§ Linear Transformations and Matrices

2-1 Linear Transformations, Null spaces, and Ranges.

13 Let V and W be vector spaces, let T: V $\rightarrow$ W be linear, and let $\{ w_1, \cdots, w_k \}$ be a linearly independent subset of R(T). Prove that if $S = \{ v_1, \cdots, v_k \}$ is chosen so that $\mathrm{T}(v_i) = w_i$ for $i = 1, 2, \cdots, k$, then S is linearly independent.

Solution. **Claim.** S is linearly independent
there exist $a_1, \cdots, a_k \in \mathrm{F} \implies$

$$\sum_{i=1}^{k} a_i v_i = 0 \implies \mathrm{T}(\sum_{i=1}^{k} a_i v_i) = 0 \implies \sum_{i=1}^{k} a_i \mathrm{T}(v_i) = 0 \implies \sum_{i=1}^{k} a_i w_i = 0$$

$\because \{ w_1, \cdots, w_k \}$ is linearly independent $\therefore \sum_{i=1}^{k} a_i v_i = 0$ only $a_i = 0, i = 1, \cdots, k$

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14 Let V and W be vector spaces and T:V $\rightarrow$ W be linear.

(a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

(b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if T(S) is linearly independent.

(c) Suppose $\beta = \{ v_1, v_2, \cdots, v_n \}$ is a basis for V and T is one-to-one and onto. Prove that T(β) = {T(v_1),T(v_2),$\cdots$,T(v_n)} is a basis for W.

Solution. (a) ($\Rightarrow$) let $\{ s_1, \cdots, s_n \}$ be a linearly independent subset of S

Claim. $\{ T(s_1), \cdots, T(s_n) \}$ is a linearly independent subset of W

$$\sum_{i=1}^{n} a_i T(s_i) = 0 \implies \sum_{i=1}^{n} T(a_i s_i) = 0 \implies T(\sum_{i=1}^{n} a_i s_i) = 0$$

$\because$ T is one-to-one $\therefore \sum_{i=1}^{n} a_i s_i = 0$ only scalars are 0

$\{ T(s_1), \cdots, T(s_n) \}$ is a linearly independent subset of W

($\Leftarrow$) let $x, y \in$V, β is a basis of V, $\beta = \{ v_1, v_2, \cdots, v_n \}$

$x = a_1 v_1 + \cdots + a_n v_n, y = b_1 v_1 + \cdots + b_n v_n$

T(x) = T(y) $\implies$ T(x)-T(y) = 0 $\implies$ T($x - y$) = 0 $\implies$ T($(a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n$) = 0

$\implies (a_1 - b_1)T(v_1) + \cdots + (a_n - b_n)T(v_n) = 0 \because \{ T(v_1), \cdots, T(v_n) \}$ is linearly independent

$\therefore (a_1 - b_1)T(v_1) + \cdots + (a_n - b_n)T(v_n) = 0$ only $a_1 - b_1 = \cdots = a_n - b_n = 0$

$\implies a_1 = b_1, \cdots, a_n = b_n \implies T(x) = T(y)$ only $x = y$

T is one-to-one

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(b) let S = $\{ s_1, s_2, \cdots, s_n \}$

($\Rightarrow$) **Claim.** $\{ T(s_1), \cdots, T(s_n) \}$ is linearly independent

$a_1T(s_1) + \cdots + a_nT(s_n) = 0 \implies T(a_1s_1) + \cdots + T(a_ns_n) = 0 \implies T(a_1s_1 + \cdots + a_ns_n) = 0$

$\because$ S is linearly independent, T is one-to-one $\therefore T(a_1s_1 + \cdots + a_ns_n) = 0$ only scalars are 0

$\{ T(s_1), \cdots, T(s_n) \}$ is linearly independent

($\Leftarrow$) **Claim.** $\{ s_1, \cdots, s_n \}$ is linearly independent

$a_1s_1 + \cdots + a_ns_n = 0 \implies T(a_1s_1 + \cdots + a_ns_n = 0) \implies a_1T(s_1) + \cdots + a_nT(s_n) = 0$

$\because \{ T(s_1), \cdots, T(s_n) \}$ is linearly independent, $\therefore a_1 = \cdots = a_n = 0$

$\{ s_1, \cdots, s_n \}$ is linearly independent

(c) **Claim.** T(β) can generate W

$\because$ T is onto, by Thm 2.2, T : V $\to$ W be linear, β is a basis of V, then R(T) = span(T(β)) = W

T(β) can generate W

Claim T(β) is a linearly independent set of W

by (b), T is one-to-one,S is a linearly independent subset of V, T(S) is linearly independent

T(S) is a basis of W

17 Let V and W be finite-dimensional vector spaces and T:V $\rightarrow$ W be linear.

(a) Prove that if dim(V) ¡ dim(W), then T cannot be onto.

(b) Prove that if dim(V) ¿ dim(W), then T cannot be one-to-one.

Solution. let dim(V)= m,dim(W)=n, β_v is a basis of V, β_w is a basis of W, $\beta_v = \{ v_1, \cdots, v_m \}$, $\beta_w = \{ w_1, \cdots, w_n \}$

(a) by dimension theorem

$\because$ rank(T) $\leq$ dim(V) ¡ dim(W) $\therefore$ R(T) $\neq$ W

(b) **Claim.** T is one-to-one

by Theorem 2.4, T is one-to-one then N(T)= $\{ 0 \}$, nullity(T) = 0

$\therefore$ R(T) is a subspace of W

dim(V) = rank(T) + nullity(T) $>$ dim(W) $\Longrightarrow$ rank(T) $>$ dim(W)$\rightarrow\leftarrow$

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21 Let V be the vector space of sequences described in Example 5 of Section 1.2. Define the functions T,U:V $\to$ V by

$$\mathrm{T}(a_1, a_2, \cdots) = (a_2, a_3, \cdots) \text{ and } \mathrm{U}(a_1, a_2, \cdots) = (0, a_1, a_2, \cdots).$$

T and U are called the **left shift** and **right shift** operators on V, respectively.

(a) Prove that T and U are linear.

(b) Prove that T is onto, but not one-to-one.

(c) Prove that U is one-to-one, but not onto.

Solution. (a) let $v_1, v_2 \in$V $c \in$ F, $v_1 = (a_1, a_2, \cdots), v_2 = (b_1, b_2, \cdots)$

Prove T is linear

$\mathrm{T}(cv_1 + v_2) = \mathrm{T}((ca_1, ca_2, \cdots) + (b_1, b_2, \cdots)) = \mathrm{T}(ca_1 + b_1, ca_2 + b_2, \cdots) = (ca_2 + b_2, ca_3 + b_3, \cdots) = c(a_2, a_3, \cdots) + (b_2, b_3, \cdots) = c\mathrm{T}(v_1) + \mathrm{T}(v_2)$

$\mathrm{T}((0, 0, \cdots)) = (0, 0, \cdots)$

$\therefore$ T is a linear function

Prove U is linear

$\mathrm{U}(cv_1 + v_2) = \mathrm{U}((ca_1, ca_2, \cdots)+(b_1, b_2, \cdots)) = \mathrm{U}(ca_1 + b_1, \cdots) = (0, ca_1 + b_1, \cdots) = c(0, a_1, \cdots) + (0, b_1, \cdots) = c\mathrm{U}(v_1) + \mathrm{U}(v_2)$

$\mathrm{U}(0, 0, \cdots) = (0, 0, \cdots)$

$\therefore$ U is a linear function

(b) let $v_1 = (a_1, a_2, \cdots) \in V, a_1, a_2, \cdots \in F$

$\exists(a_2, \cdots) = T(a_2, a_3, \cdots) \in V \therefore$ T is onto

$\mathrm{T}(1, 0, \cdots) = \mathrm{T}(2, 0, \cdots) = (0, 0, \cdots)$

T is not one-to-one

(c) $\mathrm{U}(a_1, a_2, \cdots) = (0, 0, \cdots)$ only $a_1 = a_2 = \cdots = 0 \implies N(U) = \{ 0 \}$

by Theorem 2.4, U is linear, if N(V) = $\{ 0 \}$, then U is one-to-one

$(1, 2, 3, \cdots) \in V \notin R(U)$, U is not onto

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26 Using the notation in the definition above, assume that T: V $\to$ V is the projection on W_1 along W_2.

(a) Prove that T is linear and $W_1 = \{ x \in V \mid T(x) = x \}$.

(b) Prove that W_1 =R(T) and W_2 = N(T).

Solution. (a) let $x, y \in V, x = w_1 + w_2, y = w_1' + w_2', w_1, w_1' \in W_1, w_2, w_2' \in W_2, c \in$ F

Claim. T is linear

$\mathrm{T}(cx + y) = \mathrm{T}((cw_1 + w_1') + (cw_2 + w_2')) = cw_1 + w_1' = c\mathrm{T}(x)+\mathrm{T}(y)$

$\mathrm{T}(0) = \mathrm{T}(0 + 0) = 0$ T is linear

(b) let $w_1 \in W_1, w_2 \in W_2$

$\mathrm{R(T)} = \{ T(x) \mid x \in V \} = \{ T(w_1 + w_2) | w_1 \in W_1, w_2 \in W_2 \} = \{ T(w_1) + T(w_2) \mid w_1 \in W_1, w_2 \in W_2 \} = \{ T(w_1) \mid w_1 \in W_1 \}=W_1$

let $x \in N(T) \implies T(x) = 0 \implies x \in W_2 \implies N(T) \subseteq W_2$

let $x \in W_2 \implies T(x) = 0 \implies x \in N(T) \implies W_2 \subseteq N(T)$

$\therefore N(T) = W_2$

$\because T(w_1) = 0$ only $w_1 = 0, T(w_2) = 0$ when $w_2 \in W_2$

$\therefore \mathrm{N(T)} = \{ w_2 \mid w_2 \in W_2 \} = W_2$

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35 Let V be a finite-dimensional vector space and T: V $\to$ V be linear.

(a) Suppose that V = R(T) + N(T). Prove that V = R(T) $\oplus$ N(T).

(b) Suppose that R(T) $\cap$ N(T) = $\{0\}$. Prove that V = R(T) $\oplus$ N(T).

Solution. (a) **Claim** R(T) $\cap$ N(T) = $\{0\}$

by Exercise 1.6.29 and dimension Theorem, W_1, W_2 is a subspace of V, dim($W_1 + W_2$) = dim(W_1)+dim(W_2)-dim($W_1 \cap W_2$)

dim(R(T) $\cap$ N(T)) = dim(R(T)) + dim(N(T)) - dim(V) = 0

$\because$ R(T),N(T) is a subspace, $\therefore$ R(T) $\cap$ N(T) = $\{0\}$

$\because$ R(T), N(T) is a subspace of V, R(T) $\cap$ N(T) = 0

$\therefore$ R(T) $\oplus$ N(T) = V

(b) by dimension theorem and Exercise 1.6.29

W_1, W_2 is a subspace of V, dim($W_1 + W_2$) = dim(W_1)+dim(W_2)-dim($W_1 \cap W_2$)

dim(R(T)+N(T)) = dim(R(T)) + dim(N(T)) - dim(R(T) $\cap$ N(T)) = dim(R(T)) + dim(N(T)) = dim(V)

$\because$ R(T) + N(T) is a subspace of V, dim(R(T)+N(T)) = dim(V)

$\therefore$ R(T)+N(T) = V

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