#### *0.0.1. In calculus.* IPad test

(1) Extreme Value Theorem: Every continuous function $f : [a, b] \to$ R admit both max and min value $\Rightarrow$ Compact set

(2) Intermediate value Theorem: Given continuous function $f$ : $[a, b] \to$ R for all $f(a) \leq \lambda \leq f(b) \exists c \in [a, b] \ni f(c) = \lambda \Rightarrow$ connected set

How to prove a statement: $HP$,then $Q$, $P \Rightarrow Q$

$$\begin{cases}\begin{cases}\text{Direct Proof}\\ \text{Indirect Proof}\begin{cases}\text{contrapositive } \sim Q \Rightarrow \sim P\\ \text{by contradiction}\end{cases}\end{cases}\\ \text{Mathematical Induction}\end{cases}$$

## 1. Some preliminary

### 1.1. Set Theory.

Set is a collection which has two presentation

(1) List $\{a, b, c, \cdots\}$

(2) $\{x \mid x \in \text{alphabet}\}$

We will assume that you are familiar with some basic set theory e.g. union, intersection, difference

### 1.2. The Number System.

$\mathbb{N} = \{\, 1, 2, 3, \cdots \}$ the set of all positive integers $n$ natural numbers

$\mathbb{Z} = \{\, \cdots, -2, -1, 0, -1, -2, \cdots \}$ the set of all integers called the ring of intepus

$\mathbb{Q} = \left\{ \dfrac{m}{n} \ : \ n, m \in \mathbb{Z}, n \neq 0 \right\}$ the set of all rational numbers

$\mathbb{R}$ the set all of real numbers on the real number field on real line

$\mathbb{C} = \{\, z = a + ib \mid a, b \in \mathbb{R} \}$ the set of all complex numbers or the complex number filed on complex plane, where $i = \sqrt{-1}$

and $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

**Remark.**

(1) $x+2=0$ *no root in* $\mathbb{N}$
$3x-5=0$ *no root in* $\mathbb{Z}$
$x^2+1=0$ *no root in* $\mathbb{R}$
(2) *One can construct* $\mathbb{Q}$ *from* $\mathbb{Z}$ *in algebraic way, called the fraction field of* $\mathbb{Z}$
(3) *One can construct* $\mathbb{R}$ *from* $\mathbb{Q}$ *in two ways:*
· *Using Dedekind cut which is given in the appendix of Rudin p17-21*
· *Using completion of matrix space*
(4) *One can construct* $\mathbb{C}$ *from in complex analysis*

**Example.**

(1) *Between any two rational numbers, there is another one*

*Proof.* Let $r, s \in \mathbb{Q}$ with $r < s$, then $\dfrac{r+s}{2} \in \mathbb{Q}$ and $r < \dfrac{r+s}{2} < s$

$$r = \frac{m_1}{n_1}, s = \frac{m_2}{n_2}, \frac{r+s}{2} = \frac{\frac{m_1}{n_1} + \frac{m_2}{n_2}}{2} = \frac{m_1 n_2 + n_1 m_2}{2 n_1 m_1} \in Q$$
$$s = \frac{s+s}{2} > \frac{r+s}{2} > \frac{r+r}{2} = r$$

■

(2) $x^2 = \dfrac{4}{9}$ *has exactly two rational solutions, namely,* $\pm\frac{2}{3}$
(3) $x^2 = 2$ *has exactly two real root, namely,* $\pm\sqrt{2}$
(4) *Is there any rational roots of* $x^2 = 2$*? i.e., is* $\sqrt{2}$ *rational?*

*Suppose* $r = \dfrac{m}{n} \in \mathrm{Q}$*, is a root of* $x^2 = 2$*, where* $(m, n) = 1$
*Then* $\dfrac{m^2}{n^2} = 2 \implies m^2 = 2n^2 \implies 2 \mid m^2 \implies 2 \mid m \implies 4 \mid m^2 \implies 4 \mid 2n^2$
$\implies 2 \mid n^2 \implies 2 \mid n \implies (n, m) \neq 1$

(5) *Let* $A = \{\, r \in \mathrm{Q} \mid r > 0 \;\&\; r^2 < 2 \,\}, B = \{\, r \in \mathrm{Q} \mid r > 0 \;\&\; r^2 > 2 \,\}$

*Then $A$ contains no largest numbers, i.e. max element & $B$ contains no smallest numbers, i.e. min element*

*Proof.* $A$ contains no largest numbers $\Leftrightarrow$ given $r \in A$, $\exists s \in A \ni s > r$
Now, given $r \in A$, Let $s = r - \dfrac{r^2-2}{r+2} = \dfrac{2r+2}{r+2}$ $(\star_1)$
$\implies s^2 - 2 = \dfrac{2(r^2-2)}{(r+2)^2}$ $(\star_2)$
Now, $r \in A, r^2 < 2 \implies r^2 - 2 < 0 \therefore$
$(\star_1)\&(\star_2) \implies s > r \ \& \ s^2 < 2 \implies s \in A$ ■

(6) *As you know, in calculus, the sequence* $\{1, 1.4, 1.41, 1.414, 1.4142, \cdots\}$ *does not converge in* $\mathrm{Q}$*, but it converges to* $\sqrt{2}$ *in* $\mathrm{R}$

### 1.3. Order Sets.

**Definition** (Relation).
*Let $X$ be a nonempty set $A$, relation on $X$ is a subset $\mathrm{R}$ of $X \times X = \{(x, y) \mid x, y \in X\}$*
*Let $\mathrm{R}$ be a relation on $X$, if $(x, y) \in \mathrm{R}$, then we say that $x$ is retaliated to $y$, and is written as $xRy(x \sim y)$*

**Definition** (Order Set). *An ordered set on $S$, is a relation denoted by " $<$ " on $S$, satisfy:*

(i) *The low of trichonomy*
*Given $x, y \in S$, one and only one of the following holds: $x < y, x = y, y < x$*
(ii) *Transitivity: if $x < y \& y < z$, than $x < z$*

***Notation***

(1) *$x < y$ means "$x$ is less than $y$" or "$x$ is smaller than $y$"*
(2) *$y > x$ means $x < y$*
(3) *$x \le y$ means $x < y$ or $x = y$, i.e. the negative of $x > y$*

**Definition** (bdd). *Let $S$ is an ordered set & $E \subseteq S (E \neq \emptyset)$*

- *$E$ is bounded above if $\exists\, \alpha \in S \implies x \le \alpha \ \forall\, x \in E$*
  *such $\alpha$ is called an upper bound of $E$*

- *E is bounded below if* $\exists\ \beta \in S \ni \beta \leq x, \forall\ x \in E$*, such* $\beta$ *is called a lower bdd of* $E$
- $E$ *is bdd is* $E$ *is both bdd above and below.*

**Definition** (least upper bound). *Let $S$ be an ordered set and $E \subseteq S(E \neq \emptyset)$ bdd above. An element $\alpha \in S$ is called the last upper bound or supremum of $E$ if*

*(i) $\alpha$ is an upper bound of $E$*

*(ii) $\alpha$ is the smallest such one.*

*Equivalently,*

*(i') $x \leq \alpha, \forall x \in E$*

*(ii') if $\beta < \alpha$, then $\beta$ is not an upper bdd of $E$, i.e. $\exists\, x \in E \ni x > \beta$*

*Such $\alpha$(if exists) is denoted by*

$$\alpha = sup(E)$$

*similarly, one can defined the greatest lower bdd of infimum of $E$*

**Remark.** *if* $\sup(E)$ *exists then it is unique*
*suppose $\alpha \neq \alpha'$ both lub of $E$*
$\because$ *by trichotomy ,$\alpha > \alpha'$ or $\alpha = \alpha'$ or $\alpha < \alpha'$($\rightarrow\leftarrow$)*

**Definition** (least upper bdd property). *A ordered set $S$ is said to have the least upper bdd property if $E \subseteq S,\ E \neq \emptyset$ and $E$ is bdd above, then* $\sup(E)$ *exists in $S$*

**Example.**

(1) *In* Q *with the normal ordining*
$A = \{\, r \in \mathrm{Q} \mid r > 0,\ r^2 < 2 \,\} \& B = \{\, r \in \mathrm{Q} \mid r > 0,\ r^2 > 2 \,\}$
*Then $A$ is bdd above, in fact, bdd by every element in $B$, but* $\sup(A)$ *does not exist in* Q*($\because$ by Ex1.5)*

(2) *$B$ is bdd below by every element of $A$ and* $\inf B$ *does not exists*

(3) *Note that* $\sup(E) \& \inf(E)$ *may not in $E$ even if exist*

**Remark.**

(1) *By the Example above,* Q *with the usual ordering has no l.u.b property*

(2) *In* 1.5 *we will explain that* R *with usual ordering has the l.u.b. property. However, we usually adopt the follwing*

**The Axiom of Completence or Least upper bdd property**:
Every nonempty subset $E$ of R which is bdd above has l.u.b

**Theorem** (l.u.b.p. $\to$ g.l.b.p.)**.** *Let $S$ is an ordered set if $S$ has the l.u.b. property, then $S$ has the g.l.b. property, i.e. if $\emptyset \neq B \subseteq S$ is bdd below, then $\inf(B)$ exists in $S$*

*Proof.* $(\star)$
Given $B(\neq \emptyset) \subseteq S$ which is bdd below
Let $L = \{ a \in S \mid a \text{ is a lower bdd of } B \}$

- $L \neq \emptyset (\because B \text{ is bdd below})$
- $L$ is bdd above (in fact, every element in $B$ is on upper bound of $L$)
  $\implies \forall a \in L \implies a \leq x, \ \forall x \in B \implies x$ is an upper bound of $L$
- $\sup(L) = \alpha$ exists by assumption

**Claim** $\alpha = \inf B$

(i) $\alpha$ is a lower bdd of $B$, i.e. $\alpha \leq x, \ \forall x \in B$
By $\alpha = \sup L$, if $r < \alpha$, them $r$ is not an upper bdd of $L(\because \alpha$ is the smallest one).Hence,$r \notin B(\because$ every element of $B$ is an upper bdd of $L$), so $\alpha \leq x, \forall x \in B$
We have proved $(r < \alpha \implies r \notin B) \implies (r \in B \implies r \geq \alpha)$

(ii) $\alpha$ is the greated one
if $\alpha < \beta$ and $\beta$ is a lower bdd of $B$, then $\beta \notin L$, i.e. $\beta$ is not a lower bdd of $B$, so $\alpha$ is the greatest one. Therefore, $\alpha = \inf(B)$

$\blacksquare$

**Remark.** *Let $E(\neq \emptyset) \subseteq \mathrm{R}$ be bdd below, then $\inf(E)$ exists and $\inf(E) = -\sup(-E)$, where $-E = \{ -x \mid x \in E \}$*

### 1.4. Field.

Recall the addition & multiplication in R
$+ : \mathrm{R} \times \mathrm{R} \to \mathrm{R}((a, b) \mapsto a + b)$
$\times : \mathrm{R} \times \mathrm{R} \to \mathrm{R}((a, b) \mapsto a \cdot b = ab)$

**Definition.** *Let $X$ is a nonempty set $A$, binary operation on $X$ is a function, $o : X \times X \to X$*

**Definition.** *Let* F *be a nonempty set, we say that* F *is a field* (($F, +, \cdot$) *is a field) if there are two binary operator called addition* "$+$" *and multiplication* "$\cdot$" *on* F *property*

***Axioms for*** "$+$"

*(A1) Commutative:* $\forall x, y \in \mathrm{F},\ x + y = y + x$

*(A2) Associative:* $\forall x, y, z \in F, (x + y) + z = x + (y + z)$

*(A3) Additive identity or zero element:* $\exists\ 0 \in F \implies x + 0 = 0 + x = x,\ \forall x \in \mathrm{F}$

*(A4) Additive inverse on negative: For each* $x \in X,\ \exists\ -x \in F \implies x + (-x) = (-x) + x = 0$

*i.e.* $(\mathrm{F}, +)$ *is an abelian group* ***Axioms for multiplication***

*(M1) Commutative:* $\forall x, y \in \mathrm{F},\ xy = yx$

*(M2) Associative:* $\forall x, y, z \in \mathrm{F},\ (xy)z = x(yz)$

*(M3) Muti identity:* $\exists 1 \neq 0$ *in* $\mathrm{F} \ni x1 = 1x = x$

*(M4) Multiplicative inverse: For each* $x \neq 0, \exists x^{-1} \in \mathrm{F} \implies xx^{-1} = x^{-1}x = 1$

*i.e.* $(F = F \cdot \{0\}, \cdot)$ *is an abelian group*

***Distributive Law***

*(D1)* $\forall x, y, z \in \mathrm{F},\ (x, y)z = xz + yz\ \&\ x(y + z) = xy + xz$

### Induction from Axioms

let $(\mathrm{F}, +, \cdot)$ be a field, we list a series of basic identity as you learn in high school in the real number system

(a) Cancellation law for " + " : $x + y = x + z \implies y = z$

$\because x + y = x + z \implies (-x) + (x + y) = (-x) + (x + z) \implies ((-x) + x) + y = ((-x) + x) + z$

$\implies 0 + y = 0 + z \implies y = z$

(b) 0 is "1"

suppose $0' \in$ F is another element satisfy $A_3$, then $0 = 0 + 0' = 0'$

(c) $x + y = x \implies y = 0$ by (a) $\because x + y = x + 0 \implies y = 0$

(d) negative $-x$ of $x$ is "1"

if $x' \in F$, is another negative of $x$, them $x + x' = x' + x = 0$

From $x + x' = 0 \implies (-x) + (x + x') = -x + 0 = -x$

(e) $x + y = 0 \implies y = -x$

$x+y = 0 \implies (-x)+(x+y) = (-x)+0 \implies ((-x)+x)+y = -x$

$\implies 0 + y = -x \implies y = -x$

(f) $-(-x) = x$

$-(-x) + (-x) = 0$, By (d) $x = -(x)$

(a') cancellation law

if $x \neq 0$, then $xy = xz \implies y = z, \because (x^{-1})(xy) = (x^{-1})(xz)$

$\implies (x^{-1})(xy) = (x^{-1}x)z \implies 1y = 1z \implies y = z$

(b') 1 is "1"

if $1'$ is another identity, then $1 = 11' = 1'$

(c') $x \neq 0$ & $xy = x \implies y = 1$

$xy = x1 \implies y = 1$

(d') For $x \neq 0$ in $\mathrm{F}, x^{-1}$ is "1"

if $x$ is another one, i.e. $x'x = xx' = 1 \implies (x^{-1})(xx') = (x^{-1})1 = x^{-1}$

(f') $x \neq 0 \implies (x^{-1})^{-1} = x$

$(x^{-1})^{-1}(x^{-1}) = 1 \implies x = (x^{-1})^{-1}$

(g') $0x = x0 = 0$

$(0 + 0)x = 0x + 0x \implies 0x = 0$

(h') $x \neq 0$ & $y \neq 0 \implies xy \neq 0$, equivalently $xy = 0 \implies x = 0$ or $y = 0$

$\because xy = 0$ then $(x^{-1})(xy) = ((x^{-1})x)y = 1y = y(\rightarrow\leftarrow)$

(i') $(-x)y = -(xy) = x(-y)$

$\because [(-x)+x]y = 0y = 0 = (-x)y = -(xy) \implies (-x)y = -(xy)$

(j') $(-x)(-y) = xy$

$\because (-x)(-y) = -(x(-y))$ by (i)

$= -(-(xy)) = xy$

(k) $-x = (-1)x$

$\because (1-1)x = 0x = 0 = 1x+(-1)x = x+(-1)x \implies (-1)x = -x$

**Definition** (Order Field)**.** *Let* F *is a field, we say that* F *is an order field if there is an ordering " $<$ " satisfying*

*(1) if $x < y$, then $x+z < y+z,\ \forall z \in F$*

*(2) if $x > y$ and $y > 0$, then $xy > 0$*

**Example.** Q *and* R *are order field under the usual ordering*

*Some basic properties of ordered field, let* F *be an ordered field with ordering " $<$ "*

*(a)* $x > 0 \implies -x < 0$

$\because x > 0 \implies x + (-x) > 0 + (-x) \implies 0 > -x$

*(b)* $x > y \Leftrightarrow x - y > 0$

$\because x > y \implies x + (-y) > y = (-y) \implies x - y > 0$

$x - y > 0 \implies x - y + y > y \implies x + 0 > y \implies x > y$

*(c)* $x > 0$ *and* $y < z \implies xy < xz$

$\because x > 0$ *and* $y < z \implies x > 0$ *and* $z - y < 0 \implies x(z-y) > 0 \implies xz + x(-y) > 0$

$\implies xz - xy > 0 \implies xz > xy$

*(d)* $x < 0$ *and* $y < z \implies xy > xz$

$\because x < 0$ *and* $y < z \implies -x > 0$ *and* $z - y > 0 \implies (-x)(z-y) > 0 \implies -xz + xy > 0$

$\implies xy > xz$

*(e)* $\forall x \neq 0$ *in* $\mathrm{F}, x^2 > 0$

$\because x > 0 \implies x \cdot x > x0$ *by (c) or*

$x < 0 \implies -x > 0$ *by (a)* $\implies -x > 0$ *by (a)* $\implies (-x)^2 > 0 \implies x^2 > 0$

*(f)* $1 > 0,\ -1 < 0$

$\because 1 \neq 0 \implies 1^2 > 0$ *by (e)* $\implies 1 > 0$

*(g)* $0 < x < y \implies 0 < \frac{1}{y} < \frac{1}{x}$

$\because$ *Note that* $\forall u \in \mathrm{F},\ u > 0 \implies \frac{1}{u} = u^{-1} > 0$

$\because$ *if* $\frac{1}{u} < 0$, *then* $u \cdot \frac{1}{u} < 0$ *by (e)* $\implies 1 < 0 (\rightarrow\leftarrow)\ \therefore\ \frac{1}{u} > 0$

*Now,* $\frac{1}{x}, \frac{1}{y} > 0$ *from* $x < y$ *we get* $(\frac{1}{x} \cdot \frac{1}{y})x < (\frac{1}{x} \cdot \frac{1}{y})y \implies 0 < \frac{1}{y} < \frac{1}{x}$

**Remark.** *By (e)(f), we conclude that* $\mathrm{C}$ *is not an ordered field*
$\because \mathrm{C}$ *were an ordered field, then by (e),* $i^2 > 0 \implies -1 > 0 (\rightarrow\leftarrow)$
$\therefore \mathrm{C}$ *is not an order field*

### 1.5. The Real Number Field $\mathrm{R}$.

**Theorem.** *There exists an ordered field* $\mathrm{R}$ *containing* $\mathrm{Q}$ *which has the l.u.b. property. Moreover, such* $\mathrm{R}$ *is unique up to order-isomorphism Such* $\mathrm{R}$ *is called the real number field or real number system or real line*

**Note.**if " $<$ " and " $<'$ " are two orders on R, them

$$\exists f(\mathrm{R}, <) \to (\mathrm{R}, <') \text{ such that}$$

(i) $f$ is a field isomorphism,
i.e. $\forall a, b \in \mathrm{R},\ f(a+b) = f(a) + f(b)$,
$f(ab) = f(a)f(b),\ f(1) = 1$
(ii) $f$ preserves ordering, $a < b \implies f(a) < f(b)$

Note. In most book, above theorem is taken as axiom, called the least upper bound property of $\mathbb{R}$ or the completeness axiom of $\mathbb{R}$. However, it can be constructed from $\mathbb{Q}$. There are two ways to construct it

(1) Using Dedekind cut as in the Appendix of Chapter 1.
(2) Using Cauchy sequence to get the completion of $\mathbb{Q}$.

**Theorem.**

*(a) The Archimedean property of* $\mathrm{R}$ : *Given* $x, y \in \mathrm{R}$ *with* $x > 0,\ \exists\ n \in \mathrm{N} \implies nx > y$
*(b)* $\mathrm{Q}$ *is dense in* $\mathrm{R} : \forall x, y \in \mathrm{R}$ *with* $x \le y,\ \exists\ r \in \mathrm{Q} \implies x < r < y$

*Proof.*

(a) Let $A = \{\, nx \mid n \in \mathrm{N} \,\} \subseteq \mathrm{R}$
if (a) were false, them $A$ is bdd above by $y$, since R has the l.u.b property
$\alpha = \sup A$ exists in R, since $x > 0,\ \alpha - x < \alpha \implies \alpha - x$ is not an upper bdd of $A$
$\implies \exists m \in \mathrm{N} \ni mx > \alpha - x \implies (m+1)x > \alpha (\rightarrow\leftarrow)$
(b) Since $x < y,\ y - x > 0$, by (a),$\exists n \in \mathrm{N} \implies n(y - x) > 1$

By (a) again, $\exists m_1, m_2 \in \mathrm{N} \implies m_1 = m_1 1 > n_x$ & $m_2 = m_2 \cdot 1 > -nx$

we have $-m_2 < nx < m_1$, choose $m \in \mathrm{Z} \implies -m_2 \leq m \leq m_1$ & $m - 1 \leq nx < m$

(in fact, $m = [nx] + 1$,where $[z]$ in the greatest integer of $z$)

we have $nx < m < 1 + nx < ny(\because n(y - x) > 1) \implies x < \frac{m}{n} < y$

Let $r = \frac{m}{n} \in \mathrm{Q}$, then $x < r < y$

■

An application of the density property of Q in R :
Given $x \in \mathrm{R} - \mathrm{Q}$ i.e. $x$ is an irrational numbers, i.e. $\forall\ \epsilon > 0, \exists r \in \mathrm{Q} \implies |x - r| < \epsilon$
equivalently, $\exists$ a sequence $\{ r_n \}$ in Q $\implies r_n \to x$
In fact, one may choose $\{ r_n \}$ to $\uparrow$ or $\downarrow$
$\because \forall n \geq 1,\ \exists\ r_n \in \mathrm{Q} \implies x < r_n < \frac{1}{n} + x$ by Thm.1.3(b) By squeezing lemma, $r_n \to x$ on $n \to \infty$

**Theorem** (existence of $n$th root)**.** *Given $x \in \mathrm{T}, x > 0$ & $n \in \mathrm{N}, \exists$ "1"$y > 0 \implies y^n = x$*
*Such $y$ is called the nth root of $x$ & denoted by $y = \sqrt[n]{x} = x^{\frac{1}{n}}$*

*Proof.* **not important**
"1". Suppose $y_1, y_2 > 0 \implies y_1^n = x$ & $y_2^n = x$
Bt trichotomy, we have

(i) $0 < y_1 < y_2 \implies y_1^n < y_2^n (\to\leftarrow)$
(ii) $0 < y_2 < y_1 \implies y_2^n < y_1^n (\to\leftarrow)$
(iii) $y_1 = y_2$

"$\exists$". Let $E = \{ t \in \mathrm{R} \mid t^n < x \}$
Claim:

- $E \neq \emptyset$, Let $t = \dfrac{x}{1 + x}$, then $0 < t < 1$, hence $t^n < t < x$, $\therefore t \in E$ & $E \neq \emptyset$
- $E$ is bdd above, in fact $E$ is bdd above by $1+x$ if $t > 1+x > 1$, then $t^n > t > x$, so $E$ is bdd above by $1 + 1$

  Therefore $y = \sup E$ exists & is finite
- Claim $y > 0$ & $y^n = x$, clearly, $y > 0(\because \frac{x}{1+x} \in E$ & $\frac{x}{1+x} > 0)$

by trichotomy, we have $y^n < x,\ y^n > x,\ y^n = x$

Now, to show that (i) & (ii) are impossible, do (iii) holds $y^n = x$

By the identity, $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$

(i)$y^n < x$ choose $0 < h < 1 = \alpha$ & $0 < \dfrac{x - y^n}{n(y+1)^{n-1}}$, $0 < h < \min\{\,\alpha, \beta\,\}$

put $a = y,\ b = y + h$ in $(\star)$, we obtain

$$(y+h)^n - y^n < hn(y-h)^{n-1} < hn(y+1)^{n-1} < x - y^n$$

$\implies (y+h)^n < x \implies y + h \in E\ \&y + h > y(\rightarrow\leftarrow) \therefore$ (i) fails

(ii) $y^n > x$, Let $k = \dfrac{y^n - x}{ny^{n-1}}$, Then $0 < k < y,\ k = \dfrac{y^n - x}{ny^{n-1}} < \dfrac{y^n}{ny^{n-1}} = \dfrac{y}{n} < y$
if $t > y - k > 0$, then $y^n - t^n \leq y^n - (y-k)^n < kny^{n-1}$ by $(\star) = y^n - x$
$\implies t^n > x \implies t \in E \implies E$ is bdd above by $y - k \implies \sup E \leq y - k (\rightarrow\leftarrow)$
$\therefore$ (ii) fails ■

**Corollary.** *Let $a, b \in \mathbb{R}$ with $a, b > 0,\ n \in \mathbb{N}$ Then$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$*
$\because a^{\frac{1}{n}}, b^{\frac{1}{n}} > 0$ & $(a^{\frac{1}{n}} \cdot b^{\frac{1}{n}}) = ab$, *By (1) in Thm 1.4* $(a,b)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$

**infinite in $\mathbb{R}$**
After discuss the real number $\mathbb{R}$, sometimes, we have to work with the extended real number system $\mathbb{R}^* = [-\infty, \infty] = \mathbb{R} \cup \{ +\infty, -\infty \}$ with observe, $x \in \mathbb{R}$

$$\lim_{n\to\infty} (-n) = -\infty,\ \lim_{n\to\infty} n = \infty,\ \lim_{n\to\infty} (\frac{1}{n} + n) = \infty,\ \lim_{n\to\infty} (n^2 - n) = \infty$$
$$x \pm \infty = \pm\infty,\ 0 \cdot (\pm\infty) = 0,\ \infty - \infty \text{ is not define}$$

Element in $\mathbb{R} \subseteq \mathbb{R}^*$ are called finite. Now, given any nonempty subset $E \subseteq \mathbb{R}$,

$$\sup E = \begin{cases} +\infty \text{ if } E \text{ is not bdd above} \\ \text{finite if } E \text{ is bdd above} \end{cases} \quad \& \inf E = \begin{cases} -\infty \text{ if } E \text{ is not bdd below} \\ \text{finite if } E \text{ is bdd below} \end{cases}$$

Note that if $A \subseteq B$, then $\sup A \leq \sup$ & $\inf A \geq \inf B$
$\therefore \emptyset \subseteq B,\ \forall B \subseteq \mathbb{R}$, One may define $\sup \emptyset = -\infty, \inf \emptyset = +\infty$

### 1.6. The Complex Number Field $\mathbb{C}$.

Consider the contention product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{ (a,b) \mid a, b \in \mathbb{R} \}$
Note that $(a,b) = (c,d) \Leftrightarrow a = c$ & $b = d$, From now, we can write $\mathbb{C} = \mathbb{R}^2$

**Operation on** $\mathbb{C}$ Given $(a,b),(c,d) \in \mathbb{C}$

(1) $(a,b)+(c,d)=(a+c,b+d)$

(2) $(a,b)(c,d)=(ac-bd,ad+bc)$

It is easy to see that, with these operations, $\mathbb{C}$ is a field.

**Note that**

- the zero element is $(0,0)$
- the negative of $(a,b)$ is $-(a,b) = (-a,-b)$
- the identity is $(1,0)$
- if $(a,b) \neq (0,0)$, then $(a,b)^{-1} = \left(\frac{1}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

R is a subset of C (not vary important)
consider that map

$$f : \mathbb{R} \to \mathbb{C} \text{ define by } f(a) = (a,0),\ a \in \mathbb{R}$$

we have $(1) f$ is injective $(2) f(1) = (1,0)$ $\because \forall\ a,b \in \mathbb{R}$
$f(a+b) = (a+b,0) = (a,0)+(b,0) = f(a)+f(b),\ f(a \cdot b) = (ab,0) = (a,0)\cdot(b,0)$
$f$ is a field homomorphism
$\therefore f : \mathbb{R} \to \mathbb{C}$ is an injective and isomorphism
Therefore, we identify $\mathbb{R}$ with $f(\mathbb{R})$ through the injective $f$
i.e. $a \in \mathbb{R}$ is identified with $f(a,0)$ in $\mathbb{C}$
$ab = (a,0)\cdot(b,0),\ a+b = (a,0)+(b,0)\ \forall\ a,b \in \mathbb{R}$

**Change $(a,b)$ to $a+bi$**

Now, we can transform an element $(a,b) \in \mathbb{C}$ into the normal form:
(a,b) = (a,0)+(0,b) = (a,0)(1,0)+(b,0)(0,1) = a1+bi = a+ib,
where i = (0,1)
Therefore, from new on, we write $\mathbb{C} = \{\, a+ib \mid a,b \in \mathbb{C}\,\}$
An element $z = a+ib \in \mathbb{C}$ is called a complex number
Hence, under this notification, $z = a+ib, w = c+id \in \mathbb{C}$

(1) $z+w = (a+c)+i(b+d)$
(2) $zw = (ac-bd)+i(ad+bc)$

and the $a$ is called the real part of $z$, $a = \mathrm{Re}(z)$, $b$ is called imaginary part of $z$, $b = \mathrm{Im} z$

**Some basic properties of complex numbers whose proofs are easy**

$\forall z, w \in \mathbb{C}$

- $\overline{z+w} = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z} \cdot \overline{w}$
- $\text{Re}z = \dfrac{z+\overline{z}}{2}$
- $\text{Im}z = \dfrac{z-\overline{z}}{2i}$
- $|z| = 0 \Leftrightarrow z = 0$
- Triangle inequality $|z+w| \leq |z| + |w|$
- $||z| - |w|| \leq |z - w|$
- $\mathbb{C}$ is not an ordered field
- $|z|^2 = z\overline{z}$
- $|\overline{z}| = |z|$
- $|\text{Re}z| \leq |z|, |\text{Im}z| \leq |z|$
- $|zw| = |z||w|$

*Proof.* $|z+w| \leq |z| + |w|$

$|z+w|^2 = (z+w)(\overline{z+w}) = (z+w)(\overline{z}+\overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w}$

$= |z|^2 + 2\text{Re}(z\overline{w}) + |w|^2 \leq |z|^2 + 2|z\overline{w}| + |w|^2 = |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2$

$\therefore |z+w| \leq |z| + |w|$ ■

**Theorem** (basic algebraic theorem)**.**

*(a)* $x^2+1$ *has no root in* $\mathbb{R}$

*(b)* $x^2+1$ *has two distinct roots in* $\mathbb{C}$

*Proof.*

(a) $1 > 0,\ x^2 > 0,\ \forall x \in \mathbb{R} - \{0\} \implies x^2+1 > 0\ \forall x \neq 0$

$0^2 + 1 = 1 > 0,\ \therefore x^2+1 > 0,\ \forall\ x \in \mathbb{R}$. Hence, $x^2+1 = 0$ has no root in $\mathbb{R}$

(b) $i^2 = (0,1)(0,1) = (0-1,0) = (-1,0) = -1$

$(-i)^2 = (-(0,1))^2 = (0,-1)^2 = (0,-1)(0,-1) = -1$, $\therefore \pm i$ are root of $\mathbb{C}$

■

**Conclusion:** Every non const polynomial $f(x) \in \mathbb{R}[x]$ has $n$ roots where $n = \deg f(x)$

**The complex root is even**

**no important proof**
$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x],\ a_n \neq 0,\ n \geq 1$
if $\alpha = a + ib \in \mathbb{C}$ is a root of $f(x)$,then
$0 = f(\alpha) = a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0$
$0 = f(\overline{\alpha}) = a_n\overline{\alpha}^n + a_{n-1}\overline{\alpha}^{n-1} + \cdots + a_1\overline{\alpha} + a_0$
$\therefore\ (x-\alpha)|f(x),\ (x-\overline{\alpha})|f(x) \implies (x-\alpha)(x-\overline{\alpha})|f(x) \implies (x^2 - (\alpha - \overline{\alpha})x + |\overline{\alpha}|^2) \mid f(x)$
$\implies (x^2 - 2ax + (a^2+b^2)) \mid f(x)$
$\therefore$ quadratic function must have two roots in $\mathbb{C}$

**The fundamental Theorem of Algebra**
Every non zero polynomial $f(x) \in \mathbb{C}[x]$ has at least one root in $\mathbb{C}$
Therefore, if $\deg f(x) = n$, then $f(x)$ has $n$ roots in $\mathbb{C}(C, M)$

$\because f(x) = (x-\lambda_1)^{e_1}\cdots(x-\lambda_t)^{e_t}(a_1x^2+b_1+c_1)^{l_1}\cdots(a_sx^2+b_sx+c_s)^{ls}$, where $\lambda_1, \cdots, \lambda_t \in \mathbb{R},\ a_i, b_i, c_i \in \mathbb{R}\&e_1 + \cdots + e_t + 2l_1 + \cdots + 2l_s = \deg f(x)$ which shows that all roots of $f(x)$ are in $\mathbb{C}$
In fact, we have the famous theorem: The fundamental theorem of algebra
Every non zero polynomial $f(x) \in \mathbb{C}[x]$ has at least one root in $\mathbb{C}$
$\therefore$ if $\deg f(x) = n$, then $f(x)$ has $n$ roots in $\mathbb{C}(C, M)$

**Theorem** (Cauchy-Scheming Inequal)**.** *Given* $z_1 \cdots, z_n, w_1, \cdots, w_n \in \mathbb{C}$*, we have*

$$\left|\sum_{j=1}^{n} z_j\overline{w}_j\right| \leq \left(\sum_{j=1}^{n}|z_j|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n}|w_j|^2\right)^{\frac{1}{2}}$$

*and* "$=$" *holds* $\Leftrightarrow \exists\ \lambda \in \mathbb{C} \ni\ w_j = \lambda z_j,\ 1 \leq j \leq n,$
*In patricial, if* $x_1, \cdots, x_n, y_1, \cdots, y_n \in \mathbb{R}$*, then*

$$\left|\sum_{j=1}^{n} x_jy_j\right| \leq \left(\sum_{j=1}^{n}x_j^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n}y_j^2\right)^{\frac{1}{2}}$$

*and* "$=$" *holds* $\Leftrightarrow \exists\ t \in \mathbb{R} \ni\ y_j = tx_j,\ 1 \leq j \leq n$

**Real Version**

*Proof.* $\forall t \in \mathbb{R},\ 0 \leq \sum_{i=1}^{n}(x_i + ty_i)^2 = \sum_{i=1}^{n}(x_i^2 + 2tx_iy_i + t^2y_i^2) = (\sum_{i=1}^{n} y_i^2)t^2 + t(\sum_{i=1}^{n} 2x_iy_i) + \sum_{i=1}^{n} x_i^2$ Not all $x_i = 0$ and $y_i = 0$
we use $b^2 - 4ac \implies 0 \geq (\sum_{i=1}^{n} 2x_iy_i)^2 - 4(\sum_{i=1}^{n} x_i^2)(\sum_{i=1}^{n} y_i^2) \implies (\sum_{i=1}^{n} x_iy_i)^2 \leq (\sum_{i=1}^{n} x_i^2)(\sum_{i=1}^{n} y_i)^2$ ■

**Complex Version**

*Proof.* we may assume that not all $z_i = 0$ and not all $w_j = 0$(otherwise it's trivial)

$$\begin{aligned}
\forall\lambda \in \mathbb{C}, 0 &\leq \sum_{j=1}^{n} |z_j - \lambda w_j|^2 \\
&= \sum_{j=1}^{n}(z_j - \lambda w_j)(\overline{z_j} - \overline{\lambda}\overline{w_j}) \\
&= \sum_{j=1}^{n}[|z_j|^2 - \lambda z_j\overline{w_j} - \lambda\overline{z_j}w_j + |\lambda|^2|w_j|^2]
\end{aligned}$$

$$\begin{aligned}
\text{Taking } \lambda &= \frac{\sum_{j=1}^{n} z_j\overline{w_j}}{\sum_{j=1}^{n} |w_j|^2} \\
&= \sum_{j=1}^{n} |z_j|^2 - \frac{2|\sum_{j=1}^{n} z_j\overline{w_j}|^2}{\sum_{j=1}^{n} |w_j|^2} \\
&= \sum_{j=1}^{n} |z_j|^2 - 2\frac{|\sum_{j=1}^{n} z_j\overline{w_j}|^2}{\sum_{j=1}^{n} |w_j|^2} + \left(\frac{|\sum_{i=1}^{n} z_j\overline{w_j}|}{\sum_{j=1}^{n} |w_j|^2}\right)^2 \sum_{j=1}^{n} |w_j|^2 \\
&= \sum_{j=1}^{n} |z_j|^2 - \frac{|\sum_{j=1}^{n} z_j\overline{w_j}|^2}{\sum_{j=1}^{n} |w_j|^2}
\end{aligned}$$

$$\implies |\sum_{j=1}^{n} z_j\overline{w_j}|^2 \leq (\sum_{j=1}^{n} |z_j|^2)^{\frac{1}{2}}(\sum_{j=1}^{n} |w_j|^2)^{\frac{1}{2}}$$

■

**Remark.** *From the proof, the* " = " *holds in CS inequality* $\Leftrightarrow\ \exists\lambda \in \mathbb{C} \ni z_j = \lambda w_j \forall j = 1, 2, \cdots, n,$ *i.e.* $(z_1, \cdots, z_n) = \lambda(w_1, \cdots, w_n)$

### 1.7. Euclidean Spaces $\mathbb{R}^n$.

**Definition.** *the n-dimensional Euclidean space* $\mathbb{R}^n$

$$= \{ x = (x_1, \cdots, x_n) \mid x_i \in \mathbb{R}, 1 \leq i \leq n \} = \mathbb{R} \times \cdots \times \mathbb{R}$$

*Note that*

$$(x_1, \cdots, x_n) = (y_1, \cdots, y_n) \Leftrightarrow x_i = y_i \ \forall \ 1 \leq i \leq n$$

We are going to introduce the structure of $\mathbb{R}^n$

- vector space
- inner product space
- normed linear space
- matrix space

**Definition.** *Two operation on* $\mathbb{R}^n$ *as follows:*

- *Addition* $+ : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n,\ (x, y) \mapsto x + y = (x_1 + y_1, \cdots, x_n + y + n)$
- *Scalar multiplication* $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,\ (a, x) \mapsto ax = (ax_1, \cdots, ax_n)$

**we skip space example here.**

**Observe.** The real version of the CS inequality can be write:

$$| < x, y > | = | \sum_{i=1}^{n} x_i y_i | \leq (\sum_{i=1}^{n} x^2)^{\frac{1}{2}} (\sum_{i=1}^{n} y^2)^{\frac{1}{2}} = ||x|| ||y||$$

$$\begin{aligned} \| x + y \|^2 = < x + y, x + y > &= < x, x > + 2 < x, y > + < y, y > \\ &= \| x \|^2 + 2 < x, y > + \| y \|^2 \\ &\leq \| x \|^2 + 2 \| x \| \| y \| + \| y \|^2 \\ &= (\| x \| + \| y \|)^2 \end{aligned}$$

### 1.8. Countability of Sets.

Given two nonempty set $A, B$ and a function $f : A \to B, f(A) = \{ f(a) \mid a \in A \}$ is called the image of $A$ under $f$

**Some basic things**

$E \subseteq A,\ f(E) = \{\, f(a) \mid a \in E \,\}$ the image of $E$ under $f$
$\bullet f$ is infective(one-to-one)
$x_1 \neq x_2 \implies f(x_1) \neq f(x_2) \Leftrightarrow f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$
$\bullet f$ is surjective(onto) if $f(A) = B$
$\bullet f$ is bijective if $f$ is one-to-one and onto

Given $F \subseteq B$, $f^{-1}(F) = \{\, x \in X \mid f(x) \in \mathrm{F} \,\}$ called the inverse image of $f$ under F

**Example**

$f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2,\ x \in \mathbb{R}$
$f^{-1}([0,1]) = \{\, x \in \mathbb{R} \mid f(x) \in [0,1] \,\} = \{\, x \in \mathbb{R} \mid x^2 \in [0,1] \,\} = [-1,1]$
$f^{-1}([-1,1]) = [-1,1]$

**Properties of inverse image**

- $F_1 \subseteq F_2 \subseteq B \implies f^{-1}(F_1) \subseteq f^{-1}(F_2)$
- Inverse image presences set operation
  $\forall F_\alpha \subseteq B,\ \alpha \in I,\ F \subseteq B$
  (i) $f^{-1}(\cup_{\alpha\in I} F_\alpha) = \cup_{\alpha\in I} f^{-1}(F_\alpha)$
  (ii) $f^{-1}(\cap_{\alpha\in I} F_\alpha) = \cap_{\alpha\in I} f^{-1}(F_\alpha)$
  (iii) $f^{-1}(B - F) = f^{-1}(B) - f^{-1}(F)$
- Given $S \subseteq A,\ f'(f'(S)) \supseteq S, " = " \Leftrightarrow$ one-to-one, **example:**
  $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2,\ S = [0,1],\ f(S) = [0,1],\ f^{-1}(f(S)) = f^{-1}([0,1]) = [-1,1]$
- Given $F \subseteq B,\ f(f^{-1}(F)) \subseteq F, " = " \Leftrightarrow$ "onto", **example**
  $f(x) = x^2,\ x \in \mathbb{R},\ F = [-1,1],\ f(f^{-1}([-1,1])) = f([-1,1]) = [0,1]$
- For $y \in B,\ f^{-1}(\{\, y \,\}) = f^{-1}(y) = \{\, x \in A \mid f(x) = y \,\}$ the inverse image of $y$, **example**
  $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2,\ f^{-1}(1) = \{\, 1, -1 \,\},\ f^{-1}(2) = \emptyset$

**Definition** (cardinality)**.** *Let $A, B$ are two set ew say that $A$ and $B$ have the same cardinality if $\exists$ a bijective map $f : A \to B$, which is denoted by $A \sim B$. or $R = \{(A,B) \mid \exists a\ bij\ map\ f : A \to B\} \subseteq \ni \in$*

*From now on, we write $|A|$ as the cardinality of $A$*

**Claim** " $\sim$ " is an $\equiv$ relation among all sets

(i) Reflexion: $\forall$ set $A$, $A \sim^{1A} A$, which $1_A$ is identity mapping
(ii) Symmetry: $A \sim^f B \implies B \sim^{f^{-1}} A$
(iii) Transitive: $A \sim^f B$ & $B \sim^g C \implies A \sim^{gof} C$

Properties of " $\equiv$ " classes on a set $X$. " $\sim$ " is an " $\equiv$ " relative on $X$
Given $x \in X$, $[x] = \overline{x} = \{y \in X \mid y \sim x\}$ the "$\equiv$" class determine by $x$
So we gave some property:

- Any two " $\equiv$ " class $[x]$ and $[y]$ are either disjoint or identical, i.e. $[x] = [y]$ or $[x] \cap [y] = \emptyset$
- $\underline{\overline{X}}$ is a disjoint union of " $\equiv$ " classes
  $[A] = \left\{ B \in \underline{\overline{X}} \mid B \sim A \right\}$ the " $\equiv$ " class set by $A$

Ant two element in an " $\equiv$ " class have the same cardinality
Notation For $n \in \mathbb{N}$, $\mathbb{N}_m = \{ 1, 2, \cdots, n \}$

**<u>Definition</u>.** *Let $A$ be a set*

*(a) $A$ is a finite set if $A = \emptyset$ or $A \sim \mathbb{N}_n$ for some $n \in \mathbb{N}$*
*(b) $A$ is a infinite set if $A$ is not a finite set*
*(c) $A$ is countable if $A \sim \mathbb{N}$*
*(d) $A$ is uncountable if $A$ is not countable.*
*(e) $A$ is at most countable if $A$ is finite or countable*

**Remark.**

*(1) when $A, B$ are finite sets, $A \sim B \Leftrightarrow |A| = |B|$, i.e. $A, B$ have same number.*
*(2) where $A, B$ are infinite and $A \sim B$, i.e. $|A| = |B|$, the concept is abstract.*
*(3) $\{ a, b, c \} \cup \mathbb{N} \sim \mathbb{N}$, $f : \mathbb{N} \to \{ a, b, c \} \cup \mathbb{N}$, $f(1) = a$, $f(2) = b$, $f(3) = c, \cdots$*
*(4) Any finite set can not equivalent to a proper subset, i.e. $A$ is finite, $B \subseteq A$*
*Then $A \sim B$, In fact $|B| < |A|$, but infinite different*
*(5) Any finite set $A$ can be listed an $A = \{ a_1, \cdots, a_n \}$ where $n = |A|$*

Now, we consider the case of countable set

> Recall, in calculus, a real sequence $\{a_n\}$, e.g.
>
> $$a_n = \frac{1}{n}\left\{\frac{1}{n}\right\},\ a_n = 1 - \frac{1}{n}\ \left\{1 - \frac{1}{n}\right\},\ a_n = \begin{cases} 0 \text{ if } n \text{ is odd} \\ 1 \text{ if } n \text{ is even} \end{cases}$$

**Definition.** *Let $X$ be a nonempty set, a sequence in $X$ is a function $a : \mathbb{N} \to X$. Given a sequence $=^a$ in $X, a$ is "1" determine by $a(n),\ \in \mathbb{N}$*
*We write*

$$a = \{a(1), a(2), \cdots, a(n), \cdots\} = \{a_1, a_2, \cdots, a_n, \cdots\} = \{a_n\} = \{a_n\}_{n=1}^{\infty}$$

**Remark.**

*(1) For a sequence $\{a_n\}$ in $X$, $a_n$ may not be distinct.*
*If all $a_n$ are distinct, then we say that $\{a_n\}$ is a distinct sequence in $X$.*
*(2) We usually use $\{a_n\}, \{b_n\}$ to denote sequence*
*(3) A sequence $\{a_n\}$ in $X$ in fact is a function from $\mathbb{N} \to X$, So $\{a_n \mid n \in \mathbb{N}\}$ is the image of the sequence.*
*(4) $\{a_n\}$ is a sequence, $a_n$ is called the $n^{th}$ term of the sequence.*
*(5) A sequence in $X$ may begin at $0$, i.e. $\{a_n\}_{n=0}$*
*By a changing index, we can make it from $\{b_n\}_{n=1}^{\infty},\ b_n = a_{n+1},\ n = 1, 2, \cdots$*

**Definition** (increasing)**.**
*A function $a : \mathbb{N} \to \mathbb{N}$ is increasing, $a$ is $\uparrow$, if $a(n) \le a(n+1)\ \forall\ n \ge 1$*
*$a$ is strictly increasing, $a$ is st. $\uparrow$, if $a(n) < a(n+1)\ \forall\ n \ge 1$*

Now, given a st. $\uparrow$ function $n : \mathbb{N} \to \mathbb{N}$, i.e. $n(k) < n(k+1),\ k \ge 1$ i.e. $n_k < n_{k+1},\ k \ge 1$, i.e. $n_1 < n_2 < \cdots < n_k < \cdots$, i.e. $\{n_k\}_{k=1}^{\infty}$ is a st. sequence in $\mathbb{N}$

**Definition.** *Let $\{a_n\}$ be a sequence in $X$ and $\{n_k\}$ be a st. $\uparrow$ sequence in $\mathbb{N}$, then the sequence $\{a_{n_k}\}$ is called a subsequence of $\{a_n\}$*
*In fact*

$$\mathbb{N} \to^{n}_{st.} \mathbb{N} \to^{a}_{seq} X \Rightarrow a \circ n : \mathbb{N} \to X \text{ is a function,}$$

*hence, it also a sequence in $X$*

$$a \circ n = \{ a \circ n(k) \} = \{ a(n(k)) \} = \left\{ a_{n(k)} \right\} = \{ a_{n_k} \}$$

**Remark.** *if $\{ a_{n_k} \}$ is st. $\uparrow$ in $\mathbb{N}$, then $k \le n_k \ \forall\ k \ge 1$*
$\because$ *By mathematical Induction*

- $1 \le n_1$
- *Assume it's true for $k \ge 2$, i.e. $k \le n_k$*
- *Consider $k+1$, $\ k+1 \le n_k + 1 \le n_{k+1}$*

***Example***
*Let $\{ a_n \}$ be a sequence in $X$, then $\{ a_{2k} \}$ and $\{ a_{2k-1} \}$ are subsequence of $\{ a_n \}$*

Finally, we will assume that you are familiar with the following property of the countability of sets:

**(1).** Every subset of a countable set is at most countable. The proof needs the well ordering of $\mathbb{N}$: Every nonempty subset of $\mathbb{N}$ has the smallest element.
$\because$ Let $X$ be a countable set and $A \subseteq X$ While $X = \{x_1, x_2, \cdots, x_n, \cdots\}$
Let $n_1 = \min k \mid x_k \in A \implies x_{n_1} \in A$
Let $n_2 = \min\{k \mid x_k \in A - \{x_{n_1}\}\} \implies x_{n_2} \in A - \{x_{n_1}\}$
$\vdots$
Let $n_k = \min\{k \mid x_k \in A - \{x_{n-1}, \cdots, x_{n_{k-1}}\}\} \implies$
$x_{n_k} \in A - \{x_{n_1}, \cdots, x_{n_{k-1}}\}$
$\vdots$ (The process will not stop if it is countable and finite)
$\therefore A = \{x_{n_1}, x_{n_2}, \cdots\}$ is countable
**(2).** $\mathbb{N} \times \mathbb{N}$ is countable. Hence if $X$ and $Y$ are countably then so is $X \times Y$.y induction, if $x_1, \cdots, x_n$ are countable, then so is $X_1 \times \cdots X_n$
$\because$ you can list $\mathbb{N} \times \mathbb{N}$ like below:

$$\begin{matrix} (1,1) & (1,2) & (1,3) & \cdots \\ (2,1) & (2,2) & (2,3) & \cdots \\ \vdots & \vdots & \vdots & \end{matrix}$$

**(3).** If $Y$ is countable and $f : X \to Y$ is injective, then $X$ is countable

$$\begin{aligned}\because f(X) \subseteq Y &\implies f(X) is countable by (1) \\ &\implies f : X \to f(X) \text{ is a bij function} \\ &\implies X \text{ is countable}\end{aligned}$$

**(4).** If $f : X \to Y$ is surjective and $X$ is countable, then $Y$ is countable
$\forall y \in Y, f^{-1}(y) \neq \emptyset$. Choose one $x_y \in X \ni f(x_y) = y$ and fix it.
$A = \{x_y \in X \mid y \in Y\} \subseteq X \implies A$ is countable $f : A \to Y$
$f(x_y) = y$ is bijective $\therefore Y$ is countable.
**(5).** Another way to see that $\mathbb{N}\times\mathbb{N}$ is countable. Consider the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(m, n) = 2^m 3^n,\ (m, n) \in \mathbb{N} \times \mathbb{N}$.
Claim: $f$ is injective $\because f(m, n) = f(k, l) \implies 2^m 3^n = 2^k 3^l \implies 2^{m-k} = 3^{l-m} \implies m - k = l - n = 0 \implies m = k,\ n = l \implies (m, n) = (k, l)$. By (3) $\mathbb{N} \times \mathbb{N}$ is countable.
**(6).** Countable union of countable sets is countable i.e. $X_n$ : countable $n \in \mathbb{N} \implies \bigcup_{n=1}^{\infty}$ is countable
We may assume that $\{X_n\}$ are pairwise disjoint, i.e. $X_n \cap X_m = \emptyset$ if $n \neq m$. Otherwise consider

$$\begin{aligned}\overline{X_n} &= X_n \times \{n\}\ n \in \mathbb{N} \\ &= \{(x, n) \mid x \in X_n\}\end{aligned}$$

$$X_n \sim \overline{X_n}$$

and $\overline{X_n}$ is countable
**(7).** $\mathbb{Z}$ is countable, $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup -\mathbb{N}$
**(8).** $\mathbb{Q}$ is countable, $\mathbb{Q} = \{\frac{m}{n}, n, m \in \mathbb{Z}\} \to \mathbb{Z} \times \mathbb{Z}$
or $\mathbb{Q} = \bigcup_{n=-\infty}^{\infty}\{\dfrac{m}{n} \mid m \in \mathbb{Z}\}$
**(9).** The set $\{\, a_n \mid a_n = 0 \text{ or } 1 \,\}, n \in \mathbb{N}$ is uncountable
This can be proved by Canton diagonal process
$\because$ if it is countable, then we can list it,$a_0 A = \left\{ a_1^{(1)}, a_2^{(2)}, \cdots \right\}$ where $a^{(1)} = \left\{ a_n^{(1)} \right\} = a_1^{(1)}, a_2^{(1)}, \cdots$ ; $a^{(2)} = \left\{ a_n^{(2)} \right\} = a_1^{(2)}, a_2^{(2)}, \cdots$
Now, construct a sequence $\{\, a_n \,\}$ in $A \ni \{\, a_n \,\} \neq a^{(k)}\ \forall\ k \geq 1 (\rightarrow\leftarrow)$

**(10).** Countable union of countable sets is countable

Recall, intervals in $\mathbb{R}$, $-\infty < a \leq b < \infty$, following are finite bdd interval

$(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$ open interval
$[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$ closed interval
$(a, b] = \{ x \in \mathbb{R} \mid a < x \leq b \}$ open-closed
$[a, b) = \{ x \in \mathbb{R} \mid a \leq x < b \}$ closed-open

An interval $I$ in $\mathbb{R}$ is said to be non-degenerate if the endpoint of $I$ are distinct i.e. length $> 0$. Otherwise, it is degenerate.

**Note.**

$(0, 1)$ is uncountable,
$\because (0, 1) = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{2^n} \mid a_n = 0 \text{ or } 1, n \in \mathbb{N} \right\}$
$x \in (0, 1)$ has a unique binary representation, so $(0, 1) \sim A$, where $A$ is $\{ \{ a_n \} \mid a_n = 0 \text{ or } 1 \}$ which is uncountable

If $a = b$ then it's degenerate

All non-degenerate intervals (the length is greater than 0) in $\mathbb{R}$ are uncountable.

$\because$ It sufficient to consider bdd non-degenerate interval in $\mathbb{R}$, given $\infty < a < b < \infty$

$(a, b)$ is uncountable($\because (0, 1) \sim (a, b)$)

Note that $(0, 1) \sim \mathbb{R}(\because (0, 1) \to (\frac{\pi}{-2}, \frac{\pi}{2}) \to \mathbb{R})$

# 2. Basic Point Set Topology

To know the "closeness", "limit" and "continue"

**Notation**. Let $X$ be a nonempty set. The power set of $X$ is denoted by $p(X)$ or $2^X$, i.e. $\mathscr{P}(X) = 2^X$ which is the collect of all subset, if $|X| = n$, then $|\mathscr{P}(X)| = 2^n$

## 2.1. Topological Spaces.

**Definition**. *Let $X$ be a nonempty set and $\mathscr{T} \subseteq \mathscr{P}(X)$, we say that $\mathscr{T}$ is a topology on $X$ if it satisfies*

(1) $\emptyset, X \in \mathscr{T}$

(2) *$\mathscr{T}$ is closed under arbitrary union,*
*i.e.* $U_\alpha \in \mathscr{T},\ \alpha \in I \implies \bigcup_{\alpha \in I} U_\alpha \in \mathscr{T}$

(3) *$\mathscr{T}$ is closed under finite intersection*
*i.e.* $U_1, \cdots, U_n \in \mathscr{T} \implies U_1 \cap \cdots \cap U_n \in \mathscr{T}$

*In this chapter, the pair $(X, J)$ or simply $J$ is called a topological space and members in $T$ are called open set in $X$ or open subsets of $X$*

**Remark.**

(1) *$X$: a nonempty set, there is at least two trivial topology on $X$*
- *$\mathscr{P}(x)$ is the largest topology on $X$ w.r.t inclusion $X$ with this topology is called a discrete topological space*
- *$\mathscr{T}_0 = \{\emptyset, X\}$ is the smallest topology on $X$ w.r.t inclusion $X$ with this topology is called an indiscrete topological space*

(2) *How many topology can be define on $\{a\}$, $\{a, b\}$?*

In the following $X$ is a topology space

**Definition** (neighborhood). *Let $p \in X$, a neighborhood of $P$ is an open set $U$ containing $p$*

**Definition** (Hausdorff space). *$X$ is a Hausdorff space if any two distinct points can be separated by open set, i.e. $\forall\ p \neq q$ in $X$, $\exists$ neighborhood $U$ of $p$ and $V$ of $q \in U \cap V = \emptyset$*

**Definition** (closed set). *A subset $F \subseteq X$ is said to be closed if $F^C = X - F$ is open in $X$*

**Theroem 2.1.** *The collection of all closed subsets of $X$ satisfied*

(a) $\emptyset, X$ *are closed*
(b) *Arbitrary intersection of closed set if closed*
(c) *Finite union of closed sets is closed*

*Proof.*

(a) $X - \emptyset = X$ is open $\therefore \emptyset$ is closed
$X - X = \emptyset$ is open $\therefore X$ is closed
(b) Given closed sets $F_\alpha, \alpha \in I$, $X - \bigcap_{\alpha \in I} F_\alpha = \bigcup_{\alpha - I}(X - F_\alpha)$ is open,$\therefore \bigcap_{\alpha - I} F_\alpha$ is closed.
(c) Given closed set $F_1, \cdots, F_n$, $X - \bigcup_{i=1}^n F_i = \bigcap_{i=1}^n (X - F_i)$ is open, $\therefore \bigcup_{i=1}^n F_i$ is closed.

■

**Definition.** *Let $Y \subseteq X$ and*

$$\mathscr{T}_y = \{U \cap Y \mid U \text{ is open in } X\}$$

**Theroem 2.2.** *$\mathscr{T}_Y$ is also a topology space*

*Proof.* To proof $\mathscr{T}_Y$ is a topology space, we take the topology's definition

(a) $\emptyset, Y \in \mathscr{T}_Y$ $(\because \emptyset = \emptyset \cap Y,\ Y = X \cap Y)$
(b) Given $U_\alpha \cap Y \in \mathscr{T}_Y,\ \alpha \in I$, where $U_\alpha$ is open in $X$
$\bigcup_{\alpha \in I}(U_{\alpha \cap Y}) = (\bigcup_{\alpha \in I} U_\alpha) \cap Y \implies \bigcup_{\alpha \in I}(U_\alpha \cap Y) \in \mathscr{T}_Y$
(c) Given $U_1 \cap Y, \cdots, U_n \cap Y$, where $U_i$ is open in $X$, $1 \leq i \leq n$
$\cap_{i=1}^n (U_i \cap Y) = (\cap_{i=1}^n U_i) \cap Y \implies \bigcap_{i=1}^n (U_i \cap Y) \in \mathscr{T}_Y$
$\therefore \mathscr{T}_Y$ is a topology on $Y$

■

**Definition.** *In Theorem 2.2, with the topology $\mathscr{T}_Y$ on $Y$, is called a topological subspace of $X$ and $\mathscr{T}_Y$ is called the relative topology of $Y$ in $X$. Members in $\mathscr{T}_Y$ are called open set in $Y$ or relative open sets in $Y$.*

### 2.2. Metric Spaces & Subspace.

In this chapter, we will introduce a class of topology space whose topology in induced by a metric.

**Definition.** *Let $X$ be a nonempty set. A metric or distance function in a function*

$$d : X \times X \to \mathbb{R}, \ (a, b) \mapsto d(a, b)$$

*satisfying:*

*(a)* $\forall a, b \in X, d(a, b) \geq 0$ *and* $d(a, b) = 0 \Leftrightarrow a = b$
*(b)* $\forall a, b \in X, d(a, b) = d(b, a)$ ***symmetry***
*(c)* $\forall a, b, c \in X, d(a, b) \leq d(a, c) + d(a, b)$ ***triangle inequaity***

*if $d$ is a metric on $X$, then the pair $(X, d)$ or simply $X$ is called a metric space and $\forall a, b \in X,\ d(a, b)$ is called the distance between $a$ & $b$*

**Examples**

(1) Let $X$ be a nonempty set define by

$$d(a, b) = \begin{cases} 0 \text{ if } a = b \\ 1 \text{ if } a \neq b \end{cases}$$

Then $d$ is a metric on $X$, called the discrete metric and with this metric $X$ is called a discrete metric space. In particular, any set admits a metric.

(2) The most important metric spaces are the Euclidean space $\mathbb{R}^k$, the metric $d$ is called the Euclidean or standard or usual metric on $\mathbb{R}^k$.There are other metrics on $\mathbb{R}^k$ induced the same metric topology on $\mathbb{R}^k$, in fact, they are all equivalent, e.g $\forall\, 1 \leq p \leq \infty$, We can define a metric $d_p$ on $\mathbb{R}^k$ as follows
- $1 \leq p < \infty,\ d_p(x, y) = ||x - y||_p = \left(\sum_{i=1}^k |x_i - y_i|^p\right)^{\frac{1}{p}}$
- $p = \infty,\ d_\infty(x, y) = \max_{1 \leq i \leq k} |x_i - y_i|$

Note that $d_2 = d$ is the Euclidean metric on $\mathbb{R}^k$

**Remark.** *In fact, every normed linear space*$(V, ||\cdot||)$ *is a metric space whose metric is induced by its norm*

(3) Let $(X,d)$ be a metric space and $Y \subseteq X,\ Y \neq \emptyset$. Then the restriction of $d$ to $Y \times Y$ is also a metric on $Y$, with this metric, $Y$ is called a metric subspace of $X$

**Definition** (ball)**.** *Given* $p \in X$ & $r > 0$
$B(p,r) = \{x \in X \mid d(x,p) < r\}$ : *open ball with center* $p$ *and radius* $r$
$\overline{B}(p,r) = \{x \in X \mid d(x,p) \leq r\}$ : *closed ball with center* $p$ *and radius* $r$

**Example**

(1) The discrete metric space $X:\ p \in X,\ r > 0$

$$B(p,r) = \begin{cases} \{p\} \text{ if } 0 \leq r \leq 1 \\ X \text{ if } r > 1 \end{cases}$$

$$\overline{B}(p,r) = \begin{cases} \{p\} \text{ if } 0 \leq r < 1 \\ X \text{ if } r \geq 1 \end{cases}$$

(2) In the Euclidean space $\mathbb{R}^k$, $p \in \mathbb{R}^k,\ r > 0$
$B(p,r) = \{x \in \mathbb{R} \mid \ \| x - p \| < r\}$ is a ""true"" open
$\overline{B}(p,r) = \{x \in \mathbb{R} \mid \ \| x - p \| \leq r\}$ is a "true" closed ball
In particular, for $k = 1$ in $\mathbb{R}$
$B(p,r) = (p - r, p + r)$ : a symmetric opne interval
$\overline{B}(p,r) = [p - r, p + r]$ : a symmetric close interval
However, w.r.t $d_1$ & $d_\infty$, we have, e.g. in $\mathbb{R}^2$
$B_1(0,1) = \{(x,y) \mid |x - 0| + |y - 0| < 1\}$
$B_\infty(0,1) = \{(x,y) \mid \max\{|x|,|y|\} \leq 1\}$

(3) In $\mathbb{C}$, $z \in \mathbb{C}, r > 0$

$$D(z,r) = \{w \in \mathbb{C} \mid |w - z| < r\}$$

is called an open disk with center $z$ and radius $r$

(4) What is the open balls in $S = [0,1] \subseteq \mathbb{R}$?
$B_S(0,\frac{1}{2}) = \{x \in S \mid |x - 0| < \frac{1}{2}\} = [0,\frac{1}{2}] = B(0,\frac{1}{2}) \cap [0,1]$
$B_S(0,3) = [0,1] = B(0,3) \cap [0,1]$

**Prop 2.2** Let $S$ be a metric subspace of a metric space $X$, then $\forall\ p \in S$ & $r > 0$, $B_S(p,r) = B(p,r) \cap S$

*Proof.* $B_S(p,r) = \{x \in S \mid d(x,p) < r\} = \{x \in X \mid d(x,p) < r\} \cap S$
$= B(p,r) \cap S$ ■

### 2.3. Open Sets in Metric Spaces.

We will see that every metric on a set induce a topology on $X$

**Definition** (interior point). *Let $S \subseteq X$ be a set, and $p \in S$, we say that $p$ is an interior point of $S$ if $\exists\ r > 0,\ \exists\ B(p,r) \subseteq S$*
*Denote by $S^o$ or int(S) by the set of all interior point of $S$*

**Definition** (open). *Let $S \subseteq X$, we say that $S$ is open if all points of $S$ are interior points of $S$*

**Remark.**

*(1) Every open set $S$ is a union of a open balls in $X$.*
$\because \forall\ x \in S, x$ *is an interior point of* $S$, $\exists r_x > 0 \ni B(x, r_x) \subseteq S$
$\therefore S = \cup_{x \in S} B(x, r_x)$
*(2) $S^o \subseteq S$ by definition*
*(3) $S$ is open $\Leftrightarrow S = S^o$*

**Prop 2.3**

**(a)** $S \subseteq T \implies S^o \subseteq T^o$

$$\because p \in S^o \implies \exists\ r > 0 \ni B(p,r) \subseteq S \subseteq T \implies p \in T^o$$

**(b)** Every open ball $B(p,r)$ in $X$ is open
$\because$ Give $q \in B(p,r)$, Let $\delta = r = d(p,q)$.Claim $B(q,\delta) \subseteq B(p,r)$ which says $q$ is an interior point of $B(p,r)$.Since $q \in B(p,r)$ is arbitrary, so $B(p,r)$ is open. Given $x \in B(q,\delta)$

$$d(x,p) \leq d(x,q) + d(q,p) < \delta + d(q,p) = r - d(p,q) + d(q,p) = r$$

**(c)** $\forall\ S \subseteq X,\ S^o$ is always open
$\because$ Given $p \in S^o,\ \exists\ r > 0 \ni B(p,r) \subseteq S$

$$\implies B(p,r) \subseteq S^o \implies p \text{ is a interior point of } S^o$$

$\therefore S^o$ is open
**(d)** $\forall\ S \subseteq X,\ S^{oo} = (S^o)^o = S^o$
$\because$ by definition of open set and (c)

Now, let $T = \{U \subseteq X \mid U \text{ is open in } X\}$
**Prop 2.4** $T$ is a topology on $X$. In particular, $X$ is a topology space.

*Proof.*
(**i**) $\emptyset, X \in \mathscr{T}$, $\because \emptyset = \emptyset$, $X^o = X$
(**ii**) $U_\alpha \in \mathscr{T}, a \in I$ are open $\implies \bigcup_{\alpha \in I} U_\alpha$ is open
Given an arbitrary point $p \in \bigcup_{\alpha \in I} U_\alpha \implies \exists \alpha_0 \in I \ni p \in U_{\alpha_0}$
$U_{\alpha_0}$ is open, $\exists r > 0 \ni B(p,r) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in I} U_\alpha$
$\because p$ is an interior point of $\bigcup_{\alpha \in I} U_\alpha$ $\therefore \bigcup_{\alpha \in I} U_\alpha$ is open, i.e. $\bigcup_{\alpha \in I} U_\alpha \in T$
(**iii**) $U_1, \cdots, U_n \in T \implies U_1 \cap \cdots \cap U_n \in T$
$\because$ Given $p \in U_1 \cap \cdots \cap U_n \implies p \in U_i$ $1 \leq i \leq n$. Each $U_i$ is open, $\exists r_i > 0$ $B(p, r_i) \subseteq U_i$, $1 \leq i \leq n \implies B(p,r) \subseteq U_1 \cap \cdots \cap U_n$
$\implies p$ is an interior point of $U_1 \cap \cdots \cap U_n$
p is arbitrary, so $U_1 \cap \cdots \cap U_n$ is open, i.e. $U_1 \cap \cdots \cap U_n \in T$
Therefore, $T$ is a topology on $X$

■

__**Definition**__. *Let $X$ be a metric space with metric $d$. The topology $T$ in prop 2.4 is called the metric topology(include by d)*

Let $X$ be a metric space and $Y \subseteq X$, Then $Y$ is a metric subspace of $X$, and $\forall y \in Y$, $r > 0$, $B_Y(y,r) = B(y,r) \cap Y$. In fact, we have more

**Prop 2.5** A subset $A \subseteq Y$ is open in $Y \Leftrightarrow A = U \cap Y$ for some open set $U$ in $X$, in particular, the metric topology on $T$ is just the relation topology of $Y$ on $X$

*Proof.* ($\Rightarrow$) suppose $A \subseteq Y$ is open in $Y$. Then

$$A = \bigcup_{y \in A} B_Y(y, r_y) = \bigcup_{y \in A} (B(y, r_y) \cap Y) = (\bigcup_{y \in A} B(y, r_y)) \cap Y$$

Let $U = \bigcup_{y \in Y} B(y, r_y)$, then $U$ is open in $X$ and $A = U \cap Y$
($\Leftarrow$) Suppose $A = U \cap Y$ where $U \subseteq X$ is open $\forall\, y \in A, y \in U \cap Y \implies y \in U \implies \exists r > 0 \ni B(y,r) \subseteq U \implies B(y,r) \cap Y \subseteq U \cap Y = A \implies B_Y(y,r) \subseteq A$, $\therefore A$ is open in $Y$. ■

**Prop 2.6** Every metric space $X$ is Hausdorff

*Proof.* Given $p, q \in X,\ p \neq q$. Choose $r = \frac{1}{2}d(p,q) > 0$. Then $B(p,r) \cap B(q,r) = \emptyset$. So $X$ is Hausdorff
($\because x \in B(p,r) \cap B(q,r) = d(x,p) < r \ \&\ d(x,q) < r \implies d(p,q) \leq d(p,x) + d(x,q) < r + r = 2r = d(p,q) (\rightarrow\leftarrow)$ ) ■

**Remark.** *Let $S \subseteq X$, where $X$ is a metric space. Then $S^o$ is the largest(w.r.t inclusion) open set contained in $S$. $\because \forall$ open set $U \subseteq S$, $U^o \subseteq S^o \implies U \subseteq S^o \subseteq S$. In fact, $S^\circ = \bigcup_{U \subseteq S} U$ (which is the definition of intension of $S$ in a topology space $X$)*

### 2.4. Closed Sets.

**Definition** (Closed set). *$F \subseteq X$ is closed $\Leftrightarrow F^C = X - F$ is open in $X$*

By Theorem 2.1, the collection of all close sets in $X$ has the properties
**(i)** $\emptyset, X$ are closed in $X$
**(ii)** $F_\alpha$ is closed in $X,\ \alpha \in I \implies \bigcap_{\alpha \in I} F_\alpha$ is closed in $X$
**(iii)** $F_1, \cdots, F_n$ are closed in $X \implies \bigcup_{i=1} F_i$ is closed in $X$

**Example**
Intersection of infinitely many open set may not be open
in $\mathbb{R}$ with Euclidean topology, $(-\frac{1}{n}, \frac{1}{n})$ is open in $\mathbb{R}\ \forall\ n \geq 1 \implies \bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$ is not open

**Prop 2.8** Let $X$ be a metric space and $Y \subseteq X$ and $B \subseteq Y$, Then $B$ is closed in $Y \Leftrightarrow B = F \cap Y$ for some closed set $F$ in $X$

*Proof.* ($\Rightarrow$) Suppose $B$ is close in $Y \implies Y - B$ is open in $Y \implies Y - B = U \cap Y$(by prop 2.5) for some open set $U$ in $X \implies Y - (Y - B) = Y - (U \cap Y) \implies B = (X - U) \cap Y$. where $(X - U)$ is close.
($\Leftarrow$) Suppose $B = F \cap Y$, where $F$ is closed in $X \implies Y - B = Y - (F \cap Y) = (X - F) \cap Y \implies Y - B$ is open in $Y \implies B$ is close in $Y$. ■

In metic space, one can use sequence to detect the closeness of a set

**Example**

**1.** We know that $[a, b)$ is not closed in $\mathbb{R}$, however, $\exists$ a sequence $\{x_n\}$ in $[a, b) \ni x_n \to b$ on $n \to \infty$, e.g. $b - \frac{1}{n} \to b$

**2.** $A = \{\frac{1}{n} \mid n \geq 1\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ is not close in $\mathbb{R}$
if $\mathbb{R} - A$ will open, them $\exists r > 0,\ B(0, r) \subseteq \mathbb{R} - A (\rightarrow\leftarrow)$
$A \cup \{0\}$ is closed in $\mathbb{R}$

$$R \setminus (A \cup \{0\}) = (-\infty, 0) \cup (1, \infty) \cup (\bigcup_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n})) \text{ is open}$$

$\therefore A \cup \{0\}$ is closed

**Definition** (Adherent, clousure $\cdots$)**.** *Let $X$ be a metric space with metric $d$, $T \subseteq X$ be a subset.(**important**)*

***(1.)*** *A point $p \in X$ is said to be an adherent point of $T$ if $\forall r > 0,\ B(p, r) \cap T \neq \emptyset$, equivalent, $\forall$ neighborhood $U$ of $p$, $U \cap T \neq \emptyset$*
***(2.)*** *Let $\overline{T}$ or $cl(T)$ denote the set of all adherent points of $T$, called the closure of $T$, i.e. $\overline{T} = \{p \in X \mid p \text{ is an adherent point of } T\}$*
***(3.)*** *A point $p \in X$ is said to be a limit point or accumulation point of $T$ if $\forall\ r > 0,\ B(p, r) \cap T - \{p\} \neq \emptyset$, equivalently, $\forall$ neighborhood $U$ of $p$, $U \cap T - \{p\} \neq \emptyset$*
*Denote by $T'$ the set of all accumulation points of $T$, called the devied set of $T$.*
***(4.)*** *$p \in T$ and $p \notin T'$, then $p$ is called an isolated point of $T$, i.e. $\exists r > 0 \ni B(p, r) \cap T = \{p\}$*
***(5.)*** *A subset $T \subseteq X$ is said to be perfect if $T$ is closed and every points of $T$ is an accumulated point of $T$, i.e. $T$ is closed & $T' = T$*
***(6.)*** *A subset $T \subseteq X$ is said to be bounded if $\exists R > 0$ and $p \in X \ni T \subseteq B(p, R)$*
***(7.)*** *A subset $T \subseteq X$ is said to be dense if $\overline{T} = X$, e.g. $\overline{\mathbb{Q}} = \mathbb{R}$*
***(8.)*** *A point $p \in X$ is said to be a boundary point of $T$ if $\forall\ r > 0, B(p, r) \cap T \neq \emptyset$ & $B(p, r) \cap (X \setminus T) \neq \emptyset$. Denote by $\partial T$ or bd(T) the set of all boundary points of $T$*

**Prop 2.9** Let $X$ be a metric space. All sets and point below are subset of $X$

**(1).** $S \subseteq T \implies \overline{S} \subseteq \overline{T}$ & $S' \subseteq T'$
$\because p \in \overline{S} \implies \forall\ r > 0, B(p, r) \cap S \neq \emptyset \implies B(p, r) \cap T \neq \emptyset \implies p \in \overline{T}$
$p \in S' \implies \forall r > 0, B(p, r) \cap S - \{p\} \neq \emptyset \implies B(p, r) \cap T - \{p\} \neq \emptyset$

**(2).** $\overline{T}$ is always closed in $X$

We want to know $\overline{T}$ is closed on $X \to X - \overline{T}$ is open $\to \forall\ p \in X - \overline{T}$ is an interior point $\implies \exists r > 0,\ B(p,r) \subseteq X - \overline{T}$
$\because p \notin \overline{T} \Rightarrow \exists r' > 0 \ni B(p,r') \cap T = \emptyset$
But we want to get $B(p,r') \cap \overline{T}$, so we check every point in $B(p,r')$ is not in $\overline{T}$, let $q \in B(p,r'),\ \exists\ \delta > 0\ , B(q,\delta) \subseteq B(p,r') \implies B(q,\delta) \cap T = \emptyset \implies q \notin \overline{T}$
because if $q \in \overline{T}, \forall r > 0 \ni B(q,r) \cap T \neq \emptyset$
$\implies B(p,r) \cap \overline{T} = \emptyset$

Let $p \in X - \overline{T} \implies p \notin \overline{T} \implies \exists\ r > 0 \ni B(p,r) \cap T = \emptyset \implies B(p,r) \cap \overline{T} = \emptyset$ ($\because \forall q \in B(p,r),\ \exists\ \delta > 0 \ni B(q,\delta) \subseteq B(p,r) \implies B(q,\delta) \cap T = \emptyset \implies q \notin \overline{T}$)
$\therefore B(p,r) \subseteq X - \overline{T}$, $\because p$ is an interior point of $X - \overline{T}$. Hence, $X - \overline{T}$ is open, i.e. $\overline{T}$ is closed.
**(3).** $T \subseteq \overline{T}(\because \forall\ p \in T, \forall\ r > 0, B(p,r) \cap T \neq \emptyset)$
**(4).** $p \in T' \implies \forall r > 0,\ B(p,r) \cap T - \{p\}$ is an infinite set, say $x_1, \cdots, x_n$, Let $\delta = \frac{1}{2}\min\{d(p,x_i) \mid 1 \le i \le n\}$. Then $B(p,\delta) \cap T - \{p\} = \emptyset (\to\leftarrow)$ to $p \in T'$, $x \in B(p,\delta) \cap T - \{p\} \implies d(x,p) < \delta \implies x = x_i$ for some $1 \le i \le n\&$ we get $d(x_i,p) < \delta \le \frac{1}{2}d(x_i,p)$
$\therefore$ no such $x$ i.e. $B(p,\delta) \cap T - \{p\} = \emptyset$
**(5).** Any finite subset of $X$ has no accumulation points in $X$ by (4). In particular, it is closed by (6)(c) below.
**(6).** TFAE

(a) $S$ is closed
(b) $S$ contains all it's adherent point, i.e. $\overline{S} \subseteq S$
(c) $S$ contains all it's accumulation points, i.e. $S' \subseteq S$
(d) $S = \overline{S}$

*Proof.* of (6)
(a) $\Rightarrow$ (b) Suppose $S$ is closed $\implies X \backslash S$ is open $\implies \forall\, p \in X - S \implies \exists r > 0 \ni B(p,r) \subseteq X \setminus S \implies B(p,r) \cap S = \emptyset \implies p \notin \overline{S}$
$\therefore \overline{S} \subseteq S$, i.e. (b) holds

(b) $\Rightarrow$ (c) $\because S' \subseteq \overline{S}$

(c) $\Rightarrow$ (d) Suppose $S' \subseteq S$. To prove $S = \overline{S}$ if not, then $S \subsetneqq \overline{S}$, i.e. $\exists\, p \in \overline{S}$ & $p \notin S \implies \forall r > 0,\ B(p,r) \cap S \neq \emptyset$ ($\because p \in \overline{S}$)
(d) $\Rightarrow$ (a) by (2)

$\blacksquare$

**(7).** $\overline{S}$ is the smallest closed set in $X$ containing $S$
$\because$ We know that $S \subseteq \overline{S}$, if $F$ is closed in $X$ & $F \subseteq S$, then $\overline{F} \subseteq \overline{S}$ by (1), $F = \overline{F} \subseteq \overline{S}$ by (6), $\therefore \overline{S}$ is the smallest such one.
**(8).** In fact, $\overline{S} = \bigcap_{F \subseteq S} F$
**(9).** $p \in S$ is an isolated point $\Leftrightarrow \exists\, r > 0 \ni B(p,r) \cap S = \{p\}$
($\Rightarrow$) Suppose $p \in S$ is an isolated point of $S$. Then $p \in S' \implies \exists\, r > 0 \ni B(p,r) \cap S - \{p\} = \emptyset \implies B(p,r) \cap S = \{p\}$
($\Leftarrow$) Trivial
**(10).** $S$ is dense in $X \Leftrightarrow\ \forall\, p \in X$ & $r > 0,\ B(p,r) \cap S \neq \emptyset \Leftrightarrow \forall$ open set $U \neq \emptyset,\ U \cap S \neq \emptyset$

*Proof.* ($\Rightarrow$) Suppose $S$ is dense in $X$, i.e. $\overline{S} = X$, So $\forall p \in X, p \in \overline{S} \implies \forall\, r > 0,\ B(p,r) \cap S \neq \emptyset$
($\Leftarrow$) Suppose the condition holds, $\forall p \in X$ & $r > 0,\ B(p,r) \cap S \neq \emptyset \implies p \in \overline{S} \implies X \subseteq \overline{S} \subseteq X,\ \therefore \overline{S} = X$ $\blacksquare$

**(11).** $\partial S = \partial(X - S)$ In particular, $\partial S = \overline{S} \cap \overline{(X-S)}$, In particular, $\partial S$ is closed in $X$, $\because$ It suffices to prove $\partial S = \overline{S} \cap \overline{(X-S)}$,
$\because \partial(X-S) = \overline{X-S} \cap \overline{X-(X-S)} = \overline{X-S} \cap \overline{S} = \partial S$
$\forall p \in \partial S \implies \forall r > 0, B(p,r) \cap S \neq \emptyset$ & $p \in \overline{X-S} \implies p \in \overline{S} \cap \overline{X-S}$
$\therefore \partial S \subseteq \overline{S} \cap \overline{X-S}$,
Conversely, $p \in \overline{S} \cap \overline{X-S} \implies p \in \overline{S}$ & $p \in \overline{X-S} \implies \forall r > 0,\ B(p,r) \cap S \neq \emptyset$ & $B(p,r) \cap (X-S) \neq \emptyset \implies p \in \partial S$
$\therefore \overline{S} \cap \overline{(X-S)} \subseteq \partial S \therefore \partial S = \overline{S} \cap \overline{(X-S)}$

### 2.5. Examples.

We give some simple examples of open sets, closed sets, adherent, accumulation, isolated and boundary points.

**1.** In a discrete metric space $X$, every subset of $X$ is both open and close, $\forall x \in X,\ B(p,r) \begin{cases} \{x\} \text{ if } 0 < r \leq 1 \\ X \text{ if } r > 1 \end{cases}$
$\therefore$ Every singleton is open in $X$, so every subset of $X$ is open.
**2.** In $\mathbb{R}$. Consider the set $S = [0,1) \cup \{3\}$, $S^\circ = \emptyset,\ S' = \{0\}$,
$\overline{S} = S \cup \{0\}$
**3.** In $\mathbb{R}$, consider the set $S = \{\frac{1}{n} \mid n = 1, 2, \cdots\}$, $S^\circ = \emptyset$,
$S' = \{0\},\ \overline{S} = S \cup \{0\}$
**4.** In $\mathbb{R}^2$, consider $S = \{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$, $S$ is open
$\overline{S} = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$
$\partial S = \{(x,0) \mid x \geq 0\} \cup \{(y,0) \mid y \geq 0\}$
**5.** Let $B(0,1)$ be the unit open ball in $\mathbb{R}^k$. Then $\partial B(0,1) = S^{k-1}$ is the unit $(k-1)$-sphere. In particular, for $k = 2, \partial B(0,1) = S^1$ in the unity circle in the plane $\mathbb{R}^2$. Similarly, for the closed unit ball $\overline{B}(0,1)$ in $\mathbb{R}^k$. Now, we define some special sets in $\mathbb{R}^n$

- Internals in $\mathbb{R}:\ -\infty < a \leq b < \infty$
  $[a,b]$ close interval which is closed in $\mathbb{R}$
  $(a,b)$ open interval which is closed in $\mathbb{R}$
  Infinite intervals:
  $(-\infty, b]$ : close in $\mathbb{R}$ , $(-\infty, b)$ open in $\mathbb{R}$
  whose length are $\infty$ and boundary

$$\begin{cases} \partial(a,+\infty) = \partial(a,+\infty) = \{a\} \\ \partial(-\infty,b) = \partial(-\infty,b) = \{b\} \end{cases}$$

- k-dimensional interval (rectangle or k-cell) $I$

$$I = I_1 \times \cdots \times I_k$$

  where $I_j$ is an interval in $\mathbb{R},\ 1 \leq j \leq k$
  (i) $I$ is bounded $\Leftrightarrow$ each $I_j$ is bounded
  $I$ is unbounded $\Leftrightarrow I_j \neq \emptyset$ & some $I_j$ is unbounded
  (ii) $I = [a_1, b_1] \times \cdots [a_k, b_k],\ -\infty < a_j \leq b_j < \infty,\ 1 \leq j \leq k$
  $k$-dimensional closed(compact) interval in $\mathbb{R}^k$

**6.** Convex sets in $\mathbb{R}^k$(In vector space)

**Definition.** *$S \subseteq \mathbb{R}^k$ is convex if $\forall\ x, y \in S$, $\overline{xy}$ is the line segment joining $x$ & $y$*

Given $p, q \in \mathbb{R}^k$, The parametric equation of

$$\begin{aligned} \overleftrightarrow{pq} &= \{p + t(q-p) \mid t \in \mathbb{R}\}\text{: the line in } \mathbb{R}^k \text{ throw } p \text{ and } q \\ \overrightarrow{pq} &= \{p + t(q-p) \mid t \geq 0\}\text{: the ray in } \mathbb{R}^k \text{ throw p and } q \\ \overline{pq} &= \{p + t(q-p) \mid 0 \leq t \leq 1\}\text{: the line segement throw p and q} \end{aligned}$$

$$\begin{aligned} x \in \overline{pq} \Leftrightarrow x &= p + t(q-p) \text{ for some } t \in [0,1] \\ &= p + tq - tp \\ &= (1-t)p + tq \\ &= \alpha p + \beta q, \text{where } \alpha\ \beta \geq 0 \text{ and } \alpha + \beta = 1 \end{aligned}$$

and we called $\alpha p + \beta q$ convex combination of $p$ and $q$.
Note that all open balls, closed balls, intervals are convex in $\mathbb{R}^k$

Given $B(p,r)$, $\forall x, y \in B(p,r)$ and $\alpha, \beta \geq 0, \alpha + \beta = 1$

$$\begin{aligned} \parallel \alpha x + \beta y - p \parallel &= \parallel \alpha x + \beta y - (\alpha + \beta)p \parallel \\ &= \parallel \alpha x - \alpha p + \beta y - \beta p \parallel \\ &\leq \parallel \alpha x - \alpha p \parallel + \parallel \beta y - \beta p \parallel \\ &= \alpha \parallel x - p \parallel + \beta \parallel y - p \parallel \\ &< \alpha r + \beta r = (\alpha + \beta) r = r \end{aligned}$$

$\therefore \alpha x + \beta y \in B(p, r \therefore B(p,r))$ is convex

**7.** Star-like sets in $\mathbb{R}^k$ with w.r.t some point $x_0$, $S \subseteq \mathbb{R}^k$ is star-like w.r.t. $x_0 \in S$ if $\forall\ x \in S$, $\overline{xx_0} \subseteq S$
**8.** We know that $\mathbb{Q}$ is dense in $\mathbb{R}$, hence $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$. Note that $\mathbb{Q}^k$ is countable, hence $\mathbb{R}^k$ has a countable dense subset $\mathbb{Q}^k$, i.e. $\mathbb{R}^k$ is separable.
**9.** $\partial\mathbb{Q} = \mathbb{R}$, $\partial\mathbb{Q}^k = \mathbb{R}^k$

**10.** $\mathbb{Z}$ is closed in $\mathbb{R}$ and every point in $\mathbb{Z}$ is isolated e.g.$\{n\pi \mid n \in \mathbb{Z}\}$, $\because \mathbb{R}-\mathbb{Z} = \bigcup_{n=-\infty}^{\infty}(n-1, n)$ is open $\implies$ $\mathbb{Z}$ is close. or $\mathbb{Z}' = \emptyset \subseteq \mathbb{Z}$, $\therefore \mathbb{Z}$ is close.
**11.** Let $S \subseteq \mathbb{R}$ be a nonempty set which is bounded above. Then $\alpha = \sup S$ exists. Moreover, $\alpha \in \overline{S}$. $\because \forall\ r > 0, \exists\ x_0 \in S \ni \alpha - r < x_0 \leq \alpha < \alpha - r \implies (\alpha - r, \alpha + r) \cap S \neq \emptyset \implies \alpha \in \overline{S}$

### 2.6. Compact Set in Metric Space.

- Compact sets in metric space, which is closely related to the extreme value problem.
- Compact set $\mathbb{R}^k$ will be discussed in next section.

__**Definition**__. *Let $X$ be a topology space and $S \subseteq X$. A collection $\mathscr{U} = \{U_\alpha\}_{\alpha \in I}$ of open sets in $X$ is called an open covering of $S$ if*

$$S \subseteq \bigcup \mathscr{U} = \bigcup_{\alpha \in I} U_\alpha$$

**Example**

(1) $X$ is discrete metric space. Then $\{\{x\} \mid x \in X\}$ is an open covering of $X$
(2) In $\mathbb{R}$, $\{(0, 1 - \frac{1}{n}) \mid n \in \mathbb{N}\}$ is an open covering of $(0, 1)$. In fact, $(0, 1) = \bigcup_{n=1}^{\infty}(0, 1 - \frac{1}{n})$
(3) $\{B(0, n) \mid n \in \mathbb{N}\}$ is an open covering of $\mathbb{R}^k$

__**Definition**__. *Let $X$ be a topology space, $S \subseteq X$ and $\mathscr{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of $S$. We say that $\mathscr{U}$ has a countable(finite) sub covering of $S$ if $\exists$ a countable(finite) sub collection of $\mathscr{U}$ which also covers $S$. i.e. $\mathscr{U}$ has a countable(finite) subcovering in $S$ if*
*$\exists$ a sequence $\{\alpha_n\}$ in $I \ni S \subseteq \bigcup_{n=1}^{\infty} U_{\alpha_n}$ (countable)*
*$\exists$ a sequence $\{\alpha_n\}$ in $I \ni S \subseteq U_\alpha \cup \cdots \cup U_{\alpha_n}$ (finite)*

__**Definition**__ (compact). *Let $X$ be a topology space. A subset $K \subseteq X$ is said to be compact if **every** open covering of $K$ admit a finite subcovering*

**Examples**

(1) Let $X$ be a topology space and $K \subseteq X$ be a finite set. Then $K$ is compact. $\because$ Given an arbitrary open covering $\{U_\alpha \mid \alpha \in I\}$ of $K$, i.e. $K \subseteq \bigcup_{\alpha \in I} U_\alpha \implies \exists \alpha_i \in I \ni x_i \in U_{\alpha_i}, 1 \leq i \leq n \implies K = \{x_1, \cdots, x_n\} \subseteq \bigcup_{i=1}^n U_{\alpha_i}$ $\therefore K$ is compact.
(2) In a discrete metric space $X$, a subset $K \subseteq X$ is compact $\Leftrightarrow K$ is a finite set.
(3) $(0,1)$ is not compact in $\mathbb{R}(\{0, 1 - \frac{1}{n} \mid n \in \mathbb{N}\})$, but $[0,1]$ is compact

**Theroem 2.10.** *Let $X$ be a metric space and $K \subseteq Y \subseteq X$. Then $K$ is compact in $X \Leftrightarrow K$ is compact in $Y$.*

*Proof.* ($\Rightarrow$) Suppose $K$ is compact in $X$. Given an open covering $\{V_\alpha\}_{\alpha \in I}$ of open sets in $Y$ which covers $K$. By Prop 2.5, each $V_\alpha = U_\alpha \cap Y$, where $U_\alpha$ is open in $X$. Now,

$$K \subseteq \bigcup_{\alpha \in I} V_\alpha = \bigcup_{\alpha \in I} (U_\alpha \cap Y) = (\bigcup_{\alpha \in I} U_\alpha) \cap Y \implies K \subseteq \bigcup_{\alpha \in I} U_\alpha$$

By the compactness of $K$ in $X$, $\exists \alpha_1, \cdots, \alpha_n \in I \ni K \subseteq \bigcup_{i=1}^n U_{\alpha_i} \implies K \cap Y \subseteq (\bigcup_{i=1}^n U_{\alpha_i}) \implies K \subseteq \bigcup_{i=1}^n (U_{\alpha_i} \cap Y) = \bigcup_{i=1}^n V_{\alpha_i}$
$\therefore K$ is compact in $Y$
($\Leftarrow$)Suppose $K$ is compact in $Y$. Given a open covering $\{U_\alpha\}_{\alpha \in I}$ of $K$ by open sets in $X$.

$$K \subseteq \bigcup_{\alpha \in I} U_\alpha \implies K \cap Y \subseteq (\bigcup_{\alpha \in I} U_\alpha) \cap Y \implies K \cap Y \subseteq \bigcup_{\alpha \in I} (U_\alpha \cap Y)$$

By Prop 2.5, $\{U_\alpha \cap Y \mid \alpha \in I\}$ is an open covering of $K$ by open set in $Y$. By assumption, $K$ is compact in $Y, \exists\ \alpha_1, \cdots, \alpha_n \in I \ni K \subseteq \bigcup_{i=1}^n (U_{\alpha_i} \cap Y) = (\bigcup_{i=1}^n U_{\alpha_i}) \cap Y \implies K \subseteq \bigcup_{i=1}^n U_{\alpha_i}$
$\therefore K$ is compact in $X$ ■

**Definition.** *Let $X$ be a metric space and $S \subseteq X$ be a nonempty set. The diameter of $S$ is defined to be $dia(S) = \sup\{d(x,y) \mid x, y \in S\}$ which generated the diameter of a circle in $\mathbb{R}^2$*

**Recall.** bounded set in a metric space, specially, in $\mathbb{R}^k$ $X$ : metric space

- $\star$ $S \subseteq X$ is bdd if $\exists p \in X$ and $r > 0 \ni S \subseteq B(p, r)$

⋆ In $\mathbb{R}^k$, $S \subseteq \mathbb{R}^k$ is bounded of $\exists M > 0 \ni \| x \| \leq M \forall x \in S$
$\because S \subseteq B(p,r) \implies \forall x_0 \in \mathbb{R}^k, \exists R_{x_0} > 0 \ni S \subseteq B(x_0, R^{x_0})$ for some $p \in \mathbb{R}^k$ and $r > 0$

In particular, $S \subseteq B(0,M) = \{x \in \mathbb{R}^k \mid \| x \| < M\}$ for some $M > 0$

**Theroem 2.11.** *Let $X$ be a metric space and $K \subseteq X$ be a compact set. Then $K$ is closed and bounded*

*Proof.* $K$ **is bounded**
Fix a point $p \in K$. Then $K \subseteq \bigcup_{n=1}^{\infty} B(p,n)$. $\because K$ is compact $\implies \exists N \in \mathbb{N} \ni K \subseteq B(p,1) \cup \cdots \cup B(p,N) \implies K \subseteq B(p,N)$ $\therefore K$ is bounded
**K is closed**, i.e. $X - K$ is open
Fix $p \in X - K$. Then $p \neq x, \ \forall x \in K$. Hence, $d(x,p) > 0, \ \forall x \in K$
Let $r_x = \frac{1}{2}d(x,p) > 0, x \in K$. Them $\{B(x, r_x) \mid x \in K\}$ is an open covering of $K$. $\because K$ is compact $\implies \exists x_1, \cdots, x_n \in K \ni B(x_1, r_{x_1}) \cup \cdots \cup B(x_n, r_{x_i})$. Let $V = \bigcap_{i=1}^{n} B(p, r_{x_i}) = B(p,r)$, where $r = \min\{r_{x_1}, \cdots, r_{x_n}\}$. Then as we can see that $V \subseteq X - K$, all point in $X - K$ are inner point. So $X - K$ is open, i.e. $K$ is close.

To show that $V \subseteq X - K$, i.e. $V \cap K \neq \emptyset$, it suffices to show

$$V \cap (\bigcup_{i=1}^{n} B(x_i, r_{x_i})) = \emptyset$$

Now,

$$\begin{aligned} V \cap (\bigcup_{i=1}^{n} B(x_i, r_{x_i})) &= \bigcup_{i=1}^{n} (V \cap B(r_i, r_{x_i})) \\ &\subseteq \bigcup_{i=1}^{n} (B(p, r_{x_i} \cap B(x_i, r_{x_i}))) = \emptyset \end{aligned}$$

■

**Remark.** *The converse of Thm 2.11 is false, i.e. closed & bounded may not be compact, e.g. $X$ is an infinite set with discrete metric. Then $X$ is not compact, but $X$ is closed and bounded.*

**Theroem 2.12.** *Let $X$ be a metric space, $K \subseteq X$ be compact & $L \subseteq K$ be a closed set in $X$. Then $L$ is compact.*

*Proof.* Let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of $L$. Then $\{U_\alpha\}_{\alpha \in I} \cup \{X - L\}$ is an open covering of $K$. By the compactness of $K$, $\exists \alpha_1, \cdots, \alpha_n \in I \ni K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \cup (X - L)$. By $L \subseteq K$ $\therefore L$ is compact ∎

**Corollary 2.13.**
*(a) Let $X$ be a metric space, $K \subseteq X$ be compact and $F$ be a closed set in $X$. Then $K \cap F$ is compact.*
*(b) If $X$ is a compact metric space, then every closed subset $F$ of $X$ is compact.*

*Proof.*
(a)

$$\begin{aligned} K \text{ is compact} \quad &\Longrightarrow \quad K \text{ is closed (Thm 2.11)} \\ &\Longrightarrow \quad K \cap F \text{ is closed in } X \\ &\Longrightarrow \quad K \cap F \text{ is compact} \end{aligned}$$

(b) follows (a)

∎

**Remark.** *Let $X$ be a metric space. If $K$ is closed in $X$ and $F$ is closed in $K$, then $F$ is closed in $X$. $\because F$ is closed in $F \implies F = L \cap F$, where $L$ is closed in $K \implies F = L \cap K$, where $L$ is closed in $X \implies F$ is closed in $X$.*

**Theroem 2.14.** *Let $X$ be a metric space. Then $X$ is compact **if and only if** $\forall$ collection $\{K_\alpha\}$ of closed subsets of $X$ with FIP(finite intersection property, i.e. $\forall \alpha_1, \cdots, \alpha_n \in I, K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \neq \emptyset$), $\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$*

*Proof.*
($\Rightarrow$) Suppose $X$ is compact. Given a collection $\{K_\alpha\}$ of closed subset of $X$ with FIP. We must prove that $\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$. If $\bigcap_{\alpha \in I} K_\alpha = \emptyset$, then $X = X - \bigcap_{\alpha \in I} K_\alpha = \bigcup_{\alpha \in I}(X - K_\alpha) \implies \exists \alpha_1, \cdots, \alpha_m \in I \ni X = \bigcup_{i=1}^n (X - K_{\alpha_i}) = X - \bigcap_{i=1}^n K_{\alpha_i} \implies \bigcap_{i=1}^n K_{\alpha_i} K_{\alpha_i} = \emptyset (\rightarrow\leftarrow)$ to FIP.

($\Leftarrow$) Suppose the condition holds. To prove that $X$ is compact. If $X$ were not compact, them $\exists$ an open covering $\{U_\alpha\}_{\alpha\ni I}$ of $X$ which has no finite subcovering. Let $K_\alpha = X - U_\alpha, \alpha \in I$. Then $\{K_\alpha\}_{\alpha\in I}$ is a collection of closed subsets.
Claim it satisfies FIP. GIven $\alpha_1, \cdots, \alpha_n \in I$, $\bigcap_{i=1}^n K_i = \bigcap_{i=1}^n (X - U_{\alpha_i}) = X - \bigcup_{n=1}^n U_{\alpha_i}$

$$\therefore \bigcap_{i=1}^n K_i = \emptyset \Leftrightarrow X - \bigcup_{i=1}^n U_\alpha = \emptyset \Leftrightarrow X = \bigcup_{i=1}^n U_{\alpha_i} (\rightarrow\leftarrow)$$

$$\therefore \bigcap_{i=1}^n K_{\alpha_i} = \emptyset$$

By assumption, $\bigcap_{\alpha\in I} K_\alpha \neq \emptyset$, i.e. $\bigcap_{\alpha\in I}(X - U_\alpha) \neq \emptyset \implies \mathcal{X} - \bigcup_{\alpha\in I} U_\alpha = \emptyset \implies \bigcup_{\alpha\in I} U_\alpha \neq X (\rightarrow\leftarrow)$
$\therefore X$ must be compact.

$\blacksquare$

**Corollary 2.15** (& 2.16). *Let $X$ be a metric space and $\{K_n\}_{n=1}^\infty$ be a decrease sequence of nonempty compact sets of $X$. Them $\bigcap_{n=1}^\infty \neq \emptyset$. In addition, if dia$_{n\to\infty}\infty 0$, them $\bigcap_{n=1}^\infty K_n$ is a singleton.*

*Proof.* $\forall j1, \cdots, j_k \in \mathbb{N}$, $K_{j1} \cap \cdots \cap K_{jk} \neq \emptyset$, $K_{j1} \cap \cdots \cap K_{jk} = K_t$, where $t = \max\{j_1, \cdots, j_k\}$. By Thm 2.14 $\bigcap_{n=1}^\infty K_n \neq \emptyset$, if $\lim_{n\to\infty} \mathrm{dia}(K_n) = 0$ and $p, q \in \bigcap_{n=1}^\infty K$ and $p \neq q$, them $\mathrm{dia}(K_n) \geq d(p,q)\ \forall n \geq 1 \implies \lim_{n\to\infty} \mathrm{dia}(K_n) \geq d(p,q) > 0 (\rightarrow\leftarrow) \because \bigcap_{n=1}^\infty K_n = \{p\}$ is a simpleton. $\blacksquare$

**Remark.** *The usual form of Cor 2.15, $X$ is a metric space, $\{K_n\}$ is a decrease sequence of nonempty closed sets in $X$ with $K_i$ is compact $\implies \bigcap_{n=1}^\infty K_n \neq \emptyset$*

**Example** In $\mathbb{R}$, $\{(0, \frac{1}{n} \mid n \geq 1]\}$ is decrease and every finite subcollection of $\{(0, \frac{1}{n}) \mid n \geq 1\}$ is nonempty, but $\bigcap_{n=1}^\infty (0, \frac{1}{n}) = \emptyset$, $\bigcap [0, \frac{1}{n}] = \emptyset$

**Theroem 2.17.** *Let $X$ be a metric space and $K \subseteq X$, TFAE:*
*(i) $K$ is compact*
*(ii) Every infinite subset has an accumulation point in $K$*
*(iii) $K$ is sequentially compact*

*(iv) $K$ is complete and totally bounded*

**<u>Definition</u>** (Convergence)**.** *$\{a_n\}$ converge if $\exists\ a \in X \ni \forall\ \epsilon \geq 0 \exists N \ni \mathbb{N} \ni \forall n \geq N,\ d(a_n, a) < \epsilon$. Such $a$ is called the limit of $\{a_n\}$, which is denoted by $\lim_{n\to\infty} a_n = a$ or $a_n \to a$ on $n \to \infty$.*

**<u>Definition</u>** (Cauchy)**.** *We say that $\{a_n\}$ is Cauchy if $\forall \epsilon > 0,\ \exists N \in \mathbb{N} \ni \forall n, m \geq \mathbb{N},\ d(a_n, a_m) < \epsilon$*

**<u>Definition</u>.** *A metric $X$ is said to be sequence compact if every sequence has a convergent subsequence*

**<u>Definition</u>.** *A metric space $X$ is said to be complete if every Cauchy sequence in $X$ convergence.*

**<u>Definition</u>.** *Let $X$ be a metric space & $K \subseteq X$. We say that $K$ is totally bounded if $\forall r > 0, \exists x_1, \cdots, x_n \in K \ni K \subseteq B(x_1, r) \cup \cdots \cup B(x_n, r)$*

**Remark.** *Totally bounded can implies bounded, but not converse.*
*$K$ is totally bounded, for $r = 1, \exists x_1, \cdots, x_n \in K \subseteq B(x_1, 1) \cup \cdots \cup B(x_n, 1) \implies K \subseteq B(x_1, R)$ for sime large $R$*

- *Take an **"infinite"** set $X$ with discrete metric. Then $X$ is bounded(e.g. $X \subseteq B(x_0, 2)$, where $x_0 \in X$) but for $r = \frac{1}{2},\ X \subsetneqq B(x_1, \frac{1}{2}) \cup \cdots \cup B(x_n, \frac{1}{2})\ \forall x_1, \cdots, x_n$*

**Lemma 2.18** (To prove (ii) to (i))**.**
*Suppose (ii) holds in Thm 2.17. Then $K$ is totally bounded*

*Proof.* If not, then $\exists\ r > 0, \ni$ no finite open balls with radius $r$ and center $K$ cover $K$. Choose $x_1 \in k \implies K \subsetneqq B(x_1, r) \implies \exists x_2 \in K - B(x_1, r),\ K \subsetneqq B(x_1, r) \cup B(x_2, r) \implies \exists x_3 \in K - (B(x_1, r) \cup B(x_2, r))$ By induction, counting this process, we obtain an infinite set $T = \{x_1, x_2, \cdots, x_n, \cdots\} \subseteq K$ with $d(x_i, x_j) \geq r \forall i \neq j$. By (ii), $T$ has an accumulation poion $p \in K$. In particular $B(p, \dfrac{r}{4}) \cap T - \{p\}$ is an infinite set, hence, $\exists i \neq j \ni x_i, x_j \in B(p, \dfrac{r}{4}) \cap T - \{p\} \implies d(x_i, x_j) \leq d(x_i, p) + d(x, j) < \dfrac{r}{4} + \dfrac{r}{4} = \dfrac{r}{2} < r (\rightarrow\leftarrow) \therefore K$ is totally bounded. ∎

**Lemma 2.19.** *Suppose (ii) holds in $T$ and $\{E_\alpha \mid \alpha \in I\}$ is an open covering of $K$. Then $\exists r > 0$(called a Lebegoue number w.r.t. the open covering $\{E_\alpha\}_{\alpha\in I}$) $\ni \forall x \in K, B(x,r) \subseteq E_\alpha$ for some $\alpha \in I$*

*Proof.* if $K$ is a finite set, let $K = \{x_1, \cdots, x_n\} K \subseteq \bigcup_{\alpha\in I} E_\alpha \implies x_i \in E_{\alpha_i}$ for some $\alpha_i \in I,\ 1 \le i \le n \implies \exists r_i > 0 \ni B(x_i, r_i) \subseteq E_{\alpha_i}, 1 \le i \le n$. Let $r = \min\{r_1, \cdots, r_n\}$. Then $B(x_i, r) \subseteq B(x_i, r_i) \subseteq E_{\alpha_i}, 1 \le i \le \alpha$. Now, assume that $K$ is infinite set. Assume that no such $r > 0$, i.e. $\forall r > 0, \exists x_r \in K \ni B(x_r, r) \subsetneqq E_\alpha \forall x \in I$. Now, for $r = \dfrac{1}{k}, r = 1, 2, \cdots$, we obtain a sequence $\{x_k\}$ in $K$, with $x_k = \dfrac{x_r}{k} \ni B(x_k, \dfrac{1}{k}) \subsetneqq E_\alpha \forall a \in I$. Let $T = \{x_1, x_2, \cdots, x_k, \cdots\}$. Then $T \subseteq K$ is an infinite set($\because$ For $k = 1, r = \dfrac{1}{1} = 1 \exists x_1 \in K \ni B(x_1, 1) \subsetneq E_\alpha \forall \alpha \in I$). The conclusion of Lemma 2.19 failed for $K - \{x_1\}$ ($\because if \exists s > 0 \ni \forall x \in K - \{x_1\} B(x, s) \subseteq E_\alpha$ for some $\alpha \in I,\ x_1 \in E_\alpha \implies \exists t > 0 \ni B(x_1, t) \subseteq E_\alpha$. Let $r = \min\{s, t\}$. Then $\forall x \in K, B(x, r) \subseteq E_\alpha$ for some $\alpha \in I (\rightarrow\leftarrow)$)
Then for $r = \dfrac{1}{2}, \exists x_2 \in K - \{x_1\} \ni B(x_2, \dfrac{1}{2}) \subsetneqq E_\alpha \forall \alpha \in I$. Continue this process, we conclude that $x_i \ne x_j \forall i \ne j$, so $T$ is an infinite set. By the assumption of (ii). Then an accumulation point $p \in K$. Now $K = \bigcup_{\alpha\in I} E_\alpha \implies p \in E_\alpha$ for some $\alpha \in I \implies \exists \epsilon > 0 \ni B(p, \epsilon) \subseteq E_{\alpha_0}$. Since $p \ni T', B(p, \epsilon) \cap T - \{p\}$ is an infinite set. Choose $m >> 0 \ni \dfrac{1}{m} < \dfrac{\epsilon}{2}$ & $x_m \in B(p, \dfrac{\epsilon}{2}) \cap T$. Claim $B(x_m, \dfrac{1}{m}) \subseteq B(p, \epsilon) \subseteq E_{\alpha_0} (\rightarrow\leftarrow)$ to our constrain, hence Lemma 2.19 holds.

$$y \in B(x_m, \frac{1}{m}) \Rightarrow d(y,p) \le d(y, x_m) + d(x_m, p) < \frac{1}{m} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\Rightarrow y \in B(p, \epsilon)$$ ■

*Proof.* (Thm 2.17 (i)(ii))
$(i) \Rightarrow (ii)$ Suppose $K$ is compact. Given an infinite set $T \subseteq K$. We must prove that $T$ has an accumulation point in $K$, if not, $\forall x \in K, x$ is not an accumulation point of $K, \exists r_x > 0 \ni B(x, r_n) \cap T - \{x\} = \emptyset \implies B(x, r_x) \cap T \subseteq \{x\}$. Clearly $\{B(x, r_x \mid x \in K)\}$ is an open covering of $K$. By (i), $K$ is compact

$$\begin{aligned}
\Longrightarrow \exists x_1, \cdots, x_n \in K \ni K &\subseteq B(x_1, r_{x_1}) \cup \cdots \cup B(x_n, r_{x_n}) \\
&= (T \cap B(x_1, r_{x_1})) \cup \cdots \cup (T \cap B(x_n, r_{x_n})) \\
&\subseteq \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_n\} \\
&= \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_n\} \\
&= \{x_1, \cdots, x_n\} (\rightarrow\leftarrow)
\end{aligned}$$

to $T$ is an infinite set, $\therefore$ (ii) holds.
$(ii) \Rightarrow (i)$ In Thm 2.17 i.e. we must prove that $K$ is compact under the assumption of (ii). Suppose $\mathscr{U} = \{E_\alpha\}_{\alpha \in I}$ is an open covering of $K$. By Lemma 2.19, $\exists$ a $r > 0$ *w.r.t.* $\mathscr{U}$, by Lemma 2.18, $\exists x_1, \cdots, x_k \in K \ni K \subseteq B(x_1, r) \cup \cdots \cup B(x_k, r) \subseteq E_{\alpha_1} \cup \cdots \cup E_{\alpha_k}$, where $B(x_i, r) \subseteq E_{\alpha_i}, 1 \leq i \leq k$. Therefore, $\mathscr{U}$ has a finite sub covering. Hence $K$ is compact and (i) holds. ■

**Remark.**

*(1) (ii) $\Longrightarrow$ (i) is exercise 26*
*(2) More or less, by Lemma 2.18 & 19, one can see that (i) and (ii) are also equal to (iii) and (iv)*

## 2.7. Compact Sets in Euclidean Spaces $\mathbb{R}^k$.

• We know that any compact set in a metric space is close and bounded
• Close and bounded subset my not be compact(infinite discrete)
• We will see that every closed and bounded subset of $\mathbb{R}^k$ is always compact which is the famous H.B. Theorem, i.e. $K \subseteq R^k$ is compact $\Leftrightarrow$ $K$ is closed and bounded

**Theroem 2.20.** *Let $\{I_n = [a_n, b_n]\}_{n=1}^{\infty}$ be a sequence of closed and bounded intervals in $\mathbb{R}$, if $\{I_n\}$ is decreasing i.e. $I_1 \subseteq \cdots \subseteq I_n \subseteq \cdots$, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Moreover, if $\lim_{n\to\infty}(b_n - a_n) = 0$, then $\bigcup_{n=1}^{\infty} I_n$ is a simpleton.*

*Proof.* Claim $T = \{a_n \mid n \in \mathbb{N}\}$ is bounded above and $x = \sup T$ exists.
$\because a_n \leq a_{m+n} (\because \{a_n\}$ is increasing, i.e. $[a_1, b_2] \supseteq [a_2, b_2],\ a_2 \geq a_1)$
$a_n \leq a_{m+n} \leq b_{m+n} \leq b_m (\because \{b_n\}$ is decreasing $) \implies T$ is bounded above by all $b_n \implies x = \sup T$ exists and $x \leq b_n \forall n \geq 1$.
Clearly, $a_n \leq x\ \forall n \geq 1$, $\because a_n \leq x \leq b_n, \forall n \geq 1$, i.e. $x \in [a_n, b_n] \forall n \geq 1 \therefore x \in \bigcap_{n=1}^{\infty} I_n$. Hence, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset \therefore$ The last statement follow the argument as in Corollary 2.16. ■

**Theroem 2.21.** *Let $\{I_n = [a_{n,1}, b_{n,1}] \times \cdots \times [a_{n,k}, b_{n,k}]\}$ be a decreasing sequence of closed and bounded intervals in $\mathbb{R}^k$. Then $\bigcap_{n=1}^{\infty} \neq \emptyset$. Moreover, if $\lim_{n\to\infty} dia(I_n) = 0$, then $\bigcup_{n=1}^{\infty}$ is a simpleton.*

*Proof.* $\forall 1 \leq j \leq k, \{[a_{n,j}, b_{n,j}]\}$ is a decrease sequence of closed and bounded intervals in $\mathbb{R}$. By Thm 2.20, $\exists x_j \in \bigcap_{n=1}^{\infty}[a_{n,j}, b_{n,j}]$. Set $x = (x_1, \cdots, x_k)$. Then $x \in \bigcap_{n=1}^{\infty} I_n$. Then last statement also follows from the argument in corollary 2.16. ■

**Theroem 2.22.** *Every $k$-dimensional closed and bounded interval $I = [a_1, b_1] \times \cdots \times [a_k, b_k]$ in $\mathbb{R}^k$ is compact.*

*Proof.* Put $\delta = (\sum_{i=1}^{k}(b_i - a_i)^2)^{\frac{1}{2}}$ which is the diametric of $I$. Then $\forall x, y \in I, \| x-y \| \leq \delta$. If $I$ were not compact, them $\exists$ an open covering $\{E_\alpha\}_{\alpha \in J}$ admitting not finite sub covering($\star$). Put $c_j = \dfrac{a_j + b_j}{2}$, $1 \leq j \leq k$. The intervals $[a_j, c_j]$ and $[c_j, b_j], 1 \leq j \leq k$, determines $2^k$ closed and bounded subinterval of $I$ whose union is $I$. By ($\star$), at least one of them, say $I_1$ which cannot be covered by finitely many $E_\alpha$. Continuing this process, we get a sequence $\{I_n\}$ of closed and bounded subintervals of $I$ satisfy's

a) $I \subseteq I_1 \subseteq \cdots$, i.e. $\{I_n\}_{n=1}^{\infty}$ is decreasing.
b) Each $I_n$ cannot be covered by finitely many $E_\alpha$
c) $\text{dia}(I_n) = 2^{-n}, \ \delta \to 0$ on $n \to \infty$

By Thm 2.21, $\bigcap_{n=1}^{\infty} I_n = \{x\}$ i.e. $x \in I_n \subseteq I \ \forall \ n \geq 1 \subseteq \bigcup_{\alpha=1}^{\infty} E_\alpha$
$\because x \in E_{\alpha_0}$ for some $\alpha_0 \in J, \ E_{\alpha_0}$ is open $\implies \exists r > 0 \ni B(x, r) \subseteq E_{\alpha_0}$. Choose $n_0 >> 0 \ni \dfrac{1}{2^{n_0}} < \dfrac{r}{\delta}(\because \dfrac{1}{2^n} \to 0)$. Since $x \in I_{n_0}, \forall y \in I_{n_0}, \ \| y - x \| \leq 2^{-n_0}\delta < \dfrac{r}{\delta} \cdot \delta = r \implies y \in B(x, r), \ \because I_{n_0} \subseteq B(x, r) \subseteq E_{\alpha_0} (\rightarrow\leftarrow)$.
Therefore $I$ is compact. ■

Combining Thm 2.22 and results in section 2.6, we conclude that the following sets in compact:

i) $[a, b]$ is compact in $\mathbb{R}$(Thm 2.22 with $k = 1$)
ii) $[a, b] \times [c, d]$ is compact in $\mathbb{R}^2$(Thm 2.22 with $k = 2$)
iii) Every closed ball $\overline{B}(x, r)$ in $\mathbb{R}^k$ is compact by Thm 2.12 and 2.22
iv) $\{0\} \ \cup \{\dfrac{1}{n} \mid n = 1, 2, \cdots\}$ is compact in $\mathbb{R}$, In fact, if $a_n \to a$, them teh set $\{a\} \cup \{a_n \mid n = 1, 2, \cdots\}$ is compact in $\mathbb{R}$

**Theroem 2.23.** *Every closed and bounded subset $K$ of $\mathbb{R}^k$ is compact.*

*Proof.* Choose a large closed and bounded interval $I$ in $\mathbb{R}^k$, $K \subseteq I$. By Thm 2.22, $I$ is compacted, so $K$ is a closed subset of $I$. By Thm 2.12, $K$ is compacted.
Combining Thm 2.17 and Thm 2.23, we can characterize compacted set in $\mathbb{R}^k$ ■

**Theroem 2.24.** *Let $K \subseteq R^k$ TFAE:*

*i) $K$ is closed and bounded*

*ii) $K$ is compacted*

*iii) Every infinite subset of $K$ has an accumulation point*

*iv) $K$ is sequence compact*

*v) $K$ is complete and totally bounded*

From these we can deduce:

**Theroem 2.25** (Bolzano-Weiestrace)**.** *Every bounded infinite subset $T$ in $\mathbb{R}^k$ has an accumulation point in $\mathbb{R}^k$*

*Proof.* Since $T$ is bounded, choose a large closed and bounded interval $I$ in $\mathbb{R}^k \ni T \subseteq I$. Now, $T$ become an infinite subset of the compact set $I$. By Thm 2.24 (iii), $T$ has an accumulation point in $I$. ■

**Theroem 2.26** (Cantor intersection)**.** *Let $\{\mathbb{Q}_n\}$ be a sequence of nonempty set in $\mathbb{R}^k$ satisfying:*

*a) $\{Q_n\}$ is decreasing*

*b) $Q_n$ is closed $\forall\ n \geq 1$ & $Q_1$ is compact.*

*Then $\bigcap_{n=1}^{\infty} Q_n \neq \emptyset$, Moreover, if dia$(Q_n) \to 0$ on $n \to \infty$, them $\bigcap_{n=1}^{\infty} Q_n$ is a simpleton.*

*Proof.* By (b), each $Q_n$ is compact ($\because Q_1 \supseteq Q_n \forall\ n \geq 1$). Therefore, it follows form Cor 2.16 and 2.15. ■

**2.8. Countability & Separability.**
Motivation: In $\mathbb{R}^k$, we have two facts:

• $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$ i.e. $\overline{\mathbb{Q}^k} = \mathbb{R}^k$ & $\mathbb{Q}^k$ is countable. i.e. $\mathbb{R}^k$ has a countable dense subset. i.e. $\mathbb{R}^k$ is separable.
• $\{B(x,r) \mid x \in \mathbb{Q}^k, r \in \mathbb{Q}^+\}$ is a countable collection of open ball in $\mathbb{R}^k$ satisfying: $\forall$ open set $U \subseteq \mathbb{R}^k$ and $y \in U$ $\exists B(x,r) \in \mathscr{B} \ni y \in B(x,r) \subseteq U$. In particular, $U$ is a union of some sub collection of $\mathscr{B}$. $\therefore U = \bigcup_{y\in U} B_y$, i.e. $\mathbb{R}^k$ is of $2^{nd}$ countable.
• $X$ is a metric space $x \in X$, $N_x = \{B(x, \frac{1}{n}) \mid n \in \mathbb{N}\}$ is a countable collection of nbh of $x$. Such $N_x$ satisfies: $\forall$ nbh $U$ of $x, \exists n \in \mathbb{N} \ni B(x, \frac{1}{n}) \subseteq U (\because \exists\, r > 0 \ni B(x,r) \subseteq U$, choose $n >> 0 \ni \frac{1}{n} < r)$. Then $x \in B(x, \frac{1}{n}) \subseteq B(x,r) \subseteq U$. i.e. Each point of $X$ has a countable nbh base(system) i.e. $X$ is of $1^{st}$ countable, i.e. $\mathbb{R}^k$ has a countable base.

**<u>Definition</u>.** *Let $X$ be a topological space*

*(1) $X$ is first if every point of $X$ has a countable nbh system(or base), i.e. $\forall p \in X \exists$ a countable collection $\{V_n \mid n \in \mathbb{N}\}$ of nbh of $p \ni \forall$ nbh $U$ of $p, \exists\, n \in \mathbb{N},\ V_n \subseteq U$*
*(2) $X$ is of second countable if $X$ has a countable a base, i.e. $\exists$ a countable collection $\mathscr{B} = \{B_n \mid n \in \mathbb{N}\}$ of open sets in $X \ni$ every open set $U$ is a union of some subcollection of $\mathscr{B}$ or $\forall$ open set $U$ in $X$ and $x \in U,\ \exists n \in \mathbb{N} \ni x \in B_n \subseteq U$*
*(3) $X$ is separable if $X$ has a countable dense subset, i.e. $\exists$ a countable set $D \subseteq X \ni \overline{D} = X$.*

**Remark.**

(1) *Every finite topological space is 1,2 countable and separable.*
(2) *Every $2^{nd}$ countable topology space $X$ is of $1^{st}$ countable, but not converse.*
*$\because$ Let $\mathscr{B} = \{B_1, B_2, \cdots\}$ be a countable base for $X$. Given $p \in X$, let $\mathscr{B}_p = \{B_n \mid p \in B_n\}$ Then $\mathscr{B}$, so $\mathscr{B}_p$ is countable and is a collection of open set in $X$ containing of $p$.*
*Claim $\mathscr{B}_p$ is a nbh system (or base) of $p$. Let $U$ be a nbh of $p$. Then $U = \bigcup_{n\in F} B_n,\ F \subseteq \mathbb{N}$. In particular, $p \in B_n$ for some*

*$n \in F$. Hence, $B_n \in B_p$ & $p \in B_n \subseteq U$, $\therefore \mathscr{B}_p$ is a countable nbh system of $p$. Hence, $X$ is of $1^{st}$ countable.*

*Consider $X$ an uncountable set with discrete metric. Hence, $X$ is $1^{st}$ countable. In fact, $\forall\ p \in X,\ N_p = \{\{p\}\}$ is a countable nbh base of $p$. However, $X$ is not of $2^{nd}$ countable. Note that if $\mathscr{B}$ is a base for the discrete space $X$ then $\mathscr{B} \subseteq \{\{x\} \mid x \in X\}$*

*$\because \{x\}$ is open $\implies \{x\} = \bigcup_{\alpha \in I} B_\alpha,\ B_\alpha = \{x\}\ \forall\ \alpha \in I \implies B_\alpha = \{x\} \forall x \in I \implies \{x\} \in \mathscr{B}$.*

*Now, $X$ is uncountable, so is $\mathscr{B}$. Hence, $X$ is not of $2^{nd}$ countable.*

(3) *We know that every metric space is of $1^{st}$ countable. In fact, $\forall p \in X,\ N_p = \{B(p, \frac{1}{n}) \mid n \in \mathbb{N}\}$ is a countable nbh system of $p$*

(4) *We know that $\mathbb{R}$ is separable with countable dense subset $\mathbb{Q}$. In general, $\mathbb{R}^n$ is separable with countable dense subset $\mathbb{Q}^n$ (Exercise 22)*

**Remark.** *An equivalent description of base in a topological space. Let $X$ is a topological space, $\mathscr{B}$ is a collection of open sets in $X$.*

$$\mathscr{B} \text{ is a base for } X \quad \Leftrightarrow \quad \forall \text{ open set } U \in X \ \&\ p \in U$$
$$\exists B \in \mathscr{B} \ni p \in B \subseteq U$$

*Proof.* ($\Rightarrow$) Suppose $\mathscr{B}$ is a base. Given an open set $U$ in $X$ and $p \in U$. Since $\mathscr{B}$ is a base, $U = \bigcup_{\alpha \in I} B_\alpha$, for some index set $I$ and $B_\alpha \in \mathscr{B}$, $\forall \alpha \in I$. $p \in U \implies p \in \bigcup_{\alpha \in I} B_\alpha \implies p \in B_{\alpha_0}$ for some $\alpha_0 \in I \implies p \in B_{\alpha_0} \subseteq U$ and $B_{\alpha_0} \in \mathscr{B}$

($\Leftarrow$) Suppose the condition holds. Given an open set $U$ in $X$. By assumption, $\forall x \in U \exists B_x \in \mathscr{B} \ni x \in B_x \subseteq U$. $\therefore U = \bigcup_{x \in U} B_x$, i.e. $U$ is a union of $\{B_x \mid x \in U\} \subseteq \mathscr{B}$, $\therefore \mathscr{B}$ is a base for $X$. ■

**Remark.** *Note that $X$ is a metric space. $D \subseteq X$.*

$$\begin{aligned} D \text{ is dense in } X \Leftrightarrow \overline{D} = X &\Leftrightarrow \forall \text{ nonempty open set } U \text{ in } X, U \cap D \neq \emptyset \\ &\Leftrightarrow \forall x \in X, \exists \text{ a sequence } \{a_n\} \text{ in } D \\ &\ni a_n \to x \text{ on } n \to \infty \end{aligned}$$

**Theroem 2.27.** *Every $2^{nd}$ countable topology space is separable*

*Proof.* Let $\mathscr{B} = \{B_1, B_2, \cdots, B_n, \cdots\}$ be a countable base of $X$. Choose a point $x_n \in B_n$, $n \in \mathbb{N}$ and form the set $D = \{x_1, x_2, \cdots, x_n, \cdots\}$, then $D$ is countable.
Claim: $D$ is dense in $X$, i.e. $\overline{D} = X$. Given a nonempty open set $U$ in $X$, then $\exists n \in \mathbb{N} \ni B_n \subseteq U \implies x_n \in U \implies U \cap D \neq \emptyset$. Hence, $\overline{D} = X$ by the remark above. So $X$ is separable. ∎

**Theroem 2.28.** *Every separable metric space $X$ is of $2^{nd}$ countable.*

*Proof.* Choose a countable dense subset $D$ of $X$. Form the countable collection of open ball $\mathscr{B} = \{B(x, r) \mid x \in D, r \in \mathbb{Q}^+\}$(it is countable). Claim $\mathscr{B}$ is a base for $X$. We are done.

(Note that $\mathscr{B}$ is a base for a topology space $X$
$\Leftrightarrow$ every open set in $X$ is a union of some subcollection of $\mathscr{B}$
$\Leftrightarrow \forall$ open set $U \subseteq X$ and $p \in U$, $\exists B \in \mathscr{B} \ni x \in B \subseteq U$)

($\Rightarrow$) Given an open set $U$ in $X$ and $p \in U$ by assumption $U = \bigcup_{\alpha \in I} B_\alpha$, where $B_\alpha \in \mathscr{B} \implies p \in B_{\alpha_0}$ for some $\alpha_0 \in I \implies p \in B_{\alpha_0} \subseteq U$
($\Leftarrow$) Suppose the condition holds. To prove $\mathscr{B}$ is a base for $X$. Given a nonempty open set $U$ in $X$. $\forall\, p \in U, \exists B_p \in \mathscr{B} \ni pinB_p \subseteq U$.
$\because U = \bigcup_{p \in U} B_p \therefore \mathscr{B}$ is a base for $X$. By the remark above, it is enough to show: Given a nonempty open set $U \subseteq X$ and $p \in U$, $\exists B(x, r) \in \mathscr{B} \ni p \in B(x, r) \subseteq U$. Now, $p \in U$ and $U$ is open $\implies$ $\exists t > 0 \ni B(p, r) \subseteq U$. Choose $r \in \mathbb{Q}^+ \ni \dfrac{t}{4} < r < \dfrac{t}{2}$, Since $D$ is dense in $X$, $B(p, r) \cap D \neq \emptyset$. Choose $x \in B(p, r) \cap D$. Then $B(x, r) \in \mathscr{B}$

Claim $p \in B(x,r) \subseteq U$

- $d(x,p) < r \implies p \in B(x,r)$
- $\forall y \in B(x,r),\ d(y,p) \leq d(y,x) + d(x,p) < r + r = 2r < t \implies$

$y \in B(p,t) \subseteq U \ \therefore y \in U \therefore B(x,r) \subseteq U$.

■

**Corollary 2.29.** *The Euclidean space $\mathbb{R}^k$ is of $2^{nd}$ countable.*

Note that from the proof of Thm 2.28, $\mathbb{R}^k$ has a countable base of the form:

$$
\begin{aligned}
\mathscr{B} &= \{B(x,r) \mid x \in \mathbb{Q}^k \ \&\ r \in \mathbb{Q}^+\} \\
&= = \{A_1, A_2, \cdots\}
\end{aligned}
$$

**Remark.** *$\mathbb{R}^k$ is of second countable which has a countable base $\mathscr{B} = \{B(x,r) \mid x \in \mathbb{Q}^k,\ r \in \mathbb{Q}^+\}$*

**Theroem 2.30.** *Every compact metric space $X$ is of $2^{nd}$ countable.*

*Proof.* The last statement follows from Thm 2.27 To prove $X$ is of $2^{nd}$ countable. For each $n \in \mathbb{N}$, $\{B(x, \frac{1}{n}) \mid x \in X\}$ is an open covering of $X$, i.e. $X = \bigcup_{x \in X} B(x, \frac{1}{n})$. By companies of $X$, it has a finite subcovering say $X = \bigcup_{i=1}^{l_n} B(x_{n_i}, \frac{1}{n})$. Then $\mathscr{B}$ is a countable collection of open balls in $X$.Claim $\mathscr{B}$ is a base for $X$. It suffices to show: given a nonempty open set $U$ and $p \in U, \exists B(x_{n_i}, \frac{1}{n}) \ni \mathscr{B} \ni p \in B(x_{n_i}, \frac{1}{n}) \subseteq U$. From $p \in U$ and $U$ is open, $\exists r > 0 \ni B(p,r) \subseteq U$. Choose $n >> 0 \ni \frac{2}{n} < r$. Since $X = \bigcup_{i=1}^{l_n} B(x_{n_i}, \frac{1}{n}),\ p \in B(x_{n_i}, \frac{1}{n})$ for some $1 \leq i \leq l_n$. Finally, $p \in B(x_{n_i}, \frac{1}{n}) \subseteq U$

- $d(p, x_{n_i}) < \frac{1}{n} \implies p \in B(x_{n_i}, \frac{1}{n})$
- $\forall y \in B(x_{n_i}, \frac{1}{n}), d(y,p) \leq d(y, x_{n_i}) + d(x_{n_i}, p) < \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$

$\therefore p \in B(x_{n_i}, \frac{1}{n}) \subseteq U$ ■

**Theroem 2.31** (Lindelof Covering)**.** *Let $S \subseteq \mathbb{R}^k$. Them every open converning $\mathscr{U} = \{U_\alpha \mid \alpha \in I\}$ of $S$ has a countable subcoverings.*

*Proof.* Let $\{a_1, A_2, \cdots\}$ be the countable base of $\mathbb{R}^k$ defined as above. Note that $S \subseteq \bigcup_{\alpha \in I}$ $\forall x \in S, x \in U_\alpha$ for some $\alpha \in I$. Hence $\exists n \in \mathbb{N} \ni xinA_n \subseteq U_\alpha$. Of course, there may be infinitely many such $n$. We choose one of them and fix it, say $x \in A_{m(x)} \subseteq U_\alpha$(e.g. $m(x) = \min\{n \in \mathbb{N} \mid x \in A_n \subseteq U_\alpha\}$). Then the collection $\{A_{m(x)} \mid x \in S\}$ is a countable open covering of $S$. Finally, for each $A_{m(x)}$, choose $U_{\alpha_{m(x)}} \ni A_{m(x)} \subseteq U_{\alpha_{m(x)}}$. Then $\{U_{\alpha_{m(x)}} \mid x \in X\}$ is a countable subcoverning of $S$. ■

**Corollary 2.32.** *Let $S \subseteq \mathbb{R}^k$ be open, if $S = \bigcup_{\alpha \in I} U_\alpha$ is a union of open sets in $X$, then $S = \bigcup_{n=1}^{\infty} U_{\alpha_n}$ is a countable union. $\because$ By Lindelof covering theorem.*

**2.9. Perfect Sets in Metric Spaces. Recall** a subset $E$ in a metric space $X$ is perfect if $E$ is closed in $X$ and and every point of $E$ is its accumulation point. i.e. $E' = E$

**Example**

- $-\infty < a < b < \infty,\ [a, b]$ is perfect
- $\mathbb{R}$ is perfect

**Theroem 2.33.** *Every nonempty perfect set $E$ in $\mathbb{R}^k$ is uncountable*

*Proof.* $E$ is an infinite set ($\because$ finite set has no accumulation point), Suppose $E$ were countable, write $E = \{x_1, x_2, \cdots\}$. We use induction to construct a sequence $\{V_n\}$ of open sets in $X$ as follows:
Let $V_1$ be any neighborhood of $y_1 = x_1$, e.g. $V_1 = B(x_1, r)$, it's closure is $\overline{V_1} = \overline{B}(x_1, r)$, $x_1 \in E'$, $V_1 \cap E$ is an infinite set, so $\exists y_2 \in V_1 \cap E \ni y_2 \neq y_1$. Choose a neighborhood $V_2$ of $y_2 \ni$
$(i)$ $\overline{V_2} \subseteq V_1$
$(ii)$ $x_1 \notin \overline{V_2}$
$(iii)$ $V_2 \cap E \neq \emptyset$($\because y_2 \in E = E'$ and it's also an infinite set)
Suppose that, for $n \geq 3$, $V_n$ has been chosen $\ni V_n$ is a neighborhood of some $y_n \in E \ni$

(1) $\overline{V_n} \subseteq V_{n-1}$
(2) $x_{n-1} \notin \overline{V_n}$
(3) $V_n \cap E \neq \emptyset$ is an infinite set

Since $V_n \cap E$ is an infinite set, $\exists y_{n+1} \in V_n \cap E \ni y_{n+1} \neq y_n$. Again, choose a neighborhood $V_{n+1}$ of $y_{n+1} \ni$

(1) $\overline{V_{n+1}} \subseteq V_n$
(2) $x_n \notin \overline{V_{n+1}}$
(3) $V_{n+1} \cap E \neq \emptyset$ is an infinite set.

By induction, we have constructed such sequence $\{V_n\}$. Put $K_n = \overline{V_n} \cap E$, $n \geq 1$. Then $\{K_n\}$ is a decrease sequence of nonempty compact sets in $\mathbb{R}^k$.

> $\overline{V_n}$ is closed, $E$ is closed $\implies$ $K_n = \overline{V_n} \cap E$ is closed, each $\overline{V_n}$ is bounded $\therefore K_n$ is closed and bounded by H.B. theorem, $K_n$ is compact.
>
> $\cdot\ \emptyset \neq V_n \cap E \subseteq \overline{V_n} \cap E \implies E_n = \overline{V_n} \cap E \neq \emptyset$
> $\cdot\ \overline{V_n} \cap E \supseteq \overline{V_{n+1}} \cap E = K_{n+1} \therefore \{K_n\}$ is decrease.

By Cantor's intersection theorem, $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$. Pick $y \in \bigcap_{n=1}^{\infty} K_n$, $y \in E (\because K_n \subseteq E \forall n \geq 1)$. Since $x_n \notin \overline{V_{n+1}}\ \forall n \geq 1$, so $x_n \notin K_n \forall n \geq 1 \implies y \notin E (\rightarrow\leftarrow)$ to $E = \{x_1, x_2, \cdots\}$
$\therefore E$ is uncountable. ■

**Corollary 2.34.** *Every nondegenerate intervvl is uncountable.*

*Proof.* $\because$ Every nondegenerate interval $I$ in $\mathbb{R}$ must contain a closed and bounded interval $[a, b]$ with $a < b$ which is perfect, so it is uncountable by theorem 2.31. Hence $I$ is uncountable. ■

Construction of the Cantor set $\underline{P} \subseteq [0, 1]$ in $\mathbb{R}$ which is a perfect set

$(a)$ Remove the middle third open subinterval of $[0, 1]$. There are two closed subintervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Let $C_1 = (\frac{1}{3}, \frac{2}{3})$ and $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$(b)$ Remove te middle thirds of $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ respectively. There are $2^2 = 4$ subintervals $[0, \frac{1}{3^2}]$, $[\frac{2}{3^2}, \frac{3}{3^2}]$, $[\frac{6}{3^2}, \frac{7}{3^2}]$, $[\frac{8}{3^2}, 1]$

$(c)$ Countinue this process, we get a sequence $\{C_n\}$ of open sets and a sequence $\{E_n\}$ of closed sets satisfy

$(i)$ $E_0 \supseteq E_1 \supseteq E_2 \subseteq \cdots$, i.e. $\{E_n\}$ is a decrease sequence of closed sets in $[0, 1]$

$(ii)$ Each $E_n$ is a union of $2^n$ closed intervals, each of length $3^{-n}$

$(iii)$ Each $C_n$ is a union of $2^{n-1}$ open subintervals, each of length $3^{-n}$. total length is $\frac{2^{n-1}}{3^n}$

**Definition.** *$\underline{P}$(or $C$) $= \bigcap_{n=1}^{\infty} E_n = [0,1] - \bigcup_{n=1}^{\infty} C_n( = \bigcap_{n=1}^{\infty}([0,1] - C_n))$*

Properties of Cantor set $\underline{P}$ :

(1) $\underline{P} \neq \emptyset$ by Cantor's intersection theorem
(2) $\underline{P}$ is compact ($\because$ $\underline{P}$ is closed bein $\cap$ of closed set and $\underline{P} \subseteq [0,1]$, $[0,1]$ is compact )
(3) $\underline{P}$ is nowhere dense, i.e. $\overline{\underline{P}}^{\circ} = \emptyset$, i.e. $\underline{P}^{\circ} = \emptyset$
$\because$ $\underline{P}$ contains no nonempty open subintervals($\because$ $\underline{P}^{\circ} \neq \emptyset \implies \exists x \in P^{\circ} \implies \exists \delta > 0, (x-\delta, x+\delta) \subseteq \underline{P}$).If $\alpha < \beta$ and $(\alpha, \beta) \subseteq \underline{P}$, them $(\alpha, \beta) \subseteq E_n \forall n \geq 1$.Choose $n >> 0 \ni \frac{1}{2^n} < \beta - \alpha$. Then for $n >> 0$, $E_n$ contains subinterval of length $\geq \frac{1}{2^n}(\rightarrow\leftarrow)$. Hence, $\underline{P}$ is nowhere dense.
(4) $\underline{P} = \{\sum_{n=1}^{\infty} \frac{a_n}{3^n} \mid a_n = 0 \text{ or } 2 \forall n \geq 1\}$
Recall the ternarry representation of a number $x \in [0,1]$, $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$, $a_n = 0, 1, 2 \forall n \geq 1$

> if $fraca_n 3^n$ the $a_n$ can be 1, then $\frac{1}{3} + \frac{1}{3}$ is not in $\underline{P}$
> if you want to represent $\frac{1}{3}$, you need use $0 + \frac{2}{3^2} + \frac{2}{3^3} + \cdots$, i.e. $0.1 = 0.0\overline{9}$ same things.

This can be used to prove that $\underline{P}$ is uncountable by $\underline{P} \to [0,1]$, $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \to \sum_{n=1}^{\infty} \frac{a_n/2}{2^n}$ is bijative $\therefore$ $\underline{P}$ is uncountable.
(5) $\underline{P}$ is of measure zero (i.e. the length of $\underline{P}$ is zero)
$\because$ The totally length remove in the construction of $\underline{P}$ is $\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \cdots = 1$. This also proves that $\underline{P}$ is nowhere dence.
(6) $\underline{P}$ is perfect. In particular, by theorem 2.31, $\underline{P}$ is uncountable
$\because$ Obviously $\underline{P}$ is a nonempty closed set. Let $x \in \underline{P}$. Then $\forall (\alpha, \beta) \ni x \in (\alpha, \beta)$. We prove that $(\alpha, \beta) \cap \underline{P} - \{x\} \neq \emptyset$, which says that $x$ is an accumulation points of $\underline{P}$. Hence $\underline{P}$ is perfect. By $x \in \underline{P}$, $x \in E_n$ $\forall n \geq 1$. Then $\exists$ a closed subinterval $I_n \subseteq E_n \ni x \in I_n$. Choose $n >> 0 \ni I_n \subseteq (\alpha, \beta)$. Let $x_n$ be an end point of $I_n \ni x \neq x_n$. By construction, $x_n \in \underline{P}$, so

$$x_n \in (\alpha, \beta) \cap \underline{P} - \{x\}$$

i.e. $x$ is an accumulation point of $\underline{P}$, Hence $\underline{P}$ is perfect.

**Definition.** *Let $X$ be a metric space (or topological space) $A, B \subseteq X$. We sat that $A$ and $B$ are separated if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty sets.*

**Definition.** *A subset $E \subseteq X$ is called connected if $E$ is not a union of two nonempty separated sets and $E$ is disconnected if $E$ is not connected*

**Remark.** *$X$ is connected $\Leftrightarrow$ $X$ is not a union of two nonempty separated sets.*
*$X$ is disconnected $\Leftrightarrow$ $X$ is a union of two nonempty separated sets, say, $X = A \cup B$,$A$ and $B$ are nonempty separated. i.e. $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$, i.e. $A = \overline{A}$, $B = \overline{B}$ $\therefore A$ and $B$ are closed $\therefore A$ and $B$ are both open and closed.*

**Remarks and Examples**

(1) Separated sets and disjoint
(2) $[0, 1]$ and $(1, 2)$ are not separated
(3) $(0, 1)$ and $(1, 2)$ are separated

**Theroem 2.35.** *Let $E \subseteq \mathbb{R}$ be a set. Then $E$ is connected $\Leftrightarrow$ $E$ is an interval*

*Proof.* We may assume that $E \neq \emptyset$
($\Rightarrow$) Assume that $E$ is connected. If $E$ were not an interval, then $\exists\ x < y$ in $E$ and $z \notin E \ni x < Z < y$. Let $A = (-\infty, z) \cap E$ and $B = (z, \infty) \cap E$. Then $A, B$ are nonempty, $\overline{A} \cap B = \emptyset$, $A \cap \overline{B} = \emptyset$ and

$$\begin{aligned} E &= E \cap (\mathbb{R} - \{z\}) \\ &= E \cap [(-\infty, z) \cup (z, \infty)] \\ &= (E \cap (-\infty, z)) \cup (E \cap (z, \infty)) \\ &= A \cup B \end{aligned}$$

$\therefore \{A, B\}$ is a nonempty separation of $E$ ($\rightarrow\leftarrow$) to connected. $\therefore E$ is an interval

($\Leftarrow$) Suppose $E$ is an interval. To show that $E$ is connected. If not, then $\exists$ two nonempty separated sets $A$ and $B \ni E = A \cup B$. Pick $x \in A$ and $y \in B$. Then $x \neq y(\because A \cap B = \emptyset)$. We may assume that $x < y$. Define $z = \sup(A \cap [x, y])$, By (9) in section 2.5, $z \in \overline{A \cap [x, y]} \in \overline{A}$. Hence $z \notin B(\because \overline{A} \cap B = \emptyset)$. Also $x \leq z \leq y$. But $y \in B$ and $z \notin B \implies x \leq z < y$
If $z \notin A$, then $x < z < y$ and $z \notin E(\rightarrow\leftarrow)$ to $E$ is an interval.
If $z \in A$, then $z \notin \overline{B}(\because A \cap \overline{B} =)$, since $z \notin B$, $z \in \mathbb{R} - \overline{B}$ which is open $\implies \exists \delta > 0 \ni (z - \delta, z + \delta) \subseteq \mathbb{R} - \overline{B}$. Choose $z < z_1 < z + \delta < y$, i.e. $z < z_1 < y$. Then $z, y \in E$, $z < y$ and $z_1 \notin E(\rightarrow\leftarrow)$ to $E$ is an interval ■

**Application of Connectedness**

$X$ : connected topological space (or metric space)
$P$ : a property on $X$
$D = \{x \in X \mid P \text{ holds at } x\}$
If one can prove $D$ is nonempty and closed and open, them $D = X$
$\because X = D \cup (X - D)$, $\overline{D} \cap (X - D) =$ and $D \cap \overline{(X - D)}$, i.e. $D$ and $X - D$ are separated
Since $X$ is connected and $D \neq \emptyset$, so $X - D = \emptyset$, i.e. $X = D$

## 3. Infinite Sequence & Series

• We will assume you are familiar with all operations of real(complex) sequence
• We have defined sequence in a set $X$

> Recall : let $\{a_n\}$ be a real or complex sequence, $\{a_n\}$ converges if $\exists a \in \mathbb{R}(\mathbb{C})$ satisfying $\forall \epsilon > 0, \exists N \in \mathbb{N} \ni \forall n \geq N, |a_n = a| < \epsilon$

• Now, we study the properties of a sequence in a metric space(topological space)

**3.1. Convergent Sequence.** Let $X$ be a metric space $\&\{x_n\}$ be a sequence in $X$, $x : \mathbb{N} \to X$

**Definition.** *We say that $\{x_n\}$ convergences (in $X$) if $\exists p \in X$ satisfying $\forall \epsilon > 0, \exists N \in \mathbb{N} \ni \forall n \geq N, d(x_n, p) < \epsilon$, Otherwise, $\{x_n\}$ divergences.*

**Remark.**

(1) *If $\{x_n\}$ convergences as in definition, them $p$ is called the limit of the sequence $\{x_n\}$ and is denoted by $\lim_{n\to\infty} x_n = p$ or $x_n \to p$ as $n \to \infty$*
(2) *$x_n \to p$ as $n \to \infty \Leftrightarrow$ the real sequence $\{d(x_n, p)\}$ convergences to $0$, i.e. $\lim_{n\to\infty} d(x_n, p) = 0$*
(3) *if $\{x_n\}$ convergences, them its limit is !*
(4) *The convergence of a sequence depends not only the sequence but also on the space.*
*e.g. $\lim_{n\to\infty} \frac{1}{n} = 0$ in $\mathbb{R}$, but $\{\frac{1}{n}\}$ divergences in $(0, 1)$*

**Recall** Let $\{x_n\}$ are a sequence in a set $X$ with is a function $x : \mathbb{N} \to X$. The image of the sequence = the image of the function $X$
$= \{x_n \mid n = 1, 2, \cdots\}$

**Remark.** *The range of a sequence may be finite. e.g. $\{(-1)^n\}$ in $\mathbb{R}$, whose range $\{-1, 1\}$ is finite, but $\{\frac{1}{n}\}$ has range $\{\dfrac{1}{n} \mid n = 1, 2, \cdots\}$*

**Definition.** *A sequence $\{x_n\}$ in $X$ is said to be bounded if its range is a bounded subset of $X$*

**Remark.** *A sequence $\{x_n\}$ in $X$ is said to be bounded if its range is a bounded subset of $X$*

**Example**

(1) Every const sequence $\{p\}$ in a metric space convergence, i.e. $\lim_{n\to\infty} p = p$
(2) $\lim_{n\to\infty} \frac{1}{n} = 0$ in $\mathbb{R}$ and $\{\dfrac{1}{n}\}$ is bounded (but the range is finite)
(3) $\{(-1)^n\}$ divergences, but $\{(-1)^n\}$ is bounded. (range is finite).
(4) $\{n^2\}$ divergences in $\mathbb{R}$ and is unbounded. In fact, $\lim_{n\to\infty} n^2 = +\infty$( which range is infinite)
(5) $\lim_{n\to\infty}(1 + \dfrac{(-1)^n}{n}) = 1$ and $\{1 + \dfrac{(-1)^n}{n}\}$ is bounded. (range is infinite)
(6) $\{i^n\}$ divergence and it's bounded (range is finite)
(7) Identify all convergence sequence in a discrete metric space $X$. $\{x_n\}$ convergence to $p \Leftrightarrow \forall \epsilon > 0 \exists N \in \mathbb{N} \ni \forall n \geq N, d(x_n, p) < \epsilon \Leftrightarrow \{x)n\}$ is almost constant.

In metric space, we can use sequences to characterise adherent and accumulation point

**Theroem 3.1.** *Let $\{x_n\}$ be a sequence in a metric space $X$ and $E \subseteq X$*

*(a) $x_n \to p$ as $n \to \infty \Leftrightarrow \forall$ neighborhood $U$ of $p, \exists N \in \mathbb{N} \ni \forall n \geq N, x_n \in U$*
*(b) If $\{x_n\}$ convergences, them its limit is !*
*(c) If $\{x_n\}$ convergences, them its range is bounded, but not converse*
*(d) $p \in \overline{E} \Leftrightarrow \exists$ a sequence $\{a_n\}$ in $E \ni a_n \to p$*
*(e) $p \in E' \Leftrightarrow \exists$ a distinct sequence($a_n \neq a_m \forall n \neq m$) $\{a_n\}$ in $E \ni a_n \to p$*

*Proof.*

*(a)*

$$\begin{aligned} x_n \to p &\Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N} \ni d(x_n, p) < \epsilon (\forall n \geq N) \\ &\Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N} \ni x_n \in B(p, \epsilon)(\forall n \geq N) \\ &\Leftrightarrow \forall \text{ neighborhood } U \text{ of } p, \exists N \in \mathbb{N} \ni \forall n \geq N, x_n \in U \end{aligned}$$

*(b)* Suppose $x_n \to p$, $x_n \to q$ and $p \neq q$, let $\epsilon = \dfrac{1}{2} d(p, q)$. By definition, $\exists N_1 \ni \forall n \geq N_1, d(x_n, p < \epsilon)$ and $\exists N_2 \ni \forall n \geq N_2, d(x_n, q) < \epsilon$

Let $N = \max\{N_1, N_2\}$ them $\forall n \geq N$, above holds. Hence $d(p,q) \leq d(p,x_N) + d(x_N,q) < \epsilon + \epsilon = 2\epsilon = d(p,q) (\to\leftarrow) \therefore p = q$

$(c)$ We have seen bounded sequence may not converges. If $x_n \to p$, them for $\epsilon = 1, \exists N \in \mathbb{N} \ni \forall n \geq N, d(x_N, p) < 1$, i.e. $\forall n \geq N,\ x_n \in B(p,1)$. Let $R = \max\{d(p,x_1), \cdots, d(p,x_{n-1})\} + 1$, Then $x_n \in B(p,R) \forall n \geq 1 \therefore \{x_n\}$ is bounded

$(d)$ $(\Rightarrow)$ Suppose $p \in \overline{E}$. Them $\forall n \geq 1, B(p, \dfrac{1}{n}) \cap E \neq \emptyset$. Choose $a_n \in B(p, \dfrac{1}{n}) \cap E, n \geq 1$. We get a sequence $\{a_n\}$ in $E$ and $0 \leq d(x_n, p) < \dfrac{1}{n}, \forall n \geq 1$.
By squeezing lemma, $\lim_{n\to\infty} d(a_n, p) = 0$, i.e. $a_n \to p$ as $n \to \infty$
$(\Leftarrow)$ Suppose the conditions holds. $\forall r > 0$, $\exists N \in \mathbb{N} \ni \forall n \geq N, d(a_n, p) < r \implies \forall n \geq N, a_n \in B(p, r) \implies B(p, r) \cap E \neq \emptyset \therefore p \in E$
$(e)$ It's similar to above.

■

**Theroem 3.2.** *For real or complex sequences* $\{x_n\}$ *and* $\{y_n\}$, $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $a, b \in \mathbb{R}$ *or* $\mathbb{C}$, $c \in \mathbb{R}$ *or* $\mathbb{C}$

(1) $\lim_{n\to\infty} c = c$
(2) $\lim_{n\to\infty}(ax_n + by_n) = ax + by = a \lim_{n\to\infty} x_n + b \lim_{n\to\infty} y_n$
(3) $\lim_{n\to\infty} x_n y_n = xy = \lim_{n\to\infty} x_n \lim_{n\to\infty} y_n$
(4) *If* $y \neq 0, \lim_{n\to\infty} \dfrac{x_n}{y_n} = \dfrac{x}{y} = \dfrac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$
(5) *If* $\{z_n\}$ *is a complex sequence, them* $z_n \to z$ *as* $n \to \infty \Leftrightarrow$ $Rez_n \to z$ *&* $Imz_n \to z$*(using* $|Rew|, |Imw| \leq |w| \leq |Rew| + |Imw| \forall w \in \mathbb{C}$*)*
(6) *(Squeezing Lemma) If* $\{x_n\}\{y_n\}$ *and* $\{t_n\}$ *are real sequence* $\ni$

$$x_n \leq t_n \leq y_n \; for \; n >> 0$$

*and* $\lim x_n = limy_n = l$, *them* $\lim_{n\to\infty} t_n = l$
(7) $\lim_{n\to\infty} x_n = x \implies \lim_{n\to\infty} |x_n| = |x|$*(using* $||x_n| - |x|| \leq |x_n - x|$*)*

**Examples.**

$(i)$ $\lim_{n\to\infty}(1 - \frac{i}{n}) = 1$ (Re $(1 - \dfrac{i}{n}) = 1$, Im$(1 - \dfrac{i}{n}) = \frac{1}{n}$)
$(ii)$ $\lim_{n\to\infty} \frac{1}{n} \sin \dfrac{1}{n} = 0$

$$0 \leq |\frac{1}{n} \sin \frac{1}{n}| \leq \frac{1}{n} \implies \lim_{n\to\infty} |\frac{1}{n} \sin \frac{1}{n}| = 0$$
$$\implies |\lim_{n\to p} \frac{1}{n} \sin \frac{1}{n}| = 0 \implies \lim_{n\to\infty} \frac{1}{n} \sin \frac{1}{n}$$

$(iii)$ $\{(-1)^n\}$ divergence, but $|(-1)^n| = 1 \to 1$

For sequences in $\mathbb{R}^k$ including in $\mathbb{C} \approx \mathbb{R}^2$, we have.

**Theroem 3.3.** *Let $\{x_n\}$ be a sequence in $\mathbb{R}^k$, where*

$$x_n = (x_{1,n}, x_{2,n}, \cdots, x_{k,n}), n = 1, 2, \cdots$$

*(a) $x_n \to p(p_1, \cdots, p_k)$ in $\mathbb{R}^k \Leftrightarrow x_{i,n} \to p_i \forall 1 \le i \le k$, i.e. $\lim_{n\to\infty}(x_{1,n}, \cdots, x_{k,n}) = (\lim_{n\to\infty} x_{1,n}, \cdots, \lim_{n\to\infty} x_{k,n})$ if exists*
*(b) Let $\{x_n\}\{y_n\}$ be sequences in $\mathbb{R}^k$ and $\{d_n\}$ be a sequence in $\mathbb{C}$ and $a, b \in \mathbb{R}$. If $x_n \to x, y_n \to y$ and $d_n \to d$, then*

- $ax_n + by_n \to ax + by$
- $< x_n, y_n > \to < x, y >$
- $d_n x_n \to d_x$
- $||x_n|| \to ||x||$

*If $k = 3$, them $x_n \times y_n \to x \times y$*

*Proof.*
(a)It follows from the inequation $\forall y \in \mathbb{R}^k |y_i| \le ||y|| \le \sum_{i=1}^k |y_i|$
$\because \forall 1 \le i \le k \ |x_{i,n} - p_i| \le ||x_n - p|| \le \sum_{i=1}^k |x_{j,n} - p_j| \forall n \ge 1$
$(\Rightarrow)$

$$\begin{aligned} \text{Suppose } x_n \to p &\implies ||x_n - p|| \to 0 \\ &\implies \forall 1 \le i \le k, |x_{i,n} - p_i| \to 0 \forall 1 \le k \le n \\ &\implies \forall 1 \le i \le k, x_{i,n} \to p_i \text{ as } n \to \infty \end{aligned}$$

$(\Leftarrow)$

$$\begin{aligned} \text{Suppose } x_n \to p_i, 1 \le i \le k &\implies ||x_{i,n} - p_i|| \to 0 \ \forall 1 \le k \le n \\ &\implies \sum_{i=1}^k |x_{i,n} - p_i| \to 0 \\ &\implies ||x_n - p|| \to 0 \implies x_n \to p \end{aligned}$$

(b) By (a)

$$\begin{aligned} ax_n + by_n &= (ax_{1,n}, ax_{2,n}, \cdots, ax_{k,n}) + (by_{1,n}, \cdots, by_{k,n}) \\ &= (ax_{1,n} + by_{1,n}, \cdots, ax_{k,n} + by_{k,n}) \\ &\to (ax1 + by_1, \cdots, ax_k + by_k) = ax + by \end{aligned}$$

- $< x_n, y_n >= \sum_{i=1}^{k} x_{i,n} y_{i,n} \to \sum_{i=1}^{n} x_i y_i =< x, y >$
- $d_n x_n = (d_n x_{1,n}, \cdots, d_n x_{k,n}) \to (dx_1, \cdots, dx_k) = dx$
  $||x_n|| = (\sum_{i=1}^{k} x_{i,n}^2)^{\frac{1}{2}} \implies (\sum_{i=1}^{k} x_i^2)^{\frac{1}{2}} = ||x||$
- $x_n \times y_n = (x_{2,n} y_{3,n} - x_{3,n} y_{2,n}, \cdots) \to (x_2 y_3 - x_3 y_2, \cdots) = x \times y$

■

**3.2. Subsequences.**

**Theroem 3.4.**

*(a) If $\{x_n\}$ convergences to $p$, i.e. $\lim_{n\to\infty} x_n = p$, them so is every subsequence of $\{x_n\}$*
*(b) If $X$ is compact and $\{x_n\}$ is a sequence in $X$, them $\{x_n\}$ has a convergent subsequence.*
*(c) Every bounded sequence $\{x_n\}$ in $\mathbb{R}^k$ has a converge subsequence.*

*Proof.* $(a)$ Given a subsequence $\{x_{n_k}\}$ of $\{x_n\}$( Note that $\{n_k\}$ is strictly increasing , i.e. $n_1 < n_2 < \cdots$, hence, $k \leq n_k\ \forall k \geq 1$), $d(x_{n_k}, p) < \epsilon$. This proves $x_{n_k} \to p$ as $k \to \infty$
$(b)$ Let $T = \{x_n \mid n \geq 1\}$ be the range of $\{x_n\}$
Case 1: $T$ is a finite set. In this case, some $x_{n_0}$ must appear infinitely many times in the sequence $\{x_n\}$. Choose $n_1 = n_0 \ni x_{n_1} = x_{n_0}$, and $n_2 > n_1 \ni x_{n_1} = x_{n_2}, \cdots$. In this way, we get a const subsequence $\{x_{n_k}\}$ which convergence to $x_{n_0}$
Case 2: $T$ is an infinite set. In this case, $T$ is an infinite subset of the compact metric space $X$. By Thm 2.17 (ii), $T$ has an accumulation point $p$ in $X$. By Thm 3.1 (e), $\exists$ a sequence in $T$ which converges to $p$. We may arrange such sequence to be a subsequence of $\{x_n\}$. We are done

> $y_1 = x_n$, choose $n_2 \to n_1 \ni x_{n_2}$ appears in $\{y_j\}$. Then $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and $\{y_j\}$, $\therefore x_{n_k} \to p$

$(c)$ $\because$ Since $\{x_n\}$ is bounded, we may choose a closed ball $\overline{B}(0, R)$ or a closed $n$-dimensional interval in $\mathbb{R}^k \ni \{x_n\}$ is a sequence in $K$, By (b), $\{x_n\}$ has a convergence subsequence .

■

**Remark.** *Thm 3.4(a) can be used to detect the divergence of a sequence, e.g.* $\{(-1)^n\}$ *in* $\mathbb{R}$ *which divergences,* $\because$ *It has two subsequences*
$$\begin{cases} x_{2n} \to 1 \\ x_{2n-1} \to -1 \end{cases} \quad \textit{which is different.}$$

**<u>Definition</u>.** *Let* $\{x_n\}$ *be a sequence in* $X$. *A point* $p \in X$ *is called a subsequential limit of* $\{x_n\}$ *if* $\exists$ *a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\} \ni x_{n_k} \to p$ *as* $k \to \infty$

**Examples**

(1) If $\{x_n\}$ convergences to $p$, then $\{x_n\}$ has only on subsequence limit $p$ $(E = \{p\})$
(2) $\{(-1)^n\}$ has two subsequence limit 1 and $-1$, $E = \{1, -1\}$
(3) $\{n\}$ has no subsequence limit $(E = \emptyset)$

Let $\{x_n\}$ be a sequence in $X$ and $E$ be the set of all subsequence limits of $\{x_n\}$

**Theroem 3.5.** *As above,* $E$ *is a closed subset of* $X$

*Proof.* If $E$ is a finite set, them $E$ is closed.
Now, assume that $E$ is an infinite set, to show that $E$ is closed, we must prove $E' \subseteq E$, i.e. $E$ contains all its accumulation point. Given $q \in E'$, to prove $q \in E$, i.e. $\exists$ a subsequence $\{x_{n)k}\} \ni x_{n_k} \to q$. Since $E$ is infinite, $\{x_n\}$ is not a constant sequence , so we can choose $x_{n_1} \neq q$. Let $\delta = d(x_n, q)$, We construct subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying : $d(x_{n_k}, q) \leq \frac{\delta}{2^{k-1}} \forall k \geq 1$. If it is done, them by squeezing lemma, $d(x, q) \to 0$ as $k \implies \infty$.
Now, to construct such subsequence $\{x_{n_k}\}$. By induction, suppose $k = 1$ we are done, we have found $n_1 < n_2 < \cdots < n_{k-1}, k \geq 2$. To find $x_{n_k}$. Since $q \in E', B(q, \frac{\delta}{2^k}) \cap E - \{q\} \neq \emptyset$. Choose $x \in B(q, \dfrac{\delta}{2^k}) \cap E - \{q\}$. Now, $x \in E, \exists$ a subsequence of $\{x_n\}$ which convergence to $x$. Hence $\exists n_k > n_{k-1} \ni d(x_{n_k}, x) < \frac{\delta}{2^k}$. Finally, $d(x_{n_k}, q) \leq d(x_{n_k}, x) + d(x, q) < \frac{\delta}{2^k} + \frac{\delta}{2^k} = \frac{\delta}{2^{k-1}}$ $\therefore$ By induction, such subsequence $\{x_{n_k}\}$ can be found. ∎

### 3.3. Cauchy Sequences.

**Recall** $x_n \to p \implies \forall \epsilon > 0 \; \exists N \in \mathbb{N} \ni \forall n \geq N, \; d(x_n, p) < \frac{\epsilon}{2}$.

$\therefore \forall m, n \geq N,\ d(x_m, x_n) \leq d(x_m, p) + d(p, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

**Definition.** *A sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if it satisfies the Cauchy condition:* $\forall \epsilon > 0,\ \exists N \in \mathbb{N} \ni \forall n, m \geq N, d(x_n, x_m) < \epsilon$

**Remark.**

*(i) Convergence sequence is Cauchy*

*(ii) Cauchy sequence may not convergence, e.g. in* $(0, 1)$ $\{\dfrac{1}{n}\}$ *is Cauchy, but not convergence in* $(0, 1)$

> $\forall n, m \in \mathbb{N},\ m \geq n,\ |\frac{1}{n} - \frac{1}{m}| \leq \dfrac{1}{n} + \dfrac{1}{m} \leq \dfrac{2}{n}$
> $\forall \epsilon > 0,$ *Choose* $N \in \mathbb{N} \ni \frac{2}{N} < \epsilon.$
> *Them* $\forall n, m \geq N,\ |\dfrac{1}{n} - \dfrac{1}{m}| \leq \dfrac{2}{N} < \epsilon\ \ \therefore \{\frac{1}{n}\}$ *is Cauchy.*

*(iii)* $\{x_n\}$ *is Cauchy* $\Leftrightarrow \lim_{n,m\to\infty} d(x_n, x_m) = 0$
*(iv) Let* $E_n = \{x_n, x_{n+1}, \cdots\} n \geq 1.$ *Them* $\{x_n\}$ *is Cauchy* $\Leftrightarrow \lim_{n\to\infty}$ *dia* $(E_n) = 0$

> *Recall the definition of diameter:*
> *Let* $S \subseteq V, S \neq \emptyset$. *The diameter of* $S$ *is dia(S)* $=$ *sup*$\{d(x, y) \mid x, y \in S\}$
> *Let*

*Proof.* ($\Rightarrow$) Suppose $\{x_n\}$ is Cauchy. Then $\forall \epsilon > 0,\ \exists N \in \mathbb{N} \ni \forall n, m \geq N, d(x_n, x_m) < \frac{\epsilon}{2}$. Then $\forall n \geq N$, dia $(E_n) \leq \frac{\epsilon}{2} < \epsilon \implies \lim_{n\to\infty}$ dia $(E_n) = 0$
($\Leftarrow$) Suppose $\lim_{n\to\infty}$ dia $(E_n) = 0, \forall \epsilon > 0, \exists N \in \mathbb{N} \ni \forall n \geq N$, dia$(E_n)$ $< \epsilon \implies \forall n, m \geq N, (x_n, x_m \in E_n), d(x_n, x_m \leq)$ dia $(E_N) < \epsilon$
$\therefore \{x_n\}$ is Cauchy. ■

**Remark.** *Every Cauchy seq* $\{x_n\}$ *in a metric space is bounded. For* $\epsilon = 1, \exists N \in \mathbb{N} \ni \forall m, n \geq N, d(x_n, x_m) < 1$, *In particular,* $\forall n \geq N, d(x_n, x_N) < 1$. *Let* $R = max\{d(x_i, x_N) \mid 1 \leq i \leq N - 1\} + 1$. *Then* $x_n \in B(x_N, R) \forall n \geq 1$, *i.e.* $\{x_n \mid n \geq 1\} \subseteq B(x_N, R)$. *Hence* $\{x_n\}$ *is bounded.*

**Theroem 3.6.** $(a)$ *Every Cauchy sequence in a compact metric space converges.*
$(b)$ *Every Cauchy sequence in* $\mathbb{R}^k$ *convergences.*

*Proof.* $(a)$ Let $\{x_n\}$ be a Cauchy sequence in compact metric space $X$. Since $X$ is compact, $X$ is sequencially compact, so $\{x_n\}$ has a subsequence $\{x_{n_k}\} \ni x_{n_k} \to p$ as $k \to \infty$ for some $p \in X$.
Now to prove $x_n \to p$. Given $\epsilon > 0, \exists N \in \mathbb{N} \ni \forall n, m > N, d(x_n, x_m) < \frac{\epsilon}{2}(\because \{x_n\}$ is Cauchy), $\exists k_0 \in \mathbb{N} \ni \forall k \geq k_0, d(x_{n_k}, p) < \frac{\epsilon}{2}(\because x_{n_k} \to p)$.
Hence, $\forall n \geq N, d(x_n, p) \leq d(x_n, x_{n_k}) + d(x_{n_k}, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$, where $k >> 0$
Another proof of (a)
By (2)&(4) above,

$$\lim_{n\to\infty} \text{tia}(\overline{E}_n) = \lim m \to \infty \text{dia}(E_n) = 0$$

,
where $E = \{x_n, x_{n+1}, \cdots\}, n \geq 1$
Now, each $\overline{E_n}$ is compact($\because$ closed subset of compact set $X$) and nonempty $\forall n \geq 1$, and $\{\overline{E}_n\}$ is decreasing($\because E_n \supseteq E_{n+1} \implies \overline{E}_n \supseteq \overline{E}_{n+1}$) and dia($\overline{E}_n \to 0$). By Cantor's Intersection Theorem, $\bigcap_{n=1}^{\infty} \overline{E_n} = \{p\}$ Claim: $x_n \to p$ as $n \to \infty$, $\forall \epsilon > 0, \exists N \in \mathbb{N} \to$ dia$(\overline{E}_n) < \epsilon$. Since $p \in \overline{E}_n \forall n \geq 1, \forall n \geq N \& d(x_n, p) < \text{dia}(\overline{E}_n) < \epsilon$, $\therefore x_n \to p$
$(b)$ Given a Cauchy sequence $\{x_n\}$ in $\mathbb{R}^k$. Then $\{x_n\}$ is bounded, choose a large $k$-dimensional closed interval

$$I = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

$\ni x_n \in I \forall n \geq 1$. Now, H.B. Theorem says that $I$ is compact. Therefore, $\{x_n\}$ becomes a Cauchy sequence in the compact metric space $I$. By (a) $x_n \to p$ for some $p \in I$. This proves (b).

■

**Definition.** *A metric space $X$ is said to be complete if every Cauchy sequence in $X$ convergences.*

**Rmks and Examples**

(1) In a complete metric space $X$, a sequence $\{x_n\}$ is Cauchy $\Leftrightarrow$ it convergences.
(2) By Thm. 2.6, we have two classes of complete metric spaces
- Compact metric space

• Euclidean space $\mathbb{R}^k$

In fact, $\mathbb{R}^k$ is a Banach Space(complete normed linear space) and Hilbert space

(3) A closed subset $S$ of a complete metric space $X$ is complete.

$\because$ Let $\{x_n\}$ be a Cauchy sequence in $S$. Then $\{x_n\}$ is a Cauchy sequence in $X$, hence, $x_n \to p$ for some $p \in X$. So $p \in \overline{S} = S$. Hence, $S$ is complete.

(4) Every closed subset of $\mathbb{R}^k$ is complete.

$\because$(3), In particular, every closed interval and closed ball in $\mathbb{R}^k$,

(5) $(0, 1)$ and $\mathbb{Q}$ are not complete

**<u>Definition</u>.** *Let $\{x_n\}$ be a real sequence*

*(i) We say that $\{x_n\}$ is increasing, if $x_n \leq x_{n+1} \forall n \geq 1$*

*(ii) We say that $\{x_n\}$ is strictly increasing, if $x_n < x_{n+1} \forall n \geq 1$*

*(iii) We say that $\{x_n\}$ is decreasing, if $x_n \geq x_{n+1} \forall n \geq 1$*

*(iv) We say that $\{x_n\}$ is strictly decreasing, if $x_n > x_{n+1} \forall n \geq 1$*

*(v) We say that $\{x_n\}$ is monotonic if either $\{x_n\}$ is increasing or decreasing.*

*(vi) We say that $\{x_n\}$ is strictly monotonic if either $\{x_n\}$ is strictly increasing or strictly decreasing*

**Examples**

- $\{2n + 1\}$ is increasing, $2n + 1 \to +\infty$
- $\{-n\}$ is decreasing, $-n \to -\infty$
- $\{\frac{1}{n}\}$ is decreasing, $\frac{1}{n} \to 0$
- $\{\frac{1}{-n}\}$ is increasing, $\frac{1}{-n} \to 0$

We will show that every monotonic sequence convergences in $\mathbb{R}^* = [-\infty, \infty]$

**Theroem 3.7.** *Let $\{a_n\}$ be a sequence*

*(a) Let $\{a_n\}$ be increasing*

*(i) If $\{a_n\}$ is bounded above, then $\{a_n\}$ convergence, in fact, $a_n \to sup a_n = sup\{a_n \mid n \geq 1\}$*

*(ii) If $\{a_n\}$ is not bonded above, then $a_n \to \infty$*

*(b) Let $\{a_n\}$ be decreasing*

*(a) If $\{a_n\}$ is bounded below, then $\{a_n\}$ convergences, in fact $a_n \to inf a_n = inf\{a_n \mid n \geq 1\}$*

*(b) If $\{a_n\}$ is not bounded below, them $\{a_n\} \to -\infty$*

**Remark.** *$\{a_n\}$ is increasing $\Leftrightarrow$ $\{-a_n\}$ is decreasing. So, to study monotonic sequence, it suffices to consider the case of increasing sequence.*

*Proof.* It suffices to prove (a) By same argument or considering $\{-a_n\}$, one can prove

$(a)$ $(i)$ $\{a_n\}$ is bounded above $\implies$ $\{a_n \mid n \geq 1\}$ is bounded $\implies$ $\alpha = \sup a_n$ exists and is finite
Claim: $a_n \to \alpha$ as $n \to \infty$.
Given $\epsilon > 0 \exists n_0 \in \mathbb{N} \to \alpha - \epsilon < a_{n_0}$. Then $\forall n \geq n_0$, we have $\alpha - \epsilon < a_{n_0} \leq a_n \leq \alpha < \alpha + \epsilon$, i.e. $\forall n \geq n_0$, $|a_n - \alpha| < \epsilon$. This proves $a_n \to \alpha$

$(ii)$ $\forall M > 0$, since $\{a_n\}$ is not bounded above, $\exists n_0 \in \mathbb{N} \to a_{n_0} \geq M \implies \forall n \geq M, a_n \geq a_{n_0} \geq M$. This proves $a_n \to +\infty$

$(b)$ is similar
$(i')$ $\{a_n\}$ is bounded below $\implies$ $\{-a_n\}$ is bounded above $\implies$ $\lim_{n\to\infty}(-a_n) = \sup(-a_n) \implies -\lim a_n = -\inf a_n$
$(ii')$ Similar

■

**Remark.** *Let $\{a_n\}$ be a monotonic sequence, then $\{a_n\}$ convergences $\Leftrightarrow$ $\{a_n\}$ is bounded.*

**Theroem 3.8.**

*(1) Every Cauchy sequence $\{x_n\}$ in compact metric metric space $X$ is convergent.*

*(2) In $\mathbb{R}^k$, every Cauchy sequence is convergent.*

*Proof.*

(1) Let $E_n = \{x_n, x_{n+1}, \cdots\}$ $n \geq 1$, by (2) and (4) $\lim_{n\to\infty} diam(\bar{E_n}) = \lim_{n\to\infty} diam(E_n) = 0 \ldots (\star)$. Clearly, $\{E_n\}$ is decreasing so is $\{\bar{E_n}\}$. $\bar{E_n} = \overline{\{x_n.x_{n+1}\}} \supseteq \overline{\{x_{n+1}, \cdots\}} = \bar{E}_{n+1}$. In fact,

$\{\bar{E}_n\}$ is a decreasing sequence of nonempty compact sets in $X$. Hence, by Cantor intersection theorem, $\bigcap_{n=1}^{\infty} E_n \neq \varnothing$. In fact, $\bigcap_{n=1}^{\infty} \overline{E_n} = \{p\}$ by cor 2.16.

**Claim :** $x_n \to p$. $\forall \varepsilon > 0$, by $(\star)$, $\exists\ N \in \mathbb{N}$ s.t. $diam(\overline{E_n}) - 0 < \varepsilon \therefore x_n \to p$ as $n \to \infty$.

(2) Since Cauchy sequence is bounded, $\exists\ R >> 0$ s.t. $x_n \in \bar{B}(0, R)$, i.e. $\{x_n\}$ is a Cauchy sequencein the compact set $\bar{B}(0, R)$. Hence $\{x_n\}$ converges in $\bar{B}(0, R)$.

■

**Definition.** *A metric space $X$ is said to be complete if every Cauchy sequence in $X$ is convergent.*

**Remarks and Examples :**

(1) I a complete metric space, to show that a sequence is convergent, it is enough to show that it is Cauchy.
(2) Thm 3.8 says compact metric space is complete. $\mathbb{R}^k$ is complete. Discrete metric space is complete. $X$ be a discrete metric space, $\{x_n\}$ is convergent $\Leftrightarrow$ $\{x_n\}$ is almost constant, i.e. $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$, $x_n = x_{n_0}$. $\{x_n\}$ is Cauchy $\Leftrightarrow \forall\ 0 < \varepsilon < 1$, $\exists N \in \mathbb{N}$ s.t.$\forall\ m, n \geq N\ d(x_m, x_n) < \varepsilon < 1 \Leftrightarrow \forall\ m, n \geq N, x_m = x_n = \cdots = x_N$.
(3) Complete normed linear space over $(\mathcal{F})$ is called a Banach space, so $\mathbb{R}^k$ is a Banach space.
(4) A Hilbert space is a complete inner product space over $(\mathcal{F})$.
(5) Closed subset $F$ is a complete metric space $X$ is complete. ($\because$ Given a Cauchy sequence $\{x_n\}$ in $F$, so it is also a Cauchy sequence in $X$. $X$ is complete, $x_n \to p$ as $n \to \infty$ for some $p \in X$. By thm, $p \in \bar{F} = F(\because F$ is closed) $\therefore F$ is complete ).
(6) The closeness in (5) is necessary, e.g. $(0, 1) \subseteq \mathbb{R}$ and $\mathbb{R}$ is complete, $\{1/n\}$ is a Cauchy sequence in $(0, 1)$ but it does not converge in $(0, 1)$.
(7) Every closed ball and closed bounded interval in $\mathbb{R}^k$ are complete.

(8) $(0,1)$ , $\mathbb{Q}$, $\mathbb{Q}^k$ are not complete. Now we prove a class of real sequence which has limit, namely, all nonconstant sequence has limits in $\mathbb{R}^k = [-\infty, \infty]$.

**Definition.** *Let $\{x_n\}$ be a real sequence.*

*(1) $\{x_n\}$ is incerasing, $\{x_n\}$ is $\nearrow$, if $x_n \leq x_{n+1}$, $\forall\ n \geq 1$*
*(2) $\{x_n\}$ is strictly incerasing, $\{x_n\}$ is st $\nearrow$, if $x_n < x_{n+1}$, $\forall\ n \geq 1$*
*(3) $\{x_n\}$ is decerasing, $\{x_n\}$ is $\searrow$ , if $x_n \geq x_{n+1}$, $\forall\ n \geq 1$*
*(4) $\{x_n\}$ is strictly decerasing, $\{x_n\}$ is st $\searrow$, if $x_n > x_{n+1}$, $\forall\ n \geq 1$*
*(5) We say that $\{x_n\}$ is monotonic if either $\{x_n\}$ is increasing or decreasing. (Similar description in strictly case).*

Note that $\{x_n\}$ is $\nearrow \Leftrightarrow \{-x_n\}$ is $\searrow$ , so we only discuss increasing sequence.

**Examples :**

(1) $\{2n+1\}$ is $\nearrow$ but it diverges. In fact, $\lim_{n\to\infty}(2n+1) = +\infty = \sup\{2n+1 \mid n \geq 1\}$. (not bounded above)
(2) $\{1/n\}$ is $\searrow$ and it converges, $\lim_{n\to\infty} 1/n = 0 = \inf\{1/n \mid n \geq 1\}$.

**Theroem 3.9.** *Let $\{a_n\}$ be a real sequence, we have*

*(1) If $a_n$ is $\nearrow$ , then $\lim_{n\to\infty} a_n = \sup_{n\geq 1} a_n = \sup\{a_n \mid n \geq 1\}$*
*(2) If $a_n$ is $\searrow$ , then $\lim_{n\to\infty} a_n = \inf_{n\geq 1} a_n = \inf\{a_n \mid n \geq 1\}$*
*(3) If $a_n$ is monotonic , then $\{a_n\}$ is convergent $\Leftrightarrow$ it is bounded.*

*Proof.*

(1) We have 2 cases, (i) $\{a_n\}$ is not bounded above, i.e. $\sup_{n\geq 1} a_n = +\infty$. Given $M > 0$, $\because \{a_n\}$ is not bounded above, $\exists N \in \mathbb{N}$ s.t. $a_N \geq M \implies \forall\ n \geq N$, $a_n \geq a_N \geq M$ $\therefore \lim_{n\to\infty} = +\infty$. (ii) $\{a_n\}$ is bounded above. Then $\lim_{n\geq 1} a_n = \alpha \in \mathbb{R}$. Now, we claim that $\lim_{n\to\infty} a_n = +\infty$. $\forall\ \varepsilon > 0, \exists\ N \in \mathbb{N}$ s.t. $\alpha - \varepsilon < a_N \implies \forall\, n \geq N, \alpha+\varepsilon > \alpha \geq a_n \geq a_N > \alpha-\varepsilon$ i.e. $|a_n - \alpha| < \varepsilon$, i.e. $\lim_{n\to\infty} a_n = \alpha$.
(2) First, we use (1) to prove it. $\{a_n\}$ is $\searrow \implies \{-a_n\}$ is $\nearrow$, by (1) $\implies \lim_{n\to\infty}(-a_n) = \sup_{n\geq 1}(-a_n) = -\inf_{n\geq 1} a_n$. Next, using the argument of (1). (i) $\{a_n\}$ is not bounded below i.e.

$\inf_{n\geq 1} a_n = -\infty$. Given $M > 0, \exists\ N \in \mathbb{N}$ s.t. $a_N \leq -M$, $\forall n \geq N\ a_n \leq a_N \leq -M \implies \lim_{n\to\infty} a_n = -\infty = \inf_{n\geq 1} a_n$
(ii) $\{a_n\}$ is bounded below , i.e. $\inf_{n\geq 1} a_n = \beta$. Now, we claim $\lim_{n\to\infty} a_n = \beta$. $\forall\ \varepsilon > 0 \exists\ N \in \mathbb{N}$ s.t. $a_N < \beta + \varepsilon \implies \forall\ n \geq N, \beta - \varepsilon < \beta \leq a_n \leq a_N \leq \beta + \varepsilon \implies \forall\ n \geq N, |a_n - \beta| < \varepsilon$ i.e. $\lim_{n\to\infty} a_n = \beta$.

(3) Follows from (1) and (2). $\because \{a_n\}$ is monotonic, ($\Rightarrow$) Suppose $\{a_n\}$ converges, then $\{a_n\}$ is bounded. ($\Leftarrow$) Suppose $\{a_n\}$ is bounded, so $\{a_n\}$ is bounded above and below. If $\{a_n\}$ is $\nearrow$, then by (1) $\{a_n\}$ converges. If $\{a_n\}$ is $\searrow$, then by (2) $\{a_n\}$ converges.

■

**Remark.** *In the extended real line or system $\mathbb{R}^\star = [-\infty, \infty]$, all monotonic sequence converges in $\mathbb{R}^\star$.*

### 3.4. Limit Superior and Limit inferior.

**Definition.** *Given a real sequence $\{x_n\}$. Let*

$$\begin{aligned} E &= \{x \in \mathbb{R}^\star \mid x \textit{ is a subsequence of } \{x_n\}\} \\ &= \{x \in \mathbb{R} \mid x \textit{ is a subsequence of } \{x_n\}\} \cup \{\infty\} \textit{ if } \infty \textit{ is a subsequence limit of } \{x_n\} \\ &= \{x \in \mathbb{R} \mid x \textit{ is a subsequence of } \{x_n\}\} \cup \{-\infty\} \textit{ or} \\ &= \{x \in \mathbb{R} \mid x \textit{ is a subsequence of } \{x_n\}\} \cup \{-\infty, \infty\} \end{aligned}$$

**Claim :** $E \neq \varnothing$.

- $\{x_n\}$ is not bounded above $\implies \exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ s.t. $x_{n_k} \to \infty$ as $k \to \infty$. $\forall\ k = 1, \exists\ n_1$ s.t. $x_{n_1} > 1$, $\exists\ n_2 > n_1$ s.t. $x_{n_2} > \max\{x_{n_1}, 2\} \cdots \implies x_{n_k} > k,\ \forall\ k \geq 1 \implies \lim_{n\to\infty} x_{n_k} = \infty$ , $\therefore \infty \in E$.
- $\{x_n\}$ is bounded above and below $\implies x_n \in [-a, a]\ \forall\ n \geq 1$ for some $a > 0 \implies \{x_n\}$ has a convergent subsequence say $x_n \to x \in [a, -a]$ , $\therefore x \in E$.
- Therefore in any case, $E \subseteq \mathbb{R}^\star$ is nonempty, so $\sup E$ and $\inf E$ exists in $\mathbb{R}^\star$.

**Definition.** *Define the limit superior or say upper limit of* $\{x_n\}$ *is denoted by*

$$\limsup_{n\to\infty} x_n = \overline{\lim_{n\to\infty}} x_n = x^\star = \sup E$$

*The limit inferior or lower limit of* $\{x_n\}$ *is denoted by*

$$\liminf_{n\to\infty} x_n = \underline{\lim_{n\to\infty}} = x_\star = \inf E$$

*The following are equivalent :*

$$\limsup_{n\to\infty} a_n = \inf_{n\geq 1}\left(\sup_{k\geq 1} a_k\right) = \lim_{n\to\infty}\left(\sup_{k\geq 1} a_k\right)$$

*and*

$$\liminf_{n\to\infty} a_n = \sup_{n\geq 1}\left(\inf_{k\geq 1} a_k\right) = \lim_{n\to\infty}\left(\inf_{k\geq 1} a_k\right)$$

Let $X$ be a set, $\{A_n\}$ a sequence of subset of $X$. Define

$$\limsup_{n\to\infty} A_n = \overline{\lim_{n\to\infty}} A_n = \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} A_k\right)$$

and

$$\liminf_{n\to\infty} A_n = \underline{\lim_{n\to\infty}} A_n = \bigcup_{n=1}^{\infty}\left(\bigcap_{n=k}^{\infty} A_k\right)$$

We say that the limit exists if $\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$. In this case, $\lim_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$

**Example :** Let $X = \{0,1\}$ then $A_{2n} = \{0\}$ and $A_{2n-1} = \{1\}$, $\limsup_{n\to\infty} A_n = \{0,1\}$ and $\liminf_{n\to\infty} A_n = \varnothing$.

**Facts :** $x \in \limsup_{n\to\infty} A_n \Leftrightarrow x \in \cap_{k=n}^{\infty} A_k \ \forall\ n \geq 1 \Leftrightarrow \{x \in X \mid x \in A_n$ infinitely open$\}$ and $x \in \liminf_{n\to\infty} A_n \Leftrightarrow x \in \cup_{n=1}^{\infty}(\cap_{n=k}^{\infty} A_n) \Leftrightarrow \exists\ n_0 \in \mathbb{N}$ s.t. $x \in \cap_{k=n_0}^{\infty} A_k \Leftrightarrow \{x \in X \mid x \in A_n$ almost always$\}$.

**Examples :** If $\{A_n\}$ is $\nearrow$ ($\searrow$), then $\lim_{n\to\infty} A_n = \cup_{n=1}^{\infty} A_n (\cap_{n=1}^{\infty} A_n)$ also $\limsup_{n\to\infty} A_n = \lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$.

$\{x_n\}$ : a real sequence, $E = \{x \in \mathbb{R}^\star \mid x$ is a subseq limit of $\{x_n\}\}$.
We have proved that $E \subseteq \mathbb{R}^\star$ is nonempty, so $x^\star = \sup E$ and $x_\star = \inf E$ both exist in $\mathbb{R}^\star$

**Theorem.** *Let $\{x_n\},\ E, x^\star,\ x_\star$ as above. Then $x^\star$ is the unique value in $\mathbb{R}^\star$ satisfying*

*(1)* $x^\star \in E$
*(2)* $\forall \epsilon > 0,\ \exists N \ni \forall n \geq N,\ x_n < x^\star + \epsilon$

*Similarly for $x_\star$, i.e. $x_\star$ is the "1" value in $\mathbb{R}^\star$ satisfying*

*(1')* $x_\star \in E$
*(2')* $\forall \epsilon > 0,\ \exists N \in \mathbb{N} \ni x_\star - \epsilon < x_n$

*Proof.* (a) We have three cases

($i$) $x^\star = +\infty$. In this case, $E$ is not bdd above, hence $\{x_n\}$ is not bdd above ($\because$ If $\{x_n\}$ were bdd above, then $\exists M > 0 \ni x_n \leq M\ \forall n \geq 1 \implies \forall$ subsequence $\{x_{n_k}\}$ of $\{x_n\}$, $x_{n_k} \leq M\ \forall k \geq 1 \cdots$)

($ii$) $x^\star \in \mathbb{R}$.
In this case, $E$ is bdd above and $x^\star \in \overline{E}$ by (9) in §2.5, By Thm 3.5, $x^\star \in E$

($iii$) $x^\star = -\infty$. Then $E = \{-\infty\}$, i.e. $-\infty$ is the only subseq limit of $x_n$, i.e. $\exists$ a subseq $\{x_{n_k}\}$ of $\{x_n\} \ni x_{n_k} \to -\infty$
$\therefore x^\star = -\infty \in E$

This prove (a)

(b) Suppose (b) were false, i.e. $\exists \epsilon > 0 \ni x_n \geq x^\star + \epsilon$ for infinity many n's. Hence, $\exists$ a subseq $\{x_{n_k}\}$ of $\{x_n\} \ni x_{n_k} \geq x^\star + \epsilon\ \forall k \geq 1 \implies \exists y \in E \ni y \geq x^\star + \epsilon > x^\star (\rightarrow\leftarrow)$ to the definition of $x^\star$
$\therefore (b)$ is true

Finally, to show that $x^\star$ is the "1" value satisfying (1) and (2)

Suppose $p, q \in \mathbb{R}^\star$ both satisfy (1) and (2) and $p \neq q$, We may assume that $p < q$. Choose $p < x < q$. By (2), $\exists N \in \mathbb{N} \ni \forall n \geq N,\ x_n < x \implies q \notin E (\rightarrow\leftarrow) \therefore p = q$

By a similar argument or apply the previous case to $\{-x_n\}$ ■

By definition and Thm 3.8, we have

**Theorem.** *Let $\{x_n\}$ and $\{y_n\}$ be real seqs,*

*(a)* $\liminf x_n \leq \limsup x_n (\because \in E \leq supE)$

*(b)* $\lim_{n\to\infty} x_n = x \Leftrightarrow \limsup x_n = \liminf x_n = x$

*Proof.*

($\Rightarrow$) Suppose $x_n \to x$ as $n \to \infty$. In this case, $E = \{x\} \Rightarrow x^\star = x_\star$, i.e. $\limsup x_n = \liminf x_n = x$

($\Leftarrow$) Suppose $\limsup x_n = \liminf x_n = x$. Then by Thm 3.8 (b) $\forall \epsilon > 0,\ \exists N \in \mathbb{N} \ni x - \epsilon < x_n < x + \epsilon$

∎

*(c)* $\limsup_{n\to\infty}(-x_n) = -\liminf_{n\to\infty} x_n$

*Proof.* $\because$ It suffices to prove the second one

$$\begin{aligned}
\liminf_{n\to\infty}(-x_n) &= \inf\{y \in \mathbb{R}^\star \mid y \text{ is a subseq limit of } \{-x_n\}\} \\
&= \inf\{-(-y) \in \mathbb{R}^\star \mid -y \text{ is a subseteq limit of } \{x_n\}\} \\
&= -\sup\{-y \in \mathbb{R}^\star \mid -y \text{ is a subseq limit of } \{x_n\}\} \\
&= -\sup\{x \in \mathbb{R}^\star \mid x \text{ is a subseq limit of } \{x_n\}\} \\
&= -\limsup_{n\to\infty} x_n
\end{aligned}$$

∎

*(d) If $x_n \leq y_n$ $\forall n \geq N$ for some $N \in \mathbb{N}$, then*

$$\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n$$

$$\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n$$

*Proof.* Let $E = \{x \in \mathbb{R}^\star \mid x \text{ is a sequential limit of } \{x_n\}\}$, $F = \{y \in \mathbb{R}^\star \mid y \text{ is a sequential limit of } \{y_n\}$

$\because x \in E \implies \exists$ a subseteq $\{x_{n_k}\}$ of $\{x_n\} \ni x_{n_k} \to x,\ k \to \infty$.

By assumption, $x_{n_k} \leq y_{n_k} \forall k \geq 1$, and $\{y_{n_k}\}$ is a subseteq of $\{y_n\}$

$\therefore$ $\{y_{n_k}\}$ has a sequential limit ∎