0.0.1. In calculus. IPad test

(1) Extreme Value Theorem: Every continuous function $f : [a, b] \to$ R admit both max and min value $\Rightarrow$ Compact set
(2) Intermediate value Theorem: Given continous function $f : [a, b] \to$ R for all $f(a) \leq \lambda \leq f(b) \exists c \in [a, b] \ni f(c) = \lambda \Rightarrow$ connected set

How to prove a statement: HP,then Q, $P \Rightarrow Q$

$$\begin{cases} \begin{cases} \text{Direct Proof} \\ \text{Indirect Proof} \begin{cases} \text{contrapositive} \sim Q \Rightarrow \sim P \\ \text{by contradiction} \end{cases} \end{cases} \\ Mathematical Induction \end{cases}$$

1. Some preliminary

1.1. Set Theory. We will assume that you are familiar with some basic set theory e.g. union, intersection, difference

1.2. The Number System.

$\mathbb{N} = \{ 1, 2, 3, \cdots \}$ the set of all positive integers n natural numbers
$\mathbb{Z} = \{ \cdots, -2, -1, 0, -1, -2, \cdots \}$ the set of all integers called the ring of intepus
$\mathbb{Q} = \left\{ \frac{m}{n} \ : \ n, m \in \mathbb{Z}, n \neq 0 \right\}$ the set of all rational numbers
$\mathbb{R}$ the set all of real numbers on the real number field on real line
$\mathbb{C} = \{ z = a + ib \mid a, b \in \mathbb{R} \}$ the set of all complex numbers or the complex number filed on complex plane, where $i = \sqrt{-1}$

Remark.

(1) $x + 2 = 0$ *no root in* $\mathbb{N}$
$3x - 5 = 0$ *no root in* $\mathbb{Z}$
$x^2 + 1 = 0$ *no root in* $\mathbb{R}$
(2) One can construct $\mathbb{Q}$ *from* $\mathbb{Z}$ *in algebraic way, called the fraction field of* $\mathbb{Z}$
(3) One can construct $\mathbb{R}$ *from* $\mathbb{Q}$ *in two ways:*
· *Using Dedekind cut which is given in the appendix of Rudin p17-21*
· *Using completion of matrix space*
(4) One can construct $\mathbb{C}$ *from in complex analysis*

Example.

(1) Between any two rational numbers, there is another one

Proof. Let $r, s \in \mathbb{Q}$ with $r < s$, then $\dfrac{r+s}{2} \in \mathbb{Q}$ and $r < \dfrac{r+s}{2} < s$

$$r = \frac{m_1}{n_1}, s = \frac{m_2}{n_2}, \frac{r+s}{2} = \frac{\frac{m_1}{n_1} + \frac{m_2}{n_2}}{2} = \frac{m_1 n_2 + n_1 m_2}{2 n_1 m_1} \in Q$$
$$s = \frac{s+s}{2} > \frac{r+s}{2} > \frac{r+r}{2} = r$$

■

(2) $x^2 = \dfrac{4}{9}$ *has exactly two rational solutions, namely,* $\pm\dfrac{2}{3}$

(3) $x^2 = 2$ *has exactly two real root, namely,* $\pm\sqrt{2}$

(4) Is there any rational roots of $x^2 = 2$*? i.e., is* $\sqrt{2}$ *rational?*

Suppose $r = \dfrac{m}{n} \in \mathrm{Q}$, *is a root of* $x^2 = 2$, *where* $(m, n) = 1$

Then $\dfrac{m^2}{n^2} = 2 \implies m^2 = 2n^2 \implies 2 \mid m^2 \implies 2 \mid m \implies 4 \mid m^2 \implies 4 \mid 2n^2$
$\implies 2 \mid n^2 \implies 2 \mid n \implies (n, m) \neq 1$

(5) Let $A = \left\{ r \in \mathrm{Q} \mid r > 0 \;\&\; r^2 < 2 \right\}, B = \left\{ r \in \mathrm{Q} \mid r > 0 \;\&\; r^2 > 2 \right\}$
Then A *contains no largest numbers, i.e. max element &* B *contains no smallest numbers, i.e. min element*

Proof. A contains no largest numbers $\Leftrightarrow$ given $r \in A$, $\exists s \in A \ni s > r$

Now, given $r \in A$, Let $s = r - \dfrac{r^2 - 2}{r+2} = \dfrac{2r+2}{r+2}$ $(\star_1)$

$\implies s^2 - 2 = \dfrac{2(r^2-2)}{(r+2)^2}$ $(\star_2)$

Now, $r \in A, r^2 < 2 \implies r^2 - 2 < 0 \therefore$
$(\star_1)\&(\star_2) \implies s > r \;\&\; s^2 < 2 \implies s \in A$ ■

(6) As you know, in calculus, the sequence $\{1, 1.4, 1.41, 1.414, 1.4142, \cdots\}$ *does not converge in* Q, *but it converges to* $\sqrt{2}$ *in* R

1.3. Order Sets.

<u>Definition</u> (Relation).
Let X be a nonempty set A, relation on X is a subset R *of* $X \times X = \{ (x, y) \mid x, y \in X \}$
Let R *be a relation on X, if* $(x, y) \in \mathrm{R}$*, then we say that x is retaliated to y, and is written as* $xRy(x \sim y)$

<u>Definition</u> (Order Set). *An ordered set on S, is a relation denoted by* $" < "$ *on S, satisfy:*

(i) *The low of trichonomy*
   Given $x, y \in S$, one and only one of the following holds: $x < y, x = y, y < x$
(ii) *Transitivity: if $x < y \& y < z$, than $x < z$*

> ***Notation***
>
> (1) *$x < y$ means "x is less than y" or "x is smaller than y"*
> (2) *$y > x$ means $x < y$*
> (3) *$x \leq y$ means $x < y$ or $x = y$, i.e. the negative of $x > y$*

<u>Definition</u> (bdd). *Let S is an ordered set & $E \subseteq S (E \neq \emptyset)$*

- *E is bounded above if $\exists\ \alpha \in S \implies x \leq \alpha\ \forall\ x \in E$*
  such α is called an upper bound of E
- *E is bounded below if $\exists\ \beta \in S \ni \beta \leq x, \forall\ x \in E$, such β is called a lower bdd of E*
- *E is bdd is E is both bdd above and below.*

<u>Definition</u> (least upper bound). *Let S be an ordered set and $E \subseteq S (E \neq \emptyset)$ bdd above. An element $\alpha \in S$ is called the last upper bound or supremum of E if*

(i) *α is an upper bound of E*
(ii) *α is the smallest such one.*

Equivalently,

(i') *$x \leq \alpha, \forall x \in E$*
(ii') *if $\beta < \alpha$, then β is not an upper bdd of E, i.e. $\exists\ x \in E \ni x > \beta$*

Such α(if exists) is denoted by

$$\alpha = sup(E)$$

similarly, one can defined the greatest lower bdd of infimum of E

Remark. *if* $\sup(E)$ *exists then it is unique*
suppose $\alpha \neq \alpha'$ both lub of E

$\because$ *by trichotomy* ,$\alpha > \alpha'$ *or* $\alpha = \alpha'$ *or* $\alpha < \alpha'(\rightarrow\leftarrow)$

<u>Definition</u> (least upper bdd property)**.** *A ordered set* S *is said to have the least upper bdd property if* $E \subseteq S,\ E \neq \emptyset$ *and* E *is bdd above, then* $\sup(E)$ *exists in* S

Example.

(1) In Q *with the normal ordining*
$A = \{\, r \in \mathrm{Q} \mid r > 0,\ r^2 < 2 \,\} \& B = \{\, r \in \mathrm{Q} \mid r > 0,\ r^2 > 2 \,\}$
Then A *is bdd above, in fact, bdd by every element in* B*, but* $\sup(A)$ *does not exist in* Q*(*$\because$ *by Ex1.5)*

(2) B *is bdd below by every element of* A *and* $\inf B$ *does not exists*

(3) Note that $\sup(E) \& \inf(E)$ *may not in* E *even if exist*

Remark.

(1) By the Example above, Q *with the usual ordering has no l.u.b property*

(2) In 1.5 *we will explain that* R *with usual ordering has the l.u.b. property. However, we usually adopt the follwing*

The Axiom of Completence or Least upper bdd property:
Every nonempty subset E of R which is bdd above has l.u.b

Theorem (l.u.b.p. $\rightarrow$ g.l.b.p.)**.** *Let* S *is an ordered set if* S *has the l.u.b. property, then* S *has the g.l.b. property, i.e. if* $\emptyset \neq B \subseteq S$ *is bdd below, then* $\inf(B)$ *exists in* S

Proof. $(\star)$
Given $B(\neq \emptyset) \subseteq S$ which is bdd below
Let $L = \{\, a \in S \mid a \text{ is a lower bdd of } B \,\}$

- $L \neq \emptyset(\because B$ is bdd below)
- L is bdd above (in fact, every element in B is on upper bound of L)
  $\implies \forall a \in L \implies a \leq x,\ \forall x \in B \implies x$ is an upper bound of L
- $\sup(L) = \alpha$ exists by assumption

Claim $\alpha = \inf B$

(i) α is a lower bdd of B, i.e. $\alpha \leq x,\ \forall x \in B$
By $\alpha = \sup L$, if $r < \alpha$, them r is not an upper bdd of $L(\because \alpha$ is the smallest one).Hence,$r \notin B(\because$ every element of B is an upper bdd of L), so $\alpha \leq x, \forall x \in B$
We have proved $(r < \alpha \implies r \notin B) \implies (r \in B \implies r \geq \alpha)$

(ii) α is the greated one
if $\alpha < \beta$ and β is a lower bdd of B, then $\beta \notin L$, i.e. β is not a lower bdd of B, so α is the greatest one. Therefore, $\alpha = \inf(B)$

$\blacksquare$

Remark. *Let* $E(\neq \emptyset) \subseteq \mathrm{R}$ *be bdd below, then* $\inf(E)$ *exists and* $\inf(E) = -\sup(-E)$*, where* $-E = \{ -x \mid x \in E \}$

1.4. Field.
Recall the addition & multiplication in R
$+ : \mathrm{R} \times \mathrm{R} \to \mathrm{R}((a,b) \mapsto a+b)$
$\times : \mathrm{R} \times \mathrm{R} \to \mathrm{R}((a,b) \mapsto a \cdot b = ab)$

Definition. *Let* X *is a nonempty set* A*, binary operation on* X *is a function,* $o : X \times X \to X$

Definition. *Let* F *be a nonempty set, we say that* F *is a field ((*$F, +, \cdot$*) is a field) if there are two binary operator called addition* "$+$" *and multiplication* "$\cdot$" *on* F *property*

Axioms for "$+$"

- *(A1) Commutative:* $\forall x, y \in \mathrm{F},\ x + y = y + x$
- *(A2) Associative:* $\forall x, y, z \in F, (x + y) + z = x + (y + z)$
- *(A3) Additive identity or zero element:* $\exists\ 0 \in F \implies x + 0 = 0 + x = x,\ \forall x \in \mathrm{F}$
- *(A4) Additive inverse on negative: For each* $x \in X,\ \exists\ -x \in F \implies x + (-x) = (-x) + x = 0$

i.e. $(\mathrm{F}, +)$ *is an abelian group* ***Axioms for multiplication***

- *(M1) Commutative:* $\forall x, y \in \mathrm{F},\ xy = yx$
- *(M2) Associative:* $\forall x, y, z \in \mathrm{F},\ (xy)z = x(yz)$
- *(M3) Muti identity:* $\exists 1 \neq 0$ *in* $\mathrm{F} \ni x1 = 1x = x$
- *(M4) Multiplicative inverse: For each* $x \neq 0, \exists x^{-1} \in \mathrm{F} \implies xx^{-1} = x^{-1}x = 1$

i.e. $(F = F \cdot \{ 0 \}, \cdot)$ *is an abelian group*

Distributive Law

- *(D1)* $\forall x, y, z \in \mathrm{F},\ (x, y)z = xz + yz\ \&\ x(y + z) = xy + xz$

Induction from Axioms

let $(\mathrm{F}, +, \cdot)$ be a field, we list a series of basic identity as you learn in high school in the real number system

(a) Cancellation law for " + " : $x+y=x+z \implies y=z$
$\because x+y=x+z \implies (-x)+(x+y)=(-x)+(x+z) \implies ((-x)+x)+y=((-x)+x)+z$
$\implies 0+y=0+z \implies y=z$

(b) 0 is "1"
suppose $0' \in$ F is another element satisfy A_3, then $0=0+0'=0'$

(c) $x+y=x \implies y=0$ by (a) $\because x+y=x+0 \implies y=0$

(d) negative $-x$ of x is "1"
if $x' \in F$, is another negative of x, them $x+x'=x'+x=0$
From $x+x'=0 \implies (-x)+(x+x')=-x+0=-x$

(e) $x+y=0 \implies y=-x$
$x+y=0 \implies (-x)+(x+y)=(-x)+0 \implies ((-x)+x)+y=-x$
$\implies 0+y=-x \implies y=-x$

(f) $-(-x)=x$
$-(-x)+(-x)=0$, By (d) $x=-(x)$

(a') cancellation law
if $x \neq 0$, then $xy=xz \implies y=z$, $\because (x^{-1})(xy)=(x^{-1})(xz)$
$\implies (x^{-1})(xy)=(x^{-1}x)z \implies 1y=1z \implies y=z$

(b') 1 is "1"
if $1'$ is another identity, then $1=11'=1'$

(c') $x \neq 0$ & $xy=x \implies y=1$
$xy=x1 \implies y=1$

(d') For $x \neq 0$ in F, x^{-1} is "1"
if x is another one, i.e. $x'x=xx'=1 \implies (x^{-1})(xx')=(x^{-1})1=x^{-1}$

(f') $x \neq 0 \implies (x^{-1})^{-1}=x$
$(x^{-1})^{-1}(x^{-1})=1 \implies x=(x^{-1})^{-1}$

(g') $0x=x0=0$
$(0+0)x=0x+0x \implies 0x=0$

(h') $x \neq 0$ & $y \neq 0 \implies xy \neq 0$, equivalently $xy=0 \implies x=0$ or $y=0$
$\because xy=0$ then $(x^{-1})(xy)=((x^{-1})x)y=1y=y(\rightarrow\leftarrow)$

(i') $(-x)y=-(xy)=x(-y)$
$\because [(-x)+x]y=0y=0=(-x)y=-(xy) \implies (-x)y=-(xy)$

(j') $(-x)(-y)=xy$
$\because (-x)(-y)=-(x(-y))$ by (i)
$=-(-(xy))=xy$

(k) $-x = (-1)x$

$\because (1-1)x = 0x = 0 = 1x+(-1)x = x+(-1)x \implies (-1)x = -x$

Definition (Order Field)**.** *Let* F *is a field, we say that* F *is an order field if there is an ordering "$<$" satisfying*

(1) if $x < y$, then $x + z < y + z$, $\forall z \in F$

(2) if $x > y$ and $y > 0$, then $xy > 0$

Example. Q *and* R *are order field under the usual ordering*

Some basic properties of ordered field, let F *be an ordered field with ordering "$<$"*

(a) $x > 0 \implies -x < 0$

$\because x > 0 \implies x + (-x) > 0 + (-x) \implies 0 > -x$

(b) $x > y \Leftrightarrow x - y > 0$

$\because x > y \implies x + (-y) > y = (-y) \implies x - y > 0$

$x - y > 0 \implies x - y + y > y \implies x + 0 > y \implies x > y$

(c) $x > 0$ *and* $y < z \implies xy < xz$

$\because x > 0$ *and* $y < z \implies x > 0$ *and* $z-y < 0 \implies x(z-y) > 0 \implies xz + x(-y) > 0$

$\implies xz - xy > 0 \implies xz > xy$

(d) $x < 0$ *and* $y < z \implies xy > xz$

$\because x < 0$ *and* $y < z \implies -x > 0$ *and* $z - y > 0 \implies (-x)(z - y) > 0 \implies -xz + xy > 0$

$\implies xy > xz$

(e) $\forall x \neq 0$ *in* $\mathrm{F}, x^2 > 0$

$\because x > 0 \implies x \cdot x > x0$ *by (c) or*

$x < 0 \implies -x > 0$ *by (a)* $\implies -x > 0$ *by (a)* $\implies (-x)^2 > 0 \implies x^2 > 0$

(f) $1 > 0,\ -1 < 0$

$\because 1 \neq 0 \implies 1^2 > 0$ *by (e)* $\implies 1 > 0$

(g) $0 < x < y \implies 0 < \dfrac{1}{y} < \dfrac{1}{x}$

$\because$ *Note that* $\forall u \in \mathrm{F},\ u > 0 \implies \dfrac{1}{u} = u^{-1} > 0$

$\because$ *if* $\dfrac{1}{u} < 0$, *then* $u \cdot \dfrac{1}{u} < 0$ *by (e)* $\implies 1 < 0 (\rightarrow\leftarrow)\ \therefore\ \dfrac{1}{u} > 0$

Now, $\dfrac{1}{x}, \dfrac{1}{y} > 0$ *from* $x < y$ *we get* $(\dfrac{1}{x} \cdot \dfrac{1}{y})x < (\dfrac{1}{x} \cdot \dfrac{1}{y})y \implies 0 < \dfrac{1}{y} < \dfrac{1}{x}$

Remark. *By (e)(f), we conclude that* $\mathbb{C}$ *is not an ordered field*
$\because \mathbb{C}$ *were an ordered field, then by (e),* $i^2 > 0 \implies -1 > 0 (\rightarrow\leftarrow)$
$\therefore \mathbb{C}$ *is not an order field*

1.5. The Real Number Field $\mathbb{R}$.

Theorem. *There exists an ordered field* $\mathbb{R}$ *containing* $\mathbb{Q}$ *which has the l.u.b. property. Moreover, such* $\mathbb{R}$ *is unique up to order-isomorphism i.e. if* " $<$ " *and* " $<'$ " *are two orders on* $\mathbb{R}$, *them* $\exists f_i(\mathbb{R}, <) \to (\mathbb{R}, <') \implies$

- *(i)* f *is a field isomorphism,*
  i.e. $\forall a, b \in \mathbb{R},\ f(a+b) = f(a)+f(b),\ f(ab) = f(a)f(b),\ f(1) = 1$
- *(ii)* f *preserves ordering,* $a < b \implies f(a) < f(b)$

Such $\mathbb{R}$ *is called the real number field or real number system or real line*

Theorem.

- *(a) The Archimedean property of* $\mathbb{R}$: *Given* $x, y \in \mathbb{R}$ *with* $x > 0,\ \exists\, n \in \mathbb{N} \implies nx > y$
- *(b)* $\mathbb{Q}$ *is dense in* $\mathbb{R}$: $\forall x, y \in \mathbb{R}$ *with* $x \leq y,\ \exists\, r \in \mathbb{Q} \implies x < r < y$

Proof.

- (a) Let $A = \{\, nx \mid n \in \mathbb{N} \,\} \subseteq \mathbb{R}$
  if (a) were false, them A is bdd above by y, since $\mathbb{R}$ has the l.u.b property
  $\alpha = \sup A$ exists in $\mathbb{R}$, since $x > 0,\ \alpha - x < \alpha \implies \alpha - x$ is not an upper bdd of A
  $\implies \exists m \in \mathbb{N} \ni mx > \alpha - x \implies (m+1)x > \alpha (\rightarrow\leftarrow)$
- (b) Since $x < y,\ y - x > 0$, by (a),$\exists n \in \mathbb{N} \implies n(y-x) > 1$
  By (a) again, $\exists m_1, m_2 \in \mathbb{N} \implies m_1 = m_1 1 > n_x$ & $m_2 = m_2 \cdot 1 > -nx$
  we have $-m_2 < nx < m_1$, choose $m \in \mathbb{Z} \implies -m_2 \leq m \leq m_1$ & $m - 1 \leq nx < m$
  (in fact, $m = [nx] + 1$,where $[z]$ in the greatest integer of z)
  we have $nx < m < 1 + nx < ny (\because n(y-x) > 1) \implies x < \dfrac{m}{n} < y$
  Let $r = \dfrac{m}{n} \in \mathbb{Q}$, then $x < r < y$

$\blacksquare$

An application of the density property of Q in R :
Given $x \in \mathrm{R} - \mathrm{Q}$ i.e. x is an irrational numbers, i.e. $\forall\ \epsilon > 0, \exists r \in \mathrm{Q} \implies |x - r| < \epsilon$
equivalently, $\exists$ a sequence $\{ r_n \}$ in Q $\implies r_n \to x$
In fact, one may choose $\{ r_n \}$ to $\uparrow$ or $\downarrow$
$\because \forall n \geq 1,\ \exists\, r_n \in \mathrm{Q} \implies x < r_n < \dfrac{1}{n} + x$ by Thm.1.3(b) By squeezing lemma, $r_n \to x$ on $n \to \infty$

Theorem (existence of nth root)**.** *Given* $x \in \mathrm{T}, x > 0\ \&\ n \in \mathrm{N}, \exists\, "1" y > 0 \implies y^n = x$
Such y *is called the nth root of* x & *denoted by* $y = \sqrt[n]{x} = x^{\frac{1}{n}}$

Proof. **not important**
"1". Suppose $y_1, y_2 > 0 \implies y_1^n = x\ \&\ y_2^n = x$
Bt trichotomy, we have

(i) $0 < y_1 < y_2 \implies y_1^n < y_2^n (\rightarrow\leftarrow)$
(ii) $0 < y_2 < y_1 \implies y_2^n < y_1^n (\rightarrow\leftarrow)$
(iii) $y_1 = y_2$

"$\exists$". Let $E = \{ t \in \mathrm{R} \mid t^n < x \}$
Claim:

- $E \neq \emptyset$, Let $t = \dfrac{x}{1+x}$, then $0 < t < 1$, hence $t^n < t < x$, $\therefore t \in E\ \&\ E \neq \emptyset$
- E is bdd above, in fact E is bdd above by $1+x$ if $t > 1+x > 1$, then $t^n > t > x$, so E is bdd above by $1+1$
  Therefore $y = \sup E$ exists & is finite
- Claim $y > 0\ \&\ y^n = x$, clearly, $y > 0 (\because \dfrac{x}{1+x} \in E\ \&\ \dfrac{x}{1+x} > 0)$
  by trichotomy, we have $y^n < x,\ y^n > x,\ y^n = x$

Now, to show that (i) & (ii) are impossible, do (iii) holds $y^n = x$
By the identity, $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$
(i)$y^n < x$ choose $0 < h < 1 = \alpha\ \&\ 0 < \dfrac{x - y^n}{n(y+1)^{n-1}}$, $0 < h < \min\{ \alpha, \beta \}$
put $a = y,\ b = y + h$ in $(\star)$, we obtain

$$(y+h)^n - y^n < hn(y-h)^{n-1} < hn(y+1)^{n-1} < x - y^n$$
$$\implies (y+h)^n < x \implies y + h \in E\ \& y + h > y (\rightarrow\leftarrow) \therefore \text{(i) fails}$$

(ii) $y^n > x$, Let $k = \dfrac{y^n - x}{ny^{n-1}}$, Then $0 < k < y,\ k = \dfrac{y^n - x}{ny^{n-1}} < \dfrac{y^n}{ny^{n-1}} = \dfrac{y}{n} < y$
if $t > y-k > 0$, then $y^n - t^n \leq y^n - (y-k)^n < kny^{n-1}$ by $(\star) = y^n - x$ $\implies t^n > x \implies t \in E \implies E$ is bdd above by $y-k \implies \sup E \leq y - k (\rightarrow\leftarrow)$
$\therefore$ (ii) fails ■

Corollary. *Let $a, b \in \mathbb{R}$ with $a, b > 0,\ n \in \mathbb{N}$ Then*$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$
$\because a^{\frac{1}{n}}, b^{\frac{1}{n}} > 0$ & $(a^{\frac{1}{n}} \cdot b^{\frac{1}{n}}) = ab$, *By (1) in Thm 1.4* $(a,b)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$

infinite in $\mathbb{R}$
After discuss the real number $\mathbb{R}$, sometimes, we have to work with the extended real number system $\mathbb{R}^* = [-\infty, \infty] = \mathbb{R} \cup \{+\infty, -\infty\}$ with observe, $x \in \mathbb{R}$

$$\lim_{n\to\infty} (-n) = -\infty,\ \lim_{n\to\infty} n = \infty,\ \lim_{n\to\infty} (\frac{1}{n} + n) = \infty,\ \lim_{n\to\infty} (n^2 - n) = \infty$$
$$x \pm \infty = \pm\infty,\ 0 \cdot (\pm\infty) = 0,\ \infty - \infty \text{ is not define}$$

Element in $\mathbb{R} \subseteq \mathbb{R}^*$ are called finite. Now, given any nonempty subset $E \subseteq \mathbb{R}$,

$$\sup E = \begin{cases} +\infty \text{ if } E \text{ is not bdd above} \\ \text{finite if } E \text{ is bdd above} \end{cases} \quad \& \inf E = \begin{cases} -\infty \text{ if } E \text{ is not bdd below} \\ \text{finite if } E \text{ is bdd below} \end{cases}$$

Note that if $A \subseteq B$, then $\sup A \leq \sup$ & $\inf A \geq \inf B$
$\therefore \emptyset \subseteq B,\ \forall B \subseteq \mathbb{R}$, One may define $\sup \emptyset = -\infty, \inf \emptyset = +\infty$

1.6. The Complex Number Field $\mathbb{C}$.

Consider the contention product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a, b) \mid a, b \in \mathbb{R}\}$
Note that $(a, b) = (c, d) \Leftrightarrow a = c$ & $b = d$, From now, we can write $\mathbb{C} = \mathbb{R}^2$

Operation on $\mathbb{C}$ Given $(a, b), (c, d) \in \mathbb{C}$

(1) $(a, b) + (c, d) = (a + c, b + d)$
(2) $(a, b)(c, d) = (ac - bd, ad + bc)$

It is easy to see that, with these operations, $\mathbb{C}$ is a field.

Note that

- the zero element is $(0,0)$
- the negative of (a,b) is $-(a,b)=(-a,-b)$
- the identity is $(1,0)$
- if $(a,b)\neq(0,0)$, then $(a,b)^{-1}=\left(\dfrac{1}{a^2+b^2},\dfrac{-b}{a^2+b^2}\right)$

R is a subset of C (not vary important)
consider that map

$$f:\mathbb{R}\to\mathbb{C} \text{ define by } f(a)=(a,0),\ a\in\mathbb{R}$$

we have $(1)f$ is injective $(2)f(1)=(1,0)\ \because\forall\ a,b\in\mathbb{R}$
$f(a+b)=(a+b,0)=(a,0)+(b,0)=f(a)+f(b),\ f(a\cdot b)=(ab,0)=(a,0)\cdot(b,0)$
f is a field homomorphism
$\therefore f:\mathbb{R}\to\mathbb{C}$ is an injective and isomorphism
Therefore, we identify $\mathbb{R}$ with $f(\mathbb{R})$ through the injective f
i.e. $a\in\mathbb{R}$ is identified with $f(a,0)$ in $\mathbb{C}$
$ab=(a,0)\cdot(b,0),\ a+b=(a,0)+(b,0)\ \forall\ a,b\in\mathbb{R}$

Change (a,b) to $a+bi$

Now, we can transform an element $(a,b)\in\mathbb{C}$ into the normal form:
(a,b) = (a,0)+(0,b) = (a,0)(1,0)+(b,0)(0,1) = a1+bi = a+ib, where i = (0,1)
Therefore, from new on, we write $\mathbb{C}=\{\,a+ib\mid a,b\in\mathbb{C}\,\}$
An element $z=a+ib\in\mathbb{C}$ is called a complex number
Hence, under this notification, $z=a+ib, w=c+id\in\mathbb{C}$

(1) $z+w=(a+c)+i(b+d)$
(2) $zw=(ac-bd)+i(ad+bc)$

and the a is called the real part of z, $a=\mathrm{Re}(z)$, b is called imaginary part of z, $b=\mathrm{Im}z$

Some basic properties of complex numbers whose proofs are easy
$\forall z, w \in \mathbb{C}$

- $\overline{z+w} = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z} \cdot \overline{w}$
- $\text{Re}z = \dfrac{z+\overline{z}}{2}$
- $\text{Im}z = \dfrac{z-\overline{z}}{2i}$
- $|z| = 0 \Leftrightarrow z = 0$
- Triangle inequality $|z+w| \leq |z| + |w|$
- $||z| - |w|| \leq |z - w|$
- $\mathbb{C}$ is not an ordered field
- $|z|^2 = z\overline{z}$
- $|\overline{z}| = |z|$
- $|\text{Re}z| \leq |z|, |\text{Im}z| \leq |z|$
- $|zw| = |z||w|$

Proof. $|z+w| \leq |z| + |w|$
$|z+w|^2 = (z+w)(\overline{z+w}) = (z+w)(\overline{z}+\overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w}$
$= |z|^2 + 2\text{Re}(z\overline{w}) + |w|^2 \leq |z|^2 + 2|z\overline{w}| + |w|^2 = |z|^2 + 2|z||w| + |w|^2 = (|z|+|w|)^2$
$\therefore |z+w| \leq |z| + |w|$ ∎

Theorem (basic algebraic theorem)**.**
(a) $x^2 + 1$ *has no root in* $\mathbb{R}$
(b) $x^2 + 1$ *has two distinct roots in* $\mathbb{C}$

Proof.

(a) $1 > 0,\ x^2 > 0,\ \forall x \in \mathbb{R} - \{0\} \implies x^2 + 1 > 0\ \forall x \neq 0$
$0^2 + 1 = 1 > 0,\ \therefore x^2 + 1 > 0,\ \forall\ x \in \mathbb{R}$. Hence, $x^2 + 1 = 0$ has no root in $\mathbb{R}$

(b) $i^2 = (0,1)(0,1) = (0-1,0) = (-1,0) = -1$
$(-i)^2 = (-(0,1))^2 = (0,-1)^2 = (0,-1)(0,-1) = -1$, $\therefore \pm i$ are root of $\mathbb{C}$

∎

Conclusion: Every non const polynomial $f(x) \in \mathbb{R}[x]$ has n roots where $n = \deg f(x)$
The complex root is even

no important proof
$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x],\ a_n \neq 0,\ n \geq 1$
if $\alpha = a + ib \in \mathbb{C}$ is a root of $f(x)$,then
$0 = f(\alpha) = a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0$
$0 = f(\overline{\alpha}) = a_n\overline{\alpha}^n + a_{n-1}\overline{\alpha}^{n-1} + \cdots + a_1\overline{\alpha} + a_0$
$\therefore\ (x-\alpha)|f(x),\ (x-\overline{\alpha})|f(x) \implies (x-\alpha)(x-\overline{\alpha})|f(x) \implies (x^2 - (\alpha - \overline{\alpha})x + |\overline{\alpha}|^2) \mid f(x)$
$\implies (x^2 - 2ax + (a^2+b^2)) \mid f(x)$
$\therefore$ quadratic function must have two roots in $\mathbb{C}$

The fundamental Theorem of Algebra
Every non zero polynomial $f(x) \in \mathbb{C}[x]$ has at least one root in $\mathbb{C}$
Therefore, if $\deg f(x) = n$, then $f(x)$ has n roots in $\mathbb{C}(C, M)$

$\because f(x) = (x-\lambda_1)^{e_1}\cdots(x-\lambda_t)^{e_t}(a_1x^2+b_1+c_1)^{l_1}\cdots(a_sx^2+b_sx+c_s)^{ls}$, where $\lambda_1, \cdots, \lambda_t \in \mathbb{R},\ a_i, b_i, c_i \in \mathbb{R} \& e_1 + \cdots + e_t + 2l_1 + \cdots + 2l_s = \deg f(x)$ which shows that all roots of $f(x)$ are in $\mathbb{C}$
In fact, we have the famous theorem: The fundamental theorem of algebra
Every non zero polynomial $f(x) \in \mathbb{C}[x]$ has at least one root in $\mathbb{C}$
$\therefore$ if $\deg f(x) = n$, then $f(x)$ has n roots in $\mathbb{C}(C, M)$

Theorem (Cauchy-Scheming Inequal)**.** *Given $z_1 \cdots, z_n, w_1, \cdots, w_n \in \mathbb{C}$, we have*

$$\left|\sum_{j=1}^{n} z_j \overline{w}_j\right| \leq \left(\sum_{j=1}^{n} |z_j|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} |w_j|^2\right)^{\frac{1}{2}}$$

and "$=$" holds $\Leftrightarrow \exists\ \lambda \in \mathbb{C} \ni\ w_j = \lambda z_j,\ 1 \leq j \leq n,$
In patricial, if $x_1, \cdots, x_n, y_1, \cdots, y_n \in \mathbb{R}$, then

$$\left|\sum_{j=1}^{n} x_j y_j\right| \leq \left(\sum_{j=1}^{n} x_j^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} y_j^2\right)^{\frac{1}{2}}$$

and "$=$" holds $\Leftrightarrow \exists\ t \in \mathbb{R} \ni\ y_j = tx_j,\ 1 \leq j \leq n$

The proof is too long, I am lazy

1.7. Euclidean Spaces $\mathbb{R}^n$.

Definition. *the n-dimensional Euclidean space* $\mathbb{R}^n$

$$= \{ x = (x_1, \cdots, x_n) \mid x_i \in \mathbb{R}, 1 \leq i \leq n \} = \mathbb{R} \times \cdots \times \mathbb{R}$$

Note that

$$(x_1, \cdots, x_n) = (y_1, \cdots, y_n) \Leftrightarrow x_i = y_i \ \forall \ 1 \leq i \leq n$$

We are going to introduce the structure of $\mathbb{R}^n$

- vector space
- inner product space
- normed linear space
- matrix space

Definition. *Two operation on* $\mathbb{R}^n$ *as follows:*

- *Addition* $+ : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \ (x, y) \mapsto x + y = (x_1 + y_1, \cdots, x_n + y + n)$
- *Scalar multiplication* $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \ (a, x) \mapsto ax = (ax_1, \cdots, ax_n)$

we skip space example here.

1.8. Countability of Sets.

Given two nonempty set A, B and a function $f : A \to B, f(A) = \{ f(a) \mid a \in A \}$ is called the image of A under f

Some basic things

$E \subseteq A, \ f(E) = \{ f(a) \mid a \in E \}$ the image of E under f
f is infective(one-to-one) $x_1 \neq x_2 \implies f(x_1) \neq f(x_2) \Leftrightarrow f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$
f is surjective(onto) if $f(A) = B$, f is bijective if f is one-to-one and onto

Given $F \subseteq B$, $f^{-1}(F) = \{ x \in X \mid f(x) \in \mathrm{F} \}$ called the inverse image of f under F

Example

$f : \mathbb{R} \to \mathbb{R}, \ f(x) = x^2, \ x \in \mathbb{R}$
$f^{-1}([0, 1]) = \{ x \in \mathbb{R} \mid f(x) \in [0, 1] \} = \left\{ x \in \mathbb{R} \mid x^2 \in [0, 1] \right\} = [-1, 1]$
$f^{-1}([-1, 1]) = [-1, 1]$

Properties of inverse image

- $F_1 \subseteq F_2 \subseteq B \implies f^{-1}(F_1) \subseteq f^{-1}(F_2)$
- Inverse image presences set operation
  $\forall F_\alpha \subseteq B,\ \alpha \in I,\ F \subseteq B$
  (i) $f^{-1}(\cup_{\alpha\in I} F_\alpha) = \cup_{\alpha\in I} f^{-1}(F_\alpha)$
  (ii) $f^{-1}(\cap_{\alpha\in I} F_\alpha) = \cap_{\alpha\in I} f^{-1}(F_\alpha)$
  (iii) $f^{-1}(B - F) = f^{-1}(B) - f^{-1}(F)$
- Given $S \subseteq A,\ f'(f'(S)) \supseteq S,"="\Leftrightarrow$ one-to-one, **example:**
  $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2,\ S = [0, 1],\ f(S) = [0, 1],\ f^{-1}(f(S)) = f^{-1}([0, 1]) = [-1, 1]$
- Given $F \subseteq B,\ f(f^{-1}(F)) \subseteq F,"="\Leftrightarrow$ "onto", **example**
  $f(x) = x^2,\ x \in \mathbb{R},\ F = [-1, 1],\ f(f^{-1}([-1, 1])) = f([-1, 1]) = [0, 1]$
- For $y \in B,\ f^{-1}(\{ y \}) = f^{-1}(y) = \{ x \in A \mid f(x) = y \}$ the inverse image of y, **example**
  $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2,\ f^{-1}(1) = \{ 1, -1 \},\ f^{-1}(2) = \emptyset$

<u>Definition</u> (cardinality)**.** *Let A, B are two set ew say that A and B have the same cardinality if $\exists$ a bijective map $f : A \to B$, which is denoted by $A \sim B$*
From now on, we write $|A|$ as the cardinality of A

Claim "$\sim$" is an $\equiv$ relation among all sets

(i) Reflexion: $\forall$ set $A,\ A \sim^{1A} A,$ which 1_A is identity mapping
(ii) Symmetry: $A \sim^f B \implies B \sim^{f^{-1}} A$
(iii) Transitive: $A \sim^f B$ & $B \sim^g C \implies A \sim^{gof} C$

So we gave some property:

- Any two "$\equiv$" are either disjoint or identical
- $\overline{X}$ is a disjoint union of "$\equiv$" classes
  $[A] = \left\{ B \in \overline{X} \mid B \sim A \right\}$ the "$\equiv$" class set by A

Ant two element in an "$\equiv$" class have the same cardinality
Notation For $n \in \mathbb{N},\ \mathbb{N}_m = \{ 1, 2, \cdots, n \}$

<u>Definition</u>. *Let A be a set*

(a) A is a finite set if $A = \emptyset$ or $A \sim \mathbb{N}_n$ for some $n \in \mathbb{N}$
(b) A is a infinite set if A is not a finite set
(c) A is countable if $A \sim \mathbb{N}$
(d) A is uncountable if A is not countable.
(e) A is at most countable if A is finite or countable

Remark.

(1) *when A, B are finite sets, $A \sim B \Leftrightarrow |A| = |B|$, i.e. A, B have same number.*
(2) *where A, B are infinite and $A \sim B$, i.e. $|A| = |B|$, the concept is abstract.*
(3) $\{a, b, c\} \cup \mathbb{N} \sim \mathbb{N}$, $f : \mathbb{N} \to \{a, b, c\} \cup \mathbb{N}$, $f(1) = a$, $f(2) = b$, $f(3) = c, \cdots$
(4) *Any finite set can not equivalent to a proper subset, i.e. A is finite, $B \subseteq A$*
Then $A \sim B$, In fact $|B| < |A|$, but infinite different
(5) *Any finite set A can be listed an $A = \{a_1, \cdots, a_n\}$ where $n = |A|$*

Now, we consider the case of countable set

Recall, in calculus, a real sequence $\{a_n\}$, e.g.

$$a_n = \frac{1}{n}\left\{\frac{1}{n}\right\},\ a_n = 1 - \frac{1}{n}\ \left\{1 - \frac{1}{n}\right\},\ a_n = \begin{cases} 0 \text{ if } n \text{ is odd} \\ 1 \text{ if } n \text{ is even} \end{cases}$$

Definition. *Let X be a nonempty set, a sequence in X is a function $a : \mathbb{N} \to X$*
Given a sequence $=^a$ in X, a is "1" determine by $a(n),\ \in \mathbb{N}$
We write

$$a = \{a(1), a(2), \cdots, a(n), \cdots\} = \{a_1, a_2, \cdots, a_n, \cdots\} = \{a_n\} = \{a_n\}_{n=1}^{\infty}$$

Remark.

(1) *For a sequence $\{a_n\}$ in X, a_n may not be distinct.*
If all a_n are distinct, then we say that $\{a_n\}$ is a distinct sequence in X.
(2) *We usually use $\{a_n\}, \{b_n\}$ to denote sequence*
(3) *A sequence $\{a_n\}$ in X in fact is a function from $\mathbb{N} \to X$, So $\{a_n \mid n \in \mathbb{N}\}$ is the image of the sequence.*
(4) *$\{a_n\}$ is a sequence, a_n is called the n^{th} term of the sequence.*
(5) *A sequence in X may begin at 0, i.e. $\{a_n\}_{n=0}$*
By a changing index, we can make it from $\{b_n\}_{n=1}^{\infty}$, $b_n = a_{n+1}$, $n = 1, 2, \cdots$

Definition (increasing)**.**
A function $a : \mathbb{N} \to \mathbb{N}$ is increasing, a is $\uparrow$, if $a(n) \leq a(n+1)\ \forall\ n \geq 1$
a is strictly increasing, a is st. $\uparrow$, if $a(n) < a(n+1)\ \forall\ n \geq 1$

Now, given a st. $\uparrow$ function $n : \mathbb{N} \to \mathbb{N}$, i.e. $n(k) < n(k+1),\ k \geq 1$ i.e. $n_k < n_{k+1},\ k \geq 1$, i.e. $n_1 < n_2 < \cdots < n_k < \cdots$, i.e. $\{ n_k \}_{k=1}^{\infty}$ is a st. sequence in $\mathbb{N}$

<u>Definition</u>. *Let $\{ a_n \}$ be a sequence in X and $\{ n_k \}$ be a st. $\uparrow$ sequence in $\mathbb{N}$, then the sequence $\{ a_{n_k} \}$ is called a subsequence of $\{ a_n \}$*
In fact

$$\mathbb{N} \to^{n}_{st.} \mathbb{N} \to^{a}_{seq} X \Rightarrow a \circ n : \mathbb{N} \to X \text{ is a function,}$$

hence, it also a sequence in X

$$a \circ n = \{ a \circ n(k) \} = \{ a(n(k)) \} = \left\{ a_{n(k)} \right\} = \{ a_{n_k} \}$$

Remark. *if $\{ a_{n_k} \}$ is st. $\uparrow$ in $\mathbb{N}$, then $k \leq n_k\ \forall\ k \geq 1$*
$\because$ *By mathematical Induction*

- $1 \leq n_1$
- *Assume it's true for $k \geq 2$, i.e. $k \leq n_k$*
- *Consider $k+1$, $k+1 \leq n_k + 1 \leq n_{k+1}$*

Example
Let $\{ a_n \}$ be a sequence in X, then $\{ a_{2k} \}$ and $\{ a_{2k-1} \}$ are subsequence of $\{ a_n \}$

Finally, we will assume that you are familiar with the following property of the countability of sets:

(1) Every subset of a countable set is at most countable. The proof needs the well ordering of $\mathbb{N}$: Every nonempty subset of $\mathbb{N}$ has the smallest element
(2) Countable union of countable sets is countable
(3) If $A_1, A_2, \cdots, A_n$ are countable, then so is $A_1 \times \cdots \times A_n$
(4) If A is countable, then so is $A^n \equiv A \times \cdots \times A\ \forall n \geq 1$
(5) $\mathbb{N},\ \mathbb{Z},\ \mathbb{Q},\ \mathbb{Q}^n,\ \forall n \geq 1$ are countable
(6) The set $\{ a_n \mid a_n = 0 \text{ or } 1 \}$ is uncountable
This can be proved by Canton diagonal process
$\because$ if it is countable, then we can list it,$a_0 A = \left\{ a_1^{(1)}, a_2^{(2)}, \cdots \right\}$
where
$a^{(1)} = \left\{ a_n^{(1)} \right\} = a_1^{(1)}, a_2^{(1)}, \cdots\ ;\ a^{(2)} = \left\{ a_n^{(2)} \right\} = a_1^{(2)}, a_2^{(2)}, \cdots$
Now, construct a sequence $\{ a_n \}$ in $A \ni \{ a_n \} \neq a^{(k)}\ \forall\ k \geq 1 (\rightarrow\leftarrow)$

Recall, intervals in $\mathbb{R}$, $-\infty < a \le b < \infty$, following are finite bdd interval

$(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$ open interval
$[a, b] = \{ x \in \mathbb{R} \mid a \le x \le b \}$ closed interval
$(a, b] = \{ x \in \mathbb{R} \mid a < x \le b \}$ open-closed
$[a, b) = \{ x \in \mathbb{R} \mid a \le x < b \}$ closed-open

An interval I in $\mathbb{R}$ is said to be non-degenerate if the endpoint of I are distinct i.e. length > 0. Otherwise, it is degenerate.
Note.

$(0, 1)$ is uncountable, $\because (0, 1) = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{2^n} \mid a_n = 0 \text{ or } 1, n \in \mathbb{N} \right\}$
$x \in (0, 1)$ has a unique binary representation, so $(0, 1) \sim A$, where A is $\{ \{ a_n \} \mid a_n = 0 \text{ or } 1 \}$ which is uncountable

All non-degenerate intervals in $\mathbb{R}$ are uncountable.
$\because$ It sufficient to consider bdd non-degenerate interval in $\mathbb{R}$, given $\infty < a < b < \infty$
(a, b) is uncountable($\because (0, 1) \sim (a, b)$)
Note that $(0, 1) \sim \mathbb{R}(\because (0, 1) \to (\frac{\pi}{-2}, \frac{\pi}{2}) \to \mathbb{R})$

2. Basic Point Set Topology

To know the "closeness", "limit" and "continue"

Notation. Let X be a nonempty set. The power set of X is denoted by $p(X)$ or 2^X, i.e. $\mathscr{P}(X) = 2^X$ which is the collect of all subset, if $|X| = n$, then $|\mathscr{P}(X)| = 2^n$

2.1. Topological Spaces.

Definition. *Let X be a nonempty set and $\mathscr{T} \subseteq \mathscr{P}(X)$, we say that $\mathscr{T}$ is a topology on X if it satisfies*

(1) $\emptyset, X \in \mathscr{T}$

(2) $\mathscr{T}$ *is closed under arbitrary union,*

i.e. $U_\alpha \in \mathscr{T},\ \alpha \in I \implies \bigcup_{\alpha \in I} U_\alpha \in \mathscr{T}$

(3) $\mathscr{T}$ *is closed under finite intersection*

i.e. $U_1, \cdots, U_n \in \mathscr{T} \implies U_1 \cap \cdots \cap U_n \in \mathscr{T}$

In this chapter, the pair (X, J) or simply J is called a topological space and members in T are called open set in X or open subsets of X

Remark.

(1) X*: a nonempty set, there is at least two trivial topology on X*
- $\mathscr{P}(x)$ *is the largest topology on X w.r.t inclusion X with this topology is called a discrete topological space*
- $\mathscr{T}_0 = \{\emptyset, X\}$ *is the smallest topology on X w.r.t inclusion X with this topology is called an indiscrete topological space*

(2) *How many topology can be define on $\{a\}$, $\{a, b\}$?*

In the following X is a topology space

Definition (neighborhood). *Let $p \in X$, a neighborhood of P is an open set U containing p*

Definition (Hausdorff space). *X is a Hausdorff space if any two distinct points can be separated by open set, i.e. $\forall\ p \neq q$ in X, $\exists$ neighborhood U of p and V of $q \in U \cap V = \emptyset$*

Definition (closed set). *A subset $F \subseteq X$ is said to be closed if $F^C = X - F$ is open in X*

Theroem 2.1. *The collection of all closed subsets of X satisfied*

(a) $\emptyset, X$ *are closed*

(b) *Arbitrary intersection of closed set if closed*

(c) *Finite union of closed sets is closed*

Proof.

(a) $X - \emptyset = X$ is open $\therefore \emptyset$ is closed
$X - X = \emptyset$ is open $\therefore X$ is closed

(b) Given closed sets $F_\alpha, \alpha \in I$, $X - \bigcap\limits_{\alpha \in I} F_\alpha = \bigcup\limits_{\alpha - I}(X - F_\alpha)$ is open,$\therefore \bigcap\limits_{\alpha - I} F_\alpha$ is closed.

(c) Given closed set $F_1, \cdots, F_n$, $X - \bigcup\limits_{i=1}^{n} F_i = \bigcap\limits_{i=1}^{n}(X - F_i)$ is open, $\therefore \bigcup\limits_{i=1}^{n} F_i$ is closed.

■

<u>Definition</u>. *Let $Y \subseteq X$ and*

$$\mathscr{T}_y = \{U \cap Y \mid U \textit{ is open in } X\}$$

Theroem 2.2. *$\mathscr{T}_Y$ is also a topology space*

Proof. To proof $\mathscr{T}_Y$ is a topology space, we take the topology's definition

(a) $\emptyset, Y \in \mathscr{T}_Y$ $(\because \emptyset = \emptyset \cap Y,\ Y = X \cap Y)$

(b) Given $U_\alpha \cap Y \in \mathscr{T}_Y,\ \alpha \in I$, where U_α is open in X
$$\bigcup_{\alpha \in I}(U_{\alpha \cap Y}) = (\bigcup_{\alpha \in I} U_\alpha) \cap Y \implies \bigcup_{\alpha \in I}(U_\alpha \cap Y) \in \mathscr{T}_Y$$

(c) Given $U_1 \cap Y, \cdots, U_n \cap Y$, where U_i is open in $X,\ 1 \le i \le n$
$$\cap_{i=1}^{n}(U_i \cap Y) = (\cap_{i=1}^{n} U_i) \cap Y \implies \bigcap_{i=1}^{n}(U_i \cap Y) \in \mathscr{T}_Y$$
$\therefore \mathscr{T}_Y$ is a topology on Y

■

<u>Definition</u>. *In Theorem 2.2, with the topology $\mathscr{T}_Y$ on Y, is called a topological subspace of X and $\mathscr{T}_Y$ is called the relative topology of Y in X. Members in $\mathscr{T}_Y$ are called open set in Y or relative open sets in Y.*

2.2. Metric Spaces & Subspace.

In this chapter, we will introduce a class of topology space whose topology in induced by a metric.

Definition. *Let X be a nonempty set. A metric or distance function in a function*

$$d : X \times X \to \mathbb{R},\ (a, b) \mapsto d(a, b)$$

satisfying:

(a) $\forall a, b \in X, d(a, b) \geq 0$ *and* $d(a, b) = 0 \Leftrightarrow a = b$
(b) $\forall a, b \in X, d(a, b) = d(b, a)$ ***symmetry***
(c) $\forall a, b, c \in X, d(a, b) \leq d(a, c) + d(a, b)$ ***triangle inequaity***

if d is a metric on X, then the pair (X, d) or simply X is called a metric space and $\forall a, b \in X,\ d(a, b)$ is called the distance between a & b

Examples

(1) Let X be a nonempty set define by

$$d(a, b) = \begin{cases} 0 \text{ if } a = b \\ 1 \text{ if } a \neq b \end{cases}$$

Then d is a metric on X, called the discrete metric and with this metric X is called a discrete metric space. In particular, any set admits a metric.

(2) The most important metric spaces are the Euclidean space $\mathbb{R}^k$, the metric d is called the Euclidean or standard or usual metric on $\mathbb{R}^k$.There are other metrics on $\mathbb{R}^k$ induced the same metric topology on $\mathbb{R}^k$, in fact, they are all equivalent, e.g $\forall\, 1 \leq p \leq \infty$, We can define a metric d_p on $\mathbb{R}^k$ as follows

- $1 \leq p < \infty,\ d_p(x, y) = ||x - y||_p = \left(\sum_{i=1}^{k} |x_i - y_i|^p \right)^{\frac{1}{p}}$
- $p = \infty,\ d_\infty(x, y) = \max_{1 \leq i \leq k} |x_i - y_i|$

Note that $d_2 = d$ is the Euclidean metric on $\mathbb{R}^k$

Remark. *In fact, every normed linear space$(V, ||\cdot||)$ is a metric space whose metric is induced by its norm*

(3) Let (X, d) be a metric space and $Y \subseteq X,\ Y \neq \emptyset$. Then the restriction of d to $Y \times Y$ is also a metric on Y, with this metric, Y is called a metric subspace of X

Definition (ball). *Given* $p \in X \& r > 0$
$B(p,r) = \{x \in X \mid d(x,p) < r\}$: *open ball with center* p *and radius* r
$\overline{B}(p,r) = \{x \in X \mid d(x,p) \leq r\}$: *closed ball with center* p *and radius* r

Example

(1) The discrete metric space $X: \ p \in X, \ r > 0$

$$B(p,r) = \begin{cases} \{p\} \text{ if } 0 \leq r \leq 1 \\ X \text{ if } r > 1 \end{cases}$$

$$\overline{B}(p,r) = \begin{cases} \{p\} \text{ if } 0 \leq r < 1 \\ X \text{ if } r \geq 1 \end{cases}$$

(2) In the Euclidean space $\mathbb{R}^k$, $p \in \mathbb{R}^k, \ r > 0$
$B(p,r) = \{x \in \mathbb{R} \mid \ \| x - p \| < r\}$ is a ""true"" open
$\overline{B}(p,r) = \{x \in \mathbb{R} \mid \ \| x - p \| \leq r\}$ is a "true" closed ball
In particular, for $k = 1$ in $\mathbb{R}$
$B(p,r) = (p - r, p + r)$: a symmetric opne interval
$\overline{B}(p,r) = [p - r, p + r]$: a symmetric close interval
However, w.r.t d_1 & d_∞, we have, e.g. in $\mathbb{R}^2$
$B_1(0,1) = \{(x,y) \mid |x - 0| + |y - 0| < 1\}$
$B_\infty(0,1) = \{(x,y) \mid \ \max\{|x|,|y|\} \leq 1\}$

(3) What is the open balls in $S = [0,1] \subseteq \mathbb{R}$?
$B_S(0, \frac{1}{2}) = \{x \in S \mid |x - 0| < \frac{1}{2}\} = [0, \frac{1}{2}] = B(0, \frac{1}{2}) \cap [0,1]$
$B_S(0,3) = [0,1] = B(0,3) \cap [0,1]$

Prop 2.2 Let S be a metric subspace of a metric space X, then $\forall \ p \in S \ \& \ r > 0$, $B_S(p,r) = B(p,r) \cap S$

Proof. $B_S(p,r) = \{x \in S \mid d(x,p) < r\} = \{x \in X \mid d(x,p) < r\} \cap S$
$= B(p,r) \cap S$ ∎

2.3. Open Sets in Metric Spaces.
We will see that every metric on a set induce a topology on X

Definition (interior point). *Let* $S \subseteq X$ *be a set, and* $p \in S$, *we say that* p *is an interior point of* S *if* $\exists \ r > 0, \ \exists \ B(p,r) \subseteq S$
Denote by S^o *or int(S) by the set of all interior point of* S

Definition (open). *Let* $S \subseteq X$, *we say that* S *is open if all points of* S *are interior points of* S

Remark.

(1) Every open set S is a union of a open balls in X.
$\because \forall\ x \in S, x$ *is an interior point of* S, $\exists r_x > 0 \ni B(x, r_x) \subseteq S$
$\therefore S = \cup_{x \in S} B(x, r_x)$
(2) $S^o \subseteq S$ *by definition*
(3) S *is open* $\Leftrightarrow S = S^o$

Prop 2.3

(a) $S \subseteq T \implies S^o \subseteq T^o$

$$\because p \in S^o \implies \exists\ r > 0 \ni B(p, r) \subseteq S \subseteq T \implies p \in T^o$$

(b) Every open ball $B(p, r)$ in X is open
$\because$ Give $q \in B(p, r)$, Let $\delta = r = d(p, q)$.Claim $B(q, \delta) \subseteq B(p, r)$ which says q is an interior point of $B(p, r)$.Since $q \in B(p, r)$ is arbitrary, so $B(p, r)$ is open. Given $x \in B(q, \delta)$

$$d(x, p) \leq d(x, q) + d(q, p) < \delta + d(q, p) = r - d(p, q) + d(q, p) = r$$

(c) $\forall\ S \subseteq X,\ S^o$ is always open
$\because$ Given $p \in S^o,\ \exists\ r > 0 \ni B(p, r) \subseteq S$

$$\implies B(p, r) \subseteq S^o \implies p \text{ is a interior point of } S^o$$

$\therefore S^o$ is open
(d) $\forall\ S \subseteq X,\ S^{oo} = (S^o)^o = S^o$
$\because$ by definition of open set and (c)

Now, let $T = \{U \subseteq X \mid U \text{ is open in } X\}$
Prop 2.4 T is a topology on X. In particular, X is a topology space.

Proof.
(i) $\emptyset, X \in \mathscr{T}, \because \emptyset = \emptyset,\ X^o = X$
(ii) $U_\alpha \in \mathscr{T}, a \in I$ are open $\implies \displaystyle\bigcup_{\alpha \in I} U_\alpha$ is open
Given an arbitrary point $p \in \displaystyle\bigcup_{\alpha \in I} U_\alpha \implies \exists \alpha_0 \in I \ni p \in U_{\alpha_0}$
U_{α_0} is open, $\exists r > 0 \ni B(p, r) \subseteq U_{\alpha_0} \subseteq \displaystyle\bigcup_{\alpha \in I} U_\alpha$
$\because p$ is an interior point of $\displaystyle\bigcup_{\alpha \in I} U_\alpha\ \therefore \displaystyle\bigcup_{\alpha \in I} U_\alpha$ is open, i.e. $\displaystyle\bigcup_{\alpha \in I} U_\alpha \in T$

(iii) $U_1, \cdots, U_n \in T \implies U_1 \cap \cdots \cap U_n \in T$
$\because$ Given $p \in U_1 \cap \cdots \cap U_n \implies p \in U_i \ 1 \le i \le n$. Each U_i is open, $\exists r_i > 0 \ B(p, r_i) \subseteq U_i, \ 1 \le i \le n \implies B(p, r) \subseteq U_1 \cap \cdots \cap U_n$
$\implies p$ is an interior point of $U_1 \cap \cdots \cap U_n$
p is arbitrary, so $U_1 \cap \cdots \cap U_n$ is open, i.e. $U_1 \cap \cdots \cap U_n \in T$
Therefore, T is a topology on X

■

Definition. *Let X be a metric space with metric d. The topology T in prop 2.4 is called the metric topology(include by d)*

Let X be a metric space and $Y \subseteq X$, Then Y is a metric subspace of X, and $\forall y \in Y, \ r > 0, \ B_Y(y, r) = B(y, r) \cap Y$. In fact, we have more

Prop 2.5 A subset $A \subseteq Y$ is open in $Y \Leftrightarrow A = U \cap Y$ for some open set U in X, in particular, the metric topology on T is just the relation topology of Y on X

Proof. ($\Rightarrow$) suppose $A \subseteq Y$ is open in Y. Then

$$A = \bigcup_{y \in A} B_Y(y, r_y) = \bigcup_{y \in A} (B(y, r_y) \cap Y) = (\bigcup_{y \in A} B(y, r_y)) \cap Y$$

Let $U = \bigcup_{y \in Y} B(y, r_y)$, then U is open in X and $A = U \cap Y$

($\Leftarrow$) Suppose $A = U \cap Y$ where $U \subseteq X$ is open $\forall \ y \in A, y \in U \cap Y \implies y \in U \implies \exists r > 0 \ni B(y, r) \subseteq U \implies B(y, r) \cap Y \subseteq U \cap Y = A \implies B_Y(y, r) \subseteq A, \ \therefore A$ is open in Y. ■

Prop 2.6 Every metric space X is Hausdorff

Proof. Given $p, q \in X, \ p \neq q$. Choose $r = \dfrac{1}{2} d(p, q) > 0$. Then $B(p, r) \cap B(q, r) = \emptyset$. So X is Hausdorff
($\because x \in B(p, r) \cap B(q, r) = d(x, p) < r \ \& \ d(x, q) < r \implies d(p, q) \le d(p, x) + d(x, q) < r + r = 2r = d(p, q) (\rightarrow\leftarrow)$) ■

Remark. *Let $S \subseteq X$, where X is a metric space. Then S^o is the largest(w.r.t inclusion) open set contained in S. $\because \forall$ open set $U \subseteq S$, $U^o \subseteq S^o \implies U \subseteq S^o \subseteq S$. In fact, $S^\circ = \bigcup_{U \subseteq S} U$ (which is the definition of intension of S in a topology space X)*

2.4. Closed Sets.

Definition (Closed set). *$F \subseteq X$ is closed $\Leftrightarrow F^C = X - F$ is open in X*

By Theorem 2.1, the collection of all close sets in X has the properties

(i) $\emptyset, X$ are closed in X

(ii) F_α is closed in X, $\alpha \in I \implies \bigcap_{\alpha \in I} F_\alpha$ is closed in X

(iii) $F_1, \cdots, F_n$ are closed in $X \implies \bigcup_{i=1} F_i$ is closed in X

Example

Intersection of infinitely many open set may not be open in $\mathbb{R}$ with Euclidean topology, $(-\frac{1}{n}, \frac{1}{n})$ is open in $\mathbb{R}$ $\forall\ n \geq 1 \implies \bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$ is not open

Prop 2.8 Let X be a metric space and $Y \subseteq X$ and $B \subseteq Y$, Then B is closed in $Y \Leftrightarrow B = F \cap Y$ for some closed set F in X

Proof. ($\Rightarrow$) Suppose B is close in $Y \implies Y - B$ is open in $Y \implies Y - B = U \cap Y$(by prop 2.5) for some open set U in $X \implies Y - (Y - B) = Y - (U \cap Y) \implies B = (X - U) \cap Y$. where $(X - U)$ is close.
($\Leftarrow$) Suppose $B = F \cap Y$, where F is closed in $X \implies Y - B = Y - (F \cap Y) = (X - F) \cap Y \implies Y - B$ is open in $Y \implies B$ is close in Y. ■

In metic space, one can use sequence to detect the closeness of a set

Example

1. We know that $[a, b)$ is not closed in $\mathbb{R}$, however, $\exists$ a sequence $\{x_n\}$ in $[a, b) \ni x_n \to b$ on $n \to \infty$, e.g. $b - \frac{1}{n} \to b$

2. $A = \{\frac{1}{n} \mid n \geq 1\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ is not close in $\mathbb{R}$
if $\mathbb{R} - A$ will open, them $\exists r > 0,\ B(0, r) \subseteq \mathbb{R} - A (\rightarrow\leftarrow)$
$A \cup \{0\}$ is closed in $\mathbb{R}$

$$R \setminus (A \cup \{0\}) = (-\infty, 0) \cup (1, \infty) \cup (\bigcup_{n=1}^{\infty}(\frac{1}{n+1}, \frac{1}{n})) \text{ is open}$$

$\therefore A \cup \{0\}$ is closed

Definition (Adherent, clousure $\cdots$)**.** *Let X be a metric space with metric d, $T \subseteq X$ be a subset.(**important**)*

(1.) *A point $p \in X$ is said to be an adherent point of T if $\forall r > 0,\ B(p, r) \cap T \neq \emptyset$, equivalent, $\forall$ neighborhood U of p, $U \cap T \neq \emptyset$*

(2.) *Let $\overline{T}$ or $cl(T)$ denote the set of all adherent points of T, called the closure of T, i.e. $\overline{T} = \{p \in X \mid p$ is an adherent point of $T\}$*

(3.) *A point* $p \in X$ *is said to be a limit point or accumulation point of* T *if* $\forall\ r > 0,\ B(p,r) \cap T - \{p\} \neq \emptyset$*, equivalently,* $\forall$ *neighborhood* U *of* p*,* $U \cap T - \{p\} \neq \emptyset$
Denote by T' *the set of all accumulation points of* T*, called the devied set of* T*.*
(4.) $p \in T$ *and* $p \notin T'$*, then* p *is called an isolated point of* T*, i.e.* $\exists r > 0 \ni B(p,r) \cap T = \{p\}$
(5.) *A subset* $T \subseteq X$ *is said to be perfect if* T *is closed and every points of* T *is an accumulated point of* T*, i.e.* T *is closed &* $T' = T$
(6.) *A subset* $T \subseteq X$ *is said to be bounded if* $\exists R > 0$ *and* $p \in X \ni T \subseteq B(p,R)$
(7.) *A subset* $T \subseteq X$ *is said to be dense if* $\overline{T} = X$*, e.g.* $\overline{\mathbb{Q}} = \mathbb{R}$
(8.) *A point* $p \in X$ *is said to be a boundary point of* T *if* $\forall\ r > 0, B(p,r) \cap T \neq \emptyset$ *&* $B(p,r) \cap (X \setminus T) \neq \emptyset$*. Denote by* ∂T *or bd(*T*) the set of all boundary points of* T

Prop 2.9 Let X be a metric space. All sets and point below are subset of X
(1) $S \subseteq T \implies \overline{S} \subseteq \overline{T}$ & $S' \subseteq T'$
$\because p \in \overline{S} \implies \forall\ r > 0, B(p,r) \cap S \neq \emptyset \implies B(p,r) \cap T \neq \emptyset \implies p \in \overline{T}$
$p \in S' \implies \forall r > 0, B(p,r) \cap S - \{p\} \neq \emptyset \implies B(p,r) \cap T - \{p\} \neq \emptyset$
(2) $\overline{T}$ is always closed in X

> We want to know $\overline{T}$ is closed on $X \to X - \overline{T}$ is open $\to \forall\ p \in X - \overline{T}$ is an interior point $\implies \exists r > 0,\ B(p,r) \subseteq X - \overline{T}$
> $\because p \notin \overline{T} \Rightarrow \exists r' > 0 \ni B(p,r') \cap T = \emptyset$
> But we want to get $B(p,r') \cap \overline{T}$, so we check every point in $B(p,r')$ is not in $\overline{T}$, let $q \in B(p,r'),\ \exists\ \delta > 0\ , B(q,\delta) \subseteq B(p,r') \implies B(q,\delta) \cap T = \emptyset \implies q \notin \overline{T}$
> because if $q \in \overline{T}, \forall r > 0 \ni B(q,r) \cap T \neq \emptyset$
> $\implies B(p,r) \cap \overline{T} = \emptyset$

Let $p \in X - \overline{T} \implies p \notin \overline{T} \implies \exists\ r > 0 \ni B(p,r) \cap T = \emptyset \implies B(p,r) \cap \overline{T} = \emptyset$ ($\because \forall q \in B(p,r),\ \exists\ \delta > 0 \ni B(q,\delta) \subseteq B(p,r) \implies B(q,\delta) \cap T = \emptyset \implies q \notin \overline{T}$)
$\therefore B(p,r) \subseteq X - \overline{T}$, $\because p$ is an interior point of $X - \overline{T}$. Hence, $X - \overline{T}$ is open, i.e. $\overline{T}$ is closed.
(3) $T \subseteq \overline{T}(\because \forall\ p \in T, \forall\ r > 0, B(p,r) \cap T \neq \emptyset)$

(4) $p \in T' \implies \forall r > 0,\ B(p,r) \cap T - \{p\}$ is an infinite set, say $x_1, \cdots, x_n$, Let $\delta = \dfrac{1}{2}\min\{d(p,x_i) \mid 1 \le i \le n\}$. Then $B(p,\delta) \cap T - \{p\} = \emptyset (\rightarrow\leftarrow)$ to $p \in T'$, $x \in B(p,\delta) \cap T - \{p\} \implies d(x,p) < \delta \implies x = x_i$ for some $1 \le i \le n\&$ we get $d(x_i,p) < \delta \le \dfrac{1}{2}d(x_i,p)$
$\therefore$ no such x i.e. $B(p,\delta) \cap T - \{p\} = \emptyset$
(5) Any finite subset of X has no accumulation points in X by (4). In particular, it is closed by (6)(c) below.
(6) TFAE
(a) S is closed
(b) S contains all it's adherent point, i.e. $\overline{S} \subseteq S$
(c) S contains all it's accumulation points, i.e. $S' \subseteq S$
(d) $S = \overline{S}$
(7) $\overline{\overline{S}} = \overline{S}$ by (2) and (6)

Proof. of (6)
(a) $\Rightarrow$ (b) Suppose S is closed $\implies X \backslash S$ is open $\implies \forall\, p \in X - S \implies \exists r > 0 \ni B(p,r) \subseteq X \setminus S \implies B(p,r) \cap S = \emptyset \implies p \notin \overline{S}$
$\therefore \overline{S} \subseteq S$, i.e. (b) holds

(b) $\Rightarrow$ (c) $\because S' \subseteq \overline{S}$

(c) $\Rightarrow$ (d) Suppose $S' \subseteq S$. To prove $S = \overline{S}$ if not, then $S \subsetneq \overline{S}$, i.e. $\exists\, p \in \overline{S}\ \&\ p \notin S \implies \forall r > 0,\ B(p,r) \cap S \neq \emptyset\ (\because p \in \overline{S})$
(d) $\Rightarrow$ (a) by (2)

■

(8) $\overline{S}$ is the smallest closed set in X containing S
$\because$ We know that $S \subseteq \overline{S}$, if F is closed in X & $F \subseteq S$, then $\overline{F} \subseteq \overline{S}$ by (1), $F = \overline{F} \subseteq \overline{S}$ by (6), $\therefore \overline{S}$ is the smallest such one.
(9) In fact, $\overline{S} = \bigcap\limits_{F \subseteq S} F$
(10) $p \in S$ is an isolated point $\Leftrightarrow \exists\, r > 0 \ni B(p,r) \cap S = \{p\}$
($\Rightarrow$) Suppose $p \in S$ is an isolated point of S. Then $p \in S' \implies \exists\, r > 0 \ni B(p,r) \cap S - \{p\} = \emptyset \implies B(p,r) \cap S = \{p\}$
($\Leftarrow$) Trivial
(11) S is dense in $X \Leftrightarrow\ \forall\, p \in X\ \&\ r > 0,\ B(p,r) \cap S \neq \emptyset \Leftrightarrow \forall$ open set $U \neq \emptyset,\ U \cap S \neq \emptyset$

Proof. ($\Rightarrow$) Suppose S is dense in X, i.e. $\overline{S} = X$, So $\forall p \in X, p \in \overline{S} \implies \forall\, r > 0,\ B(p,r) \cap S \neq \emptyset$
($\Leftarrow$) Suppose the condition holds, $\forall p \in X\ \&\ r > 0,\ B(p,r) \cap S \neq \emptyset \implies p \in \overline{S} \implies X \subseteq \overline{S} \subseteq X,\ \therefore \overline{S} = X$ ■

(12) $\partial S = \partial(X-S)$ In particular, $\partial S = \overline{S} \cap \overline{(X-S)}$, In particular, ∂S is closed in X, $\because$ It suffices to prove $\partial S = \overline{S} \cap \overline{(X-S)}$,
$\because \partial(X-S) = \overline{X-S} \cap \overline{X-(X-S)} = \overline{X-S} \cap \overline{S} = \partial S$
$\forall p \in \partial S \implies \forall r > 0, B(p,r) \cap S \neq \emptyset \;\&\; p \in \overline{X-S} \implies p \in \overline{S} \cap \overline{X-S}$
$\therefore \partial S \subseteq \overline{S} \cap \overline{X-S}$,
Conversely, $p \in \overline{S} \cap \overline{X-S} \implies p \in \overline{S} \;\&\; p \in \overline{X-S} \implies \forall r > 0,\ B(p,r) \cap S \neq \emptyset \;\&\; B(p,r) \cap (X-S) \neq \emptyset \implies p \in \partial S$
$\therefore \overline{S} \cap \overline{(X-S)} \subseteq \partial S \therefore \partial S = \overline{S} \cap \overline{(X-S)}$

2.5. Examples.

We give some simple examples of open sets, closed sets, adherent, accumulation, isolated and boundary points.

1. In a discrete metric space X, every subset of X is both open and close, $\forall x \in X,\ B(p,r) \begin{cases} \{x\} \text{ if } 0 < r \leq 1 \\ X \text{ if } r > 1 \end{cases}$
$\therefore$ Every singleton is open in X, so every subset of X is open.

2. In $\mathbb{R}$. Consider the set $S = [0,1) \cup \{3\}$, $S^\circ = \emptyset,\ S' = \{0\}$, $\overline{S} = S \cup \{0\}$

3. In $\mathbb{R}$, consider the set $S = \{\frac{1}{n} \mid n = 1, 2, \cdots\}$, $S^\circ = \emptyset$, $S' = \{0\},\ \overline{S} = S \cup \{0\}$

4. In $\mathbb{R}^2$, consider $S = \{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$, S is open
$\overline{S} = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$
$\partial S = \{(x,0) \mid x \geq 0\} \cup \{(y,0) \mid y \geq 0\}$

5. Let $B(0,1)$ be the unit open ball in $\mathbb{R}^k$. Then $\partial B(0,1) = S^{k-1}$ is the unit $(k-1)$-sphere. In particular, for $k = 2, \partial B(0,1) = S^1$ in the unity circle in the plane $\mathbb{R}^2$. Similarly, for the closed unit ball $\overline{B}(0,1)$ in $\mathbb{R}^k$. Now, we define some special sets in $\mathbb{R}^n$

- Internals in $\mathbb{R}$: $-\infty < a \leq b < \infty$
  $[a,b]$ close interval which is closed in $\mathbb{R}$
  (a,b) open interval which is closed in $\mathbb{R}$
  Infinite intervals:
  $(-\infty, b]$: close in $\mathbb{R}$, $(-\infty, b)$ open in $\mathbb{R}$
- k-dimensional interval (rectangle or k-cell) I

$$I = I_1 \times \cdots \times I_k$$

  where I_j is an interval in $\mathbb{R}$, $1 \leq j \leq k$
  (i) I is bounded $\Leftrightarrow$ each I_j is bounded
  I is unbounded $\Leftrightarrow I_j \neq \emptyset$ & some I_j is unbounded
  (ii) $I = [a_1, b_1] \times \cdots [a_k, b_k],\ -\infty < a_j \leq b_j < \infty,\ 1 \leq j \leq k$
  k-dimensional closed(compact) interval in $\mathbb{R}^k$

- Convex sets in $\mathbb{R}^k$
  $S \subseteq \mathbb{R}^k$ is convex if $\forall\, x, y \in S,\ \overline{xy}$ is the line segment joining x & y
  Note that all open balls, closed balls, intervals are convex in $\mathbb{R}^k$
- Star-like sets in $\mathbb{R}^k$ with w.r.t some point x_0, $S \subseteq \mathbb{R}^k$ is star-like w.r.t. $x_0 \in S$ if $\forall\, x \in S,\ \overline{xx_0} \subseteq S$

6. We know that $\mathbb{Q}$ is dense in $\mathbb{R}$, hence $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$. Note that $\mathbb{Q}^k$ is countable, hence $\mathbb{R}^k$ has a countable dense subset $\mathbb{Q}^k$, i.e. $\mathbb{R}^k$ is separable.
7. $\partial\mathbb{Q} = \mathbb{R},\ \partial\mathbb{Q}^k = \mathbb{R}^k$
8. $\mathbb{Z}$ is closed in $\mathbb{R}$, $\because \mathbb{R} - \mathbb{Z} = \displaystyle\bigcup_{n=-\infty}^{\infty} (n-1, n)$ is open $\implies \mathbb{Z}$ is close.
or $\mathbb{Z}' = \emptyset \subseteq \mathbb{Z},\ \therefore \mathbb{Z}$ is close.
9. Let $S \subseteq \mathbb{R}$ be a nonempty set which is bounded above. Then $\alpha = \sup S$ exists. Moreover, $\alpha \in \overline{S}$. $\because \forall\, r > 0, \exists\, x_0 \in S \ni \alpha - r < x_0 \leq \alpha < \alpha - r \implies (\alpha - r, \alpha + r) \cap S \neq \emptyset \implies \alpha \in \overline{S}$

2.6. Compact Set in Metric Space.

- Compact sets in metric space, which is closely related to the extreme value problem.
- Compact set $\mathbb{R}^k$ will be discussed in next section.

Definition. *Let X be a topology space and $S \subseteq X$. A collection $\mathscr{U} = \{U_\alpha\}_{\alpha \in I}$ of open sets in X is called an open covering of S if*

$$S \subseteq \bigcup \mathscr{U} = \bigcup_{\alpha \in I} U_\alpha$$

Definition. *Let X be a topology space, $S \subseteq X$ and $\mathscr{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of S. We say that $\mathscr{U}$ has a countable(finite) sub covering of S if $\exists$ a countable(finite) sub collection of $\mathscr{U}$ which also covers S. i.e. $\mathscr{U}$ has a countable(finite) subcovering in S if*

$\exists$ a sequence $\{\alpha_n\}$ in $I \ni S \subseteq \displaystyle\bigcup_{n=1}^{\infty} U_{\alpha_n}$ (countable)
$\exists$ a sequence $\{\alpha_n\}$ in $I \ni S \subseteq U_\alpha \cup \cdots \cup U_{\alpha_n}$(finite)

Example

(1) X is discrete metric space. Then $\{\{x\} \mid x \in X\}$ is an open covering of X
(2) In $\mathbb{R}$, $\{(0, 1 - \frac{1}{n}) \mid n \in \mathbb{N}\}$ is an open covering of $(0,1)$. In fact,
$$(0,1) = \bigcup_{n=1}^{\infty} (0, 1 - \frac{1}{n})$$

(3) $\{B(0,n) \mid n \in \mathbb{N}\}$ is an open covering of $\mathbb{R}^k$

Definition (compact)**.** *Let X be a topology space. A subset $K \subseteq X$ is said to be compact if* ***every*** *open covering of K admit a finite subcovering*

Examples

(1) Let X be a topology space and $K \subseteq X$ be a finite set. Then K is compact.
(2) In a discrete metric space X, a subset $K \subseteq X$ is compact $\Leftrightarrow$ K is a finite set.
(3) $(0,1)$ is not compact in $\mathbb{R}(\{0, 1 - \dfrac{1}{n} \mid n \in \mathbb{N}\})$, but $[0,1]$ is compact

Theroem 2.10. *Let X be a metric space and $K \subseteq Y \subseteq X$. Then K is compact in $X \Leftrightarrow K$ is compact in Y.*

Proof. ($\Rightarrow$) Suppose K is compact in X. Given an open covering $\{V_\alpha\}_{\alpha \in I}$ of open sets in Y which covers K. By Prop 2.5, each $V_\alpha = U_\alpha \cap Y$, where U_α is open in X. Now,

$$K \subseteq \bigcup_{\alpha \in I} V_\alpha = \bigcup_{\alpha \in I} (U_\alpha \cap Y) = (\bigcup_{\alpha \in I} U_\alpha) \cap Y \implies K \subseteq \bigcup_{\alpha \in I} U_\alpha$$

By the compactness of K in X, $\exists \alpha_1, \cdots, \alpha_n \in I \ni K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} \implies$
$K \cap Y \subseteq (\bigcup_{i=1}^{n} U_{\alpha_i}) \implies K \subseteq \bigcup_{i=1}^{n} (U_{\alpha_i} \cap Y) = \bigcup_{i=1}^{n} V_{\alpha_i}$
$\therefore K$ is compact in Y
($\Leftarrow$)Suppose K is compact in Y. Given a open covering $\{U_\alpha\}_{\alpha \in I}$ of K by open sets in X.

$$K \subseteq \bigcup_{\alpha \in I} U_\alpha \implies K \cap Y \subseteq (\bigcup_{\alpha \in I} U_\alpha) \cap Y \implies K \cap Y \subseteq \bigcup_{\alpha \in I} (U_\alpha \cap Y)$$

By Prop 2.5, $\{U_\alpha \cap Y \mid \alpha \in I\}$ is an open covering of K by open set in Y. By assumption, K is compact in $Y, \exists\ \alpha_1, \cdots, \alpha_n \in I \ni K \subseteq \bigcup_{i=1}^{n} (U_{\alpha_i} \cap Y) = (\bigcup_{i=1}^{n} U_{\alpha_i}) \cap Y \implies K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$
$\therefore K$ is compact in X ∎

Definition. *Let X be a metric space and $S \subseteq X$ be a nonempty set. The diameter of S is defined to be $dia(S) = \sup\{d(x,y) \mid x, y \in S\}$ which generated the diameter of a circle in $\mathbb{R}^2$*

Theroem 2.11. *Let X be a metric space and $K \subseteq X$ be a compact set. Then K is closed and bounded*

Proof. K **is bounded**

Fix a point $p \in K$. Then $K \subseteq \bigcup_{n=1}^{\infty} B(p,n)$. $\because K$ is compact $\implies \exists N \in \mathbb{N} \ni K \subseteq B(p,1) \cup \cdots \cup B(p,N) \implies K \subseteq B(p,N)$ $\therefore K$ is bounded
K is closed, i.e. $X - K$ is open
Fix $p \in X - K$. Then $p \neq x, \ \forall \ x \in K$. Hence, $d(x,p) > 0, \ \forall \ x \in K$
Let $r_x = \frac{1}{2}d(x,p) > 0, x \in K$. Them $\{B(x, r_x) \mid x \in K\}$ is an open covering of K. $\because K$ is compact $\implies \exists x_1, \cdots, x_n \in K \ni B(x_1, r_{x_1}) \cup \cdots \cup B(x_n, r_{x_i})$. Let $V = \bigcap_{i=1}^{n} B(p, r_{x_i}) = B(p,r)$, where $r = \min\{r_{x_1}, \cdots, r_{x_n}\}$. Then as we can see that $V \subseteq X - K$, all point in $X - K$ are inner point. So $X - K$ is open, i.e. K is close.

To show that $V \subseteq X - K$, i.e. $V \cap K \neq \emptyset$, it suffices to show

$$V \cap (\bigcup_{i=1}^{n} B(x_i, r_{x_i})) = \emptyset$$

Now,

$$\begin{aligned} V \cap (\bigcup_{i=1}^{n} B(x_i, r_{x_i})) &= \bigcup_{i=1}^{n} (V \cap B(r_i, r_{x_i})) \\ &\subseteq \bigcup_{i=1}^{n} (B(p, r_{x_i} \cap B(x_i, r_{x_i}))) = \emptyset \end{aligned}$$

■

Remark. *The converse of Thm 2.11 is false, i.e. closed & bounded may not be compact, e.g. X is an infinite set with discrete metric. Then X is not compact, but X is closed and bounded.*

Theroem 2.12. *Let X be a metric space, $K \subseteq X$ be compact & $L \subseteq K$ be a closed set in X. Then L is compact.*

Proof. Let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of L. Then $\{U_\alpha\}_{\alpha \in I} \cup \{X - L\}$ is an open covering of K. By the compactness of K, $\exists \alpha_1, \cdots, \alpha_n \in I \ni K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \cup (X - L)$. By $L \subseteq K$ $\therefore L$ is compact ■

Corollary 2.13.

(a) Let X be a metric space, $K \subseteq X$ be compact and F be a closed set in X. Then $K \cap F$ is compact.
(b) If X is a compact metric space, then every closed subset F of X is compact.

Proof.

(a)

$$\begin{aligned} K \text{ is compact} &\implies K \text{ is closed (Thm 2.11)} \\ &\implies K \cap F \text{ is closed in } X \\ &\implies K \cap F \text{ is compact} \end{aligned}$$

(b) follows (a)

■

Remark. *Let X be a metric space. If K is closed in X and F is closed in K, then F is closed in X. $\because F$ is closed in $F \implies F = L \cap F$, where L is closed in $K \implies F = L \cap K$, where L is closed in $X \implies F$ is closed in X.*

Theroem 2.14. *Let X be a metric space, $\{K_\alpha\}_{\alpha \in I}$ be a collection of compact subsets of X with the property:*

$$\forall \alpha_1, \cdots, \alpha_n \in I, K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \neq \emptyset$$

Them $\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$

Proof. Fix $\alpha_0 \in I$. Assume that $\bigcap_{\alpha \in I} K_\alpha = \emptyset \implies X - \bigcap_{\alpha \in I} K_\alpha$
$\implies X - \emptyset = X \implies X = \bigcup_{\alpha \in I} (X - K_\alpha)$
each K_α is compact $\implies K_\alpha$ is closed $\implies X - K_\alpha$ is open
so $\{X - K_\alpha\}$ is an open covering of X. Now,

$$K_{\alpha_0} \subseteq X = \bigcup_{\alpha \in I} (X - K_\alpha) \implies K_{\alpha_0} \subseteq \bigcup_{\alpha \in I} (X - K_\alpha)$$

K_α is compact $\implies \exists \alpha_1, \cdots, \alpha_n \in I - \{\alpha_0\} \ni K_{\alpha_0} \subseteq (X - K_{\alpha_1}) \cup \cdots \cup (X - K_{\alpha_n}) \implies K_{\alpha_0} \cap K_{\alpha_1} \cap \cdots \cap K_n = \emptyset (\rightarrow\leftarrow)$ ■

Corollary 2.15. *Let X be a metric space and $\{K_n\}_{n=1}^{\infty}$ be a decrease sequence of nonempty compact sets of X. Them $\bigcap_{n=1}^{\infty} \neq \emptyset$. In addition, if $dia_{n\to\infty}\infty 0$, them $\bigcap_{n=1}^{\infty} K_n$ is a singleton.*

Proof. $\forall j1, \cdots, j_k \in \mathbb{N},\ K_{j1}\cap\cdots\cap K_{jk} \neq \emptyset,\ K_{j1}\cap\cdots\cap K_{jk} = K_t$, where $t = \max\{j_1, \cdots, j_k\}$. By Thm 2.14 $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$, if $\lim_{n\to\infty} \text{dia}(K_n) = 0$ and $p, q \in \bigcap_{n=1}^{\infty} K$ and $p \neq q$, them $\text{dia}(K_n) \geq d(p,q)\ \forall n \geq 1 \implies \lim_{n\to\infty} \text{dia}(K_n) \geq d(p,q) > 0 (\rightarrow\leftarrow) \because \bigcap_{n=1}^{\infty} K_n = \{p\}$ is a simpleton. ∎

Remark. *The usual form of Cor 2.15, X is a metric space, $\{K_n\}$ is a decrease sequence of nonempty closed sets in X with K_i is compact $\implies \bigcap_{n=1}^{\infty} K_n \neq \emptyset$*

Example In $\mathbb{R}$, $\{(0, \frac{1}{n} \mid n \geq 1]\}$ is decrease and every finite subcollection of $\{(0, \frac{1}{n}) \mid n \geq 1\}$ is nonempty, but $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset$, $\bigcap [0, \frac{1}{n}] = \emptyset$

Theroem 2.17. *Let X be a metric space and $K \subseteq X$, TFAE:*
(i) K is compact
(ii) Every infinite subset has an accumulation point in K
(iii) K is sequentially compact
(iv) K is complete and totally bounded

Definition (Convergence)**.** *$\{a_n\}$ converge if $\exists\ a \in X \ni \forall\ \epsilon \geq 0 \exists N \ni \mathbb{N} \ni \forall n \geq N,\ d(a_n, a) < \epsilon$. Such a is called the limit of $\{a_n\}$, which is denoted by $\lim_{n\to\infty} a_n = a$ or $a_n \to a$ on $n \to \infty$.*

Definition (Cauchy)**.** *We say that $\{a_n\}$ is Cauchy if $\forall \epsilon > 0,\ \exists N \in \mathbb{N} \ni \forall n, m \geq \mathbb{N},\ d(a_n, a_m) < \epsilon$*

Definition. *A metric X is said to be sequence compact if every sequence has a convergent subsequence*

Definition. *A metric space X is said to be complete if every Cauchy sequence in X convergence.*

<u>**Definition.**</u> *Let X be a metric space & $K \subseteq X$. We say that K is totally bounded if $\forall r > 0, \exists x_1, \cdots, x_n \in K \ni K \subseteq B(x_1, r) \cup \cdots \cup B(x_n, r)$*

Remark. *Totally bounded can implies bounded, but not converse.*
K is totally bounded, for $r = 1, \exists x_1, \cdots, x_n \in K \subseteq B(x_1, 1) \cup \cdots \cup B(x_n, 1) \implies K \subseteq B(x_1, R)$ for sime large R

- *Take an **"infinite"** set X with discrete metric. Then X is bounded(e.g. $X \subseteq B(x_0, 2)$, where $x_0 \in X$) but for $r = \frac{1}{2}$, $X \subsetneqq B(x_1, \frac{1}{2}) \cup \cdots \cup B(x_n, \frac{1}{2})$ $\forall x_1, \cdots, x_n$*

Lemma 2.18 (To prove (ii) to (i))**.**
Suppose (ii) holds in Thm 2.17. Then K is totally bounded

Proof. If not, then $\exists\ r > 0, \ni$ no finite open balls with radius r and center K cover K. Choose $x_1 \in k \implies K \subsetneqq B(x_1, r) \implies \exists x_2 \in K - B(x_1, r)$, $K \subsetneqq B(x_1, r) \cup B(x_2, r) \implies \exists x_3 \in K - (B(x_1, r) \cup B(x_2, r))$ By induction, counting this process, we obtain an infinite set $T = \{x_1, x_2, \cdots, x_n, \cdots\} \subseteq K$ with $d(x_i, x_j) \geq r \forall i \neq j$. By (ii), T has an accumulation poion $p \in K$. In particular $B(p, \frac{r}{4}) \cap T - \{p\}$ is an infinite set, hence, $\exists i \neq j \ni x_i, x_j \in B(p, \frac{r}{4}) \cap T - \{p\} \implies d(x_i, x_j) \leq d(x_i, p) + d(x, j) < \frac{r}{4} + \frac{r}{4} = \frac{r}{2} < r (\to\leftarrow) \therefore K$ is totally bounded. ■

Lemma 2.19. *Suppose (ii) holds in T and $\{E_\alpha \mid \alpha \in I\}$ is an open covering of K. Then $\exists r > 0$(called a Lebegoue number w.r.t. the open covering $\{E_\alpha\}_{\alpha \in I}$) $\ni \forall x \in K, B(x, r) \subseteq E_\alpha$ for some $\alpha \in I$*

Proof. if K is a finite set, let $K = \{x_1, \cdots, x_n\} K \subseteq \bigcup_{\alpha \in I} E_\alpha \implies x_i \in E_{\alpha_i}$ for some $\alpha_i \in I$, $1 \leq i \leq n \implies \exists r_i > 0 \ni B(x_i, r_i) \subseteq E_{\alpha_i}, 1 \leq i \leq n$. Let $r = \min\{r_1, \cdots, r_n\}$. Then $B(x_i, r) \subseteq B(x_i, r_i) \subseteq E_{\alpha_i}, 1 \leq i \leq \alpha$. Now, assume that K is infinite set. Assume that no such $r > 0$, i.e. $\forall r > 0, \exists x_r \in K \ni B(x_r, r) \subsetneqq E_\alpha \forall x \in I$. Now, for $r = \frac{1}{k}, r = 1, 2, \cdots$, we obtain a sequence $\{x_k\}$ in K, with $x_k = \frac{x_r}{k} \ni B(x_k, \frac{1}{k}) \subsetneqq E_\alpha \forall a \in I$. Let $T = \{x_1, x_2, \cdots, x_k, \cdots\}$. Then $T \subseteq K$ is an infinite set($\because$ For $k = 1, r = \frac{1}{1} = 1 \exists x_1 \in K \ni B(x_1, 1) \subsetneq E_\alpha \forall \alpha \in I$). The conclusion of Lemma 2.19 failed for $K - \{x_1\}$ ($\because if \exists s > 0 \ni \forall x \in K - \{x_1\} B(x, s) \subseteq E_\alpha$ for some $\alpha \in I$, $x_1 \in E_\alpha \implies \exists t > 0 \ni B(x_1, t) \subseteq E_\alpha$. Let $r = \min\{s, t\}$. Then $\forall x \in K, B(x, r) \subseteq E_\alpha$ for some $\alpha \in I (\to\leftarrow)$)

Then for $r = \frac{1}{2}, \exists x_2 \in K - \{x_1\} \ni B(x_2, \frac{1}{2}) \subsetneqq E_\alpha \forall \alpha \in I$. Continue this process, we conclude that $x_i \neq x_j \forall i \neq j$, so T is an infinite set. By the assumption of (ii). Then an accumulation point $p \in K$. Now $K = \bigcup_{\alpha \in I} E_\alpha \implies p \in E_\alpha$ for some $\alpha \in I \implies \exists \epsilon > 0 \ni B(p, \epsilon) \subseteq E_{\alpha_0}$. Since $p \ni T', B(p,\epsilon) \cap T - \{p\}$ is an infinite set. Choose $m >> 0 \ni \frac{1}{m} < \frac{\epsilon}{2}$ & $x_m \in B(p, \frac{\epsilon}{2}) \cap T$. Claim $B(x_m, \frac{1}{m}) \subseteq B(p, \epsilon) \subseteq E_{\alpha_0} (\rightarrow\leftarrow)$ to our constrain, hence Lemma 2.19 holds.

$$y \in B(x_m, \frac{1}{m}) \Rightarrow d(y,p) \leq d(y, x_m) + d(x_m, p) < \frac{1}{m} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$
$$\Rightarrow y \in B(p, \epsilon)$$ ■

Proof. (Thm 2.17 (i)(ii))
$(i) \Rightarrow (ii)$ Suppose K is compact. Given an infinite set $T \subseteq K$. We must prove that T has an accumulation point in K, if not, $\forall x \in K, x$ is not an accumulation point of $K, \exists r_x > 0 \ni B(x, r_n) \cap T - \{x\} = \emptyset \implies B(x, r_x) \cap T \subseteq \{x\}$. Clearly $\{B(x, r_x \mid x \in K)\}$ is an open covering of K. By (i), K is compact

$$\begin{aligned} \implies \exists x_1, \cdots, x_n \in K \ni K &\subseteq B(x_1, r_{x_1}) \cup \cdots \cup B(x_n, r_{x_n}) \\ &= (T \cap B(x_1, r_{x_1})) \cup \cdots \cup (T \cap B(x_n, r_{x_n})) \\ &\subseteq \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_n\} \\ &= \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_n\} \\ &= \{x_1, \cdots, x_n\} (\rightarrow\leftarrow) \end{aligned}$$

to T is an infinite set, $\therefore$ (ii) holds.
$(ii) \Rightarrow (i)$ In Thm 2.17 i.e. we must prove that K is compact under the assumption of (ii). Suppose $\mathscr{U} = \{E_\alpha\}_{\alpha \in I}$ is an open covering of K. By Lemma 2.19, $\exists$ a $r > 0$ *w.r.t.* $\mathscr{U}$, by Lemma 2.18, $\exists x_1, \cdots, x_k \in K \ni K \subseteq B(x_1, r) \cup \cdots \cup B(x_k, r) \subseteq E_{\alpha_1} \cup \cdots \cup E_{\alpha_k}$, where $B(x_i, r) \subseteq E_{\alpha_i}, 1 \leq i \leq k$. Therefore, $\mathscr{U}$ has a finite sub covering. Hence K is compact and (i) holds. ■

Remark.
(1) (ii) $\implies$ (i) is exercise 26
(2) More or less, by Lemma 2.18 & 19, one can see that (i) and (ii) are also equal to (iii) and (iv)

2.7. Compact Sets in Euclidean Spaces $\mathbb{R}^k$.

- We know that any compact set in a metric space is close and bounded
- Close and bounded subset my not be compact(infinite discrete)
- We will see that every closed and bounded subset of $\mathbb{R}^k$ is always compact which is the famous H.B. Theorem, i.e. $K \subseteq R^k$ is compact $\Leftrightarrow K$ is closed and bounded

Theroem 2.20. *Let $\{I_n = [a_n, b_n]\}_{n=1}^{\infty}$ be a sequence of closed and bounded intervals in $\mathbb{R}$, if $\{I_n\}$ is decreasing i.e. $I_1 \subseteq \cdots \subseteq I_n \subseteq \cdots$, then $\bigcap\limits_{n=1}^{\infty} I_n \neq \emptyset$. Moreover, if $\lim\limits_{n\to\infty}(b_n - a_n) = 0$, then $\bigcup\limits_{n=1}^{\infty} I_n$ is a simpleton.*

Proof. Claim $T = \{a_n \mid n \in \mathbb{N}\}$ is bounded above and $x = \sup T$ exists.
$\because a_n \leq a_{m+n} (\because \{a_n\}$ is increasing, i.e. $[a_1, b_2] \supseteq [a_2, b_2],\ a_2 \geq a_1)$
$a_n \leq a_{m+n} \leq b_{m+n} \leq b_m (\because \{b_n\}$ is decreasing $) \implies T$ is bounded above by all $b_n \implies x = \sup T$ exists and $x \leq b_n \forall n \geq 1$.
Clearly, $a_n \leq x\ \forall n \geq 1$, $\because a_n \leq x \leq b_n, \forall n \geq 1$, i.e. $x \in [a_n, b_n] \forall n \geq 1 \therefore x \in \bigcap\limits_{n=1}^{\infty} I_n$. Hence, $\bigcap\limits_{n=1}^{\infty} I_n \neq \emptyset \therefore$ The last statement follow the argument as in Corollary 2.16. ■

Theroem 2.21. *Let $\{I_n = [a_{n,1}, b_{n,1}] \times \cdots \times [a_{n,k}, b_{n,k}]\}$ be a decreasing sequence of closed and bounded intervals in $\mathbb{R}^k$. Then $\bigcap\limits_{n=1}^{\infty} \neq \emptyset$. Moreover, if $\lim\limits_{n\to\infty} dia(I_n) = 0$, then $\bigcup\limits_{n=1}^{\infty}$ is a simpleton.*

Proof. $\forall 1 \leq j \leq k, \{[a_{n,j}, b_{n,j}]\}$ is a decrease sequence of closed and bounded intervals in $\mathbb{R}$. By Thm 2.20, $\exists x_j \in \bigcap\limits_{n=1}^{\infty}[a_{n,j}, b_{n,j}]$. Set $x = (x_1, \cdots, x_k)$. Then $x \in \bigcap\limits_{n=1}^{\infty} I_n$. Then last statement also follows from the argument in corollary 2.16. ■

Theroem 2.22. *Every k-dimensional closed and bounded interval $I = [a_1, b_1] \times \cdots \times [a_k, b_k]$ in $\mathbb{R}^k$ is compact.*

Proof. Put $\delta = (\sum_{i=1}^{k}(b_i - a_i)^2)^{\frac{1}{2}}$ which is the diametric of I. Then $\forall x, y \in I, \| x-y \| \leq \delta$. If I were not compact, them $\exists$ an open covering $\{E_\alpha\}_{\alpha \in J}$ admitting not finite sub covering($\star$). Put $c_j = \frac{a_j + b_j}{2}$, $1 \leq j \leq k$. The intervals $[a_j, c_j]$ and $[c_j, b_j], 1 \leq j \leq k$, determines 2^k closed and bounded subinterval of I whose union is I. By ($\star$), at least one of them, say I_1 which cannot be covered by finitely many E_α. Continuing this process, we get a sequence $\{I_n\}$ of closed and bounded subintervals of I satisfy's

a) $I \subseteq I_1 \subseteq \cdots$, i.e. $\{I_n\}_{n=1}^{\infty}$ is decreasing.
b) Each I_n cannot be covered by finitely many E_α
c) $\text{dia}(I_n) = 2^{-n}, \ \delta \to 0$ on $n \to \infty$

By Thm 2.21, $\bigcap_{n=1}^{\infty} I_n = \{x\}$ i.e. $x \in I_n \subseteq I \ \forall \ n \geq 1 \subseteq \bigcup_{\alpha=1}^{\infty} E_\alpha$

$\because x \in E_{\alpha_0}$ for some $\alpha_0 \in J, \ E_{\alpha_0}$ is open $\implies \exists r > 0 \ni B(x,r) \subseteq E_{\alpha_0}$. Choose $n_0 >> 0 \ni \frac{1}{2^{n_0}} < \frac{r}{\delta}(\because \frac{1}{2^n} \to 0)$. Since $x \in I_{n_0}, \forall y \in I_{n_0}, \ \| y - x \| \leq 2^{-n_0}\delta < \frac{r}{\delta} \cdot \delta = r \implies y \in B(x,r), \ \because I_{n_0} \subseteq B(x,r) \subseteq E_{\alpha_0}(\rightarrow\leftarrow)$. Therefore I is compact. ∎

Combining Thm 2.22 and results in section 2.6, we conclude that the following sets in compact:

i) $[a,b]$ is compact in $\mathbb{R}$(Thm 2.22 with $k = 1$)
ii) $[a,b] \times [c,d]$ is compact in $\mathbb{R}^2$(Thm 2.22 with $k = 2$)
iii) Every closed ball $\overline{B}(x,r)$ in $\mathbb{R}^k$ is compact by Thm 2.12 and 2.22
iv) $\{0\} \cup \{\frac{1}{n} \mid n = 1, 2, \cdots\}$ is compact in $\mathbb{R}$, In fact, if $a_n \to a$, them teh set $\{a\} \cup \{a_n \mid n = 1, 2, \cdots\}$ is compact in $\mathbb{R}$

Theroem 2.23. *Every closed and bounded subset K of $\mathbb{R}^k$ is compact.*

Proof. Choose a large closed and bounded interval I in $\mathbb{R}^k$, $K \subseteq I$. By Thm 2.22, I is compacted, so K is a closed subset of I. By Thm 2.12, K is compacted.
Combining Thm 2.17 and Thm 2.23, we can characterize compacted set in $\mathbb{R}^k$ ∎

Theroem 2.24. *Let $K \subseteq R^k$ TFAE:*
i) K is closed and bounded
ii) K is compacted
iii) Every infinite subset of K has an accumulation point
iv) K is sequence compact
v) K is complete and totally bounded

From these we can deduce:

Theroem 2.25 (Bolzano-Weiestrace)**.** *Every bounded infinite subset T in $\mathbb{R}^k$ has an accumulation point in $\mathbb{R}^k$*

Proof. Since T is bounded, choose a large closed and bounded interval I in $\mathbb{R}^k \ni T \subseteq I$. Now, T become an infinite subset of the compact set I. By Thm 2.24 (iii), T has an accumulation point in I. ■

Theroem 2.26 (Cantor intersection)**.** *Let $\{\mathbb{Q}_n\}$ be a sequence of nonempty set in $\mathbb{R}^k$ satisfying:*

a) $\{Q_n\}$ is decreasing
b) Q_n is closed $\forall\ n \geq 1$ & Q_1 is compact.

Then $\bigcap_{n=1}^{\infty} Q_n \neq \emptyset$, Moreover, if $dia(Q_n) \to 0$ on $n \to \infty$, them $\bigcap_{n=1}^{\infty} Q_n$ is a simpleton.

Proof. By (b), each Q_n is compact ($\because Q_1 \supseteq Q_n \forall\ n \geq 1$). Therefore, it follows form Cor 2.16 and 2.15. ■

2.8. Countability & Separability.

Motivation: In $\mathbb{R}^k$, we have two facts:
• $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$ i.e. $\overline{\mathbb{Q}^k} = \mathbb{R}^k$ & $\mathbb{Q}^k$ is countable. i.e. $\mathbb{R}^k$ has a countable dense subset. i.e. $\mathbb{R}^k$ is separable.
• $\left\{B(x,r) \mid x \in \mathbb{Q}^k, r \in \mathbb{Q}^+\right\}$ is a countable collection of open ball in $\mathbb{R}^k$ satisfying: $\forall$ open set $U \subseteq \mathbb{R}^k$ and $y \in U$ $\exists B(x,r) \in \mathscr{B} \ni y \in B(x,r) \subseteq U$. In particular, U is a union of some sub collection of $\mathscr{B}$. $\therefore U = \bigcup_{y \in U} B_y$, i.e. $\mathbb{R}^k$ is of 2^{nd} countable.

• X is a metric space $x \in X$, $N_x = \{B(x, \frac{1}{n}) \mid n \in \mathbb{N}\}$ is a countable collection of nbh of x. Such N_x satisfies: $\forall$ nbh U of $x, \exists n \in \mathbb{N} \ni B(x, \frac{1}{n}) \subseteq U (\because \exists\ r > 0 \ni B(x,r) \subseteq U$, choose $n >> 0 \ni \frac{1}{n} < r)$. Then $x \in B(x, \frac{1}{n}) \subseteq B(x,r) \subseteq U$. i.e. Each point of X has a countable nbh base(system) i.e. X is of 1^{st} countable, i.e. $\mathbb{R}^k$ has a countable base.

__**Definition**__. *Let X be a topological space*

(1) X is first if every point of X has a countable nbh system(or base), i.e. $\exists$ a countable collection $\{V_n \mid n \in \mathbb{N}\}$ of nbh of $x \ni \forall$ nbh U of $x, \exists\ n \in \mathbb{N},\ V_n \subseteq U$

(2) X is of second countable if X has a countable a base, i.e. $\exists$ a countable collection $\mathscr{B} = \{B_n \mid n \in \mathbb{N}\}$ of open sets in $X \ni$ every open set U is a union of some subcollection of $\mathscr{B}$ or $\forall$ open set U in X and $x \in U,\ \exists n \in \mathbb{N} \ni x \in B_n \subseteq U$

(3) X is separable if X has a countable dense subset, i.e. $\exists$ a countable set $D \subseteq X \ni \overline{D} = X$.

Remark.

(1) Every 2^{nd} countable topology space X is of 1^{st} countable, but not converse. $\because$ Let $\mathscr{B} = \{B_1, B_2, \cdots\}$ be a countable base for X. Given $p \in X$, let $\mathscr{B}_p = \{B_n \mid p \in B_n\}$ Then $\mathscr{B}$, so $\mathscr{B}_p$ is countable and is a collection of open set in X containing of p.

Claim $\mathscr{B}_p$ is a nbh system (or base) of p. Let U be a nbh of p. Then $U = \bigcup_{n \in F} B_n,\ F \subseteq \mathbb{N}$. In particular, $p \in B_n$ for some $n \in F$. Hence, $B_n \in B_p$ & $p \in B_n \subseteq U$, $\therefore \mathscr{B}_p$ is a countable nbh system of p. Hence, X is of 1^{st} countable.

Consider X an uncountable set with discrete metric. Hence, X is 1^{st} countable. In fact, $\forall\ p \in X,\ N_p = \{\{p\}\}$ is a countable nbh base of p. However, X is not of 2^{nd} countable. Note that if $\mathscr{B}$ is a base for the discrete space X then $\mathscr{B} \subseteq \{\{x\} \mid x \in X\}$

$\because \{x\}$ is open $\implies \{x\} = \bigcup_{\alpha \in I} B_\alpha,\ B_\alpha = \{x\}\ \forall\ \alpha \in I \implies$

$B_\alpha = \{x\} \forall x \in I \implies \{x\} \in \mathscr{B}$.

Now, X is uncountable, so is $\mathscr{B}$. Hence, X is not of 2^{nd} countable.

(2) We know that every metric space is of 1^{st} countable. In fact, $\forall p \in X,\ N_p = \{B(p, \frac{1}{n}) \mid n \in \mathbb{N}\}$ is a countable nbh system of p

(3) We know that $\mathbb{R}$ is separable with countable dense subset $\mathbb{Q}$. In general, $\mathbb{R}^n$ is separable with countable dense subset $\mathbb{Q}^n$(Exercise 22)

Theroem 2.27. *Every 2^{nd} countable topology space is separable*

Remark. *Note that X is a metric space. $D \subseteq X$.*

$$\begin{aligned} D \text{ is dense in } X \Leftrightarrow \overline{D} = X &\Leftrightarrow \forall \text{ nonempty open set } U in X, U \cap D \neq \emptyset \\ &\Leftrightarrow \forall x \in X, \exists \text{ a sequence } \{a_n\} \text{ in } D \\ &\ni a_n \to x \text{ on } n \to \infty \end{aligned}$$

Proof.
($\Rightarrow$) Suppose $\overline{D} = X$, i.e. D is dense in X, i.e. $\forall x \in X,\ x \in \overline{D}$. Now, given a nonempty open set U in X. Choose $x \in U$ So U is a nbh of x, hence $U \cap D \neq \emptyset$
($\Leftarrow$) Suppose the condition holds, them $\forall\ x \in X$ and nbh U of x, $U \cap D \neq \emptyset \implies x \in \overline{D} \implies X \subseteq \overline{D} \subseteq X \implies \overline{D} = X$, i.e. D is dense in X. ■

Proof. (Theorem 2.27) Let $\mathscr{B} = \{B_1, B_2, \cdots, B_n, \cdots\}$ be a countable base of X. Choose a point $x_n \in B_n,\ n \in \mathbb{N}$ and form the set $D = \{x_1, x_2, \cdots, x_n, \cdots\}$, then D is countable.
Claim: D is dense in X, i.e. $\overline{D} = X$. Given a nonempty open set U in X, then $\exists n \in \mathbb{N} \ni B_n \subseteq U \implies x_n \in U \implies U \cap D \neq \emptyset$. Hence, $\overline{D} = X$ by the remark above. So X is separable. ■

Theroem 2.28. *Every separable metric space X is of 2^{nd} countable.*

Proof. Choose a countable dense subset D of X. Form the countable collection of open ball $\mathscr{B} = \{B(x,r) \mid x \in D, r \in \mathbb{Q}^+\}$(it is countable). Claim $\mathscr{B}$ is a base for X. We are done.

(Note that $\mathscr{B}$ is a base for a topology space X
$\Leftrightarrow$ every open set in X is a union of some subcollection of $\mathscr{B}$
$\Leftrightarrow \forall$ open set $U \subseteq X$ and $p \in U,\ \exists B \in \mathscr{B} \ni x \in B \subseteq U$)

($\Rightarrow$) Given an open set U in X and $p \in U$ by assumption $U = \bigcup_{\alpha \in I} B_\alpha$, where $B_\alpha \in \mathscr{B} \implies p \in B_{\alpha_0}$ for some $\alpha_0 \in I \implies p \in B_{\alpha_0} \subseteq U$
($\Leftarrow$) Suppose the condition holds. To prove $\mathscr{B}$ is a base for X. Given a nonempty open set U in X. $\forall\ p \in U, \exists B_p \in \mathscr{B} \ni p in B_p \subseteq U$.
$\because\ U = \bigcup_{p \in U} B_p \ \therefore\ \mathscr{B}$ is a base for X. By the remark above, it is enough to show: Given a nonempty open set $U \subseteq X$ and $p \in U$, $\exists B(x,r) \in \mathscr{B} \ni p \in B(x,r) \subseteq U$. Now, $p \in U$ and U is open $\implies$ $\exists t > 0 \ni B(p,r) \subseteq U$. Choose $r \in \mathbb{Q}^+ \ni \dfrac{t}{4} < r < \dfrac{t}{2}$, Since D is dense in X, $B(p,r) \cap D \neq \emptyset$. Choose $x \in B(p,r) \cap D$. Then $B(x,r) \in \mathscr{B}$

Claim $p \in B(x,r) \subseteq U$
- $d(x,p) < r \implies p \in B(x,r)$
- $\forall y \in B(x,r),\ d(y,p) \leq d(y,x) + d(x,p) < r + r = 2r < t \implies$

$y \in B(p,t) \subseteq U \ \therefore y \in U \therefore B(x,r) \subseteq U.$

■

Corollary 2.29. *The Euclidean space $\mathbb{R}^k$ is of 2^{nd} countable.*

Note that from the proof of Thm 2.28, $\mathbb{R}^k$ has a countable base of the form:

$$\begin{aligned} \mathscr{B} &= \{B(x,r) \mid x \in \mathbb{Q}^k \ \&\ r \in \mathbb{Q}^+\} \\ &= = \{A_1, A_2, \cdots\} \end{aligned}$$

Theroem 2.30. *Every compact metric space X is of 2^{nd} countable.*

Proof. The last statement follows from Thm 2.27 To prove X is of 2^{nd} countable. For each $n \in \mathbb{N}$, $\{B(x, \frac{1}{n}) \mid x \in X\}$ is an open covering of X, i.e. $X = \bigcup_{x \in X} B(x, \frac{1}{n})$. By companies of X, it has a finite subcovering say $X = \bigcup_{i=1}^{l_n} B(x_{n_i}, \frac{1}{n})$. Then $\mathscr{B}$ is a countable collection of open balls in X.Claim $\mathscr{B}$ is a base for X. It suffices to show: given a nonempty open set U and $p \in U, \exists B(x_{n_i}, \frac{1}{n}) \ni \mathscr{B} \ni p \in B(x_{n_i}, \frac{1}{n}) \subseteq U$.
From $p \in U$ and U is open, $\exists r > 0 \ni B(p,r) \subseteq U$. Choose $n >> 0 \ni \frac{2}{n} < r$. Since $X = \bigcup_{i=1}^{l_n} B(x_{n_i}, \frac{1}{n}),\ p \in B(x_{n_i}, \frac{1}{n})$ for some $1 \leq i \leq l_n$.
Finally, $p \in B(x_{n_i}, \frac{1}{n}) \subseteq U$

- $d(p, x_{n_i}) < \frac{1}{n} \implies p \in B(x_{n_i}, \frac{1}{n})$
- $\forall y \in B(x_{n_i}, \frac{1}{n}), d(y,p) \leq d(y, x_{n_i}) + d(x_{n_i}, p) < \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$

$\therefore p \in B(x_{n_i}, \frac{1}{n}) \subseteq U$ ■

Theroem 2.31 (Lindelof Covering)**.** *Let $S \subseteq \mathbb{R}^k$. Them every open coverning $\mathscr{U} = \{U_\alpha \mid \alpha \in I\}$ of S has a countable subcovernings.*

Proof. Let $\{a_1, A_2, \cdots\}$ be the countable base of $\mathbb{R}^k$ defined as above. Note that $S \subseteq \bigcup_{\alpha \in I}$ $\forall x \in S, x \in U_\alpha$ for some $\alpha \in I$. Hence $\exists n \in \mathbb{N} \ni xinA_n \subseteq U_\alpha$. Of course, there may be infinitely many such n. We choose one of them and fix it, say $x \in A_{m(x)} \subseteq U_\alpha$(e.g. $m(x) = \min\{n \in \mathbb{N} \mid x \in A_n \subseteq U_\alpha\}$). Then the collection $\{A_{m(x)} \mid x \in S\}$ is a countable open covering of S. Finally, for each $A_{m(x)}$, choose $U_{\alpha_{m(x)}} \ni A_{m(x)} \subseteq U_{\alpha_{m(x)}}$. Then $\{U_{\alpha_{m(x)}} \mid x \in X\}$ is a countable subcoverning of S. ■

Corollary 2.32. *Let $S \subseteq \mathbb{R}^k$ be open, if $S = \bigcup_{\alpha \in I} U_\alpha$ is a union of open sets in X, then $S = \bigcup_{n=1}^{\infty} U_{\alpha_n}$ is a countable union. $\because$ By Lindelof covering theorem.*

2.9. Perfect Sets in Metric Spaces. Recall a subset E in a metric space X is perfect if E is closed in X and and every point of E is its accumulation point. i.e. $E' = E$

Example

- $-\infty < a < b < \infty$, $[a, b]$ is perfect
- $\mathbb{R}$ is perfect

Theroem 2.33. *Every nonempty perfect set E in $\mathbb{R}^k$ is uncountable*

Proof. E is an infinite set ($\because$ finite set has no accumulation point), Suppose E were countable, write $E = \{x_1, x_2, \cdots\}$. We use induction to construct a sequence $\{V_n\}$ of open sets in X as follows:
Let V_1 be any neighborhood of $y_1 = x_1$, e.g. $V_1 = B(x_1, r)$, it's closure is $\overline{V_1} = \overline{B}(x_1, r)$, $x_1 \in E'$, $V_1 \cap E$ is an infinite set, so $\exists y_2 \in V_1 \cap E \ni y_2 \neq y_1$. Choose a neighborhood V_2 of $y_2 \ni$
(i) $\overline{V_2} \subseteq V_1$
(ii) $x_1 \notin \overline{V_2}$
(iii) $V_2 \cap E \neq \emptyset$($\because y_2 \in E = E'$ and it's also an infinite set)
Suppose that, for $n \geq 3$, V_n has been chosen $\ni V_n$ is a neighborhood of some $y_n \in E \ni$

(1) $\overline{V_n} \subseteq V_{n-1}$
(2) $x_{n-1} \notin \overline{V_n}$
(3) $V_n \cap E \neq \emptyset$ is an infinite set

Since $V_n \cap E$ is an infinite set, $\exists y_{n+1} \in V_n \cap E \ni y_{n+1} \neq y_n$. Again, choose a neighborhood V_{n+1} of $y_{n+1} \ni$

(1) $\overline{V_{n+1}} \subseteq V_n$

(2) $x_n \notin \overline{V_{n+1}}$
(3) $V_{n+1} \cap E \neq \emptyset$ is an infinite set.

By induction, we have constructed such sequence $\{V_n\}$. Put $K_n = \overline{V_n} \cap E$, $n \geq 1$. Then $\{K_n\}$ is a decrease sequence of nonempty compact sets in $\mathbb{R}^k$.

> $\overline{V_n}$ is closed, E is closed $\implies K_n = \overline{V_n} \cap E$ is closed, each $\overline{V_n}$ is bounded $\therefore K_n$ is closed and bounded by H.B. theorem, K_n is compact.
> · $\emptyset \neq V_n \cap E \subseteq \overline{V_n} \cap E \implies E_n = \overline{V_n} \cap E \neq \emptyset$
> · $\overline{V_n} \cap E \supseteq \overline{V_{n+1}} \cap E = K_{n+1} \therefore \{K_n\}$ is decrease.

By Cantor's intersection theorem, $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$. Pick $y \in \bigcap_{n=1}^{\infty} K_n$, $y \in E (\because K_n \subseteq E \forall n \geq 1)$. Since $x_n \notin \overline{V_{n+1}}\ \forall n \geq 1$, so $x_n \notin K_n \forall n \geq 1 \implies y \notin E (\rightarrow\leftarrow)$ to $E = \{x_1, x_2, \cdots\}$
$\therefore E$ is uncountable. ■

Corollary 2.34. *Every nondegenerate intervvl is uncountable.*

Proof. $\because$ Every nondegenerate interval I in $\mathbb{R}$ must contain a closed and bounded interval $[a, b]$ with $a < b$ which is perfect, so it is uncountable by theorem 2.31. Hence I is uncountable. ■

Construction of the Cantor set $\underline{P} \subseteq [0, 1]$ in $\mathbb{R}$ which is a perfect set

(a) Remove the middle third open subinterval of $[0, 1]$. There are two closed subintervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Let $C_1 = (\frac{1}{3}, \frac{2}{3})$ and $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

(b) Remove te middle thirds of $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ respectively. There are $2^2 = 4$ subintervals $[0, \frac{1}{3^2}]$, $[\frac{2}{3^2}, \frac{3}{3^2}]$, $[\frac{6}{3^2}, \frac{7}{3^2}]$, $[\frac{8}{3^2}, 1]$

(c) Countinue this process, we get a sequence $\{C_n\}$ of open sets and a sequence $\{E_n\}$ of closed sets satisfy

(i) $E_0 \supseteq E_1 \supseteq E_2 \subseteq \cdots$, i.e. $\{E_n\}$ is a decrease sequence of closed sets in $[0, 1]$
(ii) Each E_n is a union of 2^n closed intervals, each of length 3^{-n}
(iii) Each C_n is a union of 2^{n-1} open subintervals, each of length 3^{-n}. total length is $\frac{2^{n-1}}{3^n}$

<u>Definition</u>. $\underline{P}$*(or C)* $= \bigcap_{n=1}^{\infty} E_n = [0,1] - \bigcup_{n=1}^{\infty} C_n$*(* $= \bigcap_{n=1}^{\infty}([0,1]-C_n)$*)*

Properties of Cantor set $\underline{P}$:

(1) $\underline{P} \neq \emptyset$ by Cantor's intersection theorem
(2) $\underline{P}$ is compact ($\because$ $\underline{P}$ is closed bein $\cap$ of closed set and $\underline{P} \subseteq [0,1]$, $[0,1]$ is compact)
(3) $\underline{P}$ is nowhere dense, i.e. $\overline{\underline{P}}^{\circ} = \emptyset$, i.e. $\underline{P}^{\circ} = \emptyset$
$\because$ $\underline{P}$ contains no nonempty open subintervals($\because$ $\underline{P}^{\circ} \neq \emptyset \implies \exists x \in P^{\circ} \implies \exists \delta > 0, (x-\delta, x+\delta) \subseteq \underline{P}$).If $\alpha < \beta$ and $(\alpha,\beta) \subseteq \underline{P}$, them $(\alpha,\beta) \subseteq E_n \forall n \geq 1$.Choose $n >> 0 \ni \frac{1}{2^n} < \beta - \alpha$. Then for $n >> 0,\ E_n$ contains subinterval of length $\geq \frac{1}{2^n}(\rightarrow\leftarrow)$. Hence, $\underline{P}$ is nowhere dense.
(4) $\underline{P} = \{\sum_{n=1}^{\infty} \frac{a_n}{3^n} \mid a_n = 0 \text{ or } 2 \forall n \geq 1\}$

Recall the ternarry representation of a number $x \in [0,1],\ x = \sum_{n=1}^{\infty} \frac{a_n}{3^n},\ a_n = 0,1,2 \forall n \geq 1$

> if $fraca_n 3^n$ the a_n can be 1, then $\frac{1}{3} + \frac{1}{3}$ is not in $\underline{P}$
> if you want to represent $\frac{1}{3}$, you need use $0 + \frac{2}{3^2} + \frac{2}{3^3} + \cdots$, i.e. $0.1 = 0.0\overline{9}$ same things.

This can be used to prove that $\underline{P}$ is uncountable by $\underline{P} \rightarrow [0,1],\ x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \rightarrow \sum_{n=1}^{\infty} \frac{a_n/2}{2^n}$ is bijative $\therefore$ $\underline{P}$ is uncountable.
(5) $\underline{P}$ is of measure zero (i.e. the length of $\underline{P}$ is zero)
$\because$ The totally length remove in the construction of $\underline{P}$ is $\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \cdots = 1$. This also proves that $\underline{P}$ is nowhere dence.
(6) $\underline{P}$ is perfect. In particular, by theorem 2.31, $\underline{P}$ is uncountable
$\because$ Obviously $\underline{P}$ is a nonempty closed set. Let $x \in \underline{P}$. Then $\forall(\alpha,\beta) \ni x \in (\alpha,\beta)$. We prove that $(\alpha,\beta) \cap \underline{P} - \{x\} \neq \emptyset$, which says that x is an accumulation points of $\underline{P}$. Hence $\underline{P}$ is perfect. By $x \in \underline{P}$, $x \in E_n\ \forall n \geq 1$. Then $\exists$ a closed subinterval $I_n \subseteq E_n \ni x \in I_n$. Choose $n >> 0 \ni I_n \subseteq (\alpha,\beta)$. Let x_n be an end point of $I_n \ni x \neq x_n$. By construction, $x_n \in \underline{P}$, so

$$x_n \in (\alpha, \beta) \cap \underline{P} - \{x\}$$

i.e. x is an accumulation point of $\underline{P}$, Hence $\underline{P}$ is perfect.

Definition. *Let X be a metric space (or topological space) $A, B \subseteq X$. We sat that A and B are separated if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty sets.*

Definition. *A subset $E \subseteq X$ is called connected if E is not a union of two nonempty separated sets and E is disconnected if E is not connected*

Remark. *X is connected $\Leftrightarrow$ X is not a union of two nonempty separated sets.*
X is disconnected $\Leftrightarrow$ X is a union of two nonempty separated sets, say, $X = A \cup B$,A and B are nonempty separated. i.e. $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$, i.e. $A = \overline{A}$, $B = \overline{B}$ $\therefore A$ and B are closed $\therefore A$ and B are both open and closed.

Remarks and Examples

(1) Separated sets and disjoint
(2) $[0, 1]$ and $(1, 2)$ are not separated
(3) $(0, 1)$ and $(1, 2)$ are separated

Theroem 2.35. *Let $E \subseteq \mathbb{R}$ be a set. Then E is connected $\Leftrightarrow$ E is an interval*

Proof. We may assume that $E \neq \emptyset$
($\Rightarrow$) Assume that E is connected. If E were not an interval, then $\exists\ x < y$ in E and $z \notin E \ni x < Z < y$. Let $A = (-\infty, z) \cap E$ and $B = (z, \infty) \cap E$. Then A, B are nonempty, $\overline{A} \cap B = \emptyset$, $A \cap \overline{B} = \emptyset$ and

$$\begin{aligned} E &= E \cap (\mathbb{R} - \{z\}) \\ &= E \cap [(-\infty, z) \cup (z, \infty)] \\ &= (E \cap (-\infty, z)) \cup (E \cap (z, \infty)) \\ &= A \cup B \end{aligned}$$

$\therefore \{A, B\}$ is a nonempty separation of E ($\rightarrow\leftarrow$) to connected. $\therefore E$ is an interval
($\Leftarrow$) Suppose E is an interval. To show that E is connected. If not, then $\exists$ two nonempty separated sets A and $B \ni E = A \cup B$. Pick $x \in A$ and $y \in B$. Then $x \neq y (\because A \cap B = \emptyset)$. We may assume that $x < y$. Define $z = \sup(A \cap [x, y])$, By (9) in section 2.5, $z \in \overline{A \cap [x, y]} \in \overline{A}$. Hence $z \notin B (\because \overline{A} \cap B = \emptyset)$. Also $x \leq z \leq y$. But $y \in B$ and $z \notin B \implies x \leq z < y$

If $z \notin A$, then $x < z < y$ and $z \notin E(\rightarrow\leftarrow)$ to E is an interval.
If $z \in A$, then $z \notin \overline{B}(\because A \cap \overline{B} =)$, since $z \notin B$, $z \in \mathbb{R} - \overline{B}$ which is open $\implies \exists \delta > 0 \ni (z-\delta, z+\delta) \subseteq \mathbb{R} - \overline{B}$. Choose $z < z_1 < z+\delta < y$, i.e. $z < z_1 < y$. Then $z, y \in E$, $z < y$ and $z_1 \notin E(\rightarrow\leftarrow)$ to E is an interval ■

Application of Connectedness
X : connected topological space (or metric space)
P : a property on X
$D = \{x \in X \mid P \text{ holds at } x\}$
If one can prove D is nonempty and closed and open, them $D = X$
$\because X = D \cup (X - D)$, $\overline{D} \cap (X - D) =$ and $D \cap \overline{(X - D)}$, i.e. D and $X - D$ are separated
Since X is connected and $D \neq \emptyset$, so $X - D = \emptyset$, i.e. $X = D$

3. Infinite Sequence & Series

• We will assume you are familiar with all operations of real(complex) sequence
• We have defined sequence in a set X

Recall : let $\{a_n\}$ be a real or complex sequence, $\{a_n\}$ converges if $\exists a \in \mathbb{R}(\mathbb{C})$ satisfying $\forall \epsilon > 0, \exists N \in \mathbb{N} \ni \forall n \geq N, |a_n = a| < \epsilon$

• Now, we study the properties of a sequence in a metric space(topological space)

3.1. Convergent Sequence. Let X be a metric space $\&\{x_n\}$ be a sequence in X, $x : \mathbb{N} \to X$

Definition. *We say that $\{x_n\}$ convergences (in X) if $\exists p \in X$ satisfying $\forall \epsilon > 0, \exists N \in \mathbb{N} \ni \forall n \geq N, d(x_n, p) < \epsilon$, Otherwise, $\{x_n\}$ divergences.*

Remark.

(1) *If $\{x_n\}$ convergences as in definition, them p is called the limit of the sequence $\{x_n\}$ and is denoted by $\lim\limits_{n\to\infty} x_n = p$ or $x_n \to p$ as $n \to \infty$*
(2) *$x_n \to p$ as $n \to \infty \Leftrightarrow$ the real sequence $\{d(x_n, p)\}$ convergences to 0, i.e. $\lim\limits_{n\to\infty} d(x_n, p) = 0$*
(3) *if $\{x_n\}$ convergences, them its limit is !*
(4) *The convergence of a sequence depends not only the sequence but also on the space.*
e.g. $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$ in $\mathbb{R}$, but $\{\dfrac{1}{n}\}$ divergences in $(0, 1)$

Recall Let $\{x_n\}$ are a sequence in a set X with is a function $x : \mathbb{N} \to X$. The image of the sequence = the image of the function X
$= \{x_n \mid n = 1, 2, \cdots\}$

Remark. *The range of a sequence may be finite. e.g. $\{(-1)^n\}$ in $\mathbb{R}$, whose range $\{-1, 1\}$ is finite, but $\{\dfrac{1}{n}\}$ has range $\{\dfrac{1}{n} \mid n = 1, 2, \cdots\}$*

Definition. *A sequence $\{x_n\}$ in X is said to be bounded if its range is a bounded subset of X*

Remark. *A sequence $\{x_n\}$ in X is said to be bounded if its range is a bounded subset of X*

Example

(1) Every const sequence $\{p\}$ in a metric space convergence, i.e. $\lim_{n\to\infty} p = p$

(2) $\lim_{n\to\infty} \frac{1}{n} = 0$ in $\mathbb{R}$ and $\{\frac{1}{n}\}$ is bounded (but the range is finite)

(3) $\{(-1)^n\}$ divergences, but $\{(-1)^n\}$ is bounded. (range is finite).

(4) $\{n^2\}$ divergences in $\mathbb{R}$ and is unbounded. In fact, $\lim_{n\to\infty} n^2 = +\infty($ which range is infinite)

(5) $\lim_{n\to\infty}(1+\frac{(-1)^n}{n}) = 1$ and $\{1+\frac{(-1)^n}{n}\}$ is bounded. (range is infinite)

(6) $\{i^n\}$ divergence and it's bounded (range is finite)

(7) Identify all convergence sequence in a discrete metric space X. $\{x_n\}$ convergence to $p \Leftrightarrow \forall\epsilon > 0 \exists N \in \mathbb{N} \ni \forall n \geq N, d(x_n, p) < \epsilon \Leftrightarrow \{x)n\}$ is almost constant.

In metric space, we can use sequences to characterise adherent and accumulation point

Theroem 3.1. *Let $\{x_n\}$ be a sequence in a metric space X and $E \subseteq X$*

(a) $x_n \to p$ as $n \to \infty \Leftrightarrow \forall$ neighborhood U of $p, \exists N \in \mathbb{N} \ni \forall n \geq N, x_n \in U$

(b) If $\{x_n\}$ convergences, them its limit is !

(c) If $\{x_n\}$ convergences, them its range is bounded, but not converse

(d) $p \in \overline{E} \Leftrightarrow \exists$ a sequence $\{a_n\}$ in $E \ni a_n \to p$

(e) $p \in E' \Leftrightarrow \exists$ a distinct sequence($a_n \neq a_m \forall n \neq m$) $\{a_n\}$ in $E \ni a_n \to p$

Proof.

(a)

$$\begin{aligned} x_n \to p &\Leftrightarrow \forall\epsilon > 0, \exists N \in \mathbb{N} \ni d(x_n, p) < \epsilon(\forall n \geq N) \\ &\Leftrightarrow \forall\epsilon > 0, \exists N \in \mathbb{N} \ni x_n \in B(p, \epsilon)(\forall n \geq N) \\ &\Leftrightarrow \forall \text{ neighborhood } U \text{ of } p, \exists N \in \mathbb{N} \ni \forall n \geq N, x_n \in U \end{aligned}$$

(b) Suppose $x_n \to p$, $x_n \to q$ and $p \neq q$, let $\epsilon = \frac{1}{2}d(p, q)$. By definition, $\exists N_1 \ni \forall n \geq N_1, d(x_n, p < \epsilon)$ and $\exists N_2 \ni \forall n \geq N_2, d(x_n, q) < \epsilon$

Let $N = \max\{N_1, N_2\}$ them $\forall n \geq N$, above holds. Hence $d(p, q) \leq d(p, x_N) + d(x_N, q) < \epsilon + \epsilon = 2\epsilon = d(p, q)(\rightarrow\leftarrow) \therefore p = q$

(c) We have seen bounded sequence may not converges. If $x_n \to p$, them for $\epsilon = 1, \exists N \in \mathbb{N} \ni \forall n \geq N, d(x_N, p) < 1$, i.e. $\forall n \geq N$, $x_n \in B(p, 1)$. Let $R = \max\{d(p, x_1), \cdots, d(p, x_{n-1})\} + 1$, Then $x_n \in B(p, R) \forall n \geq 1 \therefore \{x_n\}$ is bounded

(d) $(\Rightarrow)$ Suppose $p \in \overline{E}$. Them $\forall n \geq 1, B(p, \frac{1}{n}) \cap E \neq \emptyset$. Choose $a_n \in B(p, \frac{1}{n}) \cap E, n \geq 1$. We get a sequence $\{a_n\}$ in E and $0 \leq d(x_n, p) < \frac{1}{n}, \forall n \geq 1$.
By squeezing lemma, $\lim_{n\to\infty} d(a_n, p) = 0$, i.e. $a_n \to p$ as $n \to \infty$
$(\Leftarrow)$ Suppose the conditions holds. $\forall r > 0$, $\exists N \in \mathbb{N} \ni \forall n \geq N, d(a_n, p) < r \implies \forall n \geq N, a_n \in B(p, r) \implies B(p, r) \cap E \neq \emptyset \therefore p \in E$
(e) It's similar to above.

■

Theroem 3.2. *For real or complex sequences $\{x_n\}$ and $\{y_n\}$, $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $a, b \in \mathbb{R}$ or $\mathbb{C}$, $c \in \mathbb{R}$ or $\mathbb{C}$*

(1) $\lim_{n\to\infty} c = c$
(2) $\lim_{n\to\infty} (ax_n + by_n) = ax + by = a \lim_{n\to\infty} x_n + b \lim_{n\to\infty} y_n$
(3) $\lim_{n\to\infty} x_n y_n = xy = \lim_{n\to\infty} x_n \lim_{n\to\infty} y_n$
(4) *If* $y \neq 0$, $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$
(5) *If $\{z_n\}$ is a complex sequence, them $z_n \to z$ as $n \to \infty \Leftrightarrow$ $Rez_n \to z$ & $Imz_n \to z$(using $|Rew|, |Imw| \leq |w| \leq |Rew| + |Imw| \forall w \in \mathbb{C}$)*
(6) *(Squeezing Lemma) If $\{x_n\}\{y_n\}$ and $\{t_n\}$ are real sequence $\ni$*

$$x_n \leq t_n \leq y_n \textit{ for } n >> 0$$

and $\lim x_n = lim y_n = l$, *them* $\lim_{n\to\infty} t_n = l$
(7) $\lim_{n\to\infty} x_n = x \implies \lim_{n\to\infty} |x_n| = |x|$*(using* $||x_n| - |x|| \leq |x_n - x|$*)*

Examples.

(i) $\lim_{n\to\infty} (1 - \frac{i}{n}) = 1$ (Re $(1 - \frac{i}{n}) = 1$, $\mathrm{Im}(1 - \frac{i}{n}) = \frac{1}{n}$)
(ii) $\lim_{n\to\infty} \frac{1}{n} \sin \frac{1}{n} = 0$

$$0 \leq |\frac{1}{n} \sin \frac{1}{n}| \leq \frac{1}{n} \quad \implies \quad \lim_{n\to\infty} |\frac{1}{n} \sin \frac{1}{n}| = 0$$
$$\implies \quad |\lim_{n\to p} \frac{1}{n} \sin \frac{1}{n}| = 0 \implies \lim_{n\to\infty} \frac{1}{n} \sin \frac{1}{n}$$

(iii) $\{(-1)^n\}$ divergence, but $|(-1)^n| = 1 \to 1$
For sequences in $\mathbb{R}^k$ including in $\mathbb{C} \approx \mathbb{R}^2$, we have.

Theroem 3.3. *Let $\{x_n\}$ be a sequence in $\mathbb{R}^k$, where*

$$x_n = (x_{1,n}, x_{2,n}, \cdots, x_{k,n}), n = 1, 2, \cdots$$

(a) $x_n \to p(p_1, \cdots, p_k)$ in $\mathbb{R}^k \Leftrightarrow x_{i,n} \to p_i \forall 1 \le i \le k$, i.e. $\lim\limits_{n\to\infty}(x_{1,n}, \cdots, x_{k,n}) = (\lim\limits_{n\to\infty} x_{1,n}, \cdots, \lim\limits_{n\to\infty} x_{k,n})$ if exists
(b) Let $\{x_n\}\{y_n\}$ be sequences in $\mathbb{R}^k$ and $\{d_n\}$ be a sequence in $\mathbb{C}$ and $a, b \in \mathbb{R}$. If $x_n \to x, y_n \to y$ and $d_n \to d$, then

- $ax_n + by_n \to ax + by$
- $< x_n, y_n > \to < x, y >$
- $d_n x_n \to d_x$
- $||x_n|| \to ||x||$

If $k = 3$, them $x_n \times y_n \to x \times y$

Proof.

(a)It follows from the inequation $\forall y \in \mathbb{R}^k |y_i| \le ||y|| \le \sum\limits_{i=1}^{k} |y_i|$

$\because \forall 1 \le i \le k \ |x_{i,n} - p_i| \le ||x_n - p|| \le \sum\limits_{i=1}^{k} |x_{j,n} - p_j| \forall n \ge 1$

$(\Rightarrow)$

$$\begin{aligned} \text{Suppose } x_n \to p &\implies ||x_n - p|| \to 0 \\ &\implies \forall 1 \le i \le k, |x_{i,n} - p_i| \to 0 \forall 1 \le k \le n \\ &\implies \forall 1 \le i \le k, x_{i,n} \to p_i \text{ as } n \to \infty \end{aligned}$$

$(\Leftarrow)$

$$\begin{aligned} \text{Suppose } x_n \to p_i, 1 \le i \le k &\implies ||x_{i,n} - p_i|| \to 0 \ \forall 1 \le k \le n \\ &\implies \sum_{i=1}^{k} |x_{i,n} - p_i| \to 0 \\ &\implies ||x_n - p|| \to 0 \implies x_n \to p \end{aligned}$$

(b) By (a)

$$\begin{aligned} ax_n + by_n &= (ax_{1,n}, ax_{2,n}, \cdots, ax_{k,n}) + (by_{1,n}, \cdots, by_{k,n}) \\ &= (ax_{1,n} + by_{1,n}, \cdots, ax_{k,n} + by_{k,n}) \\ &\to (ax1 + by_1, \cdots, ax_k + by_k) = ax + by \end{aligned}$$

- $< x_n, y_n >= \sum\limits_{i=1}^{k} x_{i,n} y_{i,n} \to \sum\limits_{i=1}^{n} x_i y_i =< x, y >$

- $d_n x_n = (d_n x_{1,n}, \cdots, d_n x_{k,n}) \to (dx_1, \cdots, dx_k) = dx$

$$||x_n|| = (\sum_{i=1}^{k} x_{i,n}^2)^{\frac{1}{2}} \implies (\sum_{i=1}^{k} x_i^2)^{\frac{1}{2}} = ||x||$$

- $x_n \times y_n = (x_{2,n}y_{3,n} - x_{3,n}y_{2,n}, \cdots) \to (x_2y_3 - x_3y_2, \cdots) = x \times y$

■

3.2. Subsequences.

Theroem 3.4.
(a) If $\{x_n\}$ convergences to p, i.e. $\lim_{n\to\infty} x_n = p$, them so is every subsequence of $\{x_n\}$
(b) If X is compact and $\{x_n\}$ is a sequence in X, them $\{x_n\}$ has a convergent subsequence.
(c) Every bounded sequence $\{x_n\}$ in $\mathbb{R}^k$ has a converge subsequence.

Proof. (a) Given a subsequence $\{x_{n_k}\}$ of $\{x_n\}$(Note that $\{n_k\}$ is strictly increasing , i.e. $n_1 < n_2 < \cdots$, hence, $k \le n_k \ \forall k \ge 1$), $d(x_{n_k}, p) < \epsilon$. This proves $x_{n_k} \to p$ as $k \to \infty$
(b) Let $T = \{x_n \mid n \ge 1\}$ be the range of $\{x_n\}$
Case 1: T is a finite set. In this case, some x_{n_0} must appear infinitely many times in the sequence $\{x_n\}$. Choose $n_1 = n_0 \ni x_{n_1} = x_{n_0}$, and $n_2 > n_1 \ni x_{n_1} = x_{n_2}, \cdots$. In this way, we get a const subsequence $\{x_{n_k}\}$ which convergence to x_{n_0}
Case 2: T is an infinite set. In this case, T is an infinite subset of the compact metric space X. By Thm 2.17 (ii), T has an accumulation point p in X. By Thm 3.1 (e), $\exists$ a sequence in T which converges to p. We may arrange such sequence to be a subsequence of $\{x_n\}$. We are done

> $y_1 = x_n$, choose $n_2 \to n_1 \ni x_{n_2}$ appears in $\{y_j\}$. Then $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and $\{y_j\}$, $\therefore x_{n_k} \to p$

(c) $\because$ Since $\{x_n\}$ is bounded, we may choose a closed ball $\overline{B}(0, R)$ or a closed n-dimensional interval in $\mathbb{R}^k \ni \{x_n\}$ is a sequence in K, By (b), $\{x_n\}$ has a convergence subsequence .

■

Remark. *Thm 3.4(a) can be used to detect the divergence of a sequence, e.g. $\{(-1)^n\}$ in $\mathbb{R}$ which divergences, $\because$ It has two subsequences*
$$\begin{cases} x_{2n} \to 1 \\ x_{2n-1} \to -1 \end{cases} \quad \textit{which is different.}$$

Definition. *Let $\{x_n\}$ be a sequence in X. A point $p \in X$ is called a subsequential limit of $\{x_n\}$ if $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\} \ni x_{n_k} \to p$ as $k \to \infty$*

Examples

(1) If $\{x_n\}$ convergences to p, then $\{x_n\}$ has only on subsequence limit p ($E = \{p\}$)
(2) $\{(-1)^n\}$ has two subsequence limit 1 and -1, $E = \{1, -1\}$
(3) $\{n\}$ has no subsequence limit ($E = \emptyset$)

Let $\{x_n\}$ be a sequence in X and E be the set of all subsequence limits of $\{x_n\}$

Theroem 3.5. *As above, E is a closed subset of X*

Proof. If E is a finite set, them E is closed.
Now, assume that E is an infinite set, to show that E is closed, we must prove $E' \subseteq E$, i.e. E contains all its accumulation point. Given $q \in E'$, to prove $q \in E$, i.e. $\exists$ a subsequence $\{x_{n)k}\} \ni x_{n_k} \to q$. Since E is infinite, $\{x_n\}$ is not a constant sequence , so we can choose $x_{n_1} \neq q$. Let $\delta = d(x_n, q)$, We construct subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying : $d(x_{n_k}, q) \leq \dfrac{\delta}{2^{k-1}} \forall k \geq 1$. If it is done, them by squeezing lemma, $d(x, q) \to 0$ as $k \implies \infty$.
Now, to construct such subsequence $\{x_{n_k}\}$. By induction, suppose $k = 1$ we are done, we have found $n_1 < n_2 < \cdots < n_{k-1}, k \geq 2$. To find x_{n_k}. Since $q \in E', B(q, \dfrac{\delta}{2^k}) \cap E - \{q\} \neq \emptyset$. Choose $x \in B(q, \dfrac{\delta}{2^k}) \cap E - \{q\}$. Now, $x \in E, \exists$ a subsequence of $\{x_n\}$ which convergence to x. Hence $\exists n_k > n_{k-1} \ni d(x_{n_k}, x) < \dfrac{\delta}{2^k}$. Finally, $d(x_{n_k}, q) \leq d(x_{n_k}, x) + d(x, q) < \dfrac{\delta}{2^k} + \dfrac{\delta}{2^k} = \dfrac{\delta}{2^{k-1}}$ $\therefore$ By induction, such subsequence $\{x_{n_k}\}$ can be found. ■

3.3. Cauchy Sequences.

Recall $x_n \to p \implies \forall \epsilon > 0\ \exists N \in \mathbb{N} \ni \forall n \geq N,\ d(x_n, p) < \dfrac{\epsilon}{2}$.
$\therefore \forall m, n \geq N,\ d(x_m, x_n) \leq d(x_m, p) + d(p, x_n) < \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} = \epsilon$

Definition. *A sequence $\{x_n\}$ in X is called a Cauchy sequence if it satisfies the Cauchy condition: $\forall \epsilon > 0,\ \exists N \in \mathbb{N} \ni \forall n, m \geq N, d(x_n, x_m) < \epsilon$*

Remark.

(i) Convergence sequence is Cauchy

(ii) Cauchy sequence may not convergence, e.g. in $(0,1)$ $\{\frac{1}{n}\}$ *is Cauchy, but not convergence in* $(0,1)$

> $\forall n, m \in \mathbb{N},\ m \geq n,\ |\frac{1}{n} - \frac{1}{m}| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{n}$
> $\forall \epsilon > 0,$ *Choose* $N \in \mathbb{N} \ni \frac{2}{N} < \epsilon.$
> *Them* $\forall n, m \geq N,\ |\frac{1}{n} - \frac{1}{m}| \leq \frac{2}{N} < \epsilon \ \therefore \{\frac{1}{n}\}$ *is Cauchy.*

(iii) $\{x_n\}$ *is Cauchy* $\Leftrightarrow \lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$

(iv) Let $E_n = \{x_n, x_{n+1}, \cdots\} n \geq 1$. *Them* $\{x_n\}$ *is Cauchy* $\Leftrightarrow \lim\limits_{n\to\infty}$ *dia* $(E_n) = 0$

> *Recall the definition of diameter:*
> *Let* $S \subseteq V, S \neq \emptyset$. *The diameter of* S *is dia(S) = sup*$\{d(x,y) \mid x, y \in S\}$
> *Let*

Proof. $(\Rightarrow)$ Suppose $\{x_n\}$ is Cauchy. Then $\forall \epsilon > 0,\ \exists N \in \mathbb{N} \ni \forall n, m \geq N, d(x_n, x_m) < \frac{\epsilon}{2}$. Then $\forall n \geq N$, dia $(E_n) \leq \frac{\epsilon}{2} < \epsilon \implies \lim\limits_{n\to\infty}$ dia (E_n) $= 0$

$(\Leftarrow)$ Suppose $\lim\limits_{n\to\infty}$ dia $(E_n) = 0$, $\forall \epsilon > 0, \exists N \in \mathbb{N} \ni \forall n \geq N$, dia$(E_n)$ $< \epsilon \implies \forall n, m \geq N, (x_n, x_m \in E_n), d(x_n, x_m \leq)$ dia $(E_N) < \epsilon$

$\therefore \{x_n\}$ is Cauchy. ∎

Remark. *Every Cauchy seq* $\{x_n\}$ *in a metric space is bounded. For* $\epsilon = 1, \exists N \in \mathbb{N} \ni \forall m, n \geq N, d(x_n, x_m) < 1$, *In particular,* $\forall n \geq N, d(x_n, x_N) < 1$. *Let* $R = max\{d(x_i, x_N) \mid 1 \leq i \leq N-1\} + 1$. *Then* $x_n \in B(x_N, R) \forall n \geq 1$, *i.e.* $\{x_n \mid n \geq 1\} \subseteq B(x_N, R)$. *Hence* $\{x_n\}$ *is bounded.*

Theroem 3.6. (a) *Every Cauchy sequence in a compact metric space converges.*
(b) *Every Cauchy sequence in* $\mathbb{R}^k$ *convergences.*

Proof. (a) Let $\{x_n\}$ be a Cauchy sequence in compact metric space X. Since X is compact, X is sequencially compact, so $\{x_n\}$ has a subsequence $\{x_{n_k}\} \ni x_{n_k} \to p$ as $k \to \infty$ for some $p \in X$.

Now to prove $x_n \to p$. Given $\epsilon > 0, \exists N \in \mathbb{N} \ni \forall n, m > N, d(x_n, x_m) < \dfrac{\epsilon}{2}(\because \{x_n\}$ is Cauchy), $\exists k_0 \in \mathbb{N} \ni \forall k \geq k_0, d(x_{n_k}, p) < \dfrac{\epsilon}{2}(\because x_{n_k} \to p)$. Hence, $\forall n \geq N, d(x_n, p) \leq d(x_n, x_{n_k}) + d(x_{n_k}, p) < \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} < \epsilon$, where $k >> 0$

Another proof of (a)

By (2)&(4) above,

$$\lim_{n\to\infty} \text{tia}(\overline{E}_n) = \lim m \to \infty \text{dia}(E_n) = 0$$

,

where $E = \{x_n, x_{n+1}, \cdots\}, n \geq 1$

Now, each $\overline{E_n}$ is compact($\because$ closed subset of compact set X) and nonempty $\forall n \geq 1$, and $\{\overline{E}_n\}$ is decreasing($\because E_n \supseteq E_{n+1} \implies \overline{E}_n \supseteq \overline{E}_{n+1}$) and dia($\overline{E}_n \to 0$). By Cantor's Intersection Theorem, $\bigcap_{n=1}^{\infty} \overline{E_n} = \{p\}$ Claim: $x_n \to p$ as $n \to \infty$, $\forall \epsilon > 0, \exists N \in \mathbb{N} \to$ dia$(\overline{E}_n) < \epsilon$. Since $p \in \overline{E}_n \forall n \geq 1, \forall n \geq N \& d(x_n, p) <$ dia$(\overline{E}_n) < \epsilon$, $\therefore x_n \to p$

(b) Given a Cauchy sequence $\{x_n\}$ in $\mathbb{R}^k$. Then $\{x_n\}$ is bounded, choose a large k-dimensional closed interval

$$I = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

$\ni x_n \in I \forall n \geq 1$. Now, H.B. Theorem says that I is compact. Therefore, $\{x_n\}$ becomes a Cauchy sequence in the compact metric space I. By (a) $x_n \to p$ for some $p \in I$. This proves (b).

■

Definition. *A metric space X is said to be complete if every Cauchy sequence in X convergences.*

Rmks and Examples

(1) In a complete metric space X, a sequence $\{x_n\}$ is Cauchy $\Leftrightarrow$ it convergences.

(2) By Thm. 2.6, we have two classes of complete metric spaces

- Compact metric space
- Euclidean space $\mathbb{R}^k$

In fact, $\mathbb{R}^k$ is a Banach Space(complete normed linear space) and Hilbert space

(3) A closed subset S of a complete metric space X is complete.

$\because$ Let $\{x_n\}$ be a Cauchy sequence in S. Then $\{x_n\}$ is a Cauchy sequence in X, hence, $x_n \to p$ for some $p \in X$. So $p \in \overline{S} = S$. Hence, S is complete.

(4) Every closed subset of $\mathbb{R}^k$ is complete.
$\because$(3), In particular, every closed interval and closed ball in $\mathbb{R}^k$,
(5) $(0,1)$ and $\mathbb{Q}$ are not complete

<u>Definition</u>. *Let $\{x_n\}$ be a real sequence*

(i) We say that $\{x_n\}$ is increasing, if $x_n \leq x_{n+1} \forall n \geq 1$
(ii) We say that $\{x_n\}$ is strictly increasing, if $x_n < x_{n+1} \forall n \geq 1$
(iii) We say that $\{x_n\}$ is decreasing, if $x_n \geq x_{n+1} \forall n \geq 1$
(iv) We say that $\{x_n\}$ is strictly decreasing, if $x_n > x_{n+1} \forall n \geq 1$
(v) We say that $\{x_n\}$ is monotonic if either $\{x_n\}$ is increasing or decreasing.
(vi) We say that $\{x_n\}$ is strictly monotonic if either $\{x_n\}$ is strictly increasing or strictly decreasing

Examples

- $\{2n+1\}$ is increasing, $2n+1 \to +\infty$
- $\{-n\}$ is decreasing, $-n \to -\infty$
- $\{\frac{1}{n}\}$ is decreasing, $\frac{1}{n} \to 0$
- $\{\frac{1}{-n}\}$ is increasing, $\frac{1}{-n} \to 0$

We will show that every monotonic sequence convergences in $\mathbb{R}^* = [-\infty, \infty]$

Theroem 3.7. *Let $\{a_n\}$ be a sequence*

(a) Let $\{a_n\}$ be increasing
(i) If $\{a_n\}$ is bounded above, then $\{a_n\}$ convergence, in fact, $a_n \to sup a_n = sup\{a_n \mid n \geq 1\}$
(ii) If $\{a_n\}$ is not bonded above, then $a_n \to \infty$
(b) Let $\{a_n\}$ be decreasing
(a) If $\{a_n\}$ is bounded below, then $\{a_n\}$ convergences, in fact $a_n \to inf a_n = inf\{a_n \mid n \geq 1\}$
(b) If $\{a_n\}$ is not bounded below, them $\{a_n\} \to -\infty$

Remark. *$\{a_n\}$ is increasing $\Leftrightarrow$ $\{-a_n\}$ is decreasing. So, to study monotonic sequence, it suffices to consider the case of increasing sequence.*

Proof. It suffices to prove (a) By same argument or considering $\{-a_n\}$, one can prove

(a) (i) $\{a_n\}$ is bounded above $\implies$ $\{a_n \mid n \geq 1\}$ is bounded $\implies$ $\alpha = \sup a_n$ exists and is finite
Claim: $a_n \to \alpha$ as $n \to \infty$.

Given $\epsilon > 0 \exists n_0 \in \mathbb{N} \to \alpha - \epsilon < a_{n_0}$. Then $\forall n \geq n_0$, we have $\alpha - \epsilon < a_{n_0} \leq a_n \leq \alpha < \alpha + \epsilon$, i.e. $\forall n \geq n_0$, $|a_n - \alpha| < \epsilon$. This proves $a_n \to \alpha$

(ii) $\forall M > 0$, since $\{a_n\}$ is not bounded above, $\exists n_0 \in \mathbb{N} \to a_{n_0} \geq M \implies \forall n \geq M, a_n \geq a_{n_0} \geq M$. This proves $a_n \to +\infty$

(b) is similar

(i') $\{a_n\}$ is bounded below $\implies$ $\{-a_n\}$ is bounded above $\implies$ $\lim\limits_{n\to\infty}(-a_n) = \sup(-a_n) \implies -\lim a_n = -\text{inf} a_n$

(ii') Similar

■

Remark. *Let $\{a_n\}$ be a monotonic sequence, then $\{a_n\}$ convergences $\Leftrightarrow$ $\{a_n\}$ is bounded*