1. Jordan Canonical Form

1.1. Triangular Form.

<u>Definition</u>. *let $T : V \to V$ be a linear operator, a subspace $W \subseteq V$ is said to be invariant under T if $T(W) \subseteq W$*

Remark. $\{0\}, V, Ker(T), Im(T), E_\lambda$ *are T-invarient*

<u>Definition</u>. *Let $T : V \to V$ be a linear operator on a finite dimension vector space, we say that V is triangularizable $\Leftrightarrow \exists$ a order basis $\beta \ni [T]_\beta^\beta$ is upper triangular*

Example(triangularizable matrix)

Consider $\mathbb{F} = =C$, $V = \mathbb{C}^4$, let β be a order basis of V, $\beta = \{e_1, e_2, e_3, e_4\}$

$$[T]_\beta^\beta = \begin{bmatrix} 1 & 1-i & 2 & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 1-i & 3+i \\ 0 & 0 & 0 & 1-i \end{bmatrix}$$

Clearly $[T]_\beta^\beta$ is upper triangular, and let w_i be the subspace of $\mathbb{C}^4$ spanned by the first i vectors in the standard ordered basis, clearly, $T(w_i) \subseteq w_i = \{T(w) \mid w \in W\} = Im(T|w)$

Propos 1. Let V be a finite vector space, let $T : V \to V$ be a linear operator and $\beta = \{x_1, \cdots, x_n\}$ be a basis for V, then $[T]_\beta^\beta$ is upper triangle $\Leftrightarrow$ the subspace $w_i = \text{span}(x_1, \cdots, x_i)$ is T-invariant.

Proof. It is trivial. ∎

Note that the subspace w_i in **Prop 1** related follow

$$\{0\} \subseteq W_i \subseteq \cdots \subseteq W_{n-1} \subseteq W_n = V$$

we say that w_i forms an hands up sequence of subspaces. On the other hand, a given linear operator can be a upper triangle, we must to construct the $\nearrow$ sequence of T-invariant subspace

$$\{0\} \subseteq W_1 \subseteq \cdots \subseteq W_n$$

$T|_w : W \to W$ is a linear mapping, T_W where W is a T-invariant subspace

Propos 2
Let $T : V \to V, W \leq V$ is a T-invariant, where V is a finite dimension vector space. Then the character polynomial of $T|_W$ divides the c.p. of T

Proof. (Thm 5.21 from Friedberg)**not today** ■

Corollary. *Every eigenvalue of $T|_W$ is also an eigenvalue of T, i.e. the eigenvalue of $T|_W$ is a subset of the eigenvalue of T on V,*

Review(Diagonal's condition)

> Let $T : V \to V$ be a linear operator, where V is a finite dimension vector space, $\lambda_1, \cdots, \lambda_n$ are distinct eigenvalues, m_i be the multiplicity of λ_i, as a root number of the c.p. of T, Then
>
> T is diagonal $\Leftrightarrow$ $m_1 + \cdots + m_n = \dim(V)$, $dim(E_{\lambda_i}) = m_i$
>
> The proof is Thm. 5.9 $\sim$ Thm. 5.11 from Friedberg
> It means that
>
> (1) $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_n}$(From Exercise 5-2-20 Friberg)
> (2) m_i is algebraic multiplicity, $\dim(E_{\lambda_i})$ is geometric multiplicity.

Theorem (Schur)**.** *Let V be a finite dimensional vector space over $\mathbb{F}$ and $T : V \to V$ be an linear operator, then T is triangular $\Leftrightarrow$ the c.p. have $\dim(V)$ roots (counted with multiplicities) in $\mathbb{F}$*

Remark.

- ⋆ *if $\mathrm{F} = \mathbb{C}$(algebraic closure), then, by the Fundamental theorem of Algebra, every matrix $A \in M_{n\times n}(\mathbb{C})$ can be a triangularized*
- ⋆ *if $\mathrm{F} = \mathbb{R}$($x^2 + 1$ does not split on $\mathbb{R}$) consider the rotation matix R_θ where $0 < \theta < \pi$*

Lemma. *Let V be a finite dimensional vector space over $\mathbb{F}$ and $T : V \to V$ be an linear operator, and assume that the characteristic polynomial of T has $n = \dim(V)$ roots in F. If $W \subsetneqq V$ is an invariant subspace under T, then there exists a vector $x \neq 0$ in V such that $x \notin V$ is an invariant subspace under T, then there exists a vector $x \neq 0$ in V such*

⋆ we need to use this lemma to create $\nearrow$ subspaces

Proof. let $\alpha = \{x_1, \cdots, x_k\}$ be a basis for W and extend α by adjointing $\alpha' = \{x_{k+1}, \cdots, x_n\}$ to form a basis $\beta = \alpha \cup \alpha'$ for V. Let $W' = \text{span}(\alpha')$. Clearly, $V = W \oplus W'$ because we know that the fact that

> Let β_1 and β_2 be disjoint bases for subspaces W_1 and W_2, respectively, of a vector space V. If $\beta_1 \cup \beta_2$ is a basis for V, then $V = W_1 \oplus W_2$.

We define a linear operator $P : V \to V$ by

$$P(a_1x_1 + \cdots + a_nx_n) = a_1x_1 + \cdots + a_kx_k.$$

Clearly, $W' = Ker(P)$, $W = Im(P)$, and $P^2 = P$. Hence P is the projection on W with kernal W'. Moreover $I - P$ is also the projection on W' with kernel W. Since

$$\begin{aligned}
&(I - P)(a_1x_1 + \cdots + a_nx_n) \\
&= I(a_1x_1 + \cdots + a_nx_n) - P(a_1x_1 + \cdots + a_nx_n) \\
&= a_1x_1 + \cdots + a_nx_n - a_1x_1 + \cdots + a_kx_k \\
&= a_{k+1}x_{k+1} + \cdots + a_nx_n,
\end{aligned}$$

$W = Ker(I - P)$, $W' = Im(I - P)$, and $(I - P)^2 = (I - P)(I - P) = I - P^2 - P + P = I - P^2 = I - P$. If the basis of V is orthonormal (the Grame-Schmidt process), then it is clear that $W' = W^\perp$ by theorem 6.7 in Friedbreg. Since V is a finite dimensional vector space, so $(W^\perp)^\perp = W$ for all subspace W of V. It implies that P is an orthogonal projection.Let $S = (I - P) \circ T \equiv (I - P)T$. Since $Im(I - P) = W'$, so $Im(S) \subseteq Im(I - P) = W'$, that is, W' is S-invariant subspace ($S(W') \subseteq W'$). Now, we claim that the set of eigenvalues of $S|_W$ is a subset of the root of eigenvalues of T. Since W is T-invariant (by assumption), so we compute the matrix representation of $[T]_\beta^\beta$ is the form

$$[T]_\beta^\beta = \begin{bmatrix} A & B \\ O & C \end{bmatrix}.$$

Clearly, $A = [T|_W]_\alpha^\alpha$ is $k \times k$ block matrix and $C = [S|_{W'}]_{\alpha'}^{\alpha'}$ is a $(n - k) \times (n - k)$ block matrix. Hence

$$\det(T - \lambda I) = \det(T|_W - \lambda I) \cdot \det(S|_{W'} - \lambda I).$$

From Corollary 1.1, we are done. Since all the eigenvalues of T lie in the field F ($\because$ the characteristic polynomial n roots), so by previous discussion, the same is true of all the the eigenvalues of $S|_{W'}$. Then there exists a nonzero vector x in W' ($x \notin W$) such that $Sx = \lambda x$, for some $\lambda \in \mathbb{F}$. It implies that

$$\begin{aligned}(I-P)(Tx) = \lambda x &\implies Tx - PTx = \lambda x \\ &\implies Tx = \lambda x + PTx \in \operatorname{span}(\{x\}) + W.\end{aligned}$$

Finally, we show that $W + \operatorname{span}(\{x\})$ is T-invariant. For all $y \in W + \operatorname{span}(\{x\})$, there exists $z \in W$ and $\lambda \in \mathrm{F}$ such that $y = z + \lambda x$. Then

$$\begin{aligned}T(y) = T(z+\lambda x) &= T(z) + \lambda T(x) \\ &= T(z) + \lambda(x) + \lambda PT(x) \in \operatorname{span}(\{x\}) + W.\end{aligned}$$

■

We finish this lemma, and we are going to proof Schur lemma

Proof. ($\Rightarrow$) T is triangular
$\implies$ $\exists$ order basis β of V $\implies$ $[T]_\beta$ is upper triangular $\implies$ the eigenvalues of T are the diagonal entries in F $\implies$ the c.p. splits

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Proof. ($\Leftarrow$) Suppose the condition holds
let λ be eigenvalues of T, x_i is eigenvector of T correspond with λ, $W_1 = \operatorname{span}(\{\, x_i \,\})$
Clearly, W_1 is T-invariant by lemma, $\exists x \notin W$, $x \neq 0 \ni W_1 + \operatorname{span}(\{\, x_1 \,\})$ is T-invariant.
continue the processes $W_1 \subseteq \cdots \subseteq W_k$ with $W_i = \operatorname{span}(\{\, x_1, \cdots, x_i \,\})\ \forall\, i$
By lemma, $\exists x_{k+1} \notin W_k \ni W_{k+1} = W_k + \operatorname{span}(x_{k+1})$ is also T-invariant
$\therefore$ By prop.1, we are done.

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Corollary. *if $T : V \to V$ is triangular with eigenvalues λ_i and m_i is its multiplicities, then $\exists$ an order basis β for $V \ni [T]_\beta$ is upper triangular matrix, and the diagonal entries of $[T]_\beta$ are m_1, λ_1 followed by $m_2\lambda_2$'s and so on.*

Recall
In Chapter 4, If T is a linear mapping (or matrix) and $p(t) = a_n t^n + a_{n-1}t^{n-1}T^{n-1} + \cdots + a_0$ is a polynomial, we can define a new linear mapping

$$p(T) = a_n T^n + a_{n-1}T^{n-1} + \cdots + a_1 I$$

Theorem. *Let $T : V \to V$ be a linear operator on V which is a finite dimension vector space and $p(t) = \det(T - tI)$ be its c.p Assume* *that* $p(t)$ *has* $\dim(V)$ *roots in* F *over which* V *is defined, then* $p(T) = 0$*(which is a zero transformation on* V*)*

Proof. (Exercise 6-4-16 in Friedber) For all the vector S in some basis of $V \to p(T)(x) = 0$(scalar), by Schur lemma, $\exists$ order basis $\beta = \{x_1, \cdots, x_n\}$ for $V \implies w_i = \text{span}(\{x_1, \cdots, x_i\})$
$\forall\ 1 \leq i \leq n$ is T-invariant, all the eigenvalues of T lie in F, so $p(t) = \pm(t-\lambda)\cdots(t-\lambda_n)$ for some $\lambda_i \in \mathrm{F}$(not necessary distinct),if the factors here are ordered in the same fashion as the diagonal entries of $[T]_\beta^\beta$, then

$$T(x_i) = \lambda_i x_i + y_{i-1},\ y_i \in W_{i-1},\ i \geq 2,\ T(x_1) = \lambda_1 x_1$$

Now, we use the induction on i

- For $i = 1$,
  $p(T)(x_1) = \pm(T-\lambda_1 I)\cdots(T-\lambda_n I)(x_1) = \pm(T-\lambda_2 I)\cdots(I-\lambda_n I)(T-\lambda_1)(x_1) = 0$
- Suppose that: $p(T)(x_i) = 0,\ \forall\ i \leq k$
- Consider $p(T)(x_{k+1})$, clearly $(T-\lambda_1 I)\cdots(T-\lambda_k I)$ are needed to end x_i to 0, for $i \leq k$
  $p(T)(x_{k+1}) = I(T-\lambda_1 I)\cdots(T-\lambda_n I)(T-\lambda_{k+1} I)(x_{k+1})$
  $= \pm(T-\lambda_1 I)\cdots(T-\lambda_n I)(y_k) = 0$ By induction, we are done

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Suppose that $A \in M_{n\times n}(\mathrm{F})$ if A is invertible, i.e. $\det(A) \neq 0$, consider the c.p. of A

$$\det(A - tI) = (-1)^n t^n + \cdots + a_1 t + a_0,\ t = 0 \implies a_0 = \det(A) \neq 0$$

by thm 4(Cayley-Hamilton), $p(A) = (-1)^n A^n + \cdots + \det(A) I = 0$

$$\left(\frac{-1}{\det(A)}\right)\big((-1)^n A^{n-1} + \cdots + a_1 I\big) = A^{-1}$$

1.2. A Canonical form for nilpotent mappings.
We look at the linear operator $N : V \to V$, which only one distinct eigenvalue $\lambda = 0$ with multiplicity $n = dim(V)$, if N is each mapping, then by