1. Sequence, Series, and Power Series

Definition. *A sequence $\{a_n\}$ has the limit L and we write*

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L \text{ as } n \to \infty$$

If $\lim_{n\to\infty} a_n$ exists, we say the sequence converges. Otherwise, we say the sequence diverges (or is divergent).

Definition. *A sequence $\{a_n\}$ has the limit L if for every $\epsilon > 0$ there is a corresponding integer N such that if $n > N$ then $|a_n - L| < \epsilon$*

Definition. *The notation $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that if $n > N$ then $a_n > M$*

Theorem. *If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$*

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Theorem. *If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$*

Theorem. *If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$*

Definition. *A sequence $\{a_n\}$ is called*

(1) increasing if $a_n \leq a_{n+1}$ for all $n \geq 1$,
(2) decreasing if $a_n \geq a_{n+1}$ for all $n \geq 1$,
(3) monotonic of it is either increasing or decreasing

Definition. *A sequence $\{a_n\}$ is bounded above if there is a number $M \ni a_n \leq M$ for all $n \geq 1$, and is bounded below if $m \leq a_n$ for all $n \geq 1$.*
If a sequence is bounded above and below, then it is called a bounded sequence.

Theorem (Monotonic Sequence Theorem)**.** *Every bounded, monotonic sequence is convergent. In particular, a sequence that is increasing and bounded above converges, and a sequence that is decreasing and bounded below converges.*

Definition. *If the sequence $\{S_n\}$ is convergent and $\lim_{n\to\infty} S_n = S$ exists as a real number, then the series $\sum a_n$ is called "convergent" If the sequence $\{S_n\}$ is divergent, then the series is called divergent.*

Definition (Geometric Series). • *the partial sum of geometric series* $S_n = \dfrac{a(1-r^n)}{1-r}$

- *If* $|r| < 1$ *on its sum is* $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$

Theorem. *If the series* $\displaystyle\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$

Properties. If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$(where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$, and

(*roman*∗) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(*roman*∗) $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(*roman*∗) $\displaystyle\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{b_n}^{\infty}$

1.1. Integral Test & Estimates of Sum. Integral Test. Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent $\Leftrightarrow$ improper integral $\displaystyle\int_1^{\infty} f(x)dx$ is convergent

Remark. *The* p *series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ *is convergent if* $p > 1$ *and divergent if* $p \leq 1$

Estimate Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = S - S_n$, then $\displaystyle\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx \implies S_n + \int_{n+1}^{\infty} f(x)dx \leq S \leq S_n + \int_n^{\infty} f(x)dx$

Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms

(1) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also.

(2) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

The Limit Comparison Test. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is a finite number and $c > 0$, then either **both** series converge or diverge.

Alternating Series Test. If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 + \cdots (b_n > 0)$$

satisfies the conditions
(i) $b_{n+1} \leq b_n$ for all n
(ii) $\lim_{n\to\infty} b_n = 0$
then the series is convergent.

Alternating Series Estimation Theorem. If $S = \sum (-1)^{n-1}$, where $b_n > 0$, is the sum of an alternating series that satisfies
(i) $b_{n+1} \leq b_n$ and (ii) $\lim_{n\to\infty} b_n = 0$
then

$$|R_n| = |S - S_n| \leq b_{n+1}$$

<u>Definition</u>. *A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.*

<u>Definition</u>. *A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent; that is, if $\sum a_n$ converges but $\sum |a_n|$ diverges.*

Theorem. *If a series $\sum a_n$ is absolutely convergent, then it is convergent.*

Ration Test

(i) If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

(ii) If $\lim_{n\to\infty |\frac{a_{n+1}}{a_n}|} = L > 1$ or $\lim_{n\to\infty} |\frac{a_{n+1}}{n}| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent *(iii)* If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$, the ration test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

1.2. Power Series.

Definition. *A power series is a series of the form*

$$\sum_{n=1}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

and a series of the form

$$\sum_{n=1}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Theorem. *For a power series* $\sum_{n=0}^{\infty} c_n (x-a)^n$*, there are only three possibilities:*

(i) The series converges only when $x = a$

(ii) The series converges for all x

(iii) There is a positive number R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$ *and the number* R *in case (iii) is called the radius of convergence of the power series.*

Theorem. *If the power series* $\sum c_n(x-a)^n$ *has radius of convergence* $R > 0$*, then the function* f *defined By*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is differeniable on the interval $(a-R, a+R)$ *and*

$$(i)\ f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} c_n (x-a)^{n-1}$$

(ii) $\int f(x)dx = C + x_0(x-a) + c_1\dfrac{(x-a)^2}{2} + c_2\dfrac{(x-a)^3}{3} + \cdots =$
$C + \sum_{n=1}^{\infty} c_n \dfrac{(x-a)^{n+1}}{n+1}$

1.3. Taylor & Maclaurin Series.

<u>**Definition**</u>. *If f has a power series representation (expansion) at a, that is, if*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \ |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)(a)}}{n!}$$

$\implies f(x) = \sum_{n=1}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$
and it's called Taylor series of the function f at a. If $a = 0$ it's called Maclaurin series and $T_n(x) = \sum_{i=0}^{n} \dfrac{f^{(i)}(a)}{i!}(x-a)^i$ called the nth-degree Taylor polynomial of f at a and $R_n(x) = f(x) - T_n(x)$, so that $f(x) = T_n(x) + R_n(x)$, then $R_n(x)$ is called the remainder of the Taylor series.

Theorem. *If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a, and if*

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Taylor's Inequality. If $|f^{n+1}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality $|R_n(x)| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \leq d$