Chapter 1

Vector Space

<u>Definition</u> (Vector Space)**.** *A vector space (or linear space) W over a Field $\mathbb{F}$ consists of a set on which two operations (called addition and scalar multiplication, respectively) are defined so that for each pair of elements x, y, in W there is a unique element $x + y$ in W, and for each element a in F and each element x in W there is a unique element ax in W, such that the following conditions hold.*

§ Subspace

<u>Definition</u> (Subspace)**.** *A subset W of a vector space W over a field $\mathbb{F}$ is called a subspace of W if W is a vector space over F with the operations of addition and scalar multiplication defined on W.*

Remark. *Trivial subspaces of a vector space V, namely V itself and $\{0\}$. Note that empty set ϕ is not a vector space, since it does not contains a zero vector.*

Theorem 1.1. *Let V be a vector space and W a subset of V. Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V.*
(a) $0 \in W$.
(b) $x + y \in W$ *whenever* $x \in W$ *and* $y \in W$.
(c) $cx \in W$ *whenever* $c \in \mathrm{F}$ *and* $x \in W$.

Corollary 1.1.1. *Let W be a subset of vector space V. W is a subspace of V if and only if $0 \in W$ and $ax + y \in W$ whenever $a \in F$ and, $x, y \in W$.*

Theorem 1.2. *Any intersection of subspaces of a vector space W is a subspace of V.*

Theorem 1.3. *Let W_1 and W_2 be subspaces of a vector space V, then $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.*

Definition. *If S_1 and S_2 are nonempty subsets of a vector space V, then the sum of S_1 and S_2, denoted $S_1 + S_2$, is the set $\{x + y : x \in S_1$ and $y \in S_2\}$.*

Theorem 1.4. *Let W_1 and W_2 be subspaces of a vector space V.*

(a) $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(b) Any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

Definition (Direct Sum)**.** *A vector space V is called the direct sum of W_1 and W_2 if W_1 and W_2 are subspaces of V such that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. We denote that V is the direct sum of W_1 and W_2 by writing $V = W_1 \oplus W_2$.*

Theorem 1.5. *Let W_1 and W_2 be subspaces of a vector space V. V is the direct sum of W_1 and W_2 if and only if each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$*

§ Linear Combinations and Bases

Definition (Linearly Dependent)**.** *A subset S of a vector space W is called linearly dependent if there exist a finite number of distinct vectors $u_1, u_2, ..., u_n$ in S and scalars $a_1, a_2, ..., a_n$, not all zero, such that*

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0.$$

In this case we also say that the vectors of S are linearly dependent.

Definition (Linearly Independent)**.** *A subset S of a vector space that is not linearly dependent is called linearly independent. As before, we also say that the vectors of S are linearly independent.*

Remark.

1. *The empty set is linearly independent, for linearly dependent sets must be nonempty.*

2. *A set consisting of a single nonzero vector is linearly independent. For if $\{u\}$ is linearly dependent, then $au = 0$ for some nonzero scalar a. Thus*
$$u = a^{-1}(au) = a^{-1}0 = 0$$

3. *A set is linearly independent if and only if the only representations of* 0 *as linear combinations of its vectors are trivial representations.*

Theorem 1.6. *Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent.*

Corollary 1.6.1. *Let S be a linearly independent subset of a vector space V, and let v be a vector in V that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in span(S)$.*

Definition (Basis)**.** *A basis β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.*

Theorem 1.7. *Let V be a vector space and $\beta = \{u_1, u_2, ..., u_n\}$ be a subset of V. Then β is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β, that is, can be expressed in the form*

$$V = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

for unique scalars $a_1, a_2, \cdots, a_n$.

Theorem 1.8. *If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence V has a finite basis.*

Theorem 1.9 (Replacement Theorem)**.** *Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V.*

Definition (Finite-Dimensional). *A vector space is called finite-dimensional if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for V is called the dimension of V and is denoted by dim(V). A vector space that is not finite-dimensional is called infinite-dimensional.*

Corollary 1.9.1. *Let V be a vector space with dimension n.*

1. *Any finite generating set for V contains at least n vectors, and a generating set for V that contains exactly n vectors is a basis for V.*
2. *Any linearly independent subset of V that contains exactly n vectors is a basis for V.*
3. *Every linearly independent subset of V can be extended to a basis for V.*

Theorem 1.10. *Let W be a subspace of a finite-dimensional vector space V. Then W is finite-dimensional and $\dim(W) \leq \dim(V)$. Moreover, if $\dim(W) = \dim(V)$, then $V = W$.*

Propersition 1.11. *Let W_1 and W_2 be subspaces of a finite-dimensional vector space V. $W_1 \subseteq W_2$ if and only if $\dim(W_1 \cap W_2) = \dim(W_1)$*

Theorem 1.12. *Let $v_1, v_2, \cdots, v_k, v$ be vectors in a vector space V, and define $W_1 = span(\{v_1, v_2, \cdots, v_k\})$, and $W_2 = span(\{v_1, v_2, \cdots, v_k, v\})$. Then $v \in \operatorname{span}(W_1)$ if and only if $\dim(W_1) = \dim(W_2)$.*

Remark. *We may give an example for satisfying the conditions on above but $\dim(W_1) \neq \dim(W_2)$.*

Theorem 1.13. *Let W_1 and W_2 be finite-dimensional subspaces of a vector space V.*

(a) *Then the subspace $W_1 + W_2$ is finite-dimensional, and*

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

(b) *Let $V = W_1 + W_2$. Deduce that V is the direct sum of W_1 and W_2 if and only if*

$$\dim(V) = \dim(W_1) + \dim(W_2)$$

Theorem 1.14. *Let W_1 and W_2 be subspaces of a vector space V such that $V = W_1 \oplus W_2$ if and only if there exist base β_1 , β_2 of W_1 , W_2, respectively such that $\beta_1 \cup \beta_2$ is a basis for V.*

Theorem 1.15.

If W_1 is any subspace of vector space of V, then there exists a subspace W_2 of V such that

$$V = W_1 \oplus W_2$$