In calculus

1. Extreme Value Theorem: Every continuous function $f : [a, b] \to \mathrm{R}$ admit both max and min value $\Rightarrow$ Compact set
2. Intermediate value Theorem: Given continous function $f : [a, b] \to \mathrm{R}$ for all $f(a) \leq \lambda \leq f(b) \exists c \in [a, b] \ni f(c) = \lambda \Rightarrow$ connected set

How to prove a statement: HP,then Q, $P \Rightarrow Q$

$$\begin{cases} \begin{cases} \text{Direct Proof} \\ \text{Indirect Proof} \begin{cases} \text{contrapositive } \sim Q \Rightarrow \sim P \\ \text{by contradiction} \end{cases} \end{cases} \\ Mathematical Induction \end{cases}$$

1. Some preliminary

1.1. Set Theory. We will assume that you are familiar with some basic set theory e.g. union, intersection, difference

1.2. The Number System.

$\mathbb{N} = \{ 1, 2, 3, \cdots \}$ the set of all positive integers n natural numbers

$\mathbb{Z} = \{ \cdots, -2, -1, 0, -1, -2, \cdots \}$ the set of all integers called the ring of intepus

$\mathbb{Q} = \left\{ \frac{m}{n} \; : \; n, m \in \mathbb{Z}, n \neq 0 \right\}$ the set of all rational numbers of the national number field on real line

$\mathbb{R}$ the set all of real numbers on the real number field on real line

$\mathbb{C} = \{ z = a + ib \mid a, b \in \mathbb{R} \}$ the set of all complex numbers or the complex number filed on complex plane, where $i = \sqrt{-1}$

Remark.

1. $x + 2 = 0$ *no root in* $\mathbb{N}$
   $3x - 5 = 0$ *no root in* $\mathbb{Z}$
   $x^2 + 1 = 0$ *no root in* $\mathbb{R}$
2. *One can construct* $\mathbb{Q}$ *from* $\mathbb{Z}$ *in algebraic way, called the fraction field of* $\mathbb{Z}$
3. *One can construct* $\mathbb{R}$ *from* $\mathbb{Q}$ *in two ways:*
   - *Using Dedekind cut which is given in the appendix of Rudin p17-21*
   - *Using completion of matrix space*
4. *One can construct* $\mathbb{C}$ *from in complex analysis*

Example.

1. *Between any two rational numbers, there is another one*

   Proof. Let $r, s \in \mathbb{Q}$ with $r < s$, then $\dfrac{r+s}{2} \in \mathbb{Q}$ and $r < \dfrac{r+s}{2} < s$

   $r = \dfrac{m_1}{n_1}, s = \dfrac{m_2}{n_2}, \dfrac{r+s}{2} = \dfrac{\frac{m_1}{n_1} + \frac{m_2}{n_2}}{2} = \dfrac{m_1n_2 + n_1m_2}{2n_1m_1} \in Q$
   $s = \dfrac{s+s}{2} > \dfrac{r+s}{2} > \dfrac{r+r}{2} = r$

   ∎

2. $x^2 = \dfrac{4}{9}$ *has exactly two rational solutions, namely,* $\pm\dfrac{2}{3}$
3. $x^2 = 2$ *has exactly two real root, namely,* $\pm\sqrt{2}$
4. *Is there any rational roots of* $x^2 = 2$*? i.e., is* $\sqrt{2}$ *rational?*

   Suppose $r = \dfrac{m}{n} \in \mathrm{Q}$*, is a root of* $x^2 = 2$*, where* $(m, n) = 1$
   Then $\dfrac{m^2}{n^2} = 2 \implies m^2 = 2n^2 \implies 2 \mid m^2 \implies 2 \mid m \implies 4 \mid m^2 \implies 4 \mid 2n^2$
   $\implies 2 \mid n^2 \implies 2 \mid n \implies (n, m) \neq 1$

5. *Let* $A = \left\{ r \in \mathrm{Q} \mid r > 0 \;\&\; r^2 < 2 \right\}, B = \left\{ r \in \mathrm{Q} \mid r > 0 \;\&\; r^2 > 2 \right\}$
   Then A *contains no largest numbers, i.e. max element &* B *contains no smallest numbers, i.e. min element*

   Proof. A contains no largest numbers $\Leftrightarrow$ given $r \in A$, $\exists s \in A \ni s > r$
   Now, given $r \in A$, Let $s = r - \dfrac{r^2 - 2}{r+2} = \dfrac{2r+2}{r+2}$ $(\star_1)$
   $\implies s^2 - 2 = \dfrac{2(r^2-2)}{(r+2)^2}$ $(\star_2)$
   Now, $r \in A, r^2 < 2 \implies r^2 - 2 < 0 \therefore$
   $(\star_1) \& (\star_2) \implies s > r \;\&\; s^2 < 2 \implies s \in A$ ∎

6. *As you know, in calculus, the sequence* $\{1, 1.4, 1.41, 1.414, 1.4142, \cdots\}$ *does not converge in* Q*, but it converges to* $\sqrt{2}$ *in* R

1.3. Order Sets.

Definition.

Let X be a nonempty set A, relation on X is a subset R *of* $X \times X = \{ (x, y) \mid x, y \in X \}$

Let R *be a relation on X, if $(x, y) \in \mathrm{R}$, then we say that x is retaliated to y, and is written as $xRy(x \sim y)$*

Definition. *An ordered set on S, is a relation denoted by " $<$ " on S, satisfy:*

(i) *The low of trichonomy*

Given $x, y \in S$, one and only one of the following holds: $x < y, x = y, y < x$

(ii) *Transitivity: if $x < y \& y < z$, than $x < z$*

Notation

(1) *$x < y$ means "x is less than y" or "x is smaller than y"*

(2) *$y > x$ means $x < y$*

(3) *$x \leq y$ means $x < y$ or $x = y$, i.e. the negative of $x > y$*

Definition. *Let "$<$" lie an order on a set S, then pass $(S, <)$ on simply S is called an ordered set*

Definition. *Let S is an ordered set & $E \subseteq S (E \neq \emptyset)$*

- *E is bounded above if $\exists\ \alpha \in S \implies x \leq \alpha\ \forall\ x \in E$*
  such α is called an upper bound of E
- *E is bounded below if $\exists\ \beta \in S \ni \beta \leq x, \forall\ x \in E$, such β is called a lower bdd of E*
- *E is bdd is E is both bdd above and below.*

Definition. *Let S be an ordered set and $E \subseteq S (E \neq \emptyset)$ bdd above. An element $\alpha \in S$ is called the last upper bound or supremum of E if*

(i) *α is an upper bound of E*

(ii) *α is the smallest such one.*

Equivalently,

(i') *$x \leq \alpha, \forall x \in E$*

(ii') *if $\beta < \alpha$, then β is not an upper bdd of E, i.e. $\exists\ x \in E \ni x > \beta$*

Such α(if exists) is denoted by

$$\alpha = sup(E)$$

similarly, one can defined the greatest lower bdd of infimum of E

Remark. *if* $\sup(E)$ *exists then it is unique*
suppose $\alpha \neq \alpha'$ *both lub of* E
$\because$ *by trichotomy* ,$\alpha > \alpha'$ *or* $\alpha = \alpha'$ *or* $\alpha < \alpha'(\rightarrow\leftarrow)$

Definition. *A ordered set* S *is said to have the least upper bdd property if* $E \subseteq S,\ E \neq \emptyset$ *and* E *is bdd above, then* $\sup(E)$ *exists in* S

Example.

1. *In* Q *with the normal ordining*
   $A = \{\, r \in \mathrm{Q} \mid r > 0,\ r^2 < 2 \,\} \& B = \{\, r \in \mathrm{Q} \mid r > 0,\ r^2 > 2 \,\}$
   Then A *is bdd above, in fact, bdd by every element in* B, *but* $\sup(A)$ *does not exist in* Q*(*$\because$ *by Ex1.5)*
2. B *is bdd below by every element of* A *and* $\inf B$ *does not exists*
3. *Note that* $\sup(E) \& \inf(E)$ *may not in* E *even if exist*

Remark.

1. *By the Example above,* Q *with the usual ordering has no l.u.b property*
2. *In* 1.5 *we will explain that* R *with usual ordering has the l.u.b. property. However, we usually adopt the follwing*

The Axiom of Completence or Least upper bdd property:
Every nonempty subset E of R which is bdd above has l.u.b

Theorem. *Let* S *is an ordered set if* S *has the l.u.b. property, then* S *has the g.l.b. property, i.e. if* $\emptyset \neq B \subseteq S$ *is bdd below, then* $\inf(B)$ *exists in* S

Proof. Given $B(\neq \emptyset) \subseteq S$ which is bdd below
Let $L = \{\, a \in S \mid a \text{ is a lower bdd of } B \,\}$

- $L \neq \emptyset(\because B$ is bdd below)
- L is bdd above (in fact, every element in B is on upper bound of L)
  $\implies \forall a \in L \implies a \leq x,\ \forall x \in B \implies x$ is an upper bound of L
- $\sup(L) = \alpha$ exists by assumption

Claim $\alpha = \inf B$

(i) α is a lower bdd of B, i.e. $\alpha \leq x,\ \forall x \in B$
By $\alpha = \sup L$, if $r < \alpha$, them r is not an upper bdd of $L(\because \alpha$ is the smallest one).Hence,$r \notin B(\because$ every element of B is an upper bdd of L), so $\alpha \leq x, \forall x \in B$
We have proved $(r < \alpha \implies r \notin B) \implies (r \in B \implies r \geq \alpha)$

(ii) α is the greated one
if $\alpha < \beta$ and β is a lower bdd of B, then $\beta \notin L$, i.e. β is not a lower bdd of B, so α is the greatest one. Therefore, $\alpha = \inf(B)$

■

Remark. *Let* $E(\neq \emptyset) \subseteq \mathrm{R}$ *be bdd below, then* $\inf(E)$ *exists and* $\inf(E) = -\sup(-E)$*, where* $-E = \{ -x \mid x \in E \}$

1.4. Field.

Recall the addition & multiplication in R
$+ : \mathrm{R} \times \mathrm{R} \to \mathrm{R}((a,b) \mapsto a+b)$
$\times : \mathrm{R} \times \mathrm{R} \to \mathrm{R}((a,b) \mapsto a \cdot b = ab)$

Definition. *Let* X *are a nonempty set* A*, binary operation on* X *is a function,* $o : X \times X \to X$

Definition. *Let* F *be a nonempty set, we say that* F *is a field* ($(F, +, \cdot)$ *is a field) if there are two binary operator called addition* $" + "$ *and multiplication* $" \cdot "$ *on* F *property*

Axioms for $" + "$

(A1) Commutative: $\forall x, y \in \mathrm{F},\ x + y = y + x$

(A2) Associative: $\forall x, y, z \in F, (x + y) + z = x + (y + z)$

(A3) Additive identity or zero element: $\exists\ 0 \in F \implies x + 0 = 0 + x = x,\ \forall x \in \mathrm{F}$

(A4) Additive inverse on negative: For each $x \in X,\ \exists\ -x \in F \implies x + (-x) = (-x) + x = 0$

i.e. $(\mathrm{F}, +)$ *is an abelian group* ***Axioms for multiplication***

(M1) Commutative: $\forall x, y \in \mathrm{F},\ xy = yx$

(M2) Associative: $\forall x, y, z \in \mathrm{F},\ (xy)z = x(yz)$

(M3) Muti identity: $\exists 1 \neq 0$ *in* $\mathrm{F} \ni x1 = 1x = x$

(M4) Multiplicative inverse: For each $x \neq 0, \exists x^{-1} \in \mathrm{F} \implies xx^{-1} = x^{-1}x = 1$

i.e. $(F = F \cdot \{ 0 \}, \cdot)$ *is an abelian group*

Distributive Law

(D1) $\forall x, y, z \in \mathrm{F},\ (x, y)z = xz + yz\ \&\ x(y + z) = xy + xz$

let $(\mathrm{F}, +, \cdot)$ be a field, we list a series of basic identity as you learn in high school in the real number system

(a) Cancellation law for " + " : $x+y=x+z \implies y=z$

$\because x+y=x+z \implies (-x)+(x+y)=(-x)+(x+z) \implies ((-x)+x)+y=((-x)+x)+z$
$\implies 0+y=0+z \implies y=z$

(b) 0 is "1"

suppose $0' \in \mathrm{F}$ is another element satisfy A_3, then $0=0+0'=0'$

(c) $x+y=x \implies y=0$ by (a) $\because x+y=x+0 \implies y=0$

(d) negative $-x$ of x is "1"

if $x' \in F$, is another negative of x, them $x+x'=x'+x=0$

From $x+x'=0 \implies (-x)+(x+x')=-x+0=-x$

(e) $x+y=0 \implies y=-x$

$x+y=0 \implies (-x)+(x+y)=(-x)+0 \implies ((-x)+x)+y=-x$
$\implies 0+y=-x \implies y=-x$

(f) $-(-x)=x$

$-(-x)+(-x)=0$, By (d) $x=-(x)$

(a') cancellation law

if $x \neq 0$, then $xy=xz \implies y=z, \quad \because (x^{-1})(xy)=(x^{-1})(xz)$
$\implies (x^{-1})(xy)=(x^{-1}x)z \implies 1y=1z \implies y=z$

(b') 1 is "1"

if $1'$ is another identity, then $1=11'=1'$

(c') $x \neq 0$ & $xy=x \implies y=1$

$xy=x1 \implies y=1$

(d') For $x \neq 0$ in F, x^{-1} is "1"

if x is another one, i.e. $x'x=xx'=1 \implies (x^{-1})(xx')=(x^{-1})1=x^{-1}$

(f') $x \neq 0 \implies (x^{-1})^{-1}=x$

$(x^{-1})^{-1}(x^{-1})=1 \implies x=(x^{-1})^{-1}$

(g') $0x=x0=0$

$(0+0)x=0x+0x \implies 0x=0$

(h') $x \neq 0$ & $y \neq 0 \implies xy \neq 0$, equivalently $xy=0 \implies x=0$ or $y=0$

$\because xy=0$ then $(x^{-1})(xy)=((x^{-1})x)y=1y=y(\rightarrow\leftarrow)$

(i') $(-x)y = -(xy) = x(-y)$

$\because [(-x)+x]y = 0y = 0 = (-x)y = -(xy) \implies (-x)y = -(xy)$

(j') $(-x)(-y) = xy$

$\because (-x)(-y) = -(x(-y))$ by (i)

$= -(-(xy)) = xy$

(k) $-x = (-1)x$

$\because (1-1)x = 0x = 0 = 1x + (-1)x = x + (-1)x \implies (-1)x = -x$

Definition (Order Field)**.** *Let* F *is a field, we say that* F *is an order field if there is an ordering* $" < "$ *satisfying*

(1) if $x < y$*, then* $x + z < y + z, \ \forall z \in F$

(2) if $x > y$ *and* $y > 0$*, then* $xy > 0$

Example. Q *and* R *are order field under the usual ordering*

Some basic properties of ordered field, let F *be an ordered field with ordering* $" < "$

(a) $x > 0 \implies -x < 0$

$\because x > 0 \implies x + (-x) > 0 + (-x) \implies 0 > -x$

(b) $x > y \Leftrightarrow x - y > 0$

$\because x > y \implies x + (-y) > y = (-y) \implies x - y > 0$

$x - y > 0 \implies x - y + y > y \implies x + 0 > y \implies x > y$

(c) $x > 0$ *and* $y < z \implies xy < xz$

$\because x > 0$ *and* $y < z \implies x > 0$ *and* $z - y < 0 \implies x(z-y) > 0 \implies xz + x(-y) > 0$

$\implies xz - xy > 0 \implies xz > xy$

(d) $x < 0$ *and* $y < z \implies xy > xz$

$\because x < 0$ *and* $y < z \implies -x > 0$ *and* $z - y > 0 \implies (-x)(z-y) > 0 \implies -xz + xy > 0$

$\implies xy > xz$

(e) $\forall x \neq 0$ *in* $\mathrm{F}, x^2 > 0$

$\because x > 0 \implies x \cdot x > x0$ *by (c) or*

$x < 0 \implies -x > 0$ *by (a)* $\implies -x > 0$ *by (a)* $\implies (-x)^2 > 0 \implies x^2 > 0$

(f) $1 > 0, \ -1 < 0$

$\because 1 \neq 0 \implies 1^2 > 0$ *by (e)* $\implies 1 > 0$

(g) $0 < x < y \implies 0 < \frac{1}{y} < \frac{1}{x}$

$\because$ *Note that* $\forall u \in \mathrm{F},\ u > 0 \implies \frac{1}{u} = u^{-1} > 0$

$\because$ *if* $\frac{1}{u} < 0$, *then* $u \cdot \frac{1}{u} < 0$ *by (e)* $\implies 1 < 0(\rightarrow\leftarrow)\ \therefore\ \frac{1}{u} > 0$

Now, $\frac{1}{x}, \frac{1}{y} > 0$ *from* $x < y$ *we get* $(\frac{1}{x} \cdot \frac{1}{y})x < (\frac{1}{x} \cdot \frac{1}{y})y \implies 0 < \frac{1}{y} < \frac{1}{x}$

Remark. *By (e)(f), we conclude that* C *is not an ordered field*
$\because$ C *were an ordered field, then by (e),* $i^2 > 0 \implies -1 > 0(\rightarrow\leftarrow)$
$\therefore$ C *is not an order field*

1.5. The Real Number Field R.

Theorem. *There exists an ordered field* R *containing* Q *which has the l.u.b. property. Moreover, such* R *is unique up to order-isomorphism*
i.e. if " $<$ " and " $<'$ " are two orders on R*, them* $\exists f_i(\mathrm{R}, <) \to (\mathrm{R}, <') \implies$

(i) f *is a field isomorphism, i.e.* $\forall a, b \in \mathrm{R},\ f(a+b) = f(a)+f(b),\ f(ab) = f(a)f(b),\ f(1) = 1$

(ii) f *preserves ordering,* $a < b \implies f(a) < f(b)$

Such R *is called the real number field or real number system or real line*

Theorem.

(a) The Archimedean property of R : *Given* $x, y \in \mathrm{R}$ *with* $x > 0,\ \exists\ n \in \mathrm{N} \implies nx > y$

(b) Q *is dense in* $\mathrm{R} : \forall x, y \in \mathrm{R}$ *with* $x \leq y,\ \exists\ r \in \mathrm{Q} \implies x < r < y$

Proof.

(a) Let $A = \{\, nx \mid n \in \mathrm{N} \,\} \subseteq \mathrm{R}$

if (a) were false, them A is bdd above by y, since R has the l.u.b property

$\alpha = \sup A$ exists in R, since $x > 0,\ \alpha - x < \alpha \implies \alpha - x$ is not an upper bdd of A

$\implies \exists m \in \mathrm{N} \ni mx > \alpha - x \implies (m+1)x > \alpha(\rightarrow\leftarrow)$

(b) Since $x < y,\ y - x > 0$, by (a),$\exists n \in \mathrm{N} \implies n(y-x) > 1$

By (a) again, $\exists m_1, m_2 \in \mathrm{N} \implies m_1 = m_1 1 > n_x\ \&\ m_2 = m_2 \cdot 1 > -nx$

we have $-m_2 < nx < m_1$, choose $m \in \mathrm{Z} \implies -m_2 \leq m \leq m_1\ \&\ m - 1 \leq nx < m$

(in fact, $m = [nx] + 1$,where $[z]$ in the greatest integer of z)

we have $nx < m < 1 + nx < ny(\because n(y-x) > 1) \implies x < \frac{m}{n} < y$

Let $r = \frac{m}{n} \in \mathrm{Q}$, then $x < r < y$

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An application of the density property of Q in R :
Given $x \in \mathrm{R} - \mathrm{Q}$ i.e. x is an irrational numbers, i.e. $\forall\ \epsilon > 0, \exists r \in \mathrm{Q} \implies |x - r| < \epsilon$
equivalently, $\exists$ a sequence $\{ r_n \}$ in Q $\implies r_n \to x$
In fact, one may choose $\{ r_n \}$ to $\uparrow$ or $\downarrow$
$\because \forall n \geq 1,\ \exists\ r_n \in \mathrm{Q} \implies x < r_n < \frac{1}{n} + x$ by Thm.1.3(b) By squeezing lemma, $r_n \to x$ on $n \to \infty$

Theorem (existence of nth root)**.** *Given $x \in \mathrm{T}, x > 0$ & $n \in \mathrm{N}, \exists$ "1"$y > 0 \implies y^n = x$*
Such y is called the nth root of x & denoted by $y = \sqrt[n]{x} = x^{\frac{1}{n}}$

Proof. "1". Suppose $y_1, y_2 > 0 \implies y_1^n = x$ & $y_2^n = x$
Bt trichotomy, we have

(i) $0 < y_1 < y_2 \implies y_1^n < y_2^n (\rightarrow\leftarrow)$

(ii) $0 < y_2 < y_1 \implies y_2^n < y_1^n (\rightarrow\leftarrow)$

(iii) $y_1 = y_2$

"$\exists$". Let $E = \{ t \in \mathrm{R} \mid t^n < x \}$
Claim:

- $E \neq \emptyset$, Let $t = \frac{x}{1+x}$, then $0 < t < 1$, hence $t^n < t < x$, $\therefore t \in E$ & $E \neq \emptyset$
- E is bdd above, in fact E is bdd above by $1 + x$ if $t > 1 + x > 1$, then $t^n > t > x$, so E is bdd above by $1 + 1$

  Therefore $y = \sup E$ exists & is finite
- Claim $y > 0$ & $y^n = x$, clearly, $y > 0 (\because \frac{x}{1+x} \in E$ & $\frac{x}{1+x} > 0)$

  by trichotomy, we have $y^n < x,\ y^n > x,\ y^n = x$

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