1. Introduction to spectral theory

1.1. Main definitions.

Definition. *A scalar λ is called an eigenvalue of an operator $A : V \to V$ if there exists a non-zero vector $v \in V$ such that*

$$Av = \lambda v$$

The vector v is called the eigenvector of A

Theorem (From hamberger Thm 5.2)**.** *Let $A \in M_{n\times n}(F)$. Then a scalar λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.*

Proof. A scalar λ is an eigenvalue of A if and only if there exists a nonzero vector $v \in F^n$ such that $Av = \lambda v$, that is, $(A-\lambda I_n)(v) = 0$. By Theorem 2.5, this is true **if and only if** $A-\lambda I_n$ is not invertible. However, this result is equivalent to the statement that $\det(A - \lambda I_n) = 0$ ∎

Definition. *Let $A \in M_{n\times n}(\mathrm{F})$. The polynomial $f(t) = \det(A - tI_n)$ is claaed the characteristic polynomial of A*

Theorem (From hamberger Thm 5.4)**.** *Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. A vector $v \in V$ is an eigenvector of T corresponding to λ if and only if $v \neq 0$ and $v \in N(T - \lambda I)$.*

Definition. *The nullspace $N(A - \lambda I)$, i.e. the set of all eigenvectors and 0 vector, is called the eigenspace. The set of all eigenvalues of an operator A is called spectrum of A, and is usually denoted $\sigma(A)$.*

Remark.

If the matrix A is ugly, what should we do?
we can use the similar matrices
A and B are called similar if there exists an invertible matrix S such that

$$A = SBS^{-1}$$

The determinants of similar matrix is same

$$\det(A) = \det(SBS^{-1}) = \det(S)\det(B)\det(S^{-1}) = \det(B)$$

We can find $A - \lambda I$ and $B - \lambda I$ is similar

$$A - \lambda I = SBS^{-1} - \lambda SIS^{-1} = S(BS^{-1} - \lambda IS^{-1}) = S(B - \lambda I)S^{-1}$$

It same in transform
If $T : V \to V$ is a linear transform, α, β are two bases in V, then

$$[T]^{\alpha}_{\alpha} = [I]^{\alpha}_{\beta}[T]^{\beta}_{\beta}[I]^{\beta}_{\alpha}$$

Definition (algebraic mutiplicity). *The largest positive integer k such that $(x-\lambda)^k$ divides $p(x)$ is called the multiplicity of the root λ.*

If λ is an eigenvalue of an operator (matrix) A, then it is a root of the characteristic polynomial $p(z) = \det(A - zI)$. The multiplicity of this root is called the (algebraic) multiplicity of the eigenvalue λ.

Definition (geometric multiplicity). *The dimension of the eigen space $N(A - \lambda I)$ is called geometric multiplicity of the eigenvalue λ.*

1.2. Diagonalization.

Definition. *A linear operator T on a finite-dimensional vector space V is called diagonalizable if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix. A square matrix A is called diagonalizable if L_A is diagonalizable.*

Theorem. *A matrix A admits a representation $A = SDS^{-1}$, where D is a diagonal matrix and S is an invertible one* ***if and only if*** *there exists a basis in F^n of eigenvectors of A.*

Proof. Let $D = \operatorname{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, and let $b_1, \cdots, b_n$ be the columns of S (note that since S is invertible it's columns form a basis in F^n). Then the identity $A = SDS^{-1}$ means that $D = [A]$ ■