1. Introduction to spectral theory

1.1. Main definitions.

Introduce eigenvalue & eigenvector first, and the method to find these thing.

Definition. *A scalar λ is called an eigenvalue of an operator $A : V \to V$ if there exists a non-zero vector $v \in V$ such that*

$$Av = \lambda v$$

The vector v is called the eigenvector of A

Theorem (From hamberger Thm 5.2)**.** *Let $A \in M_{n\times n}(F)$. Then a scalar λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.*

Proof. A scalar λ is an eigenvalue of A if and only if there exists a nonzero vector $v \in F^n$ such that $Av = \lambda v$, that is, $(A-\lambda I_n)(v) = 0$. By Theorem 2.5, this is true **if and only if** $A-\lambda I_n$ is not invertible. However, this result is equivalent to the statement that $\det(A - \lambda I_n) = 0$ ∎

Definition. *Let $A \in M_{n\times n}(\mathrm{F})$. The polynomial $f(t) = \det(A-tI_n)$ is called the characteristic polynomial of A*

Theorem (From hamberger Thm 5.4)**.** *Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. A vector $v \in V$ is an eigenvector of T corresponding to λ if and only if $v \neq 0$ and $v \in N(T - \lambda I)$.*

Definition. *The nullspace $N(A - \lambda I)$, i.e. the set of all eigenvectors and 0 vector, is called the eigenspace. The set of all eigenvalues of an operator A is called spectrum of A, and is usually denoted $\sigma(A)$.*

Remark.
If the matrix A is ugly, what should we do?
we can use the similar matrices
A and B are called similar if there exists an invertible matrix S such that

$$A = SBS^{-1}$$

The determinants of similar matrix is same

$$\det(A) = \det(SBS^{-1}) = \det(S)\det(B)\det(S^{-1}) = \det(B)$$

We can find $A - \lambda I$ and $B - \lambda I$ is similar

$$A - \lambda I = SBS^{-1} - \lambda SIS^{-1} = S(BS^{-1} - \lambda IS^{-1}) = S(B - \lambda I)S^{-1}$$

It same in transform
If $T : V \to V$ is a linear transform, α, β are two bases in V, then

$$[T]_{\alpha}^{\alpha} = [I]_{\beta}^{\alpha}[T]_{\beta}^{\beta}[I]_{\alpha}^{\beta}$$

> Before introducing diagonal, we have these two mutiplicity,

Definition (algebraic mutiplicity)**.** *The largest positive integer k such that $(x - \lambda)^k$ divides $p(x)$ is called the multiplicity of the root λ.*
If λ is an eigenvalue of an operator (matrix) A, then it is a root of the characteristic polynomial $p(z) = \det(A - zI)$. The multiplicity of this root is called the (algebraic) multiplicity of the eigenvalue λ.

Definition (geometric multiplicity)**.** *The dimension of the eigen space $N(A - \lambda I)$ is called geometric multiplicity of the eigenvalue λ.*

1.2. Diagonalization.

Definition. *A linear operator T on a finite-dimensional vector space V is called diagonalizable if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix. A square matrix A is called diagonalizable if L_A is diagonalizable.*

> After have the definition of diagonal, we want to know want matrix can be diagonal.

Definition. *A linear operator T on a finite-dimensional vector space V is called diagonalizable if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix. A square matrix A is called diagonalizable if L_A is diagonalizable.*

Theorem. *A matrix A admits a representation $A = SDS^{-1}$, where D is a diagonal matrix and S is an invertible one* ***if and only if*** *there exists a basis in F^n of eigenvectors of A.*

> In this theorem, we will know the relation between diagonal matrix and the eigenvector

Proof. Let $D = \operatorname{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, and let $b_1, \cdots, b_n$ be the columns of S (note that since S is invertible it's columns form a basis in $\mathrm{F^n}$). Then the identity $A = SDS^{-1}$ means that $D = [A]_\beta^\beta$

Indeed, $S = [I]_S^\beta$ is the change of the coordinates matrix from β to the standard basis S so we get from $A = SDS^{-1}$ that $D = S^{-1}AS = [I]_S^\beta A[I]_\beta^S$, which means exactly that $D = [A]_\beta$ ■

Theorem. *Let $\lambda_1, \lambda_2, \cdots, \lambda_r$ be distinct eigenvalues of A, and let $v_1, v_2, \cdots, v_r$ be the corresponding eigenvectors. Then vectors $v_1, v_2, \cdots, v_r$ are linearly independent.*

Corollary. *If an operator $A : V \to V$ has exactly $n = dimV$ distinct eigenvalues, then it is diagonalizable.*

> This section we will focus the bases of subspace to build the tool to proof the criterion of diagnal.

Definition (Direct sums of subspaces)**.** *Let $V_1, V_2, \cdots, V_p$ be a subspaces of a vector space V. We say that the system of subspace is a basis in V if any vector $v \in V$ admits a unique representation as a sum*

$$v = v_1 + v_2 + \cdots + v_p = \sum_{k=1}^{p} v_k,\ \ v_k \in V_k$$

We also say, that a system of subspaces $V_1, V_2, \cdots, V_p$ is linearly independent if the equation

$$v_1 + v_2 + \cdots + v_p = 0,\ \ v_k \in V_k$$

has only trivial solution

Remark. *From the above definition, the system of eigenspaces E_k if an operator A*

$$E_k = N(A - \lambda_k I)$$

is linearly independent

Lemma. *Let $V_1, V_2, \cdots, V_p$ be a linearly independent family of subspaces, and leu us have in each subspace V_k a linearly independent system β_k of vectors. Then the union $B = \cup_k \beta_k$ is a linearly independent system.*

Theorem. *Let $V_1, V_2, \cdots, V_p$ be a basis of subspaces, and let us have in each subspace V_k a basis (of vectors) β_k. Then the union $\cup_k \beta_k$ of these bases is a basis in V.*

> After we have the tool, we can use it to check the criterion of diagonal.

Theorem. *Let an operator $A : V \to V$ has exactly $n = \dim V$ eigenvalues (counting multiplicities). Then A is diagonalizable if and only if for each eigenvalue λ, geometric multiplicity = algebraic multiplicity.*

2. Inner Product Spaces

2.1. Inner Products & Norms.

Definition. *Let V be a vector space over F. An* ***inner product*** *on V is a function that assigns, to every ordered pair of vectors x and y in V, a scalar in F, denoted $< x, y >$, such that for all x, y and z in V and all c in F, the following hold:*

(a) $\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$

(b) $\langle cx, y\rangle = c\langle x, y\rangle$

(c) $\overline{\langle x, y\rangle} = \langle y, x\rangle$

(d) $\langle x, x\rangle \ > 0$ *if* $x \neq 0$

if $a_1, a_2, \cdots, a_n \in \mathrm{F}$ *and* $y, v_1, v_2, \cdots, v_n \in V$, *then*

$$\left\langle \sum_{i=1}^{n} a_i v_i, y \right\rangle = \sum_{i=1}^{n} a_i \langle v_i, y\rangle$$

vector space V over F endowed with a specific inner product is called ***inner product space***

> **Note**: We just define the **rule** of the inner product, not define **what** is inner product.

Theorem. *Let V be an inner product space. Then for $x, y, z \in V$ and $c \in \mathrm{F}$, the following statements are true.*

(a) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$.

(b) $\langle x, cy\rangle = \overline{c}\langle x, y\rangle$

(c) $\langle x, 0\rangle = \langle 0, x\rangle = 0$

(d) $\langle x, x\rangle = 0$ *if and only if* $x = 0$

(e) If $\langle x, y\rangle = \langle x, z\rangle$ *for all* $x \in V$, *then* $y = z$

Definition. *Let V be an inner product space. For $x \in V$, we define the **norm** or **length** of x by $||x|| = \sqrt{\langle x, x\rangle}$*

Theorem. *Let V be an inner product space over F. Then for all $x, y \in V$ and $c \in F$, the following statements are true*

(a) $||cx|| = |c| \cdot ||x||$

(b) $||x|| = 0$ *if and only if* $x = 0$. *In any case,* $||x|| \geq 0$

(c) *(Cauchy-Schwarz Inequality)* $| \langle x, y\rangle | \leq ||x|| \cdot ||y||$.

(d) *(Triangle Inequality)* $||x + y|| \leq ||x|| + ||y||$.

After have the definition of inner product and norm, we can define orthogonal.

Definition. *Let V be an inner product space. Vectors x and y in V are **orthogonal(perpendicular)** if $\langle x, y\rangle = 0$. A subset S of V is **orthogonal** if ant two distinct vectors in S are orthogonal. A vector x in V is a **unit vector** if $||x|| = 1$. Finally, a subset S of V is **orthonormal** if S is orthogonal and consists entirely of unit vectors.*

2.2. Gram-Schmidt & Orthogonal complements.

The power of the orthogonal subset

Theorem. *Let V be an inner product space and $S = \{ v_1, v_2, \cdots, v_k \}$ be an orthogonal subset of V consisting of nonzero vectors. If $y \in span(S)$, then*

$$y = \sum_{i=1}^{k} \frac{\langle y, v_i\rangle}{||v_i||^2} v_i$$

Corollary. *If, in addition to the hypotheses of thm, S is orthonormal and $y \in span(S)$, then*

$$y = \sum_{i=1}^{k} \langle y, v_i\rangle \, v_i$$

.

Corollary. *Let V be an inner product space, and let S be an orthogonal subset of V consisting of nonzero vectors. Then S is linearly independent.*

Theorem (Gram-Schmidt). *Let V be an inner product space and $S = \{ w_1, w_2, \cdots, w_n \}$ be a linearly independent subset of V. Define $S' = \{ v_1, v_2, \cdots, v_n \}$, where $v_1 = w_1$ and*

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{||v_j||^2} v_j \quad \text{for } 2 \leq k \leq n$$

Then S' is an orthogonal set of nonzero vectors such that span(S') = span(S).

Theorem. *Let V be a nonzero finite-dimensional inner product space. Then V has an orthonormal basis β. Furthermore, if $\beta = \{ v_1, v_2, \cdots, v_n \}$ and $x \in V$, then*

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$$

Corollary. *Let V be finite-dimensional inner product space with an orthonormal basis $\beta = \{ v_1, v_2, \cdots, v_n \}$. Let T be a linear operator on V, and let $A = [T]_\beta$. Then for any i and j, $A_{ij} = \langle T(v_j), v_i \rangle$*

Definition. *Let β be an orthonormal subset (possibly infinite) of an inner product space V, and let $x \in V$. We define the* ***Fourier coefficients*** *of x relative to β to*

After have good tools, we are going to check the **orthogonal complement** of S.

Definition. *Let S be a nonempty subset of an inner product space V. We define $S^\perp$ to be the set of all vectors in V that are orthogonal to every vector in S; that is, $S^\perp = \{ x \in V \mid \langle x, y \rangle = 0 \text{ for all } y \in S \}$. The set $S^\perp$ is called the* ***orthogonal complement*** *of S.*

Theorem. *Let W be a finite-dimensional subspace of an inner product space V, and let $y \in V$. Then there exist unique vectors $u \in W$ and $z \in W^\perp$. i.e.*

$$V = W \oplus W^\perp$$

Furthermore, if $\{ v_1, v_2, \cdots, v_k \}$ is an orthonormal basis for W, then

$$u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$$

Corollary. *In the notation of Thm, the vector u is the unique vector in W that is "closest" to y; that is, for any $x \in W, ||y - x|| \geq ||y - u||$, and this inequality is an equality if and only if $x = u$.*

Theorem. *Suppose that $S = \{ v_1, v_2, \cdots, v_k \}$ is an orthonormal set in an n-dimensional inner product space V. Then*

(a) S can be extended to an orthonormal basis $\{ v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_n \}$ for V

(b) If $W =$ span(S), then $S_1 = \{ v_{k+1}, v_{k+2}, \cdots, v_n \}$ is an orthonormal basis for $W^\perp$

(c) If W is any subspace of V, then $\dim(V) = \dim(W) + \dim(W^\perp)$

2.3. The Adjoint of a Linear Operator.

In this section, we will introduce the adjoint linear operator and it's property.
Before know adjoint operation, we need to know the theorem first.

Theorem. *Let V be a finite-dimensional inner product space over F, and let $g : V \to F$ be a linear transformation. Then there exists a* ***unique*** *vector $y \in V$ such that $g(x) = \langle x, y \rangle$ for all $x \in V$*

this proof next time

Theorem. *Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Then there exists a unique function $T^* : V \to V$ such that $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for all $x, y \in V$. Furthermore, T^* is linear.*

We define the T^* here, and next we want to know the $*$ in the matrix presentation.

Theorem. *Let V be a finite-dimensional inner product space, and let β be an orthonormal basis for V. If T is a linear operator on V, then*

$$[T^*]_\beta = [T]_\beta^*$$

Corollary. *Let A be an "$n \times n$" matrix. Then $L_{A^*} = (L_A)^*$*

Theorem. *Let V be an inner product space, and let T and U be linear operators on V. Then*

a) $(T + U)^* = T^* + U^*$

b) $(cT)^* = \overline{c}T^*$ for any $c \in \mathrm{F}$

c) $(TU)^* = U * T*$

d) $T^{**} = T*$

e) $I^* = I$

Corollary. *Let A and B be $n \times n$ matrices. Then*

a) $(A+B)^* = A^* + B^*$

b) $(cA)^* = \overline{c}A^*$ *for all* $c \in \mathrm{F}$

c) $(AB)^* = B^*A^*$

d) $A^{**} = A$

e) $I^* = I$

we skip the minimum function here(LAMMA~Thm 6.13)

2.4. Normal & Self-Adjoint Operators.

Lemma. *Let T be a linear operator on a finite-dimensional inner product space V. If T has an eigenvector, then so does T^*.*

Theorem (Schur)**.** *Let T be a linear operator on a finite-dimension inner product space V. Suppose that the characteristic polynomial of T splits. Then there exists an orthonormal basis β for V such that the matrix $[T]_\beta$ is upper triangular.*

I don't know why read Schur here, but we are going to introduce **normal** and it's property.

Definition. *Let V be an inner product space, and let T be a linear operator on V. We say that T is* ***normal*** *if $TT^* = T^*T$. An $n \times n$ real or complex matrix A is* ***normal*** *if $AA^* = A^*A$.*

Theorem. *Let V be an inner product space, and let T be a* ***normal operator*** *on V. Then the following statements are true.*

a) $||T(x)|| = ||T^*(x)||$ *for all* $x \in V$.

b) $T - cI$ *is normal for every* $c \in \mathrm{F}$

c) *If x is an eigenvector of T, then x is also an eigenvector of T^*. In fact, if $T(x) = \lambda x$, then $T^*(x) = \overline{\lambda}x$.*

d) *If λ_1 and λ_2 are distinct eigenvalues of T with corresponding eigenvectors x_1 and x_2, then x_1 and x_2 are orthogonal.*