2. Topological Space and Continuous functions

We will introduce some basic topological space.
e.g. Order topology, Product topology, Subspace topology,
Metric topology, (Quotient topology)

§ 12 Topological Spaces.

Definition. *Let X be a nonempty set $\mathscr{P}(X) = 2^X$ power set of X. We say that $\mathscr{T} \subseteq \mathscr{P}(X)$ is a topology on X if*

(1) $\emptyset, X \in \mathscr{T}$
(2) $U_\alpha \in \mathscr{T},\ \alpha \in I \implies \bigcup_{\alpha\in I} U_\alpha \in \mathscr{T}$
(3) $U_1, \cdots, U_n \in \mathscr{T} \implies U_1 \cap \cdots \cap U_n \in \mathscr{T}$

If $\mathscr{T}$ is a topology on X, then the pair $(X, \mathscr{T})$ or simply X is called a topological space and members in $\mathscr{T}$ are called open sets in X

Example.

(1) $X = \{a, b, c\}$
  (a) The following are topological space on X, $\mathscr{T}_1 = \{\emptyset, X\}$, $\mathscr{T}_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\mathscr{T}_3 = \mathscr{P}(X)$
  (b) The following are not topology on X
  $\mathscr{A} = \{\emptyset, \{a\}, \{b\}, X\}$ $(\because \{a\} \cup \{b\} = \{a, b\} \notin \mathscr{A})$
  $\mathscr{B} = \{\emptyset, \{a, b\}, \{b, c\}, X\}$ $(\because \{a, b\} \cap \{b, c\} = \{b\} \notin \mathscr{B})$
(2) Any set with more than 1 element has at least two topology $\{\emptyset, X\}$(in discrete topology) and $\mathscr{P}(X)$(discrete)
  and former is smallest one, another is the largest one.

Definition. $\mathscr{T}_{op} = \{\mathscr{T} \mid \mathscr{T}$ *is a topology on* $X\}$ $\mathscr{T}_1 \leq \mathscr{T}_2 \Leftrightarrow \mathscr{T}_1 \subseteq \mathscr{T}_2$
Claim " $\leq$ " is a partial ordering on $\mathscr{T}_{op}$

- ⋆ *Reflexive:* $\forall \mathscr{T} \in \mathscr{T}_{op},\ \mathscr{T} \leq \mathscr{T}$
- ⋆ *Anti-symmetry:* $\forall \mathscr{T}_1, \mathscr{T}_2 \in \mathscr{T}_{op},\ \mathscr{T}_1 \leq \mathscr{T}_2$ *and* $\mathscr{T}_2 \leq \mathscr{T}_1 \implies \mathscr{T}_1 = \mathscr{T}_2$
- ⋆ *Transitive:* $\forall \mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3 \in \mathscr{T}_{op},\ \mathscr{T}_1 \leq \mathscr{T}_2$ *and* $\mathscr{T}_2 \leq \mathscr{T}_3 \implies \mathscr{T}_1 \leq \mathscr{T}_3$

Example. Let X be a set, $\mathscr{T}_f = \{U \subseteq X,\ U = \emptyset$ or $X - U$ is finite $\}$
Then $\mathscr{T}_f$ is a topology on X, called the "finite complement topology" on X

Proof.

(1) $\emptyset, X \in \mathscr{T}_f$ $(\because X - X = \emptyset)$
(2) $U_\alpha \in \mathscr{T}_f,\ \alpha \in I$
  If $\bigcup_{\alpha\in I} U_\alpha = \emptyset$, then $\bigcup_{\alpha\in I} U_\alpha \in \mathscr{T}_f$.
  If $U_{\alpha\in I} U_\alpha \neq \emptyset$, then $\exists \alpha_0 \in I \ni U_{\alpha_0} \neq \emptyset$ and $X - U_\alpha$ is finite

$X - \bigcup_{\alpha \in I} U_\alpha = \bigcap_{\alpha\in I}(X - U_\alpha) \subseteq X - U_{\alpha_0} \implies X - (\bigcup_{\alpha\in I} U_\alpha)$ is finte $\implies \bigcup_{\alpha\in I} U_\alpha \in \mathscr{T}_f$

(3) $U_1, \cdots, U_n \in \mathscr{T}_f$

If $U_1 \cap \cdots \cap U_n = \emptyset$, then $U_1 \cap \cdots \cap U_n \in \mathscr{T}_f$

If $U_1 \cap \cdots \cap U_n \neq \emptyset$, then $X - (U_1 \cap \cdots \cap U_n) = (X - U_1) \cap \cdots \cup (X - U_n)$ is finite since each $X - U_i$ is finite. Thus $U_1 \cap \cdots \cap U_n \in \mathscr{T}_f$

From (1)(2)(3), $\mathscr{T}_f$ is a topology on X. ■

Remark. *If X is a finite set, then $\mathscr{T}_f$ is the discrete topology on X*

Example. Let X be a set and $\mathscr{T}_c = \{U \subseteq X \mid U = \emptyset$ or $X - U$ is countable $\}$. Then as in example above, $\mathscr{T}_c$ is a topology on X, called the countable complement topology on X. Moreover, if X is countable, then $\mathscr{T}_c$ is just a discrete topology on X

Definition. *Let $\mathscr{T}$ and $\mathscr{T}'$ be two topologies on X. We say that $\mathscr{T}'$ is (strictly) finer then $\mathscr{T}$ or $\mathscr{T}$ is (strictly) coaser that $\mathscr{T}'$ if $\mathscr{T} \leq \mathscr{T}'(\mathscr{T} < \mathscr{T}')$, i.e. $\mathscr{T} \subseteq \mathscr{T}'(\mathscr{T} \underset{\neq}{\subseteq} \mathscr{T}')$*

Remark.

(1) Two topologies on X need not be comparable

(2) Other terminology, if $\mathscr{T}' \supset T$, $\mathscr{T}'$ is larger(stronger) than $\mathscr{T}$ and $\mathscr{T}$ is smaller(weaker) than $\mathscr{T}$

§ 13 Bases for a topology.

Definition. *Let X be a set. A base for a topology on X is a collection $\mathscr{B} \subseteq \mathscr{P}(X)$ satisfying*

(1) $U\mathscr{B} = X$ $(\bigcup \mathscr{B} = \bigcup_{B\in\mathscr{B}} B)$

(2) Given $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$ $\exists B_3 \in \mathscr{B} \ni x \in B_3 \subseteq B_1 \cap B_2$

Members in $\mathscr{B}$ are called basic open sets in X

Given a base $\mathscr{B}$ for a topology on X, we can define the smallest topology $\mathscr{T}$ on X containing $\mathscr{B}$ called the topology on X generated by $\mathscr{B}$.

Usually, there are two ways to describe it

(I) $\mathscr{T} = \{U \subseteq X, \forall x \in U \exists B \in \mathscr{B} \ni x \in B \subseteq U\}$. Clearly, $\mathscr{B} \subseteq \mathscr{T}$

(a) $\emptyset, X \in \mathscr{T}$(by the definition of bases (1))

(b) $U_\alpha \in \mathscr{T},\ \alpha \in I \implies \bigcup_{\alpha\in I} U_\alpha \in \mathscr{T}$. Given $x \in \bigcup_{\alpha\in I} U_\alpha, x \in U_{\alpha_0}$ for some $\alpha_0 \in I$, $\exists B \in \mathscr{B} \ni x \in B \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha\in I} U_\alpha$

(c) $U_1, \cdots, U_n \in \mathscr{T} \implies U_1 \cap \cdots \cap U_n \in \mathscr{T}$. By induction on n, we only prove $n = 2$. Given $x \in U_1 \cap U_2$, $x \in U_1$ and $x \in U_2$ $\implies \exists B_1, B_2 \in \mathscr{B} \ni x \in B_1 \subseteq U_1$ and X in $\mathscr{B}_2 \subseteq U_2 \implies$

$x \in B_1 \cap B_2 \subseteq U_1 \cap U_2 \implies \exists B_3 \in \mathscr{B} \ni x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2 \implies U_1 \cap U_2 \in \mathscr{T}$

(II) $\mathscr{T}' = \{\bigcup \mathscr{A} \mid \mathscr{A} \subseteq \mathscr{B}\} = \{\bigcup_{\alpha \in I} A_\alpha \mid A_\alpha \in \mathscr{B}\}$

Clearly, $\mathscr{B} \subseteq \mathscr{T}'$(only choose one element in $\mathscr{B}$)

(a) $\emptyset, X \in \mathscr{T}'$(trivial)

(b) $U_\alpha \in \mathscr{T}',\ \alpha \in I \implies \bigcup_{\alpha \in I} U_\alpha \in \mathscr{T}'$
$\forall \alpha \in I, U_\alpha = \bigcup_{\beta \in I_\alpha} A_\beta$. Then $\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} \bigcup_{\beta \in I_\alpha} A_\beta \implies \bigcup_{\alpha \in I} U_\alpha \in \mathscr{T}'$

(c) $U_1, \cdots, U_n \in \mathscr{T}' \implies U_1 \cap \cdots \cap U_n \in \mathscr{T}'$. By induction on n, we only to prove that $n = 2$. For $i = 1, 2, \cdots,\ U_i = \bigcup_{\alpha \in I_j} A_\alpha$.
$U_1 \cap U_2 = \bigcup_{\alpha \in I_2} (A^1_\beta \cap A^2_\alpha)$. $\forall x \in U_1 \cap U_2,\ x \in A'_\beta \cap A^2_\alpha \implies U_1 \cap U_2 = \bigcup_{x \in U_1 \cap U_2} B_X \in \mathscr{T}'$

(III) $\mathscr{T} = \mathscr{T}'$

($\subseteq$) Given $U \in \mathscr{T},\ \forall x \in U,\ \exists B_x \in \mathscr{B} \ni x \in B_x \subseteq U \implies U = \bigcup_{x \in U} B_x \in \mathscr{T}'$

($\supseteq$) Given $U \in \mathscr{T}'$ $U = \bigcup_{\alpha \in I} A_\alpha, A_\alpha \in \mathscr{B}$
$\forall x \in U, x \in A_\alpha$ for some $\alpha \in I$ and $A_\alpha \in \mathscr{B}$, i.e. $X \in A_\alpha \in U$ and $A_\alpha \in \mathscr{B} \implies U \in \mathscr{T}$. Hence $\mathscr{T} = \mathscr{T}'$

Example.

(1) Let $\mathscr{B}$ be the collection of all open balls in $\mathbb{R}^n$. Then $\mathscr{B}$ is a base for a topology on $\mathbb{R}^n$, namely, then Euclidean topology on $\mathbb{R}^n$
(2) Let $\mathscr{B}'$ be the collection of all $n-$dimentional open intervals in $\mathbb{R}$. Then $\mathscr{B}'$ is a base for a topology on $\mathbb{R}^n$. In fact, β and β' generate the same topology on $\mathbb{R}^n$

Lemma. *Let X be a set, let $\mathscr{B}$ be a basis for a topology $\mathscr{T}$ on X. $\mathscr{T}$ equals the collection if all unions of elements of $\mathscr{B}$.*

Lemma. *Let X be a topological space and $\mathscr{C}$ be a collection of open sets of $X \ni \forall$ open set U in X and $\forall x \in U$ $\exists C \in \mathscr{C} \ni x \in C \subseteq U$. Then $\mathscr{C}$ is a base for the topology of X.*

Proof. (1) $\bigcup \mathscr{C} = X$
Since X is open $\forall x \in X,\ \exists C_x \in \mathscr{C} \ni x \in C_x \subseteq X \implies x \in \bigcup \mathscr{C} \implies X = \bigcup \mathscr{C}$
(2) Given $C_1, C_2 \in \mathscr{C}$ and $x \in C_1 \cap C_2$. Since $C_1 \cap C_2$ is open, $\exists C \in \mathscr{C} \ni x \in C \subseteq C_1 \cap C_2, \therefore \mathscr{C}$ is a base for a topology of X ∎

Remark. *Let $\mathscr{T}$ be the original topology on X and $\mathscr{T}'$ be the topology generated by $\mathscr{C}$. Then $\mathscr{T} = \mathscr{T}'$*

Proof. ($\subseteq$) Given $U \in \mathscr{T}$, $\forall x \in U \exists C \in \mathscr{C} \ni x \in C \subseteq U \implies U \in \mathscr{T}'$

($\supseteq$) Given $v \in \mathscr{T}'$, by lemma, $V = \bigcup \mathscr{A}$ for some $A \subseteq \mathscr{C}$. Since $\mathscr{C} \subseteq \mathscr{T}$, $\mathscr{A} \subseteq \mathscr{T}$, $\therefore V = \bigcup \mathscr{A} \in \mathscr{T}$

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