

§ Linear Transformations and Matrices

2-1 Linear Transformations, Null spaces, and Ranges.

- 13 Let V and W be vector spaces, let $T: V \rightarrow W$ be linear, and let $\{w_1, \dots, w_k\}$ be a linearly independent subset of $R(T)$. Prove that if $S = \{v_1, \dots, v_k\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, 2, \dots, k$, then S is linearly independent.

Solution. Claim. S is linearly independent

there exist $a_1, \dots, a_k \in F \implies$

$$\sum_{i=1}^k a_i v_i = 0 \implies T\left(\sum_{i=1}^k a_i v_i\right) = 0 \implies \sum_{i=1}^k a_i T(v_i) = 0 \implies \sum_{i=1}^k a_i w_i = 0$$

$\because \{w_1, \dots, w_k\}$ is linearly independent $\therefore \sum_{i=1}^k a_i v_i = 0$ only $a_i = 0, i = 1, \dots, k$



14 Let V and W be vector spaces and $T:V \rightarrow W$ be linear.

- (a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W .
- (b) Suppose that T is one-to-one and that S is a subset of V . Prove that S is linearly independent if and only if $T(S)$ is linearly independent.
- (c) Suppose $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), \dots, T(v_n)\}$ is a basis for W .

Solution. (a) (\Rightarrow) let $\{s_1, \dots, s_n\}$ be a linearly independent subset of S

Claim. $\{T(s_1), \dots, T(s_n)\}$ is a linearly independent subset of W

$$\sum_{i=1}^n a_i T(s_i) = 0 \implies \sum_{i=1}^n T(a_i s_i) = 0 \implies T\left(\sum_{i=1}^n a_i s_i\right) = 0$$

$$\because T \text{ is one-to-one } \therefore \sum_{i=1}^n a_i s_i = 0 \text{ only scalars are } 0$$

$\{T(s_1), \dots, T(s_n)\}$ is a linearly independent subset of W

(\Leftarrow) let $x, y \in V$, β is a basis of V , $\beta = \{v_1, v_2, \dots, v_n\}$

$$x = a_1 v_1 + \dots + a_n v_n, y = b_1 v_1 + \dots + b_n v_n$$

$$T(x) = T(y) \implies T(x) - T(y) = 0 \implies T(x - y) = 0 \implies T((a_1 - b_1)v_1 + \dots + (a_n - b_n)v_n) = 0$$

$$\implies (a_1 - b_1)T(v_1) + \dots + (a_n - b_n)T(v_n) = 0 \because \{T(v_1), \dots, T(v_n)\} \text{ is linearly independent}$$

$$\therefore (a_1 - b_1)T(v_1) + \dots + (a_n - b_n)T(v_n) = 0 \text{ only } a_1 - b_1 = \dots = a_n - b_n = 0$$

$$\implies a_1 = b_1, \dots, a_n = b_n \implies T(x) = T(y) \text{ only } x = y$$

T is one-to-one



(b) let $S = \{s_1, s_2, \dots, s_n\}$

(\Rightarrow) **Claim.** $\{T(s_1), \dots, T(s_n)\}$ is linearly independent

$$a_1T(s_1) + \dots + a_nT(s_n) = 0 \implies T(a_1s_1) + \dots + T(a_ns_n) = 0 \implies T(a_1s_1 + \dots + a_ns_n) = 0$$

$\because S$ is linearly independent, T is one-to-one $\therefore T(a_1s_1 + \dots + a_ns_n) = 0$ only scalars are 0

$\{T(s_1), \dots, T(s_n)\}$ is linearly independent

(\Leftarrow) **Claim.** $\{s_1, \dots, s_n\}$ is linearly independent

$$a_1s_1 + \dots + a_ns_n = 0 \implies T(a_1s_1 + \dots + a_ns_n) = 0 \implies a_1T(s_1) + \dots + a_nT(s_n) = 0$$

$\because \{T(s_1), \dots, T(s_n)\}$ is linearly independent, $\therefore a_1 = \dots = a_n = 0$

$\{s_1, \dots, s_n\}$ is linearly independent

(c) **Claim.** $T(\beta)$ can generate W

$\because T$ is onto, by Thm 2.2, $T : V \rightarrow W$ be linear, β is a basis of V , then $R(T) = \text{span}(T(\beta)) = W$

$T(\beta)$ can generate W

Claim $T(\beta)$ is a linearly independent set of W

by (b), T is one-to-one, S is a linearly independent subset of V , $T(S)$ is linearly independent

$T(S)$ is a basis of W

17 Let V and W be finite-dimensional vector spaces and $T:V \rightarrow W$ be linear.

- (a) Prove that if $\dim(V) < \dim(W)$, then T cannot be onto.
- (b) Prove that if $\dim(V) < \dim(W)$, then T cannot be one-to-one.

Solution. let $\dim(V) = m, \dim(W) = n$, β_v is a basis of V , β_w is a basis of W ,
 $\beta_v = \{v_1, \dots, v_m\}$, $\beta_w = \{w_1, \dots, w_n\}$

(a) by dimension theorem

$$\because \text{rank}(T) \leq \dim(V) < \dim(W) \therefore R(T) \neq W$$

(b) **Claim.** T is one-to-one

by Theorem 2.4, T is one-to-one then $N(T) = \{0\}$, $\text{nullity}(T) = 0$

$\therefore R(T)$ is a subspace of W

$$\dim(V) = \text{rank}(T) + \text{nullity}(T) > \dim(W) \implies \text{rank}(T) > \dim(W) \rightarrow \leftarrow$$



- 21 Let V be the vector space of sequences described in Example 5 of Section 1.2. Define the functions $T, U: V \rightarrow V$ by

$$T(a_1, a_2, \dots) = (a_2, a_3, \dots) \text{ and } U(a_1, a_2, \dots) = (0, a_1, a_2, \dots).$$

T and U are called the **left shift** and **right shift** operators on V , respectively.

- (a) Prove that T and U are linear.
- (b) Prove that T is onto, but not one-to-one.
- (c) Prove that U is one-to-one, but not onto.

Solution. (a) let $v_1, v_2 \in V, c \in F, v_1 = (a_1, a_2, \dots), v_2 = (b_1, b_2, \dots)$

Prove T is linear

$$T(cv_1 + v_2) = T((ca_1, ca_2, \dots) + (b_1, b_2, \dots)) = T(ca_1 + b_1, ca_2 + b_2, \dots) = (ca_2 + b_2, ca_3 + b_3, \dots) = c(a_2, a_3, \dots) + (b_2, b_3, \dots) = cT(v_1) + T(v_2)$$

$$T((0, 0, \dots)) = (0, 0, \dots)$$

$\therefore T$ is a linear function

Prove U is linear

$$U(cv_1 + v_2) = U((ca_1, ca_2, \dots) + (b_1, b_2, \dots)) = U(ca_1 + b_1, \dots) = (0, ca_1 + b_1, \dots) = c(0, a_1, \dots) + (0, b_1, \dots) = cU(v_1) + U(v_2)$$

$$U(0, 0, \dots) = (0, 0, \dots)$$

$\therefore U$ is a linear function

- (b) let $v_1 = (a_1, a_2, \dots) \in V, a_1, a_2, \dots \in F$

$$\exists (a_2, \dots) = T(a_2, a_3, \dots) \in V \therefore T \text{ is onto}$$

$$T(1, 0, \dots) = T(2, 0, \dots) = (0, 0, \dots)$$

T is not one-to-one

- (c) $U(a_1, a_2, \dots) = (0, 0, \dots)$ only $a_1 = a_2 = \dots = 0 \implies N(U) = \{0\}$

by Theorem 2.4, U is linear, if $N(V) = \{0\}$, then U is one-to-one

$(1, 2, 3, \dots) \in V \notin R(U)$, U is not onto

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26 Using the notation in the definition above, assume that $T: V \rightarrow V$ is the projection on W_1 along W_2 .

- (a) Prove that T is linear and $W_1 = \{x \in V \mid T(x) = x\}$.
 (b) Prove that $W_1 = R(T)$ and $W_2 = N(T)$.

Solution. (a) let $x, y \in V, x = w_1 + w_2, y = w'_1 + w'_2, w_1, w'_1 \in W_1, w_2, w'_2 \in W_2, c \in F$

Claim. T is linear

$$T(cx + y) = T((cw_1 + w'_1) + (cw_2 + w'_2)) = cw_1 + w'_1 = cT(x) + T(y)$$

$$T(0) = T(0 + 0) = 0 \quad T \text{ is linear}$$

(b) let $w_1 \in W_1, w_2 \in W_2$

$$\begin{aligned} R(T) &= \{T(x) \mid x \in V\} = \{T(w_1 + w_2) \mid w_1 \in W_1, w_2 \in W_2\} = \\ &= \{T(w_1) + T(w_2) \mid w_1 \in W_1, w_2 \in W_2\} = \{T(w_1) \mid w_1 \in W_1\} = W_1 \end{aligned}$$

$$\text{let } x \in N(T) \implies T(x) = 0 \implies x \in W_2 \implies N(T) \subseteq W_2$$

$$\text{let } x \in W_2 \implies T(x) = 0 \implies x \in N(T) \implies W_2 \subseteq N(T)$$

$$\therefore N(T) = W_2$$

$$\therefore T(w_1) = 0 \text{ only } w_1 = 0, T(w_2) = 0 \text{ when } w_2 \in W_2$$

$$\therefore N(T) = \{w_2 \mid w_2 \in W_2\} = W_2$$



35 Let V be a finite-dimensional vector space and $T: V \rightarrow V$ be linear.

- (a) Suppose that $V = R(T) + N(T)$. Prove that $V = R(T) \oplus N(T)$.
 (b) Suppose that $R(T) \cap N(T) = \{0\}$. Prove that $V = R(T) \oplus N(T)$.

Solution. (a) **Claim** $R(T) \cap N(T) = \{0\}$

by Exercise 1.6.29 and dimension Theorem, W_1, W_2 is a subspace of V ,
 $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

$$\dim(R(T) \cap N(T)) = \dim(R(T)) + \dim(N(T)) - \dim(V) = 0$$

$\because R(T), N(T)$ is a subspace, $\therefore R(T) \cap N(T) = \{0\}$

$\because R(T), N(T)$ is a subspace of V , $R(T) \cap N(T) = 0$

$$\therefore R(T) \oplus N(T) = V$$

(b) by dimension theorem and Exercise 1.6.29

W_1, W_2 is a subspace of V , $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

$$\dim(R(T) + N(T)) = \dim(R(T)) + \dim(N(T)) - \dim(R(T) \cap N(T)) = \dim(R(T)) + \dim(N(T)) = \dim(V)$$

$\because R(T) + N(T)$ is a subspace of V , $\dim(R(T) + N(T)) = \dim(V)$

$$\therefore R(T) + N(T) = V$$

