高等微積分

Exercise 1 (Chapter 1)

1. Let a and b be two real numbers. If $a \leq b + \epsilon$ for any $\epsilon > 0$, then $a \leq b$.

2. (a) If $r, s \in \mathbb{Q}$, then $r + s$ and rs are rational.
   (b) If $r \in \mathbb{Q}$ with $r \neq 0$ and $x \in \mathbb{R} \setminus \mathbb{Q}$, then $r + x$ and rx are irrational.

3. Let $f : X \to Y$ be a function. If $B \subseteq Y$, we denote by $f^{-1}(B)$ the largest subset of X which f maps into B. That is,
$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$
The set $f^{-1}(B)$ is called the **inverse image** of B under f. Prove the following for arbitrary $A, A_1, A_2 \subseteq X$ and $B, B_1, B_2 \subseteq Y$.
   (a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
   (b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. Given an example such that the inclusion is strict.
   (c) $A \subseteq f^{-1}[f(A)]$ and $f[f^{-1}(B)] \subseteq B$. Given an example such that the inclusion is strict.
   (d) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$ and $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

4. Let E be a nonempty subset of an order set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

5. Let A, B be two nonempty sets of $\mathbb{R}$.
   (a) If $A \subseteq B$,then $\sup A \leq \sup B$ and $\inf A \geq \inf B$.
   (b) Show that $\inf A \leq \sup A$.
   (c) Show that $\inf(-A) = -\sup A$ and $\sup(-A) = -\inf(A)$, where $-A = \{-a \mid a \in A\}$.
   (d) Show that $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$,
   where $A + B = \{a + b \mid a \in A, b \in B\}$, the **Minkowski sum** of A and B.
   (e) If A, B be two sets of positive numbers which is bounded above.
   Let $a = \sup A$, $b = \sup B$ and $C = \{ab \mid a \in A, b \in B\}$. Prove that $\sup C = ab$.

6. Prove or disprove the following statement by given a counterexample:
   (a) $\sup(A \cap B) \leq \inf\{\sup A, \sup B\}$.
   (b) $\sup(A \cap B) = \inf\{\sup A, \sup B\}$.
   (c) $\sup(A \cap B) \geq \sup\{\sup A, \sup B\}$.
   (d) $\sup(A \cap B) = \sup\{\sup A, \sup B\}$.

7. Let $A, B \subseteq \mathbb{R}$ such that $\sup A = \sup B$ and $\inf A = \inf B$. Does $A = B$?

8. Fix $b > 1$.
   (a) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that
$$(b^m)^{1/n} = (b^p)^{1/q}.$$
   Hence, it makes sense to define $b^r = (b^m)^{1/n}$.
   (b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.
   (c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that
$$b^r = \sup B(r)$$
   when r is rational. Hence, it make sense to define
$$b^x = \sup B(x)$$
   for every real x.

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

9. Prove that no order can be defined in the complex field that turns it into an ordered field.

10. Suppose $z = a + ib,\ w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons) Does this ordered set have the least-upper-bound property ?

11. Suppose $z = a + bi,\ w = u + iv$ and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2}, \quad b = \left(\frac{|w| - u}{2}\right)^{1/2}$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\overline{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

12. Under what conditions does equality hold in the Cauchy-Schwartz inequality ?