PROBABILITY EXERCISE

WEEK 1

(1) Compute

$$\int_{-\infty}^{\infty} e^{-x^2/2} \ dx$$

(2) Compute

$$\int_0^\infty \frac{\sin x}{x} \ dx$$

(3) Show that

$$E[Y] = \int_0^\infty P\{Y > y\} \ dy - \int_0^\infty P\{Y < y\} \ dy$$

(4) Show that if X has density function f, then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x) \ dx$$

(5) Show that Z is a standard normal random variable; then, for x > 0,

(a)
$$P{Z > x} = P{Z < -x}$$

(b)
$$P\{|Z| > x\} = 2P\{X > x\}$$

(c)
$$P\{|Z| < x\} = 2P\{Z < x\} - 1$$

(6) Let f(x) denote the probability density function of a normal random variable with mean μ and variance σ^2 . Show that $\mu - \sigma$ and $\mu + \sigma$ are pints of inflection of this function. That is, show that f''(x) = 0 when $x = \mu - \sigma$ or $x = \mu + \sigma$

(7) Let Z be standard normal random variable Z, and let g be a differentiable function with derivative g'.

- (a) Show that E[g'(Z)] = E[Zg(Z)]
- (b) Show that $E[Z^{n+1}] = nE[Z^{n-1}]$
- (c) Find $E[Z^4]$

(8) Let X be a random variable with probability density function

$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1\\ 0 & \text{otherwise} \end{cases}$$

- (a) What is the value of c?
- (b) What is the cumulative distribution function of X?
- (9) The probability density function of X, the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \begin{cases} \frac{10}{x^2} & x > 10\\ 0 & x \le 10 \end{cases}$$

- (a) Find $P\{X > 20\}$
- (b) What is the cumulative distribution function of X?
- (c) What is the probability that of 6 such types of devices, at least 3 will function for at least 15 hours? What assumptions are you making?
- (10) Compute E[X] if X has a density function given by

(a)
$$f(x) = \begin{cases} \frac{1}{4}xe^{-x/2} & x > 0\\ 0 & \text{otherwise} \end{cases}$$

(b) $f(x) = \begin{cases} c(1-x^2) & -1 < x < 1\\ 0 & \text{otherwise} \end{cases}$
(c) $f(x) = \begin{cases} \frac{5}{x^2} & x > 5\\ 0 & x \le 5 \end{cases}$

(11) The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1\\ 0 & \text{otherwise} \end{cases}$$

If $E[X] = \frac{3}{5}$, find a and b.

(12) The lifetime in hours of an electronic tube is a random variable having a probability density function given by

$$f(x) = xe^{-x} \qquad x \ge 0$$

Compute the expected lifetime of such a tube.

- (13) You arrive at a bus stop at q0 A.M., knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30.
 - (a) What is the probability that you will have to wait longer that 10 minutes?
 - (b) If, at 10: 15, the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?
- (14) If X is a normal random variable with parameters $\mu=10$ and $\sigma^2=36$, compute
 - (a) $P\{X > 5\}$
 - (b) $P{4 < X < 16}$
 - (c) $P\{X < 8\}$
 - (d) $P\{X < 20\}$
 - (e) $P\{X > 16\}$