A FIRST COURSE IN STOCHASTIC PROCESSES

QSNAKE EDITION

1. 03/08/2024

Chapter 1.

(1) $(\Omega, \mathscr{F}, P)$ a sample space
(2) random variable X
(3) $P(X \in A) \in \mathscr{F}$ for all $A \in mathscrB$
(4) moment generating function $\phi_X(\lambda) = E[e^{\lambda X}]$

Proposition. *If $\phi_X(\lambda) = \phi_Y(\lambda) \Rightarrow X$ and Y has same distribution.*

Example. $X = N(0,1)$

$$\begin{aligned}
\phi_X(\lambda) = E[e^{\lambda X}] &= \int_{\mathbb{R}} e^{\lambda X} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \int_{R} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[x^2 - \frac{\lambda x}{2} + (\frac{\lambda}{2})^2] + \frac{\lambda^2}{2}} dx \\
&= \int_{\mathbb{R}} \frac{1}{2\pi} e^{-\frac{1}{2}(x - \frac{\lambda}{2})^2 + \frac{\lambda^2}{2}} dx \\
&= e^{\frac{\lambda^2}{2}} \cdots (\star)
\end{aligned}$$

If $X = N(\mu, \sigma^2)$

$$\begin{aligned}
\phi_X(\lambda) &= \int_{\mathbb{R}} e^{\lambda X} e^{-\frac{(X-\mu)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}} dx \\
&= \int_{\mathbb{R}} e^{\lambda(\mu+\sigma y)} e^{-\frac{y^2}{2}} \frac{1}{\sqrt{2\pi}} dy \qquad (\textit{let } y = \frac{X-\mu}{\sigma}) \\
&= e^{\lambda\mu} \int_{\mathbb{R}} e^{\lambda\sigma y} e^{-\frac{y^2}{2}} \frac{1}{\sqrt{2\pi}} dy \\
&= e^{\lambda\mu} \int_{\mathbb{R}} e^{\lambda\sigma y} e^{-\frac{y^2}{2}} \frac{1}{2\pi} dy \\
&= e^{\lambda\mu + \frac{\lambda^2\sigma^2}{2}} (\textit{by } (\star))
\end{aligned}$$

If we want to calculate EX^{2n} when $X = N(0,1)$

$$\int_{\mathbb{R}} x^{2n} \frac{1}{2\pi} e^{-\frac{x^2}{2}}$$

but we have moment

$$\begin{aligned}
E[e^{\lambda X}] &= e^{-\frac{x^2}{2}} = \sigma_{n=0}^{\infty} \frac{1}{n!} (\frac{\lambda^2}{2})^n \\
&= E[\sigma_{n=0}^{\infty} \frac{\lambda^n}{n!}] = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} E[X^n]
\end{aligned}$$

$\implies E[X^{2n+1}] = 0$ *and*

$$\frac{\lambda^{2^n}}{(2n)!} E[X^{2n}] = \frac{1}{n!} (\frac{\lambda^{2n}}{2^n}) \quad \textit{for all } \lambda \in \mathbb{R}$$

$\implies E[X^{2n}] = \frac{(2n)!}{n!2^n} \quad n = 1, 2, \cdots$

Example. *If X, Y are independent and standard normal (i.e. $N(0,1)$). Then (X,Y), we can define a moment generating function*

$$\begin{aligned}
\phi(a_1, a_2) &= E[e^{a_1 X + a_2 Y}] \\
&= E[e^{a_1 X}] E[e^{a_2^Y}] \\
&= \phi(a_1)\phi(a_2) \\
&= e^{\frac{a_1^2 + a_2^2}{2}}
\end{aligned}$$

On the other hand the distribution

$$\begin{aligned} P(X \le c_1, Y \le c_2) &= P(X \le c_1)P(Y \le c_2) \\ &= \left(\int_{-\infty}^{c_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx\right)\left(\int_{-\infty}^{c_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy\right) \\ &= \int_{-\infty}^{c_1}\int_{-\infty}^{c_2} \frac{1}{2\pi} e^{-(\frac{x^2+y^2}{2})} dxdy \\ &\implies f(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} \end{aligned}$$

$X_1, \cdots, X_n$ are $N(0,1)$ and independent
$\implies f(x_1, \cdots, x_n) = (\frac{1}{2\pi})^{\frac{n}{2}} e^{-(\frac{x_1^2+\cdots+x_n^2}{2})}, \quad n \ge 1$
and the MGF $\phi(a_1, \cdots, a_n) = e^{\frac{a_1^2+\cdots+a_n^2}{2}}$

Important inequality:

(1) Markov inequality: $P(X > a) \le \frac{E|X|}{a}$
(2) Chebyshev inequality: $P(X > a) \le \frac{EX^2}{a^2}$

An n dimensional boll's volume.

Chapter 2.

Remark. *$Cov(X,Y) = 0 \Leftrightarrow X$ and Y are not correlated $\not\Rightarrow$ they are independent.*

Consider a random vector $X = (X_1, \cdots, X_n) : (\Omega, \mathscr{F}, P) \to \mathbb{R}^n$
and

$$E[X] = (EX_1, \cdots, EX_n) = m = (m_1, \cdots, m_n)$$

The covariance matrix is defined as follow:

$$\mathscr{C} = (\mathscr{C})_{1 \le 1, j \le n}, \mathscr{C}_i j = E[(X_i - E[X_i])(X_j - E[X_j])], i, j = 1, 2, \cdots, n$$

Proposition. *$\mathscr{C}$ is symmetry and ≥ 0*
Consider for any $a = (a_1, \cdots, a_n)^T \in \mathbb{R}^n$

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\begin{aligned}
0 &\leq E[a_1X_1 + \cdots + a_nX_n - E(aX_1 + \cdots + a_nX_n)]^2 \\
&= E[\sum_{k=1}^{n}(a_kX_k - a_kE[x_k])(\sum_{j=1}^{n}(a_jX_j - a_jE[X_j]))]^2 \\
&= \sum_{k,j=1}^{n} a_ja_kE[(X_k - E[X_k])(X_jE[X_j])] \\
&= \sum_{k,j=1}^{n} a_ka_j\mathscr{C}_{k,j} = a^T\mathscr{C}a \implies e \geq 0
\end{aligned}
$$

Definition. *X, Y are uncorrelated if $(\mathscr{C})_{ij} = 0$ for all $i \neq j$*

Proposition. *$E[X^n] = E[Y^n]$ for all $n \in \mathbb{N} \implies X, Y$ has the same distribution.*

$$\phi(e^{\lambda X}) = \phi_X(\lambda) = \phi_Y(\lambda) = \phi(e^{\lambda Y})$$

Definition. *A n-dimensional random variable $X = (X_1, \cdots, X_n)$ is called to be Gaussian $\Leftrightarrow$ $a_1X_1 + \cdots + a_nX_n$ is Gaussian for any $a_1, \cdots, a_n \in \mathbb{R}$*

Remark. *(1) If X is Gaussian $\Rightarrow X_k$ is Gaussian for $k = 1, 2, \cdots, n$*

(2) If X and Y are independent $N(0, 1)$ then $a_1X + a_2Y$ is a $N(0, a_1^2 + a_2^2)$ for all $a_1, a_2 \in \mathbb{R} \implies (X, Y)$ is Gaussian

Lemma. *If X is an n'dimensional Gaussian vector and M is $m * n$ matrix then MX is m-dimensional Gaussian vector.*

$$\begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_m \end{pmatrix} X = \begin{pmatrix} X_1X \\ M_2X \\ \vdots \\ M_nX \end{pmatrix}$$

Proposition. *A random variable $X = (X_1, \cdots X_n)$*
X is Gaussian $\Leftrightarrow$ MGF of X is $e^{a^Tm + \frac{1}{2}a^Tea} = E[e^{a^TX}] = \phi(a_1, \cdots, a_n) = \phi(a)$
$X = N(\mu, \sigma^2) \implies \phi_X(\lambda) = e^{E(\lambda X) + \frac{E(\lambda(X - EX))^2}{2}}$

$$\begin{aligned}\phi(a) = E[e^{a^T X}] &= e^{E(a^T X) + \frac{E(a^T(X - E[X^2]))}{2}} \\ &= e^{a^T m + \frac{E(a^T X X^T a)}{2}} \\ &= e^{a^T m + \frac{a^T e a}{2}}\end{aligned}$$

note $E[a^T X - E(a^T X)]^2 = (a^T E(X - EX))^2 = a^T E(X - EX)(E(X - EX))^T a = e$

Proposition. *The covariance matrix is diagonal* $\Leftrightarrow$ *the random variable are independent.*

Proof. ($\Rightarrow$) By $e^{a^T m + \frac{1}{2} a^T \mathscr{C} a}$

$$\begin{aligned}\phi(a) &= e^{\sum_{j=1}^n m_j a_j + \frac{1}{2}\sum_{j=1}^n d_{jj}^2 a_j^2} \\ &= \prod_{j=1}^n e^{m_j a_j + \frac{d_j j a_j^2}{2}} \\ &= \prod_{j=1}^n \phi_{X_j}(a_j) \text{ where } X_j = N(m_{jj}, d_j^2) \\ \Longrightarrow & \quad f(x_1, \cdots, x_n) = \prod_{j=1}^n f_{X_j}(x_j)\end{aligned}$$

($\Leftarrow$) If $X_1, \cdots, X_n$ is independent

$\Rightarrow \mathscr{C}_{ij} = E[(X_i - EX_i)(X_j - EX_j)] = 0$ for all $i \neq j$

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Example: $Z_1, Z_2, Z_3 = N(0, 1)$ are independent

$X = Z_1 + Z_2 + Z_3$

$Y = Z_1 + Z_2$

$Z = Z_3$

(X, Y, Z) is Gaussian vector with mean $(0, 0, 0)$

covariance $\mathscr{C} = \begin{pmatrix} EX^2 & EXY & EXZ \\ EYX & EY^2 & EYZ \\ EZX & EZY & EZ^2 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$\det\mathscr{C} = 0$(degenerate Gaussan vector)

Note $X - Y - Z = 0$ independent.

Lemma 0.1. *Let $X = (X_1, \cdots, X_n)$ be a Gaussian vector. Then X is degerate (i.e. $det\mathscr{C} = 0$) $\Leftrightarrow$ the coordinates are linear dependent.*

Proposition. *$X = (X_1, \cdots, X_n)$ be a non-degenerate Gaussian vector with mean m and covariance matrix $\mathscr{C}$ then the joint distribution of X is given by the joint PDF.*

$$f(x_1, \cdots, x_n) = \frac{e^{-\frac{1}{2}(x-m)^T e^{-1}(x-m)}}{(2\pi)^{\frac{n}{2}} det(e)^{\frac{1}{2}}}$$

Example: Consider a Gaussian random variable X_1, X_2 mean 0 and covariance $\mathscr{C} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$\implies \mathscr{C}^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$ and $\det(\mathscr{C}) = 3$

$\implies f(x, y) = \frac{e^{(-\frac{x^2}{3} + \frac{1}{3}xy - \frac{1}{3}y^2)}}{2\pi\sqrt{3}}$

Proposition. *Let $X = (X_1, \cdots, X_n)$ be a Gaussian vector with mean 0. If X is Non-degenerate $\exists$ n i.i.d. $N(0,1)$ random variable $Z = (Z_1, \cdots, Z_n)$ and invertible $n * n$ matrix $A \ni X = AZ$.*

Example:
$X_1 = W_1 + W_2$
$X_2 = W_1 - W_2$
$W_1, W_2 = N(0,1)$, independent
$X = AW$ where $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
The covariance matrix of X is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
$\implies$ Let $Z_1 = \frac{X_1}{\sqrt{2}}, \; Z_2 = \frac{X_2}{\sqrt{2}}$
$\implies$ Z_1, Z_2 is $N(0,1)$

Proof. Take $Z_1 = \frac{X_1}{\sqrt{c_{11}}} \implies Z_1$ is $N(0,1)$ define $Z_2' = X_2 - E[X_2 Z_1]Z_1 \implies E[Z_2 X_2] = 0$
Let $Z_2 = \frac{Z_2'}{\sqrt{\mathrm{var}(Z_2')}} = N(0,1)$

$\vdots$

$Z = (Z_1, Z_2, \cdots, Z_n),\ Z_k = N(0,1)$ is independent

$$A = \begin{pmatrix} \frac{1}{\sqrt{var X_1}} & 0 & \cdots & 0 \\ \sqrt{X_1 Z_1} & c & \cdots & 0 \\ \vdots & \cdots & \vdots & \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \end{pmatrix}$$

$$f(x_1, \cdots, x_n) = \frac{e^{-\frac{1}{2}(x-m)^T \mathscr{C}^{-1}(x-m)}}{(2\pi)^{\frac{n}{2}} \det(\mathscr{C})^{\frac{1}{2}}}$$

WLOG let $m = \vec{0}$ and $\mathscr{C} = AA^T$

$\because \det\mathscr{C} = \det(AA^T) = \det(A)^2 > 0$

$\therefore A$ is invertible.

Let $X = AZ$ where $Z_i = N(0,1)$

Consider

$$\begin{aligned} P(X \in B) &= P(Z \in A^{-1}B) \\ &= \int_{A^{-1}B} \frac{1}{(2\pi)^{\frac{\pi}{2}}} e^{\frac{1}{2}Z^T Z} dz_1, \cdots dz_n \end{aligned}$$

change variable $x = AZ \implies Z = A^{-1}x$

$Z \in A^{-1}B \implies x \in B$

$$\begin{aligned} Z^T Z &= (A^{-1}x)^T (A^{-1}x) \\ &= x^T (A^{-1})^T A^{-1} x \\ &= x^T (A^T)^{-1} A^{-1} x \\ &= x^T (AA^T)^{-1} x \\ &= x^T \mathscr{C}^{-1} x \end{aligned}$$

$$\begin{aligned} dz_1, \cdots, dz_n &= |detA^{-1}| dx_1, \cdots, dx_n \\ &= (det\mathscr{C})^{\frac{-1}{2}} dx_1, \cdots, dx_n \end{aligned}$$

$$\therefore \int_B \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{x^T - \mathscr{C}x}{2}} \frac{1}{(det\mathscr{C})^{\frac{1}{2}}} dx_1, \cdots, dx_n$$

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