SOME RESULT

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1. When n=1.

$f_1(p) = (1-p)^9 + 3p(1-p)^8$
$g_1(p) = 2p(1-p)^8 + 2p^2(1-p)^7$
$h_1(p) = 3p^2(1-p)^7$
$t_1(p) = 2p^3(1-p)^6$

Ratios

$\alpha_1(p) = \frac{g_1(p)}{f_1(p)} = \frac{2pq+2p^2}{q^2+3pq} = \frac{2p}{q(1+2p)} = \frac{2p}{(1-p)(1+2p)}$
$\beta_1(p) = \frac{h_1(p)}{g_1(p)} = \frac{3p}{2q+2p} = \frac{3p}{2}$
$\gamma_1(p) = \frac{t_1(p)}{h_1(p)} = \frac{2p}{3q}$
Note: $q = (1-p)$

Difference

$\tilde{\epsilon_1}(p) = \alpha_1(p) - \beta_1(p) = \frac{2p}{q(1+2p)} - \frac{3p}{2} = \frac{4p}{2q(1+2p)} - \frac{3pq(1+2p)}{2q(1+2p)} = \frac{p}{q}(\frac{4-3q-6pq}{2+4p}) = \frac{p}{q}(\frac{6p^2-3p+1}{2+4p})$
$\tilde{\epsilon_1'}(p) = \beta_1(p) - \gamma_1(p) = \frac{3p}{2} - \frac{2p}{3q} = \frac{9pq-4p}{6q} = \frac{p}{q}\frac{9q-4}{6} = \frac{p}{q}\frac{5-9p}{6}$
$\tilde{\epsilon_1}(p) + \tilde{\epsilon_1'}(p) = \alpha_1(p) - \gamma_1(p) = \frac{2p}{q(1+2p)} - \frac{2p}{3q} = \frac{6p-2p(1+2p)}{3q(1+2p)} = \frac{6p-2p-4p^2}{3q(1+2p)} = \frac{p}{q}\frac{4q}{3(1+2p)}$
Define $c_p = \frac{q}{p}$
$\epsilon_1 = c_p(\alpha_1(p) - \beta_1(p)) = \frac{6p^2-3p+1}{2+4p}$
$\epsilon_1' = c_p(\beta_1(p) - \gamma_1(p)) = \frac{5-9p}{6}$
$\epsilon_1(p) + \epsilon_1'(p) = c_p(\alpha_1(p) - \gamma_1(p)) = \frac{4q}{3(1+2p)}$

2. Iterated.

In this section, we shorten the notation of (p), like $f_k(p)$ denote f_k
$f_{k+1} = f_k^3 + 6_k^2 g_k + 3f_k^2 h_k + 9f_k g_k^2 + 6f_k g_k h_k + 2g_k^3$
$g_{k+1} = f_k^2 g_k + 4f_k g_k^2 + 2f_k^2 h_k + f_k^2 t_k + 8f_k g_k h_k + 3g_k^3 + 2f_k g_k t_k + 2f_k h_k^2 + 4g_k^2 h_k$
$h_{k+1} = f_k g_k^2 + 4f_k g_k h_k + 2g_k^3 + 2f_k g_k t_k + 7g_k^2 h_k + 3f_k h_k^2 + 2g_k^2 t_k + 4g_k h_k^2 + 2f_k h_k t_k$
$t_{k+1} = g_k^3 + 6g_k^2 h_k + 3g_k^2 t_k + 9g_k h_k^2 + 6g_k h_k t_k + 2h_k^3$

Ratios

$\alpha_{n+1} = \alpha_n \frac{B_n}{A_n}$, $\beta_{n+1} = \alpha_n \frac{C_n}{B_n}$, $\gamma_{n+1} = \alpha_n \frac{D_n}{C_n}$
$A_n = 1 + 6\alpha_n + 3\alpha_n\beta_n + 9\alpha_n^2 + 6\alpha_n^2\beta_n + 2\alpha_n^3$
$B_n = 1 + 4\alpha_n + 2\beta_n + \beta_n\gamma_n + 8\alpha_n\beta_n + 3\alpha_n^2 + 2\alpha_n\beta_n\gamma_n + 2\alpha_n\beta_n^2 + 4\alpha_n^2\beta_n$
$C_n = 1 + 2\alpha_n + 4\beta_n + 2\beta_n\gamma_n + 7\alpha_n\beta_n + 3\beta_n^2 + 2\alpha_n\beta_n\gamma_n + 4\alpha_n\beta_n^2 + 2\beta_n^2\gamma_n$
$D_n = 1 + 6\beta_n + 3\beta_n\gamma_n + 9\beta_n^2 + 6\beta_n\gamma_n + 2\beta_n^3$

Difference

$$\tilde{\epsilon}_{n+1} = \alpha_{n+1} - \beta_{n+1} = \alpha_n \left(\frac{B_n}{A_n} - \frac{C_n}{B_n} \right) = \frac{\alpha_n}{A_n B_n} \left(B_n^2 - A_n C_n \right)$$

$$= \frac{1}{A_n B_n} \left[\begin{array}{l} \tilde{\epsilon}_n^2 (4\alpha_n^3 \beta_n^2 + 10\alpha_n^3 \beta_n + 2\alpha^2 \beta^2 + 5\alpha^3 + 7\alpha_n^2 \beta_n + \alpha_n \beta_n^2 + 4\alpha_n^2 + 2\alpha_n \beta_n + \alpha_n) + \\ \tilde{\epsilon}_n \tilde{\epsilon}'_n (4\alpha_n^4 \beta_n + 4\alpha_n^3 \beta_n^2 + 10\alpha_n^3 \beta_n + 6\alpha_n^2 \beta_n^2 + 8\alpha_n^2 \beta_n + 2\alpha_n \beta_n^2 + 2\alpha_n \beta_n) + \\ \tilde{\epsilon}_n'^2 (4\alpha_n^3 \beta_n^2 + 4\alpha_n^2 \beta_n^2 + \alpha_n \beta_n^2) \end{array} \right]$$

$$\tilde{\epsilon'}_{n+1} = \beta_{n+1} - \gamma_{n+1} = \alpha_n \left(\frac{C_n}{B_n} - \frac{D_n}{C_n} \right) = \alpha_n \left(\frac{C_n^2 - B_n D_n}{B_n C_n} \right)$$

$$= \frac{1}{B_n C_n} \left[\begin{array}{l} \tilde{\epsilon}_n^2 (4\alpha_n \beta_n^4 + 12\alpha_n \beta_n^3 + 13\alpha_n \beta_n^2 + 6\alpha_n \beta_n + \alpha_n) + \\ \tilde{\epsilon}_n \tilde{\epsilon}'_n (4\alpha_n^2 \beta_n^3 - 4\alpha_n^2 \beta_n^2 \gamma_n - 8\alpha_n \beta_n^4 + 18\alpha_n \gamma_n^3 + 2\alpha_n^2 \beta_n^2 + 9\alpha_n \beta_n^2 + \alpha_n^2 \beta_n + 2\alpha_n \beta_n) + \\ \tilde{\epsilon}_n'^2 (4\alpha_n \beta_n^4 + 2\alpha_n \beta_n^3 + 2\alpha_n^2 \beta_n^2 + \alpha_n \beta_n^2) \end{array} \right]$$

$$\begin{aligned} \tilde{\epsilon}_{n+1} + \tilde{\epsilon}'_{n+1} &= \alpha_{n+1} - \gamma_{n+1} = \alpha_n \left(\frac{B_n}{A_n} - \frac{D_n}{C_n} \right) = \alpha_n \left(\frac{B_n C_n - A_n D_n}{A_n C_n} \right) \\ &= \frac{\alpha_n}{A_n C_n} (2\alpha_n + 1)(\alpha_n^2 + 3\alpha_n \beta_n + 4\alpha_n + 1) \\ &\quad (4\alpha_n \beta_n^2 + 2\alpha_n \beta_n \gamma_n - 2\beta_n^3 - 4\beta_n^2 \gamma_n + 7\alpha_n \beta_n - 6\beta_n^2 - \beta_n \gamma_n + 2\alpha_n - 2\beta_n) \\ &= \frac{\alpha_n}{A_n C_n} (2\alpha_n + 1)(\alpha_n^2 + 3\alpha_n \beta_n + 4\alpha_n + 1) \\ &\quad [2\beta_n^2 \tilde{\epsilon}_n + 2\beta_n^2 (\tilde{\epsilon}_n + \tilde{\epsilon}'_n) + 2\beta_n \gamma_n \tilde{\epsilon}_n + 6\beta_n \tilde{\epsilon}_n + \beta_n (\tilde{\epsilon}_n + \tilde{\epsilon}'_n) + 2\tilde{\epsilon}_n] \\ &= \frac{\alpha_n}{A_n C_n} (2\alpha_n + 1)(\alpha_n^2 + 3\alpha_n \beta_n + 4\alpha_n + 1) \\ &\quad [\tilde{\epsilon}_n (2\beta_n^2 + 2\beta_n \gamma_n + 6\beta_n + 2) + (\tilde{\epsilon}_n + \tilde{\epsilon'}_n)(2\beta_n^2 + \beta_n)] \end{aligned}$$

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\begin{aligned}
A_nB_n =& 8\alpha_n^5\beta_n + 28\alpha_n^4\beta_n^2 + 4\alpha_n^4\beta_n\gamma_n + 12\alpha_n^3\beta_n^3 + 12\alpha_n^3\beta_n^2\gamma_n \\
&+ 6\alpha_n^5 + 70\alpha_n^4\beta_n + 78\alpha_n^3\beta_n^2 + 20\alpha_n^3\beta_n\gamma_n + 6\alpha_n^2\beta_n^3 + 12\alpha_n^2\beta_n^2\gamma_n \\
&+ 35\alpha_n^4 + 133\alpha_n^3\beta_n + 48\alpha_n^2\beta_n^2 + 21\alpha_n^2\beta_n\gamma_n + 3\alpha_n\beta_n^2\gamma_n \\
&+ 56\alpha_n^3 + 88\alpha_n^2\beta_n + 8\alpha_n\beta_n^2 + 8\alpha_n\beta_n\gamma_n + 36\alpha_n^2 + 23\alpha_n\beta_n + \beta_n\gamma_n + 10\alpha_n + 2\beta_n + 1
\end{aligned}
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\begin{aligned}
B_nC_n =& 16\alpha_n^3\beta_n^3 + 8\alpha_n^3\beta_n^2\gamma_n + 8\alpha_n^2\beta_n^4 \\
&+ 20\alpha_n^2\beta_n^3\gamma_n + 4\alpha_n^2\beta_n^2\gamma_n^2 + 4\alpha_n\beta_n^4\gamma_n + 4\alpha_n\beta_n^3\gamma_n^2 + 40\alpha_n^3\beta_n^2 + 6\alpha_n^3\beta_n\gamma_n + 58\alpha_n^2\beta_n^3 + 44\alpha_n^2\beta_n^2\gamma_n \\
&+ 6\alpha_n\beta_n^4 + 30\alpha_n\beta_n^3\gamma_n + 6\alpha_n\beta_n^2\gamma_n^2 + 2\beta_n^3\gamma_n^2 \\
&+ 29\alpha_n^3\beta_n + 101\alpha_n^2\beta_n^2 + 18\alpha_n^2\beta_n\gamma_n + 40\alpha_n\beta_n^3 \\
&+ 43\alpha_n\beta_n^2\gamma_n + 7\beta_n^3\gamma_n + 2\beta_n^2\gamma_n^2 + 6\alpha_n^3 + 60\alpha_n^2\beta_n + 64\alpha_n\beta_n^2 + 14\alpha_n\beta_n\gamma_n + 6\beta_n^3 + 10\beta_n^2\gamma_n \\
&+ 11\alpha_n^2 + 35\alpha_n\beta_n + 11\beta_n^2 + 3\beta_n\gamma_n + 6\alpha_n + 6\beta_n + 1
\end{aligned}
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$$
\begin{aligned}
\epsilon_{n+1} &= c_p(\alpha_{n+1} - \beta_{n+1}) = c_p\alpha_n\left(\frac{B_n}{A_n} - \frac{C_n}{B_n}\right) = \frac{c_p\alpha_n}{A_nB_n}\left(B_n^2 - A_nC_n\right) = c_p\tilde{\epsilon}_{n+1} \\
&= \frac{c_p}{A_nB_n}\left[\begin{array}{l}
\tilde{\epsilon}_n^2(4\alpha_n^3\beta_n^2 + 10\alpha_n^3\beta_n + 2\alpha^2\beta^2 + 5\alpha^3 + 7\alpha_n^2\beta_n + \alpha_n\beta_n^2 + 4\alpha_n^2 + 2\alpha_n\beta_n + \alpha_n)+ \\
\tilde{\epsilon}_n\tilde{\epsilon}'_n(4\alpha_n^4\beta_n + 4\alpha_n^3\beta_n^2 + 10\alpha_n^3\beta_n + 6\alpha_n^2\beta_n^2 + 8\alpha_n^2\beta_n + 2\alpha_n\beta_n^2 + 2\alpha_n\beta_n)+ \\
\tilde{\epsilon}_n'^2(4\alpha_n^3\beta_n^2 + 4\alpha_n^2\beta_n^2 + \alpha_n\beta_n^2)
\end{array}\right] \\
&= \frac{1}{c_pA_nB_n}\left[\begin{array}{l}
\epsilon_n^2(4\alpha_n^3\beta_n^2 + 10\alpha_n^3\beta_n + 2\alpha^2\beta^2 + 5\alpha^3 + 7\alpha_n^2\beta_n + \alpha_n\beta_n^2 + 4\alpha_n^2 + 2\alpha_n\beta_n + \alpha_n)+ \\
\epsilon_n\epsilon'_n(4\alpha_n^4\beta_n + 4\alpha_n^3\beta_n^2 + 10\alpha_n^3\beta_n + 6\alpha_n^2\beta_n^2 + 8\alpha_n^2\beta_n + 2\alpha_n\beta_n^2 + 2\alpha_n\beta_n)+ \\
\epsilon_n'^2(4\alpha_n^3\beta_n^2 + 4\alpha_n^2\beta_n^2 + \alpha_n\beta_n^2)
\end{array}\right] \\
&= \frac{1}{A_nB_n}\left[\begin{array}{l}
\epsilon_n(4\alpha_n^4\beta_n^2 - 4\alpha_n^3\beta_n^3 + 10\alpha_n^4\beta_n - 8\alpha_n^3\beta_n^2 + 2\alpha_n^2\beta_n^3 - 4\alpha_n^2\beta_n^2\gamma_n \\
+5\alpha_n^4 + 2\alpha_n^3\beta_n - 6\alpha_n^2\beta_n\gamma_n - \alpha_n\beta_n^2\gamma_n + 4\alpha_n^3 - 2\alpha_n^2\beta_n - 2\alpha_n\beta_n\gamma_n + \alpha_n^2 - \alpha_n\beta_n)+ \\
\epsilon'_n(4\alpha_n^5\beta_n - 4\alpha_n^3\beta_n^2\gamma_n + 10\alpha_n^4\beta_n - 8\alpha_n^3\beta_n^2 + 2\alpha_n^2\beta_n^3 - 4\alpha_n^2\beta_n^2\gamma_n \\
+2\alpha_n^3\beta_n - \alpha_n\beta_n^2\gamma_n - \alpha_n^2\beta_n^2)
\end{array}\right]
\end{aligned}
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\begin{aligned}
\epsilon'_{n+1} &= c_p(\beta_{n+1} - \gamma_{n+1}) = c_p\alpha_n\left(\frac{C_n}{B_n} - \frac{D_n}{C_n}\right) = c_p\alpha_n\left(\frac{C_n^2 - B_nD_n}{B_nC_n}\right) \\
&= \frac{1}{c_pB_nC_n}\left[\begin{array}{l}
\epsilon_n^2(4\alpha_n\beta_n^4 + 12\alpha_n\beta_n^3 + 13\alpha_n\beta_n^2 + 6\alpha_n\beta_n + \alpha_n)+ \\
\epsilon_n\epsilon'_n(4\alpha_n^2\beta_n^3 - 4\alpha_n^2\beta_n^2\gamma_n - 8\alpha_n\beta_n^4 + 18\alpha_n\gamma_n^3 + 2\alpha_n^2\beta_n^2 + 9\alpha_n\beta_n^2 + \alpha_n^2\beta_n + 2\alpha_n\beta_n)+ \\
\epsilon_n'^2(4\alpha_n\beta_n^4 + 2\alpha_n\beta_n^3 + 2\alpha_n^2\beta_n^2 + \alpha_n\beta_n^2)
\end{array}\right] \\
&= \frac{1}{B_nC_n}\left[\begin{array}{l}
\epsilon_n(4\alpha_n^2\beta_n^4 - 4\alpha_n\beta_n^3\gamma_n^2 + 14\alpha_n^2\beta_n^3 - 12\alpha_n\beta_n^3\gamma_n - 2\alpha_n^2\beta_n^2\gamma_n \\
-12\alpha_n\beta_n^3 - \alpha_n\beta_n^2\gamma_n + 13\alpha_n^2\beta_n^2 - 2\alpha_n\beta_n\gamma_n + 6\alpha_n^2\beta_n - 4\alpha_n\beta_n^2 + \alpha_n^2 - \alpha_n\beta_n)+ \\
\epsilon'_n(4\alpha_n^3\beta_n^3 - 4\alpha_n^3\beta_n^2\gamma_n + 14\alpha_n^2\beta_n^3 - 2\alpha_n\beta_n^3\gamma_n - 10\alpha_n^2\beta_n^3 \\
-2\alpha_n^2\beta_n^2\gamma_n - 7\alpha_n\beta_n^3 - \alpha_n\beta_n^2\gamma_n + \alpha_n^3\beta_n + 7\alpha_n^2\beta_n^2)
\end{array}\right]
\end{aligned}
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