PROBABILITY EXERCISE

WEEK 1

(1) Compute
$$\int_{-\infty}^{\infty} e^{-x^2/2}\ dx$$
(2) Compute
$$\int_{0}^{\infty} \frac{\sin x}{x}\ dx$$
(3) Show that
$$E[Y] = \int_0^\infty \mathrm{P}\{Y > y\}\ dy - \int_0^\infty \mathrm{P}\{Y < -y\}\ dy$$
(4) Show that if X has density function f, then
$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\ dx$$
(5) Show that Z is a standard normal random variable; then, for $x > 0$,
 (a) $P\{Z > x\} = P\{Z < -x\}$
 (b) $P\{|Z| > x\} = 2P\{X > x\}$
 (c) $P\{|Z| < x\} = 2P\{Z < x\} - 1$

(6) Let $f(x)$ denote the probability density function of a normal random variable with mean μ and variance σ^2. Show that $\mu-\sigma$ and $\mu+\sigma$ are pints of inflection of this function. That is, show that $f''(x) = 0$ when $x = \mu - \sigma$ or $x = \mu + \sigma$

(7) Let Z be standard normal random variable Z, and let g be a differentiable function with derivative g'.
 (a) Show that $E[g'(Z)] = E[Zg(Z)]$
 (b) Show that $E[Z^{n+1}] = nE[Z^{n-1}]$
 (c) Find $E[Z^4]$

(8) Let X be a random variable with probability density function

$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of c?

(b) What is the cumulative distribution function of X?

(9) The probability density function of X, the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \begin{cases} \dfrac{10}{x^2} & x > 10 \\ 0 & x \le 10 \end{cases}$$

(a) Find $P\{X > 20\}$

(b) What is the cumulative distribution function of X?

(c) What is the probability that of 6 such types of devices, at least 3 will function for at least 15 hours? What assumptions are you making?

(10) Compute $E[X]$ if X has a density function given by

(a) $f(x) = \begin{cases} \dfrac{1}{4}xe^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

(b) $f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{5}{x^2} & x > 5 \\ 0 & x \le 5 \end{cases}$

(11) The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & \textbf{otherwise} \end{cases}$$

If $E[X] = \dfrac{3}{5}$, find a and b.

(12) The lifetime in hours of an electronic tube is a random variable having a probability density function given by

$$f(x) = xe^{-x} \qquad x \geq 0$$

Compute the expected lifetime of such a tube.

(13) You arrive at a bus stop at $q0$ A.M., knowing that the bus will arrive at some time uniformly distributed between 10 and $10:30$.

(a) What is the probability that you will have to wait longer that 10 minutes?

(b) If, at $10:15$, the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

(14) If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute

(a) $P\{X > 5\}$

(b) $P\{4 < X < 16\}$

(c) $P\{X < 8\}$

(d) $P\{X < 20\}$

(e) $P\{X > 16\}$