

Advance Calculus Exercise

1. Let \mathcal{T}_1 and \mathcal{T}_2 be topologies on a set X . Prove or disprove that $\mathcal{T}_1 \cup \mathcal{T}_2$ and $\mathcal{T}_1 \cap \mathcal{T}_2$ are topologies on X .

Solution.

- (a) Claim $\mathcal{T}_1 \cup \mathcal{T}_2$ is a topology on X
- i) $\because \mathcal{T}_1, \mathcal{T}_2$ is a topology on $X, \therefore \emptyset, X \subseteq \mathcal{T}_1, \emptyset, X \subseteq \mathcal{T}_2$
 $\implies \emptyset, X \subseteq \mathcal{T}_1 \cup \mathcal{T}_2$
 - ii) choose $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}_1$ but not in \mathcal{T}_2 and $\bigcup_{\alpha' \in I'} U_{\alpha'} \in \mathcal{T}_2$ but not in \mathcal{T}_1
 $\implies U_\alpha \in \mathcal{T}_1 \cup \mathcal{T}_2$ and $U_{\alpha'} \in \mathcal{T}_1 \cup \mathcal{T}_2$
 Claim $\bigcup_{\alpha \in I} U_\alpha \cup \bigcup_{\alpha' \in I'} U_{\alpha'} \subseteq \mathcal{T}_1 \cup \mathcal{T}_2 \implies \bigcup_{\alpha \in I} U_\alpha \cup \bigcup_{\alpha' \in I'} U_{\alpha'} \subseteq \mathcal{T}_1$ or \mathcal{T}_2 ($\rightarrow \leftarrow$) with $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}_1$ but not in \mathcal{T}_2
- (b) Claim $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology on X
- i) $\because \mathcal{T}_1, \mathcal{T}_2$ is a topology on $X \therefore \emptyset, X \in \mathcal{T}_1$ and $\emptyset, X \in \mathcal{T}_2$
 $\implies \emptyset, X \in \mathcal{T}_1 \cap \mathcal{T}_2$
 - ii) let $U_\alpha \in \mathcal{T}_1 \cap \mathcal{T}_2, \alpha \in I, \because \mathcal{T}_1, \mathcal{T}_2$ are topology on X
 $\therefore \bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}_1$ and $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}_2 \implies \bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}_1 \cap \mathcal{T}_2$
 - iii) let $U_1, U_2, \dots, U_k \in \mathcal{T}_1 \cap \mathcal{T}_2 \because \mathcal{T}_1, \mathcal{T}_2$ are topology on X
 $\therefore \bigcap_{i=1}^k U_i \subseteq \mathcal{T}_1$ and $\bigcap_{i=1}^k U_i \subseteq \mathcal{T}_2 \implies \bigcap_{i=1}^k U_i \subseteq \mathcal{T}_1 \cap \mathcal{T}_2$ ■

2. In any metric space $X, \overline{B(x, r)} \subseteq \overline{B(x, r)}$ for all $r \geq 0$

Solution.

- let $p \in \overline{B(x, r)} \implies \forall R > 0, B(p, R) \cap B(x, r) \neq \emptyset$
 $\implies \exists q \in B(p, R) \cap B(x, r) \ni d(q, x) < r$ and $d(p, q) < R$
 $\implies d(p, x) \leq d(p, q) + d(q, x) < r + R$ (triangular inequality)
 $\implies d(p, x) < r + R (\forall R) > 0 \implies d(p, x) - r < R (\forall R > 0)$
 $\implies r - d(p, x) > -R (\forall R > 0) \implies r - d(p, x) \geq 0$
 $\implies r \geq d(p, x) \implies p \in \overline{B(x, r)}$ ■

3. Let X be a metric space, $S \subseteq X$. Then S is closed if and only if $\partial S \subseteq S$

Solution.

(\Rightarrow) let $p \in \partial S \Rightarrow p \in \overline{S}$ (by definition of partial S)
 $\because S$ is close $\therefore \overline{S} \subseteq S \Rightarrow p \in S$
the proof of S is close then $\overline{S} \subseteq S$
 let $q \in \overline{S}$, Claim $q \notin S \Rightarrow q \in X - S$
 $\because S$ is close $\therefore X - S$ is open
 $\Rightarrow \exists r > 0 \ni B(q, r) \subseteq X - S \Rightarrow B(q, r) \cap S = \emptyset$
 $\because q \in \overline{S} \therefore \forall R > 0, B(q, R) \cap S \neq \emptyset$
 pick $R = r$, $B(q, R) \cap S \neq \emptyset$ and $B(q, r) \subseteq X - S$
 (\Leftarrow) let $p \in X - S \Rightarrow p \notin S \Rightarrow p \notin \partial S (\because \partial S \subseteq S)$
 $\Rightarrow p \notin \overline{S} (\because p \in X - S \text{ and } (X - S) \subseteq \overline{X - S})$ so
 $p \in \overline{X - S}$ but $p \notin \partial S \Rightarrow p \notin \overline{S}$
 $\Rightarrow \exists R > 0 \ni B(p, R) \cap S = \emptyset \Rightarrow B(p, R) \subseteq X - S$
 $\Rightarrow X - S$ is open $\Rightarrow S$ is close.

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4. Show that:

- (a) $(\bigcup_{\alpha \in I} E_{\alpha})^c = \bigcap_{\alpha \in I} (E_{\alpha})^c$
 (b) $(\bigcap_{\alpha \in I} E_{\alpha})^c = \bigcup_{\alpha \in I} (E_{\alpha})^c$

It is called De Morgan's laws.

Solution.

- (a) (\subseteq) let $p \in (\bigcup_{\alpha \in I} E_{\alpha})^c \Rightarrow p \notin \bigcup_{\alpha \in I} E_{\alpha}$
 $\Rightarrow p \notin E_{\alpha} \forall \alpha \in I \Rightarrow p \in (E_{\alpha})^c \forall \alpha \in I$
 $\Rightarrow p \in \bigcap_{\alpha \in I} (E_{\alpha})^c$
 (\supseteq) let $p \in \bigcap_{\alpha \in I} (E_{\alpha})^c \Rightarrow p \in (E_{\alpha})^c$
 $\Rightarrow p \in (E_{\alpha})^c \forall \alpha \in I \Rightarrow p \notin E_{\alpha} \forall \alpha \in I$
 $\Rightarrow p \notin \bigcup_{\alpha \in I} E_{\alpha} \Rightarrow p \in (\bigcup_{\alpha \in I} E_{\alpha})^c$
 (b) (\subseteq) let $p \in (\bigcap_{\alpha \in I} E_{\alpha})^c \Rightarrow p \notin \bigcap_{\alpha \in I} E_{\alpha}$
 $\Rightarrow p \in (E_{\alpha})^c$ for some $\alpha \in I \Rightarrow p \in \bigcup_{\alpha \in I} (E_{\alpha})^c$
 (\supseteq) let $p \in \bigcup_{\alpha \in I} (E_{\alpha})^c \Rightarrow p \in (E_{\alpha})^c$ for some $\alpha \in I$
 $\Rightarrow p \notin E_{\alpha}$ for some $\alpha \in I \Rightarrow p \notin \bigcap_{\alpha \in I} E_{\alpha}$
 $\Rightarrow p \in (\bigcap_{\alpha \in I} E_{\alpha})^c$

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5. Let a function $f : X \rightarrow Y$ and $A \subseteq Y$. We define

$$f^{-1}(A) = \{x \in X \mid f(x) \in A\}$$

is called the inverse image of a set B . Then

- (a) $B \subseteq f^{-1}(f(B))$ for all $B \subseteq X$
- (b) $f(f^{-1}(B)) \subseteq B$ for all $B \subseteq Y$
- (c) $B = f^{-1}(f(B))$ for all $B \subseteq X$ if and only if f is injective
- (d) $f(f^{-1}(B)) = B$ for all $B \subseteq Y$ if and only if f is surjective.
- (e) Find an example to show that the inclusions in (a) and (b) may be strict.

Solution.

(a) let $b \in B \implies f(b) \in f(B)$ (by definition)

$\because b \in X$ ($\because B \subseteq X \therefore b \in X$) and $f(b) \in f(B)$

$\therefore b \in f^{-1}(f(B))$

(b) $f(f^{-1}(B)) = \{f(x) \mid x \in X, f(x) \in B\}$

let $p \in f(f^{-1}(B))$, by definition, $p \in B$

(c) (\Leftarrow) we proof $B \subseteq f^{-1}(f(B))$ in (a)

Claim $f^{-1}(f(B)) \subseteq B$ when f is injective

let $p \in f^{-1}(f(B)) \because f$ is injective $\therefore f(x) \in f(B)$ only $x \in B$

if not, let $y \notin B \ni f(y) \in f(B) \exists x \in B, f(x) = f(y), \because$
 $x \in B, y \notin B \therefore x \neq y, f$ is not injective

therefore $p \in B$

(\Rightarrow) We need to prove if $x, x' \in X \ni f(x) = f(x')$ then $x = x'$.

Suppose $\exists x, x' \in X$ which $x \neq x'$ and $f(x) = f(x')$

let $B = \{x\}, B' = \{x'\} \Rightarrow f(B) = \{f(x)\}, f(B') = \{f(x')\}$

by question, $B \subseteq f^{-1}(f(B)) = \{x \in X \mid f(x) \in f(B)\}$

$\implies \{p \in X \mid f(p) = f(x)\} \subseteq \{x\}$ (\star)

$\implies f^{-1}(f(B)) = \{x\}$, and by similar method

$f^{-1}(f(B')) = \{x'\}$

by (\star) $f^{-1}(f(B)) = f^{-1}(f(B')) = \{x\} (\rightarrow \leftarrow)$

$\therefore f$ is injective.



(d) (\Leftarrow) we proof $f(f^{-1}(B)) \subseteq B$ in (b), now proof $f(f^{-1}(B)) \subseteq B$ when f is surjective.

let $b \in B \because f$ is surjective $\implies \exists x \in X \ni f(x) = b$

$\therefore b \in f(f^{-1}(B))$

(\Rightarrow) we want to proof $\forall y \in Y \exists x \in X \ni f(x) = y$

let $B = Y$, we get $f(f^{-1}(Y)) \supseteq Y$ from question given

$\because f^{-1}(Y) \subseteq X \therefore f(f^{-1}(Y)) \subseteq f(X)$

$\implies Y \subseteq f(f^{-1}(Y)) \subseteq f(X) \subseteq Y \implies f(X) = Y,$

f is surjective.

(e) $X = \{1, 2, 3\}$ $Y = \{1, 2\}$ and $f(1) = 1, f(2) = 1, f(3) = 2$

and let $B = \{1\}, B' = \{1, 2\}$

$$\{1\} = B \subset f^{-1}(f(B)) = \{1, 2\}$$

and

$$\{1\} = f(f^{-1}(B')) \subset B' = \{1, 2\}$$