0.0.1. In calculus.
(1) Extreme Value Theorem: Every continuous function $f : [a, b] \to$ R admit both max and min value $\Rightarrow$ Compact set
(2) Intermediate value Theorem: Given continous function $f : [a, b] \to$ R for all $f(a) \leq \lambda \leq f(b) \exists c \in [a, b] \ni f(c) = \lambda \Rightarrow$ connected set

How to prove a statement: HP,then Q, $P \Rightarrow Q$

$$\begin{cases} \begin{cases} \text{Direct Proof} \\ \text{Indirect Proof} \begin{cases} \text{contrapositive} \sim Q \Rightarrow \sim P \\ \text{by contradiction} \end{cases} \end{cases} \\ Mathematical Induction \end{cases}$$

1. Some preliminary

1.1. Set Theory. We will assume that you are familiar with some basic set theory e.g. union, intersection, difference

1.2. The Number System.

$\mathbb{N} = \{ 1, 2, 3, \cdots \}$ the set of all positive integers n natural numbers

$\mathbb{Z} = \{ \cdots, -2, -1, 0, -1, -2, \cdots \}$ the set of all integers called the ring of intepus

$\mathbb{Q} = \left\{ \dfrac{m}{n} : n, m \in \mathbb{Z}, n \neq 0 \right\}$ the set of all rational numbers

$\mathbb{R}$ the set all of real numbers on the real number field on real line

$\mathbb{C} = \{ z = a + ib \mid a, b \in \mathbb{R} \}$ the set of all complex numbers or the complex number filed on complex plane, where $i = \sqrt{-1}$

Remark.

(1) $x + 2 = 0$ no root in $\mathbb{N}$
$3x - 5 = 0$ no root in $\mathbb{Z}$
$x^2 + 1 = 0$ no root in $\mathbb{R}$
(2) One can construct $\mathbb{Q}$ from $\mathbb{Z}$ in algebraic way, called the fraction field of $\mathbb{Z}$
(3) One can construct $\mathbb{R}$ from $\mathbb{Q}$ in two ways:
· Using Dedekind cut which is given in the appendix of Rudin p17-21
· Using completion of matrix space
(4) One can construct $\mathbb{C}$ from in complex analysis

Example.

(1) *Between any two rational numbers, there is another one*

Proof. Let $r, s \in \mathbb{Q}$ with $r < s$, then $\dfrac{r+s}{2} \in \mathbb{Q}$ and $r < \dfrac{r+s}{2} < s$

$$r = \frac{m_1}{n_1}, s = \frac{m_2}{n_2}, \frac{r+s}{2} = \frac{\frac{m_1}{n_1} + \frac{m_2}{n_2}}{2} = \frac{m_1 n_2 + n_1 m_2}{2 n_1 m_1} \in \mathbb{Q}$$

$$s = \frac{s+s}{2} > \frac{r+s}{2} > \frac{r+r}{2} = r$$

■

(2) $x^2 = \dfrac{4}{9}$ *has exactly two rational solutions, namely,* $\pm \dfrac{2}{3}$

(3) $x^2 = 2$ *has exactly two real root, namely,* $\pm\sqrt{2}$

(4) *Is there any rational roots of* $x^2 = 2$? *i.e., is* $\sqrt{2}$ *rational?*

Suppose $r = \dfrac{m}{n} \in \mathbb{Q}$, *is a root of* $x^2 = 2$, *where* $(m, n) = 1$

Then $\dfrac{m^2}{n^2} = 2 \implies m^2 = 2n^2 \implies 2 \mid m^2 \implies$

$2 \mid m \implies 4 \mid m^2 \implies 4 \mid 2n^2$

$\implies 2 \mid n^2 \implies 2 \mid n \implies (n, m) \neq 1$

(5) *Let* $A = \left\{ r \in \mathbb{Q} \mid r > 0 \ \& \ r^2 < 2 \right\}, B = \left\{ r \in \mathbb{Q} \mid r > 0 \ \& \ r^2 > 2 \right\}$
   Then A *contains no largest numbers, i.e. max element* $\&$ B
contains no smallest numbers, i.e. min element

Proof. A *contains no largest numbers* $\Leftrightarrow$ *given* $r \in A$,
$\exists s \in A \ni s > r$

Now, given $r \in A$, *Let* $s = r - \dfrac{r^2 - 2}{r + 2} = \dfrac{2r + 2}{r + 2}$ $(\star_1)$

$\implies s^2 - 2 = \dfrac{2(r^2 - 2)}{(r+2)^2}$ $(\star_2)$

Now, $r \in A, r^2 < 2 \implies r^2 - 2 < 0 \ \therefore$
$(\star_1) \& (\star_2) \implies s > r \ \& \ s^2 < 2 \implies s \in A$ ■

(6) *As you know, in calculus, the sequence* $\{ 1, 1.4, 1.41, 1.414, 1.4142, \cdots \}$
does not converge in $\mathbb{Q}$, *but it converges to* $\sqrt{2}$ *in* $\mathbb{R}$

1.3. Order Sets.

__Definition__ (Relation)__.__
Let X be a nonempty set A, relation on X is a subset R of $X \times X = \{ (x,y) \mid x,y \in X \}$
Let R be a relation on X, if $(x,y) \in$ R, then we say that x is retaliated to y, and is written as $xRy(x \sim y)$

__Definition__ (Order Set)__.__ *An ordered set on S, is a relation denoted by " $<$ " on S, satisfy:*

    (i) The low of trichonomy
        Given $x,y \in S$, one and only one of the following holds: $x < y, x = y, y < x$
    (ii) Transitivity: if $x < y \& y < z$, than $x < z$

> ### Notation
>   *(1) $x < y$ means "x is less than y" or "x is smaller than y"*
>   *(2) $y > x$ means $x < y$*
>   *(3) $x \leq y$ means $x < y$ or $x = y$, i.e. the negative of $x > y$*

__Definition__ (bdd)__.__ *Let S is an ordered set $\& E \subseteq S(E \neq \emptyset)$*

- *E is bounded above if $\exists \, \alpha \in S \implies x \leq \alpha \, \forall \, x \in E$ such α is called an upper bound of E*
- *E is bounded below if $\exists \, \beta \in S \ni \beta \leq x, \forall \, x \in E$, such β is called a lower bdd of E*
- *E is bdd is E is both bdd above and below.*

__Definition__ (least upper bound)__.__ *Let S be an ordered set and $E \subseteq S(E \neq \emptyset)$ bdd above. An element $\alpha \in S$ is called the last upper bound or supremum of E if*

    (i) α is an upper bound of E
    (ii) α is the smallest such one.
Equivalently,
    (i') $x \leq \alpha, \forall x \in E$
    (ii') if $\beta < \alpha$, then β is not an upper bdd of E, i.e. $\exists \, x \in E \ni x > \beta$
Such α (if exists) is denoted by

$$\alpha = sup(E)$$

similarly, one can defined the greatest lower bdd of infimum of E

__Remark.__ *if $\sup(E)$ exists then it is unique*
suppose $\alpha \neq \alpha'$ both lub of E

$\because$ *by trichotomy* $,\alpha > \alpha'$ *or* $\alpha = \alpha'$ *or* $\alpha < \alpha'(\rightarrow\leftarrow)$

Definition (least upper bdd property). *A ordered set S is said to have the least upper bdd property if $E \subseteq S$, $E \neq \emptyset$ and E is bdd above, then $\sup(E)$ exists in S*

Example.

 (1) *In* Q *with the normal ordining*
   $A = \left\{\, r \in Q \mid r > 0,\ r^2 < 2 \,\right\} \& B = \left\{\, r \in Q \mid r > 0,\ r^2 > 2 \,\right\}$
   Then A is bdd above, in fact, bdd by every element in B, but
  $\sup(A)$ *does not exist in* Q*($\because$ by Ex1.5)*
 (2) *B is bdd below by every element of A and $\inf B$ does not exists*
 (3) *Note that $\sup(E) \& \inf(E)$ may not in E even if exist*

Remark.

 (1) *By the Example above,* Q *with the usual ordering has no l.u.b property*
 (2) *In 1.5 we will explain that* R *with usual ordering has the l.u.b. property. However, we usually adopt the follwing*

The Axiom of Completence or Least upper bdd property:
Every nonempty subset E of R which is bdd above has l.u.b

Theorem (l.u.b.p. $\rightarrow$ g.l.b.p.). *Let S is an ordered set if S has the l.u.b. property, then S has the g.l.b. property, i.e. if $\emptyset \neq B \subseteq S$ is bdd below, then $\inf(B)$ exists in S*

Proof. ($\star$)
Given $B(\neq \emptyset) \subseteq S$ which is bdd below
Let $L = \{\, a \in S \mid a$ is a lower bdd of $B \,\}$

 • $L \neq \emptyset(\because B$ is bdd below)
 • L is bdd above (in fact, every element in B is on upper bound of L)
   $\implies \forall a \in L \implies a \leq x,\ \forall x \in B \implies x$ is an upper bound of L
 • $\sup(L) = \alpha$ exists by assumption

Claim $\alpha = \inf B$

 (i) α is a lower bdd of B, i.e. $\alpha \leq x,\ \forall x \in B$
  By $\alpha = \sup L$, if $r < \alpha$, them r is not an upper bdd of $L(\because \alpha$ is the smallest one).Hence,$r \notin B(\because$ every element of B is an upper bdd of L), so $\alpha \leq x, \forall x \in B$
  We have proved $(r < \alpha \implies r \notin B) \implies (r \in B \implies r \geq \alpha)$

(ii) α is the greated one

if $\alpha < \beta$ and β is a lower bdd of B, then $\beta \notin L$, i.e. β is not a lower bdd of B, so α is the greatest one. Therefore, $\alpha = \inf(B)$

∎

Remark. *Let $E(\neq \emptyset) \subseteq \mathrm{R}$ be bdd below, then $\inf(E)$ exists and $\inf(E) = -\sup(-E)$, where $-E = \{ -x \mid x \in E \}$*

1.4. Field.
Recall the addition & multiplication in R
$+ : \mathrm{R} \times \mathrm{R} \to \mathrm{R}((a,b) \mapsto a + b)$
$\times : \mathrm{R} \times \mathrm{R} \to \mathrm{R}((a,b) \mapsto a \cdot b = ab)$

<u>**Definition**</u>. *Let X is a nonempty set A, binary operation on X is a function, $o : X \times X \to X$*

<u>**Definition**</u>. *Let F be a nonempty set, we say that F is a field $((F, +, \cdot)$ is a field) if there are two binary operator called addition " $+$ " and multiplication " $\cdot$ " on F property*

Axioms for *" $+$ "*

   (A1) Commutative: $\forall x, y \in \mathrm{F}, \ x + y = y + x$
   (A2) Associative: $\forall x, y, z \in F, (x + y) + z = x + (y + z)$
   (A3) Additive identity or zero element: $\exists \, 0 \in F \implies x + 0 = 0 + x = x, \ \forall x \in \mathrm{F}$
   (A4) Additive inverse on negative: For each $x \in X, \ \exists \, -x \in F \implies x + (-x) = (-x) + x = 0$

i.e. $(\mathrm{F}, +)$ is an abelian group ***Axioms for multiplication***

   (M1) Commutative: $\forall x, y \in \mathrm{F}, \ xy = yx$
   (M2) Associative: $\forall x, y, z \in \mathrm{F}, \ (xy)z = x(yz)$
   (M3) Muti identity: $\exists 1 \neq 0$ in $\mathrm{F} \ni x1 = 1x = x$
   (M4) Multiplicative inverse: For each $x \neq 0, \exists x^{-1} \in \mathrm{F} \implies xx^{-1} = x^{-1}x = 1$

i.e. $(F = F \cdot \{ 0 \}, \cdot)$ is an abelian group
Distributive Law

   (D1) $\forall x, y, z \in \mathrm{F}, \ (x, y)z = xz + yz \ \& \ x(y + z) = xy + xz$

Induction from Axioms
let $(\mathrm{F}, +, \cdot)$ be a field, we list a series of basic identity as you learn in high school in the real number system

(a) Cancellation law for " $+$ " : $x + y = x + z \implies y = z$

$\because x + y = x + z \implies (-x) + (x + y) = (-x) + (x + z) \implies$
$((-x) + x) + y = ((-x) + x) + z$
$\implies 0 + y = 0 + z \implies y = z$

(b) 0 is "1"

suppose $0' \in F$ is another element satisfy A_3, then $0 = 0 + 0' = 0'$

(c) $x + y = x \implies y = 0$ by (a) $\because x + y = x + 0 \implies y = 0$

(d) negative $-x$ of x is "1"

if $x' \in F$, is another negative of x, them $x + x' = x' + x = 0$
From $x + x' = 0 \implies (-x) + (x + x') = -x + 0 = -x$

(e) $x + y = 0 \implies y = -x$

$x + y = 0 \implies (-x) + (x + y) = (-x) + 0 \implies ((-x) + x) + y = -x$
$\implies 0 + y = -x \implies y = -x$

(f) $-(-x) = x$

$-(-x) + (-x) = 0$, By (d) $x = -(x)$

(a') cancellation law

if $x \neq 0$, then $xy = xz \implies y = z$, $\because (x^{-1})(xy) = (x^{-1})(xz)$
$\implies (x^{-1})(xy) = (x^{-1}x)z \implies 1y = 1z \implies y = z$

(b') 1 is "1"

if $1'$ is another identity, then $1 = 11' = 1'$

(c') $x \neq 0$ & $xy = x \implies y = 1$

$xy = x1 \implies y = 1$

(d') For $x \neq 0$ in F, x^{-1} is "1"

if x is another one, i.e. $x'x = xx' = 1 \implies (x^{-1})(xx') = (x^{-1})1 = x^{-1}$

(f') $x \neq 0 \implies (x^{-1})^{-1} = x$

$(x^{-1})^{-1}(x^{-1}) = 1 \implies x = (x^{-1})^{-1}$

(g') $0x = x0 = 0$

$(0 + 0)x = 0x + 0x \implies 0x = 0$

(h') $x \neq 0$ & $y \neq 0 \implies xy \neq 0$, equivalently $xy = 0 \implies x = 0$ or $y = 0$

$\because xy = 0$ then $(x^{-1})(xy) = ((x^{-1})x)y = 1y = y(\to\leftarrow)$

(i') $(-x)y = -(xy) = x(-y)$

$\because [(-x) + x]y = 0y = 0 = (-x)y = -(xy) \implies (-x)y = -(xy)$

(j') $(-x)(-y) = xy$

$\because (-x)(-y) = -(x(-y))$ by (i)
$= -(-(xy)) = xy$

(k) $-x = (-1)x$

  $\because (1-1)x = 0x = 0 = 1x + (-1)x = x + (-1)x \implies (-1)x = -x$

<u>Definition</u> (Order Field). *Let* F *is a field, we say that* F *is an order field if there is an ordering " $<$ " satisfying*

  (1) if $x < y$, then $x + z < y + z$, $\forall z \in F$
  (2) if $x > y$ and $y > 0$, then $xy > 0$

Example. Q *and* R *are order field under the usual ordering*
Some basic properties of ordered field, let F *be an ordered field with ordering " $<$ "*

  (a) $x > 0 \implies -x < 0$
   $\because x > 0 \implies x + (-x) > 0 + (-x) \implies 0 > -x$
  (b) $x > y \Leftrightarrow x - y > 0$
   $\because x > y \implies x + (-y) > y = (-y) \implies x - y > 0$
   $x - y > 0 \implies x - y + y > y \implies x + 0 > y \implies x > y$
  (c) $x > 0$ and $y < z \implies xy < xz$
   $\because x > 0$ and $y < z \implies x > 0$ and $z - y < 0 \implies x(z-y) > 0 \implies xz + x(-y) > 0$
   $\implies xz - xy > 0 \implies xz > xy$
  (d) $x < 0$ and $y < z \implies xy > xz$
   $\because x < 0$ and $y < z \implies -x > 0$ and $z - y > 0 \implies (-x)(z-y) > 0 \implies -xz + xy > 0$
   $\implies xy > xz$
  (e) $\forall x \neq 0$ in F, $x^2 > 0$
   $\because x > 0 \implies x \cdot x > x0$ by (c) or
   $x < 0 \implies -x > 0$ by (a) $\implies -x > 0$ by (a) $\implies (-x)^2 > 0 \implies x^2 > 0$
  (f) $1 > 0$, $-1 < 0$
   $\because 1 \neq 0 \implies 1^2 > 0$ by (e) $\implies 1 > 0$
  (g) $0 < x < y \implies 0 < \dfrac{1}{y} < \dfrac{1}{x}$

   $\because$ *Note that $\forall u \in$ F, $u > 0 \implies \dfrac{1}{u} = u^{-1} > 0$*

   $\because$ *if $\dfrac{1}{u} < 0$, then $u \cdot \dfrac{1}{u} < 0$ by (e) $\implies 1 < 0 (\to\leftarrow) \therefore \dfrac{1}{u} > 0$*

   Now, $\dfrac{1}{x}, \dfrac{1}{y} > 0$ from $x < y$ we get $(\dfrac{1}{x} \cdot \dfrac{1}{y})x < (\dfrac{1}{x} \cdot \dfrac{1}{y})y \implies$

   $0 < \dfrac{1}{y} < \dfrac{1}{x}$

Remark. *By (e)(f), we conclude that* C *is not an ordered field*
$\because$ C *were an ordered field, then by (e),* $i^2 > 0 \implies -1 > 0(\to\leftarrow)$
$\therefore$ C *is not an order field*

1.5. The Real Number Field R.

Theorem. *There exists an ordered field* R *containing* Q *which has the l.u.b. property. Moreover, such* R *is unique up to order-isomorphism i.e. if* " $<$ " *and* " $<'$ " *are two orders on* R*, them* $\exists f_i(R, <) \to (R, <')$
$\implies$

  (i) f *is a field isomorphism,*
      i.e. $\forall a, b \in R,\ f(a+b) = f(a)+f(b),\ f(ab) = f(a)f(b),\ f(1) = 1$
  (ii) f *preserves ordering,* $a < b \implies f(a) < f(b)$

Such R *is called the real number field or real number system or real line*

Theorem.
  (a) *The Archimedean property of* R : *Given* $x, y \in R$ *with* $x > 0,\ \exists\, n \in N \implies nx > y$
  (b) Q *is dense in* R : $\forall x, y \in R$ *with* $x \leq y,\ \exists\, r \in Q \implies x < r < y$

Proof.
  (a) Let $A = \{\, nx \mid n \in N \,\} \subseteq R$
      if (a) were false, them A is bdd above by y, since R has the l.u.b property
      $\alpha = \sup A$ exists in R, since $x > 0,\ \alpha - x < \alpha \implies \alpha - x$ is not an upper bdd of A
      $\implies \exists m \in N \ni mx > \alpha - x \implies (m+1)x > \alpha(\to\leftarrow)$
  (b) Since $x < y,\ y - x > 0$, by (a),$\exists n \in N \implies n(y-x) > 1$
      By (a) again, $\exists m_1, m_2 \in N \implies m_1 = m_1 1 > n_x \ \& \ m_2 = m_2 \cdot 1 > -nx$
      we have $-m_2 < nx < m_1$, choose $m \in Z \implies -m_2 \leq m \leq m_1 \ \& \ m - 1 \leq nx < m$
      (in fact, $m = [nx] + 1$,where $[z]$ in the greatest integer of z)
      we have $nx < m < 1 + nx < ny(\because n(y-x) > 1) \implies x < \dfrac{m}{n} < y$
        Let $r = \dfrac{m}{n} \in Q$, then $x < r < y$

$\blacksquare$

An application of the density property of Q in R :

Given $x \in R - Q$ i.e. x is an irrational numbers, i.e. $\forall \epsilon > 0, \exists r \in Q \implies |x - r| < \epsilon$

equivalently, $\exists$ a sequence $\{ r_n \}$ in Q $\implies r_n \to x$

In fact, one may choose $\{ r_n \}$ to $\uparrow$ or $\downarrow$

$\because \forall n \geq 1, \exists r_n \in Q \implies x < r_n < \dfrac{1}{n} + x$ by Thm.1.3(b) By squeezing lemma, $r_n \to x$ on $n \to \infty$

Theorem (existence of nth root). *Given $x \in T, x > 0$ & $n \in N, \exists$ "1" $y > 0 \implies y^n = x$*

Such y is called the nth root of x & denoted by $y = \sqrt[n]{x} = x^{\frac{1}{n}}$

Proof. **not important**

"1". Suppose $y_1, y_2 > 0 \implies y_1^n = x$ & $y_2^n = x$

Bt trichotomy, we have

    (i) $0 < y_1 < y_2 \implies y_1^n < y_2^n (\to\leftarrow)$
    (ii) $0 < y_2 < y_1 \implies y_2^n < y_1^n (\to\leftarrow)$
    (iii) $y_1 = y_2$

"$\exists$". Let $E = \{ t \in R \mid t^n < x \}$

Claim:

- $E \neq \emptyset$, Let $t = \dfrac{x}{1 + x}$, then $0 < t < 1$, hence $t^n < t < x$, $\therefore t \in E$ & $E \neq \emptyset$

- E is bdd above, in fact E is bdd above by $1 + x$ if $t > 1 + x > 1$, then $t^n > t > x$, so E is bdd above by $1 + 1$
    Therefore $y = \sup E$ exists & is finite

- Claim $y > 0$ & $y^n = x$, clearly, $y > 0 (\because \dfrac{x}{1 + x} \in E$ & $\dfrac{x}{1 + x} > 0)$
    by trichotomy, we have $y^n < x,\ y^n > x,\ y^n = x$

Now, to show that (i) & (ii) are impossible, do (iii) holds $y^n = x$

By the identity, $b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$

(i)$y^n < x$ choose $0 < h < 1 = \alpha$ & $0 < \dfrac{x - y^n}{n(y + 1)^{n-1}}$, $0 < h < \min \{ \alpha, \beta \}$

put $a = y,\ b = y + h$ in $(\star)$, we obtain

$$(y + h)^n - y^n < hn(y - h)^{n-1} < hn(y + 1)^{n-1} < x - y^n$$
$$\implies (y + h)^n < x \implies y + h \in E\ \& y + h > y(\to\leftarrow) \therefore \text{(i) fails}$$

(ii) $y^n > x$, Let $k = \dfrac{y^n - x}{ny^{n-1}}$, Then $0 < k < y$, $k = \dfrac{y^n - x}{ny^{n-1}} < \dfrac{y^n}{ny^{n-1}} = \dfrac{y}{n} < y$

if $t > y - k > 0$, then $y^n - t^n \le y^n - (y-k)^n < kny^{n-1}$ by $(\star) = y^n - x$
$\implies t^n > x \implies t \in E \implies E$ is bdd above by $y - k \implies \sup E \le y - k(\to\leftarrow)$
$\therefore$ (ii) fails $\blacksquare$

Corollary. *Let* $a, b \in \mathbb{R}$ *with* $a, b > 0$, $n \in \mathbb{N}$ *Then*$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}$
$\because a^{\frac{1}{n}}, b^{\frac{1}{n}} > 0$ & $(a^{\frac{1}{n}} \cdot b^{\frac{1}{n}}) = ab$, *By (1) in Thm 1.4* $(a,b)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}$

infinite in $\mathbb{R}$

After discuss the real number $\mathbb{R}$, sometimes, we have to work with the extended real number system $\mathbb{R}^* = [-\infty, \infty] = \mathbb{R} \cup \{+\infty, -\infty\}$ with observe, $x \in \mathbb{R}$

$$\lim_{n\to\infty}(-n) = -\infty, \ \lim_{n\to\infty} n = \infty, \ \lim_{n\to\infty}(\frac{1}{n}+n) = \infty, \ \lim_{n\to\infty}(n^2-n) = \infty$$
$$x \pm \infty = \pm\infty, \ 0 \cdot (\pm\infty) = 0, \ \infty - \infty \text{ is not define}$$

Element in $\mathbb{R} \subseteq \mathbb{R}^*$ are called finite. Now, given any nonempty subset $E \subseteq \mathbb{R}$,

$$\sup E = \begin{cases} +\infty \text{ if } E \text{ is not bdd above} \\ \text{finite if } E \text{ is bdd above} \end{cases} \ \& \ \inf E = \begin{cases} -\infty \text{ if } E \text{ is not bdd below} \\ \text{finite if } E \text{ is bdd below} \end{cases}$$

Note that if $A \subseteq B$, then $\sup A \le \sup$ & $\inf A \ge \inf B$
$\therefore \emptyset \subseteq B$, $\forall B \subseteq \mathbb{R}$, One may define $\sup \emptyset = -\infty, \inf \emptyset = +\infty$

1.6. The Complex Number Field $\mathbb{C}$.

Consider the contention product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a,b) \mid a, b \in \mathbb{R}\}$
Note that $(a,b) = (c,d) \Leftrightarrow a = c$ & $b = d$, From now, we can write $\mathbb{C} = \mathbb{R}^2$

Operation on $\mathbb{C}$ Given $(a,b), (c,d) \in \mathbb{C}$
    (1) $(a,b) + (c,d) = (a+c, b+d)$
    (2) $(a,b)(c,d) = (ac - bd, ad + bc)$
It is easy to see that, with these operations, $\mathbb{C}$ is a field.

Note that

· the zero element is $(0,0)$
· the negative of (a,b) is $-(a,b) = (-a,-b)$
· the identity is $(1,0)$
· if $(a,b) \neq (0,0)$, then $(a,b)^{-1} = \left(\dfrac{1}{a^2+b^2}, \dfrac{-b}{a^2+b^2} \right)$

R is a subset of C (not vary important)
consider that map

$$f : \mathbb{R} \to \mathbb{C} \text{ define by } f(a) = (a,0), \ a \in \mathbb{R}$$

we have $(1) f$ is injective $(2) f(1) = (1,0) \ \because \forall \ a,b \in \mathbb{R}$
$f(a+b) = (a+b,0) = (a,0) + (b,0) = f(a) + f(b), \ f(a \cdot b) = (ab,0) = (a,0) \cdot (b,0)$
f is a field homomorphism
$\therefore f : \mathbb{R} \to \mathbb{C}$ is an injective and isomorphism
Therefore, we identify $\mathbb{R}$ with $f(\mathbb{R})$ through the injective f
i.e. $a \in \mathbb{R}$ is identified with $f(a,0)$ in $\mathbb{C}$
$ab = (a,0) \cdot (b,0), \ a+b = (a,0) + (b,0) \ \forall \ a,b \in \mathbb{R}$

Change (a,b) to $a+bi$

Now, we can transform an element $(a,b) \in \mathbb{C}$ into the normal form:
(a,b) = (a,0)+(0,b) = (a,0)(1,0)+(b,0)(0,1) = a1+bi = a+ib,
 where $i = (0,1)$
Therefore, from new on, we write $\mathbb{C} = \{\, a+ib \mid a,b \in \mathbb{C} \,\}$
An element $z = a+ib \in \mathbb{C}$ is called a complex number
Hence, under this notification, $z = a+ib, w = c+id \in \mathbb{C}$

(1) $z + w = (a+c) + i(b+d)$
(2) $zw = (ac-bd) + i(ad+bc)$

and the a is called the real part of z, $a = \mathrm{Re}(z)$, b is called imaginary part of z, $b = \mathrm{Im}z$

Some basic properties of complex numbers whose proofs are easy

$\forall z, w \in \mathbb{C}$

- $\overline{z+w} = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z} \cdot \overline{w}$
- $\operatorname{Re} z = \dfrac{z + \overline{z}}{2}$

- $\operatorname{Im} z = \dfrac{z - \overline{z}}{2i}$
- $|z| = 0 \Leftrightarrow z = 0$
- Triangle inequality $|z + w| \leq |z| + |w|$

- $||z| - |w|| \leq |z - w|$
- $\mathbb{C}$ is not an ordered field
- $|z|^2 = z\overline{z}$

- $|\overline{z}| = |z|$
- $|\operatorname{Re} z| \leq |z|, |\operatorname{Im} z| \leq |z|$
- $|zw| = |z||w|$

Proof. $|z + w| \leq |z| + |w|$
$|z + w|^2 = (z + w)(\overline{z + w}) = (z + w)(\overline{z} + \overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w}$
$= |z|^2 + 2\operatorname{Re}(z\overline{w}) + |w|^2 \leq |z|^2 + 2|z\overline{w}| + |w|^2 = |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2$
$\therefore |z + w| \leq |z| + |w|$ ∎

Theorem (basic algebraic theorem).
(a) $x^2 + 1$ has no root in $\mathbb{R}$
(b) $x^2 + 1$ has two distinct roots in $\mathbb{C}$

Proof.

(a) $1 > 0$, $x^2 > 0$, $\forall x \in \mathbb{R} - \{0\} \implies x^2 + 1 > 0 \ \forall x \neq 0$
  $0^2 + 1 = 1 > 0$, $\therefore x^2 + 1 > 0$, $\forall x \in \mathbb{R}$. Hence, $x^2 + 1 = 0$ has no root in $\mathbb{R}$
(b) $i^2 = (0,1)(0,1) = (0 - 1, 0) = (-1, 0) = -1$
  $(-i)^2 = (-(0,1))^2 = (0,-1)^2 = (0,-1)(0,-1) = -1$, $\therefore \pm i$ are root of $\mathbb{C}$

∎

Conclusion: Every non const polynomial $f(x) \in \mathbb{R}[x]$ has n roots where $n = \deg f(x)$
The complex root is even

> **no important proof**
> $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x], \ a_n \neq 0, \ n \geq 1$
> if $\alpha = a + ib \in \mathbb{C}$ is a root of $f(x)$,then
> $0 = f(\alpha) = a_n \alpha^n + a_{n-1} \alpha^{n-1} + \cdots + a_1 \alpha + a_0$
> $0 = f(\overline{\alpha}) = a_n \overline{\alpha}^n + a_{n-1} \overline{\alpha}^{n-1} + \cdots + a_1 \overline{\alpha} + a_0$
> $\therefore (x - \alpha)|f(x), \ (x - \overline{\alpha})|f(x) \implies (x - \alpha)(x - \overline{\alpha})|f(x) \implies$
> $(x^2 - (\alpha - \overline{\alpha})x + |\overline{\alpha}|^2) \mid f(x)$
> $\implies (x^2 - 2ax + (a^2 + b^2)) \mid f(x)$
> $\therefore$ quadratic function must have two roots in $\mathbb{C}$

The fundamental Theorem of Algebra
Every non zero polynomial $f(x) \in \mathbb{C}[x]$ has at least one root in $\mathbb{C}$
Therefore, if $\deg f(x) = n$, then $f(x)$ has n roots in $\mathbb{C}(C, M)$

> $\because f(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_t)^{e_t}(a_1 x^2 + b_1 + c_1)^{l_1} \cdots (a_s x^2 + b_s x + c_s)^{ls}$, where $\lambda_1, \cdots, \lambda_t \in \mathbb{R}, \ a_i, b_i, c_i \in \mathbb{R} \& e_1 + \cdots + e_t + 2l_1 + \cdots + 2l_s = \deg f(x)$ which shows that all roots of $f(x)$ are in $\mathbb{C}$
> In fact, we have the famous theorem: The fundamental theorem of algebra
> Every non zero polynomial $f(x) \in \mathbb{C}[x]$ has at least one root in $\mathbb{C}$
> $\therefore$ if $\deg f(x) = n$, then $f(x)$ has n roots in $\mathbb{C}(C, M)$

Theorem (Cauchy-Scheming Inequal). *Given* $z_1 \cdots, z_n, w_1, \cdots, w_n \in \mathbb{C}$, *we have*

$$\left| \sum_{j=1}^{n} z_j \overline{w}_j \right| \leq \left(\sum_{j=1}^{n} |z_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} |w_j|^2 \right)^{\frac{1}{2}}$$

and " = " holds $\Leftrightarrow \exists \ \lambda \in \mathbb{C} \ni w_j = \lambda z_j, \ 1 \leq j \leq n,$
In patricial, if $x_1, \cdots, x_n, y_1, \cdots, y_n \in \mathbb{R}$, *then*

$$\left| \sum_{j=1}^{n} x_j y_j \right| \leq \left(\sum_{j=1}^{n} x_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} y_j^2 \right)^{\frac{1}{2}}$$

and " = " holds $\Leftrightarrow \exists \ t \in \mathbb{R} \ni y_j = tx_j, \ 1 \leq j \leq n$

The proof is too long, I am lazy

1.7. Euclidean Spaces $\mathbb{R}^n$.

<u>Definition</u>. *the n-dimensional Euclidean space* $\mathbb{R}^n$

$$= \{\, x = (x_1, \cdots, x_n) \mid x_i \in \mathbb{R}, 1 \leq i \leq n \,\} = \mathbb{R} \times \cdots \times \mathbb{R}$$

Note that

$$(x_1, \cdots, x_n) = (y_1, \cdots, y_n) \Leftrightarrow x_i = y_i \; \forall \; 1 \leq i \leq n$$

We are going to introduce the structure of $\mathbb{R}^n$

- vector space
- normed linear space
- inner product space
- matrix space

<u>Definition</u>. *Two operation on* $\mathbb{R}^n$ *as follows:*
- *Addition* $+ : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $(x, y) \mapsto x + y = (x_1 + y_1, \cdots, x_n + y + n)$
- *Scalar multiplication* $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $(a, x) \mapsto ax = (ax_1, \cdots, ax_n)$

we skip space example here.

1.8. Countability of Sets.

Given two nonempty set A, B and a function $f : A \to B$, $f(A) = \{\, f(a) \mid a \in A \,\}$ is called the image of A under f

Some basic things

> $E \subseteq A$, $f(E) = \{\, f(a) \mid a \in E \,\}$ the image of E under f
> f is infective(one-to-one) $x_1 \neq x_2 \implies f(x_1) \neq f(x_2) \Leftrightarrow f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$
> f is surjective(onto) if $f(A) = B$, f is bijective if f is one-to-one and onto

Given $F \subseteq B$, $f^{-1}(F) = \{\, x \in X \mid f(x) \in F \,\}$ called the inverse image of f under F

Example

> $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, $x \in \mathbb{R}$
> $f^{-1}([0, 1]) = \{\, x \in \mathbb{R} \mid f(x) \in [0, 1] \,\} = \{\, x \in \mathbb{R} \mid x^2 \in [0, 1] \,\} = [-1, 1]$
> $f^{-1}([-1, 1]) = [-1, 1]$

Properties of inverse image

- $F_1 \subseteq F_2 \subseteq B \implies f^{-1}(F_1) \subseteq f^{-1}(F_2)$
- Inverse image presences set operation

  $\forall F_\alpha \subseteq B, \ \alpha \in I, \ F \subseteq B$

  (i) $f^{-1}(\cup_{\alpha \in I} F_\alpha) = \cup_{\alpha \in I} f^{-1}(F_\alpha)$

  (ii) $f^{-1}(\cap_{\alpha \in I} F_\alpha) = \cap_{\alpha \in I} f^{-1}(F_\alpha)$

  (iii) $f^{-1}(B - F) = f^{-1}(B) - f^{-1}(F)$
- Given $S \subseteq A, \ f'(f'(S)) \supseteq S,$ " $=$ " $\Leftrightarrow$ one-to-one, **example:**

  $f : \mathbb{R} \to \mathbb{R}, \ f(x) = x^2, \ S = [0,1], \ f(S) = [0,1], \ f^{-1}(f(S)) = f^{-1}([0,1]) = [-1,1]$
- Given $F \subseteq B, \ f(f^{-1}(F)) \subseteq F,$ " $=$ " $\Leftrightarrow$ "onto", **example**

  $f(x) = x^2, \ x \in \mathbb{R}, \ F = [-1,1], \ f(f^{-1}([-1,1])) = f([-1,1]) = [0,1]$
- For $y \in B, \ f^{-1}(\{y\}) = f^{-1}(y) = \{x \in A \mid f(x) = y\}$ the inverse image of y, **example**

  $f : \mathbb{R} \to \mathbb{R}, \ f(x) = x^2, \ f^{-1}(1) = \{1, -1\}, \ f^{-1}(2) = \emptyset$

Definition (cardinality). *Let A, B are two set ew say that A and B have the same cardinality if $\exists$ a bijective map $f : A \to B$, which is denoted by $A \sim B$*

From now on, we write $|A|$ as the cardinality of A

Claim " $\sim$ " is an $\equiv$ relation among all sets

(i) Reflexion: $\forall$ set $A, \ A \sim^{1A} A,$ which 1_A is identity mapping

(ii) Symmetry: $A \sim^f B \implies B \sim^{f^{-1}} A$

(iii) Transitive: $A \sim^f B \ \& \ B \sim^g C \implies A \sim^{g \circ f} C$

So we gave some property:

- Any two " $\equiv$ " are either disjoint or identical
- $\overline{X}$ is a disjoint union of " $\equiv$ " classes

  $[A] = \{B \in \overline{X} \mid B \sim A\}$ the " $\equiv$ " class set by A

Ant two element in an " $\equiv$ " class have the same cardinality

Notation For $n \in \mathbb{N}, \ \mathbb{N}_m = \{1, 2, \cdots, n\}$

Definition. *Let A be a set*

(a) A is a finite set if $A = \emptyset$ or $A \sim \mathbb{N}_n$ for some $n \in \mathbb{N}$

(b) A is a infinite set if A is not a finite set

(c) A is countable if $A \sim \mathbb{N}$

(d) A is uncountable if A is not countable.

(e) A is at most countable if A is finite or countable

Remark.

    (1) when A, B are finite sets, $A \sim B \Leftrightarrow |A| = |B|$, i.e. A, B have same number.

    (2) where A, B are infinite and $A \sim B$, i.e. $|A| = |B|$, the concept is abstract.

    (3) $\{ a, b, c \} \cup \mathbb{N} \sim \mathbb{N}$, $f : \mathbb{N} \to \{ a, b, c \} \cup \mathbb{N}$, $f(1) = a$, $f(2) = b$, $f(3) = c, \cdots$

    (4) Any finite set can not equivalent to a proper subset, i.e. A is finite, $B \subseteq A$

        Then $A \sim B$, In fact $|B| < |A|$, but infinite different

    (5) Any finite set A can be listed an $A = \{ a_1, \cdots, a_n \}$ where $n = |A|$

Now, we consider the case of countable set

> Recall, in calculus, a real sequence $\{ a_n \}$, e.g.
>
> $$a_n = \frac{1}{n} \left\{ \frac{1}{n} \right\}, \quad a_n = 1 - \frac{1}{n} \left\{ 1 - \frac{1}{n} \right\}, \quad a_n = \begin{cases} 0 \text{ if } n \text{ is odd} \\ 1 \text{ if } n \text{ is even} \end{cases}$$

<u>**Definition.**</u> *Let X be a nonempty set, a sequence in X is a function $a : \mathbb{N} \to X$*

Given a sequence $=^a$ in X, a is "1" determine by $a(n), \in \mathbb{N}$

We write

$$a = \{ a(1), a(2), \cdots, a(n), \cdots \} = \{ a_1, a_2, \cdots, a_n, \cdots \} = \{ a_n \} = \{ a_n \}_{n=1}^{\infty}$$

Remark.

    (1) For a sequence $\{ a_n \}$ in X, a_n may not be distinct.

        If all a_n are distinct, then we say that $\{ a_n \}$ is a distinct sequence in X.

    (2) We usually use $\{ a_n \}, \{ b_n \}$ to denote sequence

    (3) A sequence $\{ a_n \}$ in X in fact is a function from $\mathbb{N} \to X$, So $\{ a_n \mid n \in \mathbb{N} \}$ is the image of the sequence.

    (4) $\{ a_n \}$ is a sequence, a_n is called the nth term of the sequence.

    (5) A sequence in X may begin at 0, i.e. $\{ a_n \}_{n=0}$

        By a changing index, we can make it from $\{ b_n \}_{n=1}^{\infty}$, $b_n = a_{n+1}$, $n = 1, 2, \cdots$

<u>**Definition**</u> (increasing).

A function $a : \mathbb{N} \to \mathbb{N}$ is increasing, a is $\uparrow$, if $a(n) \leq a(n+1) \ \forall \ n \geq 1$

a is strictly increasing, a is st. $\uparrow$, if $a(n) < a(n+1) \ \forall \ n \geq 1$

Now, given a st. $\uparrow$ function $n : \mathbb{N} \to \mathbb{N}$, i.e. $n(k) < n(k+1)$, $k \geq 1$
i.e. $n_k < n_{k+1}$, $k \geq 1$, i.e. $n_1 < n_2 < \cdots < n_k < \cdots$, i.e. $\{ n_k \}_{k=1}^{\infty}$ is a
st. sequence in $\mathbb{N}$

<u>Definition.</u> *Let $\{ a_n \}$ be a sequence in X and $\{ n_k \}$ be a st. $\uparrow$ sequence
in $\mathbb{N}$, then the sequence $\{ a_{n_k} \}$ is called a subsequence of $\{ a_n \}$*
In fact

$$\mathbb{N} \to_{st.}^{n} \mathbb{N} \to_{seq}^{a} X \Rightarrow a \circ n : \mathbb{N} \to X \text{ is a function,}$$

hence, it also a sequence in X

$$a \circ n = \{ a \circ n(k) \} = \{ a(n(k)) \} = \{ a_{n(k)} \} = \{ a_{n_k} \}$$

Remark. *if $\{ a_{n_k} \}$ is st. $\uparrow$ in $\mathbb{N}$, then $k \leq n_k \ \forall \ k \geq 1$*
$\because$ *By mathematical Induction*

    $\cdot$ $1 \leq n_1$
    $\cdot$ *Assume it's true for $k \geq 2$, i.e. $k \leq n_k$*
    $\cdot$ *Consider $k + 1$, $k + 1 \leq n_k + 1 \leq n_{k+1}$*

Example
*Let $\{ a_n \}$ be a sequence in X, then $\{ a_{2k} \}$ and $\{ a_{2k-1} \}$ are subsequence
of $\{ a_n \}$*

Finally, we will assume that you are familiar with the following property
of the countability of sets:

    (1) Every subset of a countable set is at most countable. The proof
        needs the well ordering of $\mathbb{N}$: Every nonempty subset of $\mathbb{N}$ has
        the smallest element
    (2) Countable union of countable sets is countable
    (3) If $A_1, A_2, \cdots, A_n$ are countable, then so is $A_1 \times \cdots \times A_n$
    (4) If A is countable, then so is $A^n \equiv A \times \cdots \times A \ \forall n \geq 1$
    (5) $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}^n$, $\forall n \geq 1$ are countable
    (6) The set $\{ a_n \mid a_n = 0 \text{ or } 1 \}$ is uncountable
        This can be proved by Canton diagonal process
        $\because$ if it is countable, then we can list it, $a_0 A = \left\{ a_1^{(1)}, a_2^{(2)}, \cdots \right\}$
        where
        $a^{(1)} = \{ a_n^{(1)} \} = a_1^{(1)}, a_2^{(1)}, \cdots$; $a^{(2)} = \{ a_n^{(2)} \} = a_1^{(2)}, a_2^{(2)}, \cdots$
        Now, construct a sequence $\{ a_n \}$ in $A \ni \{ a_n \} \neq a^{(k)} \ \forall \ k \geq$
    $1 (\to \leftarrow)$

Recall, intervals in $\mathbb{R}$, $-\infty < a \le b < \infty$, following are finite bdd interval

$(a, b) = \{\, x \in \mathbb{R} \mid a < x < b \,\}$ open interval
$[a, b] = \{\, x \in \mathbb{R} \mid a \le x \le b \,\}$ closed interval
$(a, b] = \{\, x \in \mathbb{R} \mid a < x \le b \,\}$ open-closed
$[a, b) = \{\, x \in \mathbb{R} \mid a \le x < b \,\}$ closed-open

An interval I in $\mathbb{R}$ is said to be non-degenerate if the endpoint of I are distinct i.e. length > 0. Otherwise, it is degenerate.

Note.

$(0, 1)$ is uncountable, $\because (0, 1) = \left\{\, \sum_{n=1}^{\infty} \frac{a_n}{2^n} \mid a_n = 0 \text{ or } 1, n \in \mathbb{N} \,\right\}$

$x \in (0, 1)$ has a unique binary representation, so $(0, 1) \sim A$, where A is $\{\, \{\, a_n \,\} \mid a_n = 0 \text{ or } 1 \,\}$ which is uncountable

All non-degenerate intervals in $\mathbb{R}$ are uncountable.
$\because$ It sufficient to consider bdd non-degenerate interval in $\mathbb{R}$, given $\infty < a < b < \infty$
(a, b) is uncountable($\because (0, 1) \sim (a, b)$)
Note that $(0, 1) \sim \mathbb{R}(\because (0, 1) \to (\frac{\pi}{-2}, \frac{\pi}{2}) \to \mathbb{R})$

2. Basic Point Set Topology

To know the "closeness", "limit" and "continue"

Notation. Let X be a nonempty set. The power set of X is denoted by $p(X)$ or 2^X, i.e. $\mathscr{P}(X) = 2^X$ which is the collect of all subset, if $|X| = n$, then $|\mathscr{P}(X)| = 2^n$

2.1. Topological Spaces.

Definition. *Let X be a nonempty set and $\mathscr{T} \subseteq \mathscr{P}(X)$, we say that $\mathscr{T}$ is a topology on X if it satisfies*

(1) $\emptyset, X \in \mathscr{T}$

(2) $\mathscr{T}$ is closed under arbitrary union,

  i.e. $U_\alpha \in \mathscr{T}, \ \alpha \in I \implies \bigcup_{\alpha \in I} U_\alpha \in \mathscr{T}$

(3) $\mathscr{T}$ is closed under finite intersection

  i.e. $U_1, \cdots, U_n \in \mathscr{T} \implies U_1 \cap \cdots \cap U_n \in \mathscr{T}$

In this chapter, the pair (X, J) or simply J is called a topological space and members in T are called open set in X or open subsets of X

Remark.

(1) X: a nonempty set, there is at least two trivial topology on X
- *$\mathscr{P}(x)$ is the largest topology on X w.r.t inclusion X with this topology is called a discrete topological space*
- *$\mathscr{T}_0 = \{\emptyset, X\}$ is the smallest topology on X w.r.t inclusion X with this topology is called an indiscrete topological space*

(2) How many topology can be define on $\{a\}$, $\{a, b\}$?

In the following X is a topology space

Definition (neighborhood). *Let $p \in X$, a neighborhood of P is an open set U containing p*

Definition (Hausdorff space). *X is a Hausdorff space if any two distinct points can be separated by open set, i.e. $\forall \ p \neq q$ in X, $\exists$ neighborhood U of p and V of $q \in U \cap V = \emptyset$*

Definition (closed set). *A subset $F \subseteq X$ is said to be closed if $F^C = X - F$ is open in X*

Theroem 2.1. *The collection of all closed subsets of X satisfied*

(a) $\emptyset, X$ are closed

(b) Arbitrary intersection of closed set if closed

(c) Finite union of closed sets is closed

Proof.

(a) $X - \emptyset = X$ is open $\therefore \emptyset$ is closed
$X - X = \emptyset$ is open $\therefore X$ is closed

(b) Given closed sets $F_\alpha, \alpha \in I$, $X - \bigcap_{\alpha \in I} F_\alpha = \bigcup_{\alpha - I}(X - F_\alpha)$ is

open,$\therefore \bigcap_{\alpha - I} F_\alpha$ is closed.

(c) Given closed set $F_1, \cdots, F_n$, $X - \bigcup_{i=1}^{n} F_i = \bigcap_{i=1}^{n}(X - F_i)$ is open,

$\therefore \bigcup_{i=1}^{n} F_i$ is closed.

∎

Definition. *Let $Y \subseteq X$ and*

$$\mathscr{T}_y = \{U \cap Y \mid U \text{ is open in } X\}$$

Theroem 2.2. *$\mathscr{T}_Y$ is also a topology space*

Proof. To proof $\mathscr{T}_Y$ is a topology space, we take the topology's definition

(a) $\emptyset, Y \in \mathscr{T}_Y$ $(\because \emptyset = \emptyset \cap Y, \; Y = X \cap Y)$

(b) Given $U_\alpha \cap Y \in \mathscr{T}_Y$, $\alpha \in I$, where U_α is open in X
$$\bigcup_{\alpha \in I}(U_{\alpha \cap Y}) = (\bigcup_{\alpha \in I} U_\alpha) \cap Y \implies \bigcup_{\alpha \in I}(U_\alpha \cap Y) \in \mathscr{T}_Y$$

(c) Given $U_1 \cap Y, \cdots, U_n \cap Y$, where U_i is open in X, $1 \leq i \leq n$
$$\cap_{i=1}^{n}(U_i \cap Y) = (\cap_{i=1}^{n} U_i) \cap Y \implies \bigcap_{i=1}^{n}(U_i \cap Y) \in \mathscr{T}_Y$$

$\therefore \mathscr{T}_Y$ is a topology on Y

∎

Definition. *In Theorem 2.2, with the topology $\mathscr{T}_Y$ on Y, is called a topological subspace of X and $\mathscr{T}_Y$ is called the relative topology of Y in X. Members in $\mathscr{T}_Y$ are called open set in Y or relative open sets in Y.*

2.2. Metric Spaces & Subspace.

In this chapter, we will introduce a class of topology space whose topology in induced by a metric.

Definition. *Let X be a nonempty set. A metric or distance function in a function*

$$d : X \times X \to \mathbb{R}, \ (a, b) \mapsto d(a, b)$$

satisfying:

(a) $\forall a, b \in X, d(a, b) \geq 0$ and $d(a, b) = 0 \Leftrightarrow a = b$
*(b) $\forall a, b \in X, d(a, b) = d(b, a)$ **symmetry***
*(c) $\forall a, b, c \in X, d(a, b) \leq d(a, c) + d(a, b)$ **triangle inequaity***

if d is a metric on X, then the pair (X, d) or simply X is called a metric space and $\forall a, b \in X, \ d(a, b)$ is called the distance between a & b

Examples

(1) Let X be a nonempty set define by

$$d(a, b) = \begin{cases} 0 \text{ if } a = b \\ 1 \text{ if } a \neq b \end{cases}$$

Then d is a metric on X, called the discrete metric and with this metric X is called a discrete metric space. In particular, any set admits a metric.

(2) The most important metric spaces are the Euclidean space $\mathbb{R}^k$, the metric d is called the Euclidean or standard or usual metric on $\mathbb{R}^k$. There are other metrics on $\mathbb{R}^k$ induced the same metric topology on $\mathbb{R}^k$, in fact, they are all equivalent, e.g $\forall \ 1 \leq p \leq \infty$, We can define a metric d_p on $\mathbb{R}^k$ as follows

- $1 \leq p < \infty, \ d_p(x, y) = ||x - y||_p = \left(\sum_{i=1}^{k} |x_i - y_i|^p \right)^{\frac{1}{p}}$

- $p = \infty, \ d_\infty(x, y) = \max_{1 \leq i \leq k} |x_i - y_i|$

Note that $d_2 = d$ is the Euclidean metric on $\mathbb{R}^k$

Remark. *In fact, every normed linear space $(V, ||\cdot||)$ is a metric space whose metric is induced by its norm*

(3) Let (X, d) be a metric space and $Y \subseteq X, \ Y \neq \emptyset$. Then the restriction of d to $Y \times Y$ is also a metric on Y, with this metric, Y is called a metric subspace of X

<u>Definition</u> (ball). *Given $p \in X \& r > 0$*
$B(p, r) = \{x \in X \mid d(x, p) < r\}$: *open ball with center p and radius r*
$\overline{B}(p, r) = \{x \in X \mid d(x, p) \leq r\}$: *closed ball with center p and radius r*

Example

(1) The discrete metric space X : $p \in X$, $r > 0$
$$B(p, r) = \begin{cases} \{p\} \text{ if } 0 \leq r \leq 1 \\ X \text{ if } r > 1 \end{cases}$$
$$\overline{B}(p, r) = \begin{cases} \{p\} \text{ if } 0 \leq r < 1 \\ X \text{ if } r \geq 1 \end{cases}$$

(2) In the Euclidean space $\mathbb{R}^k$, $p \in \mathbb{R}^k$, $r > 0$
$B(p, r) = \{x \in \mathbb{R} \mid \; \| x - p \| < r\}$ is a ""true"" open
$\overline{B}(p, r) = \{x \in \mathbb{R} \mid \; \| x - p \| \leq r\}$ is a "true" closed ball
In particular, for $k = 1$ in $\mathbb{R}$
$B(p, r) = (p - r, p + r)$: a symmetric opne interval
$\overline{B}(p, r) = [p - r, p + r]$: a symmetric close interval
However, w.r.t d_1 & d_∞, we have, e.g. in $\mathbb{R}^2$
$B_1(0, 1) = \{(x, y) \mid |x - 0| + |y - 0| < 1\}$
$B_\infty(0, 1) = \{(x, y) \mid \max\{|x|, |y|\} \leq 1\}$

(3) What is the open balls in $S = [0, 1] \subseteq \mathbb{R}$?
$B_S(0, \frac{1}{2}) = \{x \in S \mid |x - 0| < \frac{1}{2}\} = [0, \frac{1}{2}] = B(0, \frac{1}{2}) \cap [0, 1]$
$B_S(0, 3) = [0, 1] = B(0, 3) \cap [0, 1]$

Prop 2.2 Let S be a metric subspace of a metric space X, then $\forall \, p \in S$ & $r > 0$, $B_S(p, r) = B(p, r) \cap S$

Proof. $B_S(p, r) = \{x \in S \mid d(x, p) < r\} = \{x \in X \mid d(x, p) < r\} \cap S$
$= B(p, r) \cap S$ ■

2.3. Open Sets in Metric Spaces.
We will see that every metric on a set induce a topology on X

<u>Definition</u> (interior point). *Let $S \subseteq X$ be a set, and $p \in S$, we say that p is an interior point of S if $\exists \, r > 0$, $\exists \, B(p, r) \subseteq S$*
Denote by S^o or int(S) by the set of all interior point of S

<u>Definition</u> (open). *Let $S \subseteq X$, we say that S is open if all points of S are interior points of S*

Remark.

(1) Every open set S is a union of a open balls in X.
$\because \forall\ x \in S, x$ *is an interior point of* $S, \exists r_x > 0 \ni B(x, r_x) \subseteq S$
$\therefore S = \cup_{x \in S} B(x, r_x)$
(2) $S^o \subseteq S$ *by definition*
(3) S *is open* $\Leftrightarrow S = S^o$

Prop 2.3

(a) $S \subseteq T \implies S^o \subseteq T^o$

$$\because p \in S^o \implies \exists\ r > 0 \ni B(p, r) \subseteq S \subseteq T \implies p \in T^o$$

(b) Every open ball $B(p, r)$ in X is open
$\because$ Give $q \in B(p, r)$, Let $\delta = r = d(p, q)$.Claim $B(q, \delta) \subseteq B(p, r)$ which says q is an interior point of $B(p, r)$.Since $q \in B(p, r)$ is arbitrary, so $B(p, r)$ is open. Given $x \in B(q, \delta)$

$$d(x, p) \leq d(x, q) + d(q, p) < \delta + d(q, p) = r - d(p, q) + d(q, p) = r$$

(c) $\forall\ S \subseteq X,\ S^o$ is always open
$\because$ Given $p \in S^o,\ \exists\ r > 0 \ni B(p, r) \subseteq S$

$$\implies\ B(p, r) \subseteq S^o \implies p \text{ is a interior point of } S^o$$

$\therefore S^o$ is open
(d) $\forall\ S \subseteq X,\ S^{oo} = (S^o)^o = S^o$
$\because$ by definition of open set and (c)

Now, let $T = \{U \subseteq X \mid U \text{ is open in } X\}$
Prop 2.4 T is a topology on X. In particular, X is a topology space.

Proof.

(i) $\emptyset, X \in \mathscr{T}, \because \emptyset = \emptyset,\ X^o = X$
(ii) $U_\alpha \in \mathscr{T}, a \in I$ are open $\implies \bigcup_{\alpha \in I} U_\alpha$ is open

Given an arbitrary point $p \in \bigcup_{\alpha \in I} U_\alpha \implies \exists \alpha_0 \in I \ni p \in U_{\alpha_0}$

U_{α_0} is open, $\exists r > 0 \ni B(p, r) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in I} U_\alpha$

$\because p$ is an interior point of $\bigcup_{\alpha \in I} U_\alpha \therefore \bigcup_{\alpha \in I} U_\alpha$ is open, i.e. $\bigcup_{\alpha \in I} U_\alpha \in T$

(iii) $U_1, \cdots, U_n \in T \implies U_1 \cap \cdots \cap U_n \in T$

$\because$ Given $p \in U_1 \cap \cdots \cap U_n \implies p \in U_i$ $1 \leq i \leq n$. Each U_i is open,

$\exists r_i > 0$ $B(p, r_i) \subseteq U_i,$ $1 \leq i \leq n \implies B(p, r) \subseteq U_1 \cap \cdots \cap U_n$

$\implies p$ is an interior point of $U_1 \cap \cdots \cap U_n$

p is arbitrary, so $U_1 \cap \cdots \cap U_n$ is open, i.e. $U_1 \cap \cdots \cap U_n \in T$

Therefore, T is a topology on X

■

Definition. *Let X be a metric space with metric d. The topology T in prop 2.4 is called the metric topology(include by d)*

Let X be a metric space and $Y \subseteq X$, Then Y is a metric subspace of X, and $\forall y \in Y,$ $r > 0,$ $B_Y(y, r) = B(y, r) \cap Y$. In fact, we have more

Prop 2.5 A subset $A \subseteq Y$ is open in $Y \Leftrightarrow A = U \cap Y$ for some open set U in X, in particular, the metric topology on T is just the relation topology of Y on X

Proof. ($\Rightarrow$) suppose $A \subseteq Y$ is open in Y. Then

$$A = \bigcup_{y \in A} B_Y(y, r_y) = \bigcup_{y \in A} (B(y, r_y) \cap Y) = (\bigcup_{y \in A} B(y, r_y)) \cap Y$$

Let $U = \bigcup_{y \in Y} B(y, r_y)$, then U is open in X and $A = U \cap Y$

($\Leftarrow$) Suppose $A = U \cap Y$ where $U \subseteq X$ is open $\forall y \in A, y \in U \cap Y \implies$ $y \in U \implies \exists r > 0 \ni B(y, r) \subseteq U \implies B(y, r) \cap Y \subseteq U \cap Y = A \implies$ $B_Y(y, r) \subseteq A,$ $\therefore A$ is open in Y. ■

Prop 2.6 Every metric space X is Hausdorff

Proof. Given $p, q \in X,$ $p \neq q$. Choose $r = \dfrac{1}{2} d(p, q) > 0$. Then $B(p, r) \cap B(q, r) = \emptyset$. So X is Hausdorff

$(\because x \in B(p, r) \cap B(q, r) = d(x, p) < r \ \& \ d(x, q) < r \implies d(p, q) \leq d(p, x) + d(x, q) < r + r = 2r = d(p, q)(\to\leftarrow))$ ■

Remark. *Let $S \subseteq X$, where X is a metric space. Then S^o is the largest(w.r.t inclusion) open set contained in S. $\because \forall$ open set $U \subseteq S,$ $U^o \subseteq S^o \implies U \subseteq S^o \subseteq S$. In fact, $S^o = \bigcup_{U \subseteq S} U$ (which is the definition of intension of S in a topology space X)*

2.4. Closed Sets.

Definition (Closed set). *$F \subseteq X$ is closed $\Leftrightarrow F^C = X - F$ is open in X*

By Theorem 2.1, the collection of all close sets in X has the properties

(i) $\emptyset, X$ are closed in X

(ii) F_α is closed in X, $\alpha \in I \implies \bigcap_{\alpha \in I} F_\alpha$ is closed in X

(iii) $F_1, \cdots, F_n$ are closed in $X \implies \bigcup_{i=1}^{n} F_i$ is closed in X

Example

Intersection of infinitely many open set may not be open

in $\mathbb{R}$ with Euclidean topology, $(-\frac{1}{n}, \frac{1}{n})$ is open in $\mathbb{R}$ $\forall\, n \geq 1$ $\implies$

$\bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$ is not open

Prop 2.8 Let X be a metric space and $Y \subseteq X$ and $B \subseteq Y$, Then B is closed in $Y \Leftrightarrow B = F \cap Y$ for some closed set F in X

Proof. ($\Rightarrow$) Suppose B is close in Y $\implies$ $Y - B$ is open in Y $\implies$ $Y - B = U \cap Y$ (by prop 2.5) for some open set U in X $\implies$ $Y - (Y - B) = Y - (U \cap Y)$ $\implies$ $B = (X - U) \cap Y$. where $(X - U)$ is close.
($\Leftarrow$) Suppose $B = F \cap Y$, where F is closed in X $\implies$ $Y - B = Y - (F \cap Y) = (X - F) \cap Y$ $\implies$ $Y - B$ is open in Y $\implies$ B is close in Y. $\blacksquare$

In metic space, one can use sequence to detect the closeness of a set

Example

1. We know that $[a, b)$ is not closed in $\mathbb{R}$, however, $\exists$ a sequence $\{x_n\}$ in $[a, b) \ni x_n \to b$ on $n \to \infty$, e.g. $b - \frac{1}{n} \to b$

2. $A = \{\frac{1}{n} \mid n \geq 1\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ is not close in $\mathbb{R}$
if $\mathbb{R} - A$ will open, them $\exists r > 0$, $B(0, r) \subseteq \mathbb{R} - A (\to \leftarrow)$
$A \cup \{0\}$ is closed in $\mathbb{R}$

$$R \setminus (A \cup \{0\}) = (-\infty, 0) \cup (1, \infty) \cup (\bigcup_{n=1}^{\infty}(\frac{1}{n+1}, \frac{1}{n})) \text{ is open}$$

$\therefore A \cup \{0\}$ is closed

<u>Definition</u> (Adherent, clousure $\cdots$). *Let X be a metric space with metric d, $T \subseteq X$ be a subset.*(**important**)
(1.) A point $p \in X$ is said to be an adherent point of T if $\forall r > 0$, $B(p, r) \cap T \neq \emptyset$, equivalent, $\forall$ neighborhood U of p, $U \cap T \neq \emptyset$
(2.) Let $\overline{T}$ or $cl(T)$ denote the set of all adherent points of T, called the closure of T, i.e. $\overline{T} = \{p \in X \mid p$ is an adherent point of $T\}$

(3.) A point $p \in X$ is said to be a limit point or accumulation point of T if $\forall \, r > 0$, $B(p,r) \cap T - \{p\} \neq \emptyset$, equivalently, $\forall$ neighborhood U of p, $U \cap T - \{p\} \neq \emptyset$
Denote by T' the set of all accumulation points of T, called the devied set of T.
(4.) $p \in T$ and $p \notin T'$, then p is called an isolated point of T, i.e. $\exists r > 0 \ni B(p,r) \cap T = \{p\}$
(5.) A subset $T \subseteq X$ is said to be perfect if T is closed and every points of T is an accumulated point of T, i.e. T is closed & $T' = T$
(6.) A subset $T \subseteq X$ is said to be bounded if $\exists R > 0$ and $p \in X \ni T \subseteq B(p,R)$
(7.) A subset $T \subseteq X$ is said to be dense if $\overline{T} = X$, e.g. $\overline{\mathbb{Q}} = \mathbb{R}$
(8.) A point $p \in X$ is said to be a boundary point of T if $\forall \, r > 0, B(p,r) \cap T \neq \emptyset$ & $B(p,r) \cap (X \setminus T) \neq \emptyset$. Denote by ∂T or bd(T) the set of all boundary points of T

Prop 2.9 Let X be a metric space. All sets and point below are subset of X

(1) $S \subseteq T \implies \overline{S} \subseteq \overline{T}$ & $S' \subseteq T'$
$\because p \in \overline{S} \implies \forall \, r > 0, B(p,r) \cap S \neq \emptyset \implies B(p,r) \cap T \neq \emptyset \implies p \in \overline{T}$
$p \in S' \implies \forall r > 0, B(p,r) \cap S - \{p\} \neq \emptyset \implies B(p,r) \cap T - \{p\} \neq \emptyset$
(2) $\overline{T}$ is always closed in X

> We want to know $\overline{T}$ is closed on $X \to X - \overline{T}$ is open $\to \forall \, p \in X - \overline{T}$ is an interior point $\implies \exists r > 0, \ B(p,r) \subseteq X - \overline{T}$
> $\because p \notin \overline{T} \Rightarrow \exists r' > 0 \ni B(p,r') \cap T = \emptyset$
> But we want to get $B(p,r') \cap \overline{T}$, so we check every point in $B(p,r')$ is not in $\overline{T}$, let $q \in B(p,r')$, $\exists \, \delta > 0$, $B(q,\delta) \subseteq B(p,r') \implies B(q,\delta) \cap T = \emptyset \implies q \notin \overline{T}$
> because if $q \in \overline{T}, \forall r > 0 \ni B(q,r) \cap T \neq \emptyset$
> $\implies B(p,r) \cap \overline{T} = \emptyset$

Let $p \in X - \overline{T} \implies p \notin \overline{T} \implies \exists \, r > 0 \ni B(p,r) \cap T = \emptyset \implies B(p,r) \cap \overline{T} = \emptyset$ ($\because \forall q \in B(p,r), \exists \, \delta > 0 \ni B(q,\delta) \subseteq B(p,r) \implies B(q,\delta) \cap T = \emptyset \implies q \notin \overline{T}$)
$\therefore B(p,r) \subseteq X - \overline{T}, \because p$ is an interior point of $X - \overline{T}$. Hence, $X - \overline{T}$ is open, i.e. $\overline{T}$ is closed.
(3) $T \subseteq \overline{T}(\because \forall \, p \in T, \forall \, r > 0, B(p,r) \cap T \neq \emptyset)$

(4) $p \in T' \implies \forall r > 0, \; B(p,r) \cap T - \{p\}$ is an infinite set, say $x_1, \cdots, x_n$, Let $\delta = \dfrac{1}{2} \min\{d(p,x_i) \mid 1 \le i \le n\}$. Then $B(p,\delta) \cap T - \{p\} = \emptyset (\to \leftarrow)$ to $p \in T'$, $x \in B(p,\delta) \cap T - \{p\} \implies d(x,p) < \delta \implies x = x_i$ for some $1 \le i \le n$& we get $d(x_i, p) < \delta \le \dfrac{1}{2} d(x_i, p)$

$\therefore$ no such x i.e. $B(p,\delta) \cap T - \{p\} = \emptyset$

(5) Any finite subset of X has no accumulation points in X by (4). In particular, it is closed by (6)(c) below.

(6) TFAE
  (a) S is closed
  (b) S contains all it's adherent point, i.e. $\overline{S} \subseteq S$
  (c) S contains all it's accumulation points, i.e. $S' \subseteq S$
  (d) $S = \overline{S}$

(7) $\overline{\overline{S}} = \overline{S}$ by (2) and (6)

Proof. of (6)
(a) $\Rightarrow$ (b) Suppose S is closed $\implies X\backslash S$ is open $\implies \forall\, p \in X - S \implies \exists r > 0 \ni B(p,r) \subseteq X \backslash S \implies B(p,r) \cap S = \emptyset \implies p \notin \overline{S}$
$\therefore \overline{S} \subseteq S$, i.e. (b) holds

(b) $\Rightarrow$ (c) $\because S' \subseteq \overline{S}$

(c) $\Rightarrow$ (d) Suppose $S' \subseteq S$. To prove $S = \overline{S}$ if not, then $S \subsetneq \overline{S}$, i.e. $\exists\, p \in \overline{S}$ & $p \notin S \implies \forall r > 0, \; B(p,r) \cap S \ne \emptyset \; (\because p \in \overline{S})$
(d) $\Rightarrow$ (a) by (2)
                                                                          ∎

(8) $\overline{S}$ is the smallest closed set in X containing S
$\because$ We know that $S \subseteq \overline{S}$, if F is closed in X & $F \subseteq S$, then $\overline{F} \subseteq \overline{S}$ by (1), $F = \overline{F} \subseteq \overline{S}$ by (6), $\therefore \overline{S}$ is the smallest such one.

(9) In fact, $\overline{S} = \bigcap_{F \subseteq S} F$

(10) $p \in S$ is an isolated point $\Leftrightarrow \exists\, r > 0 \ni B(p,r) \cap S = \{p\}$
($\Rightarrow$) Suppose $p \in S$ is an isolated point of S. Then $p \in S' \implies \exists\, r > 0 \ni B(p,r) \cap S - \{p\} = \emptyset \implies B(p,r) \cap S = \{p\}$
($\Leftarrow$) Trivial

(11) S is dense in $X \Leftrightarrow \forall\, p \in X$ & $r > 0, \; B(p,r) \cap S \ne \emptyset \Leftrightarrow \forall$ open set $U \ne \emptyset, \; U \cap S \ne \emptyset$

Proof. ($\Rightarrow$) Suppose S is dense in X, i.e. $\overline{S} = X$, So $\forall p \in X, p \in \overline{S} \implies \forall\, r > 0, \; B(p,r) \cap S \ne \emptyset$
($\Leftarrow$) Suppose the condition holds, $\forall p \in X$ & $r > 0, \; B(p,r) \cap S \ne \emptyset \implies p \in \overline{S} \implies X \subseteq \overline{S} \subseteq X, \; \therefore \overline{S} = X$ ∎

(12) $\partial S = \partial(X - S)$ In particular, $\partial S = \overline{S} \cap \overline{(X - S)}$, In particular,
∂S is closed in X, $\because$ It suffices to prove $\partial S = \overline{S} \cap \overline{(X - S)}$,
$\because \partial(X - S) = \overline{X - S} \cap \overline{X - (X - S)} = \overline{X - S} \cap \overline{S} = \partial S$
$\forall p \in \partial S \implies \forall r > 0, B(p, r) \cap S \neq \emptyset \ \& \ p \in \overline{X - S} \implies p \in \overline{S} \cap \overline{X - S}$
$\therefore \partial S \subseteq \overline{S} \cap \overline{X - S}$,
Conversely, $p \in \overline{S} \cap \overline{X - S} \implies p \in \overline{S} \ \& \ p \in \overline{X - S} \implies \forall r > 0, \ B(p, r) \cap S \neq \emptyset \ \& \ B(p, r) \cap (X - S) \neq \emptyset \implies p \in \partial S$
$\therefore \overline{S} \cap \overline{(X - S)} \subseteq \partial S \therefore \partial S = \overline{S} \cap \overline{(X - S)}$

2.5. Examples.

We give some simple examples of open sets, closed sets, adherent, accumulation, isolated and boundary points.

1. In a discrete metric space X, every subset of X is both open and close, $\forall x \in X, \ B(p, r) \begin{cases} \{x\} \text{ if } 0 < r \leq 1 \\ X \text{ if } r > 1 \end{cases}$
$\therefore$ Every singleton is open in X, so every subset of X is open.

2. In $\mathbb{R}$. Consider the set $S = [0, 1) \cup \{3\}$, $S^\circ = \emptyset$, $S' = \{0\}$, $\overline{S} = S \cup \{0\}$

3. In $\mathbb{R}$, consider the set $S = \{\dfrac{1}{n} \mid n = 1, 2, \cdots\}$, $S^\circ = \emptyset$, $S' = \{0\}$, $\overline{S} = S \cup \{0\}$

4. In $\mathbb{R}^2$, consider $S = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$, S is open
$\overline{S} = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$
$\partial S = \{(x, 0) \mid x \geq 0\} \cup \{(y, 0) \mid y \geq 0\}$

5. Let $B(0, 1)$ be the unit open ball in $\mathbb{R}^k$. Then $\partial B(0, 1) = S^{k-1}$ is the unit $(k - 1)$-sphere. In particular, for $k = 2, \partial B(0, 1) = S^1$ in the unity circle in the plane $\mathbb{R}^2$. Similarly, for the closed unit ball $\overline{B}(0, 1)$ in $\mathbb{R}^k$. Now, we define some special sets in $\mathbb{R}^n$

- Internals in $\mathbb{R}$: $-\infty < a \leq b < \infty$
  $[a, b]$ close interval which is closed in $\mathbb{R}$
  (a, b) open interval which is closed in $\mathbb{R}$
  Infinite intervals:
  $(-\infty, b]$: close in $\mathbb{R}$, $(-\infty, b)$ open in $\mathbb{R}$
- k-dimensional interval (rectangle or k-cell) I

$$I = I_1 \times \cdots \times I_k$$

  where I_j is an interval in $\mathbb{R}$, $1 \leq j \leq k$
  (i) I is bounded $\Leftrightarrow$ each I_j is bounded
  I is unbounded $\Leftrightarrow I_j \neq \emptyset \ \&$ some I_j is unbounded
  (ii) $I = [a_1, b_1] \times \cdots [a_k, b_k]$, $-\infty < a_j \leq b_j < \infty$, $1 \leq j \leq k$
  k-dimensional closed(compact) interval in $\mathbb{R}^k$

- Convex sets in $\mathbb{R}^k$
  $S \subseteq \mathbb{R}^k$ is convex if $\forall\, x, y \in S$, $\overline{xy}$ is the line segment joining x & y
  Note that all open balls, closed balls, intervals are convex in $\mathbb{R}^k$
  - Star-like sets in $\mathbb{R}^k$ with w.r.t some point x_0, $S \subseteq \mathbb{R}^k$ is star-like
    w.r.t. $x_0 \in S$ if $\forall\, x \in S$, $\overline{xx_0} \subseteq S$

6. We know that $\mathbb{Q}$ is dense in $\mathbb{R}$, hence $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$. Note that $\mathbb{Q}^k$ is countable, hence $\mathbb{R}^k$ has a countable dense subset $\mathbb{Q}^k$, i.e. $\mathbb{R}^k$ is separable.

7. $\partial \mathbb{Q} = \mathbb{R}$, $\partial \mathbb{Q}^k = \mathbb{R}^k$

8. $\mathbb{Z}$ is closed in $\mathbb{R}$, $\because \mathbb{R} - \mathbb{Z} = \bigcup_{n=-\infty}^{\infty} (n-1, n)$ is open $\implies \mathbb{Z}$ is close.

or $\mathbb{Z}' = \emptyset \subseteq \mathbb{Z}$, $\therefore \mathbb{Z}$ is close.

9. Let $S \subseteq \mathbb{R}$ be a nonempty set which is bounded above. Then $\alpha = \sup S$ exists. Moreover, $\alpha \in \overline{S}$. $\because \forall\, r > 0, \exists\, x_0 \in S \ni \alpha - r < x_0 \leq \alpha < \alpha - r \implies (\alpha - r, \alpha + r) \cap S \neq \emptyset \implies \alpha \in \overline{S}$

2.6. Compact Set in Metric Space.

- Compact sets in metric space, which is closely related to the extreme value problem.
- Compact set $\mathbb{R}^k$ will be discussed in next section.

<u>Definition.</u> *Let X be a topology space and $S \subseteq X$. A collection $\mathscr{U} = \{U_\alpha\}_{\alpha \in I}$ of open sets in X is called an open covering of S if*

$$S \subseteq \bigcup \mathscr{U} = \bigcup_{\alpha \in I} U_\alpha$$

<u>Definition.</u> *Let X be a topology space, $S \subseteq X$ and $\mathscr{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of S. We say that $\mathscr{U}$ has a countable(finite) sub covering of S if $\exists$ a countable(finite) sub collection of $\mathscr{U}$ which also covers S. i.e. $\mathscr{U}$ has a countable(finite) subcovering in S if*

$\exists$ a sequence $\{\alpha_n\}$ in $I \ni S \subseteq \bigcup_{n=1}^{\infty} U_{\alpha_n}$ (countable)

$\exists$ a sequence $\{\alpha_n\}$ in $I \ni S \subseteq U_\alpha \cup \cdots \cup U_{\alpha_n}$ (finite)

Example

(1) X is discrete metric space. Then $\{\{x\} \mid x \in X\}$ is an open covering of X

(2) In $\mathbb{R}$, $\{(0, 1 - \frac{1}{n}) \mid n \in \mathbb{N}\}$ is an open covering of $(0, 1)$. In fact,

$$(0, 1) = \bigcup_{n=1}^{\infty} (0, 1 - \frac{1}{n})$$

(3) $\{B(0, n) \mid n \in \mathbb{N}\}$ is an open covering of $\mathbb{R}^k$

Definition (compact). *Let X be a topology space. A subset $K \subseteq X$ is said to be compact if **every** open covering of K admit a finite subcovering*

Examples

(1) Let X be a topology space and $K \subseteq X$ be a finite set. Then K is compact.

(2) In a discrete metric space X, a subset $K \subseteq X$ is compact $\Leftrightarrow K$ is a finite set.

(3) $(0,1)$ is not compact in $\mathbb{R}(\{0, 1 - \dfrac{1}{n} \mid n \in \mathbb{N}\})$, but $[0,1]$ is compact

Theroem 2.10. *Let X be a metric space and $K \subseteq Y \subseteq X$. Then K is compact in $X \Leftrightarrow K$ is compact in Y.*

Proof. ($\Rightarrow$) Suppose K is compact in X. Given an open covering $\{V_\alpha\}_{\alpha \in I}$ of open sets in Y which covers K. By Prop 2.5, each $V_\alpha = U_\alpha \cap Y$, where U_α is open in X. Now,

$$K \subseteq \bigcup_{\alpha \in I} V_\alpha = \bigcup_{\alpha \in I}(U_\alpha \cap Y) = (\bigcup_{\alpha \in I} U_\alpha) \cap Y \implies K \subseteq \bigcup_{\alpha \in I} U_\alpha$$

By the compactness of K in X, $\exists \alpha_1, \cdots, \alpha_n \in I \ni K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} \implies$

$K \cap Y \subseteq (\bigcup_{i=1}^{n} U_{\alpha_i}) \implies K \subseteq \bigcup_{i=1}^{n}(U_{\alpha_i} \cap Y) = \bigcup_{i=1}^{n} V_{\alpha_i}$

$\therefore K$ is compact in Y

($\Leftarrow$) Suppose K is compact in Y. Given a open covering $\{U_\alpha\}_{\alpha \in I}$ of K by open sets in X.

$$K \subseteq \bigcup_{\alpha \in I} U_\alpha \implies K \cap Y \subseteq (\bigcup_{\alpha \in I} U_\alpha) \cap Y \implies K \cap Y \subseteq \bigcup_{\alpha \in I}(U_\alpha \cap Y)$$

By Prop 2.5, $\{U_\alpha \cap Y \mid \alpha \in I\}$ is an open covering of K by open set in Y. By assumption, K is compact in $Y, \exists\, \alpha_1, \cdots, \alpha_n \in I \ni K \subseteq$

$\bigcup_{i=1}^{n}(U_{\alpha_i} \cap Y) = (\bigcup_{i=1}^{n} U_{\alpha_i}) \cap Y \implies K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$

$\therefore K$ is compact in X ∎

Definition. *Let X be a metric space and $S \subseteq X$ be a nonempty set. The diameter of S is defined to be $dia(S) = \sup\{d(x,y) \mid x, y \in S\}$ which generated the diameter of a circle in $\mathbb{R}^2$*

Theroem 2.11. *Let X be a metric space and $K \subseteq X$ be a compact set. Then K is closed and bounded*

Proof. **K is bounded**

Fix a point $p \in K$. Then $K \subseteq \bigcup_{n=1}^{\infty} B(p,n)$. $\because K$ is compact $\implies \exists N \in \mathbb{N} \ni K \subseteq B(p,1) \cup \cdots \cup B(p,N) \implies K \subseteq B(p,N) \therefore K$ is bounded
K is closed, i.e. $X - K$ is open
Fix $p \in X - K$. Then $p \neq x$, $\forall \, x \in K$. Hence, $d(x,p) > 0$, $\forall \, x \in K$
Let $r_x = \dfrac{1}{2} d(x,p) > 0, x \in K$. Them $\{B(x, r_x) \mid x \in K\}$ is an open covering of K. $\because K$ is compact $\implies \exists x_1, \cdots, x_n \in K \ni$
$B(x_1, r_{x_1}) \cup \cdots \cup B(x_n, r_{x_i})$. Let $V = \bigcap_{i=1}^{n} B(p, r_{x_i}) = B(p,r)$, where
$r = \min\{r_{x_1}, \cdots, r_{x_n}\}$. Then as we can see that $V \subseteq X - K$, all point in $X - K$ are inner point. So $X - K$ is open, i.e. K is close.

To show that $V \subseteq X - K$, i.e. $V \cap K \neq \emptyset$, it suffices to show

$$V \cap \left(\bigcup_{i=1}^{n} B(x_i, r_{x_i})\right) = \emptyset$$

Now,

$$
\begin{aligned}
V \cap \left(\bigcup_{i=1}^{n} B(x_i, r_{x_i})\right) &= \bigcup_{i=1}^{n} (V \cap B(r_i, r_{x_i})) \\
&\subseteq \bigcup_{i=1}^{n} (B(p, r_{x_i} \cap B(x_i, r_{x_i}))) = \emptyset
\end{aligned}
$$

∎

Remark. *The converse of Thm 2.11 is false, i.e. closed & bounded may not be compact, e.g. X is an infinite set with discrete metric. Then X is not compact, but X is closed and bounded.*

Theroem 2.12. *Let X be a metric space, $K \subseteq X$ be compact & $L \subseteq K$ be a closed set in X. Then L is compact.*

Proof. Let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of L. Then $\{U_\alpha\}_{\alpha \in I} \cup \{X - L\}$ is an open covering of K. By the compactness of K, $\exists \alpha_1, \cdots, \alpha_n \in I \ni$ $K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \cup (X - L)$. By $L \subseteq K$ $\therefore L$ is compact ∎

Corollary 2.13.

(a) Let X be a metric space, $K \subseteq X$ be compact and F be a closed set in X. Then $K \cap F$ is compact.
(b) If X is a compact metric space, then every closed subset F of X is compact.

Proof.

(a)

$$
\begin{aligned}
K \text{ is compact} \quad &\Longrightarrow \quad K \text{ is closed (Thm 2.11)} \\
&\Longrightarrow \quad K \cap F \text{ is closed in } X \\
&\Longrightarrow \quad K \cap F \text{ is compact}
\end{aligned}
$$

(b) follows (a)

∎

Remark. *Let X be a metric space. If K is closed in X and F is closed in K, then F is closed in X. $\because F$ is closed in $F \implies F = L \cap F$, where L is closed in $K \implies F = L \cap K$, where L is closed in $X \implies F$ is closed in X.*

Theroem 2.14. *Let X be a metric space, $\{K_\alpha\}_{\alpha \in I}$ be a collection of compact subsets of X with the property:*

$$
\forall \alpha_1, \cdots, \alpha_n \in I, K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \neq \emptyset
$$

Them $\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$

Proof. Fix $\alpha_0 \in I$. Assume that $\bigcap_{\alpha \in I} K_\alpha = \emptyset \implies X - \bigcap_{\alpha \in I} K_\alpha$

$\implies X - \emptyset = X \implies X = \bigcup_{\alpha \in I} (X - K_\alpha)$

each K_α is compact $\implies K_\alpha$ is closed $\implies X - K_\alpha$ is open
so $\{X - K_\alpha\}$ is an open covering of X. Now,

$$
K_{\alpha_0} \subseteq X = \bigcup_{\alpha \in I} (X - K_\alpha) \implies K_{\alpha_0} \subseteq \bigcup_{\alpha \in I} (X - K_\alpha)
$$

K_α is compact $\implies \exists \alpha_1, \cdots, \alpha_n \in I - \{\alpha_0\} \ni K_{\alpha_0} \subseteq (X - K_{\alpha_1}) \cup \cdots \cup (X - K_{\alpha_n}) \implies K_{\alpha_0} \cap K_{\alpha_1} \cap \cdots \cap K_n = \emptyset (\rightarrow \leftarrow)$ ∎

Corollary 2.15. *Let X be a metric space and $\{K_n\}_{n=1}^{\infty}$ be a decrease sequence of nonempty compact sets of X. Them $\bigcap_{n=1}^{\infty} \neq \emptyset$. In addition, if $dia_{n\to\infty}\infty 0$, them $\bigcap_{n=1}^{\infty} K_n$ is a singleton.*

Proof. $\forall j1, \cdots, j_k \in \mathbb{N}$, $K_{j1} \cap \cdots \cap K_{jk} \neq \emptyset$, $K_{j1} \cap \cdots \cap K_{jk} = K_t$, where $t = \max\{j_1, \cdots, j_k\}$. By Thm 2.14 $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$, if $\lim_{n\to\infty} \text{dia}(K_n) = 0$ and $p, q \in \bigcap_{n=1}^{\infty} K$ and $p \neq q$, them $\text{dia}(K_n) \geq d(p,q) \ \forall n \geq 1 \implies \lim_{n\to\infty} \text{dia}(K_n) \geq d(p,q) > 0(\to\leftarrow) \because \bigcap_{n=1}^{\infty} K_n = \{p\}$ is a simpleton. ∎

Remark. *The usual form of Cor 2.15, X is a metric space, $\{K_n\}$ is a decrease sequence of nonempty closed sets in X with K_i is compact $\implies \bigcap_{n=1}^{\infty} K_n \neq \emptyset$*

Example In $\mathbb{R}$, $\{(0, \frac{1}{n} \mid n \geq 1]\}$ is decrease and every finite subcollection of $\{(0, \frac{1}{n}) \mid n \geq 1\}$ is nonempty, but $\bigcap_{n=1}^{\infty}(0, \frac{1}{n}) = \emptyset$, $\bigcap[0, \frac{1}{n}] = \emptyset$

Theroem 2.17. *Let X be a metric space and $K \subseteq X$, TFAE:*
(i) K is compact
(ii) Every infinite subset has an accumulation point in K
(iii) K is sequentially compact
(iv) K is complete and totally bounded

Definition (Convergence)**.** *$\{a_n\}$ converge if $\exists a \in X \ni \forall \epsilon \geq 0 \exists N \ni \mathbb{N} \ni \forall n \geq N$, $d(a_n, a) < \epsilon$. Such a is called the limit of $\{a_n\}$, which is denoted by $\lim_{n\to\infty} a_n = a$ or $a_n \to a$ on $n \to \infty$.*

Definition (Cauchy)**.** *We say that $\{a_n\}$ is Cauchy if $\forall \epsilon > 0$, $\exists N \in \mathbb{N} \ni \forall n, m \geq \mathbb{N}$, $d(a_n, a_m) < \epsilon$*

Definition. *A metric X is said to be sequence compact if every sequence has a convergent subsequence*

Definition. *A metric space X is said to be complete if every Cauchy sequence in X convergence.*

Definition. *Let X be a metric space & $K \subseteq X$. We say that K is totally bounded if $\forall r > 0, \exists x_1, \cdots, x_n \in K \ni K \subseteq B(x_1, r) \cup \cdots \cup B(x_n, r)$*

Remark. *Totally bounded can implies bounded, but not converse.*
K is totally bounded, for $r = 1, \exists x_1, \cdots, x_n \in K \subseteq B(x_1, 1) \cup \cdots \cup B(x_n, 1) \implies K \subseteq B(x_1, R)$ for sime large R

- *Take an **"infinite"** set X with discrete metric. Then X is bounded(e.g. $X \subseteq B(x_0, 2)$, where $x_0 \in X$) but for $r = \dfrac{1}{2}$, $X \not\subseteq B(x_1, \dfrac{1}{2}) \cup \cdots \cup B(x_n, \dfrac{1}{2}) \ \forall x_1, \cdots, x_n$*

Lemma 2.18 (To prove (ii) to (i))**.**
Suppose (ii) holds in Thm 2.17. Then K is totally bounded

Proof. If not, then $\exists \, r > 0, \ni$ no finite open balls with radius r and center K cover K. Choose $x_1 \in k \implies K \not\subseteq B(x_1, r) \implies \exists x_2 \in K - B(x_1, r)$, $K \not\subseteq B(x_1, r) \cup B(x_2, r) \implies \exists x_3 \in K - (B(x_1, r) \cup B(x_2, r))$ By induction, counting this process, we obtain an infinite set $T = \{x_1, x_2, \cdots, x_n, \cdots\} \subseteq K$ with $d(x_i, x_j) \geq r \forall i \neq j$. By (ii), T has an accumulation poion $p \in K$. In particular $B(p, \dfrac{r}{4}) \cap T - \{p\}$ is an infinite set, hence, $\exists i \neq j \ni x_i, x_j \in B(p, \dfrac{r}{4}) \cap T - \{p\} \implies d(x_i, x_j) \leq d(x_i, p) + d(x, j) < \dfrac{r}{4} + \dfrac{r}{4} = \dfrac{r}{2} < r (\rightarrow \leftarrow) \therefore K$ is totally bounded. $\blacksquare$

Lemma 2.19. *Suppose (ii) holds in T and $\{E_\alpha \mid \alpha \in I\}$ is an open covering of K. Then $\exists r > 0$(called a Lebegoue number w.r.t. the open covering $\{E_\alpha\}_{\alpha \in I}$) $\ni \forall x \in K, B(x, r) \subseteq E_\alpha$ for some $\alpha \in I$*

Proof. if K is a finite set, let $K = \{x_1, \cdots, x_n\} K \subseteq \bigcup_{\alpha \in I} E_\alpha \implies x_i \in E_{\alpha_i}$ for some $\alpha_i \in I$, $1 \leq i \leq n \implies \exists r_i > 0 \ni B(x_i, r_i) \subseteq E_{\alpha_i}, 1 \leq i \leq n$. Let $r = \min\{r_1, \cdots, r_n\}$. Then $B(x_i, r) \subseteq B(x_i, r_i) \subseteq E_{\alpha_i}, 1 \leq i \leq \alpha$. Now, assume that K is infinite set. Assume that no such $r > 0$, i.e. $\forall r > 0, \exists x_r \in K \ni B(x_r, r) \not\subseteq E_\alpha \forall x \in I$. Now, for $r = \dfrac{1}{k}, r = 1, 2, \cdots$, we obtain a sequence $\{x_k\}$ in K, with $x_k = \dfrac{x_r}{k} \ni B(x_k, \dfrac{1}{k}) \not\subseteq E_\alpha \forall a \in I$. Let $T = \{x_1, x_2, \cdots, x_k, \cdots\}$. Then $T \subseteq K$ is an infinite set($\because$ For $k = 1, r = \dfrac{1}{1} = 1 \exists x_1 \in K \ni B(x_1, 1) \not\subseteq E_\alpha \forall \alpha \in I$). The conclusion of Lemma 2.19 failed for $K - \{x_1\}$ ($\because if \exists s > 0 \ni \forall x \in K - \{x_1\} B(x, s) \subseteq E_\alpha$ for some $\alpha \in I$, $x_1 \in E_\alpha \implies \exists t > 0 \ni B(x_1, t) \subseteq E_\alpha$. Let $r = \min\{s, t\}$. Then $\forall x \in K, B(x, r) \subseteq E_\alpha$ for some $\alpha \in I(\rightarrow \leftarrow)$)

Then for $r = \dfrac{1}{2}, \exists x_2 \in K - \{x_1\} \ni B(x_2, \dfrac{1}{2}) \subsetneqq E_\alpha \forall \alpha \in I$. Continue this process, we conclude that $x_i \neq x_j \forall i \neq j$, so T is an infinite set. By the assumption of (ii). Then an accumulation point $p \in K$. Now $K = \bigcup_{\alpha \in I} E_\alpha \implies p \in E_\alpha$ for some $\alpha \in I \implies \exists \epsilon > 0 \ni B(p, \epsilon) \subseteq E_{\alpha_0}$. Since $p \ni T', B(p, \epsilon) \cap T - \{p\}$ is an infinite set. Choose $m >> 0 \ni \dfrac{1}{m} < \dfrac{\epsilon}{2}$ & $x_m \in B(p, \dfrac{\epsilon}{2}) \cap T$. Claim $B(x_m, \dfrac{1}{m}) \subseteq B(p, \epsilon) \subseteq E_{\alpha_0}(\to\leftarrow)$ to our constrain, hence Lemma 2.19 holds.

$$y \in B(x_m, \frac{1}{m}) \Rightarrow d(y, p) \leq d(y, x_m) + d(x_m, p) < \frac{1}{m} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$
$$\Rightarrow y \in B(p, \epsilon) \qquad\qquad\qquad\qquad\qquad \blacksquare$$

Proof. (Thm 2.17 (i)(ii))

$(i) \Rightarrow (ii)$ Suppose K is compact. Given an infinite set $T \subseteq K$. We must prove that T has an accumulation point in K, if not, $\forall x \in K, x$ is not an accumulation point of $K, \exists r_x > 0 \ni B(x, r_n) \cap T - \{x\} = \emptyset \implies B(x, r_x) \cap T \subseteq \{x\}$. Clearly $\{B(x, r_x \mid x \in K)\}$ is an open covering of K. By (i), K is compact

$$\begin{aligned}
\implies \exists x_1, \cdots, x_n \in K \ni K \ &\subseteq\ B(x_1, r_{x_1}) \cup \cdots \cup B(x_n, r_{x_n}) \\
&=\ (T \cap B(x_1, r_{x_1})) \cup \cdots \cup (T \cap B(x_n, r_{x_n})) \\
&\subseteq\ \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_n\} \\
&=\ \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_n\} \\
&=\ \{x_1, \cdots, x_n\}(\to\leftarrow)
\end{aligned}$$

to T is an infinite set, $\therefore$ (ii) holds.

$(ii) \Rightarrow (i)$ In Thm 2.17 i.e. we must prove that K is compact under the assumption of (ii). Suppose $\mathscr{U} = \{E_\alpha\}_{\alpha \in I}$ is an open covering of K. By Lemma 2.19, $\exists$ a $r > 0$ w.r.t. $\mathscr{U}$, by Lemma 2.18, $\exists x_1, \cdots, x_k \in K \ni K \subseteq B(x_1, r) \cup \cdots \cup B(x_k, r) \subseteq E_{\alpha_1} \cup \cdots \cup E_{\alpha_k}$, where $B(x_i, r) \subseteq E_{\alpha_i}, 1 \leq i \leq k$. Therefore, $\mathscr{U}$ has a finite sub covering. Hence K is compact and (i) holds. $\blacksquare$

Remark.

(1) (ii) $\implies$ (i) is exercise 26

(2) More or less, by Lemma 2.18 & 19, one can see that (i) and (ii) are also equal to (iii) and (iv)

2.7. Compact Sets in Euclidean Spaces $\mathbb{R}^k$.

• We know that any compact set in a metric space is close and bounded
• Close and bounded subset my not be compact(infinite discrete)
• We will see that every closed and bounded subset of $\mathbb{R}^k$ is always compact which is the famous H.B. Theorem, i.e. $K \subseteq R^k$ is compact $\Leftrightarrow K$ is closed and bounded

Theroem 2.20. *Let $\{I_n = [a_n, b_n]\}_{n=1}^{\infty}$ be a sequence of closed and bounded intervals in $\mathbb{R}$, if $\{I_n\}$ is decreasing i.e. $I_1 \subseteq \cdots \subseteq I_n \subseteq \cdots$, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Moreover, if $\lim_{n\to\infty}(b_n - a_n) = 0$, then $\bigcup_{n=1}^{\infty} I_n$ is a simpleton.*

Proof. Claim $T = \{a_n \mid n \in \mathbb{N}\}$ is bounded above and $x = \sup T$ exists. $\because a_n \leq a_{m+n}(\because \{a_n\}$ is increasing, i.e. $[a_1, b_2] \supseteq [a_2, b_2]$, $a_2 \geq a_1)$
$a_n \leq a_{m+n} \leq b_{m+n} \leq b_m(\because \{b_n\}$ is decreasing $) \implies T$ is bounded above by all $b_n \implies x = \sup T$ exists and $x \leq b_n \forall n \geq 1$.
Clearly, $a_n \leq x \; \forall n \geq 1$, $\because a_n \leq x \leq b_n, \forall n \geq 1$, i.e. $x \in [a_n, b_n] \forall n \geq 1 \therefore x \in \bigcap_{n=1}^{\infty} I_n$. Hence, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset \therefore$ The last statement follow the argument as in Corollary 2.16. ∎

Theroem 2.21. *Let $\{I_n = [a_{n,1}, b_{n,1}] \times \cdots \times [a_{n,k}, b_{n,k}]\}$ be a decreasing sequence of closed and bounded intervals in $\mathbb{R}^k$. Then $\bigcap_{n=1}^{\infty} \neq \emptyset$.*

Moreover, if $\lim_{n\to\infty} dia(I_n) = 0$, then $\bigcup_{n=1}^{\infty}$ is a simpleton.

Proof. $\forall 1 \leq j \leq k, \{[a_{n,j}, b_{n,j}]\}$ is a decrease sequence of closed and bounded intervals in $\mathbb{R}$. By Thm 2.20, $\exists x_j \in \bigcap_{n=1}^{\infty} [a_{n,j}, b_{n,j}]$. Set $x = (x_1, \cdots, x_k)$. Then $x \in \bigcap_{n=1}^{\infty} I_n$. Then last statement also follows from the argument in corollary 2.16. ∎

Theroem 2.22. *Every k-dimensional closed and bounded interval $I = [a_1, b_1] \times \cdots \times [a_k, b_k]$ in $\mathbb{R}^k$ is compact.*

Proof. Put $\delta = (\sum_{i=1}^{k}(b_i - a_i)^2)^{\frac{1}{2}}$ which is the diametric of I. Then $\forall x, y \in I, \| x - y \| \leq \delta$. If I were not compact, them $\exists$ an open covering $\{E_\alpha\}_{\alpha \in J}$ admitting not finite sub covering($\star$). Put $c_j = \dfrac{a_j + b_j}{2}$, $1 \leq j \leq k$. The intervals $[a_j, c_j]$ and $[c_j, b_j], 1 \leq j \leq k$, determines 2^k closed and bounded subinterval of I whose union is I. By ($\star$), at least one of them, say I_1 which cannot be covered by finitely many E_α. Continuing this process, we get a sequence $\{I_n\}$ of closed and bounded subintervals of I satisfy's

    a) $I \subseteq I_1 \subseteq \cdots$, i.e. $\{I_n\}_{n=1}^{\infty}$ is decreasing.
    b) Each I_n cannot be covered by finitely many E_α
    c) $\mathrm{dia}(I_n) = 2^{-n}$, $\delta \to 0$ on $n \to \infty$

By Thm 2.21, $\bigcap_{n=1}^{\infty} I_n = \{x\}$ i.e. $x \in I_n \subseteq I \; \forall \; n \geq 1 \subseteq \bigcup_{\alpha=1}^{\infty} E_\alpha$
$\because x \in E_{\alpha_0}$ for some $\alpha_0 \in J$, E_{α_0} is open $\implies \exists r > 0 \ni B(x, r) \subseteq E_{\alpha_0}$.
Choose $n_0 >> 0 \ni \dfrac{1}{2^{n_0}} < \dfrac{r}{\delta}(\because \dfrac{1}{2^n} \to 0)$. Since $x \in I_{n_0}, \forall y \in I_{n_0}$, $\| y - x \| \leq 2^{-n_0}\delta < \dfrac{r}{\delta} \cdot \delta = r \implies y \in B(x, r)$, $\because I_{n_0} \subseteq B(x, r) \subseteq E_{\alpha_0}(\to\leftarrow)$.
Therefore I is compact.   ■

Combining Thm 2.22 and results in section 2.6, we conclude that the following sets in compact:
i) $[a, b]$ is compact in $\mathbb{R}$(Thm 2.22 with $k = 1$)
ii) $[a, b] \times [c, d]$ is compact in $\mathbb{R}^2$(Thm 2.22 with $k = 2$)
iii) Every closed ball $\overline{B}(x, r)$ in $\mathbb{R}^k$ is compact by Thm 2.12 and 2.22
iv) $\{0\} \cup \{\dfrac{1}{n} \mid n = 1, 2, \cdots\}$ is compact in $\mathbb{R}$, In fact, if $a_n \to a$, them teh set $\{a\} \cup \{a_n \mid n = 1, 2, \cdots\}$ is compact in $\mathbb{R}$

Theroem 2.23. *Every closed and bounded subset K of $\mathbb{R}^k$ is compact.*

Proof. Choose a large closed and bounded interval I in $\mathbb{R}^k$, $K \subseteq I$. By Thm 2.22, I is compacted, so K is a closed subset of I. By Thm 2.12, K is compacted.
Combining Thm 2.17 and Thm 2.23, we can characterize compacted set in $\mathbb{R}^k$ ■

Theroem 2.24. *Let $K \subseteq R^k$ TFAE:*

i) K is closed and bounded
ii) K is compacted
iii) Every infinite subset of K has an accumulation point
iv) K is sequence compact
v) K is complete and totally bounded

From these we can deduce:

Theroem 2.25 (Bolzano-Weiestrace)**.** *Every bounded infinite subset T in $\mathbb{R}^k$ has an accumulation point in $\mathbb{R}^k$*

Proof. Since T is bounded, choose a large closed and bounded interval I in $\mathbb{R}^k \ni T \subseteq I$. Now, T become an infinite subset of the compact set I. By Thm 2.24 (iii), T has an accumulation point in I. ∎

Theroem 2.26 (Cantor intersection)**.** *Let $\{\mathbb{Q}_n\}$ be a sequence of nonempty set in $\mathbb{R}^k$ satisfying:*

   a) $\{Q_n\}$ is decreasing
   b) Q_n is closed $\forall\, n \geq 1$ & Q_1 is compact.

Then $\displaystyle\bigcap_{n=1}^{\infty} Q_n \neq \emptyset$, Moreover, if $dia(Q_n) \to 0$ on $n \to \infty$, them $\displaystyle\bigcap_{n=1}^{\infty} Q_n$ is a simpleton.

Proof. By (b), each Q_n is compact ($\because Q_1 \supseteq Q_n \forall\, n \geq 1$). Therefore, it follows form Cor 2.16 and 2.15. ∎

2.8. Countability & Separability.

Motivation: In $\mathbb{R}^k$, we have two facts:

• $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$ i.e. $\overline{\mathbb{Q}^k} = \mathbb{R}^k$ & $\mathbb{Q}^k$ is countable. i.e. $\mathbb{R}^k$ has a countable dense subset. i.e. $\mathbb{R}^k$ is separable.

• $\left\{ B(x,r) \mid x \in \mathbb{Q}^k, r \in \mathbb{Q}^+ \right\}$ is a countable collection of open ball in $\mathbb{R}^k$ satisfying: $\forall$ open set $U \subseteq \mathbb{R}^k$ and $y \in U$ $\exists B(x,r) \in \mathscr{B} \ni y \in B(x,r) \subseteq U$. In particular, U is a union of some sub collection of $\mathscr{B}$. $\therefore U = \displaystyle\bigcup_{y \in U} B_y$, i.e. $\mathbb{R}^k$ is of 2nd countable.

• X is a metric space $x \in X$, $N_x = \{B(x, \frac{1}{n}) \mid n \in \mathbb{N}\}$ is a countable collection of nbh of x. Such N_x satisfies: $\forall$ nbh U of $x, \exists n \in \mathbb{N} \ni$ $B(x, \frac{1}{n}) \subseteq U$ ($\because \exists\, r > 0 \ni B(x,r) \subseteq U$, choose $n \gg 0 \ni \frac{1}{n} < r$). Then $x \in B(x, \frac{1}{n}) \subseteq B(x,r) \subseteq U$. i.e. Each point of X has a countable nbh base(system) i.e. X is of 1st countable, i.e. $\mathbb{R}^k$ has a countable base.

Definition. *Let X be a topological space*

(1) X is first if every point of X has a countable nbh system(or base), i.e. $\exists$ a countable collection $\{V_n \mid n \in \mathbb{N}\}$ of nbh of $x \ni \forall$ nbh U of $x, \exists\, n \in \mathbb{N}$, $V_n \subseteq U$

(2) X is of second countable if X has a countable a base, i.e. $\exists$ a countable collection $\mathscr{B} = \{B_n \mid n \in \mathbb{N}\}$ of open sets in $X \ni$ every open set U is a union of some subcollection of $\mathscr{B}$ or $\forall$ open set U in X and $x \in U$, $\exists n \in \mathbb{N} \ni x \in B_n \subseteq U$

(3) X is separable if X has a countable dense subset, i.e. $\exists$ a countable set $D \subseteq X \ni \overline{D} = X$.

Remark.

(1) Every 2nd countable topology space X is of 1st countable, but not converse. $\because$ Let $\mathscr{B} = \{B_1, B_2, \cdots\}$ be a countable base for X. Given $p \in X$, let $\mathscr{B}_p = \{B_n \mid p \in B_n\}$ Then $\mathscr{B}$, so $\mathscr{B}_p$ is countable and is a collection of open set in X containing of p.

Claim $\mathscr{B}_p$ is a nbh system (or base) of p. Let U be a nbh of p. Then $U = \bigcup_{n \in F} B_n$, $F \subseteq \mathbb{N}$. In particular, $p \in B_n$ for some $n \in F$. Hence, $B_n \in \mathscr{B}_p$ & $p \in B_n \subseteq U$, $\therefore \mathscr{B}_p$ is a countable nbh system of p. Hence, X is of 1st countable.

Consider X an uncountable set with discrete metric. Hence, X is 1st countable. In fact, $\forall\, p \in X$, $N_p = \{\{p\}\}$ is a countable nbh base of p. However, X is not of 2nd countable. Note that if $\mathscr{B}$ is a base for the discrete space X then $\mathscr{B} \subseteq \{\{x\} \mid x \in X\}$

$\because \{x\}$ is open $\implies \{x\} = \bigcup_{\alpha \in I} B_\alpha$, $B_\alpha = \{x\}\ \forall\ \alpha \in I \implies$

$B_\alpha = \{x\}\forall x \in I \implies \{x\} \in \mathscr{B}$.

Now, X is uncountable, so is $\mathscr{B}$. Hence, X is not of 2nd countable.

(2) We know that every metric space is of 1st countable. In fact, $\forall p \in X$, $N_p = \{B(p, \frac{1}{n}) \mid n \in \mathbb{N}\}$ is a countable nbh system of p

(3) We know that $\mathbb{R}$ is separable with countable dense subset $\mathbb{Q}$. In general, $\mathbb{R}^n$ is separable with countable dense subset $\mathbb{Q}^n$(Exercise 22)

Theroem 2.27. *Every 2nd countable topology space is separable*

Remark. *Note that X is a metric space. $D \subseteq X$.*

$$D \text{ is dense in } X \Leftrightarrow \overline{D} = X \quad \Leftrightarrow \quad \forall \text{ nonempty open set } U \text{ in } X, U \cap D \neq \emptyset$$
$$\Leftrightarrow \quad \forall x \in X, \exists \text{ a sequence } \{a_n\} \text{ in } D$$
$$\ni \quad a_n \to x \text{ on } n \to \infty$$

Proof.
($\Rightarrow$) Suppose $\overline{D} = X$, i.e. D is dense in X, i.e. $\forall x \in X$, $x \in \overline{D}$. Now, given a nonempty open set U in X. Choose $x \in U$ So U is a nbh of x, hence $U \cap D \neq \emptyset$
($\Leftarrow$) Suppose the condition holds, them $\forall x \in X$ and nbh U of x, $U \cap D \neq \emptyset \implies x \in \overline{D} \implies X \subseteq \overline{D} \subseteq X \implies \overline{D} = X$, i.e. D is dense in X. ∎

Proof. (Theorem 2.27) Let $\mathscr{B} = \{B_1, B_2, \cdots, B_n, \cdots\}$ be a countable base of X. Choose a point $x_n \in B_n$, $n \in \mathbb{N}$ and form the set $D = \{x_1, x_2, \cdots, x_n, \cdots\}$, then D is countable.
Claim: D is dense in X, i.e. $\overline{D} = X$. Given a nonempty open set U in X, then $\exists n \in \mathbb{N} \ni B_n \subseteq U \implies x_n \in U \implies U \cap D \neq \emptyset$. Hence, $\overline{D} = X$ by the remark above. So X is separable. ∎

Theroem 2.28. *Every separable metric space X is of 2nd countable.*

Proof. Choose a countable dense subset D of X. Form the countable collection of open ball $\mathscr{B} = \{B(x,r) \mid x \in D, r \in \mathbb{Q}^+\}$(it is countable). Claim $\mathscr{B}$ is a base for X. We are done.

(Note that $\mathscr{B}$ is a base for a topology space X
$\Leftrightarrow$ every open set in X is a union of some subcollection of $\mathscr{B}$
$\Leftrightarrow \forall$ open set $U \subseteq X$ and $p \in U$, $\exists B \in \mathscr{B} \ni x \in B \subseteq U$)

($\Rightarrow$) Given an open set U in X and $p \in U$ by assumption $U = \bigcup_{\alpha \in I} B_\alpha$, where $B_\alpha \in \mathscr{B} \implies p \in B_{\alpha_0}$ for some $\alpha_0 \in I \implies p \in B_{\alpha_0} \subseteq U$
($\Leftarrow$) Suppose the condition holds. To prove $\mathscr{B}$ is a base for X. Given a nonempty open set U in X. $\forall p \in U, \exists B_p \in \mathscr{B} \ni p in B_p \subseteq U$.
$\because U = \bigcup_{p \in U} B_p \therefore \mathscr{B}$ is a base for X. By the remark above, it is enough to show: Given a nonempty open set $U \subseteq X$ and $p \in U$, $\exists B(x,r) \in \mathscr{B} \ni p \in B(x,r) \subseteq U$. Now, $p \in U$ and U is open $\implies$ $\exists t > 0 \ni B(p,r) \subseteq U$. Choose $r \in \mathbb{Q}^+ \ni \frac{t}{4} < r < \frac{t}{2}$, Since D is dense in X, $B(p,r) \cap D \neq \emptyset$. Choose $x \in B(p,r) \cap D$. Then $B(x,r) \in \mathscr{B}$

Claim $p \in B(x, r) \subseteq U$

• $d(x, p) < r \implies p \in B(x, r)$
• $\forall y \in B(x, r), \ d(y, p) \le d(y, x) + d(x, p) < r + r = 2r < t \implies$
$y \in B(p, t) \subseteq U \ \therefore y \in U \ \therefore B(x, r) \subseteq U.$

∎

Corollary 2.29. *The Euclidean space $\mathbb{R}^k$ is of 2nd countable.*

Note that from the proof of Thm 2.28, $\mathbb{R}^k$ has a countable base of the form:

$$
\begin{aligned}
\mathscr{B} &= \{B(x, r) \mid x \in \mathbb{Q}^k \ \& \ r \in \mathbb{Q}^+\} \\
&= = \{A_1, A_2, \cdots\}
\end{aligned}
$$

Theroem 2.30. *Every compact metric space X is of 2nd countable.*

Proof. The last statement follows from Thm 2.27 To prove X is of 2nd countable. For each $n \in \mathbb{N}$, $\{B(x, \frac{1}{n}) \mid x \in X\}$ is an open covering of X, i.e. $X = \bigcup_{x \in X} B(x, \frac{1}{n})$. By companies of X, it has a finite subcovering say $X = \bigcup_{i=1}^{l_n} B(x_{n_i}, \frac{1}{n})$. Then $\mathscr{B}$ is a countable collection of open balls in X. Claim $\mathscr{B}$ is a base for X. It suffices to show: given a nonempty open set U and $p \in U, \exists B(x_{n_i}, \frac{1}{n}) \ni \mathscr{B} \ni p \in B(x_{n_i}, \frac{1}{n}) \subseteq U$.
From $p \in U$ and U is open, $\exists r > 0 \ni B(p, r) \subseteq U$. Choose $n \gg 0 \ni$
$\frac{2}{n} < r$. Since $X = \bigcup_{i=1}^{l_n} B(x_{n_i}, \frac{1}{n})$, $p \in B(x_{n_i}, \frac{1}{n})$ for some $1 \le i \le l_n$.
Finally, $p \in B(x_{n_i}, \frac{1}{n}) \subseteq U$

• $d(p, x_{n_i}) < \frac{1}{n} \implies p \in B(x_{n_i}, \frac{1}{n})$
• $\forall y \in B(x_{n_i}, \frac{1}{n}), d(y, p) \le d(y, x_{n_i}) + d(x_{n_i}, p) < \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$

$\therefore p \in B(x_{n_i}, \frac{1}{n}) \subseteq U$ ∎

Theroem 2.31 (Lindelof Covering). *Let $S \subseteq \mathbb{R}^k$. Them every open covering $\mathscr{U} = \{U_\alpha \mid \alpha \in I\}$ of S has a countable subcoverings.*

Proof. Let $\{a_1, A_2, \cdots\}$ be the countable base of $\mathbb{R}^k$ defined as above. Note that $S \subseteq \bigcup_{\alpha \in I} \forall x \in S, x \in U_\alpha$ for some $\alpha \in I$. Hence $\exists n \in \mathbb{N} \ni xinA_n \subseteq U_\alpha$. Of course, there may be infinitely many such n. We choose one of them and fix it, say $x \in A_{m(x)} \subseteq U_\alpha$(e.g. $m(x) = \min\{n \in \mathbb{N} \mid x \in A_n \subseteq U_\alpha\}$). Then the collection $\{A_{m(x)} \mid x \in S\}$ is a countable open covering of S. Finally, for each $A_{m(x)}$, choose $U_{\alpha_{m(x)}} \ni A_{m(x)} \subseteq U_{\alpha_{m(x)}}$. Then $\{U_{\alpha_{m(x)}} \mid x \in X\}$ is a countable subcoverning of S. $\blacksquare$

Corollary 2.32. *Let $S \subseteq \mathbb{R}^k$ be open, if $S = \bigcup_{\alpha \in I} U_\alpha$ is a union of open sets in X, then $S = \bigcup_{n=1}^{\infty} U_{\alpha_n}$ is a countable union. $\because$ By Lindelof covering theorem.*