_n, \frac{1}{2}) \forall x_1, \dots, x_n$

**Lemma 2.18** (To prove (ii) to (i)).

Suppose (ii) holds in Thm 2.17. Then  $K$  is totally bounded

*Proof.* If not, then  $\exists r > 0, \ni$  no finite open balls with radius  $r$  and center  $K$  cover  $K$ . Choose  $x_1 \in K \implies K \not\subseteq B(x_1, r) \implies \exists x_2 \in K - B(x_1, r), K \not\subseteq B(x_1, r) \cup B(x_2, r) \implies \exists x_3 \in K - (B(x_1, r) \cup B(x_2, r))$  By induction, counting this process, we obtain an infinite set  $T = \{x_1, x_2, \dots, x_n, \dots\} \subseteq K$  with  $d(x_i, x_j) \geq r \forall i \neq j$ . By (ii),  $T$  has an accumulation point  $p \in K$ . In particular  $B(p, \frac{r}{4}) \cap T - \{p\}$  is an infinite set, hence,  $\exists i \neq j \ni x_i, x_j \in B(p, \frac{r}{4}) \cap T - \{p\} \implies d(x_i, x_j) \leq d(x_i, p) + d(x, j) < \frac{r}{4} + \frac{r}{4} = \frac{r}{2} < r (\rightarrow \leftarrow) \therefore K$  is totally bounded. ■

**Lemma 2.19.** Suppose (ii) holds in  $T$  and  $\{E_\alpha \mid \alpha \in I\}$  is an open covering of  $K$ . Then  $\exists r > 0$  (called a Lebesgue number w.r.t. the open covering  $\{E_\alpha\}_{\alpha \in I}$ )  $\ni \forall x \in K, B(x, r) \subseteq E_\alpha$  for some  $\alpha \in I$

*Proof.* if  $K$  is a finite set, let  $K = \{x_1, \dots, x_n\}$   $K \subseteq \bigcup_{\alpha \in I} E_\alpha \implies x_i \in E_{\alpha_i}$  for some  $\alpha_i \in I$ ,  $1 \leq i \leq n \implies \exists r_i > 0 \ni B(x_i, r_i) \subseteq E_{\alpha_i}$ ,  $1 \leq i \leq n$ . Let  $r = \min\{r_1, \dots, r_n\}$ . Then  $B(x_i, r) \subseteq B(x_i, r_i) \subseteq E_{\alpha_i}$ ,  $1 \leq i \leq n$ . Now, assume that  $K$  is infinite set. Assume that no such  $r > 0$ , i.e.  $\forall r > 0, \exists x_r \in K \ni B(x_r, r) \not\subseteq E_\alpha \forall \alpha \in I$ . Now, for  $r = \frac{1}{k}$ ,  $k = 1, 2, \dots$ , we obtain a sequence  $\{x_k\}$  in  $K$ , with  $x_k = \frac{x_r}{k} \ni B(x_k, \frac{1}{k}) \not\subseteq E_\alpha \forall \alpha \in I$ . Let  $T = \{x_1, x_2, \dots, x_k, \dots\}$ . Then  $T \subseteq K$  is an infinite set ( $\because$  For  $k = 1, r = \frac{1}{1} = 1 \ni x_1 \in K \ni B(x_1, 1) \not\subseteq E_\alpha \forall \alpha \in I$ ). The conclusion of Lemma 2.19 failed for  $K - \{x_1\}$  ( $\because$  if  $\exists s > 0 \ni \forall x \in K - \{x_1\} B(x, s) \subseteq E_\alpha$  for some  $\alpha \in I$ ,  $x_1 \in E_\alpha \implies \exists t > 0 \ni B(x_1, t) \subseteq E_\alpha$ . Let  $r = \min\{s, t\}$ . Then  $\forall x \in K, B(x, r) \subseteq E_\alpha$  for some  $\alpha \in I$  ( $\rightarrow \leftarrow$ )). Then for  $r = \frac{1}{2}$ ,  $\exists x_2 \in K - \{x_1\} \ni B(x_2, \frac{1}{2}) \not\subseteq E_\alpha \forall \alpha \in I$ . Continue this process, we conclude that  $x_i \neq x_j \forall i \neq j$ , so  $T$  is an infinite set. By the assumption of (ii). Then an accumulation point  $p \in K$ . Now  $K = \bigcup_{\alpha \in I} E_\alpha \implies p \in E_\alpha$  for some  $\alpha \in I \implies \exists \epsilon > 0 \ni B(p, \epsilon) \subseteq E_{\alpha_0}$ . Since  $p \in T'$ ,  $B(p, \epsilon) \cap T - \{p\}$  is an infinite set. Choose  $m \gg 0 \ni \frac{1}{m} < \frac{\epsilon}{2}$  &  $x_m \in B(p, \frac{\epsilon}{2}) \cap T$ . Claim  $B(x_m, \frac{1}{m}) \subseteq B(p, \epsilon) \subseteq E_{\alpha_0}$  ( $\rightarrow \leftarrow$ ) to our constrain, hence Lemma 2.19 holds.

$$y \in B(x_m, \frac{1}{m}) \Rightarrow d(y, p) \leq d(y, x_m) + d(x_m, p) < \frac{1}{m} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\Rightarrow y \in B(p, \epsilon) \quad \blacksquare$$

*Proof.* (Thm 2.17 (i)(ii))

(i)  $\Rightarrow$  (ii) Suppose  $K$  is compact. Given an infinite set  $T \subseteq K$ . We must prove that  $T$  has an accumulation point in  $K$ , if not,  $\forall x \in K, x$  is not an accumulation point of  $K$ ,  $\exists r_x > 0 \ni B(x, r_x) \cap T - \{x\} = \emptyset \implies B(x, r_x) \cap T \subseteq \{x\}$ . Clearly  $\{B(x, r_x) \mid x \in K\}$  is an open covering of  $K$ . By (i),  $K$  is compact

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$$\begin{aligned}
\implies \exists x_1, \dots, x_n \in K \ni K &\subseteq B(x_1, r_{x_1}) \cup \dots \cup B(x_n, r_{x_n}) \\
&= (T \cap B(x_1, r_{x_1})) \cup \dots \cup (T \cap B(x_n, r_{x_n})) \\
&\subseteq \{x_1\} \cup \{x_2\} \cup \dots \cup \{x_n\} \\
&= \{x_1\} \cup \{x_2\} \cup \dots \cup \{x_n\} \\
&= \{x_1, \dots, x_n\}(\rightarrow \leftarrow)
\end{aligned}$$

to  $T$  is an infinite set,  $\therefore$  (ii) holds.

(ii)  $\Rightarrow$  (i) In Thm 2.17 i.e. we must prove that  $K$  is compact under the assumption of (ii). Suppose  $\mathcal{U} = \{E_\alpha\}_{\alpha \in I}$  is an open covering of  $K$ . By Lemma 2.19,  $\exists$  a  $r > 0$  w.r.t.  $\mathcal{U}$ , by Lemma 2.18,  $\exists x_1, \dots, x_k \in K \ni K \subseteq B(x_1, r) \cup \dots \cup B(x_k, r) \subseteq E_{\alpha_1} \cup \dots \cup E_{\alpha_k}$ , where  $B(x_i, r) \subseteq E_{\alpha_i}$ ,  $1 \leq i \leq k$ . Therefore,  $\mathcal{U}$  has a finite sub covering. Hence  $K$  is compact and (i) holds.  $\blacksquare$

**Remark.**

- (1) (ii)  $\implies$  (i) is exercise 26
- (2) More or less, by Lemma 2.18 & 19, one can see that (i) and (ii) are also equal to (iii) and (iv)

### 2.7. Compact Sets in Euclidean Spaces $\mathbb{R}^k$ .

- We know that any compact set in a metric space is close and bounded
- Close and bounded subset may not be compact (infinite discrete)
- We will see that every closed and bounded subset of  $\mathbb{R}^k$  is always compact which is the famous H.B. Theorem, i.e.  $K \subseteq \mathbb{R}^k$  is compact  $\Leftrightarrow K$  is closed and bounded

**Theorem 2.20.** *Let  $\{I_n = [a_n, b_n]\}_{n=1}^\infty$  be a sequence of closed and bounded intervals in  $\mathbb{R}$ , if  $\{I_n\}$  is decreasing i.e.  $I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$ , then  $\bigcap_{n=1}^\infty I_n \neq \emptyset$ . Moreover, if  $\lim_{n \rightarrow \infty} (b_n - a_n) = 0$ , then  $\bigcup_{n=1}^\infty I_n$  is a singleton.*

*Proof.* Claim  $T = \{a_n \mid n \in \mathbb{N}\}$  is bounded above and  $x = \sup T$  exists.  
 $\because a_n \leq a_{m+n}$  ( $\because \{a_n\}$  is increasing, i.e.  $[a_1, b_1] \supseteq [a_2, b_2]$ ,  $a_2 \geq a_1$ )  
 $a_n \leq a_{m+n} \leq b_{m+n} \leq b_m$  ( $\because \{b_n\}$  is decreasing)  $\implies T$  is bounded above by all  $b_n \implies x = \sup T$  exists and  $x \leq b_n \forall n \geq 1$ .

Clearly,  $a_n \leq x \forall n \geq 1$ ,  $\because a_n \leq x \leq b_n, \forall n \geq 1$ , i.e.  $x \in [a_n, b_n] \forall n \geq 1 \therefore x \in \bigcap_{n=1}^\infty I_n$ . Hence,  $\bigcap_{n=1}^\infty I_n \neq \emptyset$ . The last statement follows the argument as in Corollary 2.16. ■

**Theorem 2.21.** *Let  $\{I_n = [a_{n,1}, b_{n,1}] \times \cdots \times [a_{n,k}, b_{n,k}]\}$  be a decreasing sequence of closed and bounded intervals in  $\mathbb{R}^k$ . Then  $\bigcap_{n=1}^\infty I_n \neq \emptyset$ . Moreover, if  $\lim_{n \rightarrow \infty} \text{dia}(I_n) = 0$ , then  $\bigcup_{n=1}^\infty I_n$  is a singleton.*

*Proof.*  $\forall 1 \leq j \leq k, \{[a_{n,j}, b_{n,j}]\}$  is a decrease sequence of closed and bounded intervals in  $\mathbb{R}$ . By Thm 2.20,  $\exists x_j \in \bigcap_{n=1}^\infty [a_{n,j}, b_{n,j}]$ . Set  $x = (x_1, \dots, x_k)$ . Then  $x \in \bigcap_{n=1}^\infty I_n$ . Then last statement also follows from the argument in corollary 2.16. ■

**Theorem 2.22.** *Every  $k$ -dimensional closed and bounded interval  $I = [a_1, b_1] \times \cdots \times [a_k, b_k]$  in  $\mathbb{R}^k$  is compact.*

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*Proof.* Put  $\delta = (\sum_{i=1}^k (b_i - a_i)^2)^{\frac{1}{2}}$  which is the diametric of  $I$ . Then  $\forall x, y \in I, \|x - y\| \leq \delta$ . If  $I$  were not compact, then  $\exists$  an open covering  $\{E_\alpha\}_{\alpha \in J}$  admitting not finite sub covering( $\star$ ). Put  $c_j = \frac{a_j + b_j}{2}$ ,  $1 \leq j \leq k$ . The intervals  $[a_j, c_j]$  and  $[c_j, b_j]$ ,  $1 \leq j \leq k$ , determines  $2^k$  closed and bounded subinterval of  $I$  whose union is  $I$ . By ( $\star$ ), at least one of them, say  $I_1$  which cannot be covered by finitely many  $E_\alpha$ . Continuing this process, we get a sequence  $\{I_n\}$  of closed and bounded subintervals of  $I$  satisfy's

- a)  $I \subseteq I_1 \subseteq \dots$ , i.e.  $\{I_n\}_{n=1}^\infty$  is decreasing.
- b) Each  $I_n$  cannot be covered by finitely many  $E_\alpha$
- c)  $\text{dia}(I_n) = 2^{-n}$ ,  $\delta \rightarrow 0$  on  $n \rightarrow \infty$

By Thm 2.21,  $\bigcap_{n=1}^\infty I_n = \{x\}$  i.e.  $x \in I_n \subseteq I \forall n \geq 1 \subseteq \bigcup_{\alpha=1}^\infty E_\alpha$   
 $\because x \in E_{\alpha_0}$  for some  $\alpha_0 \in J$ ,  $E_{\alpha_0}$  is open  $\implies \exists r > 0 \ni B(x, r) \subseteq E_{\alpha_0}$ .  
Choose  $n_0 \gg 0 \ni \frac{1}{2^{n_0}} < \frac{r}{\delta} (\because \frac{1}{2^n} \rightarrow 0)$ . Since  $x \in I_{n_0}, \forall y \in I_{n_0}, \|y - x\| \leq 2^{-n_0} \delta < \frac{r}{\delta} \cdot \delta = r \implies y \in B(x, r), \because I_{n_0} \subseteq B(x, r) \subseteq E_{\alpha_0} (\rightarrow \leftarrow)$ .  
Therefore  $I$  is compact. ■

Combining Thm 2.22 and results in section 2.6, we conclude that the following sets in compact:

- i)  $[a, b]$  is compact in  $\mathbb{R}$  (Thm 2.22 with  $k = 1$ )
- ii)  $[a, b] \times [c, d]$  is compact in  $\mathbb{R}^2$  (Thm 2.22 with  $k = 2$ )
- iii) Every closed ball  $\overline{B}(x, r)$  in  $\mathbb{R}^k$  is compact by Thm 2.12 and 2.22
- iv)  $\{0\} \cup \{\frac{1}{n} \mid n = 1, 2, \dots\}$  is compact in  $\mathbb{R}$ , In fact, if  $a_n \rightarrow a$ , then the set  $\{a\} \cup \{a_n \mid n = 1, 2, \dots\}$  is compact in  $\mathbb{R}$

**Theroem 2.23.** *Every closed and bounded subset  $K$  of  $\mathbb{R}^k$  is compact.*

*Proof.* Choose a large closed and bounded interval  $I$  in  $\mathbb{R}^k$ ,  $K \subseteq I$ . By Thm 2.22,  $I$  is compacted, so  $K$  is a closed subset of  $I$ . By Thm 2.12,  $K$  is compacted.

Combining Thm 2.17 and Thm 2.23, we can characterize compacted set in  $\mathbb{R}^k$  ■



**Theorem 2.24.** *Let  $K \subseteq \mathbb{R}^k$  TFAE:*

- i)  $K$  is closed and bounded*
- ii)  $K$  is compact*
- iii) Every infinite subset of  $K$  has an accumulation point*
- iv)  $K$  is sequence compact*
- v)  $K$  is complete and totally bounded*

From these we can deduce:

**Theorem 2.25** (Bolzano-Weierstrass). *Every bounded infinite subset  $T$  in  $\mathbb{R}^k$  has an accumulation point in  $\mathbb{R}^k$*

*Proof.* Since  $T$  is bounded, choose a large closed and bounded interval  $I$  in  $\mathbb{R}^k \ni T \subseteq I$ . Now,  $T$  become an infinite subset of the compact set  $I$ . By Thm 2.24 (iii),  $T$  has an accumulation point in  $I$ . ■

**Theorem 2.26** (Cantor intersection). *Let  $\{Q_n\}$  be a sequence of nonempty set in  $\mathbb{R}^k$  satisfying:*

- a)  $\{Q_n\}$  is decreasing*
- b)  $Q_n$  is closed  $\forall n \geq 1$  &  $Q_1$  is compact.*

*Then  $\bigcap_{n=1}^{\infty} Q_n \neq \emptyset$ , Moreover, if  $\text{dia}(Q_n) \rightarrow 0$  on  $n \rightarrow \infty$ , then  $\bigcap_{n=1}^{\infty} Q_n$  is a singleton.*

*Proof.* By (b), each  $Q_n$  is compact ( $\because Q_1 \supseteq Q_n \forall n \geq 1$ ). Therefore, it follows from Cor 2.16 and 2.15. ■

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### 2.8. Countability & Separability.

Motivation: In  $\mathbb{R}^k$ , we have two facts:

- $\mathbb{Q}^k$  is dense in  $\mathbb{R}^k$  i.e.  $\overline{\mathbb{Q}^k} = \mathbb{R}^k$  &  $\mathbb{Q}^k$  is countable. i.e.  $\mathbb{R}^k$  has a countable dense subset. i.e.  $\mathbb{R}^k$  is separable.
- $\{B(x, r) \mid x \in \mathbb{Q}^k, r \in \mathbb{Q}^+\}$  is a countable collection of open ball in  $\mathbb{R}^k$  satisfying:  $\forall$  open set  $U \subseteq \mathbb{R}^k$  and  $y \in U \exists B(x, r) \in \mathcal{B} \ni y \in B(x, r) \subseteq U$ . In particular,  $U$  is a union of some sub collection of  $\mathcal{B}$ .  
 $\therefore U = \bigcup_{y \in U} B_y$ , i.e.  $\mathbb{R}^k$  is of  $2^{nd}$  countable.
- $X$  is a metric space  $x \in X$ ,  $N_x = \{B(x, \frac{1}{n}) \mid n \in \mathbb{N}\}$  is a countable collection of nbh of  $x$ . Such  $N_x$  satisfies:  $\forall$  nbh  $U$  of  $x, \exists n \in \mathbb{N} \ni B(x, \frac{1}{n}) \subseteq U$  ( $\because \exists r > 0 \ni B(x, r) \subseteq U$ , choose  $n >> 0 \ni \frac{1}{n} < r$ ). Then  $x \in B(x, \frac{1}{n}) \subseteq B(x, r) \subseteq U$ . i.e. Each point of  $X$  has a countable nbh base(system) i.e.  $X$  is of  $1^{st}$  countable, i.e.  $\mathbb{R}^k$  has a countable base.

**Definition.** Let  $X$  be a topological space

- (1)  $X$  is first if every point of  $X$  has a countable nbh system(or base), i.e.  $\forall p \in X \exists$  a countable collection  $\{V_n \mid n \in \mathbb{N}\}$  of nbh of  $p \ni \forall$  nbh  $U$  of  $p, \exists n \in \mathbb{N}, V_n \subseteq U$
- (2)  $X$  is of second countable if  $X$  has a countable a base, i.e.  $\exists$  a countable collection  $\mathcal{B} = \{B_n \mid n \in \mathbb{N}\}$  of open sets in  $X \ni$  every open set  $U$  is a union of some subcollection of  $\mathcal{B}$  or  $\forall$  open set  $U$  in  $X$  and  $x \in U, \exists n \in \mathbb{N} \ni x \in B_n \subseteq U$
- (3)  $X$  is separable if  $X$  has a countable dense subset, i.e.  $\exists$  a countable set  $D \subseteq X \ni \overline{D} = X$ .

**Remark.**

- (1) Every finite topological space is 1,2 countable and separable.
- (2) Every  $2^{nd}$  countable topology space  $X$  is of  $1^{st}$  countable, but not converse.

$\therefore$  Let  $\mathcal{B} = \{B_1, B_2, \dots\}$  be a countable base for  $X$ . Given  $p \in X$ , let  $\mathcal{B}_p = \{B_n \mid p \in B_n\}$  Then  $\mathcal{B}$ , so  $\mathcal{B}_p$  is countable and is a collection of open set in  $X$  containing of  $p$ .

Claim  $\mathcal{B}_p$  is a nbh system (or base) of  $p$ . Let  $U$  be a nbh of  $p$ . Then  $U = \bigcup_{n \in F} B_n, F \subseteq \mathbb{N}$ . In particular,  $p \in B_n$  for some

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$n \in F$ . Hence,  $B_n \in \mathcal{B}_p$  &  $p \in B_n \subseteq U$ ,  $\therefore \mathcal{B}_p$  is a countable nbh system of  $p$ . Hence,  $X$  is of 1<sup>st</sup> countable.

Consider  $X$  an uncountable set with discrete metric. Hence,  $X$  is 1<sup>st</sup> countable. In fact,  $\forall p \in X$ ,  $N_p = \{\{p\}\}$  is a countable nbh base of  $p$ . However,  $X$  is not of 2<sup>nd</sup> countable. Note that if  $\mathcal{B}$  is a base for the discrete space  $X$  then  $\mathcal{B} \subseteq \{\{x\} \mid x \in X\}$   
 $\because \{x\}$  is open  $\implies \{x\} = \bigcup_{\alpha \in I} B_\alpha$ ,  $B_\alpha = \{x\} \forall \alpha \in I \implies B_\alpha = \{x\} \forall x \in I \implies \{x\} \in \mathcal{B}$ .

Now,  $X$  is uncountable, so is  $\mathcal{B}$ . Hence,  $X$  is not of 2<sup>nd</sup> countable.

- (3) We know that every metric space is of 1<sup>st</sup> countable. In fact,  
 $\forall p \in X$ ,  $N_p = \{B(p, \frac{1}{n}) \mid n \in \mathbb{N}\}$  is a countable nbh system of  $p$
- (4) We know that  $\mathbb{R}$  is separable with countable dense subset  $\mathbb{Q}$ . In general,  $\mathbb{R}^n$  is separable with countable dense subset  $\mathbb{Q}^n$  (Exercise 22)

**Remark.** An equivalent description of base in a topological space. Let  $X$  is a topological space,  $\mathcal{B}$  is a collection of open sets in  $X$ .

$$\begin{aligned} \mathcal{B} \text{ is a base for } X &\Leftrightarrow \forall \text{ open set } U \in X \text{ \& } p \in U \\ &\exists B \in \mathcal{B} \ni p \in B \subseteq U \end{aligned}$$

*Proof.* ( $\Rightarrow$ ) Suppose  $\mathcal{B}$  is a base. Given an open set  $U$  in  $X$  and  $p \in U$ . Since  $\mathcal{B}$  is a base,  $U = \bigcup_{\alpha \in I} B_\alpha$ , for some index set  $I$  and  $B_\alpha \in \mathcal{B}$ ,  $\forall \alpha \in I$ .  $p \in U \implies p \in \bigcup_{\alpha \in I} B_\alpha \implies p \in B_{\alpha_0}$  for some  $\alpha_0 \in I \implies p \in B_{\alpha_0} \subseteq U$  and  $B_{\alpha_0} \in \mathcal{B}$

( $\Leftarrow$ ) Suppose the condition holds. Given an open set  $U$  in  $X$ . By assumption,  $\forall x \in U \exists B_x \in \mathcal{B} \ni x \in B_x \subseteq U$ .  $\therefore U = \bigcup_{x \in U} B_x$ , i.e.  $U$  is a union of  $\{B_x \mid x \in U\} \subseteq \mathcal{B}$ ,  $\therefore \mathcal{B}$  is a base for  $X$ . ■

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**Remark.** Note that  $X$  is a metric space.  $D \subseteq X$ .

$$\begin{aligned} D \text{ is dense in } X &\Leftrightarrow \overline{D} = X \Leftrightarrow \forall \text{ nonempty open set } U \text{ in } X, U \cap D \neq \emptyset \\ &\Leftrightarrow \forall x \in X, \exists \text{ a sequence } \{a_n\} \text{ in } D \\ &\quad \ni a_n \rightarrow x \text{ on } n \rightarrow \infty \end{aligned}$$

**Theorem 2.27.** Every  $2^{\text{nd}}$  countable topology space is separable

*Proof.* Let  $\mathcal{B} = \{B_1, B_2, \dots, B_n, \dots\}$  be a countable base of  $X$ . Choose a point  $x_n \in B_n$ ,  $n \in \mathbb{N}$  and form the set  $D = \{x_1, x_2, \dots, x_n, \dots\}$ , then  $D$  is countable.

Claim:  $D$  is dense in  $X$ , i.e.  $\overline{D} = X$ . Given a nonempty open set  $U$  in  $X$ , then  $\exists n \in \mathbb{N} \ni B_n \subseteq U \implies x_n \in U \implies U \cap D \neq \emptyset$ . Hence,  $\overline{D} = X$  by the remark above. So  $X$  is separable. ■

**Theorem 2.28.** Every separable metric space  $X$  is of  $2^{\text{nd}}$  countable.

*Proof.* Choose a countable dense subset  $D$  of  $X$ . Form the countable collection of open ball  $\mathcal{B} = \{B(x, r) \mid x \in D, r \in \mathbb{Q}^+\}$  (it is countable). Claim  $\mathcal{B}$  is a base for  $X$ . We are done.

(Note that  $\mathcal{B}$  is a base for a topology space  $X$

$$\begin{aligned} &\Leftrightarrow \text{every open set in } X \text{ is a union of some subcollection of } \mathcal{B} \\ &\Leftrightarrow \forall \text{ open set } U \subseteq X \text{ and } p \in U, \exists B \in \mathcal{B} \ni p \in B \subseteq U \end{aligned}$$

( $\implies$ ) Given an open set  $U$  in  $X$  and  $p \in U$  by assumption  $U = \bigcup_{\alpha \in I} B_\alpha$ , where  $B_\alpha \in \mathcal{B} \implies p \in B_{\alpha_0}$  for some  $\alpha_0 \in I \implies p \in B_{\alpha_0} \subseteq U$

( $\Leftarrow$ ) Suppose the condition holds. To prove  $\mathcal{B}$  is a base for  $X$ . Given a nonempty open set  $U$  in  $X$ .  $\forall p \in U, \exists B_p \in \mathcal{B} \ni p \in B_p \subseteq U$ .

$\therefore U = \bigcup_{p \in U} B_p \therefore \mathcal{B}$  is a base for  $X$ . By the remark above, it is enough to show: Given a nonempty open set  $U \subseteq X$  and  $p \in U$ ,  $\exists B(x, r) \in \mathcal{B} \ni p \in B(x, r) \subseteq U$ . Now,  $p \in U$  and  $U$  is open  $\implies \exists t > 0 \ni B(p, t) \subseteq U$ . Choose  $r \in \mathbb{Q}^+ \ni \frac{t}{4} < r < \frac{t}{2}$ , Since  $D$  is dense in  $X$ ,  $B(p, r) \cap D \neq \emptyset$ . Choose  $x \in B(p, r) \cap D$ . Then  $B(x, r) \in \mathcal{B}$

Claim  $p \in B(x, r) \subseteq U$

- $d(x, p) < r \implies p \in B(x, r)$
- $\forall y \in B(x, r), d(y, p) \leq d(y, x) + d(x, p) < r + r = 2r < t \implies y \in B(p, t) \subseteq U \therefore y \in U \therefore B(x, r) \subseteq U.$

■

**Corollary 2.29.** *The Euclidean space  $\mathbb{R}^k$  is of  $2^{nd}$  countable.*

Note that from the proof of Thm 2.28,  $\mathbb{R}^k$  has a countable base of the form:

$$\begin{aligned} \mathcal{B} &= \{B(x, r) \mid x \in \mathbb{Q}^k \text{ \& } r \in \mathbb{Q}^+\} \\ &= \{A_1, A_2, \dots\} \end{aligned}$$

**Remark.**  $\mathbb{R}^k$  is of second countable which has a countable base  $\mathcal{B} = \{B(x, r) \mid x \in \mathbb{Q}^k, r \in \mathbb{Q}^+\}$

**Theorem 2.30.** *Every compact metric space  $X$  is of  $2^{nd}$  countable.*

*Proof.* The last statement follows from Thm 2.27 To prove  $X$  is of  $2^{nd}$  countable. For each  $n \in \mathbb{N}$ ,  $\{B(x, \frac{1}{n}) \mid x \in X\}$  is an open covering of  $X$ , i.e.  $X = \bigcup_{x \in X} B(x, \frac{1}{n})$ . By compactness of  $X$ , it has a finite subcovering say  $X = \bigcup_{i=1}^{l_n} B(x_{n_i}, \frac{1}{n})$ . Then  $\mathcal{B}$  is a countable collection of open balls in  $X$ . Claim  $\mathcal{B}$  is a base for  $X$ . It suffices to show: given a nonempty open set  $U$  and  $p \in U$ ,  $\exists B(x_{n_i}, \frac{1}{n}) \ni \mathcal{B} \ni p \in B(x_{n_i}, \frac{1}{n}) \subseteq U$ . From  $p \in U$  and  $U$  is open,  $\exists r > 0 \ni B(p, r) \subseteq U$ . Choose  $n \gg 0 \ni \frac{2}{n} < r$ . Since  $X = \bigcup_{i=1}^{l_n} B(x_{n_i}, \frac{1}{n})$ ,  $p \in B(x_{n_i}, \frac{1}{n})$  for some  $1 \leq i \leq l_n$ . Finally,  $p \in B(x_{n_i}, \frac{1}{n}) \subseteq U$

- $d(p, x_{n_i}) < \frac{1}{n} \implies p \in B(x_{n_i}, \frac{1}{n})$
- $\forall y \in B(x_{n_i}, \frac{1}{n}), d(y, p) \leq d(y, x_{n_i}) + d(x_{n_i}, p) < \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$

$\therefore p \in B(x_{n_i}, \frac{1}{n}) \subseteq U$

■

**Theorem 2.31** (Lindelof Covering). *Let  $S \subseteq \mathbb{R}^k$ . Then every open covering  $\mathcal{U} = \{U_\alpha \mid \alpha \in I\}$  of  $S$  has a countable subcoverings.*

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*Proof.* Let  $\{a_1, A_2, \dots\}$  be the countable base of  $\mathbb{R}^k$  defined as above. Note that  $S \subseteq \bigcup_{\alpha \in I} U_\alpha$   $\forall x \in S, x \in U_\alpha$  for some  $\alpha \in I$ . Hence  $\exists n \in \mathbb{N} \ni x \in A_n \subseteq U_\alpha$ . Of course, there may be infinitely many such  $n$ . We choose one of them and fix it, say  $x \in A_{m(x)} \subseteq U_\alpha$  (e.g.  $m(x) = \min\{n \in \mathbb{N} \mid x \in A_n \subseteq U_\alpha\}$ ). Then the collection  $\{A_{m(x)} \mid x \in S\}$  is a countable open covering of  $S$ . Finally, for each  $A_{m(x)}$ , choose  $U_{\alpha_{m(x)}} \ni A_{m(x)} \subseteq U_{\alpha_{m(x)}}$ . Then  $\{U_{\alpha_{m(x)}} \mid x \in S\}$  is a countable subcovering of  $S$ . ■

**Corollary 2.32.** *Let  $S \subseteq \mathbb{R}^k$  be open, if  $S = \bigcup_{\alpha \in I} U_\alpha$  is a union of open sets in  $X$ , then  $S = \bigcup_{n=1}^{\infty} U_{\alpha_n}$  is a countable union.  $\therefore$  By Lindelof covering theorem.*

**2.9. Perfect Sets in Metric Spaces.** Recall a subset  $E$  in a metric space  $X$  is perfect if  $E$  is closed in  $X$  and every point of  $E$  is its accumulation point. i.e.  $E' = E$

**Example**

- $-\infty < a < b < \infty$ ,  $[a, b]$  is perfect
- $\mathbb{R}$  is perfect

**Theorem 2.33.** *Every nonempty perfect set  $E$  in  $\mathbb{R}^k$  is uncountable*

*Proof.*  $E$  is an infinite set ( $\because$  finite set has no accumulation point), Suppose  $E$  were countable, write  $E = \{x_1, x_2, \dots\}$ . We use induction to construct a sequence  $\{V_n\}$  of open sets in  $X$  as follows:

Let  $V_1$  be any neighborhood of  $y_1 = x_1$ , e.g.  $V_1 = B(x_1, r)$ , it's closure is  $\overline{V_1} = \overline{B}(x_1, r)$ ,  $x_1 \in E'$ ,  $V_1 \cap E$  is an infinite set, so  $\exists y_2 \in V_1 \cap E \ni y_2 \neq x_1$ . Choose a neighborhood  $V_2$  of  $y_2$   $\ni$

- (i)  $\overline{V_2} \subseteq V_1$
- (ii)  $x_1 \notin \overline{V_2}$
- (iii)  $V_2 \cap E \neq \emptyset$  ( $\because y_2 \in E = E'$  and it's also an infinite set)

Suppose that, for  $n \geq 3$ ,  $V_n$  has been chosen  $\ni V_n$  is a neighborhood of some  $y_n \in E \ni$

- (1)  $\overline{V_n} \subseteq V_{n-1}$
- (2)  $x_{n-1} \notin \overline{V_n}$
- (3)  $V_n \cap E \neq \emptyset$  is an infinite set

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Since  $V_n \cap E$  is an infinite set,  $\exists y_{n+1} \in V_n \cap E \ni y_{n+1} \neq y_n$ . Again, choose a neighborhood  $V_{n+1}$  of  $y_{n+1} \ni$

- (1)  $\overline{V_{n+1}} \subseteq V_n$
- (2)  $x_n \notin \overline{V_{n+1}}$
- (3)  $V_{n+1} \cap E \neq \emptyset$  is an infinite set.

By induction, we have constructed such sequence  $\{V_n\}$ . Put  $K_n = \overline{V_n} \cap E$ ,  $n \geq 1$ . Then  $\{K_n\}$  is a decrease sequence of nonempty compact sets in  $\mathbb{R}^k$ .

$\overline{V_n}$  is closed,  $E$  is closed  $\implies K_n = \overline{V_n} \cap E$  is closed, each  $\overline{V_n}$  is bounded  $\therefore K_n$  is closed and bounded by H.B. theorem,  $K_n$  is compact.

- $\emptyset \neq V_n \cap E \subseteq \overline{V_n} \cap E \implies E_n = \overline{V_n} \cap E \neq \emptyset$
- $\overline{V_n} \cap E \supseteq \overline{V_{n+1}} \cap E = K_{n+1} \therefore \{K_n\}$  is decrease.

By Cantor's intersection theorem,  $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$ . Pick  $y \in \bigcap_{n=1}^{\infty} K_n$ ,  $y \in E (\because K_n \subseteq E \forall n \geq 1)$ . Since  $x_n \notin \overline{V_{n+1}} \forall n \geq 1$ , so  $x_n \notin K_n \forall n \geq 1 \implies y \notin E (\rightarrow \leftarrow)$  to  $E = \{x_1, x_2, \dots\}$   
 $\therefore E$  is uncountable. ■

**Corollary 2.34.** *Every nondegenerate interval is uncountable.*

*Proof.*  $\because$  Every nondegenerate interval  $I$  in  $\mathbb{R}$  must contain a closed and bounded interval  $[a, b]$  with  $a < b$  which is perfect, so it is uncountable by theorem 2.31. Hence  $I$  is uncountable. ■

Construction of the Cantor set  $P \subseteq [0, 1]$  in  $\mathbb{R}$  which is a perfect set

(a) Remove the middle third open subinterval of  $[0, 1]$ . There are two closed subintervals  $[0, \frac{1}{3}]$  and  $[\frac{2}{3}, 1]$ . Let  $C_1 = (\frac{1}{3}, \frac{2}{3})$  and  $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

(b) Remove the middle thirds of  $[0, \frac{1}{3}]$  and  $[\frac{2}{3}, 1]$  respectively. There are  $2^2 = 4$  subintervals  $[0, \frac{1}{3^2}]$ ,  $[\frac{2}{3^2}, \frac{3}{3^2}]$ ,  $[\frac{6}{3^2}, \frac{7}{3^2}]$ ,  $[\frac{8}{3^2}, 1]$

(c) Continue this process, we get a sequence  $\{C_n\}$  of open sets and a sequence  $\{E_n\}$  of closed sets satisfy

- (i)  $E_0 \supseteq E_1 \supseteq E_2 \subseteq \dots$ , i.e.  $\{E_n\}$  is a decrease sequence of closed sets in  $[0, 1]$
- (ii) Each  $E_n$  is a union of  $2^n$  closed intervals, each of length  $3^{-n}$

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- (iii) Each  $C_n$  is a union of  $2^{n-1}$  open subintervals, each of length  $3^{-n}$ .  
total length is  $\frac{2^{n-1}}{3^n}$

**Definition.**  $\underline{P}$  (or  $C$ ) =  $\bigcap_{n=1}^{\infty} E_n = [0, 1] - \bigcup_{n=1}^{\infty} C_n (= \bigcap_{n=1}^{\infty} ([0, 1] - C_n))$

Properties of Cantor set  $\underline{P}$ :

- (1)  $\underline{P} \neq \emptyset$  by Cantor's intersection theorem
- (2)  $\underline{P}$  is compact ( $\because \underline{P}$  is closed being  $\cap$  of closed set and  $\underline{P} \subseteq [0, 1]$ ,  $[0, 1]$  is compact)
- (3)  $\underline{P}$  is nowhere dense, i.e.  $\overline{\underline{P}}^\circ = \emptyset$ , i.e.  $\underline{P}^\circ = \emptyset$   
 $\because \underline{P}$  contains no nonempty open subintervals ( $\because \underline{P}^\circ \neq \emptyset \implies \exists x \in \underline{P}^\circ \implies \exists \delta > 0, (x - \delta, x + \delta) \subseteq \underline{P}$ ). If  $\alpha < \beta$  and  $(\alpha, \beta) \subseteq \underline{P}$ , then  $(\alpha, \beta) \subseteq E_n \forall n \geq 1$ . Choose  $n \gg 0 \ni \frac{1}{2^n} < \beta - \alpha$ . Then for  $n \gg 0$ ,  $E_n$  contains subinterval of length  $\geq \frac{1}{2^n} (\rightarrow \leftarrow)$ . Hence,  $\underline{P}$  is nowhere dense.
- (4)  $\underline{P} = \{\sum_{n=1}^{\infty} \frac{a_n}{3^n} \mid a_n = 0 \text{ or } 2 \forall n \geq 1\}$

Recall the ternary representation of a number  $x \in [0, 1]$ ,  $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ ,  $a_n = 0, 1, 2 \forall n \geq 1$

if  $\frac{1}{3}$  is not in  $\underline{P}$ , then  $\frac{1}{3} + \frac{1}{3}$  is not in  $\underline{P}$   
if you want to represent  $\frac{1}{3}$ , you need use  $0 + \frac{2}{3^2} + \frac{2}{3^3} + \dots$ , i.e.  $0.1 = 0.0\overline{9}$  same things.

This can be used to prove that  $\underline{P}$  is uncountable by  $\underline{P} \rightarrow [0, 1]$ ,  $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \rightarrow \sum_{n=1}^{\infty} \frac{a_n/2}{2^n}$  is bijective  $\therefore \underline{P}$  is uncountable.

- (5)  $\underline{P}$  is of measure zero (i.e. the length of  $\underline{P}$  is zero)  
 $\because$  The totally length remove in the construction of  $\underline{P}$  is  $\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \dots = 1$ . This also proves that  $\underline{P}$  is nowhere dense.
- (6)  $\underline{P}$  is perfect. In particular, by theorem 2.31,  $\underline{P}$  is uncountable  
 $\because$  Obviously  $\underline{P}$  is a nonempty closed set. Let  $x \in \underline{P}$ . Then  $\forall (\alpha, \beta) \ni x \in (\alpha, \beta)$ . We prove that  $(\alpha, \beta) \cap \underline{P} - \{x\} \neq \emptyset$ , which says that  $x$  is an accumulation point of  $\underline{P}$ . Hence  $\underline{P}$  is perfect. By  $x \in \underline{P}$ ,  $x \in E_n \forall n \geq 1$ . Then  $\exists$  a closed subinterval  $I_n \subseteq E_n \ni x \in I_n$ . Choose  $n \gg 0 \ni I_n \subseteq (\alpha, \beta)$ . Let  $x_n$  be an end point of  $I_n \ni x \neq x_n$ . By construction,  $x_n \in \underline{P}$ , so



$$x_n \in (\alpha, \beta) \cap \underline{P} - \{x\}$$

i.e.  $x$  is an accumulation point of  $\underline{P}$ , Hence  $\underline{P}$  is perfect.

**Definition.** Let  $X$  be a metric space (or topological space)  $A, B \subseteq X$ . We say that  $A$  and  $B$  are separated if both  $A \cap \overline{B}$  and  $\overline{A} \cap B$  are empty sets.

**Definition.** A subset  $E \subseteq X$  is called connected if  $E$  is not a union of two nonempty separated sets and  $E$  is disconnected if  $E$  is not connected

**Remark.**  $X$  is connected  $\Leftrightarrow X$  is not a union of two nonempty separated sets.

$X$  is disconnected  $\Leftrightarrow X$  is a union of two nonempty separated sets, say,  $X = A \cup B$ ,  $A$  and  $B$  are nonempty separated. i.e.  $\overline{A} \cap B = \emptyset$  and  $A \cap \overline{B} = \emptyset$ , i.e.  $A = \overline{A}$ ,  $B = \overline{B}$   $\therefore A$  and  $B$  are closed  $\therefore A$  and  $B$  are both open and closed.

### Remarks and Examples

- (1) Separated sets and disjoint
- (2)  $[0, 1]$  and  $(1, 2)$  are not separated
- (3)  $(0, 1)$  and  $(1, 2)$  are separated

**Theorem 2.35.** Let  $E \subseteq \mathbb{R}$  be a set. Then  $E$  is connected  $\Leftrightarrow E$  is an interval

*Proof.* We may assume that  $E \neq \emptyset$

( $\Rightarrow$ ) Assume that  $E$  is connected. If  $E$  were not an interval, then  $\exists x < y$  in  $E$  and  $z \notin E \ni x < z < y$ . Let  $A = (-\infty, z) \cap E$  and  $B = (z, \infty) \cap E$ . Then  $A, B$  are nonempty,  $\overline{A} \cap B = \emptyset$ ,  $A \cap \overline{B} = \emptyset$  and

$$\begin{aligned} E &= E \cap (\mathbb{R} - \{z\}) \\ &= E \cap [(-\infty, z) \cup (z, \infty)] \\ &= (E \cap (-\infty, z)) \cup (E \cap (z, \infty)) \\ &= A \cup B \end{aligned}$$

$\therefore \{A, B\}$  is a nonempty separation of  $E$  ( $\rightarrow \leftarrow$ ) to connected.  $\therefore E$  is an interval

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( $\Leftarrow$ ) Suppose  $E$  is an interval. To show that  $E$  is connected. If not, then  $\exists$  two nonempty separated sets  $A$  and  $B \ni E = A \cup B$ . Pick  $x \in A$  and  $y \in B$ . Then  $x \neq y (\because A \cap B = \emptyset)$ . We may assume that  $x < y$ . Define  $z = \sup(A \cap [x, y])$ , By (9) in section 2.5,  $z \in \overline{A \cap [x, y]} \in \overline{A}$ . Hence  $z \notin B (\because \overline{A} \cap B = \emptyset)$ . Also  $x \leq z \leq y$ . But  $y \in B$  and  $z \notin B \implies x \leq z < y$

If  $z \notin A$ , then  $x < z < y$  and  $z \notin E (\rightarrow \leftarrow)$  to  $E$  is an interval.

If  $z \in A$ , then  $z \notin \overline{B} (\because A \cap \overline{B} = \emptyset)$ , since  $z \notin B$ ,  $z \in \mathbb{R} - \overline{B}$  which is open  $\implies \exists \delta > 0 \ni (z - \delta, z + \delta) \subseteq \mathbb{R} - \overline{B}$ . Choose  $z < z_1 < z + \delta < y$ , i.e.  $z < z_1 < y$ . Then  $z, y \in E$ ,  $z < y$  and  $z_1 \notin E (\rightarrow \leftarrow)$  to  $E$  is an interval ■

### Application of Connectedness

$X$  : connected topological space (or metric space)

$P$  : a property on  $X$

$D = \{x \in X \mid P \text{ holds at } x\}$

If one can prove  $D$  is nonempty and closed and open, then  $D = X$

$\because X = D \cup (X - D)$ ,  $\overline{D} \cap (X - D) = \emptyset$  and  $D \cap \overline{(X - D)} = \emptyset$ , i.e.  $D$  and  $X - D$  are separated

Since  $X$  is connected and  $D \neq \emptyset$ , so  $X - D = \emptyset$ , i.e.  $X = D$

### 3. Infinite Sequence & Series

- We will assume you are familiar with all operations of real(complex) sequence
- We have defined sequence in a set  $X$

Recall : let  $\{a_n\}$  be a real or complex sequence,  $\{a_n\}$  converges if  $\exists a \in \mathbb{R}(\mathbb{C})$  satisfying  $\forall \epsilon > 0, \exists N \in \mathbb{N} \ni \forall n \geq N, |a_n - a| < \epsilon$

- Now, we study the properties of a sequence in a metric space(topological space)

**3.1. Convergent Sequence.** Let  $X$  be a metric space &  $\{x_n\}$  be a sequence in  $X$ ,  $x : \mathbb{N} \rightarrow X$

**Definition.** We say that  $\{x_n\}$  convergences (in  $X$ ) if  $\exists p \in X$  satisfying  $\forall \epsilon > 0, \exists N \in \mathbb{N} \ni \forall n \geq N, d(x_n, p) < \epsilon$ , Otherwise,  $\{x_n\}$  divergences.

**Remark.**

- (1) If  $\{x_n\}$  convergences as in definition, then  $p$  is called the limit of the sequence  $\{x_n\}$  and is denoted by  $\lim_{n \rightarrow \infty} x_n = p$  or  $x_n \rightarrow p$  as  $n \rightarrow \infty$
- (2)  $x_n \rightarrow p$  as  $n \rightarrow \infty \Leftrightarrow$  the real sequence  $\{d(x_n, p)\}$  convergences to 0, i.e.  $\lim_{n \rightarrow \infty} d(x_n, p) = 0$
- (3) if  $\{x_n\}$  convergences, then its limit is !
- (4) The convergence of a sequence depends not only the sequence but also on the space.

e.g.  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$  in  $\mathbb{R}$ , but  $\{\frac{1}{n}\}$  divergences in  $(0, 1)$

**Recall** Let  $\{x_n\}$  are a sequence in a set  $X$  with is a function  $x : \mathbb{N} \rightarrow X$ . The image of the sequence = the image of the function  $X$   
 $= \{x_n \mid n = 1, 2, \dots\}$

**Remark.** The range of a sequence may be finite. e.g.  $\{(-1)^n\}$  in  $\mathbb{R}$ , whose range  $\{-1, 1\}$  is finite, but  $\{\frac{1}{n}\}$  has range  $\{\frac{1}{n} \mid n = 1, 2, \dots\}$

**Definition.** A sequence  $\{x_n\}$  in  $X$  is said to be bounded if its range is a bounded subset of  $X$

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**Remark.** A sequence  $\{x_n\}$  in  $X$  is said to be bounded if its range is a bounded subset of  $X$

**Example**

- (1) Every const sequence  $\{p\}$  in a metric space convergence, i.e.  $\lim_{n \rightarrow \infty} p = p$
- (2)  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$  in  $\mathbb{R}$  and  $\{\frac{1}{n}\}$  is bounded (but the range is finite)
- (3)  $\{(-1)^n\}$  divergences, but  $\{(-1)^n\}$  is bounded. (range is finite).
- (4)  $\{n^2\}$  divergences in  $\mathbb{R}$  and is unbounded. In fact,  $\lim_{n \rightarrow \infty} n^2 = +\infty$  (which range is infinite)
- (5)  $\lim_{n \rightarrow \infty} (1 + \frac{(-1)^n}{n}) = 1$  and  $\{1 + \frac{(-1)^n}{n}\}$  is bounded. (range is infinite)
- (6)  $\{i^n\}$  divergence and it's bounded (range is finite)
- (7) Identify all convergence sequence in a discrete metric space  $X$ .  $\{x_n\}$  convergence to  $p \Leftrightarrow \forall \epsilon > 0 \exists N \in \mathbb{N} \ni \forall n \geq N, d(x_n, p) < \epsilon \Leftrightarrow \{x_n\}$  is almost constant.

In metric space, we can use sequences to characterise adherent and accumulation point

**Theroem 3.1.** Let  $\{x_n\}$  be a sequence in a metric space  $X$  and  $E \subseteq X$

- (a)  $x_n \rightarrow p$  as  $n \rightarrow \infty \Leftrightarrow \forall$  neighborhood  $U$  of  $p, \exists N \in \mathbb{N} \ni \forall n \geq N, x_n \in U$
- (b) If  $\{x_n\}$  convergences, then its limit is !
- (c) If  $\{x_n\}$  convergences, then its range is bounded, but not converse
- (d)  $p \in \bar{E} \Leftrightarrow \exists$  a sequence  $\{a_n\}$  in  $E \ni a_n \rightarrow p$
- (e)  $p \in E' \Leftrightarrow \exists$  a distinct sequence  $(a_n \neq a_m \forall n \neq m)$   $\{a_n\}$  in  $E \ni a_n \rightarrow p$

*Proof.*

(a)

$$\begin{aligned}
 x_n \rightarrow p &\Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N} \ni d(x_n, p) < \epsilon (\forall n \geq N) \\
 &\Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N} \ni x_n \in B(p, \epsilon) (\forall n \geq N) \\
 &\Leftrightarrow \forall \text{ neighborhood } U \text{ of } p, \exists N \in \mathbb{N} \ni \forall n \geq N, x_n \in U
 \end{aligned}$$

- (b) Suppose  $x_n \rightarrow p$ ,  $x_n \rightarrow q$  and  $p \neq q$ , let  $\epsilon = \frac{1}{2}d(p, q)$ . By definition,  $\exists N_1 \ni \forall n \geq N_1, d(x_n, p) < \epsilon$  and  $\exists N_2 \ni \forall n \geq N_2, d(x_n, q) < \epsilon$

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Let  $N = \max\{N_1, N_2\}$  then  $\forall n \geq N$ , above holds. Hence  $d(p, q) \leq d(p, x_N) + d(x_N, q) < \epsilon + \epsilon = 2\epsilon = d(p, q)(\rightarrow \leftarrow) \therefore p = q$

(c) We have seen bounded sequence may not converges. If  $x_n \rightarrow p$ , then for  $\epsilon = 1, \exists N \in \mathbb{N} \ni \forall n \geq N, d(x_n, p) < 1$ , i.e.  $\forall n \geq N, x_n \in B(p, 1)$ . Let  $R = \max\{d(p, x_1), \dots, d(p, x_{n-1})\} + 1$ , Then  $x_n \in B(p, R) \forall n \geq 1 \therefore \{x_n\}$  is bounded

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(d) ( $\Rightarrow$ ) Suppose  $p \in \overline{E}$ . Then  $\forall n \geq 1, B(p, \frac{1}{n}) \cap E \neq \emptyset$ . Choose  $a_n \in B(p, \frac{1}{n}) \cap E, n \geq 1$ . We get a sequence  $\{a_n\}$  in  $E$  and  $0 \leq d(x_n, p) < \frac{1}{n}, \forall n \geq 1$ .

By squeezing lemma,  $\lim_{n \rightarrow \infty} d(a_n, p) = 0$ , i.e.  $a_n \rightarrow p$  as  $n \rightarrow \infty$

( $\Leftarrow$ ) Suppose the conditions holds.  $\forall r > 0, \exists N \in \mathbb{N} \ni \forall n \geq N, d(a_n, p) < r \Rightarrow \forall n \geq N, a_n \in B(p, r) \Rightarrow B(p, r) \cap E \neq \emptyset \therefore p \in E$

(e) It's similar to above.

■

**Theroem 3.2.** For real or complex sequences  $\{x_n\}$  and  $\{y_n\}$ ,  $\lim_{n \rightarrow \infty} x_n = x$ ,  $\lim_{n \rightarrow \infty} y_n = y$ ,  $a, b \in \mathbb{R}$  or  $\mathbb{C}$ ,  $c \in \mathbb{R}$  or  $\mathbb{C}$

(1)  $\lim_{n \rightarrow \infty} c = c$

(2)  $\lim_{n \rightarrow \infty} (ax_n + by_n) = ax + by = a \lim_{n \rightarrow \infty} x_n + b \lim_{n \rightarrow \infty} y_n$

(3)  $\lim_{n \rightarrow \infty} x_n y_n = xy = \lim_{n \rightarrow \infty} x_n \lim_{n \rightarrow \infty} y_n$

(4) If  $y \neq 0, \lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{x}{y} = \frac{\lim_{n \rightarrow \infty} x_n}{\lim_{n \rightarrow \infty} y_n}$

(5) If  $\{z_n\}$  is a complex sequence, then  $z_n \rightarrow z$  as  $n \rightarrow \infty \Leftrightarrow$   
 $\text{Re} z_n \rightarrow z \ \& \ \text{Im} z_n \rightarrow z$  (using  $|\text{Re} w|, |\text{Im} w| \leq |w| \leq |\text{Re} w| + |\text{Im} w| \forall w \in \mathbb{C}$ )

(6) (Squeezing Lemma) If  $\{x_n\}, \{y_n\}$  and  $\{t_n\}$  are real sequence  $\ni$

$$x_n \leq t_n \leq y_n \text{ for } n \gg 0$$

and  $\lim x_n = \lim y_n = l$ , then  $\lim_{n \rightarrow \infty} t_n = l$

(7)  $\lim_{n \rightarrow \infty} x_n = x \Rightarrow \lim_{n \rightarrow \infty} |x_n| = |x|$  (using  $||x_n| - |x|| \leq |x_n - x|$ )

**Examples.**

(i)  $\lim_{n \rightarrow \infty} (1 - \frac{i}{n}) = 1$  ( $\text{Re} (1 - \frac{i}{n}) = 1, \text{Im}(1 - \frac{i}{n}) = \frac{1}{n}$ )

(ii)  $\lim_{n \rightarrow \infty} \frac{1}{n} \sin \frac{1}{n} = 0$

$$\begin{aligned} 0 \leq \left| \frac{1}{n} \sin \frac{1}{n} \right| \leq \frac{1}{n} &\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{1}{n} \sin \frac{1}{n} \right| = 0 \\ &\Rightarrow \left| \lim_{n \rightarrow \infty} \frac{1}{n} \sin \frac{1}{n} \right| = 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \sin \frac{1}{n} = 0 \end{aligned}$$

(iii)  $\{(-1)^n\}$  divergence, but  $|(-1)^n| = 1 \rightarrow 1$

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For sequences in  $\mathbb{R}^k$  including in  $\mathbb{C} \approx \mathbb{R}^2$ , we have.

**Theroem 3.3.** Let  $\{x_n\}$  be a sequence in  $\mathbb{R}^k$ , where

$$x_n = (x_{1,n}, x_{2,n}, \dots, x_{k,n}), n = 1, 2, \dots$$

(a)  $x_n \rightarrow p(p_1, \dots, p_k)$  in  $\mathbb{R}^k \Leftrightarrow x_{i,n} \rightarrow p_i \forall 1 \leq i \leq k$ , i.e.  $\lim_{n \rightarrow \infty} (x_{1,n}, \dots, x_{k,n}) = (\lim_{n \rightarrow \infty} x_{1,n}, \dots, \lim_{n \rightarrow \infty} x_{k,n})$  if exists

(b) Let  $\{x_n\}, \{y_n\}$  be sequences in  $\mathbb{R}^k$  and  $\{d_n\}$  be a sequence in  $\mathbb{C}$  and  $a, b \in \mathbb{R}$ . If  $x_n \rightarrow x, y_n \rightarrow y$  and  $d_n \rightarrow d$ , then

- $ax_n + by_n \rightarrow ax + by$
- $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$
- $d_n x_n \rightarrow d x$
- $\|x_n\| \rightarrow \|x\|$

If  $k = 3$ , then  $x_n \times y_n \rightarrow x \times y$

*Proof.*

(a) It follows from the inequation  $\forall y \in \mathbb{R}^k |y_i| \leq \|y\| \leq \sum_{i=1}^k |y_i|$   
 $\therefore \forall 1 \leq i \leq k |x_{i,n} - p_i| \leq \|x_n - p\| \leq \sum_{i=1}^k |x_{i,n} - p_i| \forall n \geq 1$   
 $(\Rightarrow)$

$$\begin{aligned} \text{Suppose } x_n \rightarrow p &\implies \|x_n - p\| \rightarrow 0 \\ &\implies \forall 1 \leq i \leq k, |x_{i,n} - p_i| \rightarrow 0 \forall 1 \leq k \leq n \\ &\implies \forall 1 \leq i \leq k, x_{i,n} \rightarrow p_i \text{ as } n \rightarrow \infty \end{aligned}$$

$(\Leftarrow)$

$$\begin{aligned} \text{Suppose } x_n \rightarrow p, 1 \leq i \leq k &\implies \|x_{i,n} - p_i\| \rightarrow 0 \forall 1 \leq k \leq n \\ &\implies \sum_{i=1}^k |x_{i,n} - p_i| \rightarrow 0 \\ &\implies \|x_n - p\| \rightarrow 0 \implies x_n \rightarrow p \end{aligned}$$

(b) By (a)

$$\begin{aligned} ax_n + by_n &= (ax_{1,n}, ax_{2,n}, \dots, ax_{k,n}) + (by_{1,n}, \dots, by_{k,n}) \\ &= (ax_{1,n} + by_{1,n}, \dots, ax_{k,n} + by_{k,n}) \\ &\rightarrow (ax_1 + by_1, \dots, ax_k + by_k) = ax + by \end{aligned}$$



- $\langle x_n, y_n \rangle = \sum_{i=1}^k x_{i,n} y_{i,n} \rightarrow \sum_{i=1}^n x_i y_i = \langle x, y \rangle$
- $d_n x_n = (d_n x_{1,n}, \dots, d_n x_{k,n}) \rightarrow (dx_1, \dots, dx_k) = dx$   
 $\|x_n\| = (\sum_{i=1}^k x_{i,n}^2)^{\frac{1}{2}} \implies (\sum_{i=1}^k x_i^2)^{\frac{1}{2}} = \|x\|$
- $x_n \times y_n = (x_{2,n} y_{3,n} - x_{3,n} y_{2,n}, \dots) \rightarrow (x_2 y_3 - x_3 y_2, \dots) = x \times y$

■

### 3.2. Subsequences.

#### Theroem 3.4.

- (a) If  $\{x_n\}$  converges to  $p$ , i.e.  $\lim_{n \rightarrow \infty} x_n = p$ , then so is every subsequence of  $\{x_n\}$
- (b) If  $X$  is compact and  $\{x_n\}$  is a sequence in  $X$ , then  $\{x_n\}$  has a convergent subsequence.
- (c) Every bounded sequence  $\{x_n\}$  in  $\mathbb{R}^k$  has a converge subsequence.

*Proof.* (a) Given a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  (Note that  $\{n_k\}$  is strictly increasing, i.e.  $n_1 < n_2 < \dots$ , hence,  $k \leq n_k \forall k \geq 1$ ),  $d(x_{n_k}, p) < \epsilon$ . This proves  $x_{n_k} \rightarrow p$  as  $k \rightarrow \infty$

(b) Let  $T = \{x_n \mid n \geq 1\}$  be the range of  $\{x_n\}$

Case 1:  $T$  is a finite set. In this case, some  $x_{n_0}$  must appear infinitely many times in the sequence  $\{x_n\}$ . Choose  $n_1 = n_0 \ni x_{n_1} = x_{n_0}$ , and  $n_2 > n_1 \ni x_{n_2} = x_{n_0}, \dots$ . In this way, we get a const subsequence  $\{x_{n_k}\}$  which convergence to  $x_{n_0}$

Case 2:  $T$  is an infinite set. In this case,  $T$  is an infinite subset of the compact metric space  $X$ . By Thm 2.17 (ii),  $T$  has an accumulation point  $p$  in  $X$ . By Thm 3.1 (e),  $\exists$  a sequence in  $T$  which converges to  $p$ . We may arrange such sequence to be a subsequence of  $\{x_n\}$ . We are done

$y_1 = x_{n_1}$ , choose  $n_2 > n_1 \ni x_{n_2}$  appears in  $\{y_j\}$ . Then  $\{x_{n_k}\}$  is a subsequence of  $\{x_n\}$  and  $\{y_j\}$ ,  $\therefore x_{n_k} \rightarrow p$

- (c)  $\therefore$  Since  $\{x_n\}$  is bounded, we may choose a closed ball  $\overline{B}(0, R)$  or a closed  $n$ -dimensional interval in  $\mathbb{R}^k \ni \{x_n\}$  is a sequence in  $K$ , By (b),  $\{x_n\}$  has a convergence subsequence .

■

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**Remark.** Thm 3.4(a) can be used to detect the divergence of a sequence, e.g.  $\{(-1)^n\}$  in  $\mathbb{R}$  which diverges,  $\because$  It has two subsequences

$$\begin{cases} x_{2n} \rightarrow 1 \\ x_{2n-1} \rightarrow -1 \end{cases} \quad \text{which is different.}$$

**Definition.** Let  $\{x_n\}$  be a sequence in  $X$ . A point  $p \in X$  is called a subsequential limit of  $\{x_n\}$  if  $\exists$  a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$   $\ni x_{n_k} \rightarrow p$  as  $k \rightarrow \infty$

#### Examples

- (1) If  $\{x_n\}$  converges to  $p$ , then  $\{x_n\}$  has only one subsequence limit  $p$  ( $E = \{p\}$ )
- (2)  $\{(-1)^n\}$  has two subsequence limits 1 and  $-1$ ,  $E = \{1, -1\}$
- (3)  $\{n\}$  has no subsequence limit ( $E = \emptyset$ )

Let  $\{x_n\}$  be a sequence in  $X$  and  $E$  be the set of all subsequence limits of  $\{x_n\}$

**Theorem 3.5.** As above,  $E$  is a closed subset of  $X$

*Proof.* If  $E$  is a finite set, then  $E$  is closed.

Now, assume that  $E$  is an infinite set, to show that  $E$  is closed, we must prove  $E' \subseteq E$ , i.e.  $E$  contains all its accumulation points. Given  $q \in E'$ , to prove  $q \in E$ , i.e.  $\exists$  a subsequence  $\{x_{n_k}\}$   $\ni x_{n_k} \rightarrow q$ . Since  $E$  is infinite,  $\{x_n\}$  is not a constant sequence, so we can choose  $x_{n_1} \neq q$ . Let  $\delta = d(x_{n_1}, q)$ , We construct subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  satisfying :  $d(x_{n_k}, q) \leq \frac{\delta}{2^{k-1}} \forall k \geq 1$ . If it is done, then by squeezing lemma,  $d(x, q) \rightarrow 0$  as  $k \rightarrow \infty$ .

Now, to construct such subsequence  $\{x_{n_k}\}$ . By induction, suppose  $k = 1$  we are done, we have found  $n_1 < n_2 < \dots < n_{k-1}$ ,  $k \geq 2$ . To find  $x_{n_k}$ . Since  $q \in E'$ ,  $B(q, \frac{\delta}{2^k}) \cap E - \{q\} \neq \emptyset$ . Choose  $x \in B(q, \frac{\delta}{2^k}) \cap E - \{q\}$ . Now,  $x \in E$ ,  $\exists$  a subsequence of  $\{x_n\}$  which converges to  $x$ . Hence  $\exists n_k > n_{k-1}$   $\ni d(x_{n_k}, x) < \frac{\delta}{2^k}$ . Finally,  $d(x_{n_k}, q) \leq d(x_{n_k}, x) + d(x, q) < \frac{\delta}{2^k} + \frac{\delta}{2^k} = \frac{\delta}{2^{k-1}} \therefore$  By induction, such subsequence  $\{x_{n_k}\}$  can be found. ■

#### 3.3. Cauchy Sequences.

**Recall**  $x_n \rightarrow p \implies \forall \epsilon > 0 \exists N \in \mathbb{N} \ni \forall n \geq N, d(x_n, p) < \frac{\epsilon}{2}$ .

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$$\therefore \forall m, n \geq N, d(x_m, x_n) \leq d(x_m, p) + d(p, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

**Definition.** A sequence  $\{x_n\}$  in  $X$  is called a Cauchy sequence if it satisfies the Cauchy condition:  $\forall \epsilon > 0, \exists N \in \mathbb{N} \ni \forall n, m \geq N, d(x_n, x_m) < \epsilon$

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**Remark.**

(i) Convergence sequence is Cauchy

(ii) Cauchy sequence may not convergence, e.g. in  $(0, 1)$   $\{\frac{1}{n}\}$  is Cauchy, but not convergence in  $(0, 1)$

$$\begin{aligned} \forall n, m \in \mathbb{N}, m \geq n, \quad & |\frac{1}{n} - \frac{1}{m}| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{n} \\ \forall \epsilon > 0, \text{ Choose } N \in \mathbb{N} \ni & \frac{2}{N} < \epsilon. \\ \text{Them } \forall n, m \geq N, \quad & |\frac{1}{n} - \frac{1}{m}| \leq \frac{2}{N} < \epsilon \therefore \{\frac{1}{n}\} \text{ is Cauchy.} \end{aligned}$$

(iii)  $\{x_n\}$  is Cauchy  $\Leftrightarrow \lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0$

(iv) Let  $E_n = \{x_n, x_{n+1}, \dots\} n \geq 1$ . Them  $\{x_n\}$  is Cauchy  $\Leftrightarrow \lim_{n \rightarrow \infty} \text{dia}(E_n) = 0$

Recall the definition of diameter:

Let  $S \subseteq V, S \neq \emptyset$ . The diameter of  $S$  is  $\text{dia}(S) = \sup\{d(x, y) \mid x, y \in S\}$   
Let

*Proof.* ( $\Rightarrow$ ) Suppose  $\{x_n\}$  is Cauchy. Then  $\forall \epsilon > 0, \exists N \in \mathbb{N} \ni \forall n, m \geq N, d(x_n, x_m) < \frac{\epsilon}{2}$ . Then  $\forall n \geq N, \text{dia}(E_n) \leq \frac{\epsilon}{2} < \epsilon \implies \lim_{n \rightarrow \infty} \text{dia}(E_n) = 0$

( $\Leftarrow$ ) Suppose  $\lim_{n \rightarrow \infty} \text{dia}(E_n) = 0, \forall \epsilon > 0, \exists N \in \mathbb{N} \ni \forall n \geq N, \text{dia}(E_n) < \epsilon \implies \forall n, m \geq N, (x_n, x_m \in E_n), d(x_n, x_m) \leq \text{dia}(E_n) < \epsilon$   
 $\therefore \{x_n\}$  is Cauchy. ■

**Remark.** Every Cauchy seq  $\{x_n\}$  in a metric space is bounded. For  $\epsilon = 1, \exists N \in \mathbb{N} \ni \forall m, n \geq N, d(x_n, x_m) < 1$ , In particular,  $\forall n \geq N, d(x_n, x_N) < 1$ . Let  $R = \max\{d(x_i, x_N) \mid 1 \leq i \leq N-1\} + 1$ . Then  $x_n \in B(x_N, R) \forall n \geq 1$ , i.e.  $\{x_n \mid n \geq 1\} \subseteq B(x_N, R)$ . Hence  $\{x_n\}$  is bounded.

**Theorem 3.6.** (a) Every Cauchy sequence in a compact metric space converges.

(b) Every Cauchy sequence in  $\mathbb{R}^k$  converges.

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*Proof.* (a) Let  $\{x_n\}$  be a Cauchy sequence in compact metric space  $X$ . Since  $X$  is compact,  $X$  is sequentially compact, so  $\{x_n\}$  has a subsequence  $\{x_{n_k}\} \ni x_{n_k} \rightarrow p$  as  $k \rightarrow \infty$  for some  $p \in X$ .

Now to prove  $x_n \rightarrow p$ . Given  $\epsilon > 0$ ,  $\exists N \in \mathbb{N} \ni \forall n, m > N, d(x_n, x_m) < \frac{\epsilon}{2}$  ( $\because \{x_n\}$  is Cauchy),  $\exists k_0 \in \mathbb{N} \ni \forall k \geq k_0, d(x_{n_k}, p) < \frac{\epsilon}{2}$  ( $\because x_{n_k} \rightarrow p$ ).

Hence,  $\forall n \geq N, d(x_n, p) \leq d(x_n, x_{n_k}) + d(x_{n_k}, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$ , where  $k \gg 0$

Another proof of (a)

By (2)&(4) above,

$$\lim_{n \rightarrow \infty} \text{dia}(\overline{E}_n) = \lim_{m \rightarrow \infty} \text{dia}(E_n) = 0$$

,

where  $E = \{x_n, x_{n+1}, \dots\}, n \geq 1$

Now, each  $\overline{E}_n$  is compact ( $\because$  closed subset of compact set  $X$ ) and nonempty  $\forall n \geq 1$ , and  $\{\overline{E}_n\}$  is decreasing ( $\because E_n \supseteq E_{n+1} \implies \overline{E}_n \supseteq \overline{E}_{n+1}$ ) and  $\text{dia}(\overline{E}_n) \rightarrow 0$ . By Cantor's Intersection Theorem,  $\bigcap_{n=1}^{\infty} \overline{E}_n = \{p\}$  Claim:  $x_n \rightarrow p$  as  $n \rightarrow \infty$ ,  $\forall \epsilon > 0, \exists N \in \mathbb{N} \rightarrow \text{dia}(\overline{E}_n) < \epsilon$ . Since  $p \in \overline{E}_n \forall n \geq 1, \forall n \geq N \& d(x_n, p) < \text{dia}(\overline{E}_n) < \epsilon, \therefore x_n \rightarrow p$

(b) Given a Cauchy sequence  $\{x_n\}$  in  $\mathbb{R}^k$ . Then  $\{x_n\}$  is bounded, choose a large  $k$ -dimensional closed interval

$$I = [a_1, b_1] \times \dots \times [a_n, b_n]$$

$\ni x_n \in I \forall n \geq 1$ . Now, H.B. Theorem says that  $I$  is compact. Therefore,  $\{x_n\}$  becomes a Cauchy sequence in the compact metric space  $I$ . By (a)  $x_n \rightarrow p$  for some  $p \in I$ . This proves (b). ■

**Definition.** A metric space  $X$  is said to be complete if every Cauchy sequence in  $X$  converges.

#### Rmks and Examples

(1) In a complete metric space  $X$ , a sequence  $\{x_n\}$  is Cauchy  $\Leftrightarrow$  it converges.

(2) By Thm. 2.6, we have two classes of complete metric spaces

- Compact metric space

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- Euclidean space  $\mathbb{R}^k$

In fact,  $\mathbb{R}^k$  is a Banach Space (complete normed linear space) and Hilbert space

(3) A closed subset  $S$  of a complete metric space  $X$  is complete.

$\therefore$  Let  $\{x_n\}$  be a Cauchy sequence in  $S$ . Then  $\{x_n\}$  is a Cauchy sequence in  $X$ , hence,  $x_n \rightarrow p$  for some  $p \in X$ . So  $p \in \overline{S} = S$ . Hence,  $S$  is complete.

(4) Every closed subset of  $\mathbb{R}^k$  is complete.

$\therefore$  (3), In particular, every closed interval and closed ball in  $\mathbb{R}^k$ ,

(5)  $(0, 1)$  and  $\mathbb{Q}$  are not complete

**Definition.** Let  $\{x_n\}$  be a real sequence

- (i) We say that  $\{x_n\}$  is increasing, if  $x_n \leq x_{n+1} \forall n \geq 1$
- (ii) We say that  $\{x_n\}$  is strictly increasing, if  $x_n < x_{n+1} \forall n \geq 1$
- (iii) We say that  $\{x_n\}$  is decreasing, if  $x_n \geq x_{n+1} \forall n \geq 1$
- (iv) We say that  $\{x_n\}$  is strictly decreasing, if  $x_n > x_{n+1} \forall n \geq 1$
- (v) We say that  $\{x_n\}$  is monotonic if either  $\{x_n\}$  is increasing or decreasing.
- (vi) We say that  $\{x_n\}$  is strictly monotonic if either  $\{x_n\}$  is strictly increasing or strictly decreasing

#### Examples

- $\{2n + 1\}$  is increasing,  $2n + 1 \rightarrow +\infty$
- $\{-n\}$  is decreasing,  $-n \rightarrow -\infty$
- $\{\frac{1}{n}\}$  is decreasing,  $\frac{1}{n} \rightarrow 0$
- $\{\frac{1}{-n}\}$  is increasing,  $\frac{1}{-n} \rightarrow 0$

We will show that every monotonic sequence converges in  $\mathbb{R}^* = [-\infty, \infty]$

**Theorem 3.7.** Let  $\{a_n\}$  be a sequence

- (a) Let  $\{a_n\}$  be increasing
  - (i) If  $\{a_n\}$  is bounded above, then  $\{a_n\}$  converges, in fact,  $a_n \rightarrow \sup a_n = \sup\{a_n \mid n \geq 1\}$
  - (ii) If  $\{a_n\}$  is not bounded above, then  $a_n \rightarrow \infty$
- (b) Let  $\{a_n\}$  be decreasing

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- (a) If  $\{a_n\}$  is bounded below, then  $\{a_n\}$  converges, in fact  $a_n \rightarrow \inf a_n = \inf\{a_n \mid n \geq 1\}$   
 (b) If  $\{a_n\}$  is not bounded below, then  $\{a_n\} \rightarrow -\infty$

**Remark.**  $\{a_n\}$  is increasing  $\Leftrightarrow \{-a_n\}$  is decreasing. So, to study monotonic sequence, it suffices to consider the case of increasing sequence.

*Proof.* It suffices to prove (a) By same argument or considering  $\{-a_n\}$ , one can prove

- (a) (i)  $\{a_n\}$  is bounded above  $\implies \{a_n \mid n \geq 1\}$  is bounded  $\implies \alpha = \sup a_n$  exists and is finite

Claim:  $a_n \rightarrow \alpha$  as  $n \rightarrow \infty$ .

Given  $\epsilon > 0 \exists n_0 \in \mathbb{N} \rightarrow \alpha - \epsilon < a_{n_0}$ . Then  $\forall n \geq n_0$ , we have  $\alpha - \epsilon < a_{n_0} \leq a_n \leq \alpha < \alpha + \epsilon$ , i.e.  $\forall n \geq n_0$ ,  $|a_n - \alpha| < \epsilon$ . This proves  $a_n \rightarrow \alpha$

- (ii)  $\forall M > 0$ , since  $\{a_n\}$  is not bounded above,  $\exists n_0 \in \mathbb{N} \rightarrow a_{n_0} \geq M \implies \forall n \geq n_0, a_n \geq a_{n_0} \geq M$ . This proves  $a_n \rightarrow +\infty$

(b) is similar

- (i')  $\{a_n\}$  is bounded below  $\implies \{-a_n\}$  is bounded above  $\implies \lim_{n \rightarrow \infty} (-a_n) = \sup(-a_n) \implies -\lim a_n = -\inf a_n$   
 (ii') Similar

■

**Remark.** Let  $\{a_n\}$  be a monotonic sequence, then  $\{a_n\}$  converges  $\Leftrightarrow \{a_n\}$  is bounded.

#### **Theroem 3.8.**

- (1) Every Cauchy sequence  $\{x_n\}$  in compact metric space  $X$  is convergent.  
 (2) In  $\mathbb{R}^k$ , every Cauchy sequence is convergent.

*Proof.*

- (1) Let  $E_n = \{x_n, x_{n+1}, \dots\}$   $n \geq 1$ , by (2) and (4)  $\lim_{n \rightarrow \infty} \text{diam}(\bar{E}_n) = \lim_{n \rightarrow \infty} \text{diam}(E_n) = 0 \dots (*)$ . Clearly,  $\{E_n\}$  is decreasing so is  $\{\bar{E}_n\}$ .  $\bar{E}_n = \{x_n, x_{n+1}\} \supseteq \{x_{n+1}, \dots\} = \bar{E}_{n+1}$ . In fact,

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$\{\bar{E}_n\}$  is a decreasing sequence of nonempty compact sets in  $X$ . Hence, by Cantor intersection theorem,  $\bigcap_{n=1}^{\infty} E_n \neq \emptyset$ . In fact,  $\bigcap_{n=1}^{\infty} \bar{E}_n = \{p\}$  by cor 2.16.

**Claim :**  $x_n \rightarrow p$ .  $\forall \varepsilon > 0$ , by  $(\star)$ ,  $\exists N \in \mathbb{N}$  s.t.  $\text{diam}(\bar{E}_n) - 0 < \varepsilon \therefore x_n \rightarrow p$  as  $n \rightarrow \infty$ .

- (2) Since Cauchy sequence is bounded,  $\exists R \gg 0$  s.t.  $x_n \in \bar{B}(0, R)$ , i.e.  $\{x_n\}$  is a Cauchy sequence in the compact set  $\bar{B}(0, R)$ . Hence  $\{x_n\}$  converges in  $\bar{B}(0, R)$ .

■

**Definition.** A metric space  $X$  is said to be complete if every Cauchy sequence in  $X$  is convergent.

#### Remarks and Examples :

- (1) In a complete metric space, to show that a sequence is convergent, it is enough to show that it is Cauchy.
- (2) Thm 3.8 says compact metric space is complete.  $\mathbb{R}^k$  is complete. Discrete metric space is complete.  $X$  be a discrete metric space,  $\{x_n\}$  is convergent  $\Leftrightarrow \{x_n\}$  is almost constant, i.e.  $\exists n_0 \in \mathbb{N}$  s.t.  $\forall n \geq n_0, x_n = x_{n_0}$ .  $\{x_n\}$  is Cauchy  $\Leftrightarrow \forall 0 < \varepsilon < 1, \exists N \in \mathbb{N}$  s.t.  $\forall m, n \geq N, d(x_m, x_n) < \varepsilon < 1 \Leftrightarrow \forall m, n \geq N, x_m = x_n = \dots = x_N$ .
- (3) Complete normed linear space over  $(\mathcal{F})$  is called a Banach space, so  $\mathbb{R}^k$  is a Banach space.
- (4) A Hilbert space is a complete inner product space over  $(\mathcal{F})$ .
- (5) Closed subset  $F$  of a complete metric space  $X$  is complete. ( $\because$  Given a Cauchy sequence  $\{x_n\}$  in  $F$ , so it is also a Cauchy sequence in  $X$ .  $X$  is complete,  $x_n \rightarrow p$  as  $n \rightarrow \infty$  for some  $p \in X$ . By thm,  $p \in \bar{F} = F$  ( $\because F$  is closed)  $\therefore F$  is complete).
- (6) The closeness in (5) is necessary, e.g.  $(0, 1) \subseteq \mathbb{R}$  and  $\mathbb{R}$  is complete,  $\{1/n\}$  is a Cauchy sequence in  $(0, 1)$  but it does not converge in  $(0, 1)$ .
- (7) Every closed ball and closed bounded interval in  $\mathbb{R}^k$  are complete.



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- (8)  $(0, 1)$ ,  $\mathbb{Q}$ ,  $\mathbb{Q}^k$  are not complete. Now we prove a class of real sequence which has limit, namely, all nonconstant sequence has limits in  $\mathbb{R}^k = [-\infty, \infty]$ .

**Definition.** Let  $\{x_n\}$  be a real sequence.

- (1)  $\{x_n\}$  is incerasing,  $\{x_n\}$  is  $\nearrow$ , if  $x_n \leq x_{n+1}$ ,  $\forall n \geq 1$
- (2)  $\{x_n\}$  is strictly incerasing,  $\{x_n\}$  is st  $\nearrow$ , if  $x_n < x_{n+1}$ ,  $\forall n \geq 1$
- (3)  $\{x_n\}$  is decerasing,  $\{x_n\}$  is  $\searrow$ , if  $x_n \geq x_{n+1}$ ,  $\forall n \geq 1$
- (4)  $\{x_n\}$  is strictly decerasing,  $\{x_n\}$  is st  $\searrow$ , if  $x_n > x_{n+1}$ ,  $\forall n \geq 1$
- (5) We say that  $\{x_n\}$  is monotonic if either  $\{x_n\}$  is increasing or decreasing. (Similar description in strictly case).

Note that  $\{x_n\}$  is  $\nearrow \Leftrightarrow \{-x_n\}$  is  $\searrow$ , so we only discuss increasing sequence.

**Examples :**

- (1)  $\{2n+1\}$  is  $\nearrow$  but it diverges. In fact,  $\lim_{n \rightarrow \infty} (2n+1) = +\infty = \sup\{2n+1 \mid n \geq 1\}$ . (not bounded above)
- (2)  $\{1/n\}$  is  $\searrow$  and it converges,  $\lim_{n \rightarrow \infty} 1/n = 0 = \inf\{1/n \mid n \geq 1\}$ .

**Theroem 3.9.** Let  $\{a_n\}$  be a real sequence, we have

- (1) If  $a_n$  is  $\nearrow$ , then  $\lim_{n \rightarrow \infty} a_n = \sup_{n \geq 1} a_n = \sup\{a_n \mid n \geq 1\}$
- (2) If  $a_n$  is  $\searrow$ , then  $\lim_{n \rightarrow \infty} a_n = \inf_{n \geq 1} a_n = \inf\{a_n \mid n \geq 1\}$
- (3) If  $a_n$  is monotonic, then  $\{a_n\}$  is convergent  $\Leftrightarrow$  it is bounded.

*Proof.*

- (1) We have 2 cases, (i)  $\{a_n\}$  is not bounded above, i.e.  $\sup_{n \geq 1} a_n = +\infty$ . Given  $M > 0$ ,  $\because \{a_n\}$  is not bounded above,  $\exists N \in \mathbb{N}$  s.t.  $a_N \geq M \implies \forall n \geq N$ ,  $a_n \geq a_N \geq M \therefore \lim_{n \rightarrow \infty} a_n = +\infty$ .  
(ii)  $\{a_n\}$  is bounded above. Then  $\lim_{n \geq 1} a_n = \alpha \in \mathbb{R}$ . Now, we claim that  $\lim_{n \rightarrow \infty} a_n = \alpha$ .  $\forall \varepsilon > 0, \exists N \in \mathbb{N}$  s.t.  $\alpha - \varepsilon < a_N \implies \forall n \geq N$ ,  $\alpha + \varepsilon > \alpha \geq a_n \geq a_N > \alpha - \varepsilon$  i.e.  $|a_n - \alpha| < \varepsilon$ , i.e.  $\lim_{n \rightarrow \infty} a_n = \alpha$ .
- (2) First, we use (1) to prove it.  $\{a_n\}$  is  $\searrow \implies \{-a_n\}$  is  $\nearrow$ , by (1)  $\implies \lim_{n \rightarrow \infty} (-a_n) = \sup_{n \geq 1} (-a_n) = -\inf_{n \geq 1} a_n$ . Next, using the argument of (1). (i)  $\{a_n\}$  is not bounded below i.e.

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$\inf_{n \geq 1} a_n = -\infty$ . Given  $M > 0, \exists N \in \mathbb{N}$  s.t.  $a_N \leq -M$ ,  
 $\forall n \geq N, a_n \leq a_N \leq -M \implies \lim_{n \rightarrow \infty} a_n = -\infty = \inf_{n \geq 1} a_n$   
(ii)  $\{a_n\}$  is bounded below, i.e.  $\inf_{n \geq 1} a_n = \beta$ . Now, we claim  
 $\lim_{n \rightarrow \infty} a_n = \beta$ .  $\forall \varepsilon > 0 \exists N \in \mathbb{N}$  s.t.  $a_N < \beta + \varepsilon \implies \forall n \geq$   
 $N, \beta - \varepsilon < \beta \leq a_n \leq a_N < \beta + \varepsilon \implies \forall n \geq N, |a_n - \beta| < \varepsilon$   
i.e.  $\lim_{n \rightarrow \infty} a_n = \beta$ .

- (3) Follows from (1) and (2).  $\because \{a_n\}$  is monotonic, ( $\implies$ ) Suppose  $\{a_n\}$  converges, then  $\{a_n\}$  is bounded. ( $\Leftarrow$ ) Suppose  $\{a_n\}$  is bounded, so  $\{a_n\}$  is bounded above and below. If  $\{a_n\}$  is  $\nearrow$ , then by (1)  $\{a_n\}$  converges. If  $\{a_n\}$  is  $\searrow$ , then by (2)  $\{a_n\}$  converges.

■

**Remark.** In the extended real line or system  $\mathbb{R}^* = [-\infty, \infty]$ , all monotonic sequence converges in  $\mathbb{R}^*$ .

#### 3.4. Limit Superior and Limit inferior.

**Definition.** Given a real sequence  $\{x_n\}$ . Let

$$\begin{aligned} E &= \{x \in \mathbb{R}^* \mid x \text{ is a subsequence of } \{x_n\}\} \\ &= \{x \in \mathbb{R} \mid x \text{ is a subsequence of } \{x_n\}\} \cup \{\infty\} \text{ if } \infty \text{ is a subsequence limit of } \{x_n\} \\ &= \{x \in \mathbb{R} \mid x \text{ is a subsequence of } \{x_n\}\} \cup \{-\infty\} \text{ or} \\ &= \{x \in \mathbb{R} \mid x \text{ is a subsequence of } \{x_n\}\} \cup \{-\infty, \infty\} \end{aligned}$$

**Claim :**  $E \neq \emptyset$ .

- $\{x_n\}$  is not bounded above  $\implies \exists$  a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  s.t.  $x_{n_k} \rightarrow \infty$  as  $k \rightarrow \infty$ .  $\forall k = 1, \exists n_1$  s.t.  $x_{n_1} > 1$ ,  
 $\exists n_2 > n_1$  s.t.  $x_{n_2} > \max\{x_{n_1}, 2\} \cdots \implies x_{n_k} > k, \forall k \geq 1 \implies \lim_{n \rightarrow \infty} x_{n_k} = \infty, \therefore \infty \in E$ .
- $\{x_n\}$  is bounded above and below  $\implies x_n \in [-a, a] \forall n \geq 1$  for some  $a > 0 \implies \{x_n\}$  has a convergent subsequence say  $x_n \rightarrow x \in [-a, a], \therefore x \in E$ .
- Therefore in any case,  $E \subseteq \mathbb{R}^*$  is nonempty, so  $\sup E$  and  $\inf E$  exists in  $\mathbb{R}^*$ .

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**Definition.** Define the limit superior or say upper limit of  $\{x_n\}$  is denoted by

$$\limsup_{n \rightarrow \infty} x_n = \overline{\lim}_{n \rightarrow \infty} x_n = x^* = \sup E$$

The limit inferior or lower limit of  $\{x_n\}$  is denoted by

$$\liminf_{n \rightarrow \infty} x_n = \lim_{\underline{n \rightarrow \infty}} x_n = x_* = \inf E$$

The following are equivalent :

$$\limsup_{n \rightarrow \infty} a_n = \inf_{n \geq 1} \left( \sup_{k \geq n} a_k \right) = \lim_{n \rightarrow \infty} \left( \sup_{k \geq n} a_k \right)$$

and

$$\liminf_{n \rightarrow \infty} a_n = \sup_{n \geq 1} \left( \inf_{k \geq n} a_k \right) = \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} a_k \right)$$

Let  $X$  be a set,  $\{A_n\}$  a sequence of subset of  $X$ . Define

$$\limsup_{n \rightarrow \infty} A_n = \overline{\lim}_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} \left( \bigcup_{k=n}^{\infty} A_k \right)$$

and

$$\liminf_{n \rightarrow \infty} A_n = \lim_{\underline{n \rightarrow \infty}} A_n = \bigcup_{n=1}^{\infty} \left( \bigcap_{k=n}^{\infty} A_k \right)$$

We say that the limit exists if  $\limsup_{n \rightarrow \infty} A_n = \liminf_{n \rightarrow \infty} A_n$ . In this case,  $\lim_{n \rightarrow \infty} A_n = \limsup_{n \rightarrow \infty} A_n = \liminf_{n \rightarrow \infty} A_n$

**Example :** Let  $X = \{0, 1\}$  then  $A_{2n} = \{0\}$  and  $A_{2n-1} = \{1\}$ ,  $\limsup_{n \rightarrow \infty} A_n = \{0, 1\}$  and  $\liminf_{n \rightarrow \infty} A_n = \emptyset$ .

**Facts :**  $x \in \limsup_{n \rightarrow \infty} A_n \Leftrightarrow x \in \bigcap_{k=n}^{\infty} A_k \ \forall n \geq 1 \Leftrightarrow \{x \in X \mid x \in A_n \text{ infinitely open}\}$  and  $x \in \liminf_{n \rightarrow \infty} A_n \Leftrightarrow x \in \bigcup_{n=1}^{\infty} (\bigcap_{k=n}^{\infty} A_k) \Leftrightarrow \exists n_0 \in \mathbb{N} \text{ s.t. } x \in \bigcap_{k=n_0}^{\infty} A_k \Leftrightarrow \{x \in X \mid x \in A_n \text{ almost always}\}$ .

**Examples :** If  $\{A_n\}$  is  $\nearrow (\searrow)$ , then  $\lim_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} A_n (\bigcap_{n=1}^{\infty} A_n)$  also  $\limsup_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} A_n = \liminf_{n \rightarrow \infty} A_n$ .