Advance Calculus Exercise

Exercise 1 (Chapter 1)

1. A complex number z is said to be algebraic if there are integers $a_0, \cdots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable.

Solution. Let A be the set of all algebraic numbers. We denoted some notation

(a) $\mathbb{Z}[x]^X$: the set of all non-zero polynomials having coefficients in $\mathbb{Z}$

(b) $Z_{p(x)}$: the set of all roots of $p(x)$, where $p(x) \in \mathbb{Z}[n]^X$

Note that

(a) $A \subseteq U_{p(x)\in\mathbb{Z}[x]^X} Z_{p(x)}$ and $Z_{p(x)}$ is finite since a polynomial of degree n has at most n roots

(b) $\mathbb{Z}[x]^X = \cup_{\infty}^{n=1} F_n$, where $F_n \subseteq \mathbb{Z}[x]^X$ the set of polynomials in $\mathbb{Z}[x]^X$ of degree n

Claim $\mathbb{Z}[x]^X$ is countable. If we are done, then $\cup_{p(x)\in\mathbb{Z}[x]} Z_{p(x)}$ is countable union of finite set, it follows that A is countable.

To prove this claim, it satisfies to show that F_n is countable $\forall n \geq 0 (F_n = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n)$

Consider the map $G : F_n \to \mathbb{Z}^{n+1}$

$$G(a_0 x^n + a_1 x^{n+1} + \cdots + a_{n-1} x + a_n) = (a_0, a_1, \cdots, a_n)$$

Clearly, G is injective. This implies F_n is countable $\forall n \geq 0$ ■

2. Prove that there exists real numbers which are not algebraic.

Solution. If not, $\forall r \in \mathbb{R}$, r is algebraic. This implies the set of all algebraic numbers is uncountable $(\rightarrow\leftarrow)$ to exercise 1. ■

3. Is the set of all irrational real numbers countable?

Solution. No! If it were, $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})$ is countable $(\rightarrow\leftarrow)$ ■

4. Construct a bounded set of real numbers with exactly three limit points.

Solution. The idea come from the set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$
In fact, $\forall x \in \mathbb{R}$, the set

$$E = \{x + \frac{1}{n} \mid n \in \mathbb{N}\}$$

has exactly one limit point, namely, x. Clearly, x is a limit point of E. We prove that E does not have any other limit point.
Given $y \in \mathbb{R}$. If $y < x$ or $y \geq x+1$, then it is clear that y cannot be limit point.
If $x < y < x+1$, then we have two cases ■

5. Let X be a metric space and $E \subset X$. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points. Do E and E' always have the same limit points?

Solution.

$\star$) E' is closed i.e. $(E')' \subseteq E'$
"Fact A is closed iff $A' \subseteq A$", given $p \in (E')'$ and $r > 0$ $B(p,r) \cap E' - \{p\} \neq \emptyset$. Choose $q \in B(p,r) \cap E' - \{p\}$. Since $q \in B(p,r)$ $\exists \delta > 0 \ni B(q,\delta) \subseteq B(p,r)$. Also $B(q,\delta) \cap E - \{q\} \subseteq B(p,r) \cap E - \{q\}$. Since $q \in E'$, we get $B(q,\delta) \cap E - \{q\} \neq \emptyset$ and it is an infinite set. This implies $B(p,r) \cap E - \{p\} \neq \emptyset \implies p \in E'$. Hence, E' is closed.

$\star$) $(E)' = (\overline{E})'$. Clearly $(E)' \subseteq (\overline{E})'$ $(\because E \subseteq \overline{E})$ $(\supseteq)$ If not, $\exists x \in (\overline{E})', x \notin E'$, i.e. $\exists r > 0, B(x,r) \cap E - \{x\} = \emptyset$. Since $x \in (\overline{E})'$, $B(x,r) \cap \overline{E} - \{x\} \neq \emptyset$. Choose $y \in B(x,r) \cap \overline{E} - \{x\}$. Since $y \in B(x,r) \exists \delta > 0 \ni B(y,\delta) \subseteq B(x,r)$
$B(y,\delta) \cap E \subseteq B(x,r) \cap E \subseteq \{x\} \implies B(y,\delta) \cap E - \{y\}$ is a finite set $\implies y \notin E'$ From above, we get $y \notin E'$ and $y \in \overline{E}(\rightarrow\leftarrow)$ to $B(x,r) \cap E - \{x\} = \emptyset$. Hence $E' = (\overline{E})'$

$\star$) E and E' may not have the same limit points, e.g.Consider $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ in $\mathbb{R}$. $E' = \{0\}$ but $(E')' = \emptyset$

$\blacksquare$

6. Let $A_1, A_2, \cdots$ be subset of a metric space.
(a) Prove that $\bigcup_{i=1}^{n} \overline{A_i} = \overline{\bigcup_{i=1}^{n} A_i}$ $\forall n \in \mathbb{N}$
(b) Prove that $\bigcup_{i=1}^{\infty} \subseteq \overline{\bigcup_{i=1}^{\infty}}$. Show by an example that this inclusion can be proper.

Solution.
(a) Since $A_i \subseteq \overline{A_i} \forall i = 1, \cdots, n$ $\bigcup_{i=1}^{n} A_i \subseteq \bigcup_{i=1}^{n} \overline{A_i} \implies \overline{\bigcup_{i=1}^{n} A_i} \subseteq \overline{\bigcup_{i=1}^{n} \overline{A_i}} = \bigcup_{i=1}^{n} \overline{A_i}$
Conversely, given $p \in \bigcup_{i=1}^{n} \overline{A_i}$, i.e. $p \in A_i$ for some $i = 1, \cdots, n$ $\forall r > 0, B(p,r) \cap A_i \neq \emptyset$
$B(p,r) \cap (\bigcup_{i=1}^{n} A_i) \neq \emptyset \implies \bigcup_{i=1}^{n}(B(p,r) \cap A_i) \neq \emptyset \implies p \in \overline{\bigcup_{i=1}^{n} A_i}$
(b) Given $p \in \bigcup_{n=1}^{\infty} \overline{A_n}$, i.e. $p \in \overline{A_n}$ for some $n \in \mathbb{N}$ $\forall r > 0, B(p,r) \cap A_n \neq \emptyset \implies B(p,r) \cap (\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty}(B(p,r) \cap A_n) \neq \emptyset \implies p \in \overline{\bigcup_{n=1}^{\infty}} A_n$
To prove that the inclusion might be strict e.g. $\bigcup_{q \in \mathbb{Q}} \{\overline{q}\} = \mathbb{Q} \subsetneqq \mathbb{R} = \overline{\mathbb{Q}} = \overline{\bigcup_{q \in \mathbb{Q}} \{q\}}$

$\blacksquare$

7. Is every point of every open set $E \subseteq \mathbb{R}^2$ a limit point of a set E? Answer the same equation for closed sets in $\mathbb{R}^2$

Solution. $\blacksquare$

8. Let X be a metric space and $E \subseteq X$
(a) Prove that E° is always open.
(b) Prove that E is open if and only if $E^\circ = E$
(c) If $G \subseteq E$ and G is open, prove that $G \subseteq E$
(d) Prove that the complement of E° is the closure of the complement of E, i.e. $(E^\circ)^c = \overline{E^c}$
(e) Do E and $\overline{E}$ always have the same interiors?
(f) Do E and E° always have the same closures?

9. Let X be an infinite set. For $p, q \in X$, define

$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}$$

Prove that this is a metric. Which subset of the resulting metric space are open? Which are closed? Which are compact?

10. For $x, y \in \mathbb{R}$, define

a) $d_1(x,y) = (x-y)^2$

b) $d_2(x,y) = \sqrt{|x-y|}$

c) $d_3(x,y) = |x^2 - y^2|$

d) $d_4(x,y) = |x - 2y|$

e) $d_5(x,y) = \dfrac{|x-y|}{1+|x-y|}$

Determine, for each of these, whether it is a metric or not.

11. Let $K = \{0\} \cup \{\frac{1}{n} \mid n \in N\} \subseteq \mathbb{R}$. Prove that K is compact directly from definition.

Solution. Give an open covering $\{U_\alpha\}_{\alpha \in I}$ of K
$0 \in K \implies 0 \in U_{\alpha_0}$ for some $\alpha_0 \in I$. Since U_{α_0} is open, $\exists r > 0 \ni (-r, r) \subseteq U_{\alpha_0}$.
By Archi, $\exists N \in \mathbb{N} \ni \frac{1}{N} < r$, $\forall n \geq N$, $\frac{1}{n} < \frac{1}{N} < r$.
This implies $\forall n \geq N$, $\frac{1}{n} \in (-r, r) \subseteq U_{\alpha_0}$. Finally, for $1 \leq i \leq N-1$, $\frac{1}{i} \in U_{\alpha_i}$ for some $\alpha \in I$. Therefore, $K \subseteq U_{i=0}^{N-1} U_{\alpha_i}$, i.e. K is compact
$\{x\} \cup \{x_n \mid n \in \mathbb{N}\}$ $x_n \to x$ as $n \to +\infty$ ■

12. Prove the following results

(a) Construct a sequence $\{K_n\}$ of closed sets in $\mathbb{R}$ with $K_{n+1} \subseteq K_n \ni \bigcap_{i=1}^{\infty} K_n = \emptyset$

(b) Construct a sequence $\{K_n\}$ of bounded sets in $\mathbb{R}$ with $K_{n+1} \subseteq K_n \ni \bigcap_{n=1}^{\infty} K_n = \emptyset$

Solution. (a) For each $n \in \mathbb{N}$, set

$$K_n = \{n, n+1, \cdots, n+k, \cdots\}$$

is closed and $K_{n+1} \subseteq K_n$. Now, to prove $\bigcap_{n=1}^{\infty} K_n = \emptyset$. If not, $\exists x \in \bigcap_{n=1}^{\infty} K_n$. Then x must be an integer but $x \notin K_{x+1}$

(b) For each $n \in \mathbb{N}$ set

$$K_n = \{\frac{1}{n}, \frac{1}{n+1}, \cdots, \frac{1}{n+k}, \cdots\}$$

is bounded and $K_{n+1} \subseteq K_n$. For the same argument, $\bigcap_{n=1}^{\infty} K_n = \emptyset$

■

13.

(a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

(b) Prove the same of disjoint open sets.

(c) Fix $p \in X, \delta > 0$. Define

$$A = \{q \in X \mid d(p,q) < \delta\}$$

and

$$B = \{q \in X \mid d(p,q) > \delta\}$$

Prove that A and B are separated

(d) Prove that every connected metric space with at least two points is uncountable.

14. Let A and B be separated subsets of some $\mathbb{R}^k$, suppose $a \in A,\ b \in B$, and define

$$p(t) = (1-t)a + tb$$

for $t \in \mathbb{R}$. Put $A_0 = p^{-1}(A),\ B_0 = p^{-1}(B)$.

(a) Prove that A_0 and B_0 are separated subsets of $\mathbb{R}$

(b) Prove that there exists $t_0 \in (0,1)$ such that $p(t_0) \notin A \cup B$.

(c) Prove that every convex subset of $\mathbb{R}^k$ is connected.

15. A metric space X is called separable if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable

Proof. Claim $\overline{\mathbb{Q}^k} = \mathbb{R}^k$

Given $x = (x_1, \cdots, x_k) \in \mathbb{R}^k$, $\forall r > 0$, we must prove that $B(x, r) \cap \mathbb{Q}^k \neq \emptyset$. Consider $x' = (x_1 + \frac{r}{2\sqrt{k}}, \cdots, x_k + \frac{r}{2\sqrt{k}})$. Then $\| x - x' \| = \frac{r}{2} < r \implies x' \in B(x, r)$. Now, for each $1 \leq i \leq k$, $\exists y \in \mathbb{Q} \ni y_i \in (x_i, x_i + \frac{r}{2\sqrt{k}})$. Thus, $y = (y_1, y_2, \cdots, y_k) \in \mathbb{Q}^k \cap B(x, r)$, i.e. $B(x, r) \cap \mathbb{Q}^k \neq \emptyset$, i.e. $x \in \overline{\mathbb{Q}^k}$ ■

16. Assume that $S \subseteq \mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is said to be a condensation point of S if every $r > 0$, $B(x, r) \cap S$ is uncountable. Prove the following statements:

(a) If for every $x \in S$, there is a $r_x > 0$ such that $B(x, r_x) \cap S$ is countable then S is countable.

(b) If S is not countable, then there exists a point $x \in S$ such that x is a condensation point of S.

17. A set in $\mathbb{R}^n$ is called perfect if $S = S'$, that is, S is a closed set which contains no isolated points. Prove the following Cantor-Bendoxon:

Every uncountable closed set F in $\mathbb{R}^n$ can be expressed in the form $F = A \cup B$, where A is perfect and B is countable.