Advance Calculus Exercise

Exercise 1 (Chapter 1)

1. Let a and b be two real numbers. If $a \leq b + \epsilon$ for any $\epsilon > 0$, then $a \leq b$.

Solution. If not, $a > b \implies a - b > 0$. For $\epsilon = \dfrac{a-b}{2} > 0$

$$a \leq b + \frac{a-b}{2} = \frac{a+b}{2}$$

$\implies a \leq b (\rightarrow\leftarrow)$ ■

2.
(a) If $r, s \in \mathbb{Q}$, then $r + s$ and rs are rational.
(b) If $r \in \mathbb{Q}$ with $r \neq 0$ and $x \in \mathbb{R} - \mathbb{Q}$, then $r + x$ and rx are irrational.

Solution.

(a) Given $r, s \in \mathbb{Q}$, $r = \dfrac{a}{b}$, $s = \dfrac{c}{d}$ where $a, b, c, d \in \mathbb{Z}$, $b, d \neq 0$. Thus,

$$r + s = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \in \mathbb{Q}$$

$$rs = \frac{ac}{bd} \in \mathbb{Q}$$

(b) If not, $r + x, rx \in \mathbb{Q}$. Since $r + x \in \mathbb{Q}$ and $r \in \mathbb{Q}$, $(r + x) - r = x \in \mathbb{Q} (\rightarrow\leftarrow)$ to $x \in \mathbb{R} - \mathbb{Q}$
Since $rx \in \mathbb{Q}$ and $r \in \mathbb{Q}, r^{-1}(rx) = x \in \mathbb{Q} (\rightarrow\leftarrow)$

■

3. Let $f : X \to Y$ be a function. If $B \subseteq Y$, we denote by $f^{-1}(B)$ the largest subset of X which f maps into B. That is,

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$

The set $f^{-1}(B)$ is called the inverse image of B under f. Prove the following for arbitrary $A, A_1, A_2 \subseteq X$ and $B, B_1, B_2 \subseteq Y$
(a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$
(b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. Given an example such that the inclusion is strict.
(c) $A \subseteq f^{-1}[f(A)]$ and $f[f^{-1}(B)] \subseteq B$. Given an example such that the inclusion is strict.

(d) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$ and $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

Solution.
(a)
($\subseteq$) Given $y \in f(A_1 \cup A_2)$, $y = f(x)$ for some $x \in A_1 \cup A_2 \implies y = f(x),\ x \in A_1$ or $y = f(x), x \in A_2 \implies y \in f(A_1)$ or $y \in f(A_2) \implies y \in f(A_1) \cup f(A_2)$
($\supseteq$)
(b)
(c) Given $x \in A, f(x) \in f(A) \implies x \in f^{-1}[f(A)]$.
Conversely, it is not true. Consider $f : \mathbb{Q} \to \mathbb{R}$ by $f(x) = x^2$, $A = \{1\}$, Clearly, $f^{-1}[f(A)] = \{-1, 1\}$. Thus, $A = \{1\} \subsetneq \{-1, 1\} = f^{-1}[f(A)]$
(d)
■

4. Let E be a nonempty subset of an order set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Solution. Suppose α is a non-empty subset of an order set. Suppose α is a lower bound of E and β is an upper bound of E. This means

$$\alpha \leq x \text{ and } x \leq \beta \forall x \in E$$

Since $E \neq \emptyset$, choose $x \in E$, then $\alpha \leq x$ and $x \leq \beta$. By transitivity, $\alpha \leq \beta$ ■

5. Let A, B be two nonempty sets of $\mathbb{R}$
(a) If $A \subseteq B$, then $\sup A \leq \sup B$ and $\inf A \geq \inf B$.
(b) Show that $\inf A \leq \sup A$.
(c) Show that $\inf(-A) = -\sup A$ and $\sup(-A) = -\inf(A)$, where $-A = \{-a \mid a \in A\}$
(d) Show that $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$, where $A + B = \{a + b \mid a \in A, b \in B\}$, the Minkowski sum of A and B.
(e) If A, B be two sets of positive numbers which is bounded above. Let $\alpha = \sup A,\ b = \sup B$ and $C = \{ab \mid a \in A,\ b \in B\}$. Prove that $\sup C = ab$.

Solution.

(a) Suppose $A \subseteq B$. To prove $\sup A \leq \sup B$

($*$) If A is unbdd above, then B is also. Thus, $\sup A = +\infty = \sup B$

($*$) If A is bdd above then we have two cases

($*_1$) If B is unbdd above, then the result follows trivilly

($*_2$) If B is bdd above, then $\sup A,\ \sup B$ exist and finite, say $\sup A = \alpha$ and $\sup B = \beta$.

Now, to prove that $\alpha \leq \beta$. Given $\epsilon > 0$, since $\sup A = \alpha,\ \exists a \in A \ni \alpha - \epsilon < a$. Since $A \subseteq B$ and $\beta = \sup B, a \in B$ and $a \leq \beta$

(b)

(c)

(d) To prove $\sup(A+B) = \alpha + \beta$ If A or B is unbdd above, then the result follows trivially

If A and B are bdd above,say $\sup A = \alpha$ and $\sup B = \beta$

Claim: $\sup(A+B) = \alpha + \beta$

(i) Given $x \in A + B, x = a + b$ for some $a \in A$ and $b \in B$. Since $\alpha = \sup A$ and $\beta = \sup B,\ a \leq \alpha$ and $b \leq \beta \implies x = a + b \leq \alpha + \beta$

(ii) Given $\epsilon > 0$, since $\alpha = \sup A$ and $\beta = \sup B, \exists a \in A, b \in B,\ \alpha - \dfrac{\epsilon}{2} < a$ and $\beta - \dfrac{\epsilon}{2} < b \implies \alpha + \beta - \epsilon < a + b$ and $a + b \in A + B$

Hence $\sup(A+B) = \alpha + \beta$

(e)

$\blacksquare$

6. Prove or disprove the following statement by given a counterexample:

(a) $\sup(A \cap B) \leq \inf\{\sup A, \sup B\}$

(b) $\sup(A \cap B) = \inf\{\sup A, \sup B\}$

(c) $\sup(A \cap B) \geq \sup\{\sup A, \sup B\}$

(d) $\sup(A \cap B) = \sup\{\sup A, \sup B\}$

Solution. $\blacksquare$

7. Let $A, B \subseteq \mathbb{R}$ such that $\sup A = \sup B$ and $\inf A = \inf B$. Does $A = B$?

Solution. $\blacksquare$

8. Fix $b > 1$.

(a) If m, n, p, q are integers, $n > 0,\ q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence, it makes sense to define $b^r = (b^m)^{1/n}$

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence, it make sense to define

$$b^x = \sup B(x)$$

for every real x.

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Solution. ■

9. Prove that no order can be defined in the complex field that truns it into an ordered field.

Solution. ■

10. Suppose $z = a + ib,\ w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a dictionary order, or lexicographic order, for obvious reasons) Does this ordered set have the least-upper-bound property?

Solution. ■

11. Suppose $z = a + bi,\ w = u + iv$ and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2},\ b = \left(\frac{|w| - u}{2}\right)^{1/2}$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\overline{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

Solution. ■

12. Under what conditions does equality hold in the Cauchy-Schwartz inequality.

Solution. ∎