2. Topological Space and Continuous functions

We will introduce some basic topological space.
e.g. Order topology, Product topology, Subspace topology,
Metric topology, (Quotient topology)

§ 12 Topological Spaces.

Definition. *Let X be a nonempty set $\mathscr{P}(X) = 2^X$ power set of X. We say that $\mathscr{T} \subseteq \mathscr{P}(X)$ is a topology on X if*

(1) $\emptyset, X \in \mathscr{T}$
(2) $U_\alpha \in \mathscr{T},\ \alpha \in I \implies \bigcup_{\alpha \in I} U_\alpha \in \mathscr{T}$
(3) $U_1, \cdots, U_n \in \mathscr{T} \implies U_1 \cap \cdots \cap U_n \in \mathscr{T}$

If $\mathscr{T}$ is a topology on X, then the pair $(X, \mathscr{T})$ or simply X is called a topological space and members in $\mathscr{T}$ are called open sets in X

Example.

(1) $X = \{a, b, c\}$
  (a) The following are topological space on X, $\mathscr{T}_1 = \{\emptyset, X\}$, $\mathscr{T}_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\mathscr{T}_3 = \mathscr{P}(X)$
  (b) The following are not topology on X
  $\mathscr{A} = \{\emptyset, \{a\}, \{b\}, X\}$ $(\because \{a\} \cup \{b\} = \{a, b\} \notin \mathscr{A})$
  $\mathscr{B} = \{\emptyset, \{a, b\}, \{b, c\}, X\}$ $(\because \{a, b\} \cap \{b, c\} = \{b\} \notin \mathscr{B})$

(2) Any set with more than 1 element has at least two topology $\{\emptyset, X\}$(in discrete topology) and $\mathscr{P}(X)$(discrete)
and former is smallest one, another is the largest one.

Definition. $\mathscr{T}_{op} = \{\mathscr{T} \mid \mathscr{T}$ *is a topology on* $X\}$ $\mathscr{T}_1 \leq \mathscr{T}_2 \Leftrightarrow \mathscr{T}_1 \subseteq \mathscr{T}_2$
Claim "$\leq$" is a partial ordering on $\mathscr{T}_{op}$

- ⋆ *Reflexive:* $\forall \mathscr{T} \in \mathscr{T}_{op},\ \mathscr{T} \leq \mathscr{T}$
- ⋆ *Anti-symmetry:* $\forall \mathscr{T}_1, \mathscr{T}_2 \in \mathscr{T}_{op},\ \mathscr{T}_1 \leq \mathscr{T}_2$ *and* $\mathscr{T}_2 \leq \mathscr{T}_1 \implies \mathscr{T}_1 = \mathscr{T}_2$
- ⋆ *Transitive:* $\forall \mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3 \in \mathscr{T}_{op},\ \mathscr{T}_1 \leq \mathscr{T}_2$ *and* $\mathscr{T}_2 \leq \mathscr{T}_3 \implies \mathscr{T}_1 \leq \mathscr{T}_3$

Example. Let X be a set, $\mathscr{T}_f = \{U \subseteq X,\ U = \emptyset$ or $X - U$ is finite $\}$
Then $\mathscr{T}_f$ is a topology on X, called the "finite complement topology" on X

Proof.

(1) $\emptyset, X \in \mathscr{T}_f$ ($\because X - X = \emptyset$)

(2) $U_\alpha \in \mathscr{T}_f,\ \alpha \in I$

If $\bigcup_{\alpha\in I} U_\alpha = \emptyset$, then $\bigcup_{\alpha\in I} U_\alpha \in \mathscr{T}_f$.

If $U_{\alpha\in I} U_\alpha \neq \emptyset$, then $\exists \alpha_0 \in I \ni U_{\alpha_0} \neq \emptyset$ and $X - U_\alpha$ is finite $X - \bigcup_{\alpha\in I} U_\alpha = \bigcap_{\alpha\in I}(X - U_\alpha) \subseteq X - U_{\alpha_0} \implies X - (\bigcup_{\alpha\in I} U_\alpha)$ is finte $\implies \bigcup_{\alpha\in I} U_\alpha \in \mathscr{T}_f$

(3) $U_1, \cdots, U_n \in \mathscr{T}_f$

If $U_1 \cap \cdots \cap U_n = \emptyset$, then $U_1 \cap \cdots \cap U_n \in \mathscr{T}_f$

If $U_1 \cap \cdots \cap U_n \neq \emptyset$, then $X - (U_1 \cap \cdots \cap U_n) = (X - U_1) \cap \cdots \cup (X - U_n)$ is finite since each $X - U_i$ is finite. Thus $U_1 \cap \cdots \cap U_n \in \mathscr{T}_f$

From (1)(2)(3), $\mathscr{T}_f$ is a topology on X. ■

Remark. *If X is a finite set, then $\mathscr{T}_f$ is the discrete topology on X*

Example. Let X be a set and $\mathscr{T}_c = \{U \subseteq X \mid U = \emptyset \text{ or } X - U \text{ is countable }\}$. Then as in example above, $\mathscr{T}_c$ is a topology on X, called the countable complement topology on X. Moreover, if X is countable, then $\mathscr{T}_c$ is just a discrete topology on X

Definition. *Let $\mathscr{T}$ and $\mathscr{T}'$ be two topologies on X. We say that $\mathscr{T}'$ is (strictly) finer then $\mathscr{T}$ or $\mathscr{T}$ is (strictly) coaser that $\mathscr{T}'$ if $\mathscr{T} \leq \mathscr{T}'(\mathscr{T} < \mathscr{T}')$, i.e. $\mathscr{T} \subseteq \mathscr{T}'(\mathscr{T} \subsetneq \mathscr{T}')$*

Remark.

(1) Two topologies on X need not be comparable

(2) Other terminology, if $\mathscr{T}' \supset T$, $\mathscr{T}'$ is larger(stronger) than $\mathscr{T}$ and $\mathscr{T}$ is smaller(weaker) than $\mathscr{T}$

§ 13 Bases for a topology.

Definition. *Let X be a set. A base for a topology on X is a collection $\mathscr{B} \subseteq \mathscr{P}(X)$ satisfying*

(1) $U\mathscr{B} = X$ ($\bigcup \mathscr{B} = \bigcup_{B\in\mathscr{B}} B$)

(2) Given $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$ $\exists B_3 \in \mathscr{B} \ni x \in B_3 \subseteq B_1 \cap B_2$

Members in $\mathscr{B}$ are called basic open sets in X

Given a base $\mathscr{B}$ for a topology on X, we can define the smallest topology $\mathscr{T}$ on X containing $\mathscr{B}$ called the topology on X generated by $\mathscr{B}$.

Usually, there are two ways to describe it

(I) $\mathscr{T} = \{U \subseteq X, \forall x \in U \exists B \in \mathscr{B} \ni x \in B \subseteq U\}$. Clearly, $\mathscr{B} \subseteq \mathscr{T}$

(a) $\emptyset, X \in \mathscr{T}$(by the definition of bases (1))

(b) $U_\alpha \in \mathscr{T},\ \alpha \in I \implies \bigcup_{\alpha \in I} U_\alpha \in \mathscr{T}$. Given $x \in \bigcup_{\alpha \in I} U_\alpha, x \in U_{\alpha_0}$ for some $\alpha_0 \in I,\ \exists B \in \mathscr{B} \ni x \in B \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in I} U_\alpha$

(c) $U_1, \cdots, U_n \in \mathscr{T} \implies U_1 \cap \cdots \cap U_n \in \mathscr{T}$. By induction on n, we only prove $n = 2$. Given $x \in U_1 \cap U_2,\ x \in U_1$ and $x \in U_2$
$\implies \exists B_1, B_2 \in \mathscr{B} \ni x \in B_1 \subseteq U_1$ and X in $\mathscr{B}_2 \subseteq U_2 \implies$
$x \in B_1 \cap B_2 \subseteq U_1 \cap U_2 \implies \exists B_3 \in \mathscr{B} \ni x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2 \implies U_1 \cap U_2 \in \mathscr{T}$

(II) $\mathscr{T}' = \{\bigcup \mathscr{A} \mid \mathscr{A} \subseteq \mathscr{B}\} = \{\bigcup_{\alpha \in I} A_\alpha \mid A_\alpha \in \mathscr{B}\}$

Clearly, $\mathscr{B} \subseteq \mathscr{T}'$(only choose one element in $\mathscr{B}$)

(a) $\emptyset, X \in \mathscr{T}'$(trivial)

(b) $U_\alpha \in \mathscr{T}',\ \alpha \in I \implies \bigcup_{\alpha \in I} U_\alpha \in \mathscr{T}'$
$\forall \alpha \in I, U_\alpha = \bigcup_{\beta \in I_\alpha} A_\beta$. Then $\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} \bigcup_{\beta \in I_\alpha} A_\beta \implies \bigcup_{\alpha \in I} U_\alpha \in \mathscr{T}'$

(c) $U_1, \cdots, U_n \in \mathscr{T}' \implies U_1 \cap \cdots \cap U_n \in \mathscr{T}'$. By induction on n, we only to prove that $n = 2$. For $i = 1, 2, \cdots,\ U_i = \bigcup_{\alpha \in I_j} A_\alpha$.
$U_1 \cap U_2 = \bigcup_{\alpha \in I_2} (A^1_\beta \cap A^2_\alpha)$. $\forall x \in U_1 \cap U_2,\ x \in A'_\beta \cap A^2_\alpha \implies U_1 \cap U_2 = \bigcup_{x \in U_1 \cap U_2} B_X \in \mathscr{T}'$

(III) $\mathscr{T} = \mathscr{T}'$

($\subseteq$) Given $U \in \mathscr{T},\ \forall x \in U,\ \exists B_x \in \mathscr{B} \ni x \in B_x \subseteq U \implies U = \bigcup_{x \in U} B_x \in \mathscr{T}'$

($\supseteq$) Given $U \in \mathscr{T}'$ $U = \bigcup_{\alpha \in I} A_\alpha, A_\alpha \in \mathscr{B}$
$\forall x \in U, x \in A_\alpha$ for some $\alpha \in I$ and $A_\alpha \in \mathscr{B}$, i.e. $X \in A_\alpha \in U$ and $A_\alpha \in \mathscr{B} \implies U \in \mathscr{T}$. Hence $\mathscr{T} = \mathscr{T}'$

Example.

(1) Let $\mathscr{B}$ be the collection of all open balls in $\mathbb{R}^n$. Then $\mathscr{B}$ is a base for a topology on $\mathbb{R}^n$, namely, then Euclidean topology on $\mathbb{R}^n$

(2) Let $\mathscr{B}'$ be the collection of all $n-$dimentional open intervals in $\mathbb{R}$. Then $\mathscr{B}'$ is a base for a topology on $\mathbb{R}^n$. In fact, β and β' generate the same topology on $\mathbb{R}^n$

Lemma. *Let X be a set, let $\mathscr{B}$ be a basis for a topology $\mathscr{T}$ on X. $\mathscr{T}$ equals the collection if all unions of elements of $\mathscr{B}$.*

Lemma. *Let X be a topological space and $\mathscr{C}$ be a collection of open sets of $X \ni \forall$ open set U in X and $\forall x \in U$ $\exists C \in \mathscr{C} \ni x \in C \subseteq U$. Then $\mathscr{C}$ is a base for the topology of X.*

Proof. (1) $\bigcup \mathscr{C} = X$

Since X is open $\forall x \in X$, $\exists C_x \in \mathscr{C} \ni x \in C_x \subseteq X \implies x \in \bigcup \mathscr{C} \implies X = \bigcup \mathscr{C}$

(2) Given $C_1, C_2 \in \mathscr{C}$ and $x \in C_1 \cap C_2$. Since $C_1 \cap C_2$ is open, $\exists C \in \mathscr{C} \ni x \in C \subseteq C_1 \cap C_2$, $\therefore \mathscr{C}$ is a base for a topology of X

■

Remark. *Let $\mathscr{T}$ be the original topology on X and $\mathscr{T}'$ be the topology generated by $\mathscr{C}$. Then $\mathscr{T} = \mathscr{T}'$*

Proof.

($\subseteq$) Given $U \in \mathscr{T}$, $\forall x \in U \exists C \in \mathscr{C} \ni x \in C \subseteq U \implies U \in \mathscr{T}'$

($\supseteq$) Given $v \in \mathscr{T}'$, by lemma, $V = \bigcup \mathscr{A}$ for some $A \subseteq \mathscr{C}$. Since $\mathscr{C} \subseteq \mathscr{T}$, $\mathscr{A} \subseteq \mathscr{T}$, $\therefore V = \bigcup \mathscr{A} \in \mathscr{T}$

■

Lemma. *Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topology $\mathscr{T}$ and $\mathscr{T}'$ on X respective TFAE*

(1) $\mathscr{T}$ is finer that $\mathscr{T}$ i.e. $\mathscr{T} \subseteq \mathscr{T}'$

(2) $\forall x \in X$ and $B \in \mathscr{B}$ with $x \in B$, $\exists B' \in \mathscr{B} \ni x \in B' \subseteq B$

Proof.

$(a) \Rightarrow (b)$ Suppose $\mathscr{T} \subseteq \mathscr{T}'$. Given $x \in X$ and $B \in \mathscr{B}$ with $x \in B$. Since $\mathscr{T} \subseteq \mathscr{T}'$, $B \in \mathscr{T}, \exists B' \in \mathscr{B} \ni x \in B' \subseteq B$

$(b) \Rightarrow (a)$ Suppose (b) holds. Given $U \in \mathscr{T}$, $\forall x \in U$, $\exists B_x \in \mathscr{B} \ni x \in B_x \subseteq U$. By (b), $\exists B'_x \in \mathscr{B} \ni x \in B'_x \subseteq B_x \subseteq U \implies U \in \mathscr{T}'$

■

Example. In §13, example 1,2
$\mathscr{B}$: all open balls in $\mathbb{R}^n$ for a topology on $\mathbb{R}^n$
$\mathscr{B}'$:all open intervals in $\mathbb{R}^n$ for a topology on $\mathbb{R}^n$
By lemma above, they generate the same Euclidean topology on $\mathbb{R}^n$
We now define 3 topologies on the real line $\mathbb{R}$

Definition.

(1) $\mathscr{B} = \{(a,b) \mid -\infty < a < b < \infty\}$*: the collection of all open intervals in* $\mathbb{R}$ *which is the base for the usual topology on* $\mathbb{R}$

(2) $\mathscr{B}' = \{[a,b) \mid -\infty < a < b < \infty\}$ *the collection of all closed-open interval in* $\mathbb{R}$*, which is also a base for a topology of* $\mathbb{R}$ *called the lower limit topology on* $\mathbb{R}$*. We denote it by* $\mathbb{R}_l$

(3) Let $K = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ *and* $\mathscr{B}'' = \{B \subseteq \mathbb{R} \mid B = (a,b)$ *or* $B = (a,b) - K$ *for* $-\infty < a < b < \infty\}$*. Claim:* $\mathscr{B}'$ *is a base for a topology on* $\mathscr{T}$

- ⋆ *Clearly,* $U\mathscr{B}'' = \mathbb{R}$
- ⋆ *Given* $B_1, B_2 \in \mathscr{B}''$ *and* $x \in B_1 \cap B_2$*. We have 4 cases:*
  - *(i)* B_1 *and* B_2 *are open intervals which is clearly.*
  - *(ii)* $B_1 = (a,b)$ *and* $B_2 = (c,d) - K$*. Let* $\alpha = \max\{a,c\}$ *and* $\beta = \min\{b,c\}$*.* $x \in (\alpha,\beta) - K \subseteq B_1 \cap B_2$ *and* $(\alpha,\beta) - K \in \mathscr{B}''$
  - *(iii) (3)(4) similarly*

The topology on $\mathbb{R}$ *generated by* B' *is called the* $K-$*topology on* $\mathbb{R}$ *and denoted* $\mathbb{R}_k$

Lemma. *The topologies of* $\mathbb{R}_l$ *and* $\mathbb{R}_k$ *are strictly finer than the Euclidean topology of* $\mathbb{R}$ *but are not comparable with one another*

Proof. Let $\mathscr{T}, \mathscr{T}'$ and $\mathscr{T}''$ be the topologies of $\mathbb{R}, \mathbb{R}_l, \mathbb{R}_k$ generated by $\mathscr{B}, \mathscr{B}', \mathscr{B}''$ respectly. We use lemma above to prove it.

- ⋆ $\mathscr{T} \subsetneqq \mathscr{T}'$ Given $(a,b) \in \mathscr{B}$ and $x \in (a,b)$. We have $[x,b) \in \mathscr{B}'$ with $x \in [x,b) \subseteq (a,b)$. By lemma, $\mathscr{T} \subseteq \mathscr{T}'$, $\forall a < b$, $[a,b) \in \mathscr{B}'$ so $[a,b) \in \mathscr{T}'$, but $[a,b)' \notin \mathscr{T}$
- ⋆ Clearly, $\mathscr{T} \subseteq \mathscr{T}''$ by $\mathscr{B} \subseteq \mathscr{B}''$. Moreover $B'' = (-1,1) - K \in \mathscr{B}''$, so $B'' \in \mathscr{T}''$ but $B'' \notin \mathscr{T}$.

- $\star$ $\mathscr{T}'$ and $\mathscr{T}''$ are not comparable
  $(-1,1)-K \in \mathscr{T}''$, but $(-1,1)-K \notin \mathscr{T}'(\because$ not $[0,c) \in \mathscr{B}' \ni 0 \in [0,c) \subseteq (-1,1)-K)$. $[0,1) \in \mathscr{T}$ but no $\mathscr{B}'' \in \mathscr{B}'' \ni 0 \in B'' \subseteq [0,1)$

$\blacksquare$

Definition. *A subbase $\mathscr{S}$ for a topology on X is a collection of subsets of X with $\bigcup \mathscr{S} = X$ and elements in $\mathscr{S}$ are calle subbasic open sets in X*

Given subbase on X

$$\mathscr{B} = \{S_1 \cap \cdots \cap S_k,\ k \in \mathbb{N}, S_1, \cdots, S_k \in S\}$$

Claim $\mathscr{B}$ is a base for a topology on X

Definition. *The topology on X generated by a subbase $\mathscr{S}$ is defined to be the topology generated by the base $\mathscr{B}$.*

§ **14 The Order Topology.** (which provides many counterexample in topology)

Definition. *A relation C on a set is called an "order relation" (or a simple order) if it satisfies*

1. *Comparable: $\forall x \neq y$ in X either xCy or yCx*
2. *Non-reflexivity: no xCx*
3. *Transitivity: xCy and $yCz \implies xCz$*

Given a simple order set $(X,<)$ and $a, b \in X$ with $a < b$(Note: $a \leq b$ means $a < b$ or $a = b$). We can define:
$(a,b) = \{x \in X \mid a < x < b\}$ open interval
$(a,b] = \{x \in X \mid a < x \leq b\}$ open interval
$[a,b) = \{x \in X \mid a \leq x < b\}$ open interval
$[a,b] = \{x \in X \mid a \leq x \leq b\}$ open interval
We assume that $|X| \geq 2$. Let $\mathscr{B}$ be the collection of all subsets of the following types

1. All open intervals (a,b) in X
2. All intervals of the forms $[a_0, b)$ where a_0 is the smallest elements of X

(3) All intervals of the forms $(a, b_0]$ where b_0 is the largest elements of X

Definition. *The topology generated by $\mathscr{B}$ is called the order topology on X*

Example.

(1) If X is an order set and $T \subseteq X$, then so is Y
(2) In $\mathbb{R}$ we give the usually ordering and the order topology on $\mathbb{R}$ is the usual topology on $\mathbb{R}$
(3) In $\mathbb{R}^\star = \mathbb{R} \cup \{-\infty, \infty\}$ with the usual ordering is an order set.
(4) In $\mathbb{R} \times \mathbb{R}$ with the dictionary order is an order set whose basis for the order topology is of the form
(5) $\mathbb{N}$ with the usual ordering is an order set with the smallest element 1. What is the order topology?
   - $\star$ $[1, b) : b \in \mathbb{N}$ and $(a, b),\ a < b$. In particular, $\{1\} = [1, 2)$ and $\{n\} = (n-1, n+1), n > 1$ are basic open sets in $\mathbb{N}$
     $\therefore$ the order topology on $\mathbb{N}$ is the discrete topology on $\mathbb{N}$
(6) The set $X = \{1, 2\} \times \mathbb{N} = \{1 \times n\}_{n=1}^{\infty} = a_n \cup b_n = \{2 \times n\}_{n=1}^{\infty}$ in the dictionary order with the smallest element 1×1. The order topology on X is not discrete topology on X
   $X : a_1, a_2, \cdots, b_1, b_2, \cdots,\ a_i < a_{i+1},\ b_j < b_j + 1,\ a_i < b_j$
   - $\star$ $\{a_1\} = [a_1, a_2)$
   - $\star$ $\{a)n\} = (a_{n-1}, a_{n+1}), n \geq 2$
   - $\star$ $\{b_n\} = (b_{n-1}, , b_{n+1}), n \geq 2$

   But $\{b_1\}$ is not open, $\because$ b_1 is not the smallest elements any basic open set in the order topology containing b_1 must of the form (a_l, b_j) for some $l \geq 1$ and $j > 1$

Definition. *Let X be an ordered set and $a \in X$. We define the rays determine by a*

- $\star$ $(a, \infty) = \{x \in X \mid x > a\}$
- $\star$ $(-\infty, a) = \{x \in X \mid x < a\}$
- $\star$ $[a, \infty) = \{x \in X \mid x \geq a\}$
- $\star$ $(\infty, a] = \{x \in X \mid x \leq a\}$

Some facts:

(1) open rays in X are open in the order topology of X. In fact, $(a,\infty) = (a,b_0]$ if X has the largest element which is a basic open set in the order topology of X. If X has no largest element, then $(a,\infty) = \bigcup_{a<x}(a,x)$ which is open in the order topology of X

(2) closed rays is close

(3) The order topology of X is contained in the topology on X generated by open rays in X. $\because (a,b) = (a,\infty) \cap (-\infty,b)$.
If X has the smallest element a_0, $[a_0,b) = (-\infty,b)$
If X has the largest element b_0, $(a,b_0] = (a,\infty)$

§ **15 The Product Topology on** $X \times Y$**.** Similarly for $X_1,\cdots,X_n$
Let X and Y be topology spaces and

$$\mathscr{B} = \{U \times V \mid U \text{ is open in } X, V \text{ is open in } Y\}$$

Claim $\mathscr{B}$ is a base for a topology on $X \times Y$

- $\bigcup \mathscr{B} = X \times Y$
- Given $U_i \times V_i \in \mathscr{B},\ i = 1,2$ and $(a,b) \in (U_1 \times V_1) \cap (U_2 \times V_2)$
  $(a,b) \in U \times V \subseteq (U_1 \times V_1) \cap (U_2 \times V_2)$ where $U = U_1 \cap U_2,\ V = V_1 \cap V_2$

<u>Definition</u>. *The topology on $X \times Y$ generate by $\mathscr{B}$ is called the product topology on $X \times Y$*

Remark. *If $X_1,\cdots,X_n$ are topological space, them*

(1) $\mathscr{B} = \{U_1 \times \cdots \times U_n \mid U_i \text{ is open in } X_i,\ 1 \le i \le n\}$ is a base for the product topology on $X_1 \times \cdots \times X_n$

(2) The product topology on $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ is the usual topology on $\mathbb{R}^n$ generate b the collection of all n-dimensional open intervals.

$$\{I_1 \times \cdots \times I_n \mid I_j \text{ is an open interval in } \mathbb{R},\ 1 \le j \le n\}$$

Theorem. *Let X and Y be topological space with bases $\mathscr{B}_X$ and $\mathscr{B}_Y$ on X and Y respectively. Then*

$$\mathscr{D} = \{B \times C \mid B \in \mathscr{B}_X,\ C \in \mathscr{B}_Y\}$$

forms a basis for the product topology on $X \times Y$

Proof. Let $\mathscr{B} = \{U \times V \mid U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$. We know that $\mathscr{B}$ is a base for the product topology on $X \times Y$
Given $U \times V \in \mathscr{B}$ with $(a, b) \in U \times V \implies a \in U,\ b \in V \implies \exists B \in \mathscr{B}_X$ and $C \in \mathscr{B}_Y \ni a \in B \subseteq U,\ b \in C \subseteq V$
$\therefore (a, b) \in B \times C \subseteq U \times V$ and $B \times C \in \mathscr{D}$ ∎

Redefine the product topology on $X_1 \times \cdots \times X_n$ by using subbase
The projection onto X_i

$$\begin{aligned} \pi_i : X_1 \times \cdots \times X_n &\rightarrow X_i \\ (x_1, \cdots, x_n) &\rightarrow x_1,\ 1 \leq i \leq n \end{aligned}$$

If $U_i \subseteq X_i$ $\pi_i^{-1}(U_i) = X_1 \times \cdots X_{i-1} \times U_i \times X_{i+1} \times \cdots \times X_n$
Let $\delta = \{\phi_i^{-1}(U_i) \mid U_i \subseteq X_i \text{ is open and } 1 \leq i \leq n\}$
Note: $\bigcup_{i=1}^{n} \pi_i^{-1}(U_i) = X_1 \times \cdots \times X_n$
$\therefore \delta$ is a subbase for a topological on $X_1 \times \cdots \times X_n$ with base

$$\{U_1 \times \cdots \times U_n \mid U_i \text{ is open in } X_i,\ 1 \leq i \leq n\}$$

Hence, the product topology on $X_1 \times \cdots \times X_n$ is generated by δ

§ **16 The subspace topolgoy.** Let X be a topology space with topology $\mathscr{T}$ and $Y \subseteq X$. Let $\mathscr{T}_Y = \{U \cap Y \mid U \in \mathscr{T}$, i.e. U is open in $X\}$

Definition. *The topology $\mathscr{T}_Y$ on Y is called the subspace topology of Y in X. With this topology, Y is called a subspace of X*

Lemma. *If $\mathscr{B}$ is a base for the topology $\mathscr{T}$ of X, then $\mathscr{B}_Y = \{B \cap Y \mid B \in \mathscr{B}\}$ is a base for the subspace topology on Y.*

Proof. Given an open set V in Y and $y \in V$. Then $y \in V = \cap Y$ for some open set in $X \implies y \in U \implies \exists B \in \mathscr{B} \ni y \in B \subseteq U \implies y \in B \cap Y \subseteq U \cap Y = V$
$\therefore \mathscr{B}_Y$ is a base for the subspace topology of Y. ∎

Lemma. *Let Y be a subspace of X. If Y is open in X and V is open in Y, then V is open in X.*

Theorem. *If A is a subspace of X and B is a subspace of Y. Then the product topology on $A \times B$ is the same as the subspace topology $A \times B$ inherits as a subspace of $X \times Y$*

Proof. Let $\mathscr{B} = \{U \times V \mid U \text{ is open in } X,\ V \text{ is open in } Y\}$. Then $\mathscr{B}$ is a base for the product topology on $X \times Y$. By lemma above, $\mathscr{B}_{A\times B} = \{(U \times V) \cap (A \times B) \mid U \times V \in \mathscr{B}\}$ is a base for the subspace topology on $A \times B$
$\mathscr{B}_{A\times B} = \{(U \cap A) \times (U \cap B) \mid U \cap A \text{ is open in } A,\ V \cap B \text{ is open in } B\}$ which is a base for the product space $A \times B$. Thus ... ■

Example.

(1) Consider $Y = [0, 1]$ in $\mathbb{R}$. The subspace topology of Y in $\mathbb{R}$ has a base of the form

$$\{(a, b) \cap Y \mid -\infty < a < b < \infty\}$$

Note that

$$(a, b) \cap Y = \begin{cases} (a, b) & \text{if } a, b \in Y \\ [0, b) & \text{if only} b \in Y \\ (0, 1] & \text{if only} a \in Y \\ \emptyset \text{ or } Y & \text{if } a, b \notin Y \end{cases}$$

The order topology on Y has a base of the form $[0, b) b \in Y,\ (a, 1] a \in Y,\ (a, b)\ a, b \in Y$

(2) Let $Y = [0, 1) \cup \{2\} \subseteq \mathbb{R}$. In the subspace topology of Y in $\mathbb{R}$. $\{2\} = (\frac{3}{2}, \frac{5}{2}) \cap Y$ is open in Y. In the order topology of Y, $\{2\}$ is not open in Y

Proof. $\because$ any basic open set in the order of Y containing 2 is of the form

$$(a, 2] = \{y \in Y \mid a < y \leq 2\} \text{ where} a \in Y$$

must contain points not equal 2, $\therefore$ The two topologies are different ■

(3) $I = [0,1]$. The dictionary order on $I \times I$ is just the restriction to $I \times I$ of the dictionary order on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

The set $V = \{\frac{1}{2}\} \times (\frac{1}{2}, 1]$ is open in the subspace topology of $I \times I$

V is not open in the order topology $I \times I$

$\because$ any basic open set in the order topology of $I \times I$ containing $\frac{1}{2} \times 1$ is of the form $(a \times b, c \times d)$

There is no basic open set B in the order topology of $I \times I$ such that $\frac{1}{2} \times 1 \in B \subseteq \{\frac{1}{2}\} \times (\frac{1}{2}, 1]$

$\therefore$ The two topologies on $I \times I$ are distinct.

Definition. *Given an order set X. A subset $Y \subseteq X$ is convex if $\forall a < b$ in Y, $(a,b) \subseteq Y$*
In fact, $[a,b] \subseteq Y$

Theorem. *Let X be an order set with order topology and $Y \subseteq X$ be a convex set of X. Then the order topology on Y and the subspace topology on Y concise.*

Proof. Let $\mathscr{T}_O$ and $\mathscr{T}_Y$ be the order topology and subspace topology on Y, respectively.
$\mathscr{T}_O \supseteq \mathscr{T}_Y$
Note that the order topology on Y is generate by the subbasic open sets of all rays in Y of the forms

$$(a, \infty) \cap Y \text{ and } (-\infty, b) \cap Y,\ a, b \in Y$$

the order topology on X is generated by subbasic open sets

$$(a, \infty) \text{ and } (-\infty, b)\ a, b \in X$$

The subbasic open sets in the subspace topology $\mathscr{T}_Y$

$$(a, \infty) \cap Y,\ (-\infty, b) \cap Y,\ a, b \in X$$

If $a \in Y$, then $(a, \infty) \cap Y$ is an open ray in Y which is a subbasic open set in the order topology $\mathscr{T}_O$ of Y, thus, $(a, \infty) \cap Y \in \mathscr{T}_O$

If $a \notin Y$, then since Y is convex, a is either a lower bound for Y or an upper bound for Y. Therefore,

$$(a,\infty)\cap Y = \begin{cases} Y & \text{if } a \text{ is a lower bound of } Y \\ X & \text{if } a \text{ is a upper bound of } Y \end{cases}$$

In any case, $(a,\infty)\cap Y \in \mathscr{T}_O \forall a \in X$. Similarly $(-\infty,b)\cap Y \in \mathscr{T}_O \forall b \in X$, $\therefore \mathscr{T}_Y \subseteq \mathscr{T}_O$
and the other way don't need convex. ■

§17 Closed Sets and Limit Points.

§*17.1.* Closed Sets

Definition. *Let X be a topological space and $A \subseteq X$, A is closed if $A^c = X - A = A$ is open in X*

Example:

(1) $\forall -\infty < a \leq b < \infty,\ [a,b],[a,\infty),(-\infty,a)$ are closed in $\mathbb{R}$
(2) $A = \{(x,y)\in\mathbb{R}^2,\ x\geq 0,\ y\geq 0\}$ is closed in $\mathbb{R}^2$
(3) In the finite complement topology on a set X. the closed set in X are X and all finite subsets of X
(4) In a discrete topological space X every subset of X is closed
(5) Consider the subspace $Y = [0,1]\cup(2,3)$ of $\mathbb{R}$, $[0,1]$ is open in Y, $(2,3)$ is open in Y and $\mathbb{R}$. Since $Y - [0,1] = (2,3)$ and $Y - (2,3) = [0,1]$, $[0,1]$ and $(2,3)$ are both open and closed in Y.

Theorem (17.1)**.** *Let X be a topological space. Then*

(1) $\emptyset, X$ are closed
(2) A_α is closed in X, $\alpha \in I \implies \bigcap_{\alpha\in I} A_\alpha$ is closed
(3) $A_1,\cdots,A_n$ are closed $\implies A_1\cup\cdots\cup A_n$ is closed.

Remark.

(1) In definition (3) of topology is false for infinitely many open set, e.g. $\bigcap_{n=1}^{\infty}\left(\frac{1}{-n},1+\frac{1}{n}\right) = [0,1]$ is not open in $\mathbb{R}$

(2) In (3) of Thm 17.1 is false for infinitely many closed set. e.g. $\bigcup_{n=1}^{\infty}[\frac{1}{n}, 1-\frac{1}{n}] = (0,1)$ *is not closed in* $\mathbb{R}$

Theorem (17.2)**.** *Let Y be a subspace of a topological space X and $A \subseteq X$. Then A is closed in Y iff $A - B \cap Y$ for some closed set B in X.*

Proof.

$$\begin{aligned} A \text{ is closed in } Y &\Leftrightarrow Y - A \text{ is open in } Y \\ &\Leftrightarrow Y - A = U \cap Y,\ U \text{ is open in } X \\ &\Leftrightarrow A = Y - (U \cap Y) = (X - U) \cap Y \text{ is closed in } X \end{aligned}$$

■

Theorem. *Let Y be a subspace of a topological space X. If A is closed in Y and Y is closed in X, then A is closed in X.*

Proof. By Thm 17.2, trivial ■

§ ***17.2.*** Closure and Interior of a set

<u>Definition</u>. *Let X be a topological space and $A \subseteq X$*

(1) The interior of A, $A^\circ = int(A) = \bigcup_{\substack{A\subseteq U \\ U \text{ is closed}}} U$

(2) The closure of A, $\overline{A} = cl(A) = \bigcap_{\substack{A\subseteq F \\ F \text{ is closed}}} F$

Remark.

(1) A° is the largest open set in X contained in A(w.r.t $\subseteq$)

(2) $A^\circ \subseteq A \subseteq \overline{A}$, A° is open in X and $\overline{A}$ is closed in X.

(3) A is open iff $A^\circ = A$. In particular, $A^{\circ\circ} = A^\circ$, A is closed iff $\overline{A} = A$. In particular $\overline{A} = \overline{\overline{A}}$

(4) Let X be a topological space and $Y \subseteq X$ be a subspace $\forall A \subseteq X$, we have the closure of A in X : $\overline{A}$, and the closure of A in Y : $\overline{A}^Y$, in general, $\overline{A} \neq \overline{A}^Y$

e.g. $X = \mathbb{R},\ Y = [0,1),\ A = (\frac{1}{2}, 1)$

$\implies \overline{A} = [\frac{1}{2}, 1),\ \overline{A}^Y = [\frac{1}{2}, 1)$

Theorem. *Let Y be a subspace of X and $A \subseteq Y$. Then $\overline{A}^Y = \overline{A} \cap Y$*

Proof. By Thm 17.2 and $\overline{A}$ is closed in X. $\overline{A} \cap Y$ is closed in Y. Since $A \subseteq Y$ is closed subset in Y containing A, $\overline{A}^Y \subseteq \overline{A} \subseteq \overline{A} \cap Y$
Conversely, $\overline{A}^Y$ is closed in $Y \implies \overline{A}^Y = F \cap Y$ for some closed set F in X. Clearly, $A \subseteq F \implies \overline{A} \subseteq \overline{F} \implies \overline{A} \subseteq F \implies \overline{A} \cap Y \subseteq F \cap Y = \overline{A}^Y \therefore \overline{A}^Y = \overline{A} \cap Y$ ■

Definition.

(1) *A set A intersects a set B if $A \cap B \neq \emptyset$*
(2) *A neighborhood of a point x is an open set containing x*

Definition. *Let X be a topological space and $A \subseteq X$. A point $x \in X$ is an adherent point of A if $\forall$ nhd U of x, $U \cap A \neq \emptyset$*

Theorem (17.5). *Let X be a topological space and $A \subseteq X$*

(1) *$x \in \overline{A}$ iff x is an adherent point of A*
(2) *Suppose the topological of X is given by a base $\mathscr{B}$. Then $x \in \overline{A}$ iff $\forall$ basic nhd B of x, $B \cap A \neq \emptyset$*

Proof.

(a) ($\Rightarrow$) Suppose $x \in \overline{A}$. If x is not an adherent point of A, then $\exists$ nhd U of $x \ni U \cap A = \emptyset$. Thus, $A \subseteq X - U$ which is closed $\implies \overline{A} \subseteq X - U \implies x \notin \overline{A} (\rightarrow\leftarrow)$

($\Leftarrow$) Suppose x is an adherent point of A. If $x \notin \overline{A}$, then $x \in X - \overline{A} \equiv U$ is a nhd of x with $U \cap A = \emptyset (\rightarrow\leftarrow)$ to x is an adherent point

(b) H.W.

■

Example. In $\mathbb{R}$ by Thm 17.5, we have

- $\overline{(0,1]} = [0,1]$
- $\overline{\{\frac{1}{n}\ n \in \mathbb{N}\}} = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$
- $\overline{\mathbb{Q}} = \mathbb{R}$, i.e. $\mathbb{Q}$ is dense in $\mathbb{R}$
- $\overline{\mathbb{N}} = \mathbb{N}$, $\overline{\mathbb{Z}} = \mathbb{Z}$
- $\overline{\mathbb{R}^+} = \mathbb{R} + \cup \{0\}$

Example. $Y = (0,1] \subseteq \mathbb{R}$, $A = (0, \frac{1}{2}) \subseteq Y$
$\overline{A}^Y = \overline{A} \cap Y = [0, \frac{1}{2}] \cap (0,1] = (0, \frac{1}{2}]$

§ ***17.3.*** Limit Points(Accumulation or cluster)

<u>**Definition**</u>. *Let X be a topological space. $A \subseteq X$ and $x \in X$. x is a limit point of A if $\forall$ nhd U of x, $U \cap A - \{x\} \neq \emptyset$ denote by A' the set of all limit points of A called the derived set of A*

Remark. *$x \in A'$, x may not in A*

Example In $\mathbb{R}$, we have

- $[0,1]' = [0,1]$
- $\{\frac{1}{n} \mid n \in \mathbb{N}\}' = \{0\}$
- $(\{0\} \cup (1,2))' = [1,2]$
- $\mathbb{Q}' = \mathbb{R}$
- $\mathbb{N}' = \mathbb{Z}' = \emptyset$
- $\mathbb{R}'+ = \mathbb{R} + \cup \{0\} = \overline{\mathbb{R}+}$

Theorem (17.6). *Let X be a topological space and $A \subseteq X$. Then $\overline{A} = A \cup A'$*

Proof. Clearly $A \subseteq \overline{A}$ and $A' \subseteq \overline{A} \implies A \cup A' \subseteq \overline{A}$. Conversely, given $x \in \overline{A}$. If $x \in A$, then $x \in A \cup A'$. If $x \notin A'$, then $\forall$ nhd U of x, $U \cap A \neq \emptyset \implies U \cap A - \{x\} \neq \emptyset (\because x \notin A) \implies x \in A' \implies x \in A \cup A'$ ■

Corollary. *A is closed in X iff $A' \subseteq A$*

Proof. $A = \overline{A} = A \cup A'$(trivial) ■

§ ***17.4.*** Hausdorff Spaces(or T_2-spaces)
Exmpale $X = \{a,b,c\}$, $\mathscr{T} = \{\emptyset, \{a,b\}, \{b\}, \{b,c\}, X\}$ which is a topology on X, $\{b\}$ is open in X but $\{b\}$ is not closed. Consider the sequence $\{x_n\}$ in X with $x_n = b \forall n \geq 1$. Then $\{x_n\}$ convergences to any point in X.

<u>**Definition**</u>. *A topological space X is called a Hausdorff space (or T_2 space) if every two distinct points in X can be separated by open sets. i.e. $\forall x_1 \neq x_2$ in X, $\exists$ nhd U_i of x_i, $i = 1,2 \ni U_1 \cap U_2 = \emptyset$*

Theorem. *Every finite set in T_2-space X is closed. In particular, every singleton is closed*

Proof. Given a finite set $F = \{x_1, \cdots, x_n\}$ Write $F = \bigcup_{i=1}^{n}\{x_i\}$. It suffices to show that every singleton $\{x\}$ is closed in X

$$\begin{aligned}
\forall y \in X - \{x\} &\implies y \neq x \\
&\implies \exists \text{ nhd } U \text{ of } x \text{ and } V \text{ of } y \ni U \cap V = \emptyset \\
&\implies y \in V \subseteq X - \{x\} \\
&\implies X - \{x\} \text{ is open in } X \\
&\implies \{x\} \text{ is closed in } X
\end{aligned}$$

■

Remark. *The converse fails, e.g. In a finite complement topological space X, where X is an infinite set, every singleton is closed in X, but X is not T_2*
$\because$ *$\forall x \neq y$ and U of x and V of y. If $U \cap V$ $X - (U \cap V) = (X - U) \cup (X - V) \implies X$ is finite$(\rightarrow\leftarrow)$.*

Definition. *A topological space X is said to be T_1 if every singleton is closed in X.*
Equivalently, $\forall x \neq y$ in X, $\exists$ a neighborhood U of x such that $y \notin U$ and $\exists$ a neighborhood of y such that $a \notin V$

Theorem. *Let X be a T_1 space and $A \subseteq X$. Then $x \in A$ iff $\forall$ neighborhood U of x, $U \cap A$ is an infinite set*

Proof.

($\Leftarrow$) trivial (By definition)

($\Rightarrow$) Suppose $x \in A$. If $\exists$ a neighborhood U of $x \ni$ $U \cap A$ is a finite set, so is $U \cap A - \{x\}$, say, $U \cap A - \{x\} = \{x_1, \cdots, x_n\}$.
Since X is $T_1\{x_1, \cdots, x_n\}$ is closed in X
$\implies X - \{x_1, \cdots, x_n\}$ is open
$\implies V \equiv U \cap (X - \{x_1, \cdots, x_n\})$ is a neighborhood of x with $V \cap A - \{x\} = \emptyset(\rightarrow\leftarrow)$ to $x \in A'$

■

Definition. *A topological space X is said to be T_0 if every two distinct points x and y in X one of them has a neighborhood not containing the other one.*

Remark. $T_1 \implies T_1 \implies T_0$, *but the converse fails, e.g.* $T_0 \to T_1$, $X = a, b$ *with topology* $\mathscr{T} = \{\emptyset, \{a\}, X\}$ *which is* T_0 *but not* T_1

Theorem. *If X is Hausdorff space, then a sequence $\{x_n\}$ in X can converge to at most on point in X.*

Proof. If $\{x_n\}$ converges to x and x', $x \neq x'$ choose neighborhood U of x and U' of $x' \ni U \cap U' = \emptyset$. Choose $N >> 0 \ni \forall n \geq N,\ x_n \in U \cap U' (\to\leftarrow)$ to $U \cap U' = \emptyset$ ■

Theorem.

(1) *Every order set X with order topology is T_2*
(2) *If X and Y are T_2, so is $X \times Y$*
(3) *If X is T_2, then so is it's subspace.*

Proof. skip in latex :). ■

§ 18 Continuous Mapping.