SOME RESULT

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Abstract. maybe later

1. Introduction.

maybe later

2. Preliminaries.

Some definition of graph, some text here

Symbolic dynamic and it's entropy, some text here

Construct Sierpinski gasket, some text here

Concept of percolation, some text here, maybe later

Let us denote the number of edges of SG_n by $|V_n|$, which is equal to 3^{n+1} for any $n \geq 0$. To define generalized model, we first consider the graph SG_n, with $\mathscr{A}_{\mathscr{M}} = \{0, 1, 2, \ldots, \mathscr{M}\}$ for a fixed positive integer $\mathscr{M} \geq 1$, and $\mathscr{F} = 00$. A symbol $x \in \{0, 1, 2, \ldots, \mathscr{M}\}$ is assigned to each vertex uniformly and independently. For each configuration, the probability distribution is

$$p^{\text{the number of vertex with symbol 0}} \times (1-p)^{\text{the number of vertex with any symbol but 0}}$$

where $p := \frac{1}{\mathscr{M}+1} \in (0, \frac{1}{2}]$ is the probability that any edge has the symbol 0, and

$$\text{the number of } \mathscr{A}_{\mathscr{F}}^{SG_n} = \sum_{C \in \mathscr{A}_{\mathscr{F}}^{SG_n}} \mathscr{M}^{I_0^C} = (\mathscr{M}+1)^{|V_n|} \sum_{C \in \mathscr{A}_{\mathscr{F}}^{SG_n}} p^{|V_n| - I_0^C} \times (1-p)^{I_0^C}$$

where I_0^C is the number of edge of a configuration $C \in \mathscr{A}_{\mathscr{F}}^{SG_n}$ that does not have the symbol 0.

Note that the range of p may be relaxed and not only in $(0, \frac{1}{2}]$ for other consideration. For example, if $\mathscr{F} = \{ij : i, j \in \{1, 2, \ldots, \mathscr{M}\}\}$, and p is the probability that each edge of $SG(n)$ with a symbol in $\{1, 2, \ldots, \mathscr{M}\}$, then $p = \frac{\mathscr{M}}{\mathscr{M}+1} \in [\frac{1}{2}, 1)$.

We now consider a more general model such that the probability p is independent of $\mathscr{M}$ and in general can be any value in $(0, 1)$, namely, it is not uniform distribution. Let $\mathscr{A}_{\mathscr{M}}$ be a set of symbol and given a subset $\mathscr{S}$ of $\mathscr{A}_{\mathscr{M}}$. Let $\mathscr{F} = \{ij : i, j \in \mathscr{S}\}$ be forbidden blocks between nearest-neighbor edges, and p the probability of each edge of SG_n with a symbol that is in $\mathscr{S}$ while $(1-p)$ is the probability of each edge of SG_n with a symbol that is not in $\mathscr{S}$. The assignment of a symbol to each edge remains independent. It follows that

$$\text{the number of } \mathscr{A}_{\mathscr{F}}^{SG_n} = (\mathscr{M}+1)^{|V_n|} \sum_{C\in\mathscr{A}_{\mathscr{F}}^{SG_n}} p^{|V_n|-I_{\mathscr{S}}^C} \times (1-p)^{I_{\mathscr{S}}^C}$$

where $I_{\mathscr{S}}^C$ is the number of edge of a configuration $C \in \mathscr{A}^{SG_n}$ that does not have the symbols in $\mathscr{S}$. The entropy per site is given by

$$z_{SG}(p,\mathscr{M}) = \log(\mathscr{M}+1) + \mathscr{E}_{SG}(p)$$

where we define $\mathscr{E}_{SG}(p) := \lim_{n\to\infty} \frac{\log m_n(p)}{|V_n|}$ and $m_n(p) := \sum_{C\in\mathscr{A}_{\mathscr{F}}^{SG_n}} p^{|V_n|-I_{S_n}^C}(1-p)^{I_{S_n}^C}$ which is a kind of random cluster model. Note that $m_n(p)$ only depends on p, and $z_{SG}(p,\mathscr{M})$ can be computed once we have $\mathscr{E}_{SG}(p)$ and $\mathscr{M}$. The purpose of the following paper is to obtain the asymptotic behavior of $m_n(p)$ as $n\to\infty$ and derive upper and lower bounds for $\mathscr{E}_{SG}(p)$.

organization some text here

3. Main result.

Definition 1. *Consider the two-dimensional Sierpinski gasket SG_n at stage n. (i) Define $f_n(p)$ as the value of $m_n(p)$ such that none of the three outermost edges have symbols that are in $\mathscr{S}$. (ii) Define $g_n(p)$ as the value of $m_n(p)$ such that exactly one of the three outermost edges has a symbol that is in $\mathscr{S}$. (iii) Define $h_n(p)$ as the value of $m_n(p)$ such that exactly two of the three outermost edges has a symbol that is in $\mathscr{S}$. (iv) Define $t_n(p)$ as the value of $m_n(p)$ such that all three outermost edges have symbols that are in $\mathscr{S}$.*

The quantities $f_n(p), g_n(p), h_n(p)$ and $t_n(p)$ are illustrated in figure, where only the outermost edges are shown. Because of rotational symmetry, there are three possible $g_n(p)$ and three possible $h_n(p)$ such that

$$m_n(p) = f_n(p) + 3g_n(p) + 3h_n(p) + t_n(p)$$

The initial values at stage one are f_1, g_1, h_1, t_1 here. so that m_n

The four quantities $f_n(p), g_n(p), h_n(p)$ and $t_n(p)$ satisfy recursion relations given in the following lemma.

Lemma 1. *For any positive integer n and $p \in (0,1)$,*

$$f_{k+1} = f_k^3 + 6_k^2 g_k + 3f_k^2 h_k + 9f_k g_k^2 + 6f_k g_k h_k + 2g_k^3 \tag{1}$$
$$g_{k+1} = f_k^2 g_k + 4f_k g_k^2 + 2f_k^2 h_k + f_k^2 t_k + 8f_k g_k h_k + 3g_k^3 + 2f_k g_k t_k + 2f_k h_k^2 + 4g_k^2 h_k \tag{2}$$
$$h_{k+1} = f_k g_k^2 + 4f_k g_k h_k + 2g_k^3 + 2f_k g_k t_k + 7g_k^2 h_k + 3f_k h_k^2 + 2g_k^2 t_k + 4g_k h_k^2 + 2f_k h_k t_k \tag{3}$$
$$t_{k+1} = g_k^3 + 6g_k^2 h_k + 3g_k^2 t_k + 9g_k h_k^2 + 6g_k h_k t_k + 2h_k^3 \tag{4}$$

Proof. Trivial. graph and proof here ∎

1. When n=1.

$f_1(p) = (1-p)^9 + 3p(1-p)^8$

$g_1(p) = 2p(1-p)^8 + 2p^2(1-p)^7$

$h_1(p) = 3p^2(1-p)^7$

$t_1(p) = 2p^3(1-p)^6$

Ratios

$\alpha_1(p) = \frac{g_1(p)}{f_1(p)} = \frac{2pq+2p^2}{q^2+3pq} = \frac{2p}{q(1+2p)} = \frac{2p}{(1-p)(1+2p)}$

$\beta_1(p) = \frac{h_1(p)}{g_1(p)} = \frac{3p}{2q+2p} = \frac{3p}{2}$

$\gamma_1(p) = \frac{t_1(p)}{h_1(p)} = \frac{2p}{3q}$

Note: $q = (1-p)$

Difference

$\tilde{\epsilon_1}(p) = \alpha_1(p) - \beta_1(p) = \frac{2p}{q(1+2p)} - \frac{3p}{2} = \frac{4p}{2q(1+2p)} - \frac{3pq(1+2p)}{2q(1+2p)} = \frac{p}{q}(\frac{4-3q-6pq}{2+4p}) = \frac{p}{q}(\frac{6p^2-3p+1}{2+4p})$

$\tilde{\epsilon_1'}(p) = \beta_1(p) - \gamma_1(p) = \frac{3p}{2} - \frac{2p}{3q} = \frac{9pq-4p}{6q} = \frac{p}{q}\frac{9q-4}{6} = \frac{p}{q}\frac{5-9p}{6}$

$\tilde{\epsilon_1}(p) + \tilde{\epsilon_1'}(p) = \alpha_1(p) - \gamma_1(p) = \frac{2p}{q(1+2p)} - \frac{2p}{3q} = \frac{6p-2p(1+2p)}{3q(1+2p)} = \frac{6p-2p-4p^2}{3q(1+2p)} = \frac{p}{q}\frac{4q}{3(1+2p)}$

Define $c_p = \frac{q}{p}$

$\epsilon_1 = c_p(\alpha_1(p) - \beta_1(p)) = \frac{6p^2-3p+1}{2+4p}$

$\epsilon_1' = c_p(\beta_1(p) - \gamma_1(p)) = \frac{5-9p}{6}$

$\epsilon_1(p) + \epsilon_1'(p) = c_p(\alpha_1(p) - \gamma_1(p)) = \frac{4q}{3(1+2p)}$

2. Iterated.

In this section, we shorten the notation of (p), like $f_k(p)$ denote f_k

$f_{k+1} = f_k^3 + 6_k^2 g_k + 3f_k^2 h_k + 9f_k g_k^2 + 6f_k g_k h_k + 2g_k^3$

$g_{k+1} = f_k^2 g_k + 4f_k g_k^2 + 2f_k^2 h_k + f_k^2 t_k + 8f_k g_k h_k + 3g_k^3 + 2f_k g_k t_k + 2f_k h_k^2 + 4g_k^2 h_k$

$h_{k+1} = f_k g_k^2 + 4f_k g_k h_k + 2g_k^3 + 2f_k g_k t_k + 7g_k^2 h_k + 3f_k h_k^2 + 2g_k^2 t_k + 4g_k h_k^2 + 2f_k h_k t_k$

$t_{k+1} = g_k^3 + 6g_k^2 h_k + 3g_k^2 t_k + 9g_k h_k^2 + 6g_k h_k t_k + 2h_k^3$

Ratios

$\alpha_{n+1} = \alpha_n \frac{B_n}{A_n}$, $\beta_{n+1} = \alpha_n \frac{C_n}{B_n}$, $\gamma_{n+1} = \alpha_n \frac{D_n}{C_n}$

$A_n = 1 + 6\alpha_n + 3\alpha_n\beta_n + 9\alpha_n^2 + 6\alpha_n^2\beta_n + 2\alpha_n^3$

$B_n = 1 + 4\alpha_n + 2\beta_n + \beta_n\gamma_n + 8\alpha_n\beta_n + 3\alpha_n^2 + 2\alpha_n\beta_n\gamma_n + 2\alpha_n\beta_n^2 + 4\alpha_n^2\beta_n$

$C_n = 1 + 2\alpha_n + 4\beta_n + 2\beta_n\gamma_n + 7\alpha_n\beta_n + 3\beta_n^2 + 2\alpha_n\beta_n\gamma_n + 4\alpha_n\beta_n^2 + 2\beta_n^2\gamma_n$

$D_n = 1 + 6\beta_n + 3\beta_n\gamma_n + 9\beta_n^2 + 6\beta_n\gamma_n + 2\beta_n^3$

Difference

$$\tilde{\epsilon}_{n+1} = \alpha_{n+1} - \beta_{n+1} = \alpha_n \left(\frac{B_n}{A_n} - \frac{C_n}{B_n} \right) = \frac{\alpha_n}{A_n B_n} \left(B_n^2 - A_n C_n \right)$$

$$= \frac{1}{A_n B_n} \left[\begin{array}{l} \tilde{\epsilon}_n^2 (4\alpha_n^3 \beta_n^2 + 10\alpha_n^3 \beta_n + 2\alpha^2 \beta^2 + 5\alpha^3 + 7\alpha_n^2 \beta_n + \alpha_n \beta_n^2 + 4\alpha_n^2 + 2\alpha_n \beta_n + \alpha_n) + \\ \tilde{\epsilon}_n \tilde{\epsilon}'_n (4\alpha_n^4 \beta_n + 4\alpha_n^3 \beta_n^2 + 10\alpha_n^3 \beta_n + 6\alpha_n^2 \beta_n^2 + 8\alpha_n^2 \beta_n + 2\alpha_n \beta_n^2 + 2\alpha_n \beta_n) + \\ \tilde{\epsilon}_n'^2 (4\alpha_n^3 \beta_n^2 + 4\alpha_n^2 \beta_n^2 + \alpha_n \beta_n^2) \end{array} \right]$$

$$\tilde{\epsilon'}_{n+1} = \beta_{n+1} - \gamma_{n+1} = \alpha_n \left(\frac{C_n}{B_n} - \frac{D_n}{C_n} \right) = \alpha_n \left(\frac{C_n^2 - B_n D_n}{B_n C_n} \right)$$

$$= \frac{1}{B_n C_n} \left[\begin{array}{l} \tilde{\epsilon}_n^2 (4\alpha_n \beta_n^4 + 12\alpha_n \beta_n^3 + 13\alpha_n \beta_n^2 + 6\alpha_n \beta_n + \alpha_n) + \\ \tilde{\epsilon}_n \tilde{\epsilon}'_n (4\alpha_n^2 \beta_n^3 - 4\alpha_n^2 \beta_n^2 \gamma_n - 8\alpha_n \beta_n^4 + 18\alpha_n \gamma_n^3 + 2\alpha_n^2 \beta_n^2 + 9\alpha_n \beta_n^2 + \alpha_n^2 \beta_n + 2\alpha_n \beta_n) + \\ \tilde{\epsilon}_n'^2 (4\alpha_n \beta_n^4 + 2\alpha_n \beta_n^3 + 2\alpha_n^2 \beta_n^2 + \alpha_n \beta_n^2) \end{array} \right]$$

$$\begin{aligned} \tilde{\epsilon}_{n+1} + \tilde{\epsilon}'_{n+1} &= \alpha_{n+1} - \gamma_{n+1} = \alpha_n \left(\frac{B_n}{A_n} - \frac{D_n}{C_n} \right) = \alpha_n \left(\frac{B_n C_n - A_n D_n}{A_n C_n} \right) \\ &= \frac{\alpha_n}{A_n C_n} (2\alpha_n + 1)(\alpha_n^2 + 3\alpha_n \beta_n + 4\alpha_n + 1) \\ &\quad (4\alpha_n \beta_n^2 + 2\alpha_n \beta_n \gamma_n - 2\beta_n^3 - 4\beta_n^2 \gamma_n + 7\alpha_n \beta_n - 6\beta_n^2 - \beta_n \gamma_n + 2\alpha_n - 2\beta_n) \\ &= \frac{\alpha_n}{A_n C_n} (2\alpha_n + 1)(\alpha_n^2 + 3\alpha_n \beta_n + 4\alpha_n + 1) \\ &\quad [2\beta_n^2 \tilde{\epsilon}_n + 2\beta_n^2 (\tilde{\epsilon}_n + \tilde{\epsilon}'_n) + 2\beta_n \gamma_n \tilde{\epsilon}_n + 6\beta_n \tilde{\epsilon}_n + \beta_n (\tilde{\epsilon}_n + \tilde{\epsilon}'_n) + 2\tilde{\epsilon}_n] \\ &= \frac{\alpha_n}{A_n C_n} (2\alpha_n + 1)(\alpha_n^2 + 3\alpha_n \beta_n + 4\alpha_n + 1) \\ &\quad [\tilde{\epsilon}_n (2\beta_n^2 + 2\beta_n \gamma_n + 6\beta_n + 2) + (\tilde{\epsilon}_n + \tilde{\epsilon'}_n)(2\beta_n^2 + \beta_n)] \end{aligned}$$

$$
\begin{aligned}
A_nB_n =&8\alpha_n^5\beta_n + 28\alpha_n^4\beta_n^2 + 4\alpha_n^4\beta_n\gamma_n + 12\alpha_n^3\beta_n^3 + 12\alpha_n^3\beta_n^2\gamma_n \\
&+ 6\alpha_n^5 + 70\alpha_n^4\beta_n + 78\alpha_n^3\beta_n^2 + 20\alpha_n^3\beta_n\gamma_n + 6\alpha_n^2\beta_n^3 + 12\alpha_n^2\beta_n^2\gamma_n \\
&+ 35\alpha_n^4 + 133\alpha_n^3\beta_n + 48\alpha_n^2\beta_n^2 + 21\alpha_n^2\beta_n\gamma_n + 3\alpha_n\beta_n^2\gamma_n \\
&+ 56\alpha_n^3 + 88\alpha_n^2\beta_n + 8\alpha_n\beta_n^2 + 8\alpha_n\beta_n\gamma_n + 36\alpha_n^2 + 23\alpha_n\beta_n + \beta_n\gamma_n + 10\alpha_n + 2\beta_n + 1
\end{aligned}
$$

$$
\begin{aligned}
B_nC_n =&16\alpha_n^3\beta_n^3 + 8\alpha_n^3\beta_n^2\gamma_n + 8\alpha_n^2\beta_n^4 \\
&+ 20\alpha_n^2\beta_n^3\gamma_n + 4\alpha_n^2\beta_n^2\gamma_n^2 + 4\alpha_n\beta_n^4\gamma_n + 4\alpha_n\beta_n^3\gamma_n^2 + 40\alpha_n^3\beta_n^2 + 6\alpha_n^3\beta_n\gamma_n + 58\alpha_n^2\beta_n^3 + 44\alpha_n^2\beta_n^2\gamma_n \\
&+ 6\alpha_n\beta_n^4 + 30\alpha_n\beta_n^3\gamma_n + 6\alpha_n\beta_n^2\gamma_n^2 + 2\beta_n^3\gamma_n^2 \\
&+ 29\alpha_n^3\beta_n + 101\alpha_n^2\beta_n^2 + 18\alpha_n^2\beta_n\gamma_n + 40\alpha_n\beta_n^3 \\
&+ 43\alpha_n\beta_n^2\gamma_n + 7\beta_n^3\gamma_n + 2\beta_n^2\gamma_n^2 + 6\alpha_n^3 + 60\alpha_n^2\beta_n + 64\alpha_n\beta_n^2 + 14\alpha_n\beta_n\gamma_n + 6\beta_n^3 + 10\beta_n^2\gamma_n \\
&+ 11\alpha_n^2 + 35\alpha_n\beta_n + 11\beta_n^2 + 3\beta_n\gamma_n + 6\alpha_n + 6\beta_n + 1
\end{aligned}
$$

$$
\begin{aligned}
\epsilon_{n+1} &= c_p(\alpha_{n+1} - \beta_{n+1}) = c_p\alpha_n\left(\frac{B_n}{A_n} - \frac{C_n}{B_n}\right) = \frac{c_p\alpha_n}{A_nB_n}\left(B_n^2 - A_nC_n\right) = c_p\tilde{\epsilon}_{n+1} \\
&= \frac{c_p}{A_nB_n}\left[\begin{array}{l}
\tilde{\epsilon}_n^2(4\alpha_n^3\beta_n^2 + 10\alpha_n^3\beta_n + 2\alpha^2\beta^2 + 5\alpha^3 + 7\alpha_n^2\beta_n + \alpha_n\beta_n^2 + 4\alpha_n^2 + 2\alpha_n\beta_n + \alpha_n) + \\
\tilde{\epsilon}_n\tilde{\epsilon}_n'(4\alpha_n^4\beta_n + 4\alpha_n^3\beta_n^2 + 10\alpha_n^3\beta_n + 6\alpha_n^2\beta_n^2 + 8\alpha_n^2\beta_n + 2\alpha_n\beta_n^2 + 2\alpha_n\beta_n) + \\
\tilde{\epsilon}_n'^2(4\alpha_n^3\beta_n^2 + 4\alpha_n^2\beta_n^2 + \alpha_n\beta_n^2)
\end{array}\right] \\
&= \frac{1}{c_pA_nB_n}\left[\begin{array}{l}
\epsilon_n^2(4\alpha_n^3\beta_n^2 + 10\alpha_n^3\beta_n + 2\alpha^2\beta^2 + 5\alpha^3 + 7\alpha_n^2\beta_n + \alpha_n\beta_n^2 + 4\alpha_n^2 + 2\alpha_n\beta_n + \alpha_n) + \\
\epsilon_n\epsilon_n'(4\alpha_n^4\beta_n + 4\alpha_n^3\beta_n^2 + 10\alpha_n^3\beta_n + 6\alpha_n^2\beta_n^2 + 8\alpha_n^2\beta_n + 2\alpha_n\beta_n^2 + 2\alpha_n\beta_n) + \\
\epsilon_n'^2(4\alpha_n^3\beta_n^2 + 4\alpha_n^2\beta_n^2 + \alpha_n\beta_n^2)
\end{array}\right] \\
&= \frac{1}{A_nB_n}\left[\begin{array}{l}
\epsilon_n(4\alpha_n^4\beta_n^2 - 4\alpha_n^3\beta_n^3 + 10\alpha_n^4\beta_n - 8\alpha_n^3\beta_n^2 + 2\alpha_n^2\beta_n^3 - 4\alpha_n^2\beta_n^2\gamma_n \\
+5\alpha_n^4 + 2\alpha_n^3\beta_n - 6\alpha_n^2\beta_n\gamma_n - \alpha_n\beta_n^2\gamma_n + 4\alpha_n^3 - 2\alpha_n^2\beta_n - 2\alpha_n\beta_n\gamma_n + \alpha_n^2 - \alpha_n\beta_n) + \\
\epsilon_n'(4\alpha_n^5\beta_n - 4\alpha_n^3\beta_n^2\gamma_n + 10\alpha_n^4\beta_n - 8\alpha_n^3\beta_n^2 + 2\alpha_n^2\beta_n^3 - 4\alpha_n^2\beta_n^2\gamma_n \\
+2\alpha_n^3\beta_n - \alpha_n\beta_n^2\gamma_n - \alpha_n^2\beta_n^2)
\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
\epsilon'_{n+1} &= c_p(\beta_{n+1} - \gamma_{n+1}) = c_p\alpha_n\left(\frac{C_n}{B_n} - \frac{D_n}{C_n}\right) = c_p\alpha_n\left(\frac{C_n^2 - B_nD_n}{B_nC_n}\right) \\
&= \frac{1}{c_pB_nC_n}\left[\begin{array}{l}
\epsilon_n^2(4\alpha_n\beta_n^4 + 12\alpha_n\beta_n^3 + 13\alpha_n\beta_n^2 + 6\alpha_n\beta_n + \alpha_n) + \\
\epsilon_n\epsilon'_n(4\alpha_n^2\beta_n^3 - 4\alpha_n^2\beta_n^2\gamma_n - 8\alpha_n\beta_n^4 + 18\alpha_n\gamma_n^3 + 2\alpha_n^2\beta_n^2 + 9\alpha_n\beta_n^2 + \alpha_n^2\beta_n + 2\alpha_n\beta_n) + \\
\epsilon_n'^2(4\alpha_n\beta_n^4 + 2\alpha_n\beta_n^3 + 2\alpha_n^2\beta_n^2 + \alpha_n\beta_n^2)
\end{array}\right] \\
&= \frac{1}{B_nC_n}\left[\begin{array}{l}
\epsilon_n(4\alpha_n^2\beta_n^4 - 4\alpha_n\beta_n^3\gamma_n^2 + 14\alpha_n^2\beta_n^3 - 12\alpha_n\beta_n^3\gamma_n - 2\alpha_n^2\beta_n^2\gamma_n \\
-12\alpha_n\beta_n^3 - \alpha_n\beta_n^2\gamma_n + 13\alpha_n^2\beta_n^2 - 2\alpha_n\beta_n\gamma_n + 6\alpha_n^2\beta_n - 4\alpha_n\beta_n^2 + \alpha_n^2 - \alpha_n\beta_n) + \\
\epsilon'_n(4\alpha_n^3\beta_n^3 - 4\alpha_n^3\beta_n^2\gamma_n + 14\alpha_n^2\beta_n^3 - 2\alpha_n\beta_n^3\gamma_n - 10\alpha_n^2\beta_n^3 \\
-2\alpha_n^2\beta_n^2\gamma_n - 7\alpha_n\beta_n^3 - \alpha_n\beta_n^2\gamma_n + \alpha_n^3\beta_n + 7\alpha_n^2\beta_n^2)
\end{array}\right]
\end{aligned}
$$