

## 1. Error Analysis

**Definition:**

let  $x$  is a value,  $\tilde{x}$  is a estimated value

(1) absolute error,  $E_a = |x - \tilde{x}|$

(2) relation error,  $E_r = \left| \frac{x - \tilde{x}}{x} \right|$

(3) percentage error,  $E_p = 100 \times \left| \frac{x - \tilde{x}}{x} \right|$

$\exists \epsilon > 0, |x - \tilde{x}| < \epsilon$ , Then  $\epsilon$  is upper limit of the absolute error measures the absolute accuracy.

### 1.1. Error in Implementation of Numerical Methods.

- (1) Round-off Error
- (2) Overflow & Underflow
- (3) Floating Point Arithmetic and Error Propagation
- (4) Truncation Error
- (5) Machine eps (Epsilon)

### (3) Floating Point Arithmetic and Error Propagation.

Let  $x_1, x_2$  are values,  $E_1, E_2$  are error of  $x_1, x_2$ , We want to check the change of error in  
" + ", " - ", " \* ", " / "  
" + "

Let  $x = x_1 + x_2$ , error of  $x$  is  $E$

Then  $x + E = x_1 + x_2 + E_1 + E_2 \implies E = E_1 + E_2$

by triangle inequality

$$\text{Absolute Error} = |E| \leq |E_1| + |E_2|$$

$$\text{Relative Error} = \frac{|E|}{|x|} \leq \frac{|E_1|}{|x|} + \frac{|E_2|}{|x|}$$

" - " (Similar " + ")

”\*”

Let  $x = x_1 * x_2$

Then  $x + E = (x_1 + E_1)(x_2 + E_2) = x_1x_2 + E_2x_1 + E_1x_2 + E_1E_2$

Absolute Error =  $|E| \leq |x_2E_1| + |x_1E_2|$

Relative Error =  $\frac{|E_1|}{|x|} \leq \frac{|E_1|}{|x_1|} + \frac{|E_2|}{|x_2|}$

”/”

Let  $x = x_1/x_2$

$x + E_x = \frac{x_1 + E_1}{x_2 + E_2} \left( \frac{x_2 - E_2}{x_2 - E_2} \right) = \frac{x_1x_2 + E_1x_2 - x_1E_2}{x_2^2 - E_2^2} + E_1E_2$

Absolute Error =  $|E_x| = \left| \frac{E_1x_2 - x_1E_2}{x_2^2} \right| \leq \frac{|E_1|}{|x_2|} + \frac{|x_1E_2|}{x_2^2}$

Relative Error =  $\frac{|E_x|}{|x|} \leq \frac{|E_1|}{|x_1|} + \frac{|E_2|}{|x_2|}$

**(4) Truncation Error.** Cause by approximation infinite with its finite terms.

Use Taylor series ( $f(x) \in P(C)$ ) as example

Let  $x = a, f(x) = f(a) + f'(a)(x - a) + f''(a)\frac{(x - a)^2}{2!} + \dots + \frac{(x - a)^n}{n!}f^n(a) + \dots + R_n$

$R_n = \int_a^x \frac{(x - t)^n}{n!} f^{(n+1)}(t) dt$

**Thm 1(First Mean Value Theorem)**

If  $g$  is continuous on  $[a, x]$ , then  $\exists \xi$  between  $a$  and  $x$  s.t.

$$\int_a^x g(t) dt = g(\xi)(x - a)$$

**Thm 2(Second Mean Value Theorem)**

If  $g, h$  is differentiable and integrable on  $[a, x]$ ,  $h$  does not change sign on  $[a, x]$  then  $\exists \xi$  that  $a \leq \xi \leq x$  s.t.

$$\int_a^x g(t)h(t) dt = g(\xi) \int_a^x h(t) dt$$

since  $t \in [a, x], h(t) = (x - t)^n \frac{1}{n!}, f^{(n+1)}(t)$  is continuous

$\exists \xi \in [a, x], R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!} f^{(x+1)}(\xi), \xi \in [a, a + h]$

(Ref. Violin page:799)

since power series convergent,  $R_n(x) \rightarrow 0, as_n \rightarrow \infty$

## Definition

Given  $\{a_n\} \{b_n\}$ ,  $b_n \geq 0$ ,  $\forall n \geq 1$   
 $a_n = O(b_n)$  if  $\exists M > 0 \rightarrow |a_n| \leq Mb_n \forall n \geq 1$   
 $R_n(x) = O(h^{n+1})$

**1.2. Condition & Stability.**

Condition number is sensitivity of the function

Stability is used to describe the sensitivity of the process

**Condition number of the  $f(x)$**

$$\text{CN} = \frac{\left| \frac{f(x) - f(\tilde{x})}{x - \tilde{x}} \right|}{\left| \frac{x - \tilde{x}}{x} \right|} = \left| \frac{f(x) - f(\tilde{x})}{x - \tilde{x}} \right| \cdot \left| \frac{x}{f(x)} \right| = \left| \frac{x}{f(x)} \cdot f'(x) \right|$$

by Mean Value Theorem,

$$\frac{f(x) - f(\tilde{x})}{x - \tilde{x}} \approx f'(x)$$

when  $\text{CN} \leq 1$  is **well condition**, other is **ill condition**

when the function is more sensitive to change, the condition number will be more big.

## 2. Methods for $f(x) = 0$

we have four way to deal this problem

- (1) Direct analytical Method
- (2) Graphical
- (3) Trial and Error Method
- (4) Iterative Method

### Thm. 3(Mean Value Theorem)

Let  $f$  be a continuous function on  $[a, b] = I$ (connected),  
if  $f(a) \leq c \leq f(b)$  that  $\exists \xi \in [a, b] \rightarrow f(\xi) = c$

### Corollary

Let  $f$  be a continuous function on  $[a, b] = I$ (connected)  
i.e.  $f(a) \cdot f(b) < 0 \Rightarrow \exists c \in (a, b) \Rightarrow f(c) = 0$   
 $c$  is a root of  $f(t)$

## Iterative Method

### 2.1. Bisection Method.

Let  $a, b$  be fixed satisfying Thm.3

$\therefore f(a) \cdot f(b) < 0, f$  is continuous on  $[a, b]$ . The first approximation is  $x_0 = \frac{a+b}{2}$   
if  $f(a) \cdot f(x_0) \leq 0$ , then By Thm. 3 the root will lie on  $(a, x_0)$  and  $x_1 = \frac{a+x_0}{2}$   
continue the process, let  $x_{n-3}, x_{n-2}, x_{n-1}$  be same step, then nth approximation  
if  $f(x_{n-1}) \cdot f(x_{n-3}) \leq 0$ , then  $x_n = \frac{x_{n-1} + x_{n-3}}{2}$   
else  $f(x_{n-1}) \cdot f(x_{n-3}) \geq 0$ , then  $x_n = \frac{x_{n-1} + x_{n-2}}{2}$   
we shall label the interval by algorithm

$$[a, b] = [a_0, b_0], [a_1, b_1], [a_2, b_2], \dots$$

by construction  $b_n a_n = \frac{1}{2}(b_{n-1} - a_{n-1})$ , Hence  $b_n - a_n = \frac{1}{2^n}[b_0 - a_0], \forall n \geq 1$

Clearly  $a_0 \leq a_1 \leq \dots \leq b, b_0 \geq b_1 \geq \dots \geq a, \{a_n\}, \{b_n\}$  is bdd and monotonic

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = f(r)$$

by assumption  $f(a_n)f(b_n) < 0, \lim_{n \rightarrow \infty} f(a_n) = f(\lim_{n \rightarrow \infty} a_n) = f(r)$

$\therefore f(b_n) = f(r), 0 \leq [f(r)]^2 \leq 0 \implies f(r) = 0$

The process is called **nested internal property**

Let  $\{C_k\}_{k=1}^\infty$  is a  $\downarrow$  sequence of nonempty closed compact subset of  $X$ , then  $\cap_k C_k \neq \emptyset$  if  $C_k \rightarrow 0$ , then  $\cap_k C_k = \{r\}$

Let  $\xi$  be the solution  $f(x) = 0$ , then  $\{x_0 - \xi\} \leq \frac{b-a}{2}, \dots, \{x_n - \xi\} \leq \frac{b-a}{2^{n+1}}$

**Definition(p-order-convergence)**

$\{x_n\} : \text{seq}, x_n \rightarrow z, s_n \rightarrow \infty$ , define  $\epsilon_n = z - x_n$ , if  $\exists c > 0, p \geq 1$

$$\lim_{n \rightarrow \infty} \frac{|\epsilon_{n+1}|}{|\epsilon_n|^p} = c$$

we call  $\{x_n\}$  is  $p$  order convergence

if  $c \leq 1$ , then it's good(only check this when it's a first order convergence)

Let  $\epsilon_n$  be the error i.e.  $\epsilon_n = |x_n - \xi|$ ,  $\epsilon_n \leq \frac{b-a}{2^{n+1}} \leq \epsilon$ , i.e.  $h \geq \frac{\ln(b-a) - \ln \epsilon}{\ln 2} - 1$

$$\epsilon_n = |x_n - \xi| \leq \frac{1}{2} \left( \frac{b-a}{2^n} \right) \approx \frac{1}{2} \epsilon_n - 1 \implies \lim_{n \rightarrow \infty} \left| \frac{\epsilon_n}{\epsilon_n - 1} \right| = \frac{1}{2}$$

Then Bisection Method is first order convergence

## 2.2. Newton-Taphson Method.

observation:

Let  $x_0$  be an initial approximate to the root of  $f(x) = 0$ , then  $x_0 + h$  is the exact root of  $f(x) = 0$ , i.e.  $f(x_0 + h) = 0$ , from Taylor series,  $f(x_0 + h) = f(x_0) + h \cdot f'(x_0) + \dots$   
i.e.  $x_0 \approx x_0 + h$

the first order approximation,  $f(x_0 + h) = f(x_0) + h \cdot f'(x_0) = 0 \implies h = \frac{-f(x_0)}{f'(x_0)}$

Let  $x_1 = x_0 + h$  be the next approximation to the root,  $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

In general  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \forall n \geq 1$

**Example**

Consider the  $f(x) = x^2 - M = 0 (M > 0)$

$$x_{n+1} = x_n - \frac{x_n^2 - M}{2x_n} = \frac{1}{2} \left( x_n + \frac{M}{x_n} \right) (\star)$$

In general, also can obtain for the  $k$ th root of  $M$ , i.e.  $\sqrt[k]{M}$  with  $f(x) = x^k - M = 0$  if  $x_1 > \sqrt[k]{M}$ , and define  $x_2, \dots$  by the interaction formula  $(\star)$ , then

(1)  $\{x_n\}$  is  $\downarrow$  (trivial) (2)  $\{x_n\}$  is bounded above ( $x_{n+1} = \frac{1}{2} \left( x_n + \frac{M}{x_n} \right) \geq \sqrt{x_n \left( \frac{M}{x_n} \right)} = \sqrt{M}$ )

By (1)(2),  $\lim_{n \rightarrow \infty} x_n = \sqrt{M}$  exists.

observation

let  $(x_0, f(x_0))$  be any point on the curve

$y = f(x)$ , then  $y - f(x_0) = f'(x_0)(x - x_0)$

**Thm. 4(The NR method is 2 order convergence)**

Let  $x$  denote the exact value of the root of  $f(x) = 0$

$x_n, x_{n+1}$  be two approximation S to the exact root  $a, (f(a) = 0)$

if  $\epsilon_n, \epsilon_{n+1}$  corresponding error  $S$ , then  $x_n = a + \epsilon_n, x_{n+1} = a + \epsilon_{n+1}$

by(NR)

$$\begin{aligned}
 a + \epsilon_{n+1} &= a + \epsilon_n - \frac{f(a + \epsilon_n)}{f'(a + \epsilon_n)} \\
 \epsilon_{n+1} &= S_n - \frac{f(a) + \epsilon_n f'(a) + \frac{\epsilon_n^2}{2!} f''(a) + \dots}{f'(a) + \epsilon_n f''(a) + \frac{\epsilon_n^2}{2!} f'''(a) + \dots} \\
 &= \epsilon_n - \frac{\epsilon_n \left( f'(a) + \epsilon_n f''(a) + \frac{\epsilon_n^2}{2!} f'''(a) + \dots \right)}{f'(a) + \epsilon_n f''(a) + \frac{\epsilon_n^2}{2!} f'''(a) + \dots} \\
 &= \frac{\epsilon_n [f'(a) + \epsilon_n f''(a) + \frac{\epsilon_n^2}{2!} f'''(a) + \dots - [f'(a) + \frac{\epsilon_n}{2!} f''(a) + \dots]]}{f'(a) + \epsilon_n f''(a) + \frac{\epsilon_n^2}{2!} f'''(a) + \dots} \\
 &= \frac{\epsilon_n [\frac{\epsilon_n}{2} f''(a) + \frac{\epsilon_n^2}{3} f'''(a) + \dots]}{f'(a) + \epsilon_n f''(a) + \frac{\epsilon_n^2}{2!} f'''(a) + \dots} \\
 &= \frac{\epsilon_n^2 [\frac{1}{2} f''(a) + \frac{\epsilon_n}{3} f'''(a) + \dots]}{f'(a) [1 + \epsilon_n \frac{f''(a)}{f'(a)} + \frac{\epsilon_n^2}{2!} \frac{f'''(a)}{f''(a)} + \dots]} \\
 \Rightarrow \frac{\epsilon_{n+1}}{\epsilon_n^2} &= \frac{\frac{1}{2} f''(a) + \frac{\epsilon_n}{3} f'''(a) + \dots}{f'(a) (1 + \epsilon_n \frac{f''(a)}{f'(a)} + \dots)}
 \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left| \frac{\epsilon_{n+1}}{\epsilon_n^2} \right| > \frac{1}{2} \left| \frac{f''(a)}{f'(a)} \right| < +\infty$$

**Remark:** if  $f(x)$  has double root  $S$

### 3. Eigen Problem

#### 3.1. Review eigenvalue & eigenvector.

$$A \in M_{n \times n}(\mathbb{R}/\mathbb{C}), AX = \lambda X = \lambda(IX) = \lambda IX \implies (A - \lambda I)X = 0$$

it's a homogeneous system of  $n$  linear equation, it determinate is 0

$$p(\lambda) = \det(A - \lambda I) = 0, \deg(p(\lambda)) = n$$

Define  $\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}, X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ ,  $X$  is a eigen vector of  $A$ ,  $\lambda$  is a eigenvalue of  $A$

the normalized eigenvector  $\hat{X} = \frac{1}{\|X\|} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$  where  $\|X\| = (X^T X)^{\frac{1}{2}} = (x_1^2 + \dots + x_n^2)^{\frac{1}{2}}$

if  $T$  is diagonalizable, then  $\exists$  order basis  $\beta$ ,  $\beta \ni [T]_{\beta} = D$ , which is a diagonal matrix  
similarly  $A$  is diagonalizable if  $L_A$  is diagonalizable

**diagonalizable**

$$\left\{ \begin{array}{l} \text{the c.p split} \left\{ \begin{array}{l} n \text{ distinct eigenvalue} \\ \text{other} \left\{ \begin{array}{l} \text{algebraic multiplicity} = \text{geometric multiplicity} \\ \text{algebraic multiplicity} \neq \text{geometric multiplicity} (\text{not diagonalizable}) \end{array} \right. \end{array} \right. \\ \text{the cp does not split (not diagonalizable)} \end{array} \right.$$
 (c.p. is charateristic polynomial)

$E_{\lambda}$  is subspace,  $E_{\lambda} = N(T - \lambda I)$ ,  $E_{\lambda}$  is  $T$ -invariant, i.e.  $T(E_{\lambda}) \subseteq E_{\lambda}, 1 \leq \dim(E_{\lambda}) \leq m$   
if  $T$  is diagonalizable, then

$$V = E_{\lambda_1} \oplus E_{\lambda_2} + \dots + E_{\lambda_n} \Leftrightarrow V = k\lambda_1 \oplus \dots \oplus k\lambda_n$$

Let any eigenvalue  $\lambda$  be repeated  $r$  times with  $k$  linearly independent eigenvector  
 $r$  is algebraic multiplicity,  $k$  is geometric multiplicity

**3.2. some introduction.**

we will learn ODE and PDE next time

$$\frac{dX}{dt} = AX, X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2, \frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2$$

$X = \chi e^{\lambda t}$  is the solution of system,  $\chi$  is column vector,  $\lambda$  is parameter to be determined

$$\frac{d\chi e^{\lambda t}}{dt} = \lambda \chi e^{\lambda t} \implies \lambda \chi e^{\lambda t} = A \chi e^{\lambda t} \implies \lambda \chi = A \chi$$

**Definition**

The spectrum of  $A$ , radius  $p$  of the smallest circle with center at the origin and contains all the spectral radius

**3.3. Power Method.****Definition**

Let  $A \in M_{n \times n}(\mathbb{C})$ , for  $1 \leq i, j \leq n$

define  $p_i(A)$  to be the sum of the abs-values of the entries of row  $i$  of  $A$  and  $r_j(A)$  to be the sum of the abs-values of the entries of column  $j$  of  $A$

$$p_i(A) = \sum_j ||A_{ij}||, r_j(A) = \sum_i ||A_{ij}||$$

$$e(A) = \max(p_i(A)), r(A) = \max(r_j(A)), 1 \leq i, j \leq n$$

**Definition**

an  $n \times n$  matrix  $A$ , we define the  $i$ th Gerschgorin disk  $c_i$  to be the disk in the complex plane with center  $A_{ii}$  and radius  $r_i = p_i(A) - |A_{ii}|$ ,  $c_i = \{z \in \mathbb{C} \mid |z - A_{ii}| < r_i\}$



**Theorem(Geisg Disk Theorem 1)**

Let  $A \in M_{n \times n}(\mathbb{C})$ , then every eigenvalue of  $A$  is contained in a Geisg Disk

pf: Let  $\lambda$  be eigenvalue of  $A$  r.t. eigenvector  $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ , clearly  $Av = \lambda v$

Then  $I_j^n = A_{ij}v_j = \lambda_{ri}$ ,  $1 \leq i \leq n(\star)$

suppose  $v_k$  is the coordinate of  $V$  having the largest absolute, ( $v_k \neq 0$ )

claim  $\lambda \in C_k$ , i.e.  $|\lambda - A_{kk}| \leq r_k$  For  $i = k$ , by  $(\star)$

$$\begin{aligned} |\lambda v_k - A_{kk}v_k| &= \left| \sum_{j=1}^n A_{kj}v_j - A_{kk}v_k \right| \\ &= \left| \sum_{j \neq k} A_{kj}v_j \right| \\ &\leq \sum_{j \neq k} |A_{kj}| |v_j| \\ &\leq \sum_{j \neq k} |A_{kj}| |v_k| = r_k |v_k| \end{aligned}$$

**Corollary 1**

Let  $\lambda$  be any eigenvalue of  $A \in M_{n \times n}(\mathbb{C})$ , then  $|\lambda| \leq p(A) = \max(p_i(A))$

pf: by Thm.  $|\lambda - A_{kk}| \leq r_k$  for some  $k$ ,  $1 \leq k \leq n$

$|\lambda| = |\lambda - A_{kk}| + |A_{kk}| \leq r_k + |A_{kk}| = p_k(A) \leq p(A)$

**Corollary 2**

$A^T \in M_{n \times n}(\mathbb{C})$ ,  $|\lambda| \leq r(A) = \max(r_j(A))$

**Corollary 3**

Let  $\lambda$  be eigenvalue of  $A \in M_{n \times n}(\mathbb{C})$ ,  $|\lambda| \leq \min \{ p(A), r(A) \}$

by corollary 1 & 2, we are done.

**Theorem(Geisg Disk Theorem 2)**

Let  $A \in M_{n \times n}(\mathbb{C})$ ,  $k$  of the disks are disjoint from the others, then exactly  $k$  eigenvalue are contained in the union of these disks.

pf: the gumltprinciple

Ref: Matrix Analysis 2/e (Horn/Johnson) P.388,389

**Rayleigh Power Method**

Let  $\lambda_1, \dots, \lambda_n$  be the eigenvalue of matrix,  $|\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$   
 our goal is to find  $|\lambda_1|$

Let  $x_1, \dots, x_n$  be eigenvectors, r.t.  $\lambda_1, \dots, \lambda_n, \implies Ax_i = \lambda_i x_i, \forall 1 \leq i \leq n$   
 if the matrix  $A$  (which is diagonalizable) has  $n$  linearly independent eigenvectors  
 then  $x = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$  for some  $c_i \in \mathbb{C}$

$$\begin{aligned} Ax &= A(c_1 x_1 + \dots + c_n x_n) \\ &= c_1 A x_1 + \dots + c_n A x_n \\ &= c_1 \lambda_1 x + \dots + c_n \lambda_n x \\ &= \lambda_1 \left( c_1 x + c_2 \left( \frac{\lambda_2}{\lambda_1} \right) x + \dots + c_n \left( \frac{\lambda_n}{\lambda_1} \right) x \right) \end{aligned}$$

$$\begin{aligned} A^2 x &= A \left( \lambda_1 \left( c_1 x + c_2 \left( \frac{\lambda_2}{\lambda_1} \right) x + \dots + c_n \left( \frac{\lambda_n}{\lambda_1} \right) x \right) \right) \\ &= \lambda_1 \left( c_1 A x + c_2 \left( \frac{\lambda_2}{\lambda_1} \right) A x + \dots + c_n \left( \frac{\lambda_n}{\lambda_1} \right) A x \right) \\ &= \lambda_1^2 \left( c_1 x + c_2 \left( \frac{\lambda_2}{\lambda_1} \right)^2 x + \dots + c_n \left( \frac{\lambda_n}{\lambda_1} \right)^2 x \right) \end{aligned}$$

Continue process

$$\begin{aligned} A^k x &= \lambda_1^k \left( c_1 x + c_2 \left( \frac{\lambda_2}{\lambda_1} \right)^k x + \dots + c_n \left( \frac{\lambda_n}{\lambda_1} \right)^k x \right) \\ A^{k+1} x &= \lambda_1^{k+1} \left( c_1 x + \dots + c_n \left( \frac{\lambda_n}{\lambda_1} \right)^{k+1} x \right) \\ \lim_{k \rightarrow \infty} \frac{A^{k+1} x}{A^k x} &= \lambda_1 \end{aligned}$$

**A stepwise procedure**

- (i)  $X^{(0)}$  is initial vector
- (ii)  $Y^{(0)} = AX^{(0)}$
- (iii)  $\lambda^{(1)}$  is the absolutely largest element, common from the vector  $Y^{(0)}$   
Let the remainly vector be  $X^{(1)}$ ,  $Y^{(0)} = \lambda^{(1)}X^{(1)}$
- (iv) repeating (ii) and (iii),  $Y^{(k)} = \lambda^{(k+1)}X^{(k+1)}$
- (v)  $|\lambda^{(k+1)}|, x^{(k+1)}$  is goal

**Example**

$$A = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 1 & -2 \\ -2 & 0 & 5 \end{pmatrix}, \quad X^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad Y^{(0)} = AX^{(0)} = \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix},$$

$$\lambda^{(1)} = 6, \quad Y^{(0)} = 6 \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} = \lambda^{(1)}X^{(1)} \implies Y^{(1)} = AX^{(1)} = A^{(1)} = \begin{pmatrix} 2 \\ 0 \\ 0.5 \end{pmatrix}, \quad \lambda^{(2)} = 2$$

**Inverse Power Method**

Let  $\lambda_i$  be an eigenvalue of matrix  $A$ , then  $\frac{i}{\lambda_i}$  is eigenvalue of the matrix  $A^{-1}$ , The eigenvector of  $A^{-1}$  is  $X_i$

pf:  $Ax_i = \lambda_i x_i \implies \frac{1}{\lambda_i} (Ax_i) = x_i \implies \frac{1}{\lambda_i} x_i = A^{-1}x_i$

**Shifted Power Method**

Let  $\lambda_i$  be an eigenvalue of matrix  $A$ , then  $(\lambda_i - k)$  is an eigenvalue of the matrix  $A - kI$  with the same eigenvector as that matrix  $A$

pf:  $Ax_i = \lambda_i x_i \implies (A - kI)x_i = AX_i - kX_i = \lambda_i x_i - kx_i = (\lambda_i - k)x_i$