1. Jordan Canonical Form

1.1. Triangular Form.

Definition. *let $T : V \to V$ be a linear operator, a subspace $W \subseteq V$ is said to be invariant under T if $T(W) \subseteq W$*

Remark. $\{0\}, V, Ker(T), Im(T), E_\lambda$ *are T-invarient*

Definition. *Let $T : V \to V$ be a linear operator on a finite dimension vector space we say that V is triangularizable $\Leftrightarrow \exists$ a order basis $\beta \ni [T]_\beta^\beta$ is upper triangular*

Example(triangularizable matrix)

Consider $\mathbb{F} = =C,\ V = \mathbb{C}^4$, let β be a order basis of V, $\beta = \{e_1, e_2, e_3, e_4\}$

$$[T]_\beta^\beta = \begin{bmatrix} 1 & 1-i & 2 & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 1-i & 3+i \\ 0 & 0 & 0 & 1-i \end{bmatrix}$$

Clearly $[T]_\beta^\beta$ is upper triangular, and let w_i be the subspace of $\mathbb{C}^4$ spanned by the first i vectors in the standard ordered basis, clearly, $T(w_i) \subseteq w_i = \{T(w) \mid w \in W\} = Im(T|w)$

Propos 1

let V be a finite vector space, let $T : V \to V$ be a linear operator and $\beta = \{x_1, \cdots, x_n\}$ be a basis for V, then

$$[T]_\beta^\beta \text{ is upper triangle} \Leftrightarrow \text{the subspace } w_i = \text{span}(x_1, \cdots, x_i) \text{ is T-invariant}$$

Note that the subspace w_i in **Prop 1** related follow

$$\{0\} \subseteq w_i \subseteq \cdots \subseteq w_{n-1} \subseteq w_n = V$$

we say that w_i forms an hands up sequence of subspaces.
on the other hand, a given linear operator can be a upper triangle, we must to construct the $\nearrow$ sequence of T-invariant subspace

$$\{0\} \subseteq w_1 \subseteq \cdots \subseteq w_n$$

$T|_w : W \to W$ is a linear mapping, T_W where W is a T-invariant subspace

Propos 2
Let $T : V \to V, W \leq V$ is a T-invariant, where V is a finite dimension vector space. Then the character polynomial of $T|_W$ divides the c.p. of T

Proof. (Thm 5.21 from Friedberg)
not today

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Corollary. *Every eigenvalue of $T|_W$ is also an eigenvalue of T, i.e. the eigenvalue of $T|_W$ is a subset of the eigenvalue of T on V,*

Review(Diagonal's condition)

Let $T : V \to V$ be a linear operator, where V is a finite dimension vector space, $\lambda_1, \cdots, \lambda_n$ are distinct eigenvalues, m_i be the multiplicity of λ_i, as a root number of the c.p. of T, Then

$$T \text{ is diagonal} \Leftrightarrow \; m_1 + \cdots + m_n = \dim(V), \; dim(E_{\lambda_i}) = m_i$$

The proof is Thm. 5.9 $\sim$ Thm. 5.11 from Friedberg
It means that

1. $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_n}$(From Exercise 5-2-20 Friberg)
2. m_i is algebraic multiplicity, $\dim(E_{\lambda_i})$ is geometric multiplicity.

Theorem (Schur)**.** *Let V be a finite dimensional vector space over $\mathbb{F}$ and $T : V \to V$ be an linear operator, then T is triangular $\Leftrightarrow$ the c.p. have $\dim(V)$ roots (counted with multiplicities) in $\mathbb{F}$*

Remark.

- ⋆ *if* $\mathrm{F} = \mathbb{C}$*(algebraic closure), then, by the Fundamental theorem of Algebra, every matrix $A \in M_{n\times n}(\mathbb{C})$ can be a triangularized*
- ⋆ *if* $\mathrm{F} = \mathbb{R}$*($x^2 + 1$ does not split on $\mathbb{R}$) consider the rotation matix R_θ where $0 < \theta < \pi$*

Lemma. *Let V be a finite dimensional vector space over $\mathbb{F}$ and $T : V \to V$ be an linear operator, and assume that the characteristic polynomial of T has $n = \dim(V)$ roots in* F*. If $W \subsetneq V$ is an invariant subspace under T, then there exists a vector $x \neq 0$ in V such that $x \notin V$ is an invariant subspace under T, then there exists a vector $x \neq 0$ in V such*

⋆ we need to use this lemma to create $\nearrow$ subspaces

Proof. **Define the P and S first**
let $\alpha = \{ x_1, \cdots, x_k \}$ be a basis for W, and extend α by adjoint $\alpha' = \{ x_{k+1}, \cdots, x_n \}$ to form a basis $\beta = \alpha \cup \alpha'$ for V, let $w' = \text{span}(\alpha')$. Define a linear operator $P : V \to V$ by

$$P(a_1x_1 + \cdots + a_nx_n) = a_1x_1 + \cdots + a_kx_k \text{ (projection, } V = W \oplus W')$$

Clearly, $W' = Ker(P),\ W = Im(P),\ P^2 = P$, Moreover $I - P$ is also the projection on W' with kernel W

$(I - P)(a_1x_1 + \cdots + a_nx_n) = I(a_1x_1 + \cdots + a_nx_n) - P(a_1x_1 + \cdots + a_nx_n)$
$= a_1x_1 + \cdots + a_nx_n - a_1x_1 + \cdots + a_kx_k = a_{k+1}x_{k+1} + \cdots + a_nx_n$
Further more $(I - P)^2 = (I - P)(I - P) = I - P^2 - P + P = I - P^2 = I - P$

if the basis is orthogonal basis, $W' = W^{\perp}$(Gramm schmit), and P is an orthogonal projection
let $S = (I - P) \circ T$, since $Im(I - P) = W'$, so $Im(S) \subseteq Im(I - P) = W'$, i.e. W' is S-invariant subspace, $\because S(W') \subseteq W'$
Claim the set of eigenvalues of $S|_W$ is a subset of the root of eigenvalues of T
First, since W is T-invariant, then $[T]_\beta^\beta = \begin{pmatrix} A & B \\ O & C \end{pmatrix}$
Clearly, $A = [T|_W]_\alpha^\alpha$ is $k \times k$ block, $C = [S|_{W'}]_{\alpha'}^{\alpha'}$ is a $(n-k) \times (n-k)$ block, Hence

$$\det(T - \lambda I) = \det(T|_W - \lambda I) \cdot \det(S|_{W'} - \lambda I)$$

by prop.2 & corollary, we down since all the eigenvalue of T lie in F(the c.p. has n roots) by claim, same is true of all the eigenvalues of $S|_{W'}$
so $\exists x \neq 0,\ x \in W' \ni Sx = \lambda x$, for some $\lambda \in \mathbb{F}$, i.e.

$$(I - P)(Tx) = \lambda x \implies Tx - PTx = \lambda x \implies Tx = \lambda x + PTx \in \text{span}(x) + W$$

Finally $W + \text{span}(x)$ is T-invariant, i.e. $y \in W + \text{span}(x) \implies T(y) \in W + \text{span}(x)$
Give $y \in W + \text{span}(x)$, then, $\exists\ z \in W_1, \lambda \in \text{F} \implies y = z + \lambda x$
$T(y) = T(z + \lambda x) = T(z) + \lambda T(x) = T(z) + \lambda(x) + \lambda PT(x)$ ∎

We finish this lemma, and we are going to proof Schur lemma

Proof. $(\Rightarrow)$ T is triangular
$\implies \exists$ order basis β of $V \implies [T]_\beta$ is upper triangular $\implies$ the eigenvalues of T are the diagonal entries in F $\implies$ the c.p. splits

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Proof. ($\Leftarrow$) Suppose the condition holds
let λ be eigenvalues of T, x_i is eigenvector of T correspond with λ, $W_1 = \text{span}(\{ x_i \})$
Clearly, W_1 is T-invariant by lemma, $\exists x \notin W,\ x \neq 0 \ni W_1 + \text{span}(\{ x_1 \})$ is T-invariant.
continue the processes $W_1 \subseteq \cdots \subseteq W_k$ with $W_i = \text{span}(\{ x_1, \cdots, x_i \})\ \forall\ i$
By lemma, $\exists x_{k+1} \notin W_k \ni W_{k+1} = W_k + \text{span}(x_{k+1})$ is also T-invariant
$\therefore$ By prop.1, we are done.

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Corollary. *if $T : V \to V$ is triangular with eigenvalues λ_i and m_i is its multiplicities, then $\exists$ an order basis β for $V \ni [T]_\beta$ is upper triangular matrix, and the diagonal entries of $[T]_\beta$ are m_1, λ_1 followed by $m_2 \lambda_2$'s and so on.*

Recall
In Chapter 4, If T is a linear mapping (or matrix) and $p(t) = a_n t^n + a_{n-1} t^{n-1} T^{n-1} + \cdots + a_0$ is a polynomial, we can define a new linear mapping

$$p(T) = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_1 I$$

Theorem. *Let $T : V \to V$ be a linear operator on V which is a finite dimension vector space and $p(t) = \det(T - tI)$ be its c.p Assume that $p(t)$ has $\dim(V)$ roots in* F *over which V is defined, then $p(T) = 0$(which is a zero transformation on V)*

Proof. (Exercise 6-4-16 in Friedber) For all the vector S in some basis of $V \to p(T)(x) =$ 0(scalar), by Schur lemma, $\exists$ order basis $\beta = \{ x_1, \cdots, x_n \}$ for $V \implies w_i = \text{span}(\{ x_1, \cdots, x_i \})$ $\forall\ 1 \leq i \leq n$ is T-invariant, all the eigenvalues of T lie in F, so $p(t) = \pm(t - \lambda) \cdots (t - \lambda_n)$ for some $\lambda_i \in$ F(not necessary distinct),if the factors here are ordered in the same fashion as the diagonal entries of $[T]_\beta^\beta$, then

$$T(x_i) = \lambda_i x_i + y_{i-1},\ y_i \in W_{i-1},\ i \geq 2,\ T(x_1) = \lambda_1 x_1$$

Now, we use the induction on i

- For $i = 1$,

  $p(T)(x_1) = \pm(T - \lambda_1 I) \cdots (T - \lambda_n I)(x_1) = \pm(T - \lambda_2 I) \cdots (I - \lambda_n I)(T - \lambda_1)(x_1) = 0$

- Suppose that: $p(T)(x_i) = 0,\ \forall\ i \leq k$

- Consider $p(T)(x_{k+1})$, clearly $(T - \lambda_1 I) \cdots (T - \lambda_k I)$ are needed to end x_i to 0, for $i \leq k$

  $p(T)(x_{k+1}) = I(T - \lambda_1 I) \cdots (T - \lambda_n I)(T - \lambda_{k+1} I)(x_{k+1})$

  $= \pm(T - \lambda_1 I) \cdots (T - \lambda_n I)(y_k) = 0$ By induction, we are done

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Suppose that $A \in M_{n\times n}(\mathrm{F})$ if A is invertible, i.e. $\det(A) \neq 0$, consider the c.p. of A

$$\det(A - tI) = (-1)^n t^n + \cdots + a_1 t + a_0,\ t = 0 \implies a_0 = \det(A) \neq 0$$

by thm 4(Cayley-Hamilton), $p(A) = (-1)^n A^n + \cdots + \det(A)I = 0$

$$\left(\frac{-1}{\det(A)}\right)\big((-1)^n A^{n-1} + \cdots + a_1 I\big) = A^{-1}$$

1.2. A Canonical form for nilpotent mappings.