Classification

order

$$\frac{dy}{dx} = y^2$$

which is 1st order, x independent variable, y dependent variable

$$\frac{d^4y}{dt^4} + 5\frac{d^2x}{dt^2} + 3x\sin t$$

which is 4th order
because above equation only have 1 independent variable, they are ordinary differential equations(ODEs).

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = r$$

which is 1st order

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

which is 2nd order
above equation have more than one independent variable, they are partial differential equations(PDEs).

nth-order ODE: $F(x, y, y', \cdots, y^{(n)}) = 0$
In certain condition on F, it can be written as

$$y^{(n)} = f(x, y, y', \cdots, y^{(n)}) = 0(\star)$$

Example $(y')^2 + y' + xy = 0$

$$y' = \frac{-1 \pm \sqrt{1-4xy}}{2}$$

Definition. *a function $\phi(x)$ is called a solution of $(\star)$ on $a < x < b$ if $\phi^{(n)}$ exists on $a < x < b$ and*

$$\phi^{(n)}(x) = f(x, \phi(x), \phi'(x), \cdots, \phi^{(n-1)}(x)) \ \forall a < x < b$$

Example. Verify that $y = e^{2x}$ is a solution of $y'' + y' - 6y = 0$

Proof. $y'' + y' - 6y = 4e^{3x} + 2e^{2x} - 6e^{3x} = 0 \ \forall -\infty < x < \infty$
$\therefore y = e^{2x}$ is a solution on $-\infty < x < \infty$ ■

Note. $y' = \dfrac{xy}{x+y+1}$ is derivative form $\Leftrightarrow$ $dy = \dfrac{xy}{x+y+1}dx$ or $xydx - (x+y+1)dy = 0$ is differential form.

Definition. *An ODE of order n is called linear if it may be written in the form*

$$b_0(x)y^{(n)} + b_1(x)y^{(n-1)} + \cdots + b_{n-1}(x)y' + b_n(x)y = R(x)$$

where $b_0 \neq 0$, An ODE that is not linear is called nonlinear ODE.

Example

linear

$$\begin{aligned} y'(x) + 5y'(x) + 6y(x) &= 0 \\ y'''(x) + x^2y''(x) + x^3y'(x) &= xe^x \end{aligned}$$

non linear

$$\begin{aligned} y''(x) + 5y'(x) + 6y^2(x) &= 0 \\ y''(x) + 5(y'(x))^3 + 6y(x) &= 0 \end{aligned}$$

Initial-Value Problem(IVP): same point (and 1st order)

$$\begin{cases} \dfrac{d^2y}{dx^2} + y = 0 \\ y(1) = 3 \\ y(1) = 2 \end{cases}$$

Boundary-Value Problem(BVP): two or more different points

$$\begin{cases} \dfrac{d^2y}{dx^2} + y = 0 \\ y(1) = 3 \\ y(2) = 2 \end{cases}$$

Theorem (Existence and uniqueness)**.** *Consider*

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

where $x_0, y_0 \in \mathbb{R}$ are given
Let $T = \{(x, y) \mid |x - x_0| \le a, |y - y_0| \le b\}$, where $a, b > 0$. Suppose that f and fy are continuous in T. Then (IVP) has a unique solution defined on $[x_0 - h, x_0 + h]$ for some $h > 0$

§ **Separable equation** $A(x)dx = B(y)dy$

Example.
(1) $\dfrac{dy}{dx} = \dfrac{2y}{x}$
Solution.

$$\begin{aligned} \frac{1}{y}dy &= \frac{2}{x}dx \\ \implies \int \frac{1}{y}dy &= \int \frac{2}{x}dx \\ \implies \ln|y| &= 2\ln|x| + C \end{aligned}$$

■

(2) $\begin{cases} (1 + y^2)dx + (1 + x^2)dy = 0 \\ y(0) = -1 \end{cases}$
Solution.

$$\begin{aligned} (1 + y^2)dx &= -(1 + x^2)dy \\ \implies \frac{dx}{-(1 + x^2)} &= \frac{dy}{(1 + y^2)} \\ \implies \int \frac{1}{1 + x^2}dx &= -\int \frac{1}{1 + y^2}dy \end{aligned}$$

you can let $x = \tan\theta \implies dx = \sec^2\theta d\theta$
$\therefore \int \dfrac{1}{1 + x^2}dx = \int \cos^2\theta \sec^2\theta d\theta = \theta + C = \tan^{-1} x + C$
$\implies \tan^{-1} x = -\tan^{-1} y + C$
$y(0) = -1 \implies 0 = \dfrac{\pi}{4} + C \implies C = \dfrac{\pi}{-4}$, $\therefore \tan^{-1} x = -\tan^{-1} y - \dfrac{\pi}{4}$

■

(3) $\begin{cases} 2x(y+1)dx - ydy = 0 \\ y(0) = -2 \end{cases}$

Solution. $\int 2xdx = \int \frac{y}{y+1}dy \implies x^2 = y - \ln|y+1| + C$
$y(0) = -2 \implies 0 = -2 + c \implies c = 2 \;\therefore x^2 + y - \ln|y+1| + 2$ ■

§ **Homogeneous equations**

Definition. *a function $f(x, y)$ is said to be homogeneous of degree k in x and y if and only if*

$$f(\lambda x, \lambda y) = \lambda^k f(x, y)$$

Example. $f(x, y) = x^2 + y^2$

$$\begin{aligned} f(\lambda x, \lambda y) &= (\lambda x)^2 + (\lambda y)^2 \\ &= \lambda^2(x^2 + y^2) \\ &= \lambda^2 f(x, y) \end{aligned}$$

$\therefore f(x, y)$ is homogeneous , $k = 2$

Theorem. *If $M(x, y)$ and $N(x, y)$ are both homogeneous and of the same degree, then $\frac{M(x,y)}{N(x,y)}$ is homogeneous of degree zero.*

Proof. Set $f(x, y) = \frac{M(x,y)}{N(x,y)}$. By definition, we assume M and N are homogeneous of degree k, so

$$M(\lambda x, \lambda y) = \lambda^k M(x, y) \text{ and } N(\lambda x, \lambda y) = \lambda^k N(x, y)$$

$\therefore f(\lambda x, \lambda y) = \frac{M(\lambda x, \lambda y)}{N(\lambda x, \lambda y)} = \frac{\lambda^k}{\lambda^k} \cdot \frac{M(x,y)}{N(x,y)} = \lambda^0 \frac{M(x,y)}{N(x,y)}$ ■