

Chapter 1

Vector Space

Definition (Vector Space). A vector space (or linear space) W over a Field \mathbb{F} consists of a set on which two operations (called addition and scalar multiplication, respectively) are defined so that for each pair of elements x, y , in W there is a unique element $x + y$ in W , and for each element a in F and each element x in W there is a unique element ax in W , such that the following conditions hold.

§ Subspace

Definition (Subspace). A subset W of a vector space W over a field \mathbb{F} is called a subspace of W if W is a vector space over F with the operations of addition and scalar multiplication defined on W .

Remark. Trivial subspaces of a vector space V , namely V itself and $\{0\}$. Note that empty set ϕ is not a vector space, since it does not contains a zero vector.

Theorem 1.1. Let V be a vector space and W a subset of V . Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V .

- (a) $0 \in W$.
- (b) $x + y \in W$ whenever $x \in W$ and $y \in W$.
- (c) $cx \in W$ whenever $c \in F$ and $x \in W$.

Proof. Suppose $W_1 \cup W_2$ is a subspace of V , we assume that $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$. Let $x \in W_1 \setminus W_2$ $y \in W_2 \setminus W_1$, then $x + y \in W_1$ or W_2 , say W_1 .

But $y = (x + y) - x \in W_1$, which is a contradiction. So we have $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. Conversely is trivial. ■

Corollary 1.1.1. *Let W be a subset of vector space V . W is a subspace of V if and only if $0 \in W$ and $ax + y \in W$ whenever $a \in F$ and, $x, y \in W$.*

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Theorem 1.2. *Any intersection of subspaces of a vector space V is a subspace of V .*

Proof. To prove (a), we first show that $W_1 + W_2$ is a subspace of V . Clearly, $0 = 0 + 0 \in W_1 + W_2$. Let $x = x_1 + x_2$, $y = y_1 + y_2$, $c \in \mathbb{F}$

$$cx + y = c(x_1 + x_2) + (y_1 + y_2) = (cx_1 + y_1) + (cx_2 + y_2) \in W_1 + W_2$$

By corollary 1.1.1, $W_1 + W_2$ is a subspace of V .

Then we show that $W_1, W_2 \subseteq W_1 + W_2$. $\forall x \in W_1, y \in W_2$, $x = x + 0 \in W_1 + W_2$, $y = 0 + y \in W_1 + W_2$, we have $W_1, W_2 \subseteq W_1 + W_2$.

To prove (b), Let W be a subspace of V contains both W_1 and W_2 , then $\forall x \in W_1, y \in W_2$, $x + y \in W$. Thus $W_1 + W_2 \subseteq W$. ■

Theorem 1.3. *Let W_1 and W_2 be subspaces of a vector space V , then $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.*

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Definition. *If S_1 and S_2 are nonempty subsets of a vector space V , then the sum of S_1 and S_2 , denoted $S_1 + S_2$, is the set $\{x + y : x \in S_1 \text{ and } y \in S_2\}$.*

Theorem 1.4. *Let W_1 and W_2 be subspaces of a vector space V .*

(a) $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2 .

(b) Any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

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Definition (Direct Sum). *A vector space V is called the direct sum of W_1 and W_2 if W_1 and W_2 are subspaces of V such that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. We denote that V is the direct sum of W_1 and W_2 by writing $V = W_1 \oplus W_2$.*

Theorem 1.5. *Let W_1 and W_2 be subspaces of a vector space V . V is the direct sum of W_1 and W_2 if and only if each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$.*

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§ Linear Combinations and Bases

Definition (Linearly Dependent). A subset S of a vector space W is called linearly dependent if there exist a finite number of distinct vectors u_1, u_2, \dots, u_n in S and scalars a_1, a_2, \dots, a_n , not all zero, such that

$$a_1u_1 + a_2u_2 + \dots + a_nu_n = 0.$$

In this case we also say that the vectors of S are linearly dependent.

Definition (Linearly Independent). A subset S of a vector space that is not linearly dependent is called linearly independent. As before, we also say that the vectors of S are linearly independent.

Remark.

1. The empty set is linearly independent, for linearly dependent sets must be nonempty.
2. A set consisting of a single nonzero vector is linearly independent. For if $\{u\}$ is linearly dependent, then $au = 0$ for some nonzero scalar a . Thus

$$u = a^{-1}(au) = a^{-1}0 = 0$$

3. A set is linearly independent if and only if the only representations of 0 as linear combinations of its vectors are trivial representations.

Theorem 1.6. Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent.

Exercise

Corollary 1.6.1. Let S be a linearly independent subset of a vector space V , and let v be a vector in V that is not in S . Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.

Proof. Suppose $S \cup v$ is linear independent, then $\exists v_1, \dots, v_n \in S \cup v$ such that $a_1u_1 + \dots + a_nu_n = 0$ for some nonzero scalars a_1, \dots, a_n . Because S is linear independent, say $u_1 = v$. We have $v = a_1^{-1}(-a_2u_2 - \dots - a_nu_n)$, which is a linear combination of u_2, \dots, u_n . Thus $v \in \text{span}(S)$. Conversely, suppose $v \in \text{span}(S)$. Then $\exists v_1, \dots, v_n \in S$ and scalars b_1, \dots, b_m such that $v = b_1v_1 + \dots + b_mv_m$. Thus $0 = b_1v_1 + \dots + b_mv_m + (-1)v$ which means $S \cup v$ is linear independent. ■

Theorem 1.7. *Let S be a linearly independent subset of a vector space V , and let v be a vector in V that is not in S . Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.*

Proof. Let β be a basis of v . Suppose $v = a_1u_1 + \dots + a_nu_n = b_1u_1 + \dots + b_nu_n$. We have $0 = (a_1 - b_1)u_1 + \dots + (a_n - b_n)u_n$. Since β is linear independent, we have $(a_1 - b_1) = \dots = (a_n - b_n)$, hence $a_1 = b_1, \dots, a_n = b_n$, which means v is uniquely expressed as a linear combination of β . Conversely, suppose v can be uniquely expressed as a linear combination of β . Clearly, β generates V and the representation of 0 is unique. So if $0 = a_1u_1 + \dots + a_nu_n$, we have $a_1 = \dots = a_n$. Thus β is linear independent and is a basis of V . ■

Definition (Basis). *A basis β for a vector space V is a linearly independent subset of V that generates V . If β is a basis for V , we also say that the vectors of β form a basis for V .*

Theorem 1.8. *Let V be a vector space and $\beta = \{u_1, u_2, \dots, u_n\}$ be a subset of V . Then β is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β , that is, can be expressed in the form*

$$v = a_1u_1 + a_2u_2 + \dots + a_nu_n$$

for unique scalars a_1, a_2, \dots, a_n .

Proof. If $S = \emptyset$ or $S = 0$, then $V = \{0\}$ and \emptyset is a subset of S which is a basis of V . Otherwise S contains a nonzero vector $\{u_1\}$. We continue choosing vectors $u_2, \dots, u_k \in S$ such that $\beta = \{u_1, \dots, u_k\}$ is linear independent and $\{u_1, \dots, u_k, v\}$ is linear dependent for some vector $v \in V \setminus S$. To show that β is a basis, it suffices to show that β generates V . Let $v \in S$. If $v \in \beta$, then clearly $\beta \in \text{span}(S)$. If $v \in S \setminus \beta$, then we have $\beta \cup v$ is linear dependent. So $v \in \text{span}(\beta)$. Thus $S \subseteq \text{span}(\beta)$. ■

Theorem 1.9. *If a vector space V is generated by a finite set S , then some subset of S is a basis for V . Hence V has a finite basis.*

Proof. We prove it by mathematical induction on m . For $m = 0$, $L = \emptyset$, so take $H = G$. Assume the theorem holds for some integer $m \geq 0$. For $m+1$, let $L = \{v_1, \dots, v_{m+1}\}$ be a linear independent subset of V containing $m+1$ vectors. Because $\{v_1, \dots, v_m\}$ is linear independent, so by the induction hypothesis, we have $m \geq n$ and \exists a subset $\{u_1, \dots, u_{n-m}\}$ of G such that $\{v_1, \dots, v_m\} \cup$

$\{u_1, \dots, u_{n-m}\}$ generates v . Thus \exists scalars $a_1, \dots, a_m, b_1, \dots, b_{n-m}$ such that $a_1v_1 + \dots + a_mv_m + b_1u_1 + \dots + b_{n-m}u_{n-m} = v$. We have $n - m > 0$ and v_{m+1} is a linear combination of $\{v_1, \dots, v_m\}$ which contradicts the assumption that L is linear independent. Hence $n > m$ that is $n \geq m + 1$. Moreover, say b_1 is nonzero, otherwise we obtain the same contradiction. We have $u_1 = (-b_1^{-1}a_1)v_1 + \dots + (-b_1^{-1}a_m)v_m + (b_1^{-1})v_{m+1} + (-b_1^{-1}b_2)u_2 + \dots + (-b_1^{-1}b_{n-m})u_{n-m}$. Let $H = \{u_2, \dots, u_{n-m}\}$. Then $u_1 \in \text{span}(L \cup H)$ and $v_1, \dots, v_m, u_2, \dots, u_{n-m} \in \text{span}(L \cup H)$, so $\{v_1, \dots, v_m, u_1, \dots, u_{n-m}\} \subseteq \text{span}(L \cup H)$. We have $\text{span}(L \cup H) = V$. Since H is a subset of G contains $(n - m) - 1 = n - (m + 1)$ vectors. So by mathematical induction, we are done. ■

Theorem 1.10 (Replacement Theorem). *Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V .*

Proof. Let $\dim(V) = n$. If $W = \{0\}$, then W is finite-dimensional and $\dim(W) = 0 \leq n$. Otherwise, W contains a nonzero vector x_1 , so $\{x_1\}$ is a linear independent set. We continue choosing vectors $x_1, \dots, x_k \in W$ such that x_1, \dots, x_k is linear independent. This process must stop at $k \leq n$ and $\{x_1, \dots, x_k\}$ is linear independent and other vector in W produces a linear dependent set. Hence it is a basis of W . Thus $\dim(W) = k \leq n$. If $\dim(W) = n$, a basis of W is a linear independent subset of V contains n vectors, so a basis of W is also a basis of V . Thus $W = V$. ■

Corollary 1.10.1. *Let V be a vector space having a finite basis. Then every basis for V contains the same number of vectors.*

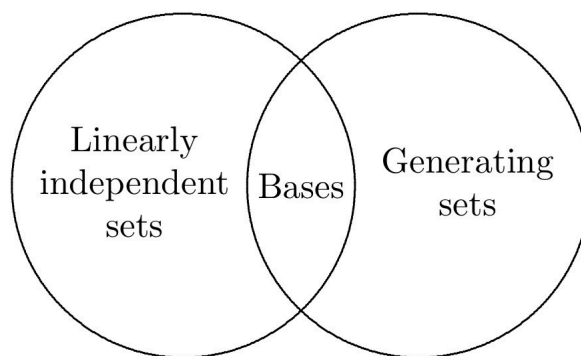
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Definition (Finite-Dimensional). *A vector space is called finite-dimensional if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for V is called the dimension of V and is denoted by $\dim(V)$. A vector space that is not finite-dimensional is called infinite-dimensional.*

Corollary 1.10.2. *Let V be a vector space with dimension n .*

1. Any finite generating set for V contains at least n vectors, and a generating set for V that contains exactly n vectors is a basis for V .
2. Any linearly independent subset of V that contains exactly n vectors is a basis for V .
3. Every linearly independent subset of V can be extended to a basis for V .

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Theorem 1.11. Let W be a subspace of a finite-dimensional vector space V . Then W is finite-dimensional and $\dim(W) \leq \dim(V)$. Moreover, if $\dim(W) = \dim(V)$, then $V = W$.

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Proposition 1.12. Let W_1 and W_2 be subspaces of a finite-dimensional vector space V . $W_1 \subseteq W_2$ if and only if $\dim(W_1 \cap W_2) = \dim(W_1)$

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Theorem 1.13. Let v_1, v_2, \dots, v_k, v be vectors in a vector space V , and define $W_1 = \text{span}(\{v_1, v_2, \dots, v_k\})$, and $W_2 = \text{span}(\{v_1, v_2, \dots, v_k, v\})$. Then $v \in \text{span}(W_1)$ if and only if $\dim(W_1) = \dim(W_2)$.

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Remark. We may give an example for satisfying the conditions on above but $\dim(W_1) \neq \dim(W_2)$.

Theorem 1.14. *Let W_1 and W_2 be finite-dimensional subspaces of a vector space V .*

(a) *Then the subspace $W_1 + W_2$ is finite-dimensional, and*

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

(b) *Let $V = W_1 + W_2$. Deduce that V is the direct sum of W_1 and W_2 if and only if*

$$\dim(V) = \dim(W_1) + \dim(W_2)$$

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Theorem 1.15. *Let W_1 and W_2 be subspaces of a vector space V such that $V = W_1 \oplus W_2$ if and only if there exist bases β_1 , β_2 of W_1 , W_2 , respectively such that $\beta_1 \cup \beta_2$ is a basis for V .*

not yet

Theorem 1.16.

If W_1 is any subspace of vector space V , then there exists a subspace W_2 of V such that

$$V = W_1 \oplus W_2$$

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Chapter 2

Linear Transformation

§ Linear Operator

Definition (Linear Transformation). Let V and W be vector spaces (over F). We call a function $T : V \rightarrow W$ a linear transformation from V to W if, for all $x, y \in V$ and $c \in F$, we have

$$(a) \quad T(x + y) = T(x) + T(y)$$

$$(b) \quad T(cx) = cT(x)$$

Definition. Let $T, U : V \rightarrow W$ be arbitrary functions, where V and W are vector spaces over F , and let $a \in F$. We define $T + U : V \rightarrow W$ by $(T + U)(x) = T(x) + U(x)$ for all $x \in V$, and $aT : V \rightarrow W$ by $(aT)(x) = aT(x)$ for all $x \in V$.

Remark. If $F = Q$ then $(a) \Rightarrow (b)$.

Remark. If $T : C \rightarrow C$ be function defined by $T(\delta) = \delta$. T is additive but not linear.

Propersition 2.1.

1. If T is linear, then $T(0) = 0$.
2. T is linear if and only if $T(cx + y) = cT(x) + T(y)$ for all $x, y \in V$ and $c \in F$.
3. If T is linear, then $T(x - y) = T(x) - T(y)$ for all $x, y \in V$

4. T is linear if and only if, for $x_1, x_2, \dots, x_n \in V$ and $a_1, a_2, \dots, a_n \in F$, we have

$$T\left(\sum_{i=1}^n a_i x_i\right) = \sum_{i=1}^n a_i T(x_i).$$

5. For all $a \in F$, $aT + U$ is linear.
6. Using the operations of addition and scalar multiplication in the preceding definition, the collection of all linear transformations from V to W is a vector space over F .

Definition (Null Space, Range). Let V and W be vector spaces and let $T : V \rightarrow W$ be linear. We define the null space (or kernel) $N(T)$ of T to be the set of all vectors x in V such that $T(x) = 0$; that is, $N(T) = \{x \in V : T(x) = 0\}$.

We define the range (or image) $R(T)$ of T to be the subset of W consisting of all images (under T) of vectors in V ; that is, $R(T) = \{T(x) : x \in V\}$

Remark. Let V and W be vector spaces, and let $I : V \rightarrow V$ and $T_0 : V \rightarrow W$ be the identity and zero transformations, respectively. Then $N(I) = \{0\}$, $R(I) = V$, $N(T_0) = V$, and $R(T_0) = \{0\}$.

Theorem 2.2. Let V and W be vector spaces and $T : V \rightarrow W$ be linear. Then $N(T)$ and $R(T)$ are subspaces of V and W , respectively.

Proof.

Let $0_V \in V, 0_W \in W$ be a zero vectors

$\because T(0_V) = 0_W \therefore 0_V \in N(T)$

Let $x, y \in N(T), c \in F$,

$\because T(x + y) = T(x) + T(y) = 0_W + 0_W = 0_W$

$\therefore x + y \in N(T)$

$\because T(cx) = cT(x) = c0_W = 0_W$

$\therefore cx \in N(T)$

$\implies N(T)$ is a subspace of V

$\because T(0_V) = 0_W \therefore 0_W \in R(T)$

Let $x, y \in R(T)$ and $c \in F$. $\exists v, w \in V, T(v) = x \in V$ and $T(w) = y$

$\because x + y = T(v) + T(w) = T(v + w)$

$\therefore x + y \in R(T)$

$\because cx = cT(v) = T(cv)$

$\therefore cx \in R(T) \implies R(T)$ is a subspace of W

■

Remark. Give an example of distinct linear transformations T and U such that $N(T) = N(U)$ and $R(T) \neq R(U)$

Theorem 2.3. Let V and W be vector spaces, and let $T : V \rightarrow W$ be linear. If $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for V , then

$$R(T) = \text{span}(T(\beta)) = \text{span}(\{T(v_1), T(v_2), \dots, T(v_n)\})$$

.

Proof.

Clearly, $T(v_i) \in R(T)$

$\therefore R(T)$ is a subspace \therefore By thm 1.5 : $\text{span}(T(\beta)) \subseteq R(T)$

Let $w \in R(T)$

$\therefore w = T(v)$ for some $v \in V$ $\therefore \beta$ is a basis of V

$$\therefore v = \sum_{i=1}^n a_i v_i \text{ for some } a_i \in F, \quad 1 \leq i \leq n$$

$$\therefore T \text{ is linear } \therefore w = T(v) = T\left(\sum_{i=1}^n a_i v_i\right) = \sum_{i=1}^n a_i T(v_i) \in \text{span}(T(\beta))$$

$$\therefore R(T) \subseteq \text{span}(T(\beta)) \implies R(T) = \text{span}(T(\beta)) = \text{span}(\{T(v_1), \dots, T(v_n)\})$$

■

Example. Prove Theorem 2.2 for the case that β is infinite, that is, $R(T) = \text{span}(\{T(v) : v \in \beta\})$.

Definition (Nullity and Rank). Let V and W be vector spaces and let $T : V \rightarrow W$ be linear. If $N(T)$ and $R(T)$ are finite-dimensional, then we define the nullity of T , denoted $\text{nullity}(T)$, and the rank of T , denoted $\text{rank}(T)$, to be the dimensions of $N(T)$ and $R(T)$, respectively.

Theorem 2.4. Let V and W be vector spaces, and let $T : V \rightarrow W$ be linear. If V is finite-dimensional, then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V).$$

Proof.

Suppose $\dim(V) = n$, $\dim(N(T)) = k$ and $\{v_i : i = 1, \dots, k\}$ is a basis of $N(T)$

\therefore Cor of thm 1.11 \therefore We can extend $\{v_i : \text{for } i = 1, \dots, k\}$ to a basis $\beta = \{v_i : \text{for } i = 1, \dots, k\}$ of V

Claim : $S = \{T(v_i) : \text{for } i = k+1, \dots, n\}$ is a basis of $R(T)$

1. Claim $\text{span}(S) = R(T)$
 \because Thm 2.2 $\therefore R(T) = \text{span}(T(\beta)) = \text{span}(\{T(v_1), \dots, T(v_n)\})$
 $\because T(v_i) = 0$ for $i = 1, \dots, k$
 $\therefore R(T) = \text{span}(\{T(v_{k+1}), \dots, T(v_n)\}) = \text{span}(S)$
2. Claim : S is linear independent
 Suppose $\sum_{i=k+1}^n b_i T(v_i) = 0 \quad \forall b_{k+1}, \dots, b_n \in F$
 $\because T$ is linear $\therefore T(\sum_{i=k+1}^n b_i v_i) = 0 \implies \sum_{i=k+1}^n b_i v_i \in N(T)$
 Hence, $\exists c_i \in F$ for $i = 1, \dots, k$

$$\implies \sum_{i=k+1}^n b_i v_i = \sum_{i=k+1}^n c_i v_i \implies \sum_{i=k+1}^n b_i v_i + \sum_{i=k+1}^n (-c_i) v_i = 0$$
 $\because \beta$ is a basis of $V \therefore b_i, \forall_i = k+1, \dots, n = 0$
 $\therefore S^n$ is linear independent
 $S = \{T(v_i) : \text{for } i = k+1, \dots, n\} \implies \text{rank}(T) = n - k$

■

Theorem 2.5. Let V and W be vector spaces, and let $T : V \rightarrow W$ be linear. Then T is one-to-one if and only if $N(T) = \{0\}$.

Proof. T is one-to-one $\iff N(T) = \{0\}$

\Rightarrow

Suppose T is one-to-one and $x \in N(T)$

$$T(x) = 0 = T(0)$$

$\because T$ is one-to-one $\therefore x = 0$

$$\implies N(T) = \{0\}$$

\Leftarrow

Suppose $N(T) = \{0\}$ and $T(x) = T(y)$

$$0 = T(x) - T(y) = T(x - y)$$

$\because N(T) = \{0\} \therefore x - y = 0 \therefore x = y$

$\implies T$ is one-to-one

$$\implies T \text{ is one-to-one} \iff N(T) = \{0\}$$

■

Propersition 2.6. Let V and W be finite-dimensional vector spaces and $T : V \rightarrow W$ be linear.

(a) Prove that if $\dim(V) < \dim(W)$, then T cannot be onto.

(b) Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.

Definition (Invertible). Let V and W be vector spaces, and let $T : V \rightarrow W$ be linear. A function $U : W \rightarrow V$ is said to be an inverse of T if $TU = I_W$ and $UT = I_V$. If T has an inverse, then T is said to be invertible. As noted in Appendix B, if T is invertible, then the inverse of T is unique and is denoted by T^{-1} .

Proposition 2.7.

1. $(TU)^{-1} = U^{-1}T^{-1}$.
2. $(T^{-1})^{-1} = T$; in particular, T^{-1} is invertible.

Definition. Let A be an $n \times n$ matrix. Then A is invertible if there exists an $n \times n$ matrix B such that $AB = BA = I$.

If A is invertible, then the matrix B such that $AB = BA = I$ is unique. (If C were another such matrix, then $C = CI = C(AB) = (CA)B = IB = B$.) The matrix B is called the inverse of A and is denoted by A^{-1} .

Definition (Isomorphism). Let V and W be vector spaces. We say that V is isomorphic to W if there exists a linear transformation $T : V \rightarrow W$ that is invertible. Such a linear transformation is called an isomorphism from V onto W .

Let \sim mean "is isomorphic to." Prove that \sim is an equivalence relation on the class of vector spaces over F .

Theorem 2.8. Let V and W be finite-dimensional vector spaces (over the same field). Then V is isomorphic to W if and only if $\dim(V) = \dim(W)$.

Corollary 2.8.1. Let V be a vector space over F . Then V is isomorphic to F^n if and only if $\dim(V) = n$.

Remark. The Linearity and Finite-dimensional is essential.

Example. Recall the definition of $P(R)$ on page 10. Define

$$T : P(R) \rightarrow P(R) \text{ by } T(f(x)) = \int_0^x f(t)dt.$$

Prove that T linear and one-to-one, but not onto.

Example. Let $T : P(R) \rightarrow P(R)$ be defined by $T(f(x)) = f'(x)$. Recall that T is linear. Prove that T is onto, but not one-to-one.

Example. Let V be the vector space of sequences described in Example 5 of Section 1.2. Define the functions $T, U : V \rightarrow V$ by

$$T(a_1, a_2, \dots) = (a_2, a_3, \dots) \quad \text{and} \quad U(a_1, a_2, \dots) = (0, a_1, a_2, \dots).$$

T and U are called the left shift and right shift operators on V , respectively.

- (a) Prove that T and U are linear.
- (b) Prove that T is onto, but not one-to-one.
- (c) Prove that U is one-to-one, but not onto.

Theorem 2.9. Let V and W be vector spaces, and suppose that V has a finite basis $\{v_1, v_2, \dots, v_n\}$. If $U, T : V \rightarrow W$ are linear and $U(v_i) = T(v_i)$ for $i = 1, 2, \dots, n$ then $U = T$.

Proof. Suppose $U(v_i) = T(v_i) = w_i$ for $i = 1, \dots, n$

Let $x \in U, V$ and $x = \sum_{i=1}^n a_i v_i$ where $a_i \in F$

$$U(x) = \sum_{i=1}^n a_i U(v_i) = \sum_{i=1}^n a_i T(v_i) = T(x)$$

$$\implies U = T$$

■

Propersition 2.10. Let V and W be vector spaces and $T : V \rightarrow W$ be linear.

- (a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W .
- (b) Suppose that T is one-to-one and that S is a subset of V . Prove that S is linearly independent if and only if $T(S)$ is linearly independent.
- (c) Suppose $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), \dots, T(v_n)\}$ is a basis for W .

§ Matrix Representation

Definition. Let V be a finite-dimensional vector space. An ordered basis for V is a basis for V endowed with a specific order; that is, an ordered basis for V is a finite sequence of linearly independent vectors in V that generates V .

Definition. Let $\beta = \{u_1, u_2, \dots, u_n\}$ be an ordered basis for a finite-dimensional vector space V . For $x \in V$, let a_1, a_2, \dots, a_n , be the unique scalars such that

$$x = \sum_{i=1}^n a_i u_i$$

We define the coordinate vector of x relative to β , denoted $[x]_\beta$, by

$$[x]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Remark. Let V be an n -dimensional vector space with an ordered basis β . Define $T : V \rightarrow F^n$ by $T(x) = [x]_\beta$. Prove that T is linear.

Definition. Using the notation above, we call the $m \times n$ matrix A defined by $A_{ij} = a_{ij}$ the matrix representation of T in the ordered bases β and γ and write $A = [T]_\beta^\gamma$. If $V = W$ and $\beta = \gamma$, then we write $A = [T]_\beta$. Notice that the j th column of A is simply $[T(v_j)]_\gamma$. Also observe that if $U : V \rightarrow W$ is a linear transformation such that $[U]_\beta^\gamma = [T]_\beta^\gamma$, then $U = T$ by the corollary to Theorem 2.6 (p. 73).

Theorem 2.11. Let V and W be finite-dimensional vector spaces with ordered bases β and γ , respectively, and let $T, U : V \rightarrow W$ be linear transformations. Then

- (a) $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$ and
- (b) $[aT]_\beta^\gamma = a[T]_\beta^\gamma$ for all scalars a .

Proof. Let $\beta = \{v_j \mid 1 \leq j \leq n\}$ and $\gamma = \{w_i \mid 1 \leq i \leq m\} \implies \exists! a_{ij}, b_{ij} \in F$ for $1 \leq j \leq n, 1 \leq i \leq m$ s.t.

$$1. T(v_j) = \sum_{i=1}^m a_{ij}w_i \text{ and } U(v_j) = \sum_{i=1}^m b_{ij}w_i, \text{ hence}$$

$$(T + U)(v_j) = \sum_{i=1}^m (a_{ij} + b_{ij})w_i$$

$$\begin{aligned} \text{Thus } ([T + U]_\beta^\gamma)_{ij} &= a_{ij} + b_{ij} = ([T]_\beta^\gamma)_{ij} + ([U]_\beta^\gamma)_{ij} = ([T]_\beta^\gamma + [U]_\beta^\gamma)_{ij} \\ \implies [T + U]_\beta^\gamma &= [T]_\beta^\gamma + [U]_\beta^\gamma \end{aligned}$$

$$\begin{aligned} 2. T(V_j) &= \sum_{i=1}^m a_{ij}w_i \text{ for } j = 1, \dots, n \text{ and } a \in F \text{ then } aT(v_j) = a \sum_{i=1}^m a_{ij}w_i = \\ &\sum_{i=1}^m aa_{ij}w_i = T(av_j) \text{ thus } ([aT]_\beta^\gamma)_{jj} = aa_{ij} = (a[T]_\beta^\gamma)_{ij} \implies [aT]_\beta^\gamma = \\ &a[T]_\beta^\gamma \end{aligned}$$

■

Theorem 2.12. Let V , W , and Z be vector spaces over the same field F , and let $T : V \rightarrow W$ and $U : W \rightarrow Z$ be linear. Then $UT : V \rightarrow Z$ is linear.

Proof. Let $x, y \in V$, $a \in F$ $UT(ax+y) = U(T(ax+y)) = U(aT(x)+T(y)) = aUT(x) + UT(y) \Rightarrow UT$ is linear. ■

Theorem 2.13. Let V be a vector space. Let $T, U_1, U_2 \in L(V)$. Then

$$(a) T(U_1 + U_2) = TU_1 + TU_2 \text{ and } (U_1 + U_2)T = U_1T + U_2T$$

$$(b) T(U_1U_2) = (TU_1)U_2$$

$$(c) TI = IT = T$$

$$(d) a(U_1U_2) = (aU_1)U_2 = U_1(aU_2) \text{ for all scalars } a.$$

Proof. Let $x \in V$

$$\begin{aligned} 1. T(U_1 + U_2)(x) &= T(U_1(x) + U_2(x)) = TU_1(x) + TU_2(x) \Rightarrow T(U_1 + U_2) = \\ &TU_1 + TU_2 \quad (U_1 + U_2)T(x) = U_1T(x) + U_2T(x) \Rightarrow (U_1 + U_2)T = U_1T + \\ &U_2T \end{aligned}$$

$$2. T(U_1U_2)(x) = T(U_1(U_2(x))) = (TU_1)(U_2(x)) \Rightarrow T(U_1U_2) = (TU_1)U_2$$

3. $TI(x) = T(x) = IT(x) \Rightarrow TI = T = IT$
4. $a(U_1U_2)(x) = a(U_1(U_2(x))) = (aU_3)(U_2(x)) = U_1(aU_2(x)) \Rightarrow a(U_1U_2) = (aU_1)U_2 = U_1(aU_2)$

■

Definition (The Product Of Two Matrices). *Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. We define the product of A and B , denoted AB , to be the $m \times p$ matrix such that*

$$(AB)_{ij} = \sum_{k=1}^n A_{ik}B_{kj} \quad \text{for } 1 \leq i \leq m, \quad 1 \leq j \leq p.$$

Theorem 2.14. *Let V , W , and Z be finite-dimensional vector spaces with ordered bases α, β , and γ , respectively. Let $T : V \rightarrow W$ and $U : W \rightarrow Z$ be linear transformations. Then*

$$[UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma}[T]_{\alpha}^{\beta}$$

.

Proof. Let $\alpha = \{v_i \mid i = 1 \cdots n\}$, $\beta = \{w_j \mid j = 1 \cdots m\}$, $\gamma = \{z_k \mid k = 1 \cdots p\}$
 $\implies \exists! a_{ij}, b_{ki} \in F$ for $j = 1, \dots, m$ $i = 1, \dots, n$ and $k = 1, \dots, p$ such that

$$T(v_i) = \sum_{j=1}^m a_{ij}w_j \quad U(w_j) = \sum_{k=1}^p b_{kj}z_k$$

$$\begin{aligned} UT(v_i) &= U(T(v_i)) = U\left(\sum_{j=1}^m a_{ij}w_j\right) = \sum_{j=1}^m a_{ij}U(w_j) = \sum_{j=1}^m a_{ij}\left(\sum_{k=1}^p b_{kj}z_k\right) \\ &= \sum_{k=1}^p \sum_{j=1}^m b_{kj}a_{ji}z_k \quad \text{for } i = 1, \dots, n \\ &\implies ([UT]_{\alpha}^{\gamma})_{ij} = \sum_{k=1}^p \sum_{j=1}^m b_{kj}a_{ji} = ([U]_{\beta}^{\gamma}[T]_{\alpha}^{\beta})_{ij} \implies [UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma}[T]_{\alpha}^{\beta} \end{aligned}$$

■

Corollary 2.14.1. *Let V be a finite-dimensional vector space with an ordered basis β . Let $T, U \in L(V)$. Then $[UT]_\beta = [U]_\beta[T]_\beta$.*

Proof. **not yet** ■

Definition. We define the Kronecker delta δ_{ij} by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The $n \times n$ identity matrix I_n is defined by $(I_n)_{ij} = \delta_{ij}$. Thus, for example,

$$I_1 = [1] \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem 2.15. *Let A be an $m \times n$ matrix, B and C be $n \times p$ matrices, and D and E be $q \times m$ matrices. Then*

$$(a) \quad A(B+C) = AB + AC \text{ and } (D+E)A = DA + EA.$$

$$(b) \quad a(AB) = (aA)B = A(aB) \text{ for any scalar } a.$$

$$(c) \quad I_m A = A = A I_n.$$

$$(d) \quad \text{If } V \text{ is an } n\text{-dimensional vector space with an ordered basis } \beta, \text{ then } [I_V]_\beta = I_n.$$

not input now

Corollary 2.15.1. *Let A be an $m \times n$ matrix, B_1, B_2, \dots, B_k be $n \times p$ matrices, C_1, C_2, \dots, C_k be $q \times m$ matrices, and a_1, a_2, \dots, a_k be scalars. Then*

$$A\left(\sum_{i=1}^k a_i B_i\right) = \sum_{i=1}^k a_i A B_i$$

and

$$\left(\sum_{i=1}^k a_i C_i\right)A = \sum_{i=1}^k a_i C_i A.$$

not input now

With this notation, we see that if

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

then $A^2 = O$ (the zero matrix) even though $A \neq O$. Thus the cancellation property for multiplication in fields is not valid for matrices. To see why, assume that the cancellation law is valid. Then, from $A \cdot A = A^2 = O = A \cdot O$, we would conclude that $A = O$, which is false.

Theorem 2.16. *Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. For each j ($1 \leq j \leq p$) let u_j and v_j denote the j th columns of AB and B , respectively. Then*

$$(a) \ u_j = Av_j$$

$$(b) \ v_j = Be_j, \text{ where } e_j \text{ is the } j\text{th standard vector of } F^p.$$

Proof. Let $A \in M_{m \times n}(F)$, $b \in M_{n \times p}(F)$ for $j = 1, \dots, p$. Let u_j, v_j be the j th columns of AB and B respectively, then

$$1. \ u_j = v_j$$

$$2. \ v_j = Be_j \text{ where } e_j \text{ is the } j\text{th standard. vector of } F^n$$

$$1. \ u_j = \begin{bmatrix} (AB)_{1j} \\ \vdots \\ (AB)_{mj} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^n A_{1k}B_{kj} \\ \vdots \\ \sum_{k=1}^n A_{mk}B_{kj} \end{bmatrix} = A \begin{bmatrix} B_{1j} \\ \vdots \\ B_{nj} \end{bmatrix} = Av_j$$

$$2. \ v_j = \begin{bmatrix} B_{1j} \\ \vdots \\ B_{nj} \end{bmatrix} = \begin{bmatrix} (BI)_{1j} \\ \vdots \\ (BI)_{nj} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^p B_{1k}I_{kj} \\ \vdots \\ \sum_{k=1}^p B_{nk}I_{kj} \end{bmatrix} = B \begin{bmatrix} I_{1j} \\ \vdots \\ I_{pj} \end{bmatrix} = Be_j$$

■

Theorem 2.17. *Assume the notation in Theorem 2.13.*

(a) *Suppose that z is a (column) vector in F^p . Use Theorem 2.13(b) to prove that Bz is a linear combination of the columns of B . In particular, if $z = (a_1, a_2, \dots, a_p)^t$, then show that*

$$Bz = \sum_{j=1}^p a_j v_j.$$

- (b) Extend (a) to prove that column j of AB is a linear combination of the columns of A with the coefficients in the linear combination being the entries of column j of B .
- (c) For any row vector $w \in F^m$, prove that wA is a linear combination of the rows of A with the coefficients in the linear combination being the coordinates of w . Hint: Use properties of the transpose operation applied to (a).
- (d) Prove the analogous result to (b) about rows: Row i of AB is a linear combination of the rows of B with the coefficients in the linear combination being the entries of row i of A .

not yet

Theorem 2.18. Let V and W be finite-dimensional vector spaces having ordered bases β and γ , respectively, and let $T : V \rightarrow W$ be linear. Then, for each $u \in V$, we have

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$$

Proof. Fix $u \in V$ define the linear transformation $T : F \rightarrow V$ by $f(a) = au$
 $g : F \rightarrow W$ by $g(a) = aT(u) \forall a \in F$ Let $\alpha = 1$ be the standard ordered basis of F , note that $g(a) = aT(u) = T(au) = T(f(a)) \implies g = Tf$, then

$$[T(u)]_\gamma = [g(1)]_\gamma = [g]_\alpha^\gamma [1]_\alpha = [g]_\alpha^\gamma = [Tf]_\alpha^\gamma = [T]_\beta^\gamma [f]_\alpha^\beta = [T]_\beta^\gamma [f(1)]_\beta = [T]_\beta^\gamma [u]_\beta$$

■

Definition. Let A be an $m \times n$ matrix with entries from a field F . We denote by L_A the mapping $L_A : F^n \rightarrow F^m$ defined by $L_A(x) = Ax$ (the matrix product of A and x) for each column vector $x \in F^n$. We call L_A a left-multiplication transformation.

Theorem 2.19. The characteristics of Left-Multiplication Transformation
 Let A be an $m \times n$ matrix with entries from F . Then the left-multiplication transformation $L_A : F^n \rightarrow F^m$ is linear. Furthermore, if B is any other $m \times n$ matrix (with entries from F) and β and γ are the standard ordered bases for F^n and F^m , respectively, then we have the following properties.

(a) $[L_A]_\beta^\gamma = A.$

- (b) $L_A = L_B$ if and only if $A = B$.
- (c) $L_{A+B} = L_A + L_B$ and $L_{aA} = aL_A$ for all $a \in F$.
- (d) If $T : F^n \rightarrow F^m$ is linear, then there exists a unique $m \times n$ matrix C such that $T = L_C$. In fact, $C = [T]_\beta^\gamma$.
- (e) If E is an $n \times p$ matrix, then $L_{AE} = L_AL_E$.
- (f) If $m = nL$, then $L_{I_n} = I_{F^n}$.

Proof. 1. The j th column of $[L_A]_\beta^\gamma = L_A(e_j) \because$ Thm 2.13 $\therefore L_A(e_j) = Ae_j$ is the j th column of $A \implies [L_A]_\beta^\gamma = A$

2. (\implies) If $L_A = L_B \implies A = [L_A]_\beta^\gamma = [L_B]_\beta^\gamma = B$
 (\impliedby) If $A = B \implies L_A(x) = Ax = Bx = L_B(x) \implies L_A = L_B$

3. $L_{A+B}(x) = (A+B)(x) = Ax + Bx = L_A(x) + L_B(x)$
 $\implies L_{A+B} = L_A + L_B$
 $L_{aA}(x) = (aA)x = aAx = aL_A(x) \implies L_{aA} = aL_A \forall a \in F$

4. Let $C = [T]_\beta^\gamma$ by Thm 2.14 $[T(x)]_\gamma = [T]_\beta^\gamma[x]_\beta \Leftrightarrow T(x) = Cx = L_C(x) \forall x \in F^n \implies T = L_C$ by (2) C is unique.

5. For $j = 1, \dots, p$ by Thm 2.13 $(AE)e_j$ is the j th column of AE , $(AE)e_j = A(Ee_j) \therefore L_{AE}(e_j) = (AE)e_j = A(Ee_j) = L_A(Ee_j) = L_A(L_E(e_j)) = L_AL_Ee_j$ By cor of Thm 2.6 $\therefore L_{AE} = L_AL_E$

6. $L_{I_n}(x) = I_nx = x \forall x \in F^n, I_{F^n}(x) = x \implies L_{I_n} = I_{F^n}$ ■

Corollary 2.19.1. Let V be a finite-dimensional vector space with an ordered basis β , and let $T : V \rightarrow V$ be linear. Then T is invertible if and only if $[T]_\beta$ is invertible. Furthermore, $[T^{-1}]_\beta = ([T]_\beta)^{-1}$

Proof. **not yet** ■

Corollary 2.19.2. Let A be an $n \times n$ matrix. Then A is invertible if and only if L_A is invertible. Furthermore, $(L_A)^{-1} = L_{A^{-1}}$

Proof. **not now** ■

Theorem 2.20. *Let A, B , and C be matrices such that $A(BC)$ is defined. Then $(AB)C$ is also defined and $A(BC) = (AB)C$; that is, matrix multiplication is associative.*

Proof. Let A, B and C be metrics $\ni A(BC)$ is defined $\implies (AB)C$ is also defined and $A(BC) = (AB)C$

Let $B \in M_{m \times n}$ $C \in M_{n \times p} \implies (BC) \in M_{m \times p} \implies A \in M_{q \times m}$

$\therefore (AB) \in M_{q \times n}$ $C \in M_{n \times p} \implies (AB)C$ is defined

\therefore By Thm 2.15.5 $\therefore L_{A(BC)} = L_A L_{BC} = L_A (L_B L_C) = (L_A L_B) L_C = L_{AB} L_C = L_{(AB)C}$

\therefore By Thm 2.15.2 $\therefore A(BC) = (AB)C$ ■

Lemma. *Let T be an invertible linear transformation from V to W . Then V is finite-dimensional if and only if W is finite-dimensional. In this case, $\dim(V) = \dim(W)$.*

Proof. Let linear transform $T : V \rightarrow W$ is invertible $\implies V$ is finite dimensional $\Leftrightarrow W$ is finite dimensional

In this case, $\dim(V) = \dim(W)$

$(\Rightarrow) V$ is finite dimension

Let $\beta = \{x_i \mid \text{for } i = 1, \dots, n\}$ be a basis of V

\therefore Thm 2.2 $\therefore R(T) = \text{span}(T(\beta))$, $\therefore T$ is invertible $\therefore R(T) = W$, \therefore Thm 1.9

$\therefore W$ is finite dimensional

$(\Leftarrow) W$ is finite dimensional. Similarly, V is finite dimensional

Suppose V, W are finite dimensional $\therefore T$ is invertible $\Leftrightarrow T$ is one-to-one and onto, $\therefore \text{nullity}(T) = 0$

\therefore By Thm 2.3 $\text{nullity}(T) + \text{rank}(T) = \dim(V)$ $\therefore \dim(V) = \text{rank}(T) = \dim(R(T)) = \dim(W)$ ■

Theorem 2.21. *Let V and W be finite-dimensional vector spaces with ordered bases β and γ , respectively. Let $T : V \rightarrow W$ be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Furthermore, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.*

Proof. Let V, W is finite dimensional vector space, β, γ be a basis of V, W , $T : V \rightarrow W$ is a linear transform $\implies T$ is invertible $\Leftrightarrow [T]_\beta^\gamma$ is invertible.

Furthermore, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$

(\Rightarrow) Suppose T is invertible \therefore lemma of Thm 2.17, we have $\dim(V) = \dim(W)$. Let $n = \dim(V)$ $\therefore [T]_\beta^\gamma \in M_{n \times n}$, $T^{-1} : W \rightarrow V$ satisfies $TT^{-1} = I_W$ and $T^{-1}T = I_V$ $\therefore I_n = [I_V]_\beta = [T^{-1}T]_\beta = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma$

$I_n = [I_W]_\gamma = [TT^{-1}]_\gamma = [T]_\beta^\gamma [T^{-1}]_\gamma^\beta$
 $\therefore [T]_\beta^\gamma [T^{-1}]_\gamma^\beta = I_n$ and $[T^{-1}]_\gamma^\beta [T]_\beta^\gamma = I_n \implies [T]_\beta^\gamma$ is invertible and $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$
 (\Leftarrow) Suppose $A = [T]_\beta^\gamma$ is invertible $\implies \exists B \in M_{n \times n} \ni AB = BA = I_n, \therefore$

By Thm 2.6 $\therefore \exists U \in \mathcal{L}(W, V) \ni U(w_j) = \sum_{i=1}^n B_{ij}v_i$ for $j = 1, \dots, n$ where

$\gamma = \{w_i \mid \text{for } i = 1, \dots, n\}$ and $\beta = \{v_i \mid \text{for } i = 1, \dots, n\} \therefore B = [U]_\gamma^\beta$

Claim $U = T^{-1}, \therefore [UT]_\beta = [U]_\gamma^\beta [T]_\beta^\gamma = BA = I_n = [I_V]_\beta$

$\therefore UT = I_V$ and $TU = I_W$ similarly $\implies U = T^{-1}$ ■

Theorem 2.22. Let β and β' be two ordered bases for a finite-dimensional vector space V , and let $Q = [I_V]_{\beta'}^\beta$. Then

(a) Q is invertible.

(b) For any $v \in V$, $[v]_\beta = Q[v]_{\beta'}$.

Proof. Let β, β' be order basis of finite dimensional vector space V . Let $Q = [I_V]_{\beta'}^\beta$

1. Q is invertible, \therefore By Thm 2.18 I_V is invertible $\implies [I_V]_{\beta'}^\beta$ is invertible $\therefore Q$ is invertible
2. $\forall v \in V, [v]_\beta = Q[v]_{\beta'} \therefore \forall v \in V$

$$[v]_\beta = [I_V(v)]_\beta = [I_V]_{\beta'}^\beta [v]_{\beta'} = Q[v]_{\beta'}$$

■

Theorem 2.23. Let T be a linear operator on a finite-dimensional vector space V , and let β and β' be ordered bases for V . Suppose that Q is the change of coordinate matrix that changes β' -coordinates into β -coordinates. Then

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q.$$

Proof. Let T be linear operator on V which is a finite dimensional vector space, β, β' be order basis of V , Q be a change coordinate matrix from β' to $\beta \implies [T]_{\beta'} = Q^{-1}[T]_\beta Q$

\therefore By Thm 2.11 $\therefore Q[T]_{\beta'} = [I]_{\beta'}^\beta [T]_{\beta'} = [IT]_{\beta'}^\beta = [T]_{\beta'} [I]_{\beta'}^\beta = [T]_\beta Q \implies [T]_{\beta'} = Q^{-1}[T]_\beta Q$ ■

Corollary 2.23.1. *Let $A \in M_{n \times n}(F)$, and let γ be an ordered basis for F^n . Then $[L_A] = Q^{-1}AQ$, where Q is the $n \times n$ matrix whose j th column is the j th vector of γ .*

Proof. Let $A \in M_{n \times n}(F)$ and γ be an order basis of $F^n \implies [L_A]_\gamma = Q^{-1}AQ$ where Q is the $n \times n$ matrix whose j th column is the j th vector of γ
 Let β be an order basis of F^n , \therefore By Thm 2.15 $\therefore [L_A]_\beta = A \therefore$ By Thm 2.23 $\therefore [L_A]_\gamma = Q^{-1}[L_A]_\beta Q = Q^{-1}AQ$ where $Q = [I]_\gamma^\beta$ ■

Definition. *Let A and B be matrices in $M_{n \times n}(F)$. We say that B is similar to A if there exists an invertible matrix Q such that $B = Q^{-1}AQ$.*

Theorem 2.24. *"is similar to" is an equivalence relation on $M_{n \times n}(F)$*

not yet

Propersition 2.25. *if A and B are similar $n \times n$ matrices, then $\text{tr}(A) = \text{tr}(B)$.*

not yet

Definition. *Let V and W be vector space over F . We denote the vector space of all linear transformations from V into W by $\mathcal{L}(V, W)$. In the case that $V = W$, we write $\mathcal{L}(V)$ instead of $\mathcal{L}(V, W)$*

Theorem 2.26. *Let V and W be finite-dimensional vector spaces over F of dimensions n and m , respectively, and let β and γ be ordered bases for V and W , respectively. Then function $\Phi : \mathcal{L}(V, W) \rightarrow M_{m \times n}(F)$, defined by $\Phi(T) = [T]_\beta^\gamma$ for $T \in \mathcal{L}(V, W)$, is an isomorphism.*

Corollary 2.26.1. *Let V and W be finite-dimensional vector spaces of dimensions n and m , respectively. Then $\mathcal{L}(V, W)$ is finite-dimensional of dimension mn .*

Lemma 2.27. *Let V and W be finite-dimensional vector spaces, and let $T : V \rightarrow W$ be a linear transformation. Suppose that β is a basis for V . Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W .*

not yet

Theorem 2.28. Let V and W be finite-dimensional vector spaces with ordered bases $\beta = \{v_1, v_2, \dots, v_n\}$ and $\gamma = \{w_1, w_2, \dots, w_m\}$, respectively. By Thm 2.6, there exist linear transformations $T_{ij} : V \rightarrow W$ such that

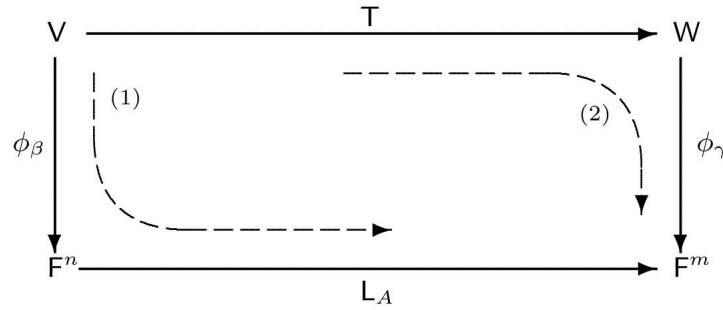
$$T_{ij}(v_k) = \begin{cases} w_i & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

First prove that $\{T_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis for $\mathcal{L}(V, W)$. Then let M^{ij} be the $m \times n$ matrix with 1 in the i th row and j th column and 0 elsewhere, and prove that $[T_{ij}]_{\beta}^{\gamma} = M^{ij}$. Again by Thm 2.6, there exists a linear transformation $\Phi : \mathcal{L}(V, W) \rightarrow M_{m \times n}(F)$ such that $\Phi(T_{ij}) = M^{ij}$. Prove that Φ is an isomorphism.

not yet

Definition. Let β be an ordered basis for an n -dimensional vector space V over the field F . The standard representation of V with respect to β is the function $\phi_{\beta} : V \rightarrow F^n$ defined by $\phi_{\beta}(x) = [x]_{\beta}$ for each $x \in V$.

Theorem 2.29. Let V and W be vector spaces and $T : V \rightarrow W$ be linear. Then $N(T)$ and $R(T)$ are subspace of V and W , respectively.



Let V and W be vector spaces of dimension n and m , respectively, and let $T : V \rightarrow W$ be a linear transformation. Define $A = [T]_{\beta}^{\gamma}$, where β and γ are arbitrary ordered bases of V and W , respectively. We are now able to use ϕ_{β} and ϕ_{γ} to study the relationship between the linear transformations T and $L_A : F^n \rightarrow F^m$. Let us first consider figure above. Notice that there are two composites of linear transformations that map V into F^m :

1. Map V into F^n with ϕ_{β} and follow this transformation with L_A ; this yields the composite $L_A \phi_{\beta}$.

2. Map V into W with T and follow it by ϕ_γ to obtain the composite $\phi_\gamma T$.

These two composites are depicted by the dashed arrows in the diagram. By a simple reformulation of Theorem 2.14 (p. 91), we may conclude that

$$L_A \phi_\beta = \phi_\gamma T$$

that is, the diagram "commutes." Heuristically, this relationship indicates that after V and W are identified with F^n and F^m via ϕ_β and ϕ_γ , respectively, we may "identify" T with L_A . This diagram allows us to transfer operations on abstract vector spaces to ones on F^n and F^m .

Theorem 2.30. *Let $T : V \rightarrow W$ be a linear transformation from an n -dimensional vector space V to an m -dimensional vector space W . Let β and γ be ordered bases for V and W , respectively. Prove that $\text{rank}(T) = \text{rank}(L_A)$ and that $\text{nullity}(T) = \text{nullity}(L_A)$, where $A = [T]_\beta^\gamma$.*

not yet