

0.0.1. In calculus.

- (1) Extreme Value Theorem: Every continuous function $f : [a, b] \rightarrow \mathbb{R}$ admit both max and min value \Rightarrow Compact set
- (2) Intermediate value Theorem: Given continuous function $f : [a, b] \rightarrow \mathbb{R}$ for all $f(a) \leq \lambda \leq f(b) \exists c \in [a, b] \ni f(c) = \lambda \Rightarrow$ connected set

How to prove a statement: HP , then $Q, P \Rightarrow Q$

$$\left\{ \begin{array}{l} \text{Direct Proof} \\ \text{Indirect Proof} \left\{ \begin{array}{l} \text{contrapositive } \sim Q \Rightarrow \sim P \\ \text{by contradiction} \end{array} \right. \\ \text{Mathematical Induction} \end{array} \right.$$

1. Some preliminary

1.1. Set Theory. We will assume that you are familiar with some basic set theory e.g. union, intersection, difference

1.2. The Number System.

$\mathbb{N} = \{1, 2, 3, \dots\}$ the set of all positive integers n natural numbers

$\mathbb{Z} = \{\dots, -2, -1, 0, -1, -2, \dots\}$ the set of all integers called the ring of integers

$\mathbb{Q} = \left\{ \frac{m}{n} : n, m \in \mathbb{Z}, n \neq 0 \right\}$ the set of all rational numbers

\mathbb{R} the set all of real numbers on the real number field on real line

$\mathbb{C} = \{z = a + ib \mid a, b \in \mathbb{R}\}$ the set of all complex numbers or the complex number field on complex plane, where $i = \sqrt{-1}$

Remark.

- (1) $x + 2 = 0$ no root in \mathbb{N}
 $3x - 5 = 0$ no root in \mathbb{Z}
 $x^2 + 1 = 0$ no root in \mathbb{R}
- (2) One can construct \mathbb{Q} from \mathbb{Z} in algebraic way, called the fraction field of \mathbb{Z}
- (3) One can construct \mathbb{R} from \mathbb{Q} in two ways:
 - Using Dedekind cut which is given in the appendix of Rudin p17-21
 - Using completion of metric space
- (4) One can construct \mathbb{C} from in complex analysis

Example.

- (1) *Between any two rational numbers, there is another one*

Proof. Let $r, s \in \mathbb{Q}$ with $r < s$, then $\frac{r+s}{2} \in \mathbb{Q}$ and $r < \frac{r+s}{2} < s$

$$r = \frac{m_1}{n_1}, s = \frac{m_2}{n_2}, \frac{r+s}{2} = \frac{\frac{m_1}{n_1} + \frac{m_2}{n_2}}{2} = \frac{m_1n_2 + n_1m_2}{2n_1n_2} \in \mathbb{Q}$$

$$s = \frac{s+s}{2} > \frac{r+s}{2} > \frac{r+r}{2} = r$$

■

- (2) $x^2 = \frac{4}{9}$ has exactly two rational solutions, namely, $\pm \frac{2}{3}$

- (3) $x^2 = 2$ has exactly two real roots, namely, $\pm\sqrt{2}$

- (4) Is there any rational roots of $x^2 = 2$? i.e., is $\sqrt{2}$ rational?

Suppose $r = \frac{m}{n} \in \mathbb{Q}$, is a root of $x^2 = 2$, where $(m, n) = 1$

Then $\frac{m^2}{n^2} = 2 \implies m^2 = 2n^2 \implies 2 \mid m^2 \implies 2 \mid m \implies 4 \mid m^2 \implies 4 \mid 2n^2 \implies 2 \mid n^2 \implies 2 \mid n \implies (n, m) \neq 1$

- (5) Let $A = \{r \in \mathbb{Q} \mid r > 0 \text{ \& } r^2 < 2\}$, $B = \{r \in \mathbb{Q} \mid r > 0 \text{ \& } r^2 > 2\}$

Then A contains no largest numbers, i.e. max element & B contains no smallest numbers, i.e. min element

Proof. A contains no largest numbers \Leftrightarrow given $r \in A$, $\exists s \in A \ni s > r$

Now, given $r \in A$, Let $s = r - \frac{r^2 - 2}{r + 2} = \frac{2r + 2}{r + 2}$ (\star_1)

$$\implies s^2 - 2 = \frac{2(r^2 - 2)}{(r + 2)^2} \quad (\star_2)$$

Now, $r \in A, r^2 < 2 \implies r^2 - 2 < 0 \therefore$

$$(\star_1) \& (\star_2) \implies s > r \text{ \& } s^2 < 2 \implies s \in A$$

■

- (6) As you know, in calculus, the sequence $\{1, 1.4, 1.41, 1.414, 1.4142, \dots\}$ does not converge in \mathbb{Q} , but it converges to $\sqrt{2}$ in \mathbb{R}

1.3. Order Sets.

Definition (Relation).

Let X be a nonempty set, relation on X is a subset R of $X \times X = \{ (x, y) \mid x, y \in X \}$

Let R be a relation on X , if $(x, y) \in R$, then we say that x is related to y , and is written as xRy ($x \sim y$)

Definition (Order Set). An ordered set on S , is a relation denoted by " $<$ " on S , satisfy:

(i) The law of trichotomy

Given $x, y \in S$, one and only one of the following holds:

$$x < y, x = y, y < x$$

(ii) Transitivity: if $x < y$ & $y < z$, then $x < z$

Notation

(1) $x < y$ means " x is less than y " or " x is smaller than y "

(2) $y > x$ means $x < y$

(3) $x \leq y$ means $x < y$ or $x = y$, i.e. the negative of $x > y$

Definition (bdd). Let S is an ordered set & $E \subseteq S$ ($E \neq \emptyset$)

- E is bounded above if $\exists \alpha \in S \implies x \leq \alpha \forall x \in E$
such α is called an upper bound of E
- E is bounded below if $\exists \beta \in S \ni \beta \leq x, \forall x \in E$, such β is called a lower bdd of E
- E is bdd if E is both bdd above and below.

Definition (least upper bound). Let S be an ordered set and $E \subseteq S$ ($E \neq \emptyset$) bdd above. An element $\alpha \in S$ is called the last upper bound or supremum of E if

(i) α is an upper bound of E

(ii) α is the smallest such one.

Equivalently,

(i') $x \leq \alpha, \forall x \in E$

(ii') if $\beta < \alpha$, then β is not an upper bdd of E , i.e. $\exists x \in E \ni x > \beta$

Such α (if exists) is denoted by

$$\alpha = \sup(E)$$

similarly, one can define the greatest lower bdd of infimum of E

Remark. if $\sup(E)$ exists then it is unique
suppose $\alpha \neq \alpha'$ both lub of E

\therefore by trichotomy, $\alpha > \alpha'$ or $\alpha = \alpha'$ or $\alpha < \alpha'$ ($\rightarrow \leftarrow$)

Definition (least upper bdd property). A ordered set S is said to have the least upper bdd property if $E \subseteq S$, $E \neq \emptyset$ and E is bdd above, then $\sup(E)$ exists in S

Example.

- (1) In \mathbb{Q} with the normal ordining
 $A = \{r \in \mathbb{Q} \mid r > 0, r^2 < 2\}$ & $B = \{r \in \mathbb{Q} \mid r > 0, r^2 > 2\}$
 Then A is bdd above, in fact, bdd by every element in B , but $\sup(A)$ does not exist in \mathbb{Q} (\therefore by Ex1.5)
- (2) B is bdd below by every element of A and $\inf B$ does not exists
- (3) Note that $\sup(E)$ & $\inf(E)$ may not in E even if exist

Remark.

- (1) By the Example above, \mathbb{Q} with the usual ordering has no l.u.b property
- (2) In 1.5 we will explain that \mathbb{R} with usual ordering has the l.u.b. property. However, we usually adopt the follwing

The Axiom of Completeness or Least upper bdd property:

Every nonempty subset E of \mathbb{R} which is bdd above has l.u.b

Theorem (l.u.b.p. \rightarrow g.l.b.p.). Let S is an ordered set if S has the l.u.b. property, then S has the g.l.b. property, i.e. if $\emptyset \neq B \subseteq S$ is bdd below, then $\inf(B)$ exists in S

Proof. (★)

Given $B(\neq \emptyset) \subseteq S$ which is bdd below

Let $L = \{a \in S \mid a \text{ is a lower bdd of } B\}$

- $L \neq \emptyset$ ($\therefore B$ is bdd below)
- L is bdd above (in fact, every element in B is on upper bound of L)
 $\implies \forall a \in L \implies a \leq x, \forall x \in B \implies x \text{ is an upper bound of } L$
- $\sup(L) = \alpha$ exists by assumption

Claim $\alpha = \inf B$

- (i) α is a lower bdd of B , i.e. $\alpha \leq x, \forall x \in B$

By $\alpha = \sup L$, if $r < \alpha$, then r is not an upper bdd of L ($\therefore \alpha$ is the smallest one). Hence, $r \notin B$ (\therefore every element of B is an upper bdd of L), so $\alpha \leq x, \forall x \in B$

We have proved $(r < \alpha \implies r \notin B) \implies (r \in B \implies r \geq \alpha)$

(ii) α is the greatest one

if $\alpha < \beta$ and β is a lower bdd of B , then $\beta \notin L$, i.e. β is not a lower bdd of B , so α is the greatest one. Therefore, $\alpha = \inf(B)$ ■

Remark. Let $E(\neq \emptyset) \subseteq \mathbb{R}$ be bdd below, then $\inf(E)$ exists and $\inf(E) = -\sup(-E)$, where $-E = \{-x \mid x \in E\}$

1.4. Field.

Recall the addition & multiplication in \mathbb{R}

$+: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}((a, b) \mapsto a + b)$

$\times: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}((a, b) \mapsto a \cdot b = ab)$

Definition. Let X is a nonempty set, A , binary operation on X is a function, $o: X \times X \rightarrow X$

Definition. Let F be a nonempty set, we say that F is a field $((F, +, \cdot)$ is a field) if there are two binary operator called addition " $+$ " and multiplication " \cdot " on F property

Axioms for " $+$ "

(A1) Commutative: $\forall x, y \in F, x + y = y + x$

(A2) Associative: $\forall x, y, z \in F, (x + y) + z = x + (y + z)$

(A3) Additive identity or zero element: $\exists 0 \in F \implies x + 0 = 0 + x = x, \forall x \in F$

(A4) Additive inverse on negative: For each $x \in X, \exists -x \in F \implies x + (-x) = (-x) + x = 0$

i.e. $(F, +)$ is an abelian group **Axioms for multiplication**

(M1) Commutative: $\forall x, y \in F, xy = yx$

(M2) Associative: $\forall x, y, z \in F, (xy)z = x(yz)$

(M3) Multi identity: $\exists 1 \neq 0$ in $F \ni x1 = 1x = x$

(M4) Multiplicative inverse: For each $x \neq 0, \exists x^{-1} \in F \implies xx^{-1} = x^{-1}x = 1$

i.e. $(F = F \cdot \{0\}, \cdot)$ is an abelian group

Distributive Law

(D1) $\forall x, y, z \in F, (x, y)z = xz + yz \ \& \ x(y + z) = xy + xz$

Induction from Axioms

let $(F, +, \cdot)$ be a field, we list a series of basic identity as you learn in high school in the real number system

- (a) Cancellation law for "+" : $x + y = x + z \implies y = z$
 $\because x + y = x + z \implies (-x) + (x + y) = (-x) + (x + z) \implies$
 $((-x) + x) + y = ((-x) + x) + z$
 $\implies 0 + y = 0 + z \implies y = z$
- (b) 0 is "1"
 suppose $0' \in F$ is another element satisfy A_3 , then $0 = 0 + 0' = 0'$
- (c) $x + y = x \implies y = 0$ by (a) $\because x + y = x + 0 \implies y = 0$
- (d) negative $-x$ of x is "1"
 if $x' \in F$, is another negative of x , then $x + x' = x' + x = 0$
 From $x + x' = 0 \implies (-x) + (x + x') = -x + 0 = -x$
- (e) $x + y = 0 \implies y = -x$
 $x + y = 0 \implies (-x) + (x + y) = (-x) + 0 \implies ((-x) + x) + y =$
 $-x$
 $\implies 0 + y = -x \implies y = -x$
- (f) $-(-x) = x$
 $-(-x) + (-x) = 0$, By (d) $x = -(-x)$
- (a') cancellation law
 if $x \neq 0$, then $xy = xz \implies y = z$, $\because (x^{-1})(xy) = (x^{-1})(xz)$
 $\implies (x^{-1})(xy) = (x^{-1}x)z \implies 1y = 1z \implies y = z$
- (b') 1 is "1"
 if $1'$ is another identity, then $1 = 11' = 1'$
- (c') $x \neq 0$ & $xy = x \implies y = 1$
 $xy = x1 \implies y = 1$
- (d') For $x \neq 0$ in F , x^{-1} is "1"
 if x is another one, i.e. $x'x = xx' = 1 \implies (x^{-1})(xx') =$
 $(x^{-1})1 = x^{-1}$
- (f') $x \neq 0 \implies (x^{-1})^{-1} = x$
 $(x^{-1})^{-1}(x^{-1}) = 1 \implies x = (x^{-1})^{-1}$
- (g') $0x = x0 = 0$
 $(0 + 0)x = 0x + 0x \implies 0x = 0$
- (h') $x \neq 0$ & $y \neq 0 \implies xy \neq 0$, equivalently $xy = 0 \implies x = 0$ or
 $y = 0$
 $\because xy = 0$ then $(x^{-1})(xy) = ((x^{-1}x)y = 1y = y(\rightarrow \leftarrow)$
- (i') $(-x)y = -(xy) = x(-y)$
 $\because [(-x) + x]y = 0y = 0 = (-x)y = -(xy) \implies (-x)y =$
 $-(xy)$
- (j') $(-x)(-y) = xy$
 $\because (-x)(-y) = -(x(-y))$ by (i)
 $= -(-(xy)) = xy$

$$\begin{aligned}
(k) \quad & -x = (-1)x \\
& \because (1-1)x = 0x = 0 = 1x + (-1)x = x + (-1)x \implies (-1)x = -x
\end{aligned}$$

Definition (Order Field). Let F is a field, we say that F is an order field if there is an ordering " $<$ " satisfying

- (1) if $x < y$, then $x + z < y + z$, $\forall z \in F$
- (2) if $x > y$ and $y > 0$, then $xy > 0$

Example. \mathbb{Q} and \mathbb{R} are order field under the usual ordering
Some basic properties of ordered field, let F be an ordered field with ordering " $<$ "

$$\begin{aligned}
(a) \quad & x > 0 \implies -x < 0 \\
& \because x > 0 \implies x + (-x) > 0 + (-x) \implies 0 > -x \\
(b) \quad & x > y \Leftrightarrow x - y > 0 \\
& \because x > y \implies x + (-y) > y + (-y) \implies x - y > 0 \\
& x - y > 0 \implies x - y + y > y \implies x + 0 > y \implies x > y \\
(c) \quad & x > 0 \text{ and } y < z \implies xy < xz \\
& \because x > 0 \text{ and } y < z \implies x > 0 \text{ and } z - y < 0 \implies x(z - y) > 0 \\
& \implies xz + x(-y) > 0 \\
& \implies xz - xy > 0 \implies xz > xy \\
(d) \quad & x < 0 \text{ and } y < z \implies xy > xz \\
& \because x < 0 \text{ and } y < z \implies -x > 0 \text{ and } z - y > 0 \implies \\
& (-x)(z - y) > 0 \implies -xz + xy > 0 \\
& \implies xy > xz \\
(e) \quad & \forall x \neq 0 \text{ in } F, x^2 > 0 \\
& \because x > 0 \implies x \cdot x > x0 \text{ by (c) or} \\
& x < 0 \implies -x > 0 \text{ by (a)} \implies -x > 0 \text{ by (a)} \implies (-x)^2 > 0 \\
& \implies x^2 > 0 \\
(f) \quad & 1 > 0, -1 < 0 \\
& \because 1 \neq 0 \implies 1^2 > 0 \text{ by (e)} \implies 1 > 0 \\
(g) \quad & 0 < x < y \implies 0 < \frac{1}{y} < \frac{1}{x} \\
& \because \text{Note that } \forall u \in F, u > 0 \implies \frac{1}{u} = u^{-1} > 0 \\
& \because \text{if } \frac{1}{u} < 0, \text{ then } u \cdot \frac{1}{u} < 0 \text{ by (e)} \implies 1 < 0 (\rightarrow \leftarrow) \therefore \frac{1}{u} > 0 \\
& \text{Now, } \frac{1}{x}, \frac{1}{y} > 0 \text{ from } x < y \text{ we get } (\frac{1}{x} \cdot \frac{1}{y})x < (\frac{1}{x} \cdot \frac{1}{y})y \implies \\
& 0 < \frac{1}{y} < \frac{1}{x}
\end{aligned}$$

Remark. By (e)(f), we conclude that \mathbb{C} is not an ordered field
 $\because \mathbb{C}$ were an ordered field, then by (e), $i^2 > 0 \implies -1 > 0 (\rightarrow \leftarrow)$
 $\therefore \mathbb{C}$ is not an order field

1.5. The Real Number Field \mathbb{R} .

Theorem. There exists an ordered field \mathbb{R} containing \mathbb{Q} which has the l.u.b. property. Moreover, such \mathbb{R} is unique up to order-isomorphism i.e. if " $<$ " and " $<'$ " are two orders on \mathbb{R} , then $\exists f_i(\mathbb{R}, <) \rightarrow (\mathbb{R}, <')$
 $) \implies$

- (i) f is a field isomorphism,
i.e. $\forall a, b \in \mathbb{R}, f(a+b) = f(a)+f(b), f(ab) = f(a)f(b), f(1) = 1$
- (ii) f preserves ordering, $a < b \implies f(a) < f(b)$

Such \mathbb{R} is called the real number field or real number system or real line

Theorem.

- (a) The Archimedean property of \mathbb{R} : Given $x, y \in \mathbb{R}$ with $x > 0$, $\exists n \in \mathbb{N} \implies nx > y$
- (b) \mathbb{Q} is dense in \mathbb{R} : $\forall x, y \in \mathbb{R}$ with $x \leq y$, $\exists r \in \mathbb{Q} \implies x < r < y$

Proof.

- (a) Let $A = \{ nx \mid n \in \mathbb{N} \} \subseteq \mathbb{R}$
if (a) were false, then A is bdd above by y , since \mathbb{R} has the l.u.b property
 $\alpha = \sup A$ exists in \mathbb{R} , since $x > 0$, $\alpha - x < \alpha \implies \alpha - x$ is not an upper bdd of A
 $\implies \exists m \in \mathbb{N} \ni mx > \alpha - x \implies (m+1)x > \alpha (\rightarrow \leftarrow)$
- (b) Since $x < y$, $y - x > 0$, by (a), $\exists n \in \mathbb{N} \implies n(y - x) > 1$
By (a) again, $\exists m_1, m_2 \in \mathbb{N} \implies m_1 = m_1 \cdot 1 > nx$ & $m_2 = m_2 \cdot 1 > -nx$
we have $-m_2 < nx < m_1$, choose $m \in \mathbb{Z} \implies -m_2 \leq m \leq m_1$ & $m - 1 \leq nx < m$
(in fact, $m = [nx] + 1$, where $[z]$ is the greatest integer of z)
we have $nx < m < 1 + nx < ny (\because n(y - x) > 1) \implies x < \frac{m}{n} < y$
Let $r = \frac{m}{n} \in \mathbb{Q}$, then $x < r < y$

■

An application of the density property of \mathbb{Q} in \mathbb{R} :

Given $x \in \mathbb{R} - \mathbb{Q}$ i.e. x is an irrational numbers, i.e. $\forall \epsilon > 0, \exists r \in \mathbb{Q} \implies |x - r| < \epsilon$

equivalently, \exists a sequence $\{r_n\}$ in $\mathbb{Q} \implies r_n \rightarrow x$

In fact, one may choose $\{r_n\}$ to \uparrow or \downarrow

$\therefore \forall n \geq 1, \exists r_n \in \mathbb{Q} \implies x < r_n < \frac{1}{n} + x$ by Thm.1.3(b) By squeezing lemma, $r_n \rightarrow x$ on $n \rightarrow \infty$

Theorem (existence of n th root). Given $x \in \mathbb{T}, x > 0$ & $n \in \mathbb{N}, \exists$ "1" $y > 0 \implies y^n = x$

Such y is called the n th root of x & denoted by $y = \sqrt[n]{x} = x^{\frac{1}{n}}$

Proof. **not important**

"1". Suppose $y_1, y_2 > 0 \implies y_1^n = x$ & $y_2^n = x$

Bt trichotomy, we have

- (i) $0 < y_1 < y_2 \implies y_1^n < y_2^n (\rightarrow \leftarrow)$
- (ii) $0 < y_2 < y_1 \implies y_2^n < y_1^n (\rightarrow \leftarrow)$
- (iii) $y_1 = y_2$

" \exists ". Let $E = \{t \in \mathbb{R} \mid t^n < x\}$

Claim:

- $E \neq \emptyset$, Let $t = \frac{x}{1+x}$, then $0 < t < 1$, hence $t^n < t < x$, $\therefore t \in E$ & $E \neq \emptyset$
- E is bdd above, in fact E is bdd above by $1+x$ if $t > 1+x > 1$, then $t^n > t > x$, so E is bdd above by $1+x$
Therefore $y = \sup E$ exists & is finite
- Claim $y > 0$ & $y^n = x$, clearly, $y > 0$ ($\because \frac{x}{1+x} \in E$ & $\frac{x}{1+x} > 0$)
by trichotomy, we have $y^n < x$, $y^n > x$, $y^n = x$

Now, to show that (i) & (ii) are impossible, do (iii) holds $y^n = x$

By the identity, $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \dots + a^{n-1})$

(i) $y^n < x$ choose $0 < h < 1 = \alpha$ & $0 < \frac{x - y^n}{n(y+1)^{n-1}}$, $0 < h < \min\{\alpha, \beta\}$

put $a = y$, $b = y + h$ in (\star) , we obtain

$$\begin{aligned} (y+h)^n - y^n &< hn(y-h)^{n-1} < hn(y+1)^{n-1} < x - y^n \\ \implies (y+h)^n &< x \implies y+h \in E \text{ \& } y+h > y (\rightarrow \leftarrow) \therefore \text{(i) fails} \end{aligned}$$

(ii) $y^n > x$, Let $k = \frac{y^n - x}{ny^{n-1}}$, Then $0 < k < y$, $k = \frac{y^n - x}{ny^{n-1}} < \frac{y^n}{ny^{n-1}} = \frac{y}{n} < y$
 if $t > y - k > 0$, then $y^n - t^n \leq y^n - (y - k)^n < kny^{n-1}$ by $(\star) = y^n - x \implies t^n > x \implies t \in E \implies E$ is bdd above by $y - k \implies \sup E \leq y - k$ ($\rightarrow \leftarrow$)
 \therefore (ii) fails ■

Corollary. Let $a, b \in \mathbb{R}$ with $a, b > 0$, $n \in \mathbb{N}$ Then $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$
 $\because a^{\frac{1}{n}}, b^{\frac{1}{n}} > 0$ & $(a^{\frac{1}{n}} \cdot b^{\frac{1}{n}})^n = ab$, By (1) in Thm 1.4 $(a, b)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$

infinite in \mathbb{R}

After discuss the real number \mathbb{R} , sometimes, we have to work with the extended real number system $\mathbb{R}^* = [-\infty, \infty] = \mathbb{R} \cup \{+\infty, -\infty\}$ with observe, $x \in \mathbb{R}$

$$\lim_{n \rightarrow \infty} (-n) = -\infty, \lim_{n \rightarrow \infty} n = \infty, \lim_{n \rightarrow \infty} \left(\frac{1}{n} + n\right) = \infty, \lim_{n \rightarrow \infty} (n^2 - n) = \infty$$

$$x \pm \infty = \pm \infty, 0 \cdot (\pm \infty) = 0, \infty - \infty \text{ is not define}$$

Element in $\mathbb{R} \subseteq \mathbb{R}^*$ are called finite. Now, given any nonempty subset $E \subseteq \mathbb{R}$,

$$\sup E = \begin{cases} +\infty & \text{if } E \text{ is not bdd above} \\ \text{finite} & \text{if } E \text{ is bdd above} \end{cases} \quad \& \quad \inf E = \begin{cases} -\infty & \text{if } E \text{ is not bdd below} \\ \text{finite} & \text{if } E \text{ is bdd below} \end{cases}$$

Note that if $A \subseteq B$, then $\sup A \leq \sup$ & $\inf A \geq \inf B$

$\therefore \emptyset \subseteq B, \forall B \subseteq \mathbb{R}$, One may define $\sup \emptyset = -\infty, \inf \emptyset = +\infty$

1.6. The Complex Number Field \mathbb{C} .

Consider the contention product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a, b) \mid a, b \in \mathbb{R}\}$

Note that $(a, b) = (c, d) \Leftrightarrow a = c$ & $b = d$, From now, we can write $\mathbb{C} = \mathbb{R}^2$

Operation on \mathbb{C} Given $(a, b), (c, d) \in \mathbb{C}$

$$(1) (a, b) + (c, d) = (a + c, b + d)$$

$$(2) (a, b)(c, d) = (ac - bd, ad + bc)$$

It is easy to see that, with these operations, \mathbb{C} is a field.

Note that

- the zero element is $(0, 0)$
- the negative of (a, b) is $-(a, b) = (-a, -b)$
- the identity is $(1, 0)$
- if $(a, b) \neq (0, 0)$, then $(a, b)^{-1} = \left(\frac{1}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right)$

\mathbb{R} is a subset of \mathbb{C} (not vary important)
consider that map

$$f : \mathbb{R} \rightarrow \mathbb{C} \text{ define by } f(a) = (a, 0), \quad a \in \mathbb{R}$$

we have (1) f is injective (2) $f(1) = (1, 0) \quad \because \forall a, b \in \mathbb{R}$

$$f(a + b) = (a + b, 0) = (a, 0) + (b, 0) = f(a) + f(b), \quad f(a \cdot b) = (ab, 0) = (a, 0) \cdot (b, 0)$$

f is a field homomorphism

$\therefore f : \mathbb{R} \rightarrow \mathbb{C}$ is an injective and isomorphism

Therefore, we identify \mathbb{R} with $f(\mathbb{R})$ through the injective f

i.e. $a \in \mathbb{R}$ is identified with $f(a, 0)$ in \mathbb{C}

$$ab = (a, 0) \cdot (b, 0), \quad a + b = (a, 0) + (b, 0) \quad \forall a, b \in \mathbb{R}$$

Change (a, b) to $a + bi$

Now, we can transform an element $(a, b) \in \mathbb{C}$ into the normal form:

$$(a, b) = (a, 0) + (0, b) = (a, 0)(1, 0) + (b, 0)(0, 1) = a1 + bi = a + ib, \quad \text{where } i = (0, 1)$$

Therefore, from now on, we write $\mathbb{C} = \{a + ib \mid a, b \in \mathbb{C}\}$

An element $z = a + ib \in \mathbb{C}$ is called a complex number

Hence, under this notation, $z = a + ib, w = c + id \in \mathbb{C}$

$$(1) \quad z + w = (a + c) + i(b + d)$$

$$(2) \quad zw = (ac - bd) + i(ad + bc)$$

and the a is called the real part of z , $a = \operatorname{Re}(z)$, b is called imaginary part of z , $b = \operatorname{Im}z$

Some basic properties of complex numbers whose proofs are easy

$\forall z, w \in \mathbb{C}$

$$\begin{aligned}
 & \cdot \quad \overline{z + w} = \bar{z} + \bar{w} \quad \cdot \quad \overline{zw} = \bar{z} \cdot \bar{w} \quad \cdot \quad \operatorname{Re} z = \frac{z + \bar{z}}{2} \\
 & \cdot \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i} \quad \cdot \quad |z| = 0 \Leftrightarrow z = 0 \quad \cdot \quad \text{Triangle inequality} \\
 & \quad \quad \quad |z + w| \leq |z| + |w| \\
 & \cdot \quad ||z| - |w|| \leq |z - w| \quad \cdot \quad \mathbb{C} \text{ is not an ordered field} \quad \cdot \quad |z|^2 = z\bar{z} \\
 & \cdot \quad |\bar{z}| = |z| \quad \cdot \quad |\operatorname{Re} z| \leq |z|, |\operatorname{Im} z| \leq |z| \quad \cdot \quad |zw| = |z||w|
 \end{aligned}$$

Proof. $|z + w| \leq |z| + |w|$
 $|z + w|^2 = (z + w)(\overline{z + w}) = (z + w)(\bar{z} + \bar{w}) = z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w}$
 $= |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2 \leq |z|^2 + 2|z\bar{w}| + |w|^2 = |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2$
 $\therefore |z + w| \leq |z| + |w|$ ■

Theorem (basic algebraic theorem).

(a) $x^2 + 1$ has no root in \mathbb{R}

(b) $x^2 + 1$ has two distinct roots in \mathbb{C}

Proof.

- (a) $1 > 0, x^2 > 0, \forall x \in \mathbb{R} - \{0\} \implies x^2 + 1 > 0 \forall x \neq 0$
 $0^2 + 1 = 1 > 0, \therefore x^2 + 1 > 0, \forall x \in \mathbb{R}$. Hence, $x^2 + 1 = 0$
has no root in \mathbb{R}
- (b) $i^2 = (0, 1)(0, 1) = (0 - 1, 0) = (-1, 0) = -1$
 $(-i)^2 = (-(0, 1))^2 = (0, -1)^2 = (0, -1)(0, -1) = -1, \therefore \pm i$
are root of \mathbb{C} ■

Conclusion: Every non const polynomial $f(x) \in \mathbb{R}[x]$ has n roots where $n = \deg f(x)$

The complex root is even

no important proof

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$, $a_n \neq 0$, $n \geq 1$

if $\alpha = a + ib \in \mathbb{C}$ is a root of $f(x)$, then

$$0 = f(\alpha) = a_n \alpha^n + a_{n-1} \alpha^{n-1} + \cdots + a_1 \alpha + a_0$$

$$0 = f(\bar{\alpha}) = a_n \bar{\alpha}^n + a_{n-1} \bar{\alpha}^{n-1} + \cdots + a_1 \bar{\alpha} + a_0$$

$$\therefore (x - \alpha) | f(x), (x - \bar{\alpha}) | f(x) \implies (x - \alpha)(x - \bar{\alpha}) | f(x) \implies (x^2 - (\alpha + \bar{\alpha})x + |\alpha|^2) | f(x)$$

$$\implies (x^2 - 2ax + (a^2 + b^2)) | f(x)$$

\therefore quadratic function must have two roots in \mathbb{C}

The fundamental Theorem of Algebra

Every non zero polynomial $f(x) \in \mathbb{C}[x]$ has at least one root in \mathbb{C}

Therefore, if $\deg f(x) = n$, then $f(x)$ has n roots in $\mathbb{C}(C, M)$

$\therefore f(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_t)^{e_t} (a_1 x^2 + b_1 x + c_1)^{l_1} \cdots (a_s x^2 + b_s x + c_s)^{l_s}$, where $\lambda_1, \dots, \lambda_t \in \mathbb{R}$, $a_i, b_i, c_i \in \mathbb{R}$ & $e_1 + \cdots + e_t + 2l_1 + \cdots + 2l_s = \deg f(x)$ which shows that all roots of $f(x)$ are in \mathbb{C}

In fact, we have the famous theorem: The fundamental theorem of algebra

Every non zero polynomial $f(x) \in \mathbb{C}[x]$ has at least one root in \mathbb{C}

\therefore if $\deg f(x) = n$, then $f(x)$ has n roots in $\mathbb{C}(C, M)$

Theorem (Cauchy-Schering Inequal). *Given $z_1, \dots, z_n, w_1, \dots, w_n \in \mathbb{C}$, we have*

$$\left| \sum_{j=1}^n z_j \bar{w}_j \right| \leq \left(\sum_{j=1}^n |z_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n |w_j|^2 \right)^{\frac{1}{2}}$$

and " $=$ " holds $\Leftrightarrow \exists \lambda \in \mathbb{C} \ni w_j = \lambda z_j, 1 \leq j \leq n$,

In patricial, if $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$, then

$$\left| \sum_{j=1}^n x_j y_j \right| \leq \left(\sum_{j=1}^n x_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n y_j^2 \right)^{\frac{1}{2}}$$

and " $=$ " holds $\Leftrightarrow \exists t \in \mathbb{R} \ni y_j = t x_j, 1 \leq j \leq n$

The proof is too long, I am lazy

1.7. Euclidean Spaces \mathbb{R}^n .

Definition. the n -dimensional Euclidean space \mathbb{R}^n

$$= \{ x = (x_1, \dots, x_n) \mid x_i \in \mathbb{R}, 1 \leq i \leq n \} = \mathbb{R} \times \dots \times \mathbb{R}$$

Note that

$$(x_1, \dots, x_n) = (y_1, \dots, y_n) \Leftrightarrow x_i = y_i \quad \forall 1 \leq i \leq n$$

We are going to introduce the structure of \mathbb{R}^n

- vector space
- inner product space
- normed linear space
- matrix space

Definition. Two operation on \mathbb{R}^n as follows:

- Addition $+$: $\mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $(x, y) \mapsto x + y = (x_1 + y_1, \dots, x_n + y_n)$
- Scalar multiplication \cdot : $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $(a, x) \mapsto ax = (ax_1, \dots, ax_n)$

we skip space example here.

1.8. Countability of Sets.

Given two nonempty set A, B and a function $f : A \rightarrow B$, $f(A) = \{ f(a) \mid a \in A \}$ is called the image of A under f

Some basic things

$E \subseteq A$, $f(E) = \{ f(a) \mid a \in E \}$ the image of E under f
 f is infective(one-to-one) $x_1 \neq x_2 \implies f(x_1) \neq f(x_2) \Leftrightarrow f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$
 f is surjective(onto) if $f(A) = B$, f is bijective if f is one-to-one and onto

Given $F \subseteq B$, $f^{-1}(F) = \{ x \in X \mid f(x) \in F \}$ called the inverse image of f under F

Example

$f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$, $x \in \mathbb{R}$
 $f^{-1}([0, 1]) = \{ x \in \mathbb{R} \mid f(x) \in [0, 1] \} = \{ x \in \mathbb{R} \mid x^2 \in [0, 1] \} = [-1, 1]$
 $f^{-1}([-1, 1]) = [-1, 1]$

Properties of inverse image

- $F_1 \subseteq F_2 \subseteq B \implies f^{-1}(F_1) \subseteq f^{-1}(F_2)$
- Inverse image preserves set operation
 - $\forall F_\alpha \subseteq B, \alpha \in I, F \subseteq B$
 - (i) $f^{-1}(\cup_{\alpha \in I} F_\alpha) = \cup_{\alpha \in I} f^{-1}(F_\alpha)$
 - (ii) $f^{-1}(\cap_{\alpha \in I} F_\alpha) = \cap_{\alpha \in I} f^{-1}(F_\alpha)$
 - (iii) $f^{-1}(B - F) = f^{-1}(B) - f^{-1}(F)$
- Given $S \subseteq A, f'(f'(S)) \supseteq S, " = " \Leftrightarrow$ one-to-one, **example:**
 $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2, S = [0, 1], f(S) = [0, 1], f^{-1}(f(S)) = f^{-1}([0, 1]) = [-1, 1]$
- Given $F \subseteq B, f(f^{-1}(F)) \subseteq F, " = " \Leftrightarrow$ "onto", **example**
 $f(x) = x^2, x \in \mathbb{R}, F = [-1, 1], f(f^{-1}([-1, 1])) = f([-1, 1]) = [0, 1]$
- For $y \in B, f^{-1}(\{y\}) = f^{-1}(y) = \{x \in A \mid f(x) = y\}$ the inverse image of y , **example**
 $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2, f^{-1}(1) = \{1, -1\}, f^{-1}(2) = \emptyset$

Definition (cardinality). Let A, B are two set ew say that A and B have the same cardinality if \exists a bijective map $f : A \rightarrow B$, which is denoted by $A \sim B$

From now on, we write $|A|$ as the cardinality of A

Claim " \sim " is an \equiv relation among all sets

- (i) Reflexion: \forall set $A, A \sim^{1_A} A$, which 1_A is identity mapping
- (ii) Symmetry: $A \sim^f B \implies B \sim^{f^{-1}} A$
- (iii) Transitive: $A \sim^f B \ \& \ B \sim^g C \implies A \sim^{g \circ f} C$

So we gave some property:

- Any two " \equiv " are either disjoint or identical
- \overline{X} is a disjoint union of " \equiv " classes
 $[A] = \{B \in \overline{X} \mid B \sim A\}$ the " \equiv " class set by A

Ant two element in an " \equiv " class have the same cardinality

Notation For $n \in \mathbb{N}, \mathbb{N}_n = \{1, 2, \dots, n\}$

Definition. Let A be a set

- (a) A is a finite set if $A = \emptyset$ or $A \sim \mathbb{N}_n$ for some $n \in \mathbb{N}$
- (b) A is a infinite set if A is not a finite set
- (c) A is countable if $A \sim \mathbb{N}$
- (d) A is uncountable if A is not countable.
- (e) A is at most countable if A is finite or countable

Remark.

- (1) when A, B are finite sets, $A \sim B \Leftrightarrow |A| = |B|$, i.e. A, B have same number.
- (2) where A, B are infinite and $A \sim B$, i.e. $|A| = |B|$, the concept is abstract.
- (3) $\{a, b, c\} \cup \mathbb{N} \sim \mathbb{N}$, $f : \mathbb{N} \rightarrow \{a, b, c\} \cup \mathbb{N}$, $f(1) = a$, $f(2) = b$, $f(3) = c, \dots$
- (4) Any finite set can not equivalent to a proper subset, i.e. A is finite, $B \subseteq A$
Then $A \sim B$, In fact $|B| < |A|$, but infinite different
- (5) Any finite set A can be listed as $A = \{a_1, \dots, a_n\}$ where $n = |A|$

Now, we consider the case of countable set

Recall, in calculus, a real sequence $\{a_n\}$, e.g.

$$a_n = \frac{1}{n} \left\{ \frac{1}{n} \right\}, \quad a_n = 1 - \frac{1}{n} \left\{ 1 - \frac{1}{n} \right\}, \quad a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

Definition. Let X be a nonempty set, a sequence in X is a function $a : \mathbb{N} \rightarrow X$

Given a sequence $=^a$ in X , a is "1" determine by $a(n)$, $\in \mathbb{N}$

We write

$$a = \{a(1), a(2), \dots, a(n), \dots\} = \{a_1, a_2, \dots, a_n, \dots\} = \{a_n\} = \{a_n\}_{n=1}^{\infty}$$

Remark.

- (1) For a sequence $\{a_n\}$ in X , a_n may not be distinct.
If all a_n are distinct, then we say that $\{a_n\}$ is a distinct sequence in X .
- (2) We usually use $\{a_n\}, \{b_n\}$ to denote sequence
- (3) A sequence $\{a_n\}$ in X in fact is a function from $\mathbb{N} \rightarrow X$, So $\{a_n \mid n \in \mathbb{N}\}$ is the image of the sequence.
- (4) $\{a_n\}$ is a sequence, a_n is called the n^{th} term of the sequence.
- (5) A sequence in X may begin at 0, i.e. $\{a_n\}_{n=0}$
By a changing index, we can make it from $\{b_n\}_{n=1}^{\infty}$, $b_n = a_{n+1}$, $n = 1, 2, \dots$

Definition (increasing).

A function $a : \mathbb{N} \rightarrow \mathbb{N}$ is increasing, a is \uparrow , if $a(n) \leq a(n+1) \forall n \geq 1$
 a is strictly increasing, a is st. \uparrow , if $a(n) < a(n+1) \forall n \geq 1$

Now, given a st. \uparrow function $n : \mathbb{N} \rightarrow \mathbb{N}$, i.e. $n(k) < n(k+1)$, $k \geq 1$
i.e. $n_k < n_{k+1}$, $k \geq 1$, i.e. $n_1 < n_2 < \dots < n_k < \dots$, i.e. $\{n_k\}_{k=1}^{\infty}$ is a
st. sequence in \mathbb{N}

Definition. Let $\{a_n\}$ be a sequence in X and $\{n_k\}$ be a st. \uparrow sequence
in \mathbb{N} , then the sequence $\{a_{n_k}\}$ is called a subsequence of $\{a_n\}$
In fact

$\mathbb{N} \xrightarrow{n_{st.}} \mathbb{N} \xrightarrow{a_{seq}} X \Rightarrow a \circ n : \mathbb{N} \rightarrow X$ is a function,
hence, it also a sequence in X

$$a \circ n = \{a \circ n(k)\} = \{a(n(k))\} = \{a_{n(k)}\} = \{a_{n_k}\}$$

Remark. if $\{a_{n_k}\}$ is st. \uparrow in \mathbb{N} , then $k \leq n_k \forall k \geq 1$
 \therefore By mathematical Induction

- $1 \leq n_1$
- Assume it's true for $k \geq 2$, i.e. $k \leq n_k$
- Consider $k+1$, $k+1 \leq n_k+1 \leq n_{k+1}$

Example

Let $\{a_n\}$ be a sequence in X , then $\{a_{2k}\}$ and $\{a_{2k-1}\}$ are subsequence
of $\{a_n\}$

Finally, we will assume that you are familiar with the following property
of the countability of sets:

- (1) Every subset of a countable set is at most countable. The proof
needs the well ordering of \mathbb{N} : Every nonempty subset of \mathbb{N} has
the smallest element
- (2) Countable union of countable sets is countable
- (3) If A_1, A_2, \dots, A_n are countable, then so is $A_1 \times \dots \times A_n$
- (4) If A is countable, then so is $A^n \equiv A \times \dots \times A \forall n \geq 1$
- (5) $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^n, \forall n \geq 1$ are countable
- (6) The set $\{a_n \mid a_n = 0 \text{ or } 1\}$ is uncountable

This can be proved by Cantor diagonal process

$$\therefore \text{ if it is countable, then we can list it, } a_0 A = \{a_1^{(1)}, a_2^{(2)}, \dots\}$$

where

$$a^{(1)} = \{a_n^{(1)}\} = a_1^{(1)}, a_2^{(1)}, \dots; a^{(2)} = \{a_n^{(2)}\} = a_1^{(2)}, a_2^{(2)}, \dots$$

Now, construct a sequence $\{a_n\}$ in $A \ni \{a_n\} \neq a^{(k)} \forall k \geq 1$
 $1(\rightarrow \leftarrow)$

Recall, intervals in \mathbb{R} , $-\infty < a \leq b < \infty$, following are finite bdd interval

$$\begin{aligned} (a, b) &= \{ x \in \mathbb{R} \mid a < x < b \} \text{ open interval} \\ [a, b] &= \{ x \in \mathbb{R} \mid a \leq x \leq b \} \text{ closed interval} \\ (a, b] &= \{ x \in \mathbb{R} \mid a < x \leq b \} \text{ open-closed} \\ [a, b) &= \{ x \in \mathbb{R} \mid a \leq x < b \} \text{ closed-open} \end{aligned}$$

An interval I in \mathbb{R} is said to be non-degenerate if the endpoint of I are distinct i.e. length > 0 . Otherwise, it is degenerate.

Note.

$$\begin{aligned} (0, 1) \text{ is uncountable, } \because (0, 1) &= \left\{ \sum_{n=1}^{\infty} \frac{a_n}{2^n} \mid a_n = 0 \text{ or } 1, n \in \mathbb{N} \right\} \\ x \in (0, 1) \text{ has a unique binary representation, so } (0, 1) &\sim A, \\ \text{where } A \text{ is } \{ \{ a_n \} \mid a_n = 0 \text{ or } 1 \} &\text{ which is uncountable} \end{aligned}$$

All non-degenerate intervals in \mathbb{R} are uncountable.

\because It sufficient to consider bdd non-degenerate interval in \mathbb{R} , given $-\infty < a < b < \infty$

(a, b) is uncountable($\because (0, 1) \sim (a, b)$)

Note that $(0, 1) \sim \mathbb{R}(\because (0, 1) \rightarrow (\frac{\pi}{-2}, \frac{\pi}{2}) \rightarrow \mathbb{R})$

2. Basic Point Set Topology

To know the "closeness", "limit" and "continue"

Notation. Let X be a nonempty set. The power set of X is denoted by $p(X)$ or 2^X , i.e. $\mathcal{P}(X) = 2^X$ which is the collect of all subset, if $|X| = n$, then $|\mathcal{P}(X)| = 2^n$

2.1. Topological Spaces.

Definition. Let X be a nonempty set and $\mathcal{T} \subseteq \mathcal{P}(X)$, we say that \mathcal{T} is a topology on X if it satisfies

- (1) $\emptyset, X \in \mathcal{T}$
- (2) \mathcal{T} is closed under arbitrary union,
i.e. $U_\alpha \in \mathcal{T}, \alpha \in I \implies \bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$
- (3) \mathcal{T} is closed under finite intersection
i.e. $U_1, \dots, U_n \in \mathcal{T} \implies U_1 \cap \dots \cap U_n \in \mathcal{T}$

In this chapter, the pair (X, J) or simply J is called a topological space and members in T are called open set in X or open subsets of X

Remark.

- (1) X : a nonempty set, there is at least two trivial topology on X
 - $\mathcal{P}(x)$ is the largest topology on X w.r.t inclusion X with this topology is called a discrete topological space
 - $\mathcal{T}_0 = \{\emptyset, X\}$ is the smallest topology on X w.r.t inclusion X with this topology is called an indiscrete topological space
- (2) How many topology can be define on $\{a\}$, $\{a, b\}$?

In the following X is a topology space

Definition (neighborhood). Let $p \in X$, a neighborhood of P is an open set U containing p

Definition (Hausdorff space). X is a Hausdorff space if any two distinct points can be separated by open set, i.e. $\forall p \neq q$ in X , \exists neighborhood U of p and V of $q \in U \cap V = \emptyset$

Definition (closed set). A subset $F \subseteq X$ is said to be closed if $F^C = X - F$ is open in X

Theorem 2.1. The collection of all closed subsets of X satisfied

- (a) \emptyset, X are closed
- (b) Arbitrary intersection of closed set if closed
- (c) Finite union of closed sets is closed

Proof.

- (a) $X - \emptyset = X$ is open $\therefore \emptyset$ is closed
 $X - X = \emptyset$ is open $\therefore X$ is closed
- (b) Given closed sets $F_\alpha, \alpha \in I$, $X - \bigcap_{\alpha \in I} F_\alpha = \bigcup_{\alpha \in I} (X - F_\alpha)$ is open, $\therefore \bigcap_{\alpha \in I} F_\alpha$ is closed.
- (c) Given closed set F_1, \dots, F_n , $X - \bigcup_{i=1}^n F_i = \bigcap_{i=1}^n (X - F_i)$ is open, $\therefore \bigcup_{i=1}^n F_i$ is closed.

■

Definition. Let $Y \subseteq X$ and

$$\mathcal{T}_Y = \{U \cap Y \mid U \text{ is open in } X\}$$

Theorem 2.2. \mathcal{T}_Y is also a topology space

Proof. To prove \mathcal{T}_Y is a topology space, we take the topology's definition

- (a) $\emptyset, Y \in \mathcal{T}_Y$ ($\because \emptyset = \emptyset \cap Y, Y = X \cap Y$)
- (b) Given $U_\alpha \cap Y \in \mathcal{T}_Y, \alpha \in I$, where U_α is open in X

$$\bigcup_{\alpha \in I} (U_\alpha \cap Y) = (\bigcup_{\alpha \in I} U_\alpha) \cap Y \implies \bigcup_{\alpha \in I} (U_\alpha \cap Y) \in \mathcal{T}_Y$$
- (c) Given $U_1 \cap Y, \dots, U_n \cap Y$, where U_i is open in $X, 1 \leq i \leq n$

$$\bigcap_{i=1}^n (U_i \cap Y) = (\bigcap_{i=1}^n U_i) \cap Y \implies \bigcap_{i=1}^n (U_i \cap Y) \in \mathcal{T}_Y$$

 $\therefore \mathcal{T}_Y$ is a topology on Y

■

Definition. In Theorem 2.2, with the topology \mathcal{T}_Y on Y , is called a topological subspace of X and \mathcal{T}_Y is called the relative topology of Y in X . Members in \mathcal{T}_Y are called open set in Y or relative open sets in Y .

2.2. Metric Spaces & Subspace.

In this chapter, we will introduce a class of topology space whose topology is induced by a metric.

Definition. Let X be a nonempty set. A metric or distance function is a function

$$d : X \times X \rightarrow \mathbb{R}, (a, b) \mapsto d(a, b)$$

satisfying:

- (a) $\forall a, b \in X, d(a, b) \geq 0$ and $d(a, b) = 0 \Leftrightarrow a = b$
- (b) $\forall a, b \in X, d(a, b) = d(b, a)$ **symmetry**
- (c) $\forall a, b, c \in X, d(a, b) \leq d(a, c) + d(c, b)$ **triangle inequality**

if d is a metric on X , then the pair (X, d) or simply X is called a metric space and $\forall a, b \in X, d(a, b)$ is called the distance between a & b

Examples

- (1) Let X be a nonempty set define by

$$d(a, b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b \end{cases}$$

Then d is a metric on X , called the discrete metric and with this metric X is called a discrete metric space. In particular, any set admits a metric.

- (2) The most important metric spaces are the Euclidean space \mathbb{R}^k , the metric d is called the Euclidean or standard or usual metric on \mathbb{R}^k . There are other metrics on \mathbb{R}^k induced the same metric topology on \mathbb{R}^k , in fact, they are all equivalent, e.g. $\forall 1 \leq p \leq \infty$, We can define a metric d_p on \mathbb{R}^k as follows

- $1 \leq p < \infty, d_p(x, y) = \|x - y\|_p = \left(\sum_{i=1}^k |x_i - y_i|^p \right)^{\frac{1}{p}}$
- $p = \infty, d_\infty(x, y) = \max_{1 \leq i \leq k} |x_i - y_i|$

Note that $d_2 = d$ is the Euclidean metric on \mathbb{R}^k

Remark. In fact, every normed linear space $(V, \|\cdot\|)$ is a metric space whose metric is induced by its norm

- (3) Let (X, d) be a metric space and $Y \subseteq X, Y \neq \emptyset$. Then the restriction of d to $Y \times Y$ is also a metric on Y , with this metric, Y is called a metric subspace of X

Definition (ball). Given $p \in X$ & $r > 0$

$B(p, r) = \{x \in X \mid d(x, p) < r\}$: open ball with center p and radius r

$\overline{B}(p, r) = \{x \in X \mid d(x, p) \leq r\}$: closed ball with center p and radius r

Example

- (1) The discrete metric space X : $p \in X$, $r > 0$

$$B(p, r) = \begin{cases} \{p\} & \text{if } 0 \leq r \leq 1 \\ X & \text{if } r > 1 \end{cases}$$

$$\overline{B}(p, r) = \begin{cases} \{p\} & \text{if } 0 \leq r < 1 \\ X & \text{if } r \geq 1 \end{cases}$$

- (2) In the Euclidean space \mathbb{R}^k , $p \in \mathbb{R}^k$, $r > 0$

$B(p, r) = \{x \in \mathbb{R}^k \mid \|x - p\| < r\}$ is a "true" open

$\overline{B}(p, r) = \{x \in \mathbb{R}^k \mid \|x - p\| \leq r\}$ is a "true" closed ball

In particular, for $k = 1$ in \mathbb{R}

$B(p, r) = (p - r, p + r)$: a symmetric open interval

$\overline{B}(p, r) = [p - r, p + r]$: a symmetric closed interval

However, w.r.t d_1 & d_∞ , we have, e.g. in \mathbb{R}^2

$B_1(0, 1) = \{(x, y) \mid |x - 0| + |y - 0| < 1\}$

$B_\infty(0, 1) = \{(x, y) \mid \max\{|x|, |y|\} \leq 1\}$

- (3) What is the open balls in $S = [0, 1] \subseteq \mathbb{R}$?

$$B_S(0, \frac{1}{2}) = \{x \in S \mid |x - 0| < \frac{1}{2}\} = [0, \frac{1}{2}] = B(0, \frac{1}{2}) \cap [0, 1]$$

$$B_S(0, 3) = [0, 1] = B(0, 3) \cap [0, 1]$$

Prop 2.2 Let S be a metric subspace of a metric space X , then $\forall p \in S$ & $r > 0$, $B_S(p, r) = B(p, r) \cap S$

Proof. $B_S(p, r) = \{x \in S \mid d(x, p) < r\} = \{x \in X \mid d(x, p) < r\} \cap S = B(p, r) \cap S$ ■

2.3. Open Sets in Metric Spaces.

We will see that every metric on a set induce a topology on X

Definition (interior point). Let $S \subseteq X$ be a set, and $p \in S$, we say that p is an interior point of S if $\exists r > 0$, $\exists B(p, r) \subseteq S$

Denote by S° or $\text{int}(S)$ by the set of all interior point of S

Definition (open). Let $S \subseteq X$, we say that S is open if all points of S are interior points of S

Remark.

(1) Every open set S is a union of open balls in X .

$\because \forall x \in S, x$ is an interior point of $S, \exists r_x > 0 \ni B(x, r_x) \subseteq S$

$\therefore S = \bigcup_{x \in S} B(x, r_x)$

(2) $S^o \subseteq S$ by definition

(3) S is open $\Leftrightarrow S = S^o$

Prop 2.3

(a) $S \subseteq T \implies S^o \subseteq T^o$

$$\because p \in S^o \implies \exists r > 0 \ni B(p, r) \subseteq S \subseteq T \implies p \in T^o$$

(b) Every open ball $B(p, r)$ in X is open

\because Give $q \in B(p, r)$, Let $\delta = r - d(p, q)$. Claim $B(q, \delta) \subseteq B(p, r)$ which says q is an interior point of $B(p, r)$. Since $q \in B(p, r)$ is arbitrary, so $B(p, r)$ is open. Given $x \in B(q, \delta)$

$$d(x, p) \leq d(x, q) + d(q, p) < \delta + d(q, p) = r - d(p, q) + d(q, p) = r$$

(c) $\forall S \subseteq X, S^o$ is always open

\because Given $p \in S^o, \exists r > 0 \ni B(p, r) \subseteq S$

$$\implies B(p, r) \subseteq S^o \implies p \text{ is a interior point of } S^o$$

$\therefore S^o$ is open

(d) $\forall S \subseteq X, S^{oo} = (S^o)^o = S^o$

\because by definition of open set and (c)

Now, let $T = \{U \subseteq X \mid U \text{ is open in } X\}$

Prop 2.4 T is a topology on X . In particular, X is a topology space.

Proof.

(i) $\emptyset, X \in \mathcal{T}, \because \emptyset = \emptyset, X^o = X$

(ii) $U_\alpha \in \mathcal{T}, \alpha \in I$ are open $\implies \bigcup_{\alpha \in I} U_\alpha$ is open

Given an arbitrary point $p \in \bigcup_{\alpha \in I} U_\alpha \implies \exists \alpha_0 \in I \ni p \in U_{\alpha_0}$

U_{α_0} is open, $\exists r > 0 \ni B(p, r) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in I} U_\alpha$

$\therefore p$ is an interior point of $\bigcup_{\alpha \in I} U_\alpha \therefore \bigcup_{\alpha \in I} U_\alpha$ is open, i.e. $\bigcup_{\alpha \in I} U_\alpha \in T$

(iii) $U_1, \dots, U_n \in T \implies U_1 \cap \dots \cap U_n \in T$
 \because Given $p \in U_1 \cap \dots \cap U_n \implies p \in U_i \ 1 \leq i \leq n$. Each U_i is open,
 $\exists r_i > 0 \ B(p, r_i) \subseteq U_i, \ 1 \leq i \leq n \implies B(p, r) \subseteq U_1 \cap \dots \cap U_n$
 $\implies p$ is an interior point of $U_1 \cap \dots \cap U_n$
 p is arbitrary, so $U_1 \cap \dots \cap U_n$ is open, i.e. $U_1 \cap \dots \cap U_n \in T$
 Therefore, T is a topology on X ■

Definition. Let X be a metric space with metric d . The topology T in prop 2.4 is called the metric topology(include by d)

Let X be a metric space and $Y \subseteq X$, Then Y is a metric subspace of X , and $\forall y \in Y, r > 0, B_Y(y, r) = B(y, r) \cap Y$. In fact, we have more

Prop 2.5 A subset $A \subseteq Y$ is open in $Y \Leftrightarrow A = U \cap Y$ for some open set U in X , in particular, the metric topology on T is just the relation topology of Y on X

Proof. (\Rightarrow) suppose $A \subseteq Y$ is open in Y . Then

$$A = \bigcup_{y \in A} B_Y(y, r_y) = \bigcup_{y \in A} (B(y, r_y) \cap Y) = \left(\bigcup_{y \in A} B(y, r_y) \right) \cap Y$$

Let $U = \bigcup_{y \in Y} B(y, r_y)$, then U is open in X and $A = U \cap Y$

(\Leftarrow) Suppose $A = U \cap Y$ where $U \subseteq X$ is open $\forall y \in A, y \in U \cap Y \implies y \in U \implies \exists r > 0 \ni B(y, r) \subseteq U \implies B(y, r) \cap Y \subseteq U \cap Y = A \implies B_Y(y, r) \subseteq A, \therefore A$ is open in Y . ■

Prop 2.6 Every metric space X is Hausdorff

Proof. Given $p, q \in X, p \neq q$. Choose $r = \frac{1}{2}d(p, q) > 0$. Then $B(p, r) \cap B(q, r) = \emptyset$. So X is Hausdorff
 $(\because x \in B(p, r) \cap B(q, r) = d(x, p) < r \ \& \ d(x, q) < r \implies d(p, q) \leq d(p, x) + d(x, q) < r + r = 2r = d(p, q) (\rightarrow \leftarrow))$ ■

Remark. Let $S \subseteq X$, where X is a metric space. Then S° is the largest(w.r.t inclusion) open set contained in S . $\because \forall$ open set $U \subseteq S, U^\circ \subseteq S^\circ \implies U \subseteq S^\circ \subseteq S$. In fact, $S^\circ = \bigcup_{U \subseteq S} U$ (which is the definition of intension of S in a topology space X)

2.4. Closed Sets.

Definition (Closed set). $F \subseteq X$ is closed $\Leftrightarrow F^C = X - F$ is open in X

By Theorem 2.1, the collection of all close sets in X has the properties

- (i) \emptyset, X are closed in X
- (ii) F_α is closed in X , $\alpha \in I \implies \bigcap_{\alpha \in I} F_\alpha$ is closed in X
- (iii) F_1, \dots, F_n are closed in $X \implies \bigcup_{i=1}^n F_i$ is closed in X

Example

Intersection of infinitely many open set may not be open

in \mathbb{R} with Euclidean topology, $(-\frac{1}{n}, \frac{1}{n})$ is open in $\mathbb{R} \forall n \geq 1 \implies$

$\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$ is not open

Prop 2.8 Let X be a metric space and $Y \subseteq X$ and $B \subseteq Y$, Then B is closed in $Y \iff B = F \cap Y$ for some closed set F in X

Proof. (\implies) Suppose B is close in $Y \implies Y - B$ is open in $Y \implies Y - B = U \cap Y$ (by prop 2.5) for some open set U in $X \implies Y - (Y - B) = Y - (U \cap Y) \implies B = (X - U) \cap Y$. where $(X - U)$ is close.

(\impliedby) Suppose $B = F \cap Y$, where F is closed in $X \implies Y - B = Y - (F \cap Y) = (X - F) \cap Y \implies Y - B$ is open in $Y \implies B$ is close in Y . ■

In metric space, one can use sequence to detect the closeness of a set

Example

1. We know that $[a, b)$ is not closed in \mathbb{R} , however, \exists a sequence $\{x_n\}$ in $[a, b) \ni x_n \rightarrow b$ on $n \rightarrow \infty$, e.g. $b - \frac{1}{n} \rightarrow b$
2. $A = \{\frac{1}{n} \mid n \geq 1\} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ is not close in \mathbb{R}
if $\mathbb{R} - A$ will open, then $\exists r > 0$, $B(0, r) \subseteq \mathbb{R} - A (\rightarrow \leftarrow)$
 $A \cup \{0\}$ is closed in \mathbb{R}

$$\mathbb{R} \setminus (A \cup \{0\}) = (-\infty, 0) \cup (1, \infty) \cup (\bigcup_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n})) \text{ is open}$$

$\therefore A \cup \{0\}$ is closed

Definition (Adherent, clousure \dots). Let X be a metric space with metric d , $T \subseteq X$ be a subset. (**important**)

(1.) A point $p \in X$ is said to be an adherent point of T if $\forall r > 0$, $B(p, r) \cap T \neq \emptyset$, equivalent, \forall neighborhood U of p , $U \cap T \neq \emptyset$

(2.) Let \bar{T} or $cl(T)$ denote the set of all adherent points of T , called the closure of T , i.e. $\bar{T} = \{p \in X \mid p \text{ is an adherent point of } T\}$

(3.) A point $p \in X$ is said to be a limit point or accumulation point of T if $\forall r > 0, B(p, r) \cap T - \{p\} \neq \emptyset$, equivalently, \forall neighborhood U of $p, U \cap T - \{p\} \neq \emptyset$

Denote by T' the set of all accumulation points of T , called the derived set of T .

(4.) $p \in T$ and $p \notin T'$, then p is called an isolated point of T , i.e. $\exists r > 0 \ni B(p, r) \cap T = \{p\}$

(5.) A subset $T \subseteq X$ is said to be perfect if T is closed and every points of T is an accumulated point of T , i.e. T is closed & $T' = T$

(6.) A subset $T \subseteq X$ is said to be bounded if $\exists R > 0$ and $p \in X \ni T \subseteq B(p, R)$

(7.) A subset $T \subseteq X$ is said to be dense if $\overline{T} = X$, e.g. $\overline{\mathbb{Q}} = \mathbb{R}$

(8.) A point $p \in X$ is said to be a boundary point of T if $\forall r > 0, B(p, r) \cap T \neq \emptyset$ & $B(p, r) \cap (X \setminus T) \neq \emptyset$. Denote by ∂T or $bd(T)$ the set of all boundary points of T

Prop 2.9 Let X be a metric space. All sets and point below are subset of X

(1) $S \subseteq T \implies \overline{S} \subseteq \overline{T}$ & $S' \subseteq T'$

$\because p \in \overline{S} \implies \forall r > 0, B(p, r) \cap S \neq \emptyset \implies B(p, r) \cap T \neq \emptyset \implies p \in \overline{T}$
 $p \in S' \implies \forall r > 0, B(p, r) \cap S - \{p\} \neq \emptyset \implies B(p, r) \cap T - \{p\} \neq \emptyset$

(2) \overline{T} is always closed in X

We want to know \overline{T} is closed on $X \rightarrow X - \overline{T}$ is open $\rightarrow \forall p \in X - \overline{T}$ is an interior point $\implies \exists r > 0, B(p, r) \subseteq X - \overline{T}$

$\because p \notin \overline{T} \implies \exists r' > 0 \ni B(p, r') \cap T = \emptyset$

But we want to get $B(p, r') \cap \overline{T}$, so we check every point in $B(p, r')$ is not in \overline{T} , let $q \in B(p, r')$, $\exists \delta > 0, B(q, \delta) \subseteq B(p, r') \implies B(q, \delta) \cap T = \emptyset \implies q \notin \overline{T}$

because if $q \in \overline{T}, \forall r > 0 \ni B(q, r) \cap T \neq \emptyset$
 $\implies B(p, r) \cap \overline{T} \neq \emptyset$

Let $p \in X - \overline{T} \implies p \notin \overline{T} \implies \exists r > 0 \ni B(p, r) \cap T = \emptyset \implies B(p, r) \cap \overline{T} = \emptyset (\because \forall q \in B(p, r), \exists \delta > 0 \ni B(q, \delta) \subseteq B(p, r) \implies B(q, \delta) \cap T = \emptyset \implies q \notin \overline{T})$

$\therefore B(p, r) \subseteq X - \overline{T}, \because p$ is an interior point of $X - \overline{T}$. Hence, $X - \overline{T}$ is open, i.e. \overline{T} is closed.

(3) $T \subseteq \overline{T} (\because \forall p \in T, \forall r > 0, B(p, r) \cap T \neq \emptyset)$

(4) $p \in T' \implies \forall r > 0, B(p, r) \cap T - \{p\}$ is an infinite set, say x_1, \dots, x_n , Let $\delta = \frac{1}{2} \min\{d(p, x_i) \mid 1 \leq i \leq n\}$. Then $B(p, \delta) \cap T - \{p\} = \emptyset (\rightarrow \leftarrow)$ to $p \in T', x \in B(p, \delta) \cap T - \{p\} \implies d(x, p) < \delta \implies x = x_i$ for some $1 \leq i \leq n$ & we get $d(x_i, p) < \delta \leq \frac{1}{2}d(x_i, p)$

\therefore no such x i.e. $B(p, \delta) \cap T - \{p\} = \emptyset$

(5) Any finite subset of X has no accumulation points in X by (4). In particular, it is closed by (6)(c) below.

(6) TFAE

(a) S is closed

(b) S contains all it's adherent point, i.e. $\overline{S} \subseteq S$

(c) S contains all it's accumulation points, i.e. $S' \subseteq S$

(d) $S = \overline{S}$

(7) $\overline{\overline{S}} = \overline{S}$ by (2) and (6)

Proof. of (6)

(a) \implies (b) Suppose S is closed $\implies X \setminus S$ is open $\implies \forall p \in X - S \implies \exists r > 0 \ni B(p, r) \subseteq X \setminus S \implies B(p, r) \cap S = \emptyset \implies p \notin \overline{S}$
 $\therefore \overline{S} \subseteq S$, i.e. (b) holds

(b) \implies (c) $\because S' \subseteq \overline{S}$

(c) \implies (d) Suppose $S' \subseteq S$. To prove $S = \overline{S}$ if not, then $S \subsetneq \overline{S}$, i.e. $\exists p \in \overline{S} \& p \notin S \implies \forall r > 0, B(p, r) \cap S \neq \emptyset (\because p \in \overline{S})$

(d) \implies (a) by (2)

■

(8) \overline{S} is the smallest closed set in X containing S

\because We know that $S \subseteq \overline{S}$, if F is closed in X & $F \subseteq S$, then $\overline{F} \subseteq \overline{S}$ by (1), $F = \overline{F} \subseteq \overline{S}$ by (6), $\therefore \overline{S}$ is the smallest such one.

(9) In fact, $\overline{S} = \bigcap_{F \subseteq S} F$

(10) $p \in S$ is an isolated point $\Leftrightarrow \exists r > 0 \ni B(p, r) \cap S = \{p\}$

(\implies) Suppose $p \in S$ is an isolated point of S . Then $p \in S' \implies \exists r > 0 \ni B(p, r) \cap S - \{p\} = \emptyset \implies B(p, r) \cap S = \{p\}$

(\Leftarrow) Trivial

(11) S is dense in $X \Leftrightarrow \forall p \in X \& r > 0, B(p, r) \cap S \neq \emptyset \Leftrightarrow \forall$ open set $U \neq \emptyset, U \cap S \neq \emptyset$

Proof. (\implies) Suppose S is dense in X , i.e. $\overline{S} = X$, So $\forall p \in X, p \in \overline{S} \implies \forall r > 0, B(p, r) \cap S \neq \emptyset$

(\Leftarrow) Suppose the condition holds, $\forall p \in X \& r > 0, B(p, r) \cap S \neq \emptyset \implies p \in \overline{S} \implies X \subseteq \overline{S} \subseteq X, \therefore \overline{S} = X$

■

(12) $\partial S = \partial(X - S)$ In particular, $\partial S = \overline{S} \cap \overline{(X - S)}$, In particular, ∂S is closed in X , \therefore It suffices to prove $\partial S = \overline{S} \cap \overline{(X - S)}$,
 $\therefore \partial(X - S) = \overline{X - S} \cap \overline{X - (X - S)} = \overline{X - S} \cap \overline{S} = \partial S$
 $\forall p \in \partial S \implies \forall r > 0, B(p, r) \cap S \neq \emptyset \ \& \ p \in \overline{X - S} \implies p \in \overline{S} \cap \overline{X - S}$
 $\therefore \partial S \subseteq \overline{S} \cap \overline{X - S}$,
 Conversely, $p \in \overline{S} \cap \overline{X - S} \implies p \in \overline{S} \ \& \ p \in \overline{X - S} \implies \forall r > 0, B(p, r) \cap S \neq \emptyset \ \& \ B(p, r) \cap (X - S) \neq \emptyset \implies p \in \partial S$
 $\therefore \overline{S} \cap \overline{(X - S)} \subseteq \partial S \therefore \partial S = \overline{S} \cap \overline{(X - S)}$

2.5. Examples.

We give some simple examples of open sets, closed sets, adherent, accumulation, isolated and boundary points.

1. In a discrete metric space X , every subset of X is both open and

close, $\forall x \in X, B(p, r) \begin{cases} \{x\} & \text{if } 0 < r \leq 1 \\ X & \text{if } r > 1 \end{cases}$

\therefore Every singleton is open in X , so every subset of X is open.

2. In \mathbb{R} . Consider the set $S = [0, 1) \cup \{3\}$, $S^\circ = \emptyset$, $S' = \{0\}$,
 $\overline{S} = S \cup \{0\}$

3. In \mathbb{R} , consider the set $S = \{\frac{1}{n} \mid n = 1, 2, \dots\}$, $S^\circ = \emptyset$,

$S' = \{0\}$, $\overline{S} = S \cup \{0\}$

4. In \mathbb{R}^2 , consider $S = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$, S is open

$\overline{S} = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$

$\partial S = \{(x, 0) \mid x \geq 0\} \cup \{(y, 0) \mid y \geq 0\}$

5. Let $B(0, 1)$ be the unit open ball in \mathbb{R}^k . Then $\partial B(0, 1) = S^{k-1}$ is the unit $(k - 1)$ -sphere. In particular, for $k = 2$, $\partial B(0, 1) = S^1$ in the unit circle in the plane \mathbb{R}^2 . Similarly, for the closed unit ball $\overline{B}(0, 1)$ in \mathbb{R}^k . Now, we define some special sets in \mathbb{R}^n

- Internals in \mathbb{R} : $-\infty < a \leq b < \infty$
 $[a, b]$ close interval which is closed in \mathbb{R}
 (a, b) open interval which is closed in \mathbb{R}
 Infinite intervals:
 $(-\infty, b]$: close in \mathbb{R} , $(-\infty, b)$ open in \mathbb{R}
- k-dimensional interval (rectangle or k-cell) I

$$I = I_1 \times \dots \times I_k$$

where I_j is an interval in \mathbb{R} , $1 \leq j \leq k$

(i) I is bounded \Leftrightarrow each I_j is bounded

I is unbounded $\Leftrightarrow I_j \neq \emptyset \ \& \ \text{some } I_j \text{ is unbounded}$

(ii) $I = [a_1, b_1] \times \dots \times [a_k, b_k]$, $-\infty < a_j \leq b_j < \infty$, $1 \leq j \leq k$
 k -dimensional closed(compact) interval in \mathbb{R}^k

- Convex sets in \mathbb{R}^k
 $S \subseteq \mathbb{R}^k$ is convex if $\forall x, y \in S$, \overline{xy} is the line segment joining x & y
 Note that all open balls, closed balls, intervals are convex in \mathbb{R}^k
- Star-like sets in \mathbb{R}^k with w.r.t some point x_0 , $S \subseteq \mathbb{R}^k$ is star-like
 w.r.t. $x_0 \in S$ if $\forall x \in S$, $\overline{xx_0} \subseteq S$
- 6. We know that \mathbb{Q} is dense in \mathbb{R} , hence \mathbb{Q}^k is dense in \mathbb{R}^k . Note that \mathbb{Q}^k is countable, hence \mathbb{R}^k has a countable dense subset \mathbb{Q}^k , i.e. \mathbb{R}^k is separable.
- 7. $\partial\mathbb{Q} = \mathbb{R}$, $\partial\mathbb{Q}^k = \mathbb{R}^k$
- 8. \mathbb{Z} is closed in \mathbb{R} , $\because \mathbb{R} - \mathbb{Z} = \bigcup_{n=-\infty}^{\infty} (n-1, n)$ is open $\implies \mathbb{Z}$ is close.

or $\mathbb{Z}' = \emptyset \subseteq \mathbb{Z}$, $\therefore \mathbb{Z}$ is close.

- 9. Let $S \subseteq \mathbb{R}$ be a nonempty set which is bounded above. Then $\alpha = \sup S$ exists. Moreover, $\alpha \in \overline{S}$. $\because \forall r > 0, \exists x_0 \in S \ni \alpha - r < x_0 \leq \alpha < \alpha + r \implies (\alpha - r, \alpha + r) \cap S \neq \emptyset \implies \alpha \in \overline{S}$

2.6. Compact Set in Metric Space.

- Compact sets in metric space, which is closely related to the extreme value problem.
- Compact set \mathbb{R}^k will be discussed in next section.

Definition. Let X be a topology space and $S \subseteq X$. A collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of open sets in X is called an open covering of S if

$$S \subseteq \bigcup \mathcal{U} = \bigcup_{\alpha \in I} U_\alpha$$

Definition. Let X be a topology space, $S \subseteq X$ and $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of S . We say that \mathcal{U} has a countable(finite) sub covering of S if \exists a countable(finite) sub collection of \mathcal{U} which also covers S . i.e. \mathcal{U} has a countable(finite) subcovering in S if

\exists a sequence $\{\alpha_n\}$ in $I \ni S \subseteq \bigcup_{n=1}^{\infty} U_{\alpha_n}$ (countable)

\exists a sequence $\{\alpha_n\}$ in $I \ni S \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$ (finite)

Example

- (1) X is discrete metric space. Then $\{\{x\} \mid x \in X\}$ is an open covering of X
- (2) In \mathbb{R} , $\{(0, 1 - \frac{1}{n}) \mid n \in \mathbb{N}\}$ is an open covering of $(0, 1)$. In fact,

$$(0, 1) = \bigcup_{n=1}^{\infty} (0, 1 - \frac{1}{n})$$

(3) $\{B(0, n) \mid n \in \mathbb{N}\}$ is an open covering of \mathbb{R}^k

Definition (compact). Let X be a topology space. A subset $K \subseteq X$ is said to be compact if **every** open covering of K admit a finite subcovering

Examples

(1) Let X be a topology space and $K \subseteq X$ be a finite set. Then K is compact.

(2) In a discrete metric space X , a subset $K \subseteq X$ is compact $\Leftrightarrow K$ is a finite set.

(3) $(0, 1)$ is not compact in $\mathbb{R}(\{0, 1 - \frac{1}{n} \mid n \in \mathbb{N}\})$, but $[0, 1]$ is compact

Theorem 2.10. Let X be a metric space and $K \subseteq Y \subseteq X$. Then K is compact in $X \Leftrightarrow K$ is compact in Y .

Proof. (\Rightarrow) Suppose K is compact in X . Given an open covering $\{V_\alpha\}_{\alpha \in I}$ of open sets in Y which covers K . By Prop 2.5, each $V_\alpha = U_\alpha \cap Y$, where U_α is open in X . Now,

$$K \subseteq \bigcup_{\alpha \in I} V_\alpha = \bigcup_{\alpha \in I} (U_\alpha \cap Y) = (\bigcup_{\alpha \in I} U_\alpha) \cap Y \Rightarrow K \subseteq \bigcup_{\alpha \in I} U_\alpha$$

By the compactness of K in X , $\exists \alpha_1, \dots, \alpha_n \in I \ni K \subseteq \bigcup_{i=1}^n U_{\alpha_i} \Rightarrow$

$$K \cap Y \subseteq (\bigcup_{i=1}^n U_{\alpha_i}) \cap Y \Rightarrow K \subseteq \bigcup_{i=1}^n (U_{\alpha_i} \cap Y) = \bigcup_{i=1}^n V_{\alpha_i}$$

$\therefore K$ is compact in Y

(\Leftarrow) Suppose K is compact in Y . Given a open covering $\{U_\alpha\}_{\alpha \in I}$ of K by open sets in X .

$$K \subseteq \bigcup_{\alpha \in I} U_\alpha \Rightarrow K \cap Y \subseteq (\bigcup_{\alpha \in I} U_\alpha) \cap Y \Rightarrow K \cap Y \subseteq \bigcup_{\alpha \in I} (U_\alpha \cap Y)$$

By Prop 2.5, $\{U_\alpha \cap Y \mid \alpha \in I\}$ is an open covering of K by open set in Y . By assumption, K is compact in $Y, \exists \alpha_1, \dots, \alpha_n \in I \ni K \subseteq$

$$\bigcup_{i=1}^n (U_{\alpha_i} \cap Y) = (\bigcup_{i=1}^n U_{\alpha_i}) \cap Y \Rightarrow K \subseteq \bigcup_{i=1}^n U_{\alpha_i}$$

$\therefore K$ is compact in X ■

Definition. Let X be a metric space and $S \subseteq X$ be a nonempty set. The diameter of S is defined to be $\text{dia}(S) = \sup\{d(x, y) \mid x, y \in S\}$ which generated the diameter of a circle in \mathbb{R}^2

Theorem 2.11. *Let X be a metric space and $K \subseteq X$ be a compact set. Then K is closed and bounded*

Proof. **K is bounded**

Fix a point $p \in K$. Then $K \subseteq \bigcup_{n=1}^{\infty} B(p, n)$. $\because K$ is compact $\implies \exists N \in \mathbb{N} \ni K \subseteq B(p, 1) \cup \dots \cup B(p, N) \implies K \subseteq B(p, N) \therefore K$ is bounded

K is closed, i.e. $X - K$ is open
Fix $p \in X - K$. Then $p \neq x, \forall x \in K$. Hence, $d(x, p) > 0, \forall x \in K$

Let $r_x = \frac{1}{2}d(x, p) > 0, x \in K$. Then $\{B(x, r_x) \mid x \in K\}$ is an open covering of K . $\because K$ is compact $\implies \exists x_1, \dots, x_n \in K \ni$

$B(x_1, r_{x_1}) \cup \dots \cup B(x_n, r_{x_n})$. Let $V = \bigcap_{i=1}^n B(p, r_{x_i}) = B(p, r)$, where

$r = \min\{r_{x_1}, \dots, r_{x_n}\}$. Then as we can see that $V \subseteq X - K$, all point in $X - K$ are inner point. So $X - K$ is open, i.e. K is close.

To show that $V \subseteq X - K$, i.e. $V \cap K \neq \emptyset$, it suffices to show

$$V \cap \left(\bigcup_{i=1}^n B(x_i, r_{x_i})\right) = \emptyset$$

Now,

$$\begin{aligned} V \cap \left(\bigcup_{i=1}^n B(x_i, r_{x_i})\right) &= \bigcup_{i=1}^n (V \cap B(x_i, r_{x_i})) \\ &\subseteq \bigcup_{i=1}^n (B(p, r_{x_i}) \cap B(x_i, r_{x_i})) = \emptyset \end{aligned}$$

■

Remark. *The converse of Thm 2.11 is false, i.e. closed & bounded may not be compact, e.g. X is an infinite set with discrete metric. Then X is not compact, but X is closed and bounded.*

Theorem 2.12. *Let X be a metric space, $K \subseteq X$ be compact & $L \subseteq K$ be a closed set in X . Then L is compact.*

Proof. Let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of L . Then $\{U_\alpha\}_{\alpha \in I} \cup \{X - L\}$ is an open covering of K . By the compactness of K , $\exists \alpha_1, \dots, \alpha_n \in I \ni K \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_n} \cup (X - L)$. By $L \subseteq K \therefore L$ is compact ■

Corollary 2.13.

(a) Let X be a metric space, $K \subseteq X$ be compact and F be a closed set in X . Then $K \cap F$ is compact.

(b) If X is a compact metric space, then every closed subset F of X is compact.

Proof.

(a)

$$\begin{aligned} K \text{ is compact} &\implies K \text{ is closed (Thm 2.11)} \\ &\implies K \cap F \text{ is closed in } X \\ &\implies K \cap F \text{ is compact} \end{aligned}$$

(b) follows (a)

■

Remark. Let X be a metric space. If K is closed in X and F is closed in K , then F is closed in X . $\because F$ is closed in $F \implies F = L \cap F$, where L is closed in $K \implies F = L \cap K$, where L is closed in $X \implies F$ is closed in X .

Theorem 2.14. Let X be a metric space, $\{K_\alpha\}_{\alpha \in I}$ be a collection of compact subsets of X with the property:

$$\forall \alpha_1, \dots, \alpha_n \in I, K_{\alpha_1} \cap \dots \cap K_{\alpha_n} \neq \emptyset$$

$$\text{Then } \bigcap_{\alpha \in I} K_\alpha \neq \emptyset$$

$$\text{Proof. Fix } \alpha_0 \in I. \text{ Assume that } \bigcap_{\alpha \in I} K_\alpha = \emptyset \implies X - \bigcap_{\alpha \in I} K_\alpha$$

$$\implies X - \emptyset = X \implies X = \bigcup_{\alpha \in I} (X - K_\alpha)$$

each K_α is compact $\implies K_\alpha$ is closed $\implies X - K_\alpha$ is open
so $\{X - K_\alpha\}$ is an open covering of X . Now,

$$K_{\alpha_0} \subseteq X = \bigcup_{\alpha \in I} (X - K_\alpha) \implies K_{\alpha_0} \subseteq \bigcup_{\alpha \in I} (X - K_\alpha)$$

$$K_{\alpha_0} \text{ is compact } \implies \exists \alpha_1, \dots, \alpha_n \in I - \{\alpha_0\} \ni K_{\alpha_0} \subseteq (X - K_{\alpha_1}) \cup \dots \cup (X - K_{\alpha_n}) \implies K_{\alpha_0} \cap K_{\alpha_1} \cap \dots \cap K_n = \emptyset (\rightarrow \leftarrow) \quad \blacksquare$$

Corollary 2.15. *Let X be a metric space and $\{K_n\}_{n=1}^{\infty}$ be a decrease sequence of nonempty compact sets of X . Then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$. In addition, if $\text{dia}_{n \rightarrow \infty} \rightarrow 0$, then $\bigcap_{n=1}^{\infty} K_n$ is a singleton.*

Proof. $\forall j_1, \dots, j_k \in \mathbb{N}$, $K_{j_1} \cap \dots \cap K_{j_k} \neq \emptyset$, $K_{j_1} \cap \dots \cap K_{j_k} = K_t$, where $t = \max\{j_1, \dots, j_k\}$. By Thm 2.14 $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$, if $\lim_{n \rightarrow \infty} \text{dia}(K_n) = 0$ and $p, q \in \bigcap_{n=1}^{\infty} K_n$ and $p \neq q$, then $\text{dia}(K_n) \geq d(p, q) \forall n \geq 1 \implies \lim_{n \rightarrow \infty} \text{dia}(K_n) \geq d(p, q) > 0 (\rightarrow \leftarrow) \therefore \bigcap_{n=1}^{\infty} K_n = \{p\}$ is a singleton. ■

Remark. *The usual form of Cor 2.15, X is a metric space, $\{K_n\}$ is a decrease sequence of nonempty closed sets in X with K_i is compact $\implies \bigcap_{n=1}^{\infty} K_n \neq \emptyset$*

Example In \mathbb{R} , $\{(0, \frac{1}{n} \mid n \geq 1]\}$ is decrease and every finite subcollection of $\{(0, \frac{1}{n}) \mid n \geq 1\}$ is nonempty, but $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset$, $\bigcap [0, \frac{1}{n}] = \emptyset$

Theorem 2.17. *Let X be a metric space and $K \subseteq X$, TFAE:*

- (i) K is compact
- (ii) Every infinite subset has an accumulation point in K
- (iii) K is sequentially compact
- (iv) K is complete and totally bounded

Definition (Convergence). $\{a_n\}$ converge if $\exists a \in X \ni \forall \epsilon \geq 0 \exists N \in \mathbb{N} \ni \forall n \geq N$, $d(a_n, a) < \epsilon$. Such a is called the limit of $\{a_n\}$, which is denoted by $\lim_{n \rightarrow \infty} a_n = a$ or $a_n \rightarrow a$ on $n \rightarrow \infty$.

Definition (Cauchy). We say that $\{a_n\}$ is Cauchy if $\forall \epsilon > 0$, $\exists N \in \mathbb{N} \ni \forall n, m \geq N$, $d(a_n, a_m) < \epsilon$

Definition. A metric X is said to be sequence compact if every sequence has a convergent subsequence

Definition. A metric space X is said to be complete if every Cauchy sequence in X convergence.

Definition. Let X be a metric space & $K \subseteq X$. We say that K is totally bounded if $\forall r > 0, \exists x_1, \dots, x_n \in K \ni K \subseteq B(x_1, r) \cup \dots \cup B(x_n, r)$

Remark. Totally bounded can implies bounded, but not converse.

K is totally bounded, for $r = 1, \exists x_1, \dots, x_n \in K \subseteq B(x_1, 1) \cup \dots \cup B(x_n, 1) \implies K \subseteq B(x_1, R)$ for sime large R

- Take an "**infinite**" set X with discrete metric. Then X is bounded (e.g. $X \subseteq B(x_0, 2)$, where $x_0 \in X$) but for $r = \frac{1}{2}$, $X \not\subseteq B(x_1, \frac{1}{2}) \cup \dots \cup B(x_n, \frac{1}{2}) \forall x_1, \dots, x_n$

Lemma 2.18 (To prove (ii) to (i)).

Suppose (ii) holds in Thm 2.17. Then K is totally bounded

Proof. If not, then $\exists r > 0, \ni$ no finite open balls with radius r and center K cover K . Choose $x_1 \in K \implies K \not\subseteq B(x_1, r) \implies \exists x_2 \in K - B(x_1, r)$, $K \not\subseteq B(x_1, r) \cup B(x_2, r) \implies \exists x_3 \in K - (B(x_1, r) \cup B(x_2, r))$. By induction, counting this process, we obtain an infinite set $T = \{x_1, x_2, \dots, x_n, \dots\} \subseteq K$ with $d(x_i, x_j) \geq r \forall i \neq j$. By (ii), T has an accumulation poion $p \in K$. In particular $B(p, \frac{r}{4}) \cap T - \{p\}$ is an infinite set, hence, $\exists i \neq j \ni x_i, x_j \in B(p, \frac{r}{4}) \cap T - \{p\} \implies d(x_i, x_j) \leq d(x_i, p) + d(x, j) < \frac{r}{4} + \frac{r}{4} = \frac{r}{2} < r (\rightarrow \leftarrow) \therefore K$ is totally bounded. ■

Lemma 2.19. Suppose (ii) holds in T and $\{E_\alpha \mid \alpha \in I\}$ is an open covering of K . Then $\exists r > 0$ (called a Lebegoue number w.r.t. the open covering $\{E_\alpha\}_{\alpha \in I}$) $\ni \forall x \in K, B(x, r) \subseteq E_\alpha$ for some $\alpha \in I$

Proof. if K is a finite set, let $K = \{x_1, \dots, x_n\} K \subseteq \bigcup_{\alpha \in I} E_\alpha \implies x_i \in E_{\alpha_i}$ for some $\alpha_i \in I, 1 \leq i \leq n \implies \exists r_i > 0 \ni B(x_i, r_i) \subseteq E_{\alpha_i}, 1 \leq i \leq n$. Let $r = \min\{r_1, \dots, r_n\}$. Then $B(x_i, r) \subseteq B(x_i, r_i) \subseteq E_{\alpha_i}, 1 \leq i \leq n$. Now, assume that K is infinite set. Assume that no such $r > 0$, i.e. $\forall r > 0, \exists x_r \in K \ni B(x_r, r) \not\subseteq E_\alpha \forall \alpha \in I$. Now, for $r = \frac{1}{k}, k = 1, 2, \dots$, we obtain a sequence $\{x_k\}$ in K , with $x_k = \frac{x_r}{k} \ni B(x_k, \frac{1}{k}) \not\subseteq E_\alpha \forall \alpha \in I$. Let $T = \{x_1, x_2, \dots, x_k, \dots\}$. Then $T \subseteq K$ is an infinite set (\because For $k = 1, r = \frac{1}{1} = 1 \exists x_1 \in K \ni B(x_1, 1) \not\subseteq E_\alpha \forall \alpha \in I$). The conclusion of Lemma 2.19 failed for $K - \{x_1\}$ (\because if $\exists s > 0 \ni \forall x \in K - \{x_1\} B(x, s) \subseteq E_\alpha$ for some $\alpha \in I, x_1 \in E_\alpha \implies \exists t > 0 \ni B(x_1, t) \subseteq E_\alpha$. Let $r = \min\{s, t\}$. Then $\forall x \in K, B(x, r) \subseteq E_\alpha$ for some $\alpha \in I (\rightarrow \leftarrow)$)

Then for $r = \frac{1}{2}$, $\exists x_2 \in K - \{x_1\} \ni B(x_2, \frac{1}{2}) \subsetneq E_\alpha \forall \alpha \in I$. Continue this process, we conclude that $x_i \neq x_j \forall i \neq j$, so T is an infinite set. By the assumption of (ii). Then an accumulation point $p \in K$. Now $K = \bigcup_{\alpha \in I} E_\alpha \implies p \in E_\alpha$ for some $\alpha \in I \implies \exists \epsilon > 0 \ni B(p, \epsilon) \subseteq E_{\alpha_0}$.

Since $p \in T'$, $B(p, \epsilon) \cap T - \{p\}$ is an infinite set. Choose $m \gg 0 \ni \frac{1}{m} < \frac{\epsilon}{2}$ & $x_m \in B(p, \frac{\epsilon}{2}) \cap T$. Claim $B(x_m, \frac{1}{m}) \subseteq B(p, \epsilon) \subseteq E_{\alpha_0} (\rightarrow \leftarrow)$ to our constrain, hence Lemma 2.19 holds.

$$\begin{aligned} y \in B(x_m, \frac{1}{m}) &\Rightarrow d(y, p) \leq d(y, x_m) + d(x_m, p) < \frac{1}{m} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\ &\Rightarrow y \in B(p, \epsilon) \quad \blacksquare \end{aligned}$$

Proof. (Thm 2.17 (i)(ii))

(i) \Rightarrow (ii) Suppose K is compact. Given an infinite set $T \subseteq K$. We must prove that T has an accumulation point in K , if not, $\forall x \in K, x$ is not an accumulation point of K , $\exists r_x > 0 \ni B(x, r_x) \cap T - \{x\} = \emptyset \implies B(x, r_x) \cap T \subseteq \{x\}$. Clearly $\{B(x, r_x) \mid x \in K\}$ is an open covering of K . By (i), K is compact

$$\begin{aligned} \implies \exists x_1, \dots, x_n \in K \ni K &\subseteq B(x_1, r_{x_1}) \cup \dots \cup B(x_n, r_{x_n}) \\ &= (T \cap B(x_1, r_{x_1})) \cup \dots \cup (T \cap B(x_n, r_{x_n})) \\ &\subseteq \{x_1\} \cup \{x_2\} \cup \dots \cup \{x_n\} \\ &= \{x_1\} \cup \{x_2\} \cup \dots \cup \{x_n\} \\ &= \{x_1, \dots, x_n\} (\rightarrow \leftarrow) \end{aligned}$$

to T is an infinite set, \therefore (ii) holds.

(ii) \Rightarrow (i) In Thm 2.17 i.e. we must prove that K is compact under the assumption of (ii). Suppose $\mathcal{U} = \{E_\alpha\}_{\alpha \in I}$ is an open covering of K . By Lemma 2.19, \exists a $r > 0$ w.r.t. \mathcal{U} , by Lemma 2.18, $\exists x_1, \dots, x_k \in K \ni K \subseteq B(x_1, r) \cup \dots \cup B(x_k, r) \subseteq E_{\alpha_1} \cup \dots \cup E_{\alpha_k}$, where $B(x_i, r) \subseteq E_{\alpha_i}, 1 \leq i \leq k$. Therefore, \mathcal{U} has a finite sub covering. Hence K is compact and (i) holds. \blacksquare

Remark.

(1) (ii) \implies (i) is exercise 26

(2) More or less, by Lemma 2.18 & 19, one can see that (i) and (ii) are also equal to (iii) and (iv)

2.7. Compact Sets in Euclidean Spaces \mathbb{R}^k .

- We know that any compact set in a metric space is close and bounded
- Close and bounded subset may not be compact (infinite discrete)
- We will see that every closed and bounded subset of \mathbb{R}^k is always compact which is the famous H.B. Theorem, i.e. $K \subseteq \mathbb{R}^k$ is compact $\Leftrightarrow K$ is closed and bounded

Theorem 2.20. *Let $\{I_n = [a_n, b_n]\}_{n=1}^\infty$ be a sequence of closed and bounded intervals in \mathbb{R} , if $\{I_n\}$ is decreasing i.e. $I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$, then $\bigcap_{n=1}^\infty I_n \neq \emptyset$. Moreover, if $\lim_{n \rightarrow \infty} (b_n - a_n) = 0$, then $\bigcup_{n=1}^\infty I_n$ is a singleton.*

Proof. Claim $T = \{a_n \mid n \in \mathbb{N}\}$ is bounded above and $x = \sup T$ exists. $\because a_n \leq a_{m+n} (\because \{a_n\} \text{ is increasing, i.e. } [a_1, b_1] \supseteq [a_2, b_2], a_2 \geq a_1)$
 $a_n \leq a_{m+n} \leq b_{m+n} \leq b_m (\because \{b_n\} \text{ is decreasing}) \implies T \text{ is bounded above by all } b_n \implies x = \sup T \text{ exists and } x \leq b_n \forall n \geq 1$.
Clearly, $a_n \leq x \forall n \geq 1$, $\because a_n \leq x \leq b_n, \forall n \geq 1$, i.e. $x \in [a_n, b_n] \forall n \geq 1$. $\therefore x \in \bigcap_{n=1}^\infty I_n$. Hence, $\bigcap_{n=1}^\infty I_n \neq \emptyset$. The last statement follows the argument as in Corollary 2.16. ■

Theorem 2.21. *Let $\{I_n = [a_{n,1}, b_{n,1}] \times \cdots \times [a_{n,k}, b_{n,k}]\}$ be a decreasing sequence of closed and bounded intervals in \mathbb{R}^k . Then $\bigcap_{n=1}^\infty I_n \neq \emptyset$.*

Moreover, if $\lim_{n \rightarrow \infty} \text{dia}(I_n) = 0$, then $\bigcup_{n=1}^\infty I_n$ is a singleton.

Proof. $\forall 1 \leq j \leq k, \{[a_{n,j}, b_{n,j}]\}$ is a decrease sequence of closed and bounded intervals in \mathbb{R} . By Thm 2.20, $\exists x_j \in \bigcap_{n=1}^\infty [a_{n,j}, b_{n,j}]$. Set $x = (x_1, \cdots, x_k)$. Then $x \in \bigcap_{n=1}^\infty I_n$. Then last statement also follows from the argument in corollary 2.16. ■

Theorem 2.22. *Every k -dimensional closed and bounded interval $I = [a_1, b_1] \times \cdots \times [a_k, b_k]$ in \mathbb{R}^k is compact.*

Proof. Put $\delta = \left(\sum_{i=1}^k (b_i - a_i)^2\right)^{\frac{1}{2}}$ which is the diametric of I . Then $\forall x, y \in I, \|x - y\| \leq \delta$. If I were not compact, then \exists an open covering $\{E_\alpha\}_{\alpha \in J}$ admitting not finite sub covering(\star). Put $c_j = \frac{a_j + b_j}{2}$, $1 \leq j \leq k$. The intervals $[a_j, c_j]$ and $[c_j, b_j]$, $1 \leq j \leq k$, determines 2^k closed and bounded subinterval of I whose union is I . By (\star), at least one of them, say I_1 which cannot be covered by finitely many E_α . Continuing this process, we get a sequence $\{I_n\}$ of closed and bounded subintervals of I satisfy's

- a) $I \subseteq I_1 \subseteq \dots$, i.e. $\{I_n\}_{n=1}^\infty$ is decreasing.
- b) Each I_n cannot be covered by finitely many E_α
- c) $\text{dia}(I_n) = 2^{-n}$, $\delta \rightarrow 0$ on $n \rightarrow \infty$

By Thm 2.21, $\bigcap_{n=1}^\infty I_n = \{x\}$ i.e. $x \in I_n \subseteq I \forall n \geq 1 \subseteq \bigcup_{\alpha=1}^\infty E_\alpha$
 $\therefore x \in E_{\alpha_0}$ for some $\alpha_0 \in J$, E_{α_0} is open $\implies \exists r > 0 \ni B(x, r) \subseteq E_{\alpha_0}$.
 Choose $n_0 \gg 0 \ni \frac{1}{2^{n_0}} < \frac{r}{\delta} (\therefore \frac{1}{2^n} \rightarrow 0)$. Since $x \in I_{n_0}, \forall y \in I_{n_0}, \|y - x\| \leq 2^{-n_0} \delta < \frac{r}{\delta} \cdot \delta = r \implies y \in B(x, r), \therefore I_{n_0} \subseteq B(x, r) \subseteq E_{\alpha_0} (\rightarrow \leftarrow)$.
 Therefore I is compact. ■

Combining Thm 2.22 and results in section 2.6, we conclude that the following sets in compact:

- i) $[a, b]$ is compact in \mathbb{R} (Thm 2.22 with $k = 1$)
- ii) $[a, b] \times [c, d]$ is compact in \mathbb{R}^2 (Thm 2.22 with $k = 2$)
- iii) Every closed ball $\overline{B}(x, r)$ in \mathbb{R}^k is compact by Thm 2.12 and 2.22
- iv) $\{0\} \cup \{\frac{1}{n} \mid n = 1, 2, \dots\}$ is compact in \mathbb{R} , In fact, if $a_n \rightarrow a$, then the set $\{a\} \cup \{a_n \mid n = 1, 2, \dots\}$ is compact in \mathbb{R}

Theorem 2.23. *Every closed and bounded subset K of \mathbb{R}^k is compact.*

Proof. Choose a large closed and bounded interval I in \mathbb{R}^k , $K \subseteq I$. By Thm 2.22, I is compacted, so K is a closed subset of I . By Thm 2.12, K is compacted.

Combining Thm 2.17 and Thm 2.23, we can characterize compacted set in \mathbb{R}^k ■

Theorem 2.24. Let $K \subseteq \mathbb{R}^k$ TFAE:

- i) K is closed and bounded
- ii) K is compact
- iii) Every infinite subset of K has an accumulation point
- iv) K is sequence compact
- v) K is complete and totally bounded

From these we can deduce:

Theorem 2.25 (Bolzano-Weierstrass). Every bounded infinite subset T in \mathbb{R}^k has an accumulation point in \mathbb{R}^k

Proof. Since T is bounded, choose a large closed and bounded interval I in $\mathbb{R}^k \ni T \subseteq I$. Now, T become an infinite subset of the compact set I . By Thm 2.24 (iii), T has an accumulation point in I . ■

Theorem 2.26 (Cantor intersection). Let $\{Q_n\}$ be a sequence of nonempty set in \mathbb{R}^k satisfying:

- a) $\{Q_n\}$ is decreasing
- b) Q_n is closed $\forall n \geq 1$ & Q_1 is compact.

Then $\bigcap_{n=1}^{\infty} Q_n \neq \emptyset$, Moreover, if $\text{dia}(Q_n) \rightarrow 0$ on $n \rightarrow \infty$, then $\bigcap_{n=1}^{\infty} Q_n$ is a singleton.

Proof. By (b), each Q_n is compact ($\because Q_1 \supseteq Q_n \forall n \geq 1$). Therefore, it follows from Cor 2.16 and 2.15. ■

2.8. Countability & Separability.

Motivation: In \mathbb{R}^k , we have two facts:

- \mathbb{Q}^k is dense in \mathbb{R}^k i.e. $\overline{\mathbb{Q}^k} = \mathbb{R}^k$ & \mathbb{Q}^k is countable. i.e. \mathbb{R}^k has a countable dense subset. i.e. \mathbb{R}^k is separable.
- $\{B(x, r) \mid x \in \mathbb{Q}^k, r \in \mathbb{Q}^+\}$ is a countable collection of open ball in \mathbb{R}^k satisfying: \forall open set $U \subseteq \mathbb{R}^k$ and $y \in U \exists B(x, r) \in \mathcal{B} \ni y \in B(x, r) \subseteq U$. In particular, U is a union of some sub collection of \mathcal{B} .
 $\therefore U = \bigcup_{y \in U} B_y$, i.e. \mathbb{R}^k is of 2^{nd} countable.

- X is a metric space $x \in X$, $N_x = \{B(x, \frac{1}{n}) \mid n \in \mathbb{N}\}$ is a countable collection of nbh of x . Such N_x satisfies: \forall nbh U of $x, \exists n \in \mathbb{N} \ni B(x, \frac{1}{n}) \subseteq U$ ($\because \exists r > 0 \ni B(x, r) \subseteq U$, choose $n \gg 0 \ni \frac{1}{n} < r$). Then $x \in B(x, \frac{1}{n}) \subseteq B(x, r) \subseteq U$. i.e. Each point of X has a countable nbh base(system) i.e. X is of 1^{st} countable, i.e. \mathbb{R}^k has a countable base.

Definition. Let X be a topological space

- (1) X is first if every point of X has a countable nbh system (or base), i.e. \exists a countable collection $\{V_n \mid n \in \mathbb{N}\}$ of nbh of $x \ni \forall$ nbh U of $x, \exists n \in \mathbb{N}, V_n \subseteq U$
- (2) X is of second countable if X has a countable base, i.e. \exists a countable collection $\mathcal{B} = \{B_n \mid n \in \mathbb{N}\}$ of open sets in $X \ni$ every open set U is a union of some subcollection of \mathcal{B} or \forall open set U in X and $x \in U, \exists n \in \mathbb{N} \ni x \in B_n \subseteq U$
- (3) X is separable if X has a countable dense subset, i.e. \exists a countable set $D \subseteq X \ni \overline{D} = X$.

Remark.

- (1) Every 2^{nd} countable topology space X is of 1^{st} countable, but not converse. \therefore Let $\mathcal{B} = \{B_1, B_2, \dots\}$ be a countable base for X . Given $p \in X$, let $\mathcal{B}_p = \{B_n \mid p \in B_n\}$. Then \mathcal{B}_p is countable and is a collection of open set in X containing of p .

Claim \mathcal{B}_p is a nbh system (or base) of p . Let U be a nbh of p . Then $U = \bigcup_{n \in F} B_n, F \subseteq \mathbb{N}$. In particular, $p \in B_n$ for some $n \in F$. Hence, $B_n \in \mathcal{B}_p$ & $p \in B_n \subseteq U, \therefore \mathcal{B}_p$ is a countable nbh system of p . Hence, X is of 1^{st} countable.

Consider X an uncountable set with discrete metric. Hence, X is 1^{st} countable. In fact, $\forall p \in X, N_p = \{\{p\}\}$ is a countable nbh base of p . However, X is not of 2^{nd} countable. Note that if \mathcal{B} is a base for the discrete space X then $\mathcal{B} \subseteq \{\{x\} \mid x \in X\}$

$$\therefore \{x\} \text{ is open} \implies \{x\} = \bigcup_{\alpha \in I} B_\alpha, B_\alpha = \{x\} \forall \alpha \in I \implies$$

$$B_\alpha = \{x\} \forall x \in I \implies \{x\} \in \mathcal{B}.$$

Now, X is uncountable, so is \mathcal{B} . Hence, X is not of 2^{nd} countable.

- (2) We know that every metric space is of 1^{st} countable. In fact, $\forall p \in X, N_p = \{B(p, \frac{1}{n}) \mid n \in \mathbb{N}\}$ is a countable nbh system of p

- (3) We know that \mathbb{R} is separable with countable dense subset \mathbb{Q} . In general, \mathbb{R}^n is separable with countable dense subset \mathbb{Q}^n (Exercise 22)

Theorem 2.27. Every 2^{nd} countable topology space is separable

Remark. Note that X is a metric space. $D \subseteq X$.

$$\begin{aligned} D \text{ is dense in } X &\Leftrightarrow \overline{D} = X \Leftrightarrow \forall \text{ nonempty open set } U \text{ in } X, U \cap D \neq \emptyset \\ &\Leftrightarrow \forall x \in X, \exists \text{ a sequence } \{a_n\} \text{ in } D \\ &\quad \ni a_n \rightarrow x \text{ on } n \rightarrow \infty \end{aligned}$$

Proof.

(\Rightarrow) Suppose $\overline{D} = X$, i.e. D is dense in X , i.e. $\forall x \in X, x \in \overline{D}$. Now, given a nonempty open set U in X . Choose $x \in U$. So U is a nbh of x , hence $U \cap D \neq \emptyset$

(\Leftarrow) Suppose the condition holds, then $\forall x \in X$ and nbh U of x , $U \cap D \neq \emptyset \Rightarrow x \in \overline{D} \Rightarrow X \subseteq \overline{D} \subseteq X \Rightarrow \overline{D} = X$, i.e. D is dense in X . ■

Proof. (Theorem 2.27) Let $\mathcal{B} = \{B_1, B_2, \dots, B_n, \dots\}$ be a countable base of X . Choose a point $x_n \in B_n$, $n \in \mathbb{N}$ and form the set $D = \{x_1, x_2, \dots, x_n, \dots\}$, then D is countable.

Claim: D is dense in X , i.e. $\overline{D} = X$. Given a nonempty open set U in X , then $\exists n \in \mathbb{N} \ni B_n \subseteq U \Rightarrow x_n \in U \Rightarrow U \cap D \neq \emptyset$. Hence, $\overline{D} = X$ by the remark above. So X is separable. ■

Theorem 2.28. Every separable metric space X is of 2^{nd} countable.

Proof. Choose a countable dense subset D of X . Form the countable collection of open ball $\mathcal{B} = \{B(x, r) \mid x \in D, r \in \mathbb{Q}^+\}$ (it is countable). Claim \mathcal{B} is a base for X . We are done.

(Note that \mathcal{B} is a base for a topology space X

\Leftrightarrow every open set in X is a union of some subcollection of \mathcal{B}

$\Leftrightarrow \forall$ open set $U \subseteq X$ and $p \in U$, $\exists B \in \mathcal{B} \ni p \in B \subseteq U$)

(\Rightarrow) Given an open set U in X and $p \in U$ by assumption $U = \bigcup_{\alpha \in I} B_\alpha$,

where $B_\alpha \in \mathcal{B} \Rightarrow p \in B_{\alpha_0}$ for some $\alpha_0 \in I \Rightarrow p \in B_{\alpha_0} \subseteq U$

(\Leftarrow) Suppose the condition holds. To prove \mathcal{B} is a base for X . Given a nonempty open set U in X . $\forall p \in U, \exists B_p \in \mathcal{B} \ni p \in B_p \subseteq U$.

$\therefore U = \bigcup_{p \in U} B_p$ $\therefore \mathcal{B}$ is a base for X . By the remark above, it is

enough to show: Given a nonempty open set $U \subseteq X$ and $p \in U$, $\exists B(x, r) \in \mathcal{B} \ni p \in B(x, r) \subseteq U$. Now, $p \in U$ and U is open \Rightarrow

$\exists t > 0 \ni B(p, r) \subseteq U$. Choose $r \in \mathbb{Q}^+ \ni \frac{t}{4} < r < \frac{t}{2}$. Since D is dense in X , $B(p, r) \cap D \neq \emptyset$. Choose $x \in B(p, r) \cap D$. Then $B(x, r) \in \mathcal{B}$

Claim $p \in B(x, r) \subseteq U$

- $d(x, p) < r \implies p \in B(x, r)$
- $\forall y \in B(x, r), d(y, p) \leq d(y, x) + d(x, p) < r + r = 2r < t \implies y \in B(p, t) \subseteq U \therefore y \in U \therefore B(x, r) \subseteq U.$

■

Corollary 2.29. *The Euclidean space \mathbb{R}^k is of 2^{nd} countable.*

Note that from the proof of Thm 2.28, \mathbb{R}^k has a countable base of the form:

$$\begin{aligned} \mathcal{B} &= \{B(x, r) \mid x \in \mathbb{Q}^k \text{ \& } r \in \mathbb{Q}^+\} \\ &= \{A_1, A_2, \dots\} \end{aligned}$$

Theroem 2.30. *Every compact metric space X is of 2^{nd} countable.*

Proof. The last statement follows from Thm 2.27 To prove X is of 2^{nd} countable. For each $n \in \mathbb{N}$, $\{B(x, \frac{1}{n}) \mid x \in X\}$ is an open covering of X ,

i.e. $X = \bigcup_{x \in X} B(x, \frac{1}{n})$. By compactness of X , it has a finite subcovering

say $X = \bigcup_{i=1}^{l_n} B(x_{n_i}, \frac{1}{n})$. Then \mathcal{B} is a countable collection of open balls in X . Claim \mathcal{B} is a base for X . It suffices to show: given a nonempty open set U and $p \in U$, $\exists B(x_{n_i}, \frac{1}{n}) \ni \mathcal{B} \ni p \in B(x_{n_i}, \frac{1}{n}) \subseteq U$.

From $p \in U$ and U is open, $\exists r > 0 \ni B(p, r) \subseteq U$. Choose $n \gg 0 \ni \frac{2}{n} < r$. Since $X = \bigcup_{i=1}^{l_n} B(x_{n_i}, \frac{1}{n})$, $p \in B(x_{n_i}, \frac{1}{n})$ for some $1 \leq i \leq l_n$.

Finally, $p \in B(x_{n_i}, \frac{1}{n}) \subseteq U$

- $d(p, x_{n_i}) < \frac{1}{n} \implies p \in B(x_{n_i}, \frac{1}{n})$
- $\forall y \in B(x_{n_i}, \frac{1}{n}), d(y, p) \leq d(y, x_{n_i}) + d(x_{n_i}, p) < \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$

$\therefore p \in B(x_{n_i}, \frac{1}{n}) \subseteq U$

■

Theroem 2.31 (Lindelof Covering). *Let $S \subseteq \mathbb{R}^k$. Then every open covering $\mathcal{U} = \{U_\alpha \mid \alpha \in I\}$ of S has a countable subcoverings.*

Proof. Let $\{A_1, A_2, \dots\}$ be the countable base of \mathbb{R}^k defined as above. Note that $S \subseteq \bigcup_{\alpha \in I} U_\alpha$ for some $\alpha \in I$. Hence $\exists n \in \mathbb{N} \ni x \in A_n \subseteq U_\alpha$. Of course, there may be infinitely many such n . We choose one of them and fix it, say $x \in A_{m(x)} \subseteq U_\alpha$ (e.g. $m(x) = \min\{n \in \mathbb{N} \mid x \in A_n \subseteq U_\alpha\}$). Then the collection $\{A_{m(x)} \mid x \in S\}$ is a countable open covering of S . Finally, for each $A_{m(x)}$, choose $U_{\alpha_{m(x)}} \ni A_{m(x)} \subseteq U_{\alpha_{m(x)}}$. Then $\{U_{\alpha_{m(x)}} \mid x \in S\}$ is a countable subcovering of S . ■

Corollary 2.32. *Let $S \subseteq \mathbb{R}^k$ be open, if $S = \bigcup_{\alpha \in I} U_\alpha$ is a union of*

open sets in X , then $S = \bigcup_{n=1}^{\infty} U_{\alpha_n}$ is a countable union. \therefore By Lindelof covering theorem.

2.9. Perfect Sets in Metric Spaces. Recall a subset E in a metric space X is perfect if E is closed in X and every point of E is its accumulation point. i.e. $E' = E$

Example

- $-\infty < a < b < \infty$, $[a, b]$ is perfect
- \mathbb{R} is perfect

Theorem 2.33. *Every nonempty perfect set E in \mathbb{R}^k is uncountable*

Proof. E is an infinite set (\therefore finite set has no accumulation point), Suppose E were countable, write $E = \{x_1, x_2, \dots\}$. We use induction to construct a sequence $\{V_n\}$ of open sets in X as follows:

Let V_1 be any neighborhood of $y_1 = x_1$, e.g. $V_1 = B(x_1, r)$, its closure is $\overline{V_1} = \overline{B}(x_1, r)$, $x_1 \in E'$, $V_1 \cap E$ is an infinite set, so $\exists y_2 \in V_1 \cap E \ni y_2 \neq x_1$. Choose a neighborhood V_2 of y_2 \ni

- (i) $\overline{V_2} \subseteq V_1$
- (ii) $x_1 \notin \overline{V_2}$
- (iii) $V_2 \cap E \neq \emptyset$ ($\therefore y_2 \in E = E'$ and it's also an infinite set)

Suppose that, for $n \geq 3$, V_n has been chosen $\ni V_n$ is a neighborhood of some $y_n \in E \ni$

- (1) $\overline{V_n} \subseteq V_{n-1}$
- (2) $x_{n-1} \notin \overline{V_n}$
- (3) $V_n \cap E \neq \emptyset$ is an infinite set

Since $V_n \cap E$ is an infinite set, $\exists y_{n+1} \in V_n \cap E \ni y_{n+1} \neq y_n$. Again, choose a neighborhood V_{n+1} of y_{n+1} \ni

- (1) $\overline{V_{n+1}} \subseteq V_n$

- (2) $x_n \notin \overline{V_{n+1}}$
- (3) $V_{n+1} \cap E \neq \emptyset$ is an infinite set.

By induction, we have constructed such sequence $\{V_n\}$. Put $K_n = \overline{V_n} \cap E$, $n \geq 1$. Then $\{K_n\}$ is a decrease sequence of nonempty compact sets in \mathbb{R}^k .

$\overline{V_n}$ is closed, E is closed $\implies K_n = \overline{V_n} \cap E$ is closed, each $\overline{V_n}$ is bounded $\therefore K_n$ is closed and bounded by H.B. theorem, K_n is compact.

- $\emptyset \neq V_n \cap E \subseteq \overline{V_n} \cap E \implies E_n = \overline{V_n} \cap E \neq \emptyset$
- $\overline{V_n} \cap E \supseteq \overline{V_{n+1}} \cap E = K_{n+1} \therefore \{K_n\}$ is decrease.

By Cantor's intersection theorem, $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$. Pick $y \in \bigcap_{n=1}^{\infty} K_n$, $y \in E$ ($\because K_n \subseteq E \forall n \geq 1$). Since $x_n \notin \overline{V_{n+1}} \forall n \geq 1$, so $x_n \notin K_n \forall n \geq 1 \implies y \notin E$ ($\rightarrow \leftarrow$) to $E = \{x_1, x_2, \dots\}$
 $\therefore E$ is uncountable. ■

Corollary 2.34. *Every nondegenerate interval is uncountable.*

Proof. \because Every nondegenerate interval I in \mathbb{R} must contain a closed and bounded interval $[a, b]$ with $a < b$ which is perfect, so it is uncountable by theorem 2.31. Hence I is uncountable. ■

Construction of the Cantor set $\underline{P} \subseteq [0, 1]$ in \mathbb{R} which is a perfect set

(a) Remove the middle third open subinterval of $[0, 1]$. There are two closed subintervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Let $C_1 = (\frac{1}{3}, \frac{2}{3})$ and $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

(b) Remove the middle thirds of $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ respectively. There are $2^2 = 4$ subintervals $[0, \frac{1}{3^2}]$, $[\frac{2}{3^2}, \frac{3}{3^2}]$, $[\frac{6}{3^2}, \frac{7}{3^2}]$, $[\frac{8}{3^2}, 1]$

(c) Continue this process, we get a sequence $\{C_n\}$ of open sets and a sequence $\{E_n\}$ of closed sets satisfy

- (i) $E_0 \supseteq E_1 \supseteq E_2 \subseteq \dots$, i.e. $\{E_n\}$ is a decrease sequence of closed sets in $[0, 1]$
- (ii) Each E_n is a union of 2^n closed intervals, each of length 3^{-n}
- (iii) Each C_n is a union of 2^{n-1} open subintervals, each of length 3^{-n} .
total length is $\frac{2^{n-1}}{3^n}$

Definition. $\underline{P}(\text{or } C) = \bigcap_{n=1}^{\infty} E_n = [0, 1] - \bigcup_{n=1}^{\infty} C_n (= \bigcap_{n=1}^{\infty} ([0, 1] - C_n))$

Properties of Cantor set \underline{P} :

- (1) $\underline{P} \neq \emptyset$ by Cantor's intersection theorem
- (2) \underline{P} is compact ($\because \underline{P}$ is closed being \cap of closed set and $\underline{P} \subseteq [0, 1]$, $[0, 1]$ is compact)
- (3) \underline{P} is nowhere dense, i.e. $\overline{\underline{P}}^\circ = \emptyset$, i.e. $\underline{P}^\circ = \emptyset$
 $\because \underline{P}$ contains no nonempty open subintervals ($\because \underline{P}^\circ \neq \emptyset \implies \exists x \in \underline{P}^\circ \implies \exists \delta > 0, (x - \delta, x + \delta) \subseteq \underline{P}$). If $\alpha < \beta$ and $(\alpha, \beta) \subseteq \underline{P}$, then $(\alpha, \beta) \subseteq E_n \forall n \geq 1$. Choose $n \gg 0 \ni \frac{1}{2^n} < \beta - \alpha$. Then for $n \gg 0$, E_n contains subinterval of length $\geq \frac{1}{2^n} (\rightarrow \leftarrow)$. Hence, \underline{P} is nowhere dense.
- (4) $\underline{P} = \{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mid a_n = 0 \text{ or } 2 \forall n \geq 1 \}$

Recall the ternary representation of a number $x \in [0, 1]$, $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$, $a_n = 0, 1, 2 \forall n \geq 1$

if $\frac{1}{3}$ the a_n can be 1, then $\frac{1}{3} + \frac{1}{3}$ is not in \underline{P}
 if you want to represent $\frac{1}{3}$, you need use $0 + \frac{2}{3^2} + \frac{2}{3^3} + \dots$, i.e.
 $0.1 = 0.0\overline{9}$ same things.

This can be used to prove that \underline{P} is uncountable by $\underline{P} \rightarrow [0, 1]$, $x =$

$\sum_{n=1}^{\infty} \frac{a_n}{3^n} \rightarrow \sum_{n=1}^{\infty} \frac{a_n/2}{2^n}$ is bijective $\therefore \underline{P}$ is uncountable.

- (5) \underline{P} is of measure zero (i.e. the length of \underline{P} is zero)

\because The totally length remove in the construction of \underline{P} is $\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \dots = 1$. This also proves that \underline{P} is nowhere dense.

- (6) \underline{P} is perfect. In particular, by theorem 2.31, \underline{P} is uncountable
 \because Obviously \underline{P} is a nonempty closed set. Let $x \in \underline{P}$. Then $\forall (\alpha, \beta) \ni x \in (\alpha, \beta)$. We prove that $(\alpha, \beta) \cap \underline{P} - \{x\} \neq \emptyset$, which says that x is an accumulation point of \underline{P} . Hence \underline{P} is perfect. By $x \in \underline{P}$, $x \in E_n \forall n \geq 1$. Then \exists a closed subinterval $I_n \subseteq E_n \ni x \in I_n$. Choose $n \gg 0 \ni I_n \subseteq (\alpha, \beta)$. Let x_n be an end point of $I_n \ni x \neq x_n$. By construction, $x_n \in \underline{P}$, so

$$x_n \in (\alpha, \beta) \cap \underline{P} - \{x\}$$

i.e. x is an accumulation point of \underline{P} , Hence \underline{P} is perfect.

Definition. Let X be a metric space (or topological space) $A, B \subseteq X$. We say that A and B are separated if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty sets.

Definition. A subset $E \subseteq X$ is called connected if E is not a union of two nonempty separated sets and E is disconnected if E is not connected

Remark. X is connected $\Leftrightarrow X$ is not a union of two nonempty separated sets.

X is disconnected $\Leftrightarrow X$ is a union of two nonempty separated sets, say, $X = A \cup B$, A and B are nonempty separated. i.e. $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$, i.e. $A = \overline{A}$, $B = \overline{B}$ $\therefore A$ and B are closed $\therefore A$ and B are both open and closed.

Remarks and Examples

- (1) Separated sets and disjoint
- (2) $[0, 1]$ and $(1, 2)$ are not separated
- (3) $(0, 1)$ and $(1, 2)$ are separated

Theorem 2.35. Let $E \subseteq \mathbb{R}$ be a set. Then E is connected $\Leftrightarrow E$ is an interval

Proof. We may assume that $E \neq \emptyset$

(\Rightarrow) Assume that E is connected. If E were not an interval, then $\exists x < y$ in E and $z \notin E \ni x < z < y$. Let $A = (-\infty, z) \cap E$ and $B = (z, \infty) \cap E$. Then A, B are nonempty, $\overline{A} \cap B = \emptyset$, $A \cap \overline{B} = \emptyset$ and

$$\begin{aligned} E &= E \cap (\mathbb{R} - \{z\}) \\ &= E \cap [(-\infty, z) \cup (z, \infty)] \\ &= (E \cap (-\infty, z)) \cup (E \cap (z, \infty)) \\ &= A \cup B \end{aligned}$$

$\therefore \{A, B\}$ is a nonempty separation of E ($\rightarrow \leftarrow$) to connected. $\therefore E$ is an interval

(\Leftarrow) Suppose E is an interval. To show that E is connected. If not, then \exists two nonempty separated sets A and $B \ni E = A \cup B$. Pick $x \in A$ and $y \in B$. Then $x \neq y$ ($\because A \cap B = \emptyset$). We may assume that $x < y$. Define $z = \sup(A \cap [x, y])$, By (9) in section 2.5, $z \in \overline{A \cap [x, y]} \in \overline{A}$. Hence $z \notin B$ ($\because \overline{A} \cap B = \emptyset$). Also $x \leq z \leq y$. But $y \in B$ and $z \notin B \implies x \leq z < y$

If $z \notin A$, then $x < z < y$ and $z \notin E(\rightarrow\leftarrow)$ to E is an interval.

If $z \in A$, then $z \notin \overline{B}(\because A \cap \overline{B} = \emptyset)$, since $z \notin B$, $z \in \mathbb{R} - \overline{B}$ which is open $\implies \exists \delta > 0 \ni (z - \delta, z + \delta) \subseteq \mathbb{R} - \overline{B}$. Choose $z < z_1 < z + \delta < y$, i.e. $z < z_1 < y$. Then $z, y \in E$, $z < y$ and $z_1 \notin E(\rightarrow\leftarrow)$ to E is an interval ■

Application of Connectedness

X : connected topological space (or metric space)

P : a property on X

$D = \{x \in X \mid P \text{ holds at } x\}$

If one can prove D is nonempty and closed and open, then $D = X$

$\because X = D \cup (X - D)$, $\overline{D} \cap (X - D) = \emptyset$ and $D \cap \overline{(X - D)} = \emptyset$, i.e. D and $X - D$ are separated

Since X is connected and $D \neq \emptyset$, so $X - D = \emptyset$, i.e. $X = D$

3. Infinite Sequence & Series

- We will assume you are familiar with all operations of real(complex) sequence
- We have defined sequence in a set X

Recall : let $\{a_n\}$ be a real or complex sequence, $\{a_n\}$ converges if $\exists a \in \mathbb{R}(\mathbb{C})$ satisfying $\forall \epsilon > 0, \exists N \in \mathbb{N} \ni \forall n \geq N, |a_n - a| < \epsilon$

- Now, we study the properties of a sequence in a metric space(topological space)

3.1. Convergent Sequence. Let X be a metric space & $\{x_n\}$ be a sequence in X , $x : \mathbb{N} \rightarrow X$

Definition. We say that $\{x_n\}$ converges (in X) if $\exists p \in X$ satisfying $\forall \epsilon > 0, \exists N \in \mathbb{N} \ni \forall n \geq N, d(x_n, p) < \epsilon$, Otherwise, $\{x_n\}$ diverges.

Remark.

- (1) If $\{x_n\}$ converges as in definition, then p is called the limit of the sequence $\{x_n\}$ and is denoted by $\lim_{n \rightarrow \infty} x_n = p$ or $x_n \rightarrow p$ as $n \rightarrow \infty$
- (2) $x_n \rightarrow p$ as $n \rightarrow \infty \Leftrightarrow$ the real sequence $\{d(x_n, p)\}$ converges to 0, i.e. $\lim_{n \rightarrow \infty} d(x_n, p) = 0$
- (3) if $\{x_n\}$ converges, then its limit is !
- (4) The convergence of a sequence depends not only the sequence but also on the space.

e.g. $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ in \mathbb{R} , but $\{\frac{1}{n}\}$ diverges in $(0, 1)$

Recall Let $\{x_n\}$ are a sequence in a set X with is a function $x : \mathbb{N} \rightarrow X$. The image of the sequence = the image of the function X
 $= \{x_n \mid n = 1, 2, \dots\}$

Remark. The range of a sequence may be finite. e.g. $\{(-1)^n\}$ in \mathbb{R} , whose range $\{-1, 1\}$ is finite, but $\{\frac{1}{n}\}$ has range $\{\frac{1}{n} \mid n = 1, 2, \dots\}$

Definition. A sequence $\{x_n\}$ in X is said to be bounded if its range is a bounded subset of X

Remark. A sequence $\{x_n\}$ in X is said to be bounded if its range is a bounded subset of X

Example

- (1) Every const sequence $\{p\}$ in a metric space convergence, i.e. $\lim_{n \rightarrow \infty} p = p$
- (2) $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ in \mathbb{R} and $\{\frac{1}{n}\}$ is bounded (but the range is finite)
- (3) $\{(-1)^n\}$ divergences, but $\{(-1)^n\}$ is bounded. (range is finite).
- (4) $\{n^2\}$ divergences in \mathbb{R} and is unbounded. In fact, $\lim_{n \rightarrow \infty} n^2 = +\infty$ (which range is infinite)
- (5) $\lim_{n \rightarrow \infty} (1 + \frac{(-1)^n}{n}) = 1$ and $\{1 + \frac{(-1)^n}{n}\}$ is bounded. (range is infinite)
- (6) $\{i^n\}$ divergence and it's bounded (range is finite)
- (7) Identify all convergence sequence in a discrete metric space X . $\{x_n\}$ convergence to $p \Leftrightarrow \forall \epsilon > 0 \exists N \in \mathbb{N} \ni \forall n \geq N, d(x_n, p) < \epsilon \Leftrightarrow \{x_n\}$ is almost constant.

In metric space, we can use sequences to characterise adherent and accumulation point

Theorem 3.1. Let $\{x_n\}$ be a sequence in a metric space X and $E \subseteq X$

- (a) $x_n \rightarrow p$ as $n \rightarrow \infty \Leftrightarrow \forall$ neighborhood U of $p, \exists N \in \mathbb{N} \ni \forall n \geq N, x_n \in U$
- (b) If $\{x_n\}$ convergences, then its limit is !
- (c) If $\{x_n\}$ convergences, then its range is bounded, but not converse
- (d) $p \in \overline{E} \Leftrightarrow \exists$ a sequence $\{a_n\}$ in $E \ni a_n \rightarrow p$
- (e) $p \in E' \Leftrightarrow \exists$ a distinct sequence $(a_n \neq a_m \forall n \neq m)$ $\{a_n\}$ in $E \ni a_n \rightarrow p$

Proof.

(a)

$$\begin{aligned}
 x_n \rightarrow p &\Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N} \ni d(x_n, p) < \epsilon (\forall n \geq N) \\
 &\Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N} \ni x_n \in B(p, \epsilon) (\forall n \geq N) \\
 &\Leftrightarrow \forall \text{ neighborhood } U \text{ of } p, \exists N \in \mathbb{N} \ni \forall n \geq N, x_n \in U
 \end{aligned}$$

(b) Suppose $x_n \rightarrow p$, $x_n \rightarrow q$ and $p \neq q$, let $\epsilon = \frac{1}{2}d(p, q)$. By definition, $\exists N_1 \ni \forall n \geq N_1, d(x_n, p) < \epsilon$ and $\exists N_2 \ni \forall n \geq N_2, d(x_n, q) < \epsilon$

Let $N = \max\{N_1, N_2\}$ then $\forall n \geq N$, above holds. Hence $d(p, q) \leq d(p, x_n) + d(x_n, q) < \epsilon + \epsilon = 2\epsilon = d(p, q) (\rightarrow \leftarrow) \therefore p = q$

(c) We have seen bounded sequence may not converges. If $x_n \rightarrow p$, then for $\epsilon = 1, \exists N \in \mathbb{N} \ni \forall n \geq N, d(x_n, p) < 1$, i.e. $\forall n \geq N, x_n \in B(p, 1)$. Let $R = \max\{d(p, x_1), \dots, d(p, x_{N-1})\} + 1$, Then $x_n \in B(p, R) \forall n \geq 1 \therefore \{x_n\}$ is bounded

(d) (\Rightarrow) Suppose $p \in \overline{E}$. Then $\forall n \geq 1, B(p, \frac{1}{n}) \cap E \neq \emptyset$. Choose $a_n \in B(p, \frac{1}{n}) \cap E, n \geq 1$. We get a sequence $\{a_n\}$ in E and $0 \leq d(x_n, p) < \frac{1}{n}, \forall n \geq 1$.

By squeezing lemma, $\lim_{n \rightarrow \infty} d(a_n, p) = 0$, i.e. $a_n \rightarrow p$ as $n \rightarrow \infty$

(\Leftarrow) Suppose the conditions holds. $\forall r > 0, \exists N \in \mathbb{N} \ni \forall n \geq N, d(a_n, p) < r \Rightarrow \forall n \geq N, a_n \in B(p, r) \Rightarrow B(p, r) \cap E \neq \emptyset \therefore p \in E$

(e) It's similar to above.

■

Theorem 3.2. For real or complex sequences $\{x_n\}$ and $\{y_n\}$, $\lim_{n \rightarrow \infty} x_n = x$, $\lim_{n \rightarrow \infty} y_n = y$, $a, b \in \mathbb{R}$ or \mathbb{C} , $c \in \mathbb{R}$ or \mathbb{C}

$$(1) \lim_{n \rightarrow \infty} c = c$$

$$(2) \lim_{n \rightarrow \infty} (ax_n + by_n) = ax + by = a \lim_{n \rightarrow \infty} x_n + b \lim_{n \rightarrow \infty} y_n$$

$$(3) \lim_{n \rightarrow \infty} x_n y_n = xy = \lim_{n \rightarrow \infty} x_n \lim_{n \rightarrow \infty} y_n$$

$$(4) \text{ If } y \neq 0, \lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{x}{y} = \frac{\lim_{n \rightarrow \infty} x_n}{\lim_{n \rightarrow \infty} y_n}$$

$$(5) \text{ If } \{z_n\} \text{ is a complex sequence, then } z_n \rightarrow z \text{ as } n \rightarrow \infty \Leftrightarrow \text{Re} z_n \rightarrow z \text{ \& } \text{Im} z_n \rightarrow z \text{ (using } |\text{Re} w|, |\text{Im} w| \leq |w| \leq |\text{Re} w| + |\text{Im} w| \forall w \in \mathbb{C})}$$

$$(6) \text{ (Squeezing Lemma) If } \{x_n\} \{y_n\} \text{ and } \{t_n\} \text{ are real sequence } \ni$$

$$x_n \leq t_n \leq y_n \text{ for } n \gg 0$$

$$\text{and } \lim x_n = \lim y_n = l, \text{ then } \lim_{n \rightarrow \infty} t_n = l$$

$$(7) \lim_{n \rightarrow \infty} x_n = x \Rightarrow \lim_{n \rightarrow \infty} |x_n| = |x| \text{ (using } ||x_n| - |x|| \leq |x_n - x|)$$

Examples.

$$(i) \lim_{n \rightarrow \infty} (1 - \frac{i}{n}) = 1 \text{ (Re } (1 - \frac{i}{n}) = 1, \text{ Im}(1 - \frac{i}{n}) = \frac{1}{n})$$

$$(ii) \lim_{n \rightarrow \infty} \frac{1}{n} \sin \frac{1}{n} = 0$$

$$\begin{aligned} 0 \leq \left| \frac{1}{n} \sin \frac{1}{n} \right| \leq \frac{1}{n} &\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{1}{n} \sin \frac{1}{n} \right| = 0 \\ &\Rightarrow \left| \lim_{n \rightarrow \infty} \frac{1}{n} \sin \frac{1}{n} \right| = 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \sin \frac{1}{n} = 0 \end{aligned}$$

$$(iii) \{(-1)^n\} \text{ divergence, but } |(-1)^n| = 1 \rightarrow 1$$

For sequences in \mathbb{R}^k including in $\mathbb{C} \approx \mathbb{R}^2$, we have.

Theroem 3.3. Let $\{x_n\}$ be a sequence in \mathbb{R}^k , where

$$x_n = (x_{1,n}, x_{2,n}, \dots, x_{k,n}), n = 1, 2, \dots$$

(a) $x_n \rightarrow p(p_1, \dots, p_k)$ in $\mathbb{R}^k \Leftrightarrow x_{i,n} \rightarrow p_i \forall 1 \leq i \leq k$, i.e. $\lim_{n \rightarrow \infty} (x_{1,n}, \dots, x_{k,n}) = (\lim_{n \rightarrow \infty} x_{1,n}, \dots, \lim_{n \rightarrow \infty} x_{k,n})$ if exists

(b) Let $\{x_n\}, \{y_n\}$ be sequences in \mathbb{R}^k and $\{d_n\}$ be a sequence in \mathbb{C} and $a, b \in \mathbb{R}$. If $x_n \rightarrow x, y_n \rightarrow y$ and $d_n \rightarrow d$, then

- $ax_n + by_n \rightarrow ax + by$
- $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$
- $d_n x_n \rightarrow d x$
- $\|x_n\| \rightarrow \|x\|$

If $k = 3$, then $x_n \times y_n \rightarrow x \times y$

Proof.

(a) It follows from the inequation $\forall y \in \mathbb{R}^k |y_i| \leq \|y\| \leq \sum_{i=1}^k |y_i|$

$$\because \forall 1 \leq i \leq k |x_{i,n} - p_i| \leq \|x_n - p\| \leq \sum_{i=1}^k |x_{i,n} - p_i| \forall n \geq 1$$

(\Rightarrow)

$$\begin{aligned} \text{Suppose } x_n \rightarrow p &\Rightarrow \|x_n - p\| \rightarrow 0 \\ &\Rightarrow \forall 1 \leq i \leq k, |x_{i,n} - p_i| \rightarrow 0 \forall 1 \leq k \leq n \\ &\Rightarrow \forall 1 \leq i \leq k, x_{i,n} \rightarrow p_i \text{ as } n \rightarrow \infty \end{aligned}$$

(\Leftarrow)

$$\begin{aligned} \text{Suppose } x_n \rightarrow p_i, 1 \leq i \leq k &\Rightarrow \|x_{i,n} - p_i\| \rightarrow 0 \forall 1 \leq k \leq n \\ &\Rightarrow \sum_{i=1}^k |x_{i,n} - p_i| \rightarrow 0 \\ &\Rightarrow \|x_n - p\| \rightarrow 0 \Rightarrow x_n \rightarrow p \end{aligned}$$

(b) By (a)

$$\begin{aligned} ax_n + by_n &= (ax_{1,n}, ax_{2,n}, \dots, ax_{k,n}) + (by_{1,n}, \dots, by_{k,n}) \\ &= (ax_{1,n} + by_{1,n}, \dots, ax_{k,n} + by_{k,n}) \\ &\rightarrow (ax_1 + by_1, \dots, ax_k + by_k) = ax + by \end{aligned}$$

$$\bullet \langle x_n, y_n \rangle = \sum_{i=1}^k x_{i,n} y_{i,n} \rightarrow \sum_{i=1}^n x_i y_i = \langle x, y \rangle$$

- $d_n x_n = (d_n x_{1,n}, \dots, d_n x_{k,n}) \rightarrow (dx_1, \dots, dx_k) = dx$
- $\|x_n\| = \left(\sum_{i=1}^k x_{i,n}^2\right)^{\frac{1}{2}} \implies \left(\sum_{i=1}^k x_i^2\right)^{\frac{1}{2}} = \|x\|$
- $x_n \times y_n = (x_{2,n}y_{3,n} - x_{3,n}y_{2,n}, \dots) \rightarrow (x_2y_3 - x_3y_2, \dots) = x \times y$

■

3.2. Subsequences.

Theroem 3.4.

- (a) If $\{x_n\}$ converges to p , i.e. $\lim_{n \rightarrow \infty} x_n = p$, then so is every subsequence of $\{x_n\}$
- (b) If X is compact and $\{x_n\}$ is a sequence in X , then $\{x_n\}$ has a convergent subsequence.
- (c) Every bounded sequence $\{x_n\}$ in \mathbb{R}^k has a converge subsequence.

Proof. (a) Given a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ (Note that $\{n_k\}$ is strictly increasing, i.e. $n_1 < n_2 < \dots$, hence, $k \leq n_k \forall k \geq 1$), $d(x_{n_k}, p) < \epsilon$.

This proves $x_{n_k} \rightarrow p$ as $k \rightarrow \infty$

(b) Let $T = \{x_n \mid n \geq 1\}$ be the range of $\{x_n\}$

Case 1: T is a finite set. In this case, some x_{n_0} must appear infinitely many times in the sequence $\{x_n\}$. Choose $n_1 = n_0 \ni x_{n_1} = x_{n_0}$, and $n_2 > n_1 \ni x_{n_2} = x_{n_0}, \dots$. In this way, we get a const subsequence $\{x_{n_k}\}$ which convergence to x_{n_0}

Case 2: T is an infinite set. In this case, T is an infinite subset of the compact metric space X . By Thm 2.17 (ii), T has an accumulation point p in X . By Thm 3.1 (e), \exists a sequence in T which converges to p . We may arrange such sequence to be a subsequence of $\{x_n\}$. We are done

$y_1 = x_n$, choose $n_2 \rightarrow n_1 \ni x_{n_2}$ appears in $\{y_j\}$. Then $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and $\{y_j\}$, $\therefore x_{n_k} \rightarrow p$

- (c) \because Since $\{x_n\}$ is bounded, we may choose a closed ball $\overline{B}(0, R)$ or a closed n -dimensional interval in $\mathbb{R}^k \ni \{x_n\}$ is a sequence in K , By (b), $\{x_n\}$ has a convergence subsequence .

■

Remark. Thm 3.4(a) can be used to detect the divergence of a sequence, e.g. $\{(-1)^n\}$ in \mathbb{R} which diverges, \because It has two subsequences

$$\begin{cases} x_{2n} \rightarrow 1 \\ x_{2n-1} \rightarrow -1 \end{cases} \quad \text{which is different.}$$

Definition. Let $\{x_n\}$ be a sequence in X . A point $p \in X$ is called a subsequential limit of $\{x_n\}$ if \exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ $\ni x_{n_k} \rightarrow p$ as $k \rightarrow \infty$

Examples

- (1) If $\{x_n\}$ converges to p , then $\{x_n\}$ has only one subsequential limit p ($E = \{p\}$)
- (2) $\{(-1)^n\}$ has two subsequential limits 1 and -1 , $E = \{1, -1\}$
- (3) $\{n\}$ has no subsequential limit ($E = \emptyset$)

Let $\{x_n\}$ be a sequence in X and E be the set of all subsequential limits of $\{x_n\}$

Theorem 3.5. As above, E is a closed subset of X

Proof. If E is a finite set, then E is closed.

Now, assume that E is an infinite set, to show that E is closed, we must prove $E' \subseteq E$, i.e. E contains all its accumulation points. Given $q \in E'$, to prove $q \in E$, i.e. \exists a subsequence $\{x_{n_k}\}$ $\ni x_{n_k} \rightarrow q$. Since E is infinite, $\{x_n\}$ is not a constant sequence, so we can choose $x_{n_1} \neq q$. Let $\delta = d(x_{n_1}, q)$. We construct subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying $d(x_{n_k}, q) \leq \frac{\delta}{2^{k-1}} \forall k \geq 1$. If it is done, then by squeezing lemma, $d(x_{n_k}, q) \rightarrow 0$ as $k \rightarrow \infty$.

Now, to construct such subsequence $\{x_{n_k}\}$. By induction, suppose $k = 1$ we are done, we have found $n_1 < n_2 < \dots < n_{k-1}$, $k \geq 2$. To find x_{n_k} . Since $q \in E'$, $B(q, \frac{\delta}{2^k}) \cap E - \{q\} \neq \emptyset$. Choose $x \in B(q, \frac{\delta}{2^k}) \cap E - \{q\}$. Now, $x \in E$, \exists a subsequence of $\{x_n\}$ which converges to x . Hence $\exists n_k > n_{k-1}$ $\ni d(x_{n_k}, x) < \frac{\delta}{2^k}$. Finally, $d(x_{n_k}, q) \leq d(x_{n_k}, x) + d(x, q) < \frac{\delta}{2^k} + \frac{\delta}{2^k} = \frac{\delta}{2^{k-1}} \therefore$ By induction, such subsequence $\{x_{n_k}\}$ can be found. ■

3.3. Cauchy Sequences.

Recall $x_n \rightarrow p \implies \forall \epsilon > 0 \exists N \in \mathbb{N} \ni \forall n \geq N, d(x_n, p) < \frac{\epsilon}{2}$.

$\therefore \forall m, n \geq N, d(x_m, x_n) \leq d(x_m, p) + d(p, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Definition. A sequence $\{x_n\}$ in X is called a Cauchy sequence if it satisfies the Cauchy condition: $\forall \epsilon > 0, \exists N \in \mathbb{N} \ni \forall n, m \geq N, d(x_n, x_m) < \epsilon$

Remark.

(i) Convergence sequence is Cauchy

(ii) Cauchy sequence may not convergence, e.g. in $(0, 1)$ $\{\frac{1}{n}\}$ is Cauchy, but not convergence in $(0, 1)$

$$\forall n, m \in \mathbb{N}, m \geq n, \left| \frac{1}{n} - \frac{1}{m} \right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{n}$$

$$\forall \epsilon > 0, \text{ Choose } N \in \mathbb{N} \ni \frac{2}{N} < \epsilon.$$

$$\text{Them } \forall n, m \geq N, \left| \frac{1}{n} - \frac{1}{m} \right| \leq \frac{2}{N} < \epsilon \therefore \left\{ \frac{1}{n} \right\} \text{ is Cauchy.}$$

(iii) $\{x_n\}$ is Cauchy $\Leftrightarrow \lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0$

(iv) Let $E_n = \{x_n, x_{n+1}, \dots\}$ $n \geq 1$. Them $\{x_n\}$ is Cauchy $\Leftrightarrow \lim_{n \rightarrow \infty} \text{dia}(E_n) = 0$

Recall the definition of diameter:

Let $S \subseteq V, S \neq \emptyset$. The diameter of S is $\text{dia}(S) = \sup\{d(x, y) \mid x, y \in S\}$

Let

Proof. (\Rightarrow) Suppose $\{x_n\}$ is Cauchy. Then $\forall \epsilon > 0, \exists N \in \mathbb{N} \ni \forall n, m \geq N, d(x_n, x_m) < \frac{\epsilon}{2}$. Then $\forall n \geq N, \text{dia}(E_n) \leq \frac{\epsilon}{2} < \epsilon \implies \lim_{n \rightarrow \infty} \text{dia}(E_n) = 0$

(\Leftarrow) Suppose $\lim_{n \rightarrow \infty} \text{dia}(E_n) = 0, \forall \epsilon > 0, \exists N \in \mathbb{N} \ni \forall n \geq N, \text{dia}(E_n) < \epsilon \implies \forall n, m \geq N, (x_n, x_m \in E_n), d(x_n, x_m) \leq \text{dia}(E_n) < \epsilon$
 $\therefore \{x_n\}$ is Cauchy. ■