Classification

order

$$\frac{dy}{dx} = y^2$$

which is 1st order, x independent variable, y dependent variable

$$\frac{d^4y}{dt^4} + 5\frac{d^2x}{dt^2} + 3x\sin t$$

which is 4th order
because above equation only have 1 independent variable, they are ordinary differential equations(ODEs).

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = r$$

which is 1st order

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

which is 2nd order
above equation have more than one independent variable, they are partial differential equations(PDEs).

nth-order ODE: $F(x, y, y', \cdots, y^{(n)}) = 0$
In certain condition on F, it can be written as

$$y^{(n)} = f(x, y, y', \cdots, y^{(n)}) = 0(\star)$$

Example $(y')^2 + y' + xy = 0$

$$y' = \frac{-1 \pm \sqrt{1 - 4xy}}{2}$$

Definition. *a function $\phi(x)$ is called a solution of $(\star)$ on $a < x < b$ if $\phi^{(n)}$ exists on $a < x < b$ and*

$$\phi^{(n)}(x) = f(x, \phi(x), \phi'(x), \cdots, \phi^{(n-1)}(x)) \ \forall a < x < b$$

Example. Verify that $y = e^{2x}$ is a solution of $y'' + y' - 6y = 0$

Proof. $y'' + y' - 6y = 4e^{3x} + 2e^{2x} - 6e^{3x} = 0 \ \forall - \infty < x < \infty$
$\therefore y = e^{2x}$ is a solution on $-\infty < x < \infty$ ∎

Note. $y' = \dfrac{xy}{x+y+1}$ is derivative form $\Leftrightarrow$ $dy = \dfrac{xy}{x+y+1}dx$ or $xydx - (x+y+1)dy = 0$ is differential form.

<u>Definition</u>. *An ODE of order n is called linear if it may be written in the form*

$$b_0(x)y^{(n)} + b_1(x)y^{(n-1)} + \cdots + b_{n-1}(x)y' + b_n(x)y = R(x)$$

where $b_0 \neq 0$, An ODE that is not linear is called nonlinear ODE.

Example

linear

$$\begin{aligned} y'(x) + 5y'(x) + 6y(x) &= 0 \\ y'''(x) + x^2y''(x) + x^3y'(x) &= xe^x \end{aligned}$$

non linear

$$\begin{aligned} y''(x) + 5y'(x) + 6y^2(x) &= 0 \\ y''(x) + 5(y'(x))^3 + 6y(x) &= 0 \end{aligned}$$

Initial-Value Problem(IVP): same point (and 1st order)

$$\begin{cases} \dfrac{d^2y}{dx^2} + y = 0 \\ y(1) = 3 \\ y(1) = 2 \end{cases}$$

Boundary-Value Problem(BVP): two or more different points

$$\begin{cases} \dfrac{d^2y}{dx^2} + y = 0 \\ y(1) = 3 \\ y(2) = 2 \end{cases}$$

Theorem (Existence and uniqueness)**.** *Consider*

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

where $x_0, y_0 \in \mathbb{R}$ are given
Let $T = \{(x,y) \mid |x-x_0| \le a, |y-y_0| \le b\}$, where $a,b > 0$. Suppose that f and fy are continuous in T. Then (IVP) has a unique solution defined on $[x_0-h, x_0+h]$ for some $h > 0$

§ **Separable equation** $A(x)dx = B(y)dy$

Example.
(1) $\dfrac{dy}{dx} = \dfrac{2y}{x}$
Solution.

$$\begin{aligned} \frac{1}{y}dy &= \frac{2}{x}dx \\ \implies \int \frac{1}{y}dy &= \int \frac{2}{x}dx \\ \implies \ln|y| &= 2\ln|x| + C \end{aligned}$$

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(2) $\begin{cases} (1+y^2)dx + (1+x^2)dy = 0 \\ y(0) = -1 \end{cases}$
Solution.

$$\begin{aligned} (1+y^2)dx &= -(1+x^2)dy \\ \implies \frac{dx}{-(1+x^2)} &= \frac{dy}{(1+y^2)} \\ \implies \int \frac{1}{1+x^2}dx &= -\int \frac{1}{1+y^2}dy \end{aligned}$$

you can let $x = \tan\theta \implies dx = \sec^2\theta d\theta$
$\therefore \int \dfrac{1}{1+x^2}dx = \int \cos^2\theta \sec^2\theta d\theta = \theta + C = \tan^{-1}x + C$
$\implies \tan^{-1}x = -\tan^{-1}y + C$
$y(0) = -1 \implies 0 = \dfrac{\pi}{4} + C \implies C = \dfrac{\pi}{-4}$, $\therefore \tan^{-1}x = -\tan^{-1}y - \dfrac{\pi}{4}$

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(3) $\begin{cases} 2x(y+1)dx - ydy = 0 \\ y(0) = -2 \end{cases}$

Solution. $\int 2xdx = \int \dfrac{y}{y+1}dy \implies x^2 = y - \ln|y+1| + C$

$y(0) = -2 \implies 0 = -2 + c \implies c = 2 \ \therefore x^2 + y - \ln|y+1| + 2$ ∎

§ Homogeneous equations

<u>Definition</u>. *a function $f(x,y)$ is said to be homogeneous of degree k in x and y if and only if*

$$f(\lambda x, \lambda y) = \lambda^k f(x,y)$$

Example. $f(x,y) = x^2 + y^2$

$$\begin{aligned} f(\lambda x, \lambda y) &= (\lambda x)^2 + (\lambda y)^2 \\ &= \lambda^2(x^2 + y^2) \\ &= \lambda^2 f(x,y) \end{aligned}$$

$\therefore f(x,y)$ is homogeneous , $k = 2$

Theorem. *If $M(x,y)$ and $N(x,y)$ are both homogeneous and of the same degree, then $\dfrac{M(x,y)}{N(x,y)}$ is homogeneous of degree zero.*

Proof. Set $f(x,y) = \dfrac{M(x,y)}{N(x,y)}$. By definition, we assume M and N are homogeneous of degree k, so

$$M(\lambda x, \lambda y) = \lambda^k M(x,y) \text{ and } N(\lambda x, \lambda y) = \lambda^k N(x,y)$$

$\therefore f(\lambda x, \lambda y) = \dfrac{M(\lambda x, \lambda y)}{N(\lambda x, \lambda y)} = \dfrac{\lambda^k}{\lambda^k} \cdot \dfrac{M(x,y)}{N(x,y)} = \lambda^0 \dfrac{M(x,y)}{N(x,y)}$ ∎

Theorem. *If $f(x,y)$ is homogeneous of degree zero in x and y, then $f(x,y) = g(\frac{y}{x})$ for some function y.*

Proof. By assumption,

$$f(\lambda x, \lambda y) = \lambda^0 f(x,y) = f(x,y)$$

Take $\lambda = \frac{1}{x}$. Then $f(x,y) = f(1, \frac{y}{x}) = g(\frac{y}{x})$, where $g(v) = f(1,v)$. ∎

Corollary. *If $M(x,y)$ and $N(x,y)$ are both homogeneous and of the same degree, then $\dfrac{M(x,y)}{N(x,y)} = g(\frac{y}{x})$ for some function g.*

<u>Definition</u>. *$M(x,y)+N(x,y)dy=0$ is said to be homogeneous if it can be written as the form $\dfrac{dy}{dx}=g(\frac{y}{x})$ for some function g*

Example. $(x^2-3y^2)dx+2xydy=0(\star)$

$$\frac{dy}{dx}=-\frac{x^2-3y^2}{2xy}=-\frac{1-3(\frac{y}{x})^2}{2\cdot\frac{y}{x}}=g(\frac{y}{x}) \text{ where } g(v)=\frac{1-3v^2}{-2v}$$

Remark. *If $M(x,y)$ and $N(x,y)$ are homogeneous of the same degree, then $M(x,y)dx+N(x,y)dy=0$ is homogeneous.*

Proof. By assumption and corollary, $\dfrac{M(x,y)}{N(x,y)}=g(\dfrac{y}{x})$ for some function g. $\therefore Mdx+Ndy=0 \implies \dfrac{dy}{dx}=-\dfrac{M(x,y)}{N(x,y)}=-g(\dfrac{y}{x})$
$\therefore Mdx+Ndy$ is homogeneous. ∎

How to solve homogeneous equation
Suppose$M(x,y)dx+N(x,y)dy=0(\star)$ is homogeneous.
Let $y=vx \implies \dfrac{dy}{dx}=\dfrac{dv}{dx}x+v(1)$
$\because(\star)$ is homogeneous $\therefore$ By definition, $(\star)\Leftrightarrow\dfrac{dy}{dx}=g(\dfrac{y}{x})(2)$
where g is a function.

$$\begin{aligned}\text{put (1) to (2)} \quad &\implies \frac{dv}{dx}x+v=g(v)\\ &\implies \frac{dv}{dx}x=g(v)-v\\ &\implies \frac{1}{g(v)-v}dv=\frac{1}{x}dx, \text{ which is separable}\end{aligned}$$

$\therefore$ The solution is $\int\dfrac{1}{g(v)-b}dv=\int\dfrac{1}{x}dx$
Example. $(x^2-xy+y^2)dx-xydy=0$—(1)

Solution.

$$\begin{aligned}M(\lambda x,\lambda y)&=(\lambda x)^2-(\lambda x)(\lambda y)+(\lambda y)^2\\&=\lambda^2(x^2-xy+y^2)\\&=\lambda^2M(x,y)\end{aligned}$$

$$\begin{aligned} N(\lambda x, \lambda y) &= -(\lambda x)(\lambda y) \\ &= -\lambda^2 xy \\ &= \lambda^2 N(x,y) \end{aligned}$$

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so (1) is homogeneous.
Let $y = vx \implies \dfrac{dy}{dx} = \dfrac{dv}{dx}x + v$

$$\begin{aligned} (1) &\implies \frac{dy}{dx} = -\frac{x^2 - xy + y^2}{xy} = \frac{1 - \frac{y}{x} + (\frac{y}{x})^2}{\frac{y}{x}} \\ &\implies \frac{dv}{dx}x + v = \frac{1 - v + v^2}{v} \\ &\implies \frac{dv}{dx}x = \frac{1 - v + v^2}{v} - v = \frac{1-v}{v} \\ &\implies \int \frac{v}{1-v} dv = \int \frac{1}{x} \\ &\implies \frac{v - 1 + 1}{1 - v} = -1 - \frac{1}{v-1} \\ &\implies -v - \ln|v-1| = \ln|x| + c \end{aligned}$$

Example. $xydx + (x^2 + y^2)dy = 0$

Solution. $\dfrac{dy}{dx} = \dfrac{xy}{-(x^2+y^2)} = -\dfrac{\frac{y}{x}}{1 + (\frac{y}{x})^2} = g(\frac{y}{x})$ —(1),
where $g(v) = \dfrac{-v}{1+v^2}$, so the equation is homogeneous.
Let $y = vx \implies \dfrac{dy}{dx} = \dfrac{dv}{dx}x + v$

$$\begin{aligned}
(1) \quad &\Longrightarrow \quad \frac{dv}{dx}x + v = -\frac{v}{1+v^2} \\
&\Longrightarrow \quad \frac{dv}{dx}\cdot x = -\frac{v}{1+v^2} - v = -\frac{2v+v^3}{1+v} \\
&\Longrightarrow \quad \int \frac{1+v^2}{2v+v^3}dv = -\int \frac{1}{x}dx \\
&\Longrightarrow \quad \int (\frac{0.5}{v} + \frac{0.5v+0}{2+v^2}) = -\int \frac{1}{x}dx \\
&\therefore \quad 0.5\int \frac{1}{v} + \frac{v}{2+v^2}dv = -\ln|x| + c \\
&\Longrightarrow \quad 0.5\ln|v| + 0.25\ln|2+v^2| + -\ln|x| + c \\
&\Longrightarrow \quad 0.5\ln|\frac{y}{x}| + 0.25\ln|2+\frac{y^2}{x^2}| = -\ln|x| + c
\end{aligned}$$

■

§ **Exact equation**

<u>Definition</u>. *$M(x,y)dx + N(x,y)dy = 0$ is called an exact equation if there exists a function $F(x,y)$ such that $F_x = M$ and $F_y = N$*

Example.
$y^2dx + 2xydy = 0$ Set $F(x,y) = xy^2 \implies Fx = y^2$ and $Fy = 2xy$
$\therefore$ exact equation.
How to solve homogeneous equation
Suppose $M(x,y)dx + N(x,y) = 0$ —$(\star)$ is exact
$\implies \exists$ a function $F(x,y)$ such that $Fx = M$ and $Fy = N$

$$\begin{aligned}
(\star) \quad &\Longrightarrow \quad Fxdx + Fydy = 0 \\
&\Longrightarrow \quad dF = 0 \\
&\Longrightarrow \quad F = C, \text{ where } C \text{ is an arbitrary constant}
\end{aligned}$$

Theorem. *Suppose M, N, My, Nx are continuous. Then $Mdx+Ndy = 0$ is an exact equation $\Leftrightarrow$ $My = Nx$*

Proof. $(\Leftarrow)$ Suppose $My = Nx$. Claim $(\star)$ is exact

$$\begin{cases} Fx = M\text{—}(1) \\ Fy = N\text{—}(2) \end{cases}$$

(1) $\Leftrightarrow F(x,y) = \int M(x,y)\partial x + \phi(y)$ for some function ϕ

$$\begin{aligned}(1)(2) \quad &\Leftrightarrow \quad \frac{\partial}{\partial y}\int M(x,y)\partial x + \phi'(y) = N(x,y)\\ &\Leftrightarrow \quad \phi'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\partial x = N(x,y) - \int My(x,y)\partial x\end{aligned}$$

We compute

$$\frac{\partial}{\partial}\left[N(x,y) - \int My(x,y)\partial x\right] = Nx(x,y) - My(x,y) = 0$$

This implies $N(x,y) - \int My(x,y)\partial x$ is independent of x
$\therefore \phi(y) = \int \left[N(x,y) - \int My(x,y)\partial x\right] dy$
$\therefore F(x,y) = \int M(x,y)\partial x + \int \left[N(x,y) - \int My(x,y)\partial x\right] dy$ satisfy (1)(2)
$\therefore Mdx + Ndy = 0$ is exact.

■

Example. $3x(xy-2)dx + (x^3+2y)dy = 0$

Solution. $\therefore My = 3x^2 = Nx \ \therefore$ exact
Find a function F s.t. $\begin{cases} Fx = M = 3x^2y - 6x \text{— (1)} \\ Fy = N = x^3 + 2y \ \text{— (2)} \end{cases}$

$$\begin{aligned}(1) \implies F \quad &= \quad \int (2x^3 - xy^2 - 2y + 3)\partial x + \phi(y)\\ &= \quad \frac{1}{3}x^4 - \frac{1}{2}x^2y^2 - 2xy + 3x + \phi(y)\\ &\implies \quad -x^2y - 2x = Fy = -x^2y - 3x + \phi'(y) \implies \phi'(y) = c\\ &\implies \quad \phi(y) = 0 \implies F = x^4 - \frac{1}{2}x^2y^2 - 2xy + 3x\end{aligned}$$

$\therefore \frac{1}{2}x^4 - \frac{1}{2}x^2y^2 - 2xy + 3x = C$ is the general solution ■

§ **Linear equations**: $A(x)y' + B(x)y = c(x)$, where $A(x) \neq 0$
Suppose $c(x) = 0$

$$\begin{aligned}&\implies \quad A(x)y' + B(x)y = 0\\ &\implies \quad A(x)y' = -B(x)y \implies \frac{1}{y}dy = \frac{B(x)}{-A(x)}dx, \text{ which is separable}\end{aligned}$$

Suppose $B(x) = 0 \implies$ it's easy to solve
Suppose $c(x) \neq 0$ and $B(x) \neq 0$

$A(x)y' + B(x)y = c(x) \implies y' + p(x)y = Q(x)$, where $p(x) = \dfrac{B(x)}{A(x)}$ and $Q(x) = \dfrac{C(x)}{A(x)}$

$\Leftrightarrow dF = 0 \Leftrightarrow Fxdx + Fydy = 0$

$\Leftrightarrow (P(x)y - Q(x))dx + 1dy = 0$—(1) is not exact.

Let $v = v(x)$, (1) $\times v \implies v(P(x)y - Q(x))dx + v(x)dy$ — (2)

$$\begin{aligned}
(2) \text{ is exact} \quad &\Leftrightarrow \quad \frac{\partial}{\partial y}\left[v(x)P(x)y - Q(x)\right] = v'(x) \\
&\Leftrightarrow \quad v(x)P(x) = v'(x) \Leftrightarrow P(x)dx = \frac{1}{v}dv \\
&\Leftrightarrow \quad \ln|v| = \int P(x)dx \Leftrightarrow |v| = e
\end{aligned}$$

Let $v(x) = e^{\int p(x)dx}$, which is called the integrating factor of (1)