Advance Calculus Exercise

Exercise 1 (Chapter 1)

1. Let a and b be two real numbers. If $a \leq b + \epsilon$ for any $\epsilon > 0$, then $a \leq b$.

Solution. If not, $a > b \implies a - b > 0$. For $\epsilon = \dfrac{a-b}{2} > 0$

$$a \leq b + \frac{a-b}{2} = \frac{a+b}{2}$$

$\implies a \leq b (\rightarrow\leftarrow)$ ∎

2.
(a) If $r, s \in \mathbb{Q}$, then $r + s$ and rs are rational.
(b) If $r \in \mathbb{Q}$ with $r \neq 0$ and $x \in \mathbb{R} - \mathbb{Q}$, then $r + x$ and rx are irrational.

Solution.

(a) Given $r, s \in \mathbb{Q}$, $r = \dfrac{a}{b}$, $s = \dfrac{c}{d}$ where $a, b, c, d \in \mathbb{Z}$, $b, d \neq 0$. Thus,

$$r + s = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \in \mathbb{Q}$$

$$rs = \frac{ac}{bd} \in \mathbb{Q}$$

(b) If not, $r + x, rx \in \mathbb{Q}$. Since $r + x \in \mathbb{Q}$ and $r \in \mathbb{Q}$, $(r + x) - r = x \in \mathbb{Q} (\rightarrow\leftarrow)$ to $x \in \mathbb{R} - \mathbb{Q}$
Since $rx \in \mathbb{Q}$ and $r \in \mathbb{Q}, r^{-1}(rx) = x \in \mathbb{Q} (\rightarrow\leftarrow)$

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3. Let $f : X \to Y$ be a function. If $B \subseteq Y$, we denote by $f^{-1}(B)$ the largest subset of X which f maps into B. That is,

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$

The set $f^{-1}(B)$ is called the inverse image of B under f. Prove the following for arbitrary $A, A_1, A_2 \subseteq X$ and $B, B_1, B_2 \subseteq Y$

(a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

(b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. Given an example such that the inclusion is strict.

(c) $A \subseteq f^{-1}[f(A)]$ and $f[f^{-1}(B)] \subseteq B$. Given an example such that the inclusion is strict.

(d) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$ and $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

Solution.
(a)
($\subseteq$) Given $y \in f(A_1 \cup A_2)$, $y = f(x)$ for some $x \in A_1 \cup A_2 \implies y = f(x),\ x \in A_1$ or $y = f(x), x \in A_2 \implies y \in f(A_1)$ or $y \in f(A_2) \implies y \in f(A_1) \cup f(A_2)$
($\supseteq$)
(b)
(c) Given $x \in A, f(x) \in f(A) \implies x \in f^{-1}[f(A)]$.
Conversely, it is not true. Consider $f : \mathbb{Q} \to \mathbb{R}$ by $f(x) = x^2$, $A = \{1\}$, Clearly, $f^{-1}[f(A)] = \{-1, 1\}$. Thus, $A = \{1\} \subsetneqq \{-1, 1\} = f^{-1}[f(A)]$
(d)

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4. Let E be a nonempty subset of an order set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Solution. Suppose α is a non-empty subset of an order set. Suppose α is a lower bound of E and β is an upper bound of E. This means

$$\alpha \leq x \text{ and } x \leq \beta \forall x \in E$$

Since $E \neq \emptyset$, choose $x \in E$, then $\alpha \leq x$ and $x \leq \beta$. By transitivity, $\alpha \leq \beta$ ■

5. Let A, B be two nonempty sets of $\mathbb{R}$

(a) If $A \subseteq B$, then $\sup A \leq \sup B$ and $\inf A \geq \inf B$.
(b) Show that $\inf A \leq \sup A$.
(c) Show that $\inf(-A) = -\sup A$ and $\sup(-A) = -\inf(A)$, where $-A = \{-a \mid a \in A\}$
(d) Show that $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$, where $A + B = \{a + b \mid a \in A, b \in B\}$, the Minkowski sum of A and B.
(e) If A, B be two sets of positive numbers which is bounded above. Let $\alpha = \sup A$, $b = \sup B$ and $C = \{ab \mid a \in A,\ b \in B\}$. Prove that $\sup C = ab$.

Solution.
(a) Suppose $A \subseteq B$. To prove $\sup A \leq \sup B$

(∗) If A is unbdd above, then B is also. Thus, $\sup A = +\infty = \sup B$
(∗) If A is bdd above then we have two cases
  ($*_1$) If B is unbdd above, then the result follows trivilly
  ($*_2$) If B is bdd above, then $\sup A$, $\sup B$ exist and finite, say $\sup A = \alpha$ and $\sup B = \beta$.
  Now, to prove that $\alpha \leq \beta$. Given $\epsilon > 0$, since $\sup A = \alpha$, $\exists a \in A \ni \alpha - \epsilon < a$. Since $A \subseteq B$ and $\beta = \sup B, a \in B$ and $a \leq \beta$

(b)
(c)
(d) To prove $\sup(A + B) = \alpha + \beta$ If A or B is unbdd above, then the result follows trivially
If A and B are bdd above,say $\sup A = \alpha$ and $\sup B = \beta$
Claim: $\sup(A + B) = \alpha + \beta$
  (i) Given $x \in A+B, x = a+b$ for some $a \in A$ and $b \in B$. Since $\alpha = \sup A$ and $\beta = \sup B,\ a \leq \alpha$ and $b \leq \beta \implies x = a + b \leq \alpha + \beta$
  (ii) Given $\epsilon > 0$, since $\alpha = \sup A$ and $\beta = \sup B, \exists a \in A, b \in B,\ \alpha - \frac{\epsilon}{2} < a$ and $\beta - \frac{\epsilon}{2} < b \implies \alpha + \beta - \epsilon < a + b$ and $a + b \in A + B$
Hence $\sup(A + B) = \alpha + \beta$
(e) Given $c \in C,\ c = xy$ for some $x \in A, y \in B$. Then $0 < x \leq a$ and $0 < y \leq b \implies xy \leq ab$. Hence $\sup C \leq ab$. Now to prove $\sup C \geq ab$. $\forall x \in A, y \in B$

$$\begin{aligned}
\sup C \geq xy &\implies \frac{\sup C}{x} \geq y \implies \frac{\sup C}{x} \geq b \\
&\implies \frac{\sup C}{b} \geq x \implies \frac{\sup C}{b} \geq a \implies \sup C \geq ab
\end{aligned}$$

Hence, $\sup C = ab$

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6. Prove or disprove the following statement by given a counterexample:
(a) $\sup(A \cap B) \leq \inf\{\sup A, \sup B\}$
(b) $\sup(A \cap B) = \inf\{\sup A, \sup B\}$
(c) $\sup(A \cap B) \geq \sup\{\sup A, \sup B\}$
(d) $\sup(A \cap B) = \sup\{\sup A, \sup B\}$

Solution. ■

7. Let $A, B \subseteq \mathbb{R}$ such that $\sup A = \sup B$ and $\inf A = \inf B$. Does $A = B$?

Solution. ■

8. Fix $b > 1$.

(a) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence, it makes sense to define $b^r = (b^m)^{1/n}$

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence, it make sense to define

$$b^x = \sup B(x)$$

for every real x.

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Solution. **Recall.** Thm (Existence of n^{th} root): $\forall x \in \mathbb{R}, x > 0$ and $n\ in \mathbb{N} \exists 1 y > 0 \ni y^n = x (y = x^{\frac{1}{n}})$, $x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m = (x^m)^{\frac{1}{n}}$

(a) If $m = 0$ or $p = 1$ then the result trivial. Suppose $m \neq 0$ and $p \neq 0$. Since $\frac{m}{n} = \frac{p}{q}$, $mq = np$. By the existence of n^{th} root, it satisfies to show that $((b^p)^{\frac{1}{q}})^n = b^m$

$$((b^p)^{\frac{1}{q}})^n = ((b^{\frac{1}{q}})^p)^n = (b^{\frac{1}{q}})^{pn} = (b^{\frac{1}{q}})^{mq} = ((b^{\frac{1}{q}})^q)^m = b^m$$

(b) Given $r, s \in \mathbb{Q}$. We can write $r = \frac{m}{n}$ and $s = \frac{p}{q}$, where $n, m, p, q \in \mathbb{Z}$ with $n, q > 0$. Then $b^{r+s} = b^{\frac{np+mq}{nq}}$. This means b^{r+s} is the unique real number $\ni (b^{r+s})^{nq} = b^{np+mq}$.
Claim: $(b^r b^s)^n q = b^{np+mq}$
$(b^r b^s)^{nq} = b^{rnq} b^{snq} = b^{mq} b^{np} = b^{np+mq}$. Thus, $b^{r+s} = b^r b^s$

(c) First, we prove that if $r < s$, then $b^r < b^s \forall r, s \in \mathbb{Q}$

- $\star_1$ If $r = 0$, then $b^r = 1$ write $s = \frac{m}{n} > r = 0$, where $m, n \in \mathbb{N}$ with $n \neq 0$. Since b^s is the unique real number such that $(b^s)^n = b^m$ and $b > 1$ we get $b^r = 1 < b^m = (b^s)^n \Rightarrow b^r < b^s$
- $\star_2$ If $s = 0$, then consider the same result of $\star_1$
- $\star_3$ If $r < 0 < s$, then $b^r < 1$ and $b^s > 1$. Hence $b^r < b^s$

$\star_4$ If $0 < r < s$. Write $r = \frac{p}{q}$ and $s = \frac{m}{n}$, where $n, m, p, q \in \mathbb{N}$. Then $r = \frac{np}{nq} < \frac{mq}{nq} = s$ By (a), we have $b^r = b^{\frac{p}{q}} = b^{\frac{np}{nq}} = (b^{\frac{1}{nq}})^{np} < (b^{\frac{1}{n1}})^{mq} = b^s$

$\star_5$ If $r < s < 0$, then consider $0 < -s < -r$. By $\star_4$, we are done.

Hence, in any case, we have b^r is the upper bound of $B(r)$ where $r \in \mathbb{Q}$ Finally, to prove that b^r is the smallest on. We need some facts.

(i) $\forall n \in \mathbb{N},\ b^n - 1 \geq n(b-1)$

(ii) If $t > 1$ and $n > \frac{(b-1)}{(t-1)}$, then $b^{\frac{1}{n}} < t$. By replacing $b = b^{\frac{1}{n}}$ in (i), we get $\forall n \in \mathbb{N}, b - 1 \geq n(b^{\frac{1}{n}} - 1)$ Thus, $n > \frac{b-1}{t-1} \geq \frac{n(b^{\frac{1}{n}}-1)}{t-1} \implies b^{\frac{1}{n}} - 1 < t - 1 \implies b^{\frac{1}{n}} \leq t$.

If $\alpha < b^r$, we must find $q \in \mathbb{Q}$ and $qleqr \ni b^q > \alpha$.

If $\alpha \leq 0$, then trivial

If $0 < \alpha < b^r$ Changing $t = \frac{b^r}{\alpha} > 1$ in (ii) and choose $n \in \mathbb{N} \ni n > \frac{(b-1)}{(t-1)}$. We get $b^{\frac{1}{n}} < \frac{b^r}{\alpha} \implies \alpha < b^{r-\frac{1}{n}}$ Choose $q = r - \frac{1}{n}$ and we are done.

(d) Prove that $b^{x+y} = b^x b^y \forall x, y \in \mathbb{R}$.

From (c), $b^x = \sup\{b^r \mid r \in \mathbb{Q}, r \leq X\},\ b^y = \sup\{b^s \mid s \in \mathbb{Q}, s \leq y\}$.Thus, $b^x b^y = \sup\{b^{r+s} \mid r, sin\mathbb{Q}, r+s \leq x+y\} = b^{x+y}$

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9. Prove that no order can be defined in the complex field that truns it into an ordered field.

Solution. ■

10. Suppose $z = a + ib,\ w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a dictionary order, or lexicographic order, for obvious reasons) Does this ordered set have the least-upper-bound property?

Solution. (a) Given $z \neq w \in \mathbb{C}$. We must prove $z < w$ or $w < z$ Write $z = a + ib,\ w = c + id$.

If $a \neq c$, then if $a < c$, then $z < w$, if $c < a$, then $w < z$

If $a = c$, then $b \neq d$ if $b < d$, then $z < w$ If $d < b$ then $w < z$

(b) $\forall z, w, u \in \mathbb{C}$ with $z < w$ and $w < u \implies z < u$. Write $z = a + ib,\ w = c + id$ and $u = e + if$. Since $z < w$ and $w < u$ we have $a \leq c \leq e$. If $a = e$, then $a = c = e \implies b < d < f \implies z < u$

If $a < e$ then $z < u$

Finally, to prove $\mathbb{C}$ doesn't have l.u.b property:

Consider the set $S = \{1 + iy \mid y \in \mathbb{R}\}$. Then S is bdd above by 2. If $\sup S$ exists, say $a + ib \in \mathbb{C}$, then $a \geq 1$

If $a = 1$, then $b \geq y\ \forall y \in \mathbb{R} (\rightarrow\leftarrow)$

If $a > 1$, then $\dfrac{a+1}{2}$ is an upper bound of $S (\rightarrow\leftarrow)$ ■

11. Suppose $z = a + bi,\ w = u + iv$ and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2},\ b = \left(\frac{|w| - u}{2}\right)^{1/2}$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\overline{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

Solution. ■

12. Under what conditions does equality hold in the Cauchy-Schwartz inequality.

Solution. (⇒) Suppose $(a_1, \cdots, a_n) = k(b_1, \cdots, b_n)$

$$|\sum_{i=1}^{n} a_i\overline{b_i}| = |k|^2 \left|\sum_{i=1}^{n} |b_i|^2\right| = \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2$$

(⇐) Let $A = \sum_{i=1}^{n} |a_i|^2,\ B = \sum_{i=1}^{n} |b_i|^2$, and $C = \sum_{i=1}^{n} a_i\overline{b_i}$

Suppose $|C|^2 = AB$. If $B = 0$, then the result follows.

Recall the proof of Cauchy-Schwartz inequality

$\forall \lambda \in \mathbb{C},\ 0 \leq \sum_{i=1}^{n} |a_i - \lambda b_i|^2$

Take $\lambda = \dfrac{\sum_{i=1}^{n} a_i\overline{b_i}}{\sum_{i=1}^{n} |b_i|^2} = \dfrac{C}{B}$

$$\begin{aligned} 0 \leq \sum_{i=1}^{n} |a_i - \frac{C}{B} b_i|^2 &= A - 2\frac{|C|^2}{B} + \frac{|C|^2}{B} \\ &= A - \frac{|C|^2}{B} = \frac{BA - |C|^2}{B} \end{aligned}$$

Thus $a_i - \dfrac{C}{B} b_i = 0 \implies a_i = \dfrac{C}{B} b_i\ \forall 1 \leq i \leq n$

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