0.0.1. In calculus.

(1) Extreme Value Theorem: Every continuous function $f : [a, b] \to$ R admit both max and min value $\Rightarrow$ Compact set
(2) Intermediate value Theorem: Given continous function $f : [a, b] \to$ R for all $f(a) \leq \lambda \leq f(b) \exists c \in [a, b] \ni f(c) = \lambda \Rightarrow$ connected set

How to prove a statement: HP,then Q, $P \Rightarrow Q$

$$\begin{cases} \begin{cases} \text{Direct Proof} \\ \text{Indirect Proof} \begin{cases} \text{contrapositive } \sim Q \Rightarrow \sim P \\ \text{by contradiction} \end{cases} \end{cases} \\ Mathematical Induction \end{cases}$$

1. Some preliminary

1.1. Set Theory. We will assume that you are familiar with some basic set theory e.g. union, intersection, difference

1.2. The Number System.
$\mathbb{N} = \{ 1, 2, 3, \cdots \}$ the set of all positive integers n natural numbers
$\mathbb{Z} = \{ \cdots, -2, -1, 0, -1, -2, \cdots \}$ the set of all integers called the ring of intepus
$\mathbb{Q} = \left\{ \dfrac{m}{n} \ : \ n, m \in \mathbb{Z}, n \neq 0 \right\}$ the set of all rational numbers
$\mathbb{R}$ the set all of real numbers on the real number field on real line
$\mathbb{C} = \{ z = a + ib \mid a, b \in \mathbb{R} \}$ the set of all complex numbers or the complex number filed on complex plane, where $i = \sqrt{-1}$

Remark.

(1) $x + 2 = 0$ *no root in* $\mathbb{N}$
$3x - 5 = 0$ *no root in* $\mathbb{Z}$
$x^2 + 1 = 0$ *no root in* $\mathbb{R}$
(2) One can construct $\mathbb{Q}$ *from* $\mathbb{Z}$ *in algebraic way, called the fraction field of* $\mathbb{Z}$
(3) One can construct $\mathbb{R}$ *from* $\mathbb{Q}$ *in two ways:*
· *Using Dedekind cut which is given in the appendix of Rudin p17-21*
· *Using completion of matrix space*
(4) One can construct $\mathbb{C}$ *from in complex analysis*

Example.

(1) *Between any two rational numbers, there is another one*

Proof. Let $r, s \in \mathbb{Q}$ with $r < s$, then $\dfrac{r+s}{2} \in \mathbb{Q}$ and $r < \dfrac{r+s}{2} < s$

$$r = \frac{m_1}{n_1}, s = \frac{m_2}{n_2}, \frac{r+s}{2} = \frac{\frac{m_1}{n_1} + \frac{m_2}{n_2}}{2} = \frac{m_1 n_2 + n_1 m_2}{2 n_1 m_1} \in Q$$
$$s = \frac{s+s}{2} > \frac{r+s}{2} > \frac{r+r}{2} = r$$

∎

(2) $x^2 = \dfrac{4}{9}$ *has exactly two rational solutions, namely,* $\pm\dfrac{2}{3}$

(3) $x^2 = 2$ *has exactly two real root, namely,* $\pm\sqrt{2}$

(4) *Is there any rational roots of* $x^2 = 2$*? i.e., is* $\sqrt{2}$ *rational?*

Suppose $r = \dfrac{m}{n} \in \mathrm{Q}$*, is a root of* $x^2 = 2$*, where* $(m, n) = 1$

Then $\dfrac{m^2}{n^2} = 2 \implies m^2 = 2n^2 \implies 2 \mid m^2 \implies 2 \mid m \implies 4 \mid m^2 \implies 4 \mid 2n^2$
$\implies 2 \mid n^2 \implies 2 \mid n \implies (n, m) \neq 1$

(5) *Let* $A = \left\{ r \in \mathrm{Q} \mid r > 0 \ \& \ r^2 < 2 \right\}, B = \left\{ r \in \mathrm{Q} \mid r > 0 \ \& \ r^2 > 2 \right\}$
Then A contains no largest numbers, i.e. max element & B contains no smallest numbers, i.e. min element

Proof. A contains no largest numbers $\Leftrightarrow$ given $r \in A$, $\exists s \in A \ni s > r$

Now, given $r \in A$, Let $s = r - \dfrac{r^2 - 2}{r+2} = \dfrac{2r+2}{r+2}$ $(\star_1)$

$\implies s^2 - 2 = \dfrac{2(r^2-2)}{(r+2)^2}$ $(\star_2)$

Now, $r \in A, r^2 < 2 \implies r^2 - 2 < 0 \therefore$
$(\star_1)\&(\star_2) \implies s > r \ \& \ s^2 < 2 \implies s \in A$ ∎

(6) *As you know, in calculus, the sequence* $\{ 1, 1.4, 1.41, 1.414, 1.4142, \cdots \}$ *does not converge in* Q*, but it converges to* $\sqrt{2}$ *in* R

1.3. Order Sets.

Definition (Relation).
Let X be a nonempty set A, relation on X is a subset R *of* $X \times X = \{ (x, y) \mid x, y \in X \}$
Let R *be a relation on X, if* $(x, y) \in \mathrm{R}$, *then we say that x is retaliated to y, and is written as* $xRy(x \sim y)$

Definition (Order Set). *An ordered set on S, is a relation denoted by "$<$" on S, satisfy:*

(i) *The low of trichonomy*
   Given $x, y \in S$, one and only one of the following holds: $x < y, x = y, y < x$
(ii) *Transitivity: if $x < y \& y < z$, than $x < z$*

> ***Notation***
>
> (1) *$x < y$ means "x is less than y" or "x is smaller than y"*
> (2) *$y > x$ means $x < y$*
> (3) *$x \leq y$ means $x < y$ or $x = y$, i.e. the negative of $x > y$*

Definition (bdd). *Let S is an ordered set & $E \subseteq S(E \neq \emptyset)$*

- *E is bounded above if $\exists\ \alpha \in S \implies x \leq \alpha\ \forall\ x \in E$*
  such α is called an upper bound of E
- *E is bounded below if $\exists\ \beta \in S \ni \beta \leq x, \forall\ x \in E$, such β is called a lower bdd of E*
- *E is bdd is E is both bdd above and below.*

Definition (least upper bound). *Let S be an ordered set and $E \subseteq S(E \neq \emptyset)$ bdd above. An element $\alpha \in S$ is called the last upper bound or supremum of E if*

(i) *α is an upper bound of E*
(ii) *α is the smallest such one.*

Equivalently,

(i') *$x \leq \alpha, \forall x \in E$*
(ii') *if $\beta < \alpha$, then β is not an upper bdd of E, i.e. $\exists\ x \in E \ni x > \beta$*

Such α(if exists) is denoted by

$$\alpha = sup(E)$$

similarly, one can defined the greatest lower bdd of infimum of E

Remark. *if $\sup(E)$ exists then it is unique*
suppose $\alpha \neq \alpha'$ both lub of E

$\because$ *by trichotomy* ,$\alpha > \alpha'$ *or* $\alpha = \alpha'$ *or* $\alpha < \alpha'(\to\leftarrow)$

Definition (least upper bdd property). *A ordered set S is said to have the least upper bdd property if $E \subseteq S,\ E \neq \emptyset$ and E is bdd above, then* $\sup(E)$ *exists in S*

Example.

(1) *In* Q *with the normal ordining*
$A = \{\, r \in \mathrm{Q} \mid r > 0,\ r^2 < 2 \,\} \& B = \{\, r \in \mathrm{Q} \mid r > 0,\ r^2 > 2 \,\}$
Then A is bdd above, in fact, bdd by every element in B, but $\sup(A)$ *does not exist in* Q*($\because$ by Ex1.5)*

(2) *B is bdd below by every element of A and* $\inf B$ *does not exists*

(3) *Note that* $\sup(E) \& \inf(E)$ *may not in E even if exist*

Remark.

(1) *By the Example above,* Q *with the usual ordering has no l.u.b property*

(2) *In* 1.5 *we will explain that* R *with usual ordering has the l.u.b. property. However, we usually adopt the follwing*

The Axiom of Completence or Least upper bdd property:
Every nonempty subset E of R which is bdd above has l.u.b

Theorem (l.u.b.p. $\to$ g.l.b.p.). *Let S is an ordered set if S has the l.u.b. property, then S has the g.l.b. property, i.e. if $\emptyset \neq B \subseteq S$ is bdd below, then* $\inf(B)$ *exists in S*

Proof. ($\star$)
Given $B(\neq \emptyset) \subseteq S$ which is bdd below
Let $L = \{\, a \in S \mid a \text{ is a lower bdd of } B \,\}$

- $L \neq \emptyset(\because B$ is bdd below)
- L is bdd above (in fact, every element in B is on upper bound of L)
  $\implies \forall a \in L \implies a \le x,\ \forall x \in B \implies x$ is an upper bound of L
- $\sup(L) = \alpha$ exists by assumption

Claim $\alpha = \inf B$

(i) α is a lower bdd of B, i.e. $\alpha \le x,\ \forall x \in B$
By $\alpha = \sup L$, if $r < \alpha$, them r is not an upper bdd of $L(\because \alpha$ is the smallest one).Hence,$r \notin B(\because$ every element of B is an upper bdd of L), so $\alpha \le x, \forall x \in B$
We have proved $(r < \alpha \implies r \notin B) \implies (r \in B \implies r \geq \alpha)$

(ii) α is the greated one

if $\alpha < \beta$ and β is a lower bdd of B, then $\beta \notin L$, i.e. β is not a lower bdd of B, so α is the greatest one. Therefore, $\alpha = \inf(B)$

∎

Remark. *Let* $E(\neq \emptyset) \subseteq \mathrm{R}$ *be bdd below, then* $\inf(E)$ *exists and* $\inf(E) = -\sup(-E)$*, where* $-E = \{ -x \mid x \in E \}$

1.4. Field.

Recall the addition & multiplication in R

$+ : \mathrm{R} \times \mathrm{R} \to \mathrm{R}((a, b) \mapsto a + b)$

$\times : \mathrm{R} \times \mathrm{R} \to \mathrm{R}((a, b) \mapsto a \cdot b = ab)$

Definition. *Let X is a nonempty set A, binary operation on X is a function, $o : X \times X \to X$*

Definition. *Let* F *be a nonempty set, we say that* F *is a field ($(F, +, \cdot)$ is a field) if there are two binary operator called addition "$+$" and multiplication "$\cdot$" on* F *property*

Axioms for "$+$"

- *(A1) Commutative:* $\forall x, y \in \mathrm{F},\ x + y = y + x$
- *(A2) Associative:* $\forall x, y, z \in F, (x + y) + z = x + (y + z)$
- *(A3) Additive identity or zero element:* $\exists\ 0 \in F \implies x + 0 = 0 + x = x,\ \forall x \in \mathrm{F}$
- *(A4) Additive inverse on negative: For each* $x \in X,\ \exists\ -x \in F \implies x + (-x) = (-x) + x = 0$

i.e. $(\mathrm{F}, +)$ *is an abelian group* ***Axioms for multiplication***

- *(M1) Commutative:* $\forall x, y \in \mathrm{F},\ xy = yx$
- *(M2) Associative:* $\forall x, y, z \in \mathrm{F},\ (xy)z = x(yz)$
- *(M3) Muti identity:* $\exists 1 \neq 0$ *in* $\mathrm{F} \ni x1 = 1x = x$
- *(M4) Multiplicative inverse: For each* $x \neq 0, \exists x^{-1} \in \mathrm{F} \implies xx^{-1} = x^{-1}x = 1$

i.e. $(F = F \cdot \{ 0 \}, \cdot)$ *is an abelian group*

Distributive Law

- *(D1)* $\forall x, y, z \in \mathrm{F},\ (x, y)z = xz + yz\ \&\ x(y + z) = xy + xz$

Induction from Axioms

let $(\mathrm{F}, +, \cdot)$ be a field, we list a series of basic identity as you learn in high school in the real number system

(a) Cancellation law for " + " : $x+y=x+z \implies y=z$
$\because x+y=x+z \implies (-x)+(x+y)=(-x)+(x+z) \implies ((-x)+x)+y=((-x)+x)+z$
$\implies 0+y=0+z \implies y=z$

(b) 0 is "1"
suppose $0' \in$ F is another element satisfy A_3, then $0=0+0'=0'$

(c) $x+y=x \implies y=0$ by (a) $\because x+y=x+0 \implies y=0$

(d) negative $-x$ of x is "1"
if $x' \in F$, is another negative of x, them $x+x'=x'+x=0$
From $x+x'=0 \implies (-x)+(x+x')=-x+0=-x$

(e) $x+y=0 \implies y=-x$
$x+y=0 \implies (-x)+(x+y)=(-x)+0 \implies ((-x)+x)+y=-x$
$\implies 0+y=-x \implies y=-x$

(f) $-(-x)=x$
$-(-x)+(-x)=0$, By (d) $x=-(x)$

(a') cancellation law
if $x \neq 0$, then $xy=xz \implies y=z$, $\because (x^{-1})(xy)=(x^{-1})(xz)$
$\implies (x^{-1})(xy)=(x^{-1}x)z \implies 1y=1z \implies y=z$

(b') 1 is "1"
if $1'$ is another identity, then $1=11'=1'$

(c') $x \neq 0$ & $xy=x \implies y=1$
$xy=x1 \implies y=1$

(d') For $x \neq 0$ in F, x^{-1} is "1"
if x is another one, i.e. $x'x=xx'=1 \implies (x^{-1})(xx')=(x^{-1})1=x^{-1}$

(f') $x \neq 0 \implies (x^{-1})^{-1}=x$
$(x^{-1})^{-1}(x^{-1})=1 \implies x=(x^{-1})^{-1}$

(g') $0x=x0=0$
$(0+0)x=0x+0x \implies 0x=0$

(h') $x \neq 0$ & $y \neq 0 \implies xy \neq 0$, equivalently $xy=0 \implies x=0$ or $y=0$
$\because xy=0$ then $(x^{-1})(xy)=((x^{-1})x)y=1y=y(\rightarrow\leftarrow)$

(i') $(-x)y=-(xy)=x(-y)$
$\because [(-x)+x]y=0y=0=(-x)y=-(xy) \implies (-x)y=-(xy)$

(j') $(-x)(-y)=xy$
$\because (-x)(-y)=-(x(-y))$ by (i)
$=-(-(xy))=xy$

(k) $-x = (-1)x$

$\because (1-1)x = 0x = 0 = 1x + (-1)x = x + (-1)x \implies (-1)x = -x$

<u>Definition</u> (Order Field). *Let* F *is a field, we say that* F *is an order field if there is an ordering " $<$ " satisfying*

(1) if $x < y$, then $x + z < y + z$, $\forall z \in F$

(2) if $x > y$ and $y > 0$, then $xy > 0$

Example. Q *and* R *are order field under the usual ordering*

Some basic properties of ordered field, let F *be an ordered field with ordering " $<$ "*

(a) $x > 0 \implies -x < 0$

$\because x > 0 \implies x + (-x) > 0 + (-x) \implies 0 > -x$

(b) $x > y \Leftrightarrow x - y > 0$

$\because x > y \implies x + (-y) > y = (-y) \implies x - y > 0$

$x - y > 0 \implies x - y + y > y \implies x + 0 > y \implies x > y$

(c) $x > 0$ *and* $y < z \implies xy < xz$

$\because x > 0$ *and* $y < z \implies x > 0$ *and* $z - y < 0 \implies x(z-y) > 0 \implies xz + x(-y) > 0$

$\implies xz - xy > 0 \implies xz > xy$

(d) $x < 0$ *and* $y < z \implies xy > xz$

$\because x < 0$ *and* $y < z \implies -x > 0$ *and* $z - y > 0 \implies (-x)(z-y) > 0 \implies -xz + xy > 0$

$\implies xy > xz$

(e) $\forall x \neq 0$ *in* $\mathrm{F}, x^2 > 0$

$\because x > 0 \implies x \cdot x > x0$ *by (c) or*

$x < 0 \implies -x > 0$ *by (a)* $\implies -x > 0$ *by (a)* $\implies (-x)^2 > 0 \implies x^2 > 0$

(f) $1 > 0, \ -1 < 0$

$\because 1 \neq 0 \implies 1^2 > 0$ *by (e)* $\implies 1 > 0$

(g) $0 < x < y \implies 0 < \frac{1}{y} < \frac{1}{x}$

$\because$ *Note that* $\forall u \in \mathrm{F}, \ u > 0 \implies \frac{1}{u} = u^{-1} > 0$

$\because$ *if* $\frac{1}{u} < 0$, *then* $u \cdot \frac{1}{u} < 0$ *by (e)* $\implies 1 < 0 (\rightarrow\leftarrow) \ \therefore \ \frac{1}{u} > 0$

Now, $\frac{1}{x}, \frac{1}{y} > 0$ *from* $x < y$ *we get* $(\frac{1}{x} \cdot \frac{1}{y})x < (\frac{1}{x} \cdot \frac{1}{y})y \implies 0 < \frac{1}{y} < \frac{1}{x}$

Remark. *By (e)(f), we conclude that* $\mathbb{C}$ *is not an ordered field*
$\because \mathbb{C}$ *were an ordered field, then by (e),* $i^2 > 0 \implies -1 > 0 (\rightarrow\leftarrow)$
$\therefore \mathbb{C}$ *is not an order field*

1.5. The Real Number Field $\mathbb{R}$.

Theorem. *There exists an ordered field* $\mathbb{R}$ *containing* $\mathbb{Q}$ *which has the l.u.b. property. Moreover, such* $\mathbb{R}$ *is unique up to order-isomorphism i.e. if* " $<$ " *and* " $<'$ " *are two orders on* $\mathbb{R}$*, them* $\exists f_i(\mathbb{R}, <) \to (\mathbb{R}, <') \implies$

(i) f *is a field isomorphism,*
i.e. $\forall a, b \in \mathbb{R},\ f(a+b) = f(a)+f(b),\ f(ab) = f(a)f(b),\ f(1) = 1$

(ii) f *preserves ordering,* $a < b \implies f(a) < f(b)$

Such $\mathbb{R}$ *is called the real number field or real number system or real line*

Theorem.

(a) The Archimedean property of $\mathbb{R}$: *Given* $x, y \in \mathbb{R}$ *with* $x > 0,\ \exists\, n \in \mathbb{N} \implies nx > y$

(b) $\mathbb{Q}$ *is dense in* $\mathbb{R}$: $\forall x, y \in \mathbb{R}$ *with* $x \le y,\ \exists\, r \in \mathbb{Q} \implies x < r < y$

Proof.

(a) Let $A = \{\, nx \mid n \in \mathbb{N} \,\} \subseteq \mathbb{R}$
if (a) were false, them A is bdd above by y, since $\mathbb{R}$ has the l.u.b property
$\alpha = \sup A$ exists in $\mathbb{R}$, since $x > 0,\ \alpha - x < \alpha \implies \alpha - x$ is not an upper bdd of A
$\implies \exists m \in \mathbb{N} \ni mx > \alpha - x \implies (m+1)x > \alpha (\rightarrow\leftarrow)$

(b) Since $x < y,\ y - x > 0$, by (a),$\exists n \in \mathbb{N} \implies n(y-x) > 1$
By (a) again, $\exists m_1, m_2 \in \mathbb{N} \implies m_1 = m_1 1 > n_x$ & $m_2 = m_2 \cdot 1 > -nx$
we have $-m_2 < nx < m_1$, choose $m \in \mathbb{Z} \implies -m_2 \le m \le m_1$ & $m - 1 \le nx < m$
(in fact, $m = [nx] + 1$,where $[z]$ in the greatest integer of z)
we have $nx < m < 1 + nx < ny (\because n(y-x) > 1) \implies x < \dfrac{m}{n} < y$
Let $r = \dfrac{m}{n} \in \mathbb{Q}$, then $x < r < y$

■

An application of the density property of Q in R :
Given $x \in \mathrm{R}-\mathrm{Q}$ i.e. x is an irrational numbers, i.e. $\forall\ \epsilon>0, \exists r \in \mathrm{Q} \implies |x-r|<\epsilon$
equivalently, $\exists$ a sequence $\{\, r_n \,\}$ in Q $\implies r_n \to x$
In fact, one may choose $\{\, r_n \,\}$ to $\uparrow$ or $\downarrow$
$\because \forall n \geq 1,\ \exists\, r_n \in \mathrm{Q} \implies x < r_n < \dfrac{1}{n} + x$ by Thm.1.3(b) By squeezing lemma, $r_n \to x$ on $n \to \infty$

Theorem (existence of nth root)**.** *Given $x \in \mathrm{T}, x>0$ & $n \in \mathrm{N}, \exists$ "1" $y > 0 \implies y^n = x$*
Such y is called the nth root of x & denoted by $y = \sqrt[n]{x} = x^{\frac{1}{n}}$

Proof. **not important**
"1". Suppose $y_1, y_2 > 0 \implies y_1^n = x$ & $y_2^n = x$
Bt trichotomy, we have

(i) $0 < y_1 < y_2 \implies y_1^n < y_2^n (\rightarrow\leftarrow)$
(ii) $0 < y_2 < y_1 \implies y_2^n < y_1^n (\rightarrow\leftarrow)$
(iii) $y_1 = y_2$

"$\exists$". Let $E = \{\, t \in \mathrm{R} \mid t^n < x \,\}$
Claim:

- $E \neq \emptyset$, Let $t = \dfrac{x}{1+x}$, then $0<t<1$, hence $t^n < t < x$, $\therefore t \in E$ & $E \neq \emptyset$
- E is bdd above, in fact E is bdd above by $1+x$ if $t > 1+x > 1$, then $t^n > t > x$, so E is bdd above by $1+1$
  Therefore $y = \sup E$ exists & is finite
- Claim $y>0$ & $y^n = x$, clearly, $y > 0 (\because \dfrac{x}{1+x} \in E$ & $\dfrac{x}{1+x} > 0)$
  by trichotomy, we have $y^n < x,\ y^n > x,\ y^n = x$

Now, to show that (i) & (ii) are impossible, do (iii) holds $y^n = x$
By the identity, $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$
(i)$y^n < x$ choose $0 < h < 1 = \alpha$ & $0 < \dfrac{x-y^n}{n(y+1)^{n-1}}$, $0 < h < \min\{\, \alpha, \beta \,\}$
put $a = y,\ b = y+h$ in $(\star)$, we obtain

$$(y+h)^n - y^n < hn(y-h)^{n-1} < hn(y+1)^{n-1} < x - y^n$$
$$\implies (y+h)^n < x \implies y+h \in E\ \& y+h > y (\rightarrow\leftarrow) \therefore \text{(i) fails}$$

(ii) $y^n > x$, Let $k = \dfrac{y^n - x}{ny^{n-1}}$, Then $0 < k < y,\ k = \dfrac{y^n - x}{ny^{n-1}} < \dfrac{y^n}{ny^{n-1}} = \dfrac{y}{n} < y$
if $t > y-k > 0$, then $y^n - t^n \leq y^n - (y-k)^n < kny^{n-1}$ by $(\star) = y^n - x$ $\implies t^n > x \implies t \in E \implies E$ is bdd above by $y-k \implies \sup E \leq y - k (\rightarrow\leftarrow)$
$\therefore$ (ii) fails ■

Corollary. *Let $a, b \in \mathbb{R}$ with $a, b > 0,\ n \in \mathbb{N}$ Then$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$*
$\because a^{\frac{1}{n}}, b^{\frac{1}{n}} > 0$ & $(a^{\frac{1}{n}} \cdot b^{\frac{1}{n}}) = ab$, *By (1) in Thm 1.4* $(a,b)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$

infinite in $\mathbb{R}$
After discuss the real number $\mathbb{R}$, sometimes, we have to work with the extended real number system $\mathbb{R}^* = [-\infty, \infty] = \mathbb{R} \cup \{ +\infty, -\infty \}$ with observe, $x \in \mathbb{R}$

$$\lim_{n\to\infty}(-n) = -\infty,\ \lim_{n\to\infty} n = \infty,\ \lim_{n\to\infty}(\frac{1}{n}+n) = \infty,\ \lim_{n\to\infty}(n^2-n) = \infty$$
$$x \pm \infty = \pm\infty,\ 0 \cdot (\pm\infty) = 0,\ \infty - \infty \text{ is not define}$$

Element in $\mathbb{R} \subseteq \mathbb{R}^*$ are called finite. Now, given any nonempty subset $E \subseteq \mathbb{R}$,

$$\sup E = \begin{cases} +\infty \text{ if } E \text{ is not bdd above} \\ \text{finite if } E \text{ is bdd above} \end{cases} \quad \& \inf E = \begin{cases} -\infty \text{ if } E \text{ is not bdd below} \\ \text{finite if } E \text{ is bdd below} \end{cases}$$

Note that if $A \subseteq B$, then $\sup A \leq \sup$ & $\inf A \geq \inf B$
$\therefore \emptyset \subseteq B,\ \forall B \subseteq \mathbb{R}$, One may define $\sup \emptyset = -\infty, \inf \emptyset = +\infty$

1.6. The Complex Number Field $\mathbb{C}$.

Consider the contention product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{ (a,b) \mid a, b \in \mathbb{R} \}$
Note that $(a,b) = (c,d) \Leftrightarrow a = c$ & $b = d$, From now, we can write $\mathbb{C} = \mathbb{R}^2$

Operation on $\mathbb{C}$ Given $(a,b), (c,d) \in \mathbb{C}$

(1) $(a,b) + (c,d) = (a+c, b+d)$
(2) $(a,b)(c,d) = (ac - bd, ad + bc)$

It is easy to see that, with these operations, $\mathbb{C}$ is a field.

Note that

- the zero element is $(0, 0)$
- the negative of (a, b) is $-(a, b) = (-a, -b)$
- the identity is $(1, 0)$
- if $(a, b) \neq (0, 0)$, then $(a, b)^{-1} = \left(\dfrac{1}{a^2+b^2}, \dfrac{-b}{a^2+b^2} \right)$

R is a subset of C (not vary important)
consider that map

$$f : \mathbb{R} \to \mathbb{C} \text{ define by } f(a) = (a, 0),\ a \in \mathbb{R}$$

we have $(1) f$ is injective $(2) f(1) = (1, 0)\ \because \forall\ a, b \in \mathbb{R}$
$f(a + b) = (a + b, 0) = (a, 0) + (b, 0) = f(a) + f(b),\ f(a \cdot b) = (ab, 0) = (a, 0) \cdot (b, 0)$
f is a field homomorphism
$\therefore f : \mathbb{R} \to \mathbb{C}$ is an injective and isomorphism
Therefore, we identify $\mathbb{R}$ with $f(\mathbb{R})$ through the injective f
i.e. $a \in \mathbb{R}$ is identified with $f(a, 0)$ in $\mathbb{C}$
$ab = (a, 0) \cdot (b, 0),\ a + b = (a, 0) + (b, 0)\ \forall\ a, b \in \mathbb{R}$

Change (a, b) **to** $a + bi$

Now, we can transform an element $(a, b) \in \mathbb{C}$ into the normal form:
(a,b) = (a,0)+(0,b) = (a,0)(1,0)+(b,0)(0,1) = a1+bi = a+ib, where i = (0,1)
Therefore, from new on, we write $\mathbb{C} = \{ a + ib \mid a, b \in \mathbb{C} \}$
An element $z = a + ib \in \mathbb{C}$ is called a complex number
Hence, under this notification, $z = a + ib, w = c + id \in \mathbb{C}$

(1) $z + w = (a + c) + i(b + d)$
(2) $zw = (ac - bd) + i(ad + bc)$

and the a is called the real part of z, $a = \mathrm{Re}(z)$, b is called imaginary part of z, $b = \mathrm{Im} z$

Some basic properties of complex numbers whose proofs are easy
$\forall z, w \in \mathbb{C}$

- $\overline{z+w} = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z} \cdot \overline{w}$
- $\text{Re}z = \dfrac{z+\overline{z}}{2}$
- $\text{Im}z = \dfrac{z-\overline{z}}{2i}$
- $|z| = 0 \Leftrightarrow z = 0$
- Triangle inequality $|z+w| \leq |z| + |w|$
- $||z| - |w|| \leq |z - w|$
- $\mathbb{C}$ is not an ordered field
- $|z|^2 = z\overline{z}$
- $|\overline{z}| = |z|$
- $|\text{Re}z| \leq |z|, |\text{Im}z| \leq |z|$
- $|zw| = |z||w|$

Proof. $|z+w| \leq |z| + |w|$
$|z+w|^2 = (z+w)(\overline{z+w}) = (z+w)(\overline{z}+\overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w}$
$= |z|^2 + 2\text{Re}(z\overline{w}) + |w|^2 \leq |z|^2 + 2|z\overline{w}| + |w|^2 = |z|^2 + 2|z||w| + |w|^2 = (|z|+|w|)^2$
$\therefore |z+w| \leq |z| + |w|$ ■

Theorem (basic algebraic theorem)**.**
(a) x^2+1 *has no root in* $\mathbb{R}$
(b) x^2+1 *has two distinct roots in* $\mathbb{C}$

Proof.

(a) $1 > 0,\ x^2 > 0,\ \forall x \in \mathbb{R} - \{\,0\,\} \implies x^2+1 > 0\ \forall x \neq 0$
$0^2 + 1 = 1 > 0,\ \therefore x^2+1 > 0,\ \forall\ x \in \mathbb{R}$. Hence, $x^2+1 = 0$ has no root in $\mathbb{R}$

(b) $i^2 = (0,1)(0,1) = (0-1,0) = (-1,0) = -1$
$(-i)^2 = (-(0,1))^2 = (0,-1)^2 = (0,-1)(0,-1) = -1$, $\therefore \pm i$ are root of $\mathbb{C}$

■

Conclusion: Every non const polynomial $f(x) \in \mathbb{R}[x]$ has n roots where $n = \deg f(x)$
The complex root is even

no important proof
$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x],\ a_n \neq 0,\ n \geq 1$
if $\alpha = a + ib \in \mathbb{C}$ is a root of $f(x)$,then
$0 = f(\alpha) = a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0$
$0 = f(\overline{\alpha}) = a_n\overline{\alpha}^n + a_{n-1}\overline{\alpha}^{n-1} + \cdots + a_1\overline{\alpha} + a_0$
$\therefore\ (x-\alpha)|f(x),\ (x-\overline{\alpha})|f(x) \implies (x-\alpha)(x-\overline{\alpha})|f(x) \implies (x^2 - (\alpha - \overline{\alpha})x + |\overline{\alpha}|^2) \mid f(x)$
$\implies (x^2 - 2ax + (a^2+b^2)) \mid f(x)$
$\therefore$ quadratic function must have two roots in $\mathbb{C}$

The fundamental Theorem of Algebra
Every non zero polynomial $f(x) \in \mathbb{C}[x]$ has at least one root in $\mathbb{C}$
Therefore, if $\deg f(x) = n$, then $f(x)$ has n roots in $\mathbb{C}(C, M)$

$\because f(x) = (x-\lambda_1)^{e_1}\cdots(x-\lambda_t)^{e_t}(a_1x^2+b_1+c_1)^{l_1}\cdots(a_sx^2+b_sx+c_s)^{l_s}$, where $\lambda_1, \cdots, \lambda_t \in \mathbb{R},\ a_i, b_i, c_i \in \mathbb{R} \& e_1 + \cdots + e_t + 2l_1 + \cdots + 2l_s = \deg f(x)$ which shows that all roots of $f(x)$ are in $\mathbb{C}$
In fact, we have the famous theorem: The fundamental theorem of algebra
Every non zero polynomial $f(x) \in \mathbb{C}[x]$ has at least one root in $\mathbb{C}$
$\therefore$ if $\deg f(x) = n$, then $f(x)$ has n roots in $\mathbb{C}(C, M)$

Theorem (Cauchy-Scheming Inequal)**.** *Given $z_1 \cdots, z_n, w_1, \cdots, w_n \in \mathbb{C}$, we have*

$$\left|\sum_{j=1}^{n} z_j\overline{w}_j\right| \leq \left(\sum_{j=1}^{n}|z_j|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n}|w_j|^2\right)^{\frac{1}{2}}$$

and " $=$ " holds $\Leftrightarrow \exists\ \lambda \in \mathbb{C} \ni\ w_j = \lambda z_j,\ 1 \leq j \leq n$,
In patricial, if $x_1, \cdots, x_n, y_1, \cdots, y_n \in \mathbb{R}$, then

$$\left|\sum_{j=1}^{n} x_j y_j\right| \leq \left(\sum_{j=1}^{n} x_j^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n} y_j^2\right)^{\frac{1}{2}}$$

and " $=$ " holds $\Leftrightarrow \exists\ t \in \mathbb{R} \ni\ y_j = tx_j,\ 1 \leq j \leq n$

The proof is too long, I am lazy

1.7. Euclidean Spaces $\mathbb{R}^n$.

Definition. *the n-dimensional Euclidean space* $\mathbb{R}^n$

$$= \{ x = (x_1, \cdots, x_n) \mid x_i \in \mathbb{R}, 1 \le i \le n \} = \mathbb{R} \times \cdots \times \mathbb{R}$$

Note that

$$(x_1, \cdots, x_n) = (y_1, \cdots, y_n) \Leftrightarrow x_i = y_i \ \forall\ 1 \le i \le n$$

We are going to introduce the structure of $\mathbb{R}^n$

- vector space
- inner product space
- normed linear space
- matrix space

Definition. *Two operation on* $\mathbb{R}^n$ *as follows:*

- *Addition* $+ : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n,\ (x, y) \mapsto x + y = (x_1 + y_1, \cdots, x_n + y + n)$
- *Scalar multiplication* $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,\ (a, x) \mapsto ax = (ax_1, \cdots, ax_n)$

we skip space example here.

1.8. Countability of Sets.

Given two nonempty set A, B and a function $f : A \to B, f(A) = \{ f(a) \mid a \in A \}$ is called the image of A under f

Some basic things

$E \subseteq A,\ f(E) = \{ f(a) \mid a \in E \}$ the image of E under f
f is infective(one-to-one) $x_1 \ne x_2 \implies f(x_1) \ne f(x_2) \Leftrightarrow f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$
f is surjective(onto) if $f(A) = B$, f is bijective if f is one-to-one and onto

Given $F \subseteq B$, $f^{-1}(F) = \{ x \in X \mid f(x) \in \mathrm{F} \}$ called the inverse image of f under F

Example

$f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2,\ x \in \mathbb{R}$
$f^{-1}([0, 1]) = \{ x \in \mathbb{R} \mid f(x) \in [0, 1] \} = \left\{ x \in \mathbb{R} \mid x^2 \in [0, 1] \right\} = [-1, 1]$
$f^{-1}([-1, 1]) = [-1, 1]$

Properties of inverse image

- $F_1 \subseteq F_2 \subseteq B \implies f^{-1}(F_1) \subseteq f^{-1}(F_2)$
- Inverse image presences set operation
  $\forall F_\alpha \subseteq B,\ \alpha \in I,\ F \subseteq B$
  (i) $f^{-1}(\cup_{\alpha\in I} F_\alpha) = \cup_{\alpha\in I} f^{-1}(F_\alpha)$
  (ii) $f^{-1}(\cap_{\alpha\in I} F_\alpha) = \cap_{\alpha\in I} f^{-1}(F_\alpha)$
  (iii) $f^{-1}(B-F) = f^{-1}(B) - f^{-1}(F)$
- Given $S \subseteq A,\ f'(f'(S)) \supseteq S,"="$ $\Leftrightarrow$ one-to-one, **example:**
  $f:\mathbb{R}\to\mathbb{R},\ f(x)=x^2,\ S=[0,1],\ f(S)=[0,1],\ f^{-1}(f(S)) = f^{-1}([0,1]) = [-1,1]$
- Given $F \subseteq B,\ f(f^{-1}(F)) \subseteq F,"="$ $\Leftrightarrow$ "onto", **example**
  $f(x)=x^2,\ x\in\mathbb{R},\ F=[-1,1],\ f(f^{-1}([-1,1])) = f([-1,1]) = [0,1]$
- For $y \in B,\ f^{-1}(\{y\}) = f^{-1}(y) = \{x \in A \mid f(x)=y\}$ the inverse image of y, **example**
  $f:\mathbb{R}\to\mathbb{R},\ f(x)=x^2,\ f^{-1}(1)=\{1,-1\},\ f^{-1}(2)=\emptyset$

Definition (cardinality)**.** *Let A, B are two set ew say that A and B have the same cardinality if $\exists$ a bijective map $f: A \to B$, which is denoted by $A \sim B$*
From now on, we write $|A|$ as the cardinality of A

Claim " $\sim$ " is an $\equiv$ relation among all sets

(i) Reflexion: $\forall$ set $A,\ A \sim^{1A} A,$ which 1_A is identity mapping
(ii) Symmetry: $A \sim^f B \implies B \sim^{f^{-1}} A$
(iii) Transitive: $A \sim^f B$ & $B \sim^g C \implies A \sim^{gof} C$

So we gave some property:

- Any two " $\equiv$ " are either disjoint or identical
- $\overline{X}$ is a disjoint union of " $\equiv$ " classes
  $[A] = \left\{ B \in \overline{X} \mid B \sim A \right\}$ the " $\equiv$ " class set by A

Ant two element in an " $\equiv$ " class have the same cardinality
Notation For $n \in \mathbb{N},\ \mathbb{N}_m = \{1, 2, \cdots, n\}$

Definition. *Let A be a set*

(a) A is a finite set if $A = \emptyset$ or $A \sim \mathbb{N}_n$ for some $n \in \mathbb{N}$
(b) A is a infinite set if A is not a finite set
(c) A is countable if $A \sim \mathbb{N}$
(d) A is uncountable if A is not countable.
(e) A is at most countable if A is finite or countable

Remark.

(1) *when A, B are finite sets, $A \sim B \Leftrightarrow |A| = |B|$, i.e. A, B have same number.*
(2) *where A, B are infinite and $A \sim B$, i.e. $|A| = |B|$, the concept is abstract.*
(3) $\{ a, b, c \} \cup \mathbb{N} \sim \mathbb{N},\ f : \mathbb{N} \to \{ a, b, c \} \cup \mathbb{N},\ f(1) = a,\ f(2) = b,\ f(3) = c, \cdots$
(4) *Any finite set can not equivalent to a proper subset, i.e. A is finite, $B \subseteq A$*
Then $A \sim B$, In fact $|B| < |A|$, but infinite different
(5) *Any finite set A can be listed an $A = \{ a_1, \cdots, a_n \}$ where $n = |A|$*

Now, we consider the case of countable set

Recall, in calculus, a real sequence $\{ a_n \}$, e.g.

$$a_n = \frac{1}{n} \left\{ \frac{1}{n} \right\},\ a_n = 1 - \frac{1}{n}\ \left\{ 1 - \frac{1}{n} \right\},\ a_n = \begin{cases} 0 \text{ if } n \text{ is odd} \\ 1 \text{ if } n \text{ is even} \end{cases}$$

<u>Definition</u>. *Let X be a nonempty set, a sequence in X is a function $a : \mathbb{N} \to X$*
Given a sequence $=^a$ in X, a is "1" determine by $a(n),\ \in \mathbb{N}$
We write

$$a = \{ a(1), a(2), \cdots, a(n), \cdots \} = \{ a_1, a_2, \cdots, a_n, \cdots \} = \{ a_n \} = \{ a_n \}_{n=1}^{\infty}$$

Remark.

(1) *For a sequence $\{ a_n \}$ in X, a_n may not be distinct.*
If all a_n are distinct, then we say that $\{ a_n \}$ is a distinct sequence in X.
(2) *We usually use $\{ a_n \}, \{ b_n \}$ to denote sequence*
(3) *A sequence $\{ a_n \}$ in X in fact is a function from $\mathbb{N} \to X$, So $\{ a_n \mid n \in \mathbb{N} \}$ is the image of the sequence.*
(4) *$\{ a_n \}$ is a sequence, a_n is called the n^{th} term of the sequence.*
(5) *A sequence in X may begin at 0, i.e. $\{ a_n \}_{n=0}$*
By a changing index, we can make it from $\{ b_n \}_{n=1}^{\infty},\ b_n = a_{n+1},\ n = 1, 2, \cdots$

<u>Definition</u> (increasing)**.**
A function $a : \mathbb{N} \to \mathbb{N}$ is increasing, a is $\uparrow$, if $a(n) \leq a(n+1)\ \forall\ n \geq 1$
a is strictly increasing, a is st. $\uparrow$, if $a(n) < a(n+1)\ \forall\ n \geq 1$

Now, given a st. $\uparrow$ function $n : \mathbb{N} \to \mathbb{N}$, i.e. $n(k) < n(k+1),\ k \geq 1$ i.e. $n_k < n_{k+1},\ k \geq 1$, i.e. $n_1 < n_2 < \cdots < n_k < \cdots$, i.e. $\{\, n_k \,\}_{k=1}^{\infty}$ is a st. sequence in $\mathbb{N}$

<u>Definition</u>. *Let $\{\, a_n \,\}$ be a sequence in X and $\{\, n_k \,\}$ be a st. $\uparrow$ sequence in $\mathbb{N}$, then the sequence $\{\, a_{n_k} \,\}$ is called a subsequence of $\{\, a_n \,\}$*
In fact

$$\mathbb{N} \to^{n}_{st.} \mathbb{N} \to^{a}_{seq} X \Rightarrow a \circ n : \mathbb{N} \to X \ \textit{is a function,}$$

hence, it also a sequence in X

$$a \circ n = \{\, a \circ n(k) \,\} = \{\, a(n(k)) \,\} = \left\{\, a_{n(k)} \,\right\} = \{\, a_{n_k} \,\}$$

Remark. *if $\{\, a_{n_k} \,\}$ is st. $\uparrow$ in $\mathbb{N}$, then $k \leq n_k\ \forall\ k \geq 1$*
$\because$ *By mathematical Induction*

- $1 \leq n_1$
- *Assume it's true for $k \geq 2$, i.e. $k \leq n_k$*
- *Consider $k+1$, $\ k+1 \leq n_k + 1 \leq n_{k+1}$*

Example
Let $\{\, a_n \,\}$ be a sequence in X, then $\{\, a_{2k} \,\}$ and $\{\, a_{2k-1} \,\}$ are subsequence of $\{\, a_n \,\}$

Finally, we will assume that you are familiar with the following property of the countability of sets:

(1) Every subset of a countable set is at most countable. The proof needs the well ordering of $\mathbb{N}$: Every nonempty subset of $\mathbb{N}$ has the smallest element
(2) Countable union of countable sets is countable
(3) If $A_1, A_2, \cdots, A_n$ are countable, then so is $A_1 \times \cdots \times A_n$
(4) If A is countable, then so is $A^n \equiv A \times \cdots \times A\ \forall n \geq 1$
(5) $\mathbb{N},\ \mathbb{Z},\ \mathbb{Q},\ \mathbb{Q}^n,\ \forall n \geq 1$ are countable
(6) The set $\{\, a_n \mid a_n = 0 \text{ or } 1 \,\}$ is uncountable
    This can be proved by Canton diagonal process
    $\because$ if it is countable, then we can list it,$a_0 A = \left\{\, a_1^{(1)}, a_2^{(2)}, \cdots \right\}$
  where
    $a^{(1)} = \left\{\, a_n^{(1)} \,\right\} = a_1^{(1)}, a_2^{(1)}, \cdots\ ;\ a^{(2)} = \left\{\, a_n^{(2)} \,\right\} = a_1^{(2)}, a_2^{(2)}, \cdots$
    Now, construct a sequence $\{\, a_n \,\}$ in $A \ni \{\, a_n \,\} \neq a^{(k)}\ \forall\ k \geq 1 (\rightarrow\leftarrow)$

Recall, intervals in $\mathbb{R}$, $-\infty < a \leq b < \infty$, following are finite bdd interval

$(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$ open interval
$[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$ closed interval
$(a, b] = \{ x \in \mathbb{R} \mid a < x \leq b \}$ open-closed
$[a, b) = \{ x \in \mathbb{R} \mid a \leq x < b \}$ closed-open

An interval I in $\mathbb{R}$ is said to be non-degenerate if the endpoint of I are distinct i.e. length > 0. Otherwise, it is degenerate.

Note.

$(0, 1)$ is uncountable, $\because (0, 1) = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{2^n} \mid a_n = 0 \text{ or } 1, n \in \mathbb{N} \right\}$

$x \in (0, 1)$ has a unique binary representation, so $(0, 1) \sim A$, where A is $\{ \{ a_n \} \mid a_n = 0 \text{ or } 1 \}$ which is uncountable

All non-degenerate intervals in $\mathbb{R}$ are uncountable.
$\because$ It sufficient to consider bdd non-degenerate interval in $\mathbb{R}$, given $\infty < a < b < \infty$
(a, b) is uncountable($\because (0, 1) \sim (a, b)$)
Note that $(0, 1) \sim \mathbb{R}(\because (0, 1) \to (\frac{\pi}{-2}, \frac{\pi}{2}) \to \mathbb{R})$

2. Basic Point Set Topology

To know the "closeness", "limit" and "continue"

Notation. Let X be a nonempty set. The power set of X is denoted by $p(X)$ or 2^X, i.e. $\mathscr{P}(X) = 2^X$ which is the collect of all subset, if $|X| = n$, then $|\mathscr{P}(X)| = 2^n$

2.1. Topological Spaces.

Definition. *Let X be a nonempty set and $\mathscr{T} \subseteq \mathscr{P}(X)$, we say that $\mathscr{T}$ is a topology on X if it satisfies*

(1) $\emptyset, X \in \mathscr{T}$

(2) $\mathscr{T}$ is closed under arbitrary union,

i.e. $U_\alpha \in \mathscr{T},\ \alpha \in I \implies \bigcup_{\alpha \in I} U_\alpha \in \mathscr{T}$

(3) $\mathscr{T}$ is closed under finite intersection

i.e. $U_1, \cdots, U_n \in \mathscr{T} \implies U_1 \cap \cdots \cap U_n \in \mathscr{T}$

In this chapter, the pair (X, J) or simply J is called a topological space and members in T are called open set in X or open subsets of X

Remark.

(1) X: a nonempty set, there is at least two trivial topology on X
- *$\mathscr{P}(x)$ is the largest topology on X w.r.t inclusion X with this topology is called a discrete topological space*
- *$\mathscr{T}_0 = \{\emptyset, X\}$ is the smallest topology on X w.r.t inclusion X with this topology is called an indiscrete topological space*

(2) How many topology can be define on $\{a\}$, $\{a, b\}$?

In the following X is a topology space

Definition (neighborhood). *Let $p \in X$, a neighborhood of P is an open set U containing p*

Definition (Hausdorff space). *X is a Hausdorff space if any two distinct points can be separated by open set, i.e. $\forall\ p \neq q$ in X, $\exists$ neighborhood U of p and V of $q \in U \cap V = \emptyset$*

Definition (closed set). *A subset $F \subseteq X$ is said to be closed if $F^C = X - F$ is open in X*

Theroem 2.1. *The collection of all closed subsets of X satisfied*

(a) $\emptyset, X$ are closed

(b) Arbitrary intersection of closed set if closed

(c) Finite union of closed sets is closed

Proof.

(a) $X - \emptyset = X$ is open $\therefore \emptyset$ is closed
$X - X = \emptyset$ is open $\therefore X$ is closed

(b) Given closed sets $F_\alpha, \alpha \in I$, $X - \bigcap_{\alpha \in I} F_\alpha = \bigcup_{\alpha - I}(X - F_\alpha)$ is open,$\therefore \bigcap_{\alpha - I} F_\alpha$ is closed.

(c) Given closed set $F_1, \cdots, F_n$, $X - \bigcup_{i=1}^{n} F_i = \bigcap_{i=1}^{n}(X - F_i)$ is open, $\therefore \bigcup_{i=1}^{n} F_i$ is closed.

■

<u>Definition</u>. *Let $Y \subseteq X$ and*

$$\mathscr{T}_y = \{U \cap Y \mid U \text{ is open in } X\}$$

Theroem 2.2. *$\mathscr{T}_Y$ is also a topology space*

Proof. To proof $\mathscr{T}_Y$ is a topology space, we take the topology's definition

(a) $\emptyset, Y \in \mathscr{T}_Y$ $(\because \emptyset = \emptyset \cap Y,\ Y = X \cap Y)$

(b) Given $U_\alpha \cap Y \in \mathscr{T}_Y,\ \alpha \in I$, where U_α is open in X

$$\bigcup_{\alpha \in I}(U_{\alpha \cap Y}) = (\bigcup_{\alpha \in I} U_\alpha) \cap Y \implies \bigcup_{\alpha \in I}(U_\alpha \cap Y) \in \mathscr{T}_Y$$

(c) Given $U_1 \cap Y, \cdots, U_n \cap Y$, where U_i is open in X, $1 \leq i \leq n$

$$\cap_{i=1}^{n}(U_i \cap Y) = (\cap_{i=1}^{n} U_i) \cap Y \implies \bigcap_{i=1}^{n}(U_i \cap Y) \in \mathscr{T}_Y$$

$\therefore \mathscr{T}_Y$ is a topology on Y

■

<u>Definition</u>. *In Theorem 2.2, with the topology $\mathscr{T}_Y$ on Y, is called a topological subspace of X and $\mathscr{T}_Y$ is called the relative topology of Y in X. Members in $\mathscr{T}_Y$ are called open set in Y or relative open sets in Y.*

2.2. Metric Spaces & Subspace.

In this chapter, we will introduce a class of topology space whose topology in induced by a metric.

__Definition.__ *Let X be a nonempty set. A metric or distance function in a function*

$$d : X \times X \to \mathbb{R},\ (a, b) \mapsto d(a, b)$$

satisfying:

(a) $\forall a, b \in X, d(a, b) \geq 0$ *and* $d(a, b) = 0 \Leftrightarrow a = b$
(b) $\forall a, b \in X, d(a, b) = d(b, a)$ ***symmetry***
(c) $\forall a, b, c \in X, d(a, b) \leq d(a, c) + d(a, b)$ ***triangle inequaity***

if d is a metric on X, then the pair (X, d) or simply X is called a metric space and $\forall a, b \in X,\ d(a, b)$ is called the distance between a & b

Examples

(1) Let X be a nonempty set define by

$$d(a, b) = \begin{cases} 0 \text{ if } a = b \\ 1 \text{ if } a \neq b \end{cases}$$

Then d is a metric on X, called the discrete metric and with this metric X is called a discrete metric space. In particular, any set admits a metric.

(2) The most important metric spaces are the Euclidean space $\mathbb{R}^k$, the metric d is called the Euclidean or standard or usual metric on $\mathbb{R}^k$.There are other metrics on $\mathbb{R}^k$ induced the same metric topology on $\mathbb{R}^k$, in fact, they are all equivalent, e.g $\forall\, 1 \leq p \leq \infty$, We can define a metric d_p on $\mathbb{R}^k$ as follows

- $1 \leq p < \infty,\ d_p(x, y) = ||x - y||_p = \left(\sum_{i=1}^{k} |x_i - y_i|^p \right)^{\frac{1}{p}}$
- $p = \infty,\ d_\infty(x, y) = \max_{1 \leq i \leq k} |x_i - y_i|$

Note that $d_2 = d$ is the Euclidean metric on $\mathbb{R}^k$

Remark. *In fact, every normed linear space$(V, ||\cdot||)$ is a metric space whose metric is induced by its norm*

(3) Let (X, d) be a metric space and $Y \subseteq X,\ Y \neq \emptyset$. Then the restriction of d to $Y \times Y$ is also a metric on Y, with this metric, Y is called a metric subspace of X

Definition (ball). *Given $p \in X \& r > 0$*
$B(p,r) = \{x \in X \mid d(x,p) < r\}$: *open ball with center p and radius r*
$\overline{B}(p,r) = \{x \in X \mid d(x,p) \leq r\}$: *closed ball with center p and radius r*

Example

(1) The discrete metric space $X: \ p \in X, \ r > 0$

$$B(p,r) = \begin{cases} \{p\} \text{ if } 0 \leq r \leq 1 \\ X \text{ if } r > 1 \end{cases}$$

$$\overline{B}(p,r) = \begin{cases} \{p\} \text{ if } 0 \leq r < 1 \\ X \text{ if } r \geq 1 \end{cases}$$

(2) In the Euclidean space $\mathbb{R}^k$, $p \in \mathbb{R}^k, \ r > 0$
$B(p,r) = \{x \in \mathbb{R} \mid \ \| x - p \| < r\}$ is a ""true"" open
$\overline{B}(p,r) = \{x \in \mathbb{R} \mid \ \| x - p \| \leq r\}$ is a "true" closed ball
In particular, for $k = 1$ in $\mathbb{R}$
$B(p,r) = (p-r, p+r)$: a symmetric opne interval
$\overline{B}(p,r) = [p-r, p+r]$: a symmetric close interval
However, w.r.t d_1 & d_∞, we have, e.g. in $\mathbb{R}^2$
$B_1(0,1) = \{(x,y) \mid |x-0| + |y-0| < 1\}$
$B_\infty(0,1) = \{(x,y) \mid \max\{|x|,|y|\} \leq 1\}$

(3) What is the open balls in $S = [0,1] \subseteq \mathbb{R}$?
$B_S(0, \frac{1}{2}) = \{x \in S \mid |x - 0| < \frac{1}{2}\} = [0, \frac{1}{2}] = B(0, \frac{1}{2}) \cap [0,1]$
$B_S(0,3) = [0,1] = B(0,3) \cap [0,1]$

Prop 2.2 Let S be a metric subspace of a metric space X, then $\forall \ p \in S \ \& \ r > 0$, $B_S(p,r) = B(p,r) \cap S$

Proof. $B_S(p,r) = \{x \in S \mid d(x,p) < r\} = \{x \in X \mid d(x,p) < r\} \cap S$
$= B(p,r) \cap S$ ∎

2.3. Open Sets in Metric Spaces.

We will see that every metric on a set induce a topology on X

Definition (interior point). *Let $S \subseteq X$ be a set, and $p \in S$, we say that p is an interior point of S if $\exists \ r > 0, \ \exists \ B(p,r) \subseteq S$*
Denote by S^o or int(S) by the set of all interior point of S

Definition (open). *Let $S \subseteq X$, we say that S is open if all points of S are interior points of S*

Remark.

(1) Every open set S is a union of a open balls in X.
$\because \forall\ x \in S, x$ *is an interior point of* S, $\exists r_x > 0 \ni B(x, r_x) \subseteq S$
$\therefore S = \cup_{x \in S} B(x, r_x)$
(2) $S^o \subseteq S$ *by definition*
(3) S *is open* $\Leftrightarrow S = S^o$

Prop 2.3

(a) $S \subseteq T \implies S^o \subseteq T^o$

$$\because p \in S^o \implies \exists\ r > 0 \ni B(p, r) \subseteq S \subseteq T \implies p \in T^o$$

(b) Every open ball $B(p, r)$ in X is open
$\because$ Give $q \in B(p, r)$, Let $\delta = r = d(p, q)$.Claim $B(q, \delta) \subseteq B(p, r)$ which says q is an interior point of $B(p, r)$.Since $q \in B(p, r)$ is arbitrary, so $B(p, r)$ is open. Given $x \in B(q, \delta)$

$$d(x, p) \leq d(x, q) + d(q, p) < \delta + d(q, p) = r - d(p, q) + d(q, p) = r$$

(c) $\forall\ S \subseteq X,\ S^o$ is always open
$\because$ Given $p \in S^o,\ \exists\ r > 0 \ni B(p, r) \subseteq S$

$$\implies B(p, r) \subseteq S^o \implies p \text{ is a interior point of } S^o$$

$\therefore S^o$ is open
(d) $\forall\ S \subseteq X,\ S^{oo} = (S^o)^o = S^o$
$\because$ by definition of open set and (c)

Now, let $T = \{U \subseteq X \mid U \text{ is open in } X\}$
Prop 2.4 T is a topology on X. In particular, X is a topology space.

Proof.
(i) $\emptyset, X \in \mathscr{T}, \because \emptyset = \emptyset,\ X^o = X$
(ii) $U_\alpha \in \mathscr{T}, a \in I$ are open $\implies \bigcup_{\alpha \in I} U_\alpha$ is open
Given an arbitrary point $p \in \bigcup_{\alpha \in I} U_\alpha \implies \exists \alpha_0 \in I \ni p \in U_{\alpha_0}$
U_{α_0} is open, $\exists r > 0 \ni B(p, r) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in I} U_\alpha$
$\because p$ is an interior point of $\bigcup_{\alpha \in I} U_\alpha$ $\therefore \bigcup_{\alpha \in I} U_\alpha$ is open, i.e. $\bigcup_{\alpha \in I} U_\alpha \in T$

(iii) $U_1, \cdots, U_n \in T \implies U_1 \cap \cdots \cap U_n \in T$
$\because$ Given $p \in U_1 \cap \cdots \cap U_n \implies p \in U_i$ $1 \leq i \leq n$. Each U_i is open, $\exists r_i > 0$ $B(p, r_i) \subseteq U_i,\ 1 \leq i \leq n \implies B(p, r) \subseteq U_1 \cap \cdots \cap U_n$
$\implies$ p is an interior point of $U_1 \cap \cdots \cap U_n$
p is arbitrary, so $U_1 \cap \cdots \cap U_n$ is open, i.e. $U_1 \cap \cdots \cap U_n \in T$
Therefore, T is a topology on X

■

Definition. *Let X be a metric space with metric d. The topology T in prop 2.4 is called the metric topology(include by d)*

Let X be a metric space and $Y \subseteq X$, Then Y is a metric subspace of X, and $\forall y \in Y,\ r > 0,\ B_Y(y, r) = B(y, r) \cap Y$. In fact, we have more

Prop 2.5 A subset $A \subseteq Y$ is open in $Y \Leftrightarrow A = U \cap Y$ for some open set U in X, in particular, the metric topology on T is just the relation topology of Y on X

Proof. ($\Rightarrow$) suppose $A \subseteq Y$ is open in Y. Then

$$A = \bigcup_{y \in A} B_Y(y, r_y) = \bigcup_{y \in A} (B(y, r_y) \cap Y) = (\bigcup_{y \in A} B(y, r_y)) \cap Y$$

Let $U = \bigcup_{y \in Y} B(y, r_y)$, then U is open in X and $A = U \cap Y$
($\Leftarrow$) Suppose $A = U \cap Y$ where $U \subseteq X$ is open $\forall\ y \in A, y \in U \cap Y \implies y \in U \implies \exists r > 0 \ni B(y, r) \subseteq U \implies B(y, r) \cap Y \subseteq U \cap Y = A \implies B_Y(y, r) \subseteq A,\ \therefore A$ is open in Y. ■

Prop 2.6 Every metric space X is Hausdorff

Proof. Given $p, q \in X,\ p \neq q$. Choose $r = \dfrac{1}{2}d(p, q) > 0$. Then $B(p, r) \cap B(q, r) = \emptyset$. So X is Hausdorff
($\because$ $x \in B(p, r) \cap B(q, r) = d(x, p) < r$ & $d(x, q) < r \implies d(p, q) \leq d(p, x) + d(x, q) < r + r = 2r = d(p, q) (\rightarrow\leftarrow)$) ■

Remark. *Let $S \subseteq X$, where X is a metric space. Then S^o is the largest(w.r.t inclusion) open set contained in S. $\because \forall$ open set $U \subseteq S$, $U^o \subseteq S^o \implies U \subseteq S^o \subseteq S$. In fact, $S^\circ = \bigcup_{U \subseteq S} U$ (which is the definition of intension of S in a topology space X)*

2.4. Closed Sets.

Definition (Closed set). *$F \subseteq X$ is closed $\Leftrightarrow F^C = X - F$ is open in X*

By Theorem 2.1, the collection of all close sets in X has the properties

(i) $\emptyset, X$ are closed in X

(ii) F_α is closed in X, $\alpha \in I \implies \bigcap_{\alpha\in I} F_\alpha$ is closed in X

(iii) $F_1, \cdots, F_n$ are closed in $X \implies \bigcup_{i=1} F_i$ is closed in X

Example

Intersection of infinitely many open set may not be open in $\mathbb{R}$ with Euclidean topology, $(-\frac{1}{n}, \frac{1}{n})$ is open in $\mathbb{R}$ $\forall\ n \geq 1 \implies \bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$ is not open

Prop 2.8 Let X be a metric space and $Y \subseteq X$ and $B \subseteq Y$, Then B is closed in $Y \Leftrightarrow B = F \cap Y$ for some closed set F in X

Proof. ($\Rightarrow$) Suppose B is close in $Y \implies Y - B$ is open in $Y \implies Y - B = U \cap Y$(by prop 2.5) for some open set U in $X \implies Y - (Y - B) = Y - (U \cap Y) \implies B = (X - U) \cap Y$. where $(X - U)$ is close.
($\Leftarrow$) Suppose $B = F \cap Y$, where F is closed in $X \implies Y - B = Y - (F \cap Y) = (X - F) \cap Y \implies Y - B$ is open in $Y \implies B$ is close in Y. ■

In metic space, one can use sequence to detect the closeness of a set

Example

1. We know that $[a, b)$ is not closed in $\mathbb{R}$, however, $\exists$ a sequence $\{x_n\}$ in $[a, b) \ni x_n \to b$ on $n \to \infty$, e.g. $b - \frac{1}{n} \to b$

2. $A = \{\frac{1}{n} \mid n \geq 1\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ is not close in $\mathbb{R}$

if $\mathbb{R} - A$ will open, them $\exists r > 0,\ B(0, r) \subseteq \mathbb{R} - A (\rightarrow\leftarrow)$

$A \cup \{0\}$ is closed in $\mathbb{R}$

$$R \setminus (A \cup \{0\}) = (-\infty, 0) \cup (1, \infty) \cup (\bigcup_{n=1}^{\infty}(\frac{1}{n+1}, \frac{1}{n})) \text{ is open}$$

$\therefore A \cup \{0\}$ is closed

<u>**Definition**</u> (Adherent, clousure $\cdots$)**.** *Let X be a metric space with metric d, $T \subseteq X$ be a subset.(**important**)*

(1.) *A point $p \in X$ is said to be an adherent point of T if $\forall r > 0,\ B(p, r) \cap T \neq \emptyset$, equivalent, $\forall$ neighborhood U of p, $U \cap T \neq \emptyset$*

(2.) *Let $\overline{T}$ or $cl(T)$ denote the set of all adherent points of T, called the closure of T, i.e. $\overline{T} = \{p \in X \mid p \text{ is an adherent point of } T\}$*

(3.) *A point $p \in X$ is said to be a limit point or accumulation point of T if $\forall\ r > 0,\ B(p,r) \cap T - \{p\} \neq \emptyset$, equivalently, $\forall$ neighborhood U of p, $U \cap T - \{p\} \neq \emptyset$*
Denote by T' the set of all accumulation points of T, called the devied set of T.
(4.) *$p \in T$ and $p \notin T'$, then p is called an isolated point of T, i.e. $\exists r > 0 \ni B(p,r) \cap T = \{p\}$*
(5.) *A subset $T \subseteq X$ is said to be perfect if T is closed and every points of T is an accumulated point of T, i.e. T is closed & $T' = T$*
(6.) *A subset $T \subseteq X$ is said to be bounded if $\exists R > 0$ and $p \in X \ni T \subseteq B(p,R)$*
(7.) *A subset $T \subseteq X$ is said to be dense if $\overline{T} = X$, e.g. $\overline{\mathbb{Q}} = \mathbb{R}$*
(8.) *A point $p \in X$ is said to be a boundary point of T if $\forall\ r > 0, B(p,r) \cap T \neq \emptyset$ & $B(p,r) \cap (X \setminus T) \neq \emptyset$. Denote by ∂T or bd(T) the set of all boundary points of T*

Prop 2.9 Let X be a metric space. All sets and point below are subset of X
(1) $S \subseteq T \implies \overline{S} \subseteq \overline{T}$ & $S' \subseteq T'$
$\because p \in \overline{S} \implies \forall\ r > 0, B(p,r) \cap S \neq \emptyset \implies B(p,r) \cap T \neq \emptyset \implies p \in \overline{T}$
$p \in S' \implies \forall r > 0, B(p,r) \cap S - \{p\} \neq \emptyset \implies B(p,r) \cap T - \{p\} \neq \emptyset$
(2) $\overline{T}$ is always closed in X

> We want to know $\overline{T}$ is closed on $X \to X - \overline{T}$ is open $\to \forall\ p \in X - \overline{T}$ is an interior point $\implies \exists r > 0,\ B(p,r) \subseteq X - \overline{T}$
> $\because p \notin \overline{T} \Rightarrow \exists r' > 0 \ni B(p,r') \cap T = \emptyset$
> But we want to get $B(p,r') \cap \overline{T}$, so we check every point in $B(p,r')$ is not in $\overline{T}$, let $q \in B(p,r'),\ \exists\ \delta > 0\ , B(q,\delta) \subseteq B(p,r') \implies B(q,\delta) \cap T = \emptyset \implies q \notin \overline{T}$
> because if $q \in \overline{T}, \forall r > 0 \ni B(q,r) \cap T \neq \emptyset$
> $\implies B(p,r) \cap \overline{T} = \emptyset$

Let $p \in X - \overline{T} \implies p \notin \overline{T} \implies \exists\ r > 0 \ni B(p,r) \cap T = \emptyset \implies B(p,r) \cap \overline{T} = \emptyset$ ($\because \forall q \in B(p,r),\ \exists\ \delta > 0 \ni B(q,\delta) \subseteq B(p,r) \implies B(q,\delta) \cap T = \emptyset \implies q \notin \overline{T}$)
$\therefore B(p,r) \subseteq X - \overline{T}$, $\because p$ is an interior point of $X - \overline{T}$. Hence, $X - \overline{T}$ is open, i.e. $\overline{T}$ is closed.
(3) $T \subseteq \overline{T}(\because \forall\ p \in T, \forall\ r > 0, B(p,r) \cap T \neq \emptyset)$

(4) $p \in T' \implies \forall r > 0,\ B(p,r) \cap T - \{p\}$ is an infinite set, say $x_1, \cdots, x_n$, Let $\delta = \dfrac{1}{2}\min\{d(p,x_i) \mid 1 \leq i \leq n\}$. Then $B(p,\delta) \cap T - \{p\} = \emptyset (\to\leftarrow)$ to $p \in T'$, $x \in B(p,\delta) \cap T - \{p\} \implies d(x,p) < \delta \implies x = x_i$ for some $1 \leq i \leq n \&$ we get $d(x_i,p) < \delta \leq \dfrac{1}{2}d(x_i,p)$
$\therefore$ no such x i.e. $B(p,\delta) \cap T - \{p\} = \emptyset$
(5) Any finite subset of X has no accumulation points in X by (4). In particular, it is closed by (6)(c) below.
(6) TFAE
(a) S is closed
(b) S contains all it's adherent point, i.e. $\overline{S} \subseteq S$
(c) S contains all it's accumulation points, i.e. $S' \subseteq S$
(d) $S = \overline{S}$
(7) $\overline{\overline{S}} = \overline{S}$ by (2) and (6)

Proof. of (6)
(a) $\Rightarrow$ (b) Suppose S is closed $\implies X \backslash S$ is open $\implies \forall\, p \in X - S \implies \exists r > 0 \ni B(p,r) \subseteq X \setminus S \implies B(p,r) \cap S = \emptyset \implies p \notin \overline{S}$
$\therefore \overline{S} \subseteq S$, i.e. (b) holds

(b) $\Rightarrow$ (c) $\because S' \subseteq \overline{S}$

(c) $\Rightarrow$ (d) Suppose $S' \subseteq S$. To prove $S = \overline{S}$ if not, then $S \subsetneqq \overline{S}$, i.e. $\exists\, p \in \overline{S}\ \&\ p \notin S \implies \forall r > 0,\ B(p,r) \cap S \neq \emptyset\ (\because p \in \overline{S})$
(d) $\Rightarrow$ (a) by (2)

■

(8) $\overline{S}$ is the smallest closed set in X containing S
$\because$ We know that $S \subseteq \overline{S}$, if F is closed in X & $F \subseteq S$, then $\overline{F} \subseteq \overline{S}$ by (1), $F = \overline{F} \subseteq \overline{S}$ by (6), $\therefore \overline{S}$ is the smallest such one.
(9) In fact, $\overline{S} = \bigcap\limits_{F \subseteq S} F$
(10) $p \in S$ is an isolated point $\Leftrightarrow \exists\, r > 0 \ni B(p,r) \cap S = \{p\}$
($\Rightarrow$) Suppose $p \in S$ is an isolated point of S. Then $p \in S' \implies \exists\, r > 0 \ni B(p,r) \cap S - \{p\} = \emptyset \implies B(p,r) \cap S = \{p\}$
($\Leftarrow$) Trivial
(11) S is dense in $X \Leftrightarrow\ \forall\, p \in X\ \&\ r > 0,\ B(p,r) \cap S \neq \emptyset \Leftrightarrow \forall$ open set $U \neq \emptyset,\ U \cap S \neq \emptyset$

Proof. ($\Rightarrow$) Suppose S is dense in X, i.e. $\overline{S} = X$, So $\forall p \in X, p \in \overline{S} \implies \forall\, r > 0,\ B(p,r) \cap S \neq \emptyset$
($\Leftarrow$) Suppose the condition holds, $\forall p \in X\ \&\ r > 0,\ B(p,r) \cap S \neq \emptyset \implies p \in \overline{S} \implies X \subseteq \overline{S} \subseteq X,\ \therefore \overline{S} = X$ ■

(12) $\partial S = \partial(X-S)$ In particular, $\partial S = \overline{S} \cap \overline{(X-S)}$, In particular, ∂S is closed in X, $\because$ It suffices to prove $\partial S = \overline{S} \cap \overline{(X-S)}$,
$\because \partial(X-S) = \overline{X-S} \cap \overline{X-(X-S)} = \overline{X-S} \cap \overline{S} = \partial S$
$\forall p \in \partial S \implies \forall r > 0, B(p,r) \cap S \neq \emptyset \;\&\; p \in \overline{X-S} \implies p \in \overline{S} \cap \overline{X-S}$
$\therefore \partial S \subseteq \overline{S} \cap \overline{X-S}$,
Conversely, $p \in \overline{S} \cap \overline{X-S} \implies p \in \overline{S} \;\&\; p \in \overline{X-S} \implies \forall r > 0,\ B(p,r) \cap S \neq \emptyset \;\&\; B(p,r) \cap (X-S) \neq \emptyset \implies p \in \partial S$
$\therefore \overline{S} \cap \overline{(X-S)} \subseteq \partial S \therefore \partial S = \overline{S} \cap \overline{(X-S)}$

2.5. Examples.
We give some simple examples of open sets, closed sets, adherent, accumulation, isolated and boundary points.

1. In a discrete metric space X, every subset of X is both open and close, $\forall x \in X,\ B(p,r) \begin{cases} \{x\} \text{ if } 0 < r \leq 1 \\ X \text{ if } r > 1 \end{cases}$
$\therefore$ Every singleton is open in X, so every subset of X is open.
2. In $\mathbb{R}$. Consider the set $S = [0,1) \cup \{3\}$, $S^\circ = \emptyset,\ S' = \{0\}$,
$\overline{S} = S \cup \{0\}$
3. In $\mathbb{R}$, consider the set $S = \{\frac{1}{n} \mid n = 1, 2, \cdots\}$, $S^\circ = \emptyset$,
$S' = \{0\},\ \overline{S} = S \cup \{0\}$
4. In $\mathbb{R}^2$, consider $S = \{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$, S is open
$\overline{S} = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$
$\partial S = \{(x,0) \mid x \geq 0\} \cup \{(y,0) \mid y \geq 0\}$
5. Let $B(0,1)$ be the unit open ball in $\mathbb{R}^k$. Then $\partial B(0,1) = S^{k-1}$ is the unit $(k-1)$-sphere. In particular, for $k = 2, \partial B(0,1) = S^1$ in the unity circle in the plane $\mathbb{R}^2$. Similarly, for the closed unit ball $\overline{B}(0,1)$ in $\mathbb{R}^k$. Now, we define some special sets in $\mathbb{R}^n$

- Internals in $\mathbb{R}:\ -\infty < a \leq b < \infty$
  $[a,b]$ close interval which is closed in $\mathbb{R}$
  (a,b) open interval which is closed in $\mathbb{R}$
  Infinite intervals:
  $(-\infty, b]$: close in $\mathbb{R}$, $(-\infty, b)$ open in $\mathbb{R}$
- k-dimensional interval (rectangle or k-cell) I

$$I = I_1 \times \cdots \times I_k$$

  where I_j is an interval in $\mathbb{R},\ 1 \leq j \leq k$
  (i) I is bounded $\Leftrightarrow$ each I_j is bounded
  I is unbounded $\Leftrightarrow I_j \neq \emptyset \;\&\;$ some I_j is unbounded
  (ii) $I = [a_1, b_1] \times \cdots [a_k, b_k],\ -\infty < a_j \leq b_j < \infty,\ 1 \leq j \leq k$
  k-dimensional closed(compact) interval in $\mathbb{R}^k$

- Convex sets in $\mathbb{R}^k$
  $S \subseteq \mathbb{R}^k$ is convex if $\forall\, x, y \in S,\ \overline{xy}$ is the line segment joining x & y
  Note that all open balls, closed balls, intervals are convex in $\mathbb{R}^k$
- Star-like sets in $\mathbb{R}^k$ with w.r.t some point x_0, $S \subseteq \mathbb{R}^k$ is star-like w.r.t. $x_0 \in S$ if $\forall\, x \in S,\ \overline{xx_0} \subseteq S$

6. We know that $\mathbb{Q}$ is dense in $\mathbb{R}$, hence $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$. Note that $\mathbb{Q}^k$ is countable, hence $\mathbb{R}^k$ has a countable dense subset $\mathbb{Q}^k$, i.e. $\mathbb{R}^k$ is separable.

7. $\partial\mathbb{Q} = \mathbb{R},\ \partial\mathbb{Q}^k = \mathbb{R}^k$

8. $\mathbb{Z}$ is closed in $\mathbb{R}$, $\because \mathbb{R} - \mathbb{Z} = \displaystyle\bigcup_{n=-\infty}^{\infty} (n-1, n)$ is open $\implies \mathbb{Z}$ is close.
or $\mathbb{Z}' = \emptyset \subseteq \mathbb{Z},\ \therefore \mathbb{Z}$ is close.

9. Let $S \subseteq \mathbb{R}$ be a nonempty set which is bounded above. Then $\alpha = \sup S$ exists. Moreover, $\alpha \in \overline{S}$. $\because \forall\, r > 0, \exists\, x_0 \in S \ni \alpha - r < x_0 \leq \alpha < \alpha - r \implies (\alpha - r, \alpha + r) \cap S \neq \emptyset \implies \alpha \in \overline{S}$

2.6. Compact Set in Metric Space.

- Compact sets in metric space, which is closely related to the extreme value problem.
- Compact set $\mathbb{R}^k$ will be discussed in next section.

Definition. *Let X be a topology space and $S \subseteq X$. A collection $\mathscr{U} = \{U_\alpha\}_{\alpha \in I}$ of open sets in X is called an open covering of S if*

$$S \subseteq \bigcup \mathscr{U} = \bigcup_{\alpha \in I} U_\alpha$$

Definition. *Let X be a topology space, $S \subseteq X$ and $\mathscr{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of S. We say that $\mathscr{U}$ has a countable(finite) sub covering of S if $\exists$ a countable(finite) sub collection of $\mathscr{U}$ which also covers S. i.e. $\mathscr{U}$ has a countable(finite) subcovering in S if*

$\exists$ *a sequence $\{\alpha_n\}$ in $I \ni S \subseteq \displaystyle\bigcup_{n=1}^{\infty} U_{\alpha_n}$ (countable)*

$\exists$ *a sequence $\{\alpha_n\}$ in $I \ni S \subseteq U_\alpha \cup \cdots \cup U_{\alpha_n}$ (finite)*

Example

(1) X is discrete metric space. Then $\{\{x\} \mid x \in X\}$ is an open covering of X

(2) In $\mathbb{R}$, $\{(0, 1 - \frac{1}{n}) \mid n \in \mathbb{N}\}$ is an open covering of $(0, 1)$. In fact,

$$(0,1) = \bigcup_{n=1}^{\infty} (0, 1 - \frac{1}{n})$$

(3) $\{B(0,n) \mid n \in \mathbb{N}\}$ is an open covering of $\mathbb{R}^k$

Definition (compact)**.** *Let X be a topology space. A subset $K \subseteq X$ is said to be compact if* ***every*** *open covering of K admit a finite subcovering*

Examples

(1) Let X be a topology space and $K \subseteq X$ be a finite set. Then K is compact.
(2) In a discrete metric space X, a subset $K \subseteq X$ is compact $\Leftrightarrow$ K is a finite set.
(3) $(0,1)$ is not compact in $\mathbb{R}(\{0, 1-\dfrac{1}{n} \mid n \in \mathbb{N}\})$, but $[0,1]$ is compact

Theroem 2.10. *Let X be a metric space and $K \subseteq Y \subseteq X$. Then K is compact in $X \Leftrightarrow K$ is compact in Y.*

Proof. ($\Rightarrow$) Suppose K is compact in X. Given an open covering $\{V_\alpha\}_{\alpha \in I}$ of open sets in Y which covers K. By Prop 2.5, each $V_\alpha = U_\alpha \cap Y$, where U_α is open in X. Now,

$$K \subseteq \bigcup_{\alpha \in I} V_\alpha = \bigcup_{\alpha \in I}(U_\alpha \cap Y) = (\bigcup_{\alpha \in I} U_\alpha) \cap Y \implies K \subseteq \bigcup_{\alpha \in I} U_\alpha$$

By the compactness of K in X, $\exists \alpha_1, \cdots, \alpha_n \in I \ni K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} \implies$ $K \cap Y \subseteq (\bigcup_{i=1}^{n} U_{\alpha_i}) \implies K \subseteq \bigcup_{i=1}^{n}(U_{\alpha_i} \cap Y) = \bigcup_{i=1}^{n} V_{\alpha_i}$
$\therefore K$ is compact in Y
($\Leftarrow$)Suppose K is compact in Y. Given a open covering $\{U_\alpha\}_{\alpha \in I}$ of K by open sets in X.

$$K \subseteq \bigcup_{\alpha \in I} U_\alpha \implies K \cap Y \subseteq (\bigcup_{\alpha \in I} U_\alpha) \cap Y \implies K \cap Y \subseteq \bigcup_{\alpha \in I}(U_\alpha \cap Y)$$

By Prop 2.5, $\{U_\alpha \cap Y \mid \alpha \in I\}$ is an open covering of K by open set in Y. By assumption, K is compact in $Y, \exists\ \alpha_1, \cdots, \alpha_n \in I \ni K \subseteq$ $\bigcup_{i=1}^{n}(U_{\alpha_i} \cap Y) = (\bigcup_{i=1}^{n} U_{\alpha_i}) \cap Y \implies K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$
$\therefore K$ is compact in X ■

Definition. *Let X be a metric space and $S \subseteq X$ be a nonempty set. The diameter of S is defined to be dia$(S) = \sup\{d(x,y) \mid x, y \in S\}$ which generated the diameter of a circle in $\mathbb{R}^2$*

Theroem 2.11. *Let X be a metric space and $K \subseteq X$ be a compact set. Then K is closed and bounded*

Proof. K **is bounded**

Fix a point $p \in K$. Then $K \subseteq \bigcup_{n=1}^{\infty} B(p, n)$. $\because K$ is compact $\implies \exists N \in \mathbb{N} \ni K \subseteq B(p, 1) \cup \cdots \cup B(p, N) \implies K \subseteq B(p, N)$ $\therefore K$ is bounded

K is closed, i.e. $X - K$ is open

Fix $p \in X - K$. Then $p \neq x, \ \forall \ x \in K$. Hence, $d(x, p) > 0, \ \forall \ x \in K$

Let $r_x = \dfrac{1}{2} d(x, p) > 0, x \in K$. Them $\{B(x, r_x) \mid x \in K\}$ is an open covering of K. $\because K$ is compact $\implies \exists x_1, \cdots, x_n \in K \ni B(x_1, r_{x_1}) \cup \cdots \cup B(x_n, r_{x_i})$. Let $V = \bigcap_{i=1}^{n} B(p, r_{x_i}) = B(p, r)$, where $r = \min\{r_{x_1}, \cdots, r_{x_n}\}$. Then as we can see that $V \subseteq X - K$, all point in $X - K$ are inner point. So $X - K$ is open, i.e. K is close.

To show that $V \subseteq X - K$, i.e. $V \cap K \neq \emptyset$, it suffices to show

$$V \cap (\bigcup_{i=1}^{n} B(x_i, r_{x_i})) = \emptyset$$

Now,

$$\begin{aligned} V \cap (\bigcup_{i=1}^{n} B(x_i, r_{x_i})) &= \bigcup_{i=1}^{n} (V \cap B(r_i, r_{x_i})) \\ &\subseteq \bigcup_{i=1}^{n} (B(p, r_{x_i} \cap B(x_i, r_{x_i}))) = \emptyset \end{aligned}$$

■

Remark. *The converse of Thm 2.11 is false, i.e. closed & bounded may not be compact, e.g. X is an infinite set with discrete metric. Then X is not compact, but X is closed and bounded.*

Theroem 2.12. *Let X be a metric space, $K \subseteq X$ be compact & $L \subseteq K$ be a closed set in X. Then L is compact.*

Proof. Let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of L. Then $\{U_\alpha\}_{\alpha \in I} \cup \{X - L\}$ is an open covering of K. By the compactness of K, $\exists \alpha_1, \cdots, \alpha_n \in I \ni K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \cup (X - L)$. By $L \subseteq K$ $\therefore L$ is compact ■

Corollary 2.13.

(a) Let X be a metric space, $K \subseteq X$ be compact and F be a closed set in X. Then $K \cap F$ is compact.
(b) If X is a compact metric space, then every closed subset F of X is compact.

Proof.

(a)

$$\begin{aligned} K \text{ is compact} \quad &\implies \quad K \text{ is closed (Thm 2.11)} \\ &\implies \quad K \cap F \text{ is closed in } X \\ &\implies \quad K \cap F \text{ is compact} \end{aligned}$$

(b) follows (a)

■

Remark. *Let X be a metric space. If K is closed in X and F is closed in K, then F is closed in X. $\because$ F is closed in F $\implies$ $F = L \cap F$, where L is closed in K $\implies$ $F = L \cap K$, where L is closed in X $\implies$ F is closed in X.*

Theroem 2.14. *Let X be a metric space, $\{K_\alpha\}_{\alpha \in I}$ be a collection of compact subsets of X with the property:*

$$\forall \alpha_1, \cdots, \alpha_n \in I, K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \neq \emptyset$$

Them $\displaystyle\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$

Proof. Fix $\alpha_0 \in I$. Assume that $\displaystyle\bigcap_{\alpha \in I} K_\alpha = \emptyset \implies X - \bigcap_{\alpha \in I} K_\alpha$
$\displaystyle\implies X - \emptyset = X \implies X = \bigcup_{\alpha \in I} (X - K_\alpha)$
each K_α is compact $\implies$ K_α is closed $\implies$ $X - K_\alpha$ is open
so $\{X - K_\alpha\}$ is an open covering of X. Now,

$$K_{\alpha_0} \subseteq X = \bigcup_{\alpha \in I} (X - K_\alpha) \implies K_{\alpha_0} \subseteq \bigcup_{\alpha \in I} (X - K_\alpha)$$

K_α is compact $\implies$ $\exists \alpha_1, \cdots, \alpha_n \in I - \{\alpha_0\} \ni K_{\alpha_0} \subseteq (X - K_{\alpha_1}) \cup \cdots \cup (X - K_{\alpha_n}) \implies K_{\alpha_0} \cap K_{\alpha_1} \cap \cdots \cap K_n = \emptyset (\rightarrow\leftarrow)$ ■

Corollary 2.15. *Let X be a metric space and $\{K_n\}_{n=1}^{\infty}$ be a decrease sequence of nonempty compact sets of X. Them $\bigcap\limits_{n=1}^{\infty} \neq \emptyset$. In addition, if $dia_{n\to\infty}\infty 0$, them $\bigcap\limits_{n=1}^{\infty} K_n$ is a singleton.*

Proof. $\forall j1, \cdots, j_k \in \mathbb{N},\ K_{j1}\cap\cdots\cap K_{jk} \neq \emptyset,\ K_{j1}\cap\cdots\cap K_{jk} = K_t$, where $t = \max\{j_1, \cdots, j_k\}$. By Thm 2.14 $\bigcap\limits_{n=1}^{\infty} K_n \neq \emptyset$, if $\lim\limits_{n\to\infty} \mathrm{dia}(K_n) = 0$ and $p, q \in \bigcap\limits_{n=1}^{\infty} K$ and $p \neq q$, them $\mathrm{dia}(K_n) \geq d(p,q)\ \forall n \geq 1 \implies \lim\limits_{n\to\infty} \mathrm{dia}(K_n) \geq d(p,q) > 0 (\rightarrow\leftarrow) \because \bigcap\limits_{n=1}^{\infty} K_n = \{p\}$ is a simpleton. ∎

Remark. *The usual form of Cor 2.15, X is a metric space, $\{K_n\}$ is a decrease sequence of nonempty closed sets in X with K_i is compact $\implies \bigcap\limits_{n=1}^{\infty} K_n \neq \emptyset$*

Example In $\mathbb{R}$, $\{(0, \frac{1}{n} \mid n \geq 1]\}$ is decrease and every finite subcollection of $\{(0, \frac{1}{n}) \mid n \geq 1\}$ is nonempty, but $\bigcap\limits_{n=1}^{\infty}(0, \frac{1}{n}) = \emptyset$, $\bigcap [0, \frac{1}{n}] = \emptyset$

Theroem 2.17. *Let X be a metric space and $K \subseteq X$, TFAE:*
(i) K is compact
(ii) Every infinite subset has an accumulation point in K
(iii) K is sequentially compact
(iv) K is complete and totally bounded

Definition (Convergence)**.** *$\{a_n\}$ converge if $\exists\ a \in X \ni \forall\ \epsilon \geq 0 \exists N \ni \mathbb{N} \ni \forall n \geq N,\ d(a_n, a) < \epsilon$. Such a is called the limit of $\{a_n\}$, which is denoted by $\lim\limits_{n\to\infty} a_n = a$ or $a_n \to a$ on $n \to \infty$.*

Definition (Cauchy)**.** *We say that $\{a_n\}$ is Cauchy if $\forall \epsilon > 0,\ \exists N \in \mathbb{N} \ni \forall n, m \geq \mathbb{N},\ d(a_n, a_m) < \epsilon$*

Definition. *A metric X is said to be sequence compact if every sequence has a convergent subsequence*

Definition. *A metric space X is said to be complete if every Cauchy sequence in X convergence.*

__**Definition.**__ *Let X be a metric space & $K \subseteq X$. We say that K is totally bounded if $\forall r > 0, \exists x_1, \cdots, x_n \in K \ni K \subseteq B(x_1, r) \cup \cdots \cup B(x_n, r)$*

Remark. *Totally bounded can implies bounded, but not converse.*
K is totally bounded, for $r = 1, \exists x_1, \cdots, x_n \in K \subseteq B(x_1, 1) \cup \cdots \cup B(x_n, 1) \implies K \subseteq B(x_1, R)$ for sime large R

- *Take an* ***"infinite"*** *set X with discrete metric. Then X is bounded(e.g. $X \subseteq B(x_0, 2)$, where $x_0 \in X$) but for $r = \frac{1}{2}$, $X \subsetneqq B(x_1, \frac{1}{2}) \cup \cdots \cup B(x_n, \frac{1}{2})$ $\forall x_1, \cdots, x_n$*

Proof. (Thm 2.17 (i)(ii))
$(i) \Rightarrow (ii)$ Suppose K is compact. Given an infinite set $T \subseteq K$. We must prove that T has an accumulation point in K, if not, $\forall x \in K, x$ is not an accumulation point of $K, \exists r_x > 0 \ni B(x, r_n) \cap T - \{x\} = \emptyset \implies B(x, r_x) \cap T \subseteq \{x\}$. Clearly $\{B(x, r_x \mid x \in K)\}$ is an open covering of K. By (i), K is compact

$$\begin{aligned} \implies \exists x_1, \cdots, x_n \in K \ni K &\subseteq B(x_1, r_{x_1}) \cup \cdots \cup B(x_n, r_{x_n}) \\ &= (T \cap B(x_1, r_{x_1})) \cup \cdots \cup (T \cap B(x_n, r_{x_n})) \\ &\subseteq \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_n\} \\ &= \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_n\} \\ &= \{x_1, \cdots, x_n\} (\rightarrow\leftarrow) \end{aligned}$$

to T is an infinite set, $\therefore$ (ii) holds. ∎