Advance Calculus Exercise

1. Let $\mathscr{T}_1$ and T_2 be topologies on a set X. Prove or disprove that $\mathscr{T}_1 \cup \mathscr{T}_2$ and $\mathscr{T}_1 \cap \mathscr{T}_2$ are topologies on X.

Solution.

(a) Claim $\mathscr{T}_1 \cup \mathscr{T}_2$ is a topology on X

$i)$ $\because \mathscr{T}_1, \mathscr{T}_2$ is a topology on X, $\therefore \emptyset, X \subseteq \mathscr{T}_1\ \emptyset, X \subseteq \mathscr{T}_2$
$\implies \emptyset, X \subseteq \mathscr{T}_1 \cup \mathscr{T}_2$

$ii)$ choose $\bigcup_{\alpha\in I} \in \mathscr{T}_1$ but not in $\mathscr{T}_2$ and $\bigcup_{\alpha'\in I'}$ but not in T_1
$\implies U_\alpha \in \mathscr{T}_1 \cup \mathscr{T}_2$ and $U_{\alpha'} \in \mathscr{T}_1 \cup \mathscr{T}_2$
Claim $\bigcup_{\alpha\in I} U_\alpha \cup \bigcup_{\alpha'\in I'} \subseteq \mathscr{T}_1 \cup \mathscr{T}_2 \implies \bigcup_{\alpha\in I} U_\alpha \cup \bigcup_{\alpha'\in I'} \subseteq$ $\mathscr{T}_1$ or $\mathscr{T}_2(\rightarrow\leftarrow)$ with $\bigcup_{\alpha\in I} \in T_1$ but not in $\mathscr{T}_2$

(b) Claim $\mathscr{T}_1 \cap \mathscr{T}_2$ is a topology on X

$i)$ $\because \mathscr{T}_1, \mathscr{T}_2$ is a topology on X $\therefore \emptyset, X \in \mathscr{T}_1$ and $\emptyset, X \in \mathscr{T}_2$
$\implies \emptyset, X \in \mathscr{T}_1 \cap \mathscr{T}_2$

$ii)$ let $U\alpha \in \mathscr{T}_1 \cap \mathscr{T}_2,\ \alpha \in I, \because \mathscr{T}_1, \mathscr{T}_2$ are topology on X
$\therefore \bigcup_{\alpha\in I} U_\alpha \in \mathscr{T}_1$ and $\bigcup_{\alpha\in I} U_\alpha \in \mathscr{T}_2 \implies \bigcup_{\alpha\in I} U_\alpha \in \mathscr{T}_1 \cap \mathscr{T}_2$

$iii)$ let $U_1, U_2, \cdots, U_k \in \mathscr{T}_1 \cap \mathscr{T}_2\ \because \mathscr{T}_1, T_2$ are topology on X
$\therefore \bigcap_{i=1}^k U_i \subseteq \mathscr{T}_1$ and $\bigcap_{i=1}^k U_i \subseteq \mathscr{T}_2 \implies \bigcap_{i=1}^k U_i \subseteq \mathscr{T}_1 \cap \mathscr{T}_2$

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2. In any metric space X, $\overline{B(x,r)} \subseteq \overline{B}(x,r)$ for all $r \geq 0$

Solution.

let $p \in \overline{B(x,r)} \implies \forall R > 0,\ B(p,R) \cap B(x,r) \neq \emptyset$
$\implies \exists q \in B(p,R) \cap B(x,r) \ni d(q,x) < r$ and $d(p,q) < R$
$\implies d(p,x) \leq d(p,q) + d(q,x) < r + R$ (triangular inequality)
$\implies d(p,x) < r + R(\ \forall\ R) > 0 \implies d(p,x) - r < R(\ \forall R > 0)$
$\implies r - d(p,x) > -R(\ \forall R > 0) \implies r - d(p,x) \geq 0$
$\implies r \geq d(p,x) \implies p \in \overline{B}(x,r)$ ■

3. Let X be a metric space, $S \subseteq X$. Then S is closed if and only if $\partial S \subseteq S$

Solution.

($\Rightarrow$) let $p \in \partial S \implies p \in \overline{S}$(by definition of partial S)

$\because S$ is close $\therefore \overline{S} \subseteq S \implies p \in S$

the proof of S is close then $\overline{S} \subseteq S$

let $q \in \overline{S}$, Claim $q \notin S \implies q \in X - S$

$\because S$ is close $\therefore X - S$ is open

$\implies \exists r > 0 \ni B(q, r) \subseteq X - S \implies B(q, r) \cap S = \emptyset$

$\because q \in \overline{S}\ \therefore \forall R > 0,\ B(q, R) \cap S \neq \emptyset$

pick $R = r$, $B(q, R) \cap S \neq \emptyset$ and $B(q, r) \subseteq X - S$

($\Leftarrow$) let $p \in X - S \implies p \notin S \implies p \notin \partial S(\because \partial S \subseteq S)$

$\implies p \notin \overline{S}(\because p \in X - S$ and $(X - S) \subseteq \overline{X - S}$ so $p \in \overline{X - S}$ but $p \notin \partial S \implies p \notin \overline{S})$

$\implies \exists R > 0 \ni B(p, R) \cap S = \emptyset \implies B(p, R) \in X - S$

$\implies X - S$ is open $\implies S$ is close.

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4. Show that:

(a) $\left(\bigcup_{\alpha \in I} E_\alpha\right)^c = \bigcap_{\alpha \in I} (E_\alpha)^c$

(b) $\left(\bigcap_{\alpha \in I} E_\alpha\right)^c = \bigcup_{\alpha \in I} (E_\alpha)^c$

It is called De Morgan's laws.

Solution.

(a) ($\subseteq$) let $p \in \left(\bigcup_{\alpha \in I} E_\alpha\right)^c \implies p \notin \bigcup_{\alpha \in I} E_\alpha$

$\implies p \notin E_\alpha\ \forall \alpha \in I \implies p \in (E_\alpha)^c\ \forall \alpha \in I$

$\implies p \in \bigcap_{\alpha \in I} (E_\alpha)^c$

($\supseteq$) let $p \in \bigcap_{\alpha \in I} (E_\alpha)^c \implies p \in (E_\alpha)^c$

$\implies p \in (E_\alpha)^c \alpha \in I \implies p \notin E_\alpha\ \forall \alpha \in I$

$\implies p \notin \bigcup_{\alpha \in I} E_\alpha \implies p \in \left(\bigcup_{\alpha \in I} E_\alpha\right)^c$

(b) ($\subseteq$) let $p \in \left(\bigcap_{\alpha \in I} E_\alpha\right)^c \implies p \notin \bigcap_{\alpha \in I} E_\alpha$

$\implies p \in (E_\alpha)^c$ for some $\alpha \in I \implies p \in \bigcup_{\alpha \in I} (E_\alpha)^c$

($\supseteq$) let $p \in \bigcup_{\alpha \in I} (E_\alpha)^c \implies p \in (E_\alpha)^c$ for some $\alpha \in I$

$\implies p \notin E_\alpha$ for some $\alpha \in I \implies p \notin \bigcap_{\alpha \in I} E_\alpha$

$\implies p \in \left(\bigcap_{\alpha \in I} E_\alpha\right)^c$

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5. Let a function $f : X \to Y$ and $A \subseteq Y$. We define

$$f^{-1}(A) = \{x \in X \mid f(x) \in A\}$$

is called the inverse image of a set B. Then

(a) $B \subseteq f^{-1}(f(B))$ for all $B \subseteq X$
(b) $f(f^{-1}(B))$ for all $B \subseteq Y$
(c) $B = f^{-1}(f(B))$ for all $B \subseteq X$ if and only if f is injective
(d) $f(f^{-1}(B)) = B$ for all $B \subseteq Y$ if and only if f is surjective.
(e) Find an example to show that the inclusions in (a) and (b) may be strict.

Solution.

(a) let $b \in B \implies f(b) \in f(B)$(by definition)
$\because b \in X$ ($\because B \subseteq X \therefore b \in X$) and $f(b) \in f(B)$
$\therefore b \in f^{-1}(f(B))$
(b) $f(f^{-1}(B)) = \{f(x) \mid x \in X,\ f(x) \in B\}$
let $p \in f(f^{-1}(B))$, by definition, $p \in B$
(c) ($\Leftarrow$) we proof $B \subseteq f^{-1}(f(B))$ in (a)

Claim $f^{-1}(f(B)) \subseteq B$ when f is injective
let $p \in f^{-1}(f(B)) \because f$ is injective $\therefore f(x) \in f(B)$ only $x \in B$
if not, let $y \notin B \ni f(y) \in f(B)\ \exists x \in B,\ f(x) = f(y), \because$ $x \in B,\ y \notin B \therefore x \neq y, f$ is not injective
therefore $p \in B$

($\Rightarrow$) We need to prove if $x, x' \in X \ni f(x) = f(x')$ then $x = x'$.
Suppose $\exists x, x' \in X$ which $x \neq x'$ and $f(x) = f(x')$
let $B = \{x\}, b' = \{x'\} \Rightarrow f(B) = \{f(x)\}, f(B') = \{f(x')\}$
by question, $B \subseteq f^{-1}(f(B)) = \{x \in X \mid f(x) \in f(B)\}$
$\implies \{p \in X \mid f(x) = f(x)\} \subseteq \{x\}$ $(\star)$
$\implies f^{-1}(f(B)) = \{x\}$, and by similar method
$f^{-1}(f(B')) = \{x\}$
by $(\star)$ $f^{-1}(f(B)) = f^{-1}(f(B')) = \{x\}(\rightarrow\leftarrow)$
$\therefore f$ is injective.

$\blacksquare$

(d) ($\Leftarrow$) we proof $f(f^{-1}(B)) \subseteq B$ in (b), now proof $f(f^{-1}(B)) \subseteq B$ when f is surjective.
let $b \in B$ $\because f$ is surjective $\implies \exists x \in X \ni f(x) = b$
$\therefore b \in f(f^{-1}(B))$
($\Rightarrow$) we want to proof $\forall y \in Y \; \exists \; x \in X \ni f(x) = y$
let $B = Y$, we get $f(f^{-1}(Y)) \supseteq Y$ from question given
$\because f^{-1}(Y) \subseteq X \therefore f(f^{-1}(Y)) \subseteq f(X)$
$\implies Y \subseteq f(f^{-1}(Y)) \subseteq f(X) \subseteq Y \implies f(X) = Y$,
f is surjective.
(e) $X = \{1, 2, 3\}$ $Y = \{1, 2\}$ and $f(1) = 1, \; f(2) = 1, \; f(3) = 2$ and let $B = \{1\}, \; B' = \{1, 2\}$

$$\{1\} = B \subset f^{-1}(f(B)) = \{1, 2\}$$

and

$$\{1\} = f(f^{-1}(B')) \subset B' = \{1, 2\}$$