PROBABILITY EXERCISE

WEEK 1

(1) Two fair dice are rolled. Let X equal the product of the 2 dice. Compute $P\{X = i\}$ for $i = 1, \cdots, 36$

(2) Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed n times. What are the possible values of X?

(3) Suppose that the distribution function of X is given by

$$F(b) = \begin{cases} 0 & b < 0 \\ \frac{b}{4} & 0 \leq b < 1 \\ \frac{1}{2} + \frac{b-1}{4} & 1 \ \leq b < 2 \\ \frac{11}{12} & 2 \leq b < 3 \\ 1 & 3 \leq b \end{cases}$$

(a) Find $P\{X = i\},\ i = 1, 2, 3$

(b) Find $P\{\frac{1}{2} < X < \frac{3}{2}\}$

(4) Four buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, $40, 33, 25$, and 50 students. One of the students is randomly selected. Let X denote the number of students who were on the bus carrying the randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on her bus.

(a) Which of $E[X]$ or $E[Y]$ do you think is larger? Why?

(b) Compute $E[X]$ and $E[Y]$

(5) Two coins are to be flipped. The first coin will land on heads with probability 0.6, the second with probability 0.7. Assume that the results of the flips are independent, and let X equal the total number of heads that result.
   (a) Find $P\{X = 1\}$
   (b) Determine $E[X]$

(6) If $E[X] = 1$ and $Var(X) = 5$, find
   (a) $E[(2 + X)^2]$
   (b) $Var(4 + 3X)$

(7) Compare the Poisson approximation with the correct binomial probability for the following cases:
   (a) $P\{X = 2\}$ when $n = 8, p = 0.1$;
   (b) $P\{X = 9\}$ when $n = 10, p = 0.95$
   (c) $P\{X = 0\}$ when $n = 10, p = 0.1$
   (d) $P\{X = 4\}$ when $n = 9, p = 0.2$

(8) Let N be a nonnegative integer-valued random variable. For nonnegative values $a_j, j \geq 1$, show that

$$\sum_{j=1}^{\infty}(a_1 + \cdots + a_j)P\{N = j\} = \sum_{i=1}^{\infty} a_i P\{N \geq i\}$$

Then show that

$$E[N] = \sum_{i=1}^{\infty} P\{N \geq i\}$$

and

$$E[N(N + 1)] = 2\sum_{i=1}^{\infty} iP\{N \geq i\}$$

(9) Let X be a Poisson random variable with parameter λ.
   (a) Show that

$$P\{X \text{ is even}\} = \frac{1}{2}[1 + e^{-2\lambda}]$$

By using the result of Theoretical Exercise 15 and the relationship between Poisson and binomial random variables.

(b) Verify the formula in part(a) directly by making use of the expansion of $e^{-\lambda} + e^{\lambda}$

(10) Show that X is a Poisson random variable with parameter λ, then

$$E[X^n] = \lambda E[(X+1)^{n-1}]$$

Now use this result to compute $E[X^3]$

(11) Prove

$$\sum_{i=0}^{n} e^{-\lambda} \frac{\lambda^i}{i!} = \frac{1}{n!} \int_{\lambda}^{\infty} e^{-x} x^n dx$$

Hint: Use integration by parts.