2. Topological Space and Continuous functions

We will introduce some basic topological space.
e.g. Order topology, Product topology, Subspace topology,
Metric topology, (Quotient topology)

§ 12 Topological Spaces.

Definition. *Let X be a nonempty set $\mathscr{P}(X) = 2^X$ power set of X. We say that $\mathscr{T} \subseteq \mathscr{P}(X)$ is a topology on X if*

(1) $\emptyset, X \in \mathscr{T}$
(2) $U_\alpha \in \mathscr{T},\ \alpha \in I \implies \bigcup_{\alpha\in I} U_\alpha \in \mathscr{T}$
(3) $U_1, \cdots, U_n \in \mathscr{T} \implies U_1 \cap \cdots \cap U_n \in \mathscr{T}$

If $\mathscr{T}$ is a topology on X, then the pair $(X, \mathscr{T})$ or simply X is called a topological space and members in $\mathscr{T}$ are called open sets in X

Example.

(1) $X = \{a, b, c\}$
  (a) The following are topological space on X, $\mathscr{T}_1 = \{\emptyset, X\}$, $\mathscr{T}_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\mathscr{T}_3 = \mathscr{P}(X)$
  (b) The following are not topology on X
  $\mathscr{A} = \{\emptyset, \{a\}, \{b\}, X\}$ ($\because \{a\} \cup \{b\} = \{a, b\} \notin \mathscr{A}$)
  $\mathscr{B} = \{\emptyset, \{a, b\}, \{b, c\}, X\}$ ($\because \{a, b\} \cap \{b, c\} = \{b\} \notin \mathscr{B}$)
(2) Any set with more than 1 element has at least two topology $\{\emptyset, X\}$(in discrete topology) and $\mathscr{P}(X)$(discrete)
and former is smallest one, another is the largest one.

Definition. $\mathscr{T}_{op} = \{\mathscr{T} \mid \mathscr{T}$ *is a topology on* $X\}$ $\mathscr{T}_1 \leq \mathscr{T}_2 \Leftrightarrow \mathscr{T}_1 \subseteq \mathscr{T}_2$
Claim "$\leq$" is a partial ordering on $\mathscr{T}_{op}$

- ⋆ *Reflexive:* $\forall \mathscr{T} \in \mathscr{T}_{op},\ \mathscr{T} \leq \mathscr{T}$
- ⋆ *Anti-symmetry:* $\forall \mathscr{T}_1, \mathscr{T}_2 \in \mathscr{T}_{op},\ \mathscr{T}_1 \leq \mathscr{T}_2$ *and* $\mathscr{T}_2 \leq \mathscr{T}_1 \implies \mathscr{T}_1 = \mathscr{T}_2$
- ⋆ *Transitive:* $\forall \mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3 \in \mathscr{T}_{op},\ \mathscr{T}_1 \leq \mathscr{T}_2$ *and* $\mathscr{T}_2 \leq \mathscr{T}_3 \implies \mathscr{T}_1 \leq \mathscr{T}_3$

Example. Let X be a set, $\mathscr{T}_f = \{U \subseteq X,\ U = \emptyset$ or $X - U$ is finite $\}$
Then $\mathscr{T}_f$ is a topology on X, called the "finite complement topology" on X

Proof.

(1) $\emptyset, X \in \mathscr{T}_f$ ($\because X - X = \emptyset$)
(2) $U_\alpha \in \mathscr{T}_f,\ \alpha \in I$
If $\bigcup_{\alpha\in I} U_\alpha = \emptyset$, then $\bigcup_{\alpha\in I} U_\alpha \in \mathscr{T}_f$.
If $U_{\alpha\in I} U_\alpha \neq \emptyset$, then $\exists \alpha_0 \in I \ni U_{\alpha_0} \neq \emptyset$ and $X - U_\alpha$ is finite

$X - \bigcup_{\alpha \in I} U_\alpha = \bigcap_{\alpha \in I}(X - U_\alpha) \subseteq X - U_{\alpha_0} \implies X - (\bigcup_{\alpha \in I} U_\alpha)$ is finte $\implies \bigcup_{\alpha \in I} U_\alpha \in \mathscr{T}_f$

(3) $U_1, \cdots, U_n \in \mathscr{T}_f$
If $U_1 \cap \cdots \cap U_n = \emptyset$, then $U_1 \cap \cdots \cap U_n \in \mathscr{T}_f$
If $U_1 \cap \cdots \cap U_n \neq \emptyset$, then $X - (U_1 \cap \cdots \cap U_n) = (X - U_1) \cap \cdots \cup (X - U_n)$ is finite since each $X - U_i$ is finite. Thus $U_1 \cap \cdots \cap U_n \in \mathscr{T}_f$

From (1)(2)(3), $\mathscr{T}_f$ is a topology on X. ■

Remark. *If X is a finite set, then $\mathscr{T}_f$ is the discrete topology on X*

Example. Let X be a set and $\mathscr{T}_c = \{U \subseteq X \mid U = \emptyset$ or $X - U$ is countable $\}$. Then as in example above, $\mathscr{T}_c$ is a topology on X, called the countable complement topology on X. Moreover, if X is countable, then $\mathscr{T}_c$ is just a discrete topology on X

Definition. *Let $\mathscr{T}$ and $\mathscr{T}'$ be two topologies on X. We say that $\mathscr{T}'$ is (strictly) finer then $\mathscr{T}$ or $\mathscr{T}$ is (strictly) coaser that $\mathscr{T}'$ if $\mathscr{T} \leq \mathscr{T}'(\mathscr{T} < \mathscr{T}')$, i.e. $\mathscr{T} \subseteq \mathscr{T}'(\mathscr{T} \subsetneqq \mathscr{T}')$*

Remark.

(1) Two topologies on X need not be comparable
(2) Other terminology, if $\mathscr{T}' \supset T$, $\mathscr{T}'$ is larger(stronger) than $\mathscr{T}$ and $\mathscr{T}$ is smaller(weaker) than $\mathscr{T}$

§ 13 Bases for a topology.

Definition. *Let X be a set. A base for a topology on X is a collection $\mathscr{B} \subseteq \mathscr{P}(X)$ satisfying*

(1) $U\mathscr{B} = X$ $(\bigcup \mathscr{B} = \bigcup_{B \in \mathscr{B}} B)$
(2) Given $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$ $\exists B_3 \in \mathscr{B} \ni x \in B_3 \subseteq B_1 \cap B_2$

Members in $\mathscr{B}$ are called basic open sets in X

Given a base $\mathscr{B}$ for a topology on X, we can define the smallest topology $\mathscr{T}$ on X containing $\mathscr{B}$ called the topology on X generated by $\mathscr{B}$.
Usually, there are two ways to describe it

(I) $\mathscr{T} = \{U \subseteq X, \forall x \in U \exists B \in \mathscr{B} \ni x \in B \subseteq U\}$. Clearly, $\mathscr{B} \subseteq \mathscr{T}$
(a) $\emptyset, X \in \mathscr{T}$(by the definition of bases (1))
(b) $U_\alpha \in \mathscr{T},\ \alpha \in I \implies \bigcup_{\alpha \in I} U_\alpha \in \mathscr{T}$. Given $x \in \bigcup_{\alpha \in I} U_\alpha, x \in U_{\alpha_0}$ for some $\alpha_0 \in I$, $\exists B \in \mathscr{B} \ni x \in B \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in I} U_\alpha$
(c) $U_1, \cdots, U_n \in \mathscr{T} \implies U_1 \cap \cdots \cap U_n \in \mathscr{T}$. By induction on n, we only prove $n = 2$. Given $x \in U_1 \cap U_2$, $x \in U_1$ and $x \in U_2$ $\implies \exists B_1, B_2 \in \mathscr{B} \ni x \in B_1 \subseteq U_1$ and X in $\mathscr{B}_2 \subseteq U_2 \implies$

$x \in B_1 \cap B_2 \subseteq U_1 \cap U_2 \implies \exists B_3 \in \mathscr{B} \ni x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2 \implies U_1 \cap U_2 \in \mathscr{T}$

(II) $\mathscr{T}' = \{\bigcup \mathscr{A} \mid \mathscr{A} \subseteq \mathscr{B}\} = \{\bigcup_{\alpha \in I} A_\alpha \mid A_\alpha \in \mathscr{B}\}$

Clearly, $\mathscr{B} \subseteq \mathscr{T}'$(only choose one element in $\mathscr{B}$)

(a) $\emptyset, X \in \mathscr{T}'$(trivial)

(b) $U_\alpha \in \mathscr{T}',\ \alpha \in I \implies \bigcup_{\alpha \in I} U_\alpha \in \mathscr{T}'$
$\forall \alpha \in I, U_\alpha = \bigcup_{\beta \in I_\alpha} A_\beta$. Then $\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} \bigcup_{\beta \in I_\alpha} A_\beta \implies \bigcup_{\alpha \in I} U_\alpha \in \mathscr{T}'$

(c) $U_1, \cdots, U_n \in \mathscr{T}' \implies U_1 \cap \cdots \cap U_n \in \mathscr{T}'$. By induction on n, we only to prove that $n = 2$. For $i = 1, 2, \cdots,\ U_i = \bigcup_{\alpha \in I_j} A_\alpha$.
$U_1 \cap U_2 = \bigcup_{\alpha \in I_2} (A^1_\beta \cap A^2_\alpha)$. $\forall x \in U_1 \cap U_2,\ x \in A'_\beta \cap A^2_\alpha \implies U_1 \cap U_2 = \bigcup_{x \in U_1 \cap U_2} B_X \in \mathscr{T}'$

(III) $\mathscr{T} = \mathscr{T}'$

($\subseteq$) Given $U \in \mathscr{T},\ \forall x \in U,\ \exists B_x \in \mathscr{B} \ni x \in B_x \subseteq U \implies U = \bigcup_{x \in U} B_x \in \mathscr{T}'$

($\supseteq$) Given $U \in \mathscr{T}'$ $U = \bigcup_{\alpha \in I} A_\alpha, A_\alpha \in \mathscr{B}$
$\forall x \in U, x \in A_\alpha$ for some $\alpha \in I$ and $A_\alpha \in \mathscr{B}$, i.e. $X \in A_\alpha \in U$ and $A_\alpha \in \mathscr{B} \implies U \in \mathscr{T}$. Hence $\mathscr{T} = \mathscr{T}'$

Example.

(1) Let $\mathscr{B}$ be the collection of all open balls in $\mathbb{R}^n$. Then $\mathscr{B}$ is a base for a topology on $\mathbb{R}^n$, namely, then Euclidean topology on $\mathbb{R}^n$

(2) Let $\mathscr{B}'$ be the collection of all $n-$dimentional open intervals in $\mathbb{R}$. Then $\mathscr{B}'$ is a base for a topology on $\mathbb{R}^n$. In fact, β and β' generate the same topology on $\mathbb{R}^n$

Lemma. *Let X be a set, let $\mathscr{B}$ be a basis for a topology $\mathscr{T}$ on X. $\mathscr{T}$ equals the collection if all unions of elements of $\mathscr{B}$.*

Lemma. *Let X be a topological space and $\mathscr{C}$ be a collection of open sets of $X \ni \forall$ open set U in X and $\forall x \in U$ $\exists C \in \mathscr{C} \ni x \in C \subseteq U$. Then $\mathscr{C}$ is a base for the topology of X.*

Proof. (1) $\bigcup \mathscr{C} = X$
Since X is open $\forall x \in X,\ \exists C_x \in \mathscr{C} \ni x \in C_x \subseteq X \implies x \in \bigcup \mathscr{C} \implies X = \bigcup \mathscr{C}$

(2) Given $C_1, C_2 \in \mathscr{C}$ and $x \in C_1 \cap C_2$. Since $C_1 \cap C_2$ is open, $\exists C \in \mathscr{C} \ni x \in C \subseteq C_1 \cap C_2, \therefore \mathscr{C}$ is a base for a topology of X ∎

Remark. *Let $\mathscr{T}$ be the original topology on X and $\mathscr{T}'$ be the topology generated by $\mathscr{C}$. Then $\mathscr{T} = \mathscr{T}'$*

Proof.

($\subseteq$) Given $U \in \mathscr{T}$, $\forall x \in U \exists C \in \mathscr{C} \ni x \in C \subseteq U \implies U \in \mathscr{T}'$

($\supseteq$) Given $v \in \mathscr{T}'$, by lemma, $V = \bigcup \mathscr{A}$ for some $A \subseteq \mathscr{C}$. Since $\mathscr{C} \subseteq \mathscr{T}$, $\mathscr{A} \subseteq \mathscr{T}$, $\therefore V = \bigcup \mathscr{A} \in \mathscr{T}$

■

Lemma. *Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topology $\mathscr{T}$ and $\mathscr{T}'$ on X respective TFAE*

(1) $\mathscr{T}$ is finer that $\mathscr{T}$ i.e. $\mathscr{T} \subseteq \mathscr{T}'$

(2) $\forall x \in X$ and $B \in \mathscr{B}$ with $x \in B$, $\exists B' \in \mathscr{B} \ni x \in B' \subseteq B$

Proof.

$(a) \Rightarrow (b)$ Suppose $\mathscr{T} \subseteq \mathscr{T}'$. Given $x \in X$ and $B \in \mathscr{B}$ with $x \in B$. Since $\mathscr{T} \subseteq \mathscr{T}'$, $B \in \mathscr{T}, \exists B' \in \mathscr{B} \ni x \in B' \subseteq B$

$(b) \Rightarrow (a)$ Suppose (b) holds. Given $U \in \mathscr{T}$, $\forall x \in U$, $\exists B_x \in \mathscr{B} \ni x \in B_x \subseteq U$. By (b), $\exists B'_x \in \mathscr{B} \ni x \in B'_x \subseteq B_x \subseteq U \implies U \in \mathscr{T}'$

■

Example. In §13, example 1,2

$\mathscr{B}$: all open balls in $\mathbb{R}^n$ for a topology on $\mathbb{R}^n$

$\mathscr{B}'$:all open intervals in $\mathbb{R}^n$ for a topology on $\mathbb{R}^n$

By lemma above, they generate the same Euclidean topology on $\mathbb{R}^n$

We now define 3 topologies on the real line $\mathbb{R}$

Definition.

(1) $\mathscr{B} = \{(a, b) \mid -\infty < a < b < \infty\}$: the collection of all open intervals in $\mathbb{R}$ which is the base for the usual topology on $\mathbb{R}$

(2) $\mathscr{B}' = \{[a, b) \mid -\infty < a < b < \infty\}$ the collection of all closed-open interval in $\mathbb{R}$, which is also a base for a topology of $\mathbb{R}$ called the lower limit topology on $\mathbb{R}$. We denote it by $\mathbb{R}_l$

(3) Let $K = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and $\mathscr{B}'' = \{B \subseteq \mathbb{R} \mid B = (a, b)$ or $B = (a, b) - K$ for $-\infty < a < b < \infty\}$. Claim: $\mathscr{B}'$ is a base for a topology on $\mathscr{T}$

- *$\star$ Clearly, $U\mathscr{B}'' = \mathbb{R}$*
- *$\star$ Given $B_1, B_2 \in \mathscr{B}''$ and $x \in B_1 \cap B_2$. We have 4 cases:*
  - *(i) B_1 and B_2 are open intervals which is clearly.*
  - *(ii) $B_1 = (a, b)$ and $B_2 = (c, d) - K$. Let $\alpha = \max\{a, c\}$ and $\beta = \min\{b, c\}$. $x \in (\alpha, \beta) - K \subseteq B_1 \cap B_2$ and $(\alpha, \beta) - K \in \mathscr{B}''$*
  - *(iii) (3)(4) similarly*

The topology on $\mathbb{R}$ generated by B' is called the $K-$topology on $\mathbb{R}$ and denoted $\mathbb{R}_k$

Lemma. *The topologies of $\mathbb{R}_l$ and $\mathbb{R}_k$ are strictly finer than the Euclidean topology of $\mathbb{R}$ but are not comparable with one another*

Proof. Let $\mathscr{T}, \mathscr{T}'$ and $\mathscr{T}''$ be the topologies of $\mathbb{R}, \mathbb{R}_l, \mathbb{R}_k$ generated by $\mathscr{B}, \mathscr{B}', \mathscr{B}''$ respectly. We use lemma above to prove it.

- ⋆ $\mathscr{T} \subsetneqq \mathscr{T}'$ Given $(a, b) \in \mathscr{B}$ and $x \in (a, b)$. We have $[x, b) \in \mathscr{B}'$ with $x \in [x, b) \subseteq (a, b)$. By lemma, $\mathscr{T} \subseteq \mathscr{T}'$, $\forall a < b,\ [a, b) \in \mathscr{B}'$ so $[a, b) \in \mathscr{T}'$, but $[a, b)' \notin \mathscr{T}$
- ⋆ Clearly, $\mathscr{T} \subseteq \mathscr{T}''$ by $\mathscr{B} \subseteq \mathscr{B}''$. Moreover $B'' = (-1, 1) - K \in \mathscr{B}''$, so $B'' \in \mathscr{T}''$ but $B'' \notin \mathscr{T}$.
- ⋆ $\mathscr{T}'$ and $\mathscr{T}''$ are not comparable
  $(-1, 1) - K \in \mathscr{T}''$, but $(-1, 1) - K \notin \mathscr{T}'(\because$ not $[0, c) \in \mathscr{B}' \ni 0 \in [0, c) \subseteq (-1, 1) - K)$. $[0, 1) \in \mathscr{T}$ but no $\mathscr{B}'' \in \mathscr{B}'' \ni 0 \in B'' \subseteq [0, 1)$

■

Definition. *A subbase $\mathscr{S}$ for a topology on X is a collection of subsets of X with $\bigcup \mathscr{S} = X$ and elements in $\mathscr{S}$ are calle subbasic open sets in X*

Given subbase on X

$$\mathscr{B} = \{S_1 \cap \cdots \cap S_k,\ k \in \mathbb{N}, S_1, \cdots, S_k \in S\}$$

Claim $\mathscr{B}$ is a base for a topology on X

Definition. *The topology on X generated by a subbase $\mathscr{S}$ is defined to be the topology generated by the base $\mathscr{B}$.*

The Order Topology. (which provides many counterexample in topology)

Definition. *A relation C on a set is called an "order relation" (or a simple order) if it satisfies*

(1) Comparable: $\forall x \neq y$ in X either xCy or yCx
(2) Non-reflexivity: no xCx
(3) Transitivity: xCy and $yCz \implies xCz$

Given a simple order set $(X, <)$ and $a, b \in X$ with $a < b$(Note: $a \leq b$ means $a < b$ or $a = b$). We can define:
$(a, b) = \{x \in X \mid a < x < b\}$ open interval
$(a, b] = \{x \in X \mid a < x \leq b\}$ open interval
$[a, b) = \{x \in X \mid a \leq x < b\}$ open interval
$[a, b] = \{x \in X \mid a \leq x \leq b\}$ open interval
We assume that $|X| \geq 2$. Let $\mathscr{B}$ be the collection of all subsets of the following types

(1) All open intervals (a, b) in X

(2) All intervals of the forms $[a_0, b)$ where a_0 is the smallest elements of X
(3) All intervals of the forms $(a, b_0]$ where b_0 is the largest elements of X

Definition. *The topology generated by $\mathscr{B}$ is called the order topology on X*

Example.

(1) If X is an order set and $T \subseteq X$, then so is Y
(2) In $\mathbb{R}$ we give the usually ordering and the order topology on $\mathbb{R}$ is the usual topology on $\mathbb{R}$
(3) In $\mathbb{R}^\star = \mathbb{R} \cup \{-\infty, \infty\}$ with the usual ordering is an order set.
(4) In $\mathbb{R} \times \mathbb{R}$ with the dictionary order is an order set whose basis for the order topology is of the form
(5) $\mathbb{N}$ with the usual ordering is an order set with the smallest element 1. What is the order topology?
   - $\star$ $[1, b) : b \in \mathbb{N}$ and $(a, b),\ a < b$. In particular, $\{1\} = [1, 2)$ and $\{n\} = (n-1, n+1), n > 1$ are basic open sets in $\mathbb{N}$ $\therefore$ the order topology on $\mathbb{N}$ is the discrete topology on $\mathbb{N}$
(6) The set $X = \{1, 2\} \times \mathbb{N} = \{1 \times n\}_{n=1}^{\infty} = a_n \cup b_n = \{2 \times n\}_{n=1}^{\infty}$ in the dictionary order with the smallest element 1×1. The order topology on X is not discrete topology on X
   $X : a_1, a_2, \cdots, b_1, b_2, \cdots,\ a_i < a_{i+1},\ b_j < b_j + 1,\ a_i < b_j$
   $\star$ $\{a_1\} = [a_1, a_2)$
   $\star$ $\{a)n\} = (a_{n-1}, a_{n+1}), n \geq 2$
   $\star$ $\{b_n\} = (b_{n-1}, , b_{n+1}), n \geq 2$
   But $\{b_1\}$ is not open, $\because$ b_1 is not the smallest elements any basic open set in the order topology containing b_1 must of the form (a_l, b_j) for some $l \geq 1$ and $j > 1$

Definition. *Let X be an ordered set and $a \in X$. We define the rays determine by a*

- $\star$ $(a, \infty) = \{x \in X \mid x > a\}$
- $\star$ $(-\infty, a) = \{x \in X \mid x < a\}$
- $\star$ $[a, \infty) = \{x \in X \mid x \geq a\}$
- $\star$ $(\infty, a] = \{x \in X \mid x \leq a\}$

Some facts:

(1) open rays in X are open in the order topology of X. In fact, $(a, \infty) = (a, b_0]$ if X has the largest element which is a basic open set in the order topology of X. If X has no largest element, then $(a, \infty) = \bigcup_{a<x}(a, x)$ which is open in the order topology of X

(2) closed rays is close
(3) The order topology of X is contained in the topology on X generated by open rays in X. $\because (a, b) = (a, \infty) \cap (-\infty, b)$.
If X has the smallest element a_0, $[a_0, b) = (-\infty, b)$
If X has the largest element b_0, $(a, b_0] = (a, \infty)$