高等微積分

Exercise 1 (chapter 1 to 3)

1. Given two real number a and b such that $a \leq b + \epsilon$ for any $\epsilon > 0$, then $a \leq b$.

2. (a) If $r, s \in \mathbb{Q}$ then $r + s$ and rs are rational.
   (b) If $r \in \mathbb{Q}$ with $r \neq 0$ and $x \in \mathbb{R} \setminus \mathbb{Q}$, then $r + x$ and rx are irrational.

3. Let $f : X \to Y$ be a function. If $B \subseteq Y$, we denote by $f^{-1}(B)$ the largest subset of X which f maps into B. That is,
$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$
The set $f^{-1}(B)$ is called the **inverse image** of B under f. Prove the following for arbitrary $A, A_1, A_2 \subseteq X$ and $B, B_1, B_2 \subseteq Y$.
   (a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
   (b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. Given an example such that the inclusion is strict.
   (c) $A \subseteq f^{-1}[f(A)]$ and $f[f^{-1}(B)] \subseteq B$. Given an example such that the inclusion is strict.
   (d) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$ and $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

4. (a) Show that $\mathbb{N}$ is unbounded above.
   (b) Show that for any real number x, there exists a positive integer n such that $n > x$.
   (c) Using (b) to prove the following **Archimedean property**:
   If $x > 0$ and $y \in \mathbb{R}$, then there exists a positive integer n such that $nx > y$.
   (d) Using (c) to prove the denseness of $\mathbb{Q}$ in $\mathbb{R}$:
   Let $a < b \in \mathbb{R}$ be distinct real numbers, then there exists a rational number $q \in \mathbb{Q}$ such that $a < q < b$.

5. Let A, B be two nonempty sets of $\mathbb{R}$.
   (a) If $A \subseteq B$,then $\sup A \leq \sup B$ and $\inf A \geq \inf B$.
   (b) How to define $\sup \phi$ and $\inf \phi$ in $\mathbb{R}$?
   (c) Show that $\inf A \leq \sup A$.
   (d) Show that $\inf(-A) = -\sup A$ and $\sup(-A) = -\inf(A)$, where $-A = \{-a \mid a \in A\}$.
   (e) Show that $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$,
   where $A + B = \{a + b \mid a \in A, b \in B\}$, the **Minkowski sum** of A and B.
   (f) If A, B be two sets of positive numbers which is bounded above.
   Let $a = \sup A$, $b = \sup B$ and $C = \{ab \mid a \in A, b \in B\}$. Prove that $\sup C = ab$.

6. Prove or disprove the following statement by given a counterexample:
   (a) $\sup(A \cap B) \leq \inf\{\sup A, \sup B\}$.
   (b) $\sup(A \cap B) = \inf\{\sup A, \sup B\}$.
   (c) $\sup(A \cap B) \geq \sup\{\sup A, \sup B\}$.
   (d) $\sup(A \cap B) = \sup\{\sup A, \sup B\}$.

7. Let $A, B \subseteq \mathbb{R}$ such that $\sup A = \sup B$ and $\inf A = \inf B$. Does $A = B$?

8. Determine all the accumulation points of the following sets in $\mathbb{R}$ and decide whether the sets are open or closed (or neither).
   (a) Intervals $(a, b), (a, b], [a, b), [a, b]$.
   (b) $\mathbb{Z}$: the set of all integers.
   (c) $\{\frac{1}{n} \mid n = 1, 2, \cdots\}$.

9. Determine all the accumulation points of the following sets in $\mathbb{R}^2$ and decide whether the sets are open or closed (or neither).

(a) All complex z such that $|z| \geq 1$.
(b) All points (x, y) such that $x^2 - y^2 < 1$.
(c) All points (x, y) such that $x > 0$.

10. Prove that every non-empty open set in $\mathbb{R}$ contains both rational and irrational numbers.

11. Prove that a non-empty bounded closed set S in $\mathbb{R}$ is either a closed interval or that S can be obtained from a closed interval be removing a countable disjoint collection of open intervals whose endpoints belong to S.

12. If $S \subseteq \mathbb{R}^n$, prove that
    (a) S° is the union of all open subsets of $\mathbb{R}^n$ which are contained in S. i.e S° is the largest open set contained in S.
    (b) $\overline{S}$ is the intersection of all closed subsets of $\mathbb{R}^n$ which containing S. i.e $\overline{S}$ is the smallest closed set containing S.

13. If S and T are subsets of $\mathbb{R}^n$, prove that
    (a) $S^\circ \cap T^\circ = (S \cap T)^\circ$.
    (b) $S^\circ \cup T^\circ \subseteq (S \cup T)^\circ$. Give an example such that the inclusion is strict.
    (c) S' is closed in $\mathbb{R}^n$; that is $(S')' \subseteq S'$.
    (d) If $S \subseteq T$, then $S' \subseteq T'$.
    (e) $(S \cup T)' = S' \cup T'$.
    (f) $(\overline{S})' = S'$.
    (g) $\overline{S}$ is closed in $\mathbb{R}^n$.
    (h) $\overline{S \cap T} \subseteq \overline{S} \cap \overline{T}$.
    (i) If S is open, then $S \cap \overline{T} \subseteq \overline{S \cap T}$.

14. A set $S \subseteq \mathbb{R}^n$ is called **convex** if $\forall\ x, y \in S$ and for any $\lambda \in (0, 1)$, we have $\lambda x + (1 - \lambda)y \in S$. Prove that
    (a) All open balls and closed balls in $\mathbb{R}^n$ are convex.
    (b) Every n-dimensional open interval in $\mathbb{R}^n$ is convex.
    (c) The interior of a convex set is convex.
    (d) The closure of a convex set is convex.

15. Let F be a collection of sets in $\mathbb{R}^n$, and let $S = \bigcup_{A \in F} A$ and $T = \bigcap_{A \in F} A$. For each of the following statements, either give a proof or exhibit a counterexample.
    (a) If x is an accumulation point of T, then x is an accumulation point of each set A in F.
    (b) If x is an accumulation point of S, then x is an accumulation point of at least one set A in F.

16. If $S \subseteq \mathbb{R}^n$, prove that the collection of isolated points of S is countable.

17. The collection of F of open intervals of the form $(\frac{1}{n}, \frac{2}{n})$, where $n = 1, 2, \cdots$, is an open covering of the open interval $(0, 1)$. Prove, by the definition of compactness, that F has no finite subcovering covers $(0, 1)$.

18. Assume that $S \subseteq \mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is said to be a **condensation point** of S if every $r > 0$, $B(x, r) \cap S$ is uncountable. Prove the following statements.
    (a) If for every x in S, there is a $r_x > 0$ such that $B(x, r_x) \cap S$ is countable then S is countable.
    (b) If S is not countable, then there exists a point x in S such that x is a condensation point of S.

19. A set in $\mathbb{R}^n$ is called **perfect** if $S = S'$, that is, S is a closed set which contains no isolated points. Prove the following **Cantor-Bendoxon theorem**:

    Every uncountable closed set F in $\mathbb{R}^n$ can be expressed in the form $F = A \cup B$, where A is perfect and B is countable.

20. A set $A, B \subseteq \mathbb{R}^n$ be two sets. Prove or disprove the following statements by counterexample.
    (a) If A, B are open, then $A + B$ is open.
    (b) If A, B are closed, then $A + B$ is closed.

21. Consider the following two metrics in $\mathbb{R}^n$:

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|, \quad d_2(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

(a) Show that d_1 and d_2 are metrics in $\mathbb{R}^n$.

(b) Prove the following inequalities for all $x, y \in \mathbb{R}^n$:

$$d_2(x,y) \le ||x-y|| \le d_1(x,y) \text{ and } d_1(x,y) \le \sqrt{n}||x-y|| \le n d_2(x,y).$$

22. Let (M,d) be a metric space. Show that

$$\hat{d}(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

is also a metric for M.

23. Let (M,d) be a metric space and for any $x \in M$, $r > 0$. Prove that $\overline{B(x,r)} \subseteq \overline{B}(x,r)$. Give an exampe of a metric space such that the inclusion is strict.

24. In a metric space M, if $A, S \subseteq M$ satisfies that $A \subseteq S \subseteq \overline{A}$, then A is said to be **dense** in S. Show that if A is dense in S and S is dense in T, then A is dense in T.

25. A metric space M is said to be **separable** if M has a countable dense subset. Prove that $\mathbb{R}^n$ is separable for every $n \in \mathbb{N}$.

26. Prove that if a metric space is separable, then it has the Lindelöf property.

27. Let (M,d) be a metric space and $S, T \subseteq M$.
    (a) Assume that $S \subseteq T \subseteq M$. Then S is compact in (M,d) if and only if S is compact in (T,d).
    (b) If S is closed and T is compact, then $S \cap T$ is compact.
    (c) The intersection of arbitrary collection of compact sets of M is compact.

28. Let M be a metric space and $A, B \subseteq M$ be subsets.
    (a) $A^\circ = M \setminus \overline{M \setminus A}$.
    (b) $(M \setminus A)^\circ = M \setminus \overline{A}$.
    (c) $(\bigcap_{i=1}^{n} A_i)^\circ = \bigcap_{i=1}^{n} A_i^\circ$.
    (d) $(\bigcap_{A \in F} A)^\circ \subseteq \bigcap_{A \in F} A^\circ$, where F is an infinite collection of subsets of M. Give an example such that the inclusion is strict.
    (e) $\bigcup_{A \in F} A^\circ \subseteq (\bigcup_{A \in F} A)^\circ$, where F is an infinite collection of subsets of M. Give an example such that the inclusion is strict.
    (f) $\partial A = \overline{A} \cap \overline{M \setminus A}$ and $\partial A = \partial(M \setminus A)$.
    (g) If A is open or closed in M, then $(\partial A)^\circ = \phi$.
    (h) Give an example that $(\partial A)^\circ = M$.
    (i) If $A^\circ = B^\circ = \phi$ and A is closed, then $(A \cup B)^\circ = \phi$. Give an example in which $A^\circ = B^\circ = \phi$ but $(A \cup B)^\circ = M$.
    (j) If $\overline{A} \cap \overline{B} = \phi$, then $\partial(A \cup B) = \partial A \cup \partial B$.

29. Prove the following three important inequalities:
    (a) **(Young)** Let $a, b \ge 0$ and $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

    (b) **(Hölder)** Let $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$, and $1 < p,\ q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\sum_{j=1}^{n} |x_j y_j| \le \Big(\sum_{j=1}^{n} |x_j|^p\Big)^{1/p} \Big(\sum_{j=1}^{n} |y_j|^q\Big)^{1/q}.$$

    (c) **(Minkowski)** Let $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$, and $p \ge 1$. Then

$$\Big(\sum_{j=1}^{n} |x_j + y_j|^p\Big)^{1/p} \le \Big(\sum_{j=1}^{n} |x_j|^p\Big)^{1/p} + \Big(\sum_{j=1}^{n} |y_j|^p\Big)^{1/p}.$$

30. For $1 \le p \le \infty$, write $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, define p-norm $||\cdot||_p : \mathbb{R}^n \to \mathbb{R}$ by

$$||x||_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}, \text{ if } 1 \le p < \infty, \text{ and } ||x||_\infty = \max_{1 \le j \le n} |x_j|, \text{ if } p = \infty.$$

Show that p-norm is indeed a norm on $\mathbb{R}^n$, $1 \le p \le \infty$.