1. Error Analysis

Definition:

let x is a value, $\tilde{x}$ is a estimated value

(1) absolute error, $E_a = |x - \tilde{x}|$

(2) relation error, $E_r = |\frac{x - \tilde{x}}{x}|$

(3) percentage error, $E_p = 100 \times |\frac{x - \tilde{x}}{x}|$

$\exists \epsilon > 0, |x - \tilde{x}| < \epsilon$, Then ϵ is upper limit of the absolute error measures the absolute accuracy.

1.1. Error in Implementation of Numerical Methods.

(1) Round-off Error

(2) Overflow & Underflow

(3) Floating Point Arithmetic and Error Propagation

(4) Truncation Error

(5) Machine eps (Epsilon)

(3)Floating Point Arithmetic and Error Propagation.

Let x_1, x_2 are values, E_1, E_2 are error of x_1, x_2,We want to check the change of error in $" + "," - "," * ","/"$

"+"

Let $x = x_1 + x_2$, error of x is E
Then $x + E = x_1 + x_2 + E_1 + E_2 \implies E = E_1 + E_2$
by triangle inequality
Absolute Error $= |E| \leq |E_1| + |E_2|$
Relative Error $= \frac{|E|}{|x|} \leq \frac{|E_1|}{|x|} + \frac{|E_2|}{|x|}$

"-"(Similar "+")

”*”

Let $x = x_1 * x_2$
Then $x + E = (x_1 + E_1)(x_2 + E_2) = x_1x_2 + E_2x_1 + E_1x_2 + E_1E_2$
Absolute Error $= |E| \leq |x_2E_1| + |x_1E_2|$
Relative Error $= \dfrac{|E_1|}{|x|} \leq \dfrac{|E_1|}{|x_1|} + \dfrac{|E_2|}{|x_2|}$

”/”

Let $x = x_1/x_2$
$x + E_x = \dfrac{x_1 + E_1}{x_2 + E_2}\left(\dfrac{x_2 - E_2}{x_2 - E_2}\right) = \dfrac{x_1x_2 + E_1x_2 - x_1E_2}{x_2^2 - E_2^2} + E_1E_2$
Absolute Error $= |E_x| = |\dfrac{E_1x_2 - x_1E_2}{x_2^2}| \leq \dfrac{|E_1|}{|x_2|} + \dfrac{|x_1E_2|}{x_2^2}$
Relative Error $= \dfrac{|E_x|}{|x|} \leq \dfrac{|E_1|}{|x_1|} + \dfrac{|E_2|}{|x_2|}$

(4)Truncation Error. Cause by approximation infinite with its finite terms.
Use Taylor series ($f(x) \in P(C)$) as example
Let $x = a, f(x) = f(a) + f'(a)(x-a) + f''(a)\dfrac{(x-a)^2}{2!} + \cdots + \dfrac{(x-a)^n}{n!}f^n(a) + \cdots + Rn$
$Rn = \displaystyle\int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt$

Thm 1(First Mean Value Theorem)

If g is continuous on $[a, x]$, then $\exists\ \xi$ between a and x s.t.

$$\int_a^x g(t)\ dt = g(\xi)(x-a)$$

Thm 2(Second Mean Value Theorem)

If g, h is differentiable and integrable on $[a, x]$, h does not change sign on $[a, x]$
then $\exists\ \xi$ that $a \leq \xi \leq x$ s.t.

$$\int_a^x g(t)h(t)\ dt = g(\xi)\int_a^x h(t)\ dt$$

since $t \in [a, x], h(t) = (x-t)^n\dfrac{1}{n!},\ f^{(n+1)}(t)$ is continuous
$\exists\ \xi \in [a, x],\ R_n = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}f^{(x+1)}(\xi),\ \xi \in [a, a+h]$
(Ref. Violin page:799)

since power series convergent, $R_n(x) \to 0,\ as_n \to \infty$

Definition

Given $\{\, a_n \,\}\{\, b_n \,\}\,,\ b_n \geq 0,\ \forall n \geq 1$
$a_n = O(b_n)$ if $\exists\, M > 0 \to |a_w| \leq M b_n \;\forall\, n \geq 1$
$R_n(x) = O(h^{n+1})$

1.2. Condition & Stability.

Condition number is sensitivit of the function
Stability is used to describle the sensitity of the process
Condition number of the $f(n)$

$$\text{CN} = \frac{|\frac{f(x)-f(\tilde{x})}{x-\tilde{x}}|}{|\frac{x-\tilde{x}}{x}|} = |\frac{f(x)-f(\tilde{x})}{x-\tilde{x}}| \cdot |\frac{x}{f(x)}| = |\frac{x}{f(x)} \cdot f'(x)|$$

by Mean Value Theorem,

$$\frac{f(x)-f(\tilde{x})}{x-\tilde{x}} \approx f'(x)$$

when CN ≤ 1 is **well condition**, other is **ill condition**
when the function is more sensitive to change, the condition number will be more big.

2. Methods for $f(x) = 0$

we have four way to deal this problem

(1) Direct analytical Method

(2) Graphical

(3) Trial and Error Method

(4) Iterative Method

Thm. 3(Mean Value Theorem)

Let f be a continuous function on $[a, b] = I$(connected),
if $f(a) \le c \le f(b)$ that $\exists\ \xi \in [a, b] \to f(\xi) = c$

Corollary

Let f be a continuous function on $[a, b] = I$(connected)
i.e. $f(a) \cdot f(b) < 0\ \ni\ \exists c \in (a, b)\ \ni\ f(c) = 0$
c is a root of f(t)

Iterative Method

2.1. Bisection Method.

Let a, b be fixed satisfying Thm.3
$\therefore f(a) \cdot f(b) < 0, f$ is continuous on $[a, b]$. The first approximation is $x_0 = \dfrac{a+b}{2}$
if $f(a) \cdot f(x_0) \le 0$, then By Thm. 3 the root will lie on (a, x_0) and $x_1 = \dfrac{a+x_0}{2}$
continue the process, let $x_{n-3}, x_{n-2}, x_{n-1}$ be same step, then nth approximation
if $f(x_n - 1) \cdot f(x_{n-3}) \le 0$, then $x_n = \dfrac{x_{n-1} + x_{n-2}}{2}$
else $f(x_n - 1) \cdot f(x_{n-3}) \ge 0$, then $x_n = \dfrac{x_{n-1} + x_{n-3}}{2}$
we shall label the interval by algorithm

$$[a, b] = [a_0, b_0], [a_1, b_1][a_2, b_2], \cdots$$

by construction $b_n a_n = \dfrac{1}{2}(b_{n-1} - a_{n-1})$, Hence $b_n - a_n = \dfrac{1}{2^n}[b_0 - a_0],\ \forall n \ge 1$
Clearly $a_0 \le a_1 \le \cdots \le b, b_0 \ge b_1 \ge \cdots \ge a, \{\, a_n \,\}, \{\, b_n \,\}$ is bdd and monotonic

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = f(r)$$

by assumption $f(a_n)f(b_n) < 0$, $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(r)$

$\therefore f(b_n) = f(r), 0 \leq [f(r)]^2 \leq 0 \implies f(r) = 0$

The process is called **nested internal property**

Let $\{ C_k \}_{k=1}^{\infty}$ is a $\downarrow$ sequence of nonempty closed compact subset of X, then $\cap k \subset k \neq \varnothing$
if $c_k \to 0$, then $\cap_k c_k = \{ r \}$
Let ξ be the solution $f(x) = 0$, then $\{ x_0 - \xi \} \leq \dfrac{b-a}{2}, \cdots, \{ x_n - \xi \} \leq \dfrac{b-a}{2^{n+1}}$
Definition(p-order-convergence)

$\{ x_n \}$: seq, $x_n \to z, s_n \to \infty$, define $\epsilon_n = z - x_n$, if $\exists c > 0, p \geq 1$

$$\lim_{n \to \infty} \frac{|\epsilon_{n+1}|}{|\epsilon_n|^p} = c$$

we call $\{ x_n \}$ is p order convergence
if $c \leq 1$, then it's good(only check this when it's a first order convergence)

Let ϵ_n be the error i.e. $\epsilon_n = |x_n - \xi|$, $\epsilon_n \leq \dfrac{b-a}{2^{n+1}} \leq \epsilon$, i.e. $h \geq \dfrac{ln(b-a) - ln\epsilon}{ln2} - 1$
$\epsilon_n = |x_n - \xi| \leq \dfrac{1}{2}(\dfrac{b-a}{2^n}) \approx \dfrac{1}{2}\epsilon_n - 1 \implies \lim\limits_{n \to \infty} |\dfrac{\epsilon_n}{\epsilon_n - 1}| = \dfrac{1}{2}$
Then Bisection Method is first order convergence

2.2. Newton-Taphson Method.

observation:
Let x_0 be an initial approximate to the root of $f(x) = 0$, then $x_0 + h$ is the exact root of $f(x) = 0$, i.e. $f(x_0 + h) = 0$, from Taylor series, $f(x_0 + h) = f(x_0) + h \cdot f(x_0) + \cdots$
i.e. $x_0 \approx x_0 + h$
the first order approximation, $f(x_0 + h) = f(x_0) + h \cdot f'(x_0) = 0 \implies h = \dfrac{-f(x_0)}{f'(x_0)}$
Let $x_1 = x_0 + h$ be the next approximation to the root, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$
Ingeneral $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ $\forall n \geq 1$
Example

Consider the $f(x) = x^2 - M = 0 (M > 0)$

$$x_{n+1} = x_n - \frac{x_n^2 - M}{2x_n} = \frac{1}{2}(x_n + \frac{M}{x_n})(\star)$$

Ingeneral, also can obtain for the kth root of M, i.e. $\sqrt[k]{M}$ with $f(x) = x^k - M = 0$
if $x_1 > \sqrt{M}$, and define $x_2, \cdots$ by the interaction formula $(\star)$, then
(1)$\{ x_n \}$ is $\downarrow$(trivial) (2)$\{ x_n \}$ is bounded above($x_{n+1} = \frac{1}{2}(x_n + \frac{M}{x_n}) \geq \sqrt{x_n(\frac{M}{x_n})} = \sqrt{M}$)
By (1)(2), $\lim_{n \to \infty} x_n = \sqrt{M}$ exists.

observation
let $(x_0, f(x_0))$ be any point on the curve
$y = f(x)$, then $y - f(x_0) = f'(x_0)(x - x_0)$

Thm. 4(The NR method is 2 order convergence)

Let x denote the exact value of the root of $f(x) = 0$
x_n, x_{n+1} be two approximation S to the exact root $a, (f(a) = 0)$
if $\epsilon_n, \epsilon_{n+1}$ corresponding error S, then $x_n = a + \epsilon_n, x_{n+1} = a + \epsilon_{n+1}$
by(NR)

$$
\begin{aligned}
a + \epsilon_{n+1} &= a + \epsilon_n - \frac{f(a - \epsilon)}{f'(a + \epsilon_n)} \\
\epsilon_{n+1} &= S_n - \frac{f(a) + \epsilon_n f'(a) + \frac{\epsilon_n^2}{2!} f''(a) + \cdots}{f'(a) + \epsilon f''(a) + \frac{\epsilon_n^2}{2!} f'''(a) + \cdots} \\
&= \epsilon_n - \frac{\epsilon_n \left(f'(a) + \epsilon f'(a) + \frac{\epsilon_n^2}{2!} f''(a) + \cdots \right)}{f'(a) + \epsilon_n f''(a) + \frac{\epsilon^2}{2!} f''(a) + \cdots} \\
&= \frac{\epsilon [f'(a) + \epsilon_n f''(a) + \frac{\epsilon_n^2}{2!} f(a) + \cdots - [f'(a) + \frac{\epsilon_n}{2!} f''(a) + \cdots]]}{f'(a) + \epsilon_n f''(a) + \frac{\epsilon_n^2}{2!} f'''(a) + \cdots} \\
&= \frac{\epsilon_n [\frac{\epsilon_n}{2} f'(a) + \frac{\epsilon_n^2}{3} f''(a) + \cdots]}{f'(a) + \epsilon_n f''(a) + \frac{\epsilon_n^2}{2!} f'''(a) + \cdots} \\
&= \frac{\epsilon_n^2 [\frac{1}{2} f'(a) + \frac{\epsilon_n}{3} f''(x) + \cdots]}{f'(a)[1 + \epsilon_n \frac{f''(a)}{f'(a)} + \frac{\epsilon_n^2}{2!} \frac{f'''(a)}{f''(a)} + \cdots]} \\
\implies \frac{e_{n+1}}{\epsilon_n^2} &= \frac{\frac{1}{2} f''(a) + \frac{\epsilon_n}{3} f'''(a) + \cdots}{f'(a)(1 + \epsilon_n \frac{f''(a)}{f'(a)} + \cdots)}
\end{aligned}
$$

$\lim_{n\to\infty} |\frac{\epsilon_{n+1}}{\epsilon^n}| > \frac{1}{2} |\frac{f''(a)}{f'(a)}| < +\infty$
Remark: if f(x) has double root S

3. Eigen Problem

3.1. Review eigenvalue & eigenvector.

$A \in M_{n\times n}(\mathrm{R}/C),\ AX = \lambda X = \lambda(IX) = \lambda IX \implies (A-\lambda I)X = 0$
it's a homogeneous system of n linear equation, it determinate is 0
$p(\lambda) = det(A-\lambda I) = 0,\ deg(p(\lambda)) = n$

Define $\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}, X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, X is a eigen vector of A, λ is a eigenvalue of A

the normalized eigenvector $\hat{X} = \dfrac{1}{||X||}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ where $||X|| = (X^TX)^{\frac{1}{2}} = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$

if T is diagonalizable, then $\exists$ order basis β, $\beta \ni [T]_\beta = D$, which is a diagonal matrix
similarly A is diagonalizable if L_A is diagonalizable

diagonalizable

$$\begin{cases} \text{the c.p split} \begin{cases} n \text{ distinct eigenvalue} \\ \text{other} \begin{cases} \text{algebraic multiplicity} = \text{geometric multiplicity} \\ \text{algebraic multiplicity} \neq \text{geometric multiplicity}\textbf{(not diagonalizable)} \end{cases} \end{cases} \\ \text{the cp does not split (not diagonalizable)} \end{cases}$$

(c.p. is charateristic polynomial)

E_λ is subspace ,$E_\lambda = N(T-\lambda I)$, E_λ is T-invariant, i.e. $T(E_\lambda) \subseteq E_\lambda$,$1 \le \dim(E_\lambda) \le m$
if T is diagonalizable, then

$$V = E_{\lambda_1} \oplus E_{\lambda_2} + \cdots + E_{\lambda_n} \Leftrightarrow V = k\lambda_1 \oplus \cdots \oplus k\lambda_n$$

Let any eigenvalue λ be repeated r times with k linearly independent eigenvector
r is algebraic multiplicity, k is geometric multiplicity

3.2. some introduction.

we will learn ODE and PDE next time
$\frac{dX}{dt} = AX$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\frac{dx_1}{dt} = a_{11}x_1 + a_{12}$, $\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2$
$X = \chi e^{\lambda t}$ is the solution of system, χ is column vector, λ is parameter to be determind
$\frac{d\chi e^{\lambda t}}{dt} = \lambda \chi e^{\lambda t} \implies \lambda \chi e^{\lambda t} = A \chi e^{\lambda t} \implies \lambda \chi = A \chi$

Definition

The spectrum of A, radius p of the smallest circle with center at the origin and contains all the spactual radius

3.3. Power Method.

Definition

Let $A \in M_{n\times m}(\mathrm{C})$, for $1 \leq i, j \leq n$
define $p_i(A)$ to be the sum of the abs-values of the entries of row i of A and $r_i(A)$ to be the sum of the abs-values of the entries of column j of A
$p_i(A) = \sum_j^n ||A_i j||$, $r_j(A) = \sum_i^n ||A_i j||$
$e(A) = \max(p_i(A))$, $r(A) = \max(r_j(A))$, $1 \leq i, j \leq n$

Definition

an $n \times n$ matrix A, we define the ith Geisg disk c_i to be the disk in the complex plain with center A_{ii} an radius $r_i = p_i(A) - |A_{ii}|$, $c_i = \{\, z \in \mathrm{C} \mid |z - A_{ii}| < r_i \,\}$

Theorem(Geisg Disk Theorem 1)

Let $A \in M_{n\times n}(\mathrm{C})$, then every eigenvalue of A is contained in a Geisg Disk

pf: Let λ be eigenvalue of A r.t. eigenvector $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, clearly $Av = \lambda v$

Then $I_j^n = A_{ij}v_j = \lambda_{ri}$, $1 \leq i \leq n(\star)$
suppose v_k is the coordinate of V having the largest abs-solute,($v_k \neq 0$)
claim $\lambda \in C_k$, i.e. $|\lambda - A_{kk}| \leq r_k$ For $i = k$,by $(\star)$

$$\begin{aligned} |\lambda v_k - A_{kk}v_k| &= |\sum_{j=1}^{n} A_{kj}v_j - A_{kk}v_k| \\ &= |\sum_{j\neq k} A_{kj}v_j| \\ &\leq \sum_{j\neq k} |A_{kj}||v_j| \\ &\leq \sum_{j\neq k} |A_kj||v_k| = r_k|v_k| \end{aligned}$$

Corollary 1

Let λ be any eigenvalue of $A \in M_{n\times n}(\mathrm{C})$, then $|\lambda| \leq p(A) = \max(p_i(A))$
pf: by Thm. $|\lambda - A_{kk}| \leq r_k$ for some k, $1 \leq k \leq n$
$|\lambda| = |\lambda - A_{kk}| + |A_{kk}| \leq r_k + |A_{kk}| = p_k(A) \leq p(A)$

Corollary 2

$A^T \in M_{n\times n}(\mathrm{C}),\ |\lambda| \leq r(A) = \max(r_j(A))$

Corollary 3

Let λ be eigenvalue of $A \in M_{n\times n}(\mathrm{C})$, $|\lambda| \leq \min\{\, p(A), r(A)\,\}$
by corollary 1 & 2, we are done.

Theorem(Geisg Disk Theorem 2)

Let $A \in M_{n\times n}(\mathrm{C})$, k of the disks are disjoint from the others, then exactly k eigenvalue are contained in the union of these disks.
pf: the gumltprinciple
Ref:Matrix Analysis 2/e (Horn/Johnson) P.388,389

Rayleign Power Method

Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalue of matrix, $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$
our goal is to find $|\lambda_1|$

Let $x_1, \cdots, x_n$ be eigenvectors, r.t. $\lambda_1, \cdots, \lambda_n, \implies Ax_i = \lambda_i x_i, \ \forall 1 \leq i \leq n$
if the matrix A(which is diagonalizable) has n linearly independent eigenvectors
then $x = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ for some $c_i \in \mathrm{C}$

$$\begin{aligned} Ax &= A(c_1x_1 + \cdots + c_nx_n) \\ &= c_1Ax_1 + \cdots + c_nAx_n \\ &= c_1\lambda_1 x + \cdots + c_n\lambda_n x \\ &= \lambda_1 \left(c_1x + c_2\left(\frac{\lambda_2}{\lambda_1}\right) x + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)\right) \end{aligned}$$

$$\begin{aligned} A^2x &= A\left(\lambda_1 \left(c_1x + c_2\left(\frac{\lambda_2}{\lambda_1}\right) x + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)\right)\right) \\ &= \lambda\left(c_1Ax + c_2\left(\frac{\lambda_2}{\lambda_1}\right) Ax + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right) Ax\right) \\ &= \lambda_1^2\left(ax + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^2 x + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^2 x\right) \end{aligned}$$

Continue process

$$\begin{aligned} A^kx &= \lambda_1^k\left(c_1x_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k x_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k x_n\right) \\ A^{k+1}x &= \lambda_1^{k+1}\left(c_1x_1 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^{k+1} x_n\right) \end{aligned}$$

$$\lim_{k\to\infty} \frac{A^{k+1}x}{A^kx} = \lambda_1$$

A stepwise procedure

(i) $X^{(0)}$ is initial vector

(ii) $Y^{(0)} = AX^0$

(iii) $\lambda^{(1)}$ is the absolutely largest element, common from the vector $Y^{(0)}$

Let the remainly vector be X^1, $Y^(0) = \lambda^{(1)}X^{(1)}$

(iv) reapeating (ii) and (iii), $Y^{(k)} = \lambda^{(k+1)}X^{k+1}$

(v) $|\lambda^{(k+1)}|, x^{(k+1)}$ is goal

Example

$$A = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 1 & -2 \\ -2 & 0 & 5 \end{pmatrix}, \quad X^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad Y^0 = AX^0 = \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix},$$

$$\lambda^{(1)} = 6, \; Y^{(0)} = 6 \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} = \lambda^{(1)}X^{(1)} \implies Y^{(1)} = AX^{(1)} = A^{(1)} = \begin{pmatrix} 2 \\ 0 \\ 0.5 \end{pmatrix}, \lambda^{(2)} = 2$$

Inverse Power Method

Let λ_i be an eigenvalue of matrix A, then $\frac{i}{\lambda_i}$ is eigenvalue of the matrix A^{-1},The eigenvector of A^{-1} is X_i

pf: $Ax_i = \lambda_i x_i \implies \frac{1}{\lambda_i}(Ax_i) = x_i \implies \frac{1}{\lambda_i}x_i = A^{-1}x_i$

Shifted Power Method

Let λ_i be an eigenvalue of matrix A, then $(\lambda_i - k)$ is an eigenvalue of the matrix $A - kI$ with the same eigenvector as that matrix A

pf: $Ax_i = \lambda_i x_i \implies (A - kI)x_i = AX_i - kX_i = \lambda_i x_i - kx_i = (\lambda_i - k)x_i$

4. Review Linear Algebra

4.1. Lagrange polynomials. Let $T : P_n(\mathrm{F} \to \mathrm{F}^{n_1}$ be linear transform defined by $T(f) = (f(c_0), \cdots, f(c_n))$, which $c_0, c_1, \cdots, c_n$ are distinct scalars in an infinite field F,β be the stander order basis for $P_n(\mathrm{F})$, γ be the stander order basis for F^{n+1}

Claim 1:

$$[T]_\beta^\gamma = M = \begin{bmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_0 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{bmatrix}, \ \beta = \{\, 1, x, \cdots, x^n \,\}, \ \gamma = \{\, (1, \cdots, 0), \cdots, (0, \cdots, 1) \,\}$$

$T(1) = (1, \cdots, 1), T(x) = (c_0, \cdots, c_n), \cdots, T(x^n) = (c_0^n, \cdots, c_n^n)$

M is called a Vandemonde Matrix

Claim2: $\det(M) \neq 0$

$\because \dim(P_n(F)) = \dim(\mathrm{F}^{\mathrm{n+1}}) = n + 1,\ T$ is linear **check**,T is one-to-one **check**,

$\therefore T$ is invertible, $\therefore [T]_\beta^\gamma$ is invertible $\Leftrightarrow \det([T]_\beta^\gamma \neq 0 \implies \det(M) \neq 0$

Claim3: $\det(M) = \displaystyle\prod_{0 \le i < j \le n} (c_j - c_i)$

Proof. we use the induction on $n = \deg(P_n(\mathrm{F}))$

$$n = 1, \det \begin{bmatrix} 1 & c_0 \\ 1 & c_1 \end{bmatrix} = c_1 - c_0$$

Suppose the statement holds for n

$$\det \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix} = \det \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 0 & c_1 - c_0 & c_1^2 - c_0^2 & \cdots & c_1^n - c_0^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & c_n - c_0 & c_n^2 - c_0^2 & \cdots & c_n^n - c_0^n \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & c_1 - c_0 & c_1^2 - c_1 c_0 & \cdots & c_1^n - c_0 c_1^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & c_n - c_0 & c_n^2 - c_0 c_n & \cdots & c_n^n - c_0 c_n^{n-1} \end{pmatrix} = \det \begin{pmatrix} c_1 - c_0 & c_1(c_1 - c_0) & \cdots & c_1^{n-1}(c_1 - c_0) \\ \vdots & \vdots & \vdots & \vdots \\ c_n - c_0 & c_n(c_n - c_0) & \cdots & c_n^{n-1}(c_n - c_0) \end{pmatrix}$$

$$= (c_1 - c_0) \cdots (c_n - c_0) \cdot \det \begin{pmatrix} 1 & c_1 & \cdots & c_n^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & c_n & \cdots & c_n^{n-1} \end{pmatrix} = (c_1 - c_0) \cdots (c_n - c_0) \prod_{1 \le i < j \le n} (c_j - c_i)$$

$$= \prod_{0 \le i < j \le n} (c_j - c_i)$$

■

Let $P_n(X) = a_0 + a_1 x + \cdots + a_n x^n$, where $a_0, \cdots, a_n \in F$

$P_n(X)$ is a polynomial s.t. it interpolated the $n + 1$ points

$P_n(x_0) = a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_n x_0^n = y_0$

$$\vdots$$

$$P_n(x_n) = a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_n x_n^n = y_n$$

In matrix form

$$\begin{bmatrix}1 & x_0 & x_0^2 & \cdots & x_0^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \cdots & x_n^n & \end{bmatrix}\begin{bmatrix}a_0 \\ \vdots \\ a_n\end{bmatrix} = \begin{bmatrix}y_0 \\ \vdots \\ y_n\end{bmatrix}$$

Now we define the Lagrange polynomials of degree $l_0(x), \cdots, l_n(x)$ as

$$l_i(x_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

The $P_n(x) = y_0 l_0(x) + \cdots + y_n l_n(x) = \sum_{i=0}^{n} y_i l_i(x)$

$l_i(x)$ is an n-degree polynomial with roots says

$l_i(x) = c_i(x - x_0)\cdots(x - x_n) = c_i \prod_{j\neq i}(x - x_j)$

$l_i(x_i) = 1 = 1c_i \prod_{j\neq i}(x_i - x_j),\ c_i = \frac{1}{\prod_{j\neq i}(x_i - x_j)},\ l_i(x) = \frac{\prod_{j\neq i}(x - x_j)}{\prod_{j\neq i}(x_i - x_j)}$

4.2. special matrix.

Theorem (Shor's Lemma)**.** *Let T is a linear operator on V which is a finite dimension inner product space, Suppose the characteristic polynomial splits, Then $\exists$ order normal basis $\beta \implies [T]_\beta$ is uppertriangle.*

Note

normal : $AA^* = A^*A(TT^* = T * T)$
self-adjoint : $A* = A(T^* = T)$

Theorem (Spectral Theorem)**.** *Let T be a linear operator on V which is a finite dimensional inner product space*
$\mathbb{C}$: *T is normal $\Leftrightarrow \exists$ order normal basis β containing eigenvectors $\Leftrightarrow T$ is diagonal over $\mathbb{C}$*
$\mathbb{R}$: *T is self-adjoint $\Leftrightarrow \exists$ order normal basis β containing eigenvector $\Leftrightarrow T$ is diagonal over $\mathbb{R}$*

T is diagonal $\implies \exists$ order normal basis $\implies [T]\beta$ is diagonal,T is normal(over C)
$\implies [T]^*_\beta[T]_\beta = [T]_\beta[T]^* \implies [T^*]_\beta[T]_\beta = [T]_\beta[T^*]_\beta \implies [T^*T]_\beta = [TT^*]_\beta$

T is diagonalizable $\begin{cases} (C) \Leftrightarrow T \text{ is normal(unitary equivalent)(by Shur's Lemma)} \\ (R) \Leftrightarrow T \text{ is self-adjoint(orthogonal equivalent)(eigenvalue is real + Shur's)} \end{cases}$

Property

T is unitory $\Leftrightarrow$ every row(column) vectors is orthonormal basis

unitary equivalent $A \sim B \Leftrightarrow \exists$ unitary matrix $Q \implies A = Q^*BQ$
orthogonally equivalent $A \sim B \Leftrightarrow \exists$ orthogonal matrix $P \implies A = P^*BP$

Define: Let V be a vector space, $W_1, W_2 \leq V \implies V = W_1 \oplus W_2$
A function $T : V \to V$ is called projection on W_1 along W_2 if for $x = x_1 + x_2$,
$x_1 \in W_1, x_2 \in W_2,\ T(x) = x_1$

Property

$R(T) = W_1,\ N(T) = W_2,\ V = R(T) \oplus N(T)$

Proof. Claim: $R(T) = W_1$
$(\supseteq) x \in W_1 \implies T(x) = x \in R(T)$
$(\subseteq) x \in R(T) \implies \exists y \in V \implies T(y) = x$
$\because V = W_1 \oplus W_2 \therefore y = x_1 + x_2$ for some $x_1 \in W_1,\ x_2 \in W_2 \implies T(y) = x_1 = x \in W_1$
Claim: $N(T) = W_2$ **exercise**
Claim: $V = R(T) \oplus N(T), \because$(1),(2), it's trivial ■

Property

T is projection $\Leftrightarrow T = T^2$

Proof.

$(\Rightarrow)$ T is projection$(V = W_1 \oplus W_2)$
given $y \in V, \because V = W_1 \oplus W_2,\ \therefore \exists x_1 \in W_1, x_2 \in W_2 \ni y = x_1 + x_2$
$\therefore T(y) = T(x_1 + x_2) = x_1 = T(x_1) = T(T(y))$
$(\Leftarrow)$ $T = T^2$(use the previous proposition to build $R(T), N(T)$
$\implies V = R(T) \oplus N(T) \implies T$ is projection)
Given $x \in V$
(1) $x = T(x) + [x - T(x)]$
(2)$T^2(x) = T(x)$(assumption)

$$\begin{cases} (i) T(x) \in R(T) \\ (ii) x - T(x) \in N(T) \\ (iii) R(T) \cap N(T) = \{0\} \\ (iv) V = R(T) + N(R) \end{cases}$$

$(i) T(T(x)) = T(x) \in R(T),\ (ii) T(x - T(x)) = T(x) - T(T(x)) = T(x) - T(x) = 0,$
(iv)trivial($\because$(1))
(iii) Suppose $N(T) \cap R(T) = \{v\}, v \neq 0, v \in N(T) \implies T(v) = 0$
$v \in R(T) \implies \exists y \in V \implies T(y) = v \implies T(T(y)) = T(v) = 0, y \in N(T), v = 0 \to\leftarrow$

■

Property: every projections is uniquely determined by the range & kernal
Let $T, U : V \to V, R(T) = R(U) = W_1,\ N(T) = N(U) = W_2$
$\forall x \in V$, let $y' = T(x), y = U(x) \in W_1$

$T(x-y') = T(x) - T(y') = y' - y' = 0,\ x - y' \in W_2 \implies x \in y' + W_2(\text{coset})$
$\implies \exists z' \in W_2 \ni x = y' + z' \implies y = U(x) = U(y'+z') = U(y') + U(z') = y' + 0 = y' = T(x)$

Theorem. *orthogonal projection* T, $V = R(T) \oplus N(T)$, $R(T)^{\perp} = N(T)$, $N(T)^{\perp} = R(T)$
T *is orthogonal projection* $\Leftrightarrow T = T^2 = T^*$

Theorem. *Let matrix* A *normal(*$\mathbb{C}$*), self-adjoint(*$\mathbb{R}$*)*
A *is unitary equivalent to a diagonal matrix*
$u_1, \cdots, u_n$ *:eigenvectors(orthonormal),* $\lambda_1, \cdots, \lambda_n$ *:eigenvalues*

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} u_1^T \\ \vdots \\ u_n^T \end{bmatrix} = A = \sum_{i=1}^{n} \lambda_i u_i u_i^T \textit{ (spectral decomposition)}$$

Check A is normal
$A = u_i u_i^*, L_A = u_i u_i^*,\ L_A L_A = L_A^2 = (u_i u_i^*)(u_i u_i^*) = u_i u_i^* = L_A$
$L_A^* = (L_A)^* = (u_i u_i^*)^* = u_I^{*^*} u_i^* = u_i u_i^* = L_A$

example

$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$,c.p. of $A = (1-t)(-2-t) - 4 = t^2 + t - 6 = (t+3)(t-2)$
the eigenvector are $\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
$\because$ they are distinct eigenvalue $\implies$ orthogonal
$r_1 = \frac{1}{\sqrt{5}} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix}, r_2 = \frac{1}{\sqrt{5}} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
$\begin{bmatrix} r_1 & r_2 \end{bmatrix} \begin{bmatrix} -3 & - \\ 0 & 2 \end{bmatrix} \begin{bmatrix} r_1^T \\ r_2^T \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$
$A = -3 r_1 r_1^T + 2 r_2 r_2^T$

5. Integral & Differential

we have two usually problem in math

ODE (ordinary differential equation)

Example $\begin{cases} y' = f(x,y) \\ f(x_0) = y_0 \end{cases}$

PDE (partial value problem)

Example $\begin{cases} \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial f}{\partial y^2} = 0 \end{cases}$

where $\triangle = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

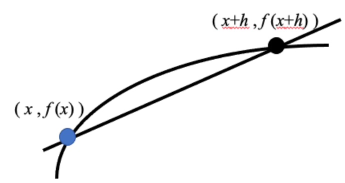
5.1. Differentials.

$f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, correction: $h > 0$(time step)

The symbol example: $D_a{}^c{}_b$ where a is converge nation, b is method, c is first order

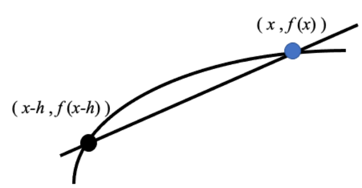
Forward

$$D_1{}^1{}_f(f,x,h) = \frac{f(x+h) - f(x)}{h}$$



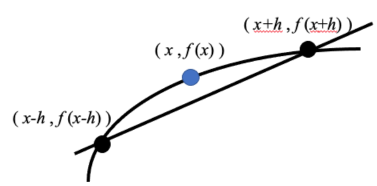
Backward

$$D_1{}^1{}_b(f,x,h) = \frac{f(x) - f(x-h)}{h}$$



Centr

$$D^1_{2\ c}(f,x,h) = \frac{f(x+h) - f(x-h)}{2h}$$



Consider convergence from Taylor formula(expansion)

$$\begin{aligned}
f(x+h) &= f(x)+\frac{f'(x)}{1!}\cdot h+\frac{f''(x)}{2}\cdot h^2+\cdots \\
&= f(x)+\frac{f'(x)}{1!}\cdot h+\frac{f''(\xi)}{2!}\cdot h^2,\ \text{for some } \xi\in(x,x+h),\ h \text{ small} \\
\Longrightarrow&\ f(x+h)-f(x)+\frac{f''(\xi)}{2!}\cdot h^2 = h\cdot f'(x) \\
\Longrightarrow&\ \frac{f(x+h)-f(x)}{h}+\frac{f''(\xi))}{2!} = f'(x)
\end{aligned}$$

which $\dfrac{f(x+h)-f(x)}{h}$ is approximate value, $\dfrac{f''(\xi))}{2!}$ is truncation error, $f'(x)$ is true error

$D_1{}^1{}_f(f,x,h)-f'(x)=\dfrac{f''(\xi)}{2!}\cdot h = O(h)$

Convergence of Central Method

(1) $f(x+h)=f(x)+f'(x)+\dfrac{1}{2}f''(x)h^2+\dfrac{1}{6}f^{(3)}(\xi_1)h^3$

(2) $f(x-h)=f(x)-f'(x)\cdot h+\dfrac{1}{2}f''(x)\cdot h^2-\dfrac{1}{6}f^{(3)}(\xi_2)h^3$

(1) - (2) $\Longrightarrow f(x+h)-f(x-h)=2f'(x)\cdot h+\dfrac{1}{6}[f^{(3)}(\xi_1)+f^{(3)}(\xi_2)]h^3$

$\Longrightarrow 2f'(x)\cdot h+\dfrac{1}{6}\cdot 2f^{(3)}(\xi)\cdot h^3$, where ξ is closed to ξ_1,ξ_2

$\dfrac{f(x+h)-f(x-h)}{2h}-f'(x)=\dfrac{1}{3}f^3(\xi)\cdot h^2=O(h^2)$

Total error of Differentials

total error = rounding + truncation

$$f'(x)\approx\frac{f(x+h)-f(x)}{h}-\frac{f''(\xi)}{2!}\cdot h$$

we can change $f(x+h)$ to $y(x+h)+e_1$, $f(x)$ to $y(x)+e_2$, where e_1,e_2 are error

$$\begin{aligned}
f'(x) &\approx \frac{y(x+h)+e_1-[y(x)+e_2]}{h}-\frac{f''(\xi)}{2!}\cdot h \\
&= \frac{[y(x+h)-y(x)]}{h}+\left(\frac{e_1-e_2}{h}+\frac{f''(\xi)}{2!}\right)
\end{aligned}$$

and the error is a polynomial $\rightarrow$ differ to find minimum

$$
\begin{aligned}
|E(h)| &= |\frac{e_1 - e_2}{h} - \frac{f''(\xi)}{2!} \cdot h| \leq |\frac{e_1 - e_2}{h}| + |\frac{f''(\xi)}{2!} h| \\
&\leq \frac{2e}{h} + |\frac{f''(\xi)}{2!} \cdot h| \equiv T(h) \quad \text{where} \quad e = max\{\,|e_1|, |e_2|\,\} \\
T'(h) &= \frac{-2e}{h^2} + \frac{f''(\xi)}{2!} = 0 \\
h^2 &= \frac{4e}{|f''(\xi)|},\ h = 2\sqrt{\frac{e}{|f''(\xi)|}} \approx 10^{-8}
\end{aligned}
$$

But need to find ξ

5.2. Integral.

Middle point

Let $x_0 = a,\ x_n = 0,\ x_i = x_0 + ih,\ h = \dfrac{b-a}{h},\ x_i^\star = \dfrac{x_{i-1} + x_i}{2}$

$I_m(f : a = b : h) = \displaystyle\sum_{i=1}^{n} h \cdot f(x_i^\star)$

Trapezoid

Let $x_0 = a,\ x_n = 0,\ x_i = x_0 + ih,\ h = \dfrac{b-a}{h},\ x_i^\star = \dfrac{x_{i-1} + x_i}{2}$

$I_i(f, a, b, h) = \dfrac{1}{2}(f(x_0) + f(x_1)) \cdot h + \cdots + \dfrac{1}{2}[f(x_{n-1}) + f(x_n)] \cdot h$

$= \dfrac{1}{2} f(x_0) + f(x_1) \cdot h + \cdots + f(x_n - 1)h + \dfrac{1}{2} f(x_n)h$

$= \displaystyle\sum_{1}^{n} w_i f(x_i)$, where $w_1 = \begin{cases} \frac{1}{2},\ i = 0, n \\ 1,\ i \neq 0, n \end{cases}$

Simpson Rule($\sim O(h^2)$)

Let $P_2(x) = ax^2 + bx + c,\ \begin{cases} f_0 = c \\ f_1 = ah^2 + bh + f_0 \\ f_{-1} = ah^2 - bh + f_0 \end{cases},\ b = \dfrac{f_1 - f_2}{h},\ a = \dfrac{f_1 + f_{-1} - 2f_0}{2h^2}$

Consider the integration, we get that

$$
\begin{aligned}
\int_{-h}^{h} ax^2 + bx + c &= \int_{-h}^{h} ax^2 dx + \int_{-h}^{h} bx dx + \int_{-h}^{h} c dx \\
&= \frac{1}{3} 2ah^3 + 2ch \\
&= \frac{n}{3}(f_{-1} + 4f_0 + f_1)
\end{aligned}
$$

Newton Cotes Quadrature Formula

Let $x_0 = a,\ x_n = b,\ x_i = x_0 + ih,\ \forall\ 1 \le i \le n$, then

$$I = \int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_n} (P_n(x) + \epsilon_n(x))dx,\ \text{where } \epsilon_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$\begin{aligned}
\prod_{n+1}(x) &= (x-x_0)\cdots(x-x_n) \\
&= \int_{x_0}^{x_n} f(x_0)L_0(x) + f(x_1)L_1(x) + \cdots + f(x_n)L_n(x)dx + \int_{x_0}^{x_n} \epsilon_n(x)dx \\
&= \int_{x_0}^{x_n} \int_{i=1}^{n} f(x_i)L_i(x)dx + \int_{x_0}^{x_n} \epsilon_n(x)dx \\
&= \sum_{i=0}^{n} f(x_i) \int_{x_0}^{x_n} L_i(x)dx + \int_{x_0}^{x_n} \epsilon_n(x)dx
\end{aligned}$$

$\because P_n(x)$ is the Lagrange polynomial(interpolating polynomial)

$\therefore \lambda_i$ are the constants to be determined

Let $x = x_0 + sh,\ x_i = x_0 + ih,\ \forall\ 1 \le i \le n$, then

$$\begin{aligned}
L_i(x) &= \frac{(x-x_0)(x-x_1)\cdots\widehat{(x-x_i)}\cdots(x-x_n)}{(x_i-x_0)\cdots\widehat{x_i-x_i}\cdots(x_i-x_n)} \\
&= \frac{(sh)((s-1)\cdot h)\cdots((s-n)\cdot h)}{(ih)((i-1)h)\cdots((i-n)\cdot h)} \\
&= \frac{s(s-1)\cdots(s-i+1)(s-i-1)\cdots(s-n)}{i(i-1)\cdots(-1)(1)\cdots(i-n)} \\
&= \frac{s(s-1)\cdots(s-i+1)(s-i-1)\cdots(s-n)}{i!(-1)^{n-1}(n-i)!} \\
\Longrightarrow \lambda_i &= \int_{x_0}^{x_n} \frac{s(s-1)\cdots(s-i+1)(s-i-1)\cdots(s-n)}{i!(-1)^{n-1}(x-i)!} \cdot hds
\end{aligned}$$

we use this formula, we have

1. Simpson's $\frac{1}{3}$ (Rule $n = 2$)
2. Simpson's $\frac{3}{8}$ (Rule $n = 3$)
3. boole rule ($n = 4$)
4. Weddle rule ($n = 6$)