MTH 108 Fall 09-10 Exam 2,

Show all work. Justify procedures and calculations except when they are obvious as, for example, when you integrate both sides of an equation. When doing so, use complete, grammatically correct sentences. No books, notes, calculators.

1. (a) Find a solution of the third order ODE in the half-line $0 < x < \infty$,

$$u''' + 8u = 0,$$

which satisfies the conditions that $u(0) = 5$ and that $u(x)$ decays to zero as x tends to infinity. Do you think the solution is unique?

(b) Find the general solution of the third order ODE

$$u''' + 4u' = 0$$

2. Let the real valued functions $f(x)$ and $g(x)$ be defined, in $0 < x < 1$, by

$$f(x) = a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + \cdots,$$

$$g(x) = b_1\phi_1(x) + b_2\phi_2(x) + b_3\phi_3(x) + \cdots,$$

where ϕ_1, ϕ_2, $\phi_3, \cdots$ is a given orthonormal basis of real valued functions in the interval $0 < x < 1$, with inner (dot) product

$$(f, g) = \int_0^1 f(x)g(x)dx.$$

(a) Prove that

$$\int_0^1 f(x)^2 dx = a_1^2 + a_2^2 + a_3^2 + \cdots.$$

(b) Suppose that you are given the numbers $a_1, a_2, a_3, \cdots$. Calculate the inner products (f, ϕ_9) and (f, ϕ_{15}).

(c) Suppose that you are given the numbers $a_1, a_2, a_3, \cdots$ and $b_1, b_2, b_3, \cdots$. Derive a formula for the inner product (f, g) that utilizes these numbers only.

3. (a) You are given the initial-boundary problem for the forced diffusion equation in the interval $0 < x < \pi$,

$$u_t = u_{xx} + q(x), \quad 0 < x < \pi,$$

with boundary conditions

$$u_x(0, t) = 0 \text{ and } u(\pi, t) = 0$$

and initial condition

$$u(x, 0) = f(x),$$

where the $q(x)$ and $f(x)$ are known functions.

(a) Determine the solution $u(x, t)$.

(b) Given $q(x) = x$, calculate the limit of the solution $u(x, t)$ as time tends to positive infinity.

4. (a) Verify that the function $u(x,t) = f(x-at) + g(x+at)$, where f and g are arbitrary functions, satisfies the wave equation $u_{tt} - a^2 u_{xx} = 0$.

   (b) Use part (a) to find the solution of the initial value problem of the wave equation on the full line $-\infty < x < \infty$

$$u_{tt} - a^2 u_{xx} = 0, \quad -\infty < x < \infty,$$

   with initial conditions

$$u(x,0) = v(x), \quad u_t(x,0) = 0.$$

   (c) Determine the function $h(x)$ so that replacing the second initial condition ($u_t(x,0) = 0$) with the condition $u_t(x,0) = h(x)$ will cause the solution to be the shifting of $v(x)$ to the right with constant speed.

5. Let the function $u(x,y)$ satisfy Laplace's equation

$$u_{xx} + u_{yy} = 0,$$

   in the interior of a rectangle, with the boundary condition that u equals the functions f_U, f_D, f_L, f_R, along the sides of the rectangle up, down,left and right, respectively. Describe how to break down this problem to problems with simpler boundary conditions that are solvable by the eigenfunction expansion method. Show that the break-down is legitimate. DO NOT carry out any calculations toward the solution of the simpler problems.

6. Effect of the shifting of a periodic function on its Fourier coefficients: Let the function $f(x)$ be periodic with period 2π. Let its Fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\pi x + b_n \sin n\pi x \right),$$

   and let the Fourier series of the shifted function $f(x-c)$ be

$$f(x-c) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos n\pi x + B_n \sin n\pi x \right)$$

   Given the coefficients $a_0, a_1, a_2, a_3, \cdots$ and $b_1, b_2, b_3, \cdots$ derive formulae for the coefficients A_n and B_n for arbitrary $n = 1, 2, 3 \cdots$, as well as for A_0.

   Hint:

   $\cos(y+z) = \cos y \cos z - \sin y \sin z$ and

   $\sin(y+z) = \sin y \cos z + \cos y \sin z$