Advance Calculus Exercise

Exercise 1 (Chapter 1)

1. A complex number z is said to be algebraic if there are integers $a_0, \cdots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable.

Solution. Let A be the set of all algebraic numbers. We denoted some notation

(a) $\mathbb{Z}[x]^X$: the set of all non-zero polynomials having coefficients in $\mathbb{Z}$

(b) $Z_{p(x)}$: the set of all roots of $p(x)$, where $p(x) \in \mathbb{Z}[n]^X$

Note that

(a) $A \subseteq U_{p(x) \in \mathbb{Z}[x]^X} Z_{p(x)}$ and $Z_{p(x)}$ is finite since a polynomial of degree n has at most n roots

(b) $\mathbb{Z}[x]^X = \cup_{\infty}^{n=1} F_n$, where $F_n \subseteq \mathbb{Z}[x]^X$ the set of polynomials in $\mathbb{Z}[x]^X$ of degree n

Claim $\mathbb{Z}[x]^X$ is countable. If we are done, then $\cup_{p(x) \in \mathbb{Z}[x]} Z_{p(x)}$ is countable union of finite set, it follows that A is countable.

To prove this claim, it satisfies to show that F_n is countable $\forall n \geq 0 (F_n = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n)$

Consider the map $G : F_n \to \mathbb{Z}^{n+1}$

$$G(a_0 x^n + a_1 x^{n+1} + \cdots + a_{n-1} x + a_n) = (a_0, a_1, \cdots, a_n)$$

Clearly, G is injective. This implies F_n is countable $\forall n \geq 0$ ■

2. Prove that there exists real numbers which are not algebraic.

Solution. If not, $\forall r \in \mathbb{R}$, r is algebraic. This implies the set of all algebraic numbers is uncountable $(\rightarrow\leftarrow)$ to exercise 1. ■

3. Is the set of all irrational real numbers countable?

Solution. No! If it were, $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})$ is countable $(\rightarrow\leftarrow)$ ■

4. Construct a bounded set of real numbers with exactly three limit points.

Solution. The idea come from the set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$
In fact, $\forall x \in \mathbb{R}$, the set

$$E = \{x + \frac{1}{n} \mid n \in \mathbb{N}\}$$

has exactly one limit point, namely, x. Clearly, x is a limit point of E. We prove that E does not have any other limit point.
Given $y \in \mathbb{R}$. If $y < x$ or $y \geq x+1$, then it is clear that y cannot be limit point.
If $x < y < x+1$, then we have two cases ■

5. Let X be a metric space and $E \subset X$. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points. Do E and E' always have the same limit points?

Solution.

⋆) E' is closed i.e. $(E')' \subseteq E'$
"Fact A is closed iff $A' \subseteq A$", given $p \in (E')'$ and $r > 0$ $B(p,r) \cap E' - \{p\} \neq \emptyset$. Choose $q \in B(p,r) \cap E' - \{p\}$. Since $q \in B(p,r)$ $\exists \delta > 0 \ni B(q,\delta) \subseteq B(p,r)$. Also $B(q,\delta) \cap E - \{q\} \subseteq B(p,r) \cap E - \{q\}$. Since $q \in E'$, we get $B(q,\delta) \cap E - \{q\} \neq \emptyset$ and it is an infinite set. This implies $B(p,r) \cap E - \{p\} \neq \emptyset \implies p \in E'$. Hence, E' is closed.

⋆) $(E)' = (\overline{E})'$. Clearly $(E)' \subseteq (\overline{E})'$ ($\because E \subseteq \overline{E}$) ($\supseteq$) If not, $\exists x \in (\overline{E})', x \notin E'$, i.e. $\exists r > 0, B(x,r) \cap E - \{x\} = \emptyset$. Since $x \in (\overline{E})'$, $B(x,r) \cap \overline{E} - \{x\} \neq \emptyset$. Choose $y \in B(x,r) \cap \overline{E} - \{x\}$. Since $y \in B(x,r) \exists \delta > 0 \ni B(y,\delta) \subseteq B(x,r)$
$B(y,\delta) \cap E \subseteq B(x,r) \cap E \subseteq \{x\} \implies B(y,\delta) \cap E - \{y\}$ is a finite set $\implies y \notin E'$ From above, we get $y \notin E'$ and $y \in \overline{E}(\rightarrow\leftarrow)$ to $B(x,r) \cap E - \{x\} = \emptyset$. Hence $E' = (\overline{E})'$

⋆) E and E' may not have the same limit points, e.g.Consider $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ in $\mathbb{R}$. $E' = \{0\}$ but $(E')' = \emptyset$

■

6. Let $A_1, A_2, \cdots$ be subset of a metric space.
(a) Prove that $\bigcup_{i=1}^{n} \overline{A_i} = \overline{\bigcup_{i=1}^{n} A_i}$ $\forall n \in \mathbb{N}$
(b) Prove that $\bigcup_{i=1}^{\infty} \subseteq \overline{\bigcup_{i=1}^{\infty}}$. Show by an example that this inclusion can be proper.

Solution.

(a) Since $A_i \subseteq \overline{A_i} \forall i = 1, \cdots, n$ $\bigcup_{i=1}^{n} A_i \subseteq \bigcup_{i=1}^{n} \overline{A_i} \implies \overline{\bigcup_{i=1}^{n} A_i} \subseteq \overline{\bigcup_{i=1}^{n} \overline{A_i}} = \bigcup_{i=1}^{n} \overline{A_i}$

Conversely, given $p \in \bigcup_{i=1}^{n} \overline{A_i}$, i.e. $p \in A_i$ for some $i = 1, \cdots, n$ $\forall r > 0, B(p,r) \cap A_i \neq \emptyset$
$B(p,r) \cap (\bigcup_{i=1}^{n} A_i) \neq \emptyset \implies \bigcup_{i=1}^{n}(B(p,r) \cap A_i) \neq \emptyset \implies p \in \overline{\bigcup_{i=1}^{n} A_i}$

(b) Given $p \in \bigcup_{n=1}^{\infty} \overline{A_n}$, i.e. $p \in \overline{A_n}$ for some $n \in \mathbb{N}$ $\forall r > 0, B(p,r) \cap A_n \neq \emptyset \implies B(p,r) \cap (\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty}(B(p,r) \cap A_n) \neq \emptyset \implies p \in \overline{\bigcup_{n=1}^{\infty}} A_n$
To prove that the inclusion might be strict e.g. $\bigcup_{q\in\mathbb{Q}}\{\overline{q}\} = \mathbb{Q} \subsetneq \mathbb{R} = \overline{\mathbb{Q}} = \overline{\bigcup_{q\in\mathbb{Q}}\{q\}}$

■

7. Let $A, B \subseteq \mathbb{R}$ such that $\sup A = \sup B$ and $\inf A = \inf B$. Does $A = B$?

Solution. ■

8. Fix $b > 1$.
(a) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence, it makes sense to define $b^r = (b^m)^{1/n}$
(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.
(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence, it make sense to define

$$b^x = \sup B(x)$$

for every real x.
(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Solution. **Recall.** Thm (Existence of n^{th} root): $\forall x \in \mathbb{R}, x > 0$ and $n\ in \mathbb{N} \exists 1 y > 0 \ni y^n = x (y = x^{\frac{1}{n}})$, $x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m = (x^m)^{\frac{1}{n}}$
(a) If $m = 0$ or $p = 1$ then the result trivial. Suppose $m \neq 0$ and $p \neq 0$. Since $\frac{m}{n} = \frac{p}{q}$, $mq = np$. By the existence of n^{th} root, it satisfies to show that $((b^p)^{\frac{1}{q}})^n = b^m$

$$((b^p)^{\frac{1}{q}})^n = ((b^{\frac{1}{q}})^p)^n = (b^{\frac{1}{q}})^{pn} = (b^{\frac{1}{q}})^{mq} = ((b^{\frac{1}{q}})^q)^m = b^m$$

(b) Given $r, s \in \mathbb{Q}$. We can write $r = \frac{m}{n}$ and $s = \frac{p}{q}$, where $n, m, p, q \in \mathbb{Z}$ with $n, q > 0$. Then $b^{r+s} = b^{\frac{np+mq}{nq}}$. This means b^{r+s} is the unique real number $\ni (b^{r+s})^{nq} = b^{np+mq}$.
Claim: $(b^r b^s)^n q = b^{np+mq}$
$(b^r b^s)^{nq} = b^{rnq} b^{snq} = b^{mq} b^{np} = b^{np+mq}$. Thus, $b^{r+s} = b^r b^s$

(c) First, we prove that if $r < s$, then $b^r < b^s \forall r, s \in \mathbb{Q}$
- $\star_1$ If $r = 0$, then $b^r = 1$ write $s = \frac{m}{n} > r = 0$, where $m, n \in \mathbb{N}$ with $n \neq 0$. Since b^s is the unique real number such that $(b^s)^n = b^m$ and $b > 1$ we get $b^r = 1 < b^m = (b^s)^n \Rightarrow b^r < b^s$
- $\star_2$ If $s = 0$, then consider the same result of $\star_1$
- $\star_3$ If $r < 0 < s$, then $b^r < 1$ and $b^s > 1$. Hence $b^r < b^s$
- $\star_4$ If $0 < r < s$. Write $r = \frac{p}{q}$ and $s = \frac{m}{n}$, where $n, m, p, q \in \mathbb{N}$. Then $r = \frac{np}{nq} < \frac{mq}{nq} = s$ By (a), we have $b^r = b^{\frac{p}{q}} = b^{\frac{np}{nq}} = (b^{\frac{1}{nq}})^{np} < (b^{\frac{1}{n1}})^{mq} = b^s$
- $\star_5$ If $r < s < 0$, then consider $0 < -s < -r$. By $\star_4$, we are done.

Hence, in any case, we have b^r is the upper bound of $B(r)$ where $r \in \mathbb{Q}$ Finally, to prove that b^r is the smallest on. We need some facts.
- (i) $\forall n \in \mathbb{N},\ b^n - 1 \geq n(b-1)$
- (ii) If $t > 1$ and $n > \frac{(b-1)}{(t-1)}$, then $b^{\frac{1}{n}} < t$. By replacing $b = b^{\frac{1}{n}}$ in (i), we get $\forall n \in \mathbb{N}, b - 1 \geq n(b^{\frac{1}{n}} - 1)$ Thus, $n > \frac{b-1}{t-1} \geq \frac{n(b^{\frac{1}{n}}-1)}{t-1} \implies b^{\frac{1}{n}} - 1 < t - 1 \implies b^{\frac{1}{n}} \leq t$.
  If $\alpha < b^r$, we must find $q \in \mathbb{Q}$ and $qleqr \ni b^q > \alpha$.
  If $\alpha \leq 0$, then trivial
  If $0 < \alpha < b^r$ Changing $t = \frac{b^r}{\alpha} > 1$ in (ii) and choose $n \in \mathbb{N} \ni n > \frac{(b-1)}{(t-1)}$. We get $b^{\frac{1}{n}} < \frac{b^r}{\alpha} \implies \alpha < b^{r-\frac{1}{n}}$ Choose $q = r - \frac{1}{n}$ and we are done.

(d) Prove that $b^{x+y} = b^x b^y \forall x, y \in \mathbb{R}$.
From (c), $b^x = \sup\{b^r \mid r \in \mathbb{Q}, r \leq X\},\ b^y = \sup\{b^s \mid s \in \mathbb{Q}, s \leq y\}$.Thus, $b^x b^y = \sup\{b^{r+s} \mid r, sin\mathbb{Q}, r + s \leq x + y\} = b^{x+y}$

■

9. Prove that no order can be defined in the complex field that truns it into an ordered field.

Solution. ■

10. Suppose $z = a + ib,\ w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers

into an ordered set. (This type of order relation is called a dictionary order, or lexicographic order, for obvious reasons) Does this ordered set have the least-upper-bound property?

Solution. ■

11. Suppose $z = a + bi,\ w = u + iv$ and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2},\ b = \left(\frac{|w| - u}{2}\right)^{1/2}$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\overline{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

Solution. ■

12. Under what conditions does equality hold in the Cauchy-Schwartz inequality.

Solution. ■

13.
14.
15.
16.
17.