<u>Definition</u>. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$. Define the graph of f to be the subset of $\mathbb{R}^{n+1}$ consisting of all the points*

$$(x_1, \cdots, x+n, f(x_1, \cdots, x_n))$$

in $\mathbb{R}^{n+1}$ for $(x_1, \cdots, x_n)$ in U. In symbols,

$$\mathit{graph} f = \{(x_1, \cdots, x_n, f(x_1, \cdots, x_n)) \in \mathbb{R}^{n+1} \mid (x_1, \cdots, x_n) \in U\}$$

<u>Definition</u>. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ and let $c \in \mathbb{R}$. Then the level set of value c is defined to be the set of those points $x \in U$ at which $f(x) = c$. In symbols, the level set of value c is written*

$$\{x \in U \mid f(x) = c\} \subset \mathbb{R}^n$$

Note that the level set is always in the domain space.

<u>Definition</u>. *Let $U \subset \mathbb{R}^n$ be an open set and suppose $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is a real-valued function. Then, $\dfrac{\partial f}{\partial x_1}, \cdots, \dfrac{\partial f}{\partial x_n}$, the partial derivatives of f with respect to the first, second, $\cdots$, nth variable, are the real-valued functions of n variables, which at the point $(x_1, \cdots, x_n) = x$, are defined by*

$$\begin{aligned} \frac{\partial f}{\partial x_j}(x_1, \cdots, x_n) &= \lim_{h \to 0} \frac{f(x_1, x_2, \cdots, x_j + h, \cdots, x_n) - f(x_1, \cdots, x+n)}{h} \\ &= \lim_{h \to 0} \frac{f(x + he_j) - f(x)}{h} \end{aligned}$$

<u>Definition</u>. *Let U be an open set in $\mathbb{R}^n$ and let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a given function. We say that f is differentiable at $x_0 \in U$ if the partial derivatives of f exist at x_0 and if*

$$\lim_{x \to x_0} \frac{\| f(x) - f(x_0) - T(x - x_0) \|}{\| x - x_0 \|} = 0,$$

where $T = Df(x_0)$ is the $m \times n$ matrix with elements $\dfrac{\partial f_i}{\partial x_j}$ evaluated at x_0 and $T(x - x_0)$ means the derivative of f at x_0.

Remark. *In the case where $m = 1$, the matrix T is just the row matrix*

$$[\frac{\partial f}{\partial x_1}(x_0) \cdots \frac{\partial f}{\partial x_n}(x_0))]$$

For the general case of f mapping a subset of $\mathbb{R}^n$ to $\mathbb{R}^m$, the derivative is the $m \times n$ matrix given by

$$Df(x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

where $\frac{\partial f_i}{\partial x_j}$ *is evaluated at* x_0*. The matrix* $Df(x_0)$ *is called the matrix of partial derivatives of* f *at* x_0*.*

Definition. *Consider the special case* $f : U \subset \mathbb{R}^n \to \mathbb{R}$*. Here* $Df(x)$ *is a* $1 \times n$ *matrix:*

$$Df(x) = [\frac{\partial f}{\partial x_1} \;\cdots\; \frac{\partial f}{\partial x_n}].$$

We can form the corresponding "vector" $(\frac{\partial f}{\partial x_1}, \cdots, \partial x_n)$*, called the gradient of* f *and denoted by* ∇f*, or grad* f*.*

Remark. • *for* $\mathbb{R}^3 \to \mathbb{R}, \nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

• *In terms of inner products, we can write the derivative of* f *as* $Df(x)(h) = \nabla f(x) \cdot h$

Definition. • *A path in* $\mathbb{R}^n$ *is a map* $c : [a, b] \to \mathbb{R}^n$

• *The collection* C *of points* $c(t)$ *as* t *varies in* $[a, b]$ *is called a curve, and* $c(a)$ *and* $c(b)$ *are its endpoints.*

• *The path c is said to parametrize the curve* C*. We also say* $c(t)$ *traces out* C *as* t *varies.*

• *If c is a path in* $\mathbb{R}^3$*, we can write* $c(t) = (x(t), y(t), z(t))$*, and we also say* $c(t)$ *traces out* C *as* t *varies.*

• *If c is a path in* $\mathbb{R}^3$*, we can write* $c(t) = (x(t), y(t), z(t))$*, and we call* $x(t), y(t),$ *and* $z(t)$ *the component functions of c.*

Definition. *If c is a path and it is differentiable, we say c is a differentiable path. The "velocity" of c at time t is defined by*

$$c'(t) = \lim_{h \to 0} \frac{c(t+h) - c(t)}{h}$$

and the speed of the path $c(t)$ *is* $s = \| c'(t) \|$*, the length of the velocity vector.*

Remark. *if* $c(t) = (x(t), y(t), z(t))$ *in* $\mathbb{R}^3$*, then*

$$c'(t) = (x'(t), y'(t), z'(t)) = x'(t)i + y'(t)j + z'(t)k$$

Definition. *The velocity $c'(t)$ is a "vector tangent" to the path $c(t)$ at time t. If C is a curve traced out by c and if $c'(t)$ is not equal to $0 \in \mathbb{R}^n$, then $c'(t)$ is a vector tangent to the curve C at the point $c(t)$.*

Definition. *If $c(t)$ is a path, and if $c'(t_0) \neq 0$, the equation of its tangent line at the point $c(t_0)$ is*

$$l(t) = c(t_0) + (t - t_0)c'(t_0)$$

If C is the curve traced out by c, then the line traced out by l is the tangent line to the curve C at $c(t_0)$

Definition. *If $f : \mathbb{R}^3 \to \mathbb{R}$, the directional derivative of f at x along the vector v is given by*

$$\frac{d}{dt} f(x + tv)|_{t=0}$$

if this exists.

Theorem. *If $f : R^3 \to \mathbb{R}$ is differentiable, then all directional derivatives exist. The directional derivative at x in the direction v is given by*

$$Df(x)v = gradf(x) \cdot v = \nabla f(x) \cdot v = [\frac{\partial f}{\partial x}(x)]v_1 + [\frac{\partial f}{\partial y}(x)]v_2 + [\frac{\partial f}{\partial z}(x)]v_3$$

where $v = (v_1, v_2, v_3)$.

Theorem. *Assume $\nabla f(x) \neq 0$. Then $\nabla f(a)$ points in the direction along which f is increasing the fastest.*

Theorem. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^1 map and let (x_0, y_0, z_0) lie on the level surface S defined by $f(x, y, z) = k$, for k a constant. Then $\nabla f(x_0, y_0, z_0)$ is normal to the level surface in the following sense: If v is the tangent vector at $t = 0$ of a path $c(t)$ in S with $c(0) = (x_0, y_0, z_0)$, then $\nabla f(x_0, y_0, z_0) \cdot v = 0$*

Definition. *Let S be the surface consisting of those (x, y, z) such that $f(x, y, z) = k$ for k a constant. The tangent plane of S at a point (x_0, y_0, z_0) of S is defined by the equation.*

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

if $\nabla f(x_0, y_0, z_0) \neq 0$. That is, the tangent plane is the set of points (x, y, z) that satisfy equation.

Theorem. *If $f(x, y)$ is of class C^2(is twice continuously differentiable.), then the mixed partial derivatives are equal, that is,*

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Definition. • *If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is a given scalar function, a point $x_0 \in U$ is called a local minimum of f if there is a neighborhood V of x_0 such that for all points x in V, $f(x) \geq f(x_0)$. Same as local maximum.*
• *The point $x_0 \in U$ is said to be a local, or relative, extremum if it is either a local minimum or a local maximum.*
• *A point x_0 is a critical point of f if either f is not differentiable at x_0, or if it is, $Df(x_0) = 0$.*
• *A critical point that is not a local extremum is called a saddle point.*

Theorem. *If $U \subset \mathbb{R}^n$ is open, the function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable, and $x_0 \in U$ is a local extremum, then $DF(x_0) = 0$*

Definition. *Suppose that $f : U \subset \mathbb{R}^n \to \mathbb{R}$ has second-order continuous derivatives $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)(x_0)$ for $i, j = 1, \cdots, n$, at a point $x_0 \in U$. The Hessian of f at x_0 is the quadratic function defined by*

$$\begin{aligned} Hf(x_0)(h) &= \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) h_i h_j \\ &= \frac{1}{2}[h_1, \cdots, h_n] \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} \end{aligned}$$

Theorem. *If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is of class C^3, $x_0 \in U$ is a critical point of f, and the Hessian $Hf(x_0)$ is positive-definite, then x_0 is a relative minimum of f. Similarly, if $Hf(x_0)$ is negative-definite, then x_0 is a relative maximum.*

Theorem. *Let $f(x, y)$ be of class C^2 on an open set U in $\mathbb{R}^2$. A point (x_0, y_0) is a local minimum of f provided the following three conditions holds*

i) $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$

ii) $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) = 0$

iii) $D = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$ *at* (x_0, y_0)

Definition. *Suppose $f : A \to \mathbb{R}$ is a function defined on a set A in $\mathbb{R}^2$ or $\mathbb{R}^3$.*
A point $x_0 \in A$ is said to be an absolute maximum point of f if $f(x) \leq f(x_0)$ for all $x \in A$

Theorem. *Let D be closed and bounded in $\mathbb{R}^n$ and let $f : D \to \mathbb{R}$ be continuous. Then f assumes its absolute maximum and minimum values at some points x_0 and x_1 of D.*

Strategy Let f be a continuous function of two variables defined on a closed and bounded region D in $\mathbb{R}^2$, which is bounded by a smooth closed curve. To find the absolute maximum and minimum of f on D:

i) Locate all critical points for f in U
ii) Find all the critical points of f viewed as a function only on ∂U
iii) Compute the value of f at all of these critical points.
iv) Compare all these values and select the largest and the smallest.

Theorem. *Suppose that $f : U\mathbb{R}^n \to \mathbb{R}$ and $g : U \subset \mathbb{R}^n \to \mathbb{R}$ are given C^1 real-valued function. Let $x_0 \in U$ and $g(x_0) = c$, and let S be the level set for g with value c. Assume $\nabla g(x_0) \neq 0$.*
If $f|S$, which denotes "f restricted S", has a local maximum or minimum on S at x_0, then there is a real number λ(which might be zero) such that

$$\nabla f(x_0) = \lambda \nabla g(x_0).$$

Theorem. *If f, when constrained to a surface S, has a maximum or minimum at x_0, then $\nabla f(x_0)$ is perpendicular to S at x_0.*

Definition. *Let U be an open region in $\mathbb{R}^n$ with boundary ∂U. We say that ∂U is smooth if ∂U is the level set of a smooth function g whose gradient ∇g never vanishes (i.e., $\partial g \neq 0$).*

Starategy Let f be a differentiable function on a closed and bounded region $D = U \cup \partial U$, U open in $\mathbb{R}^n$, with smooth boundary ∂U. To find the absolute maximum and minimum of f on D

i) Locate all critical points of f in U
ii) Use the method of Lagrange multiplier to locate all the critical points of $f|\partial U$
iii) Compute the values of f at all these critical points
iv) Select the largest and the smallest

Definition. *The length of the path $c(t) = (x(t), y(t), z(t))$ for $t_0 \leq t \leq t_1$, is*

$$L(c) = \int^{t_1} t_0 \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}dt$$

Definition. *An infinitesimal displacement of a particle following a path* $c(t) = x(t)i + y(t)j + z(t)k$ *is*

$$ds = dxi + dyj + dzk = \left(\frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k\right) dt,$$

and its length

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

is the differential of arc length.

Definition. *Let* $c : [t_0, t_1] \to \mathbb{R}^n$ *be a piecewise* C^1 *path. Its length is defined to be*

$$L(c) = \int_{t_0}^{t_1} \parallel c'(t) \parallel dt$$

The integrand is the square root of the sum of the squares of the coordinate functions of $c'(t)$ *If*

$$c(t) = (x_1(t), x_2(t), \cdots, x_n(t))$$

then

$$L(c) = \int_{t_0}^{t_1} \sqrt{(x_1'(t))^2 + \cdots + (x_n'(t))^2}dt$$

Definition. *A vector field in* $\mathbb{R}^n$ *is a map* $F : A \subset \mathbb{R}^n \to \mathbb{R}^n$ *that assigns to each point* x *in its domain* A *a vector* $F(x)$.

Definition. *If* F *is a vector field a "flow line" for* F *is a path* $c(t)$ *such that*

$$c'(t) = F(c(t))$$

Definition. *If* $F = F_1i + F_2j + F_3k$, *the divergence of* F *is the scalar field*

$$divF = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Definition. *If $F = F_1 i + F_2 j + F_3 k$, the curl of F is the vector field.*

$$
\begin{aligned}
curlF = \nabla \times F &= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F)3 \end{vmatrix} \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) k
\end{aligned}
$$

Theorem. *For any C^2 function f,*

$$\nabla \times (\nabla f) = 0$$

That is, the curl of any gradient is the zero vector.

Theorem. *For any C^2 vector field F,*

$$div\ curl\ = \nabla \cdot (\nabla \times F) = 0$$

That is, the divergence of any curl is zero.

Definition. *The volume of the region above R and under the graph of a nonnegative function f is called the (dobel) integral of f over R and is denoted by*

$$\int\int_R f(x,y)dA \ or \ \int\int_R f(x,y)dxdy$$

Definition. *If the sequence $\{S_n\}$ converges to a limit S as $n \to \infty$ and if the limit S is the same for any choice of points c_{jk} in the rectangles R_{jk}, then we say that f is integrable over R and we write*

$$\int\int_R f(x,y)dA, \int\int_R f(x,y)dxdy \ or \ \int\int_R f dxdy$$

for the limit S.

Theorem. *Any continuous function defined on a closed rectangle R is integrable.*

Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded real-valued function on the rectangle R, and suppose that the set of points where f is discontinuous lies on a finite union of graphs of continuous function. The f is integrable over R*

Theorem. *Let f be a continuous function with a rectangular domain $R = [a,b] \times [c,d]$. Then*

$$\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy = \int\int_R f(x,y)dA$$

Definition. *If D is an elementary region in the plane, choose a rectangle R that contains D. Given $f : D \to \mathbb{R}$, where f is continuous(and hence bounded), defin $\int\int_D f(x,y)dA$, the integral of f over the set D, as follows: Extend f to a function d^* defined on all of R by*

$$f^*(x,y) = \begin{cases} f(x,y) \ \textit{if} \ (x,y) \in D \\ 0 \ \textit{if} \ (x,y) \notin D \ \textit{and} \ (x,y) \in R \end{cases}$$

then

$$\int\int_D f(x,y)dA = \int\int_R f^*(x,y)dA$$

Theorem. *Suppose that D is the set of points (x,y) such that $y \in [c,d]$ and $\Phi_1(y) \leq x \leq \Phi_2(y)$ If f is continuous on D, then*

$$\int\int_D f(x,y)dA = \int_c^d \left[\int_{\Phi_1(y)}^{\Phi_2(y)} f(x,y)dx\right] dy$$

Theorem. *Suppose $f : D \to \mathbb{R}$ is continuous and D is an elementary region. Then for some point (x_0, y_0) in D we have*

$$\int\int_D f(x,y)dA = f(x_0,y_0)A(D),$$

where $A(D)$ denotes the area of D.

Theorem. *Let A be a 2×2 matrix with $\det A \neq 0$ and let T be the linear mapping of $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $T(x) = Ax$. Then T transforms parallelograms into parallelograms and vertices into vertices.*

Definition. *Let $T : D^* \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 transformation given by $x = x(u,v)$ and $y = y(u,v)$. The Jacobian determinant of T, written $\frac{\partial(x,y)}{\partial(u,v)}$, is the determinant of the derivative matrix $DT(u,v)$ of T :*

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Theorem. *Let D and D^* be elementary regions in the plane and let $T : D^* \to D$ be of class C^1; suppose that T is one-to-one on D^*. Furthermore, suppose that $D = T(D^*)$. Then for any integrable function $f : D \to \mathbb{R}$, we have.*

$$\int\int_D f(x,y)dxdy = \int\int D^* f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

Definition. *The path integral, or the integral of $f(x,y,z)$ along the path c, is defined when $c : I = [a,b] \to \mathbb{R}^3$ is of class C^1 and when the composition $t \mapsto f(x(t), y(t), z(t))$ is continuous on I. We define this integral by the equation*

$$\int_c f ds = \int_a^b f(x(t), y(t), z(t)) \parallel c'(t) \parallel dt$$

or denote

$$\int_c f(x,y,z)ds$$

or

$$\int_a^b f(c(t)) \parallel c'(t) \parallel dt$$

If $c(t)$ is only piecewise C^1 or $f(c(t))$ is piecewise continuous, we define $\int_c f ds$ by breaking $[a,b]$ into pieces over which $f(c(t)) \parallel c'(t) \parallel$ is continuous and summing the integrals over the pieces.

Definition. *Let F be a vector field on $\mathbb{R}^3$ that is continuous on the C^1 path $c : [a,b] \to \mathbb{R}^3$. We define $\int_c F \cdots ds$, the line integral of F along c, by the formula*

$$\int F \cdots ds = \int_a^b F(c(t)) \cdot c'(t)dt$$

that is, we integrate the dot product of F with c' over the interval $[a,b]$ As is the case with scalar functions, we can also define $\int_c F \cdot ds$ if $F(c(t)) - c'(t)$ is only piecewise continuous.

Definition. *Let $h : I \to I_1$ be a C^1 real-valued function that is a one-to-one map of an interval $I = [a,b]$ onto another interval $I_1 = [a_1, b_1]$. Let $c : I_1 \to \mathbb{R}^3$ be a piecewise C^1 path. Then we call the composition*

$$p = c \circ h : I \to \mathbb{R}^3$$

a reparametrization of c