

# The theoretical values of the physical constants

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## THE THEORETICAL VALUES OF THE PHYSICAL CONSTANTS

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**ABSTRACT.** The principal physical constants are calculated by the theory developed in *Relativity Theory of Protons and Electrons* and subsequent papers, and are compared with the values given by Birge (1942) in *Reports on Progress in Physics*, **8**. There is satisfactory agreement with Birge's values as they stand. But the theory also indicates that small corrections are required in the computation of certain constants from experimental data; when these corrections are included, the small discordance between the spectroscopic and deflection values of  $e/m_e c$  disappears, and the agreement of observation and theory is complete. It is concluded that there ought to be no difference between the direct and indirect values of  $h/e$ . The calculated constant of gravitation is  $6.6665.10^{-8}$ ; and it is pointed out that the expected agreement of the calculated and observed values is not affected by the mean chemical constitution of the universe or the free radiant energy in space—a point previously left doubtful. As the purpose is not to justify or explain the theory but to compare it with observation, theoretical explanation is limited to the points which arise in adapting the theory to practical comparisons.

### § 1. INTRODUCTION

FROM the theory developed in *Relativity Theory of Protons and Electrons* (hereafter quoted as *P. & E.*) and subsequent papers (Eddington, 1937, 1940, 1940 a) it should be possible to determine precise values of the general physical constants. The recent publication of R. T. Birge's comprehensive survey of the experimental data (Birge, 1941) provides a suitable opportunity for tabulating the results and comparing them with observation.

A complication arises because the experimental values are expressed in three traditional units—the centimetre, gram and second—which have no relation to any theory. It is therefore necessary to select three measured quantities which, taken together, determine the centimetre, gram and second, to be used as "conversion constants". They must be quantities which occur in the most

fundamental part of physics, where the theoretical development is believed to be exact (a condition which excludes the characteristics of elements other than hydrogen); subject to this, they should be the most accurately ascertained constants available. Happily there is no doubt as to the most suitable choice. The three quantities are the velocity of light ( $c$ ), the Rydberg constant for hydrogen ( $\mathfrak{R}$ ) and the Faraday constant for hydrogen ( $\mathfrak{F}$ ). The last-named is the ordinary Faraday constant divided by the atomic weight of hydrogen, so as to reduce it to the scale  ${}^1\text{H} = 1$  instead of  ${}^{16}\text{O} = 16$ . Birge gives—

Table 1

$$\begin{aligned} c &= (2.99776 \pm .00004) \cdot 10^{10} \text{ cm. sec.}^{-1} \\ \mathfrak{R} &= 109677.58 \pm .05 \text{ cm.}^{-1} \\ \mathfrak{F} &= 9573.56 \pm 1.0 \text{ e.m.u.g.}^{-1} \end{aligned}$$

From table 1 it would be possible to reconstruct the C.G.S. system if the original standards were destroyed. We accordingly adopt it as a *verbal definition* of C.G.S. units which is in as close agreement as possible with the primary *gesticulatory definition*.\*

The calculated values of the physical constants, listed later, involve no observational data other than the three conversion constants. The uncertainty of any constant is therefore immediately evident from its dimensions in  $c$ - $\mathfrak{R}$ - $\mathfrak{F}$  units. The dimensions of the principal quantities are found to be

length	$\mathfrak{R}^{-1}$	angular momentum	$\mathfrak{R}^{-2} \mathfrak{F}^{-2} c$
time	$\mathfrak{R}^{-1} c^{-1}$	electrostatic charge	$\mathfrak{R}^{-1} \mathfrak{F}^{-1} c$
mass	$\mathfrak{R}^{-1} \mathfrak{F}^{-2}$	constant of gravitation	$\mathfrak{F}^2 c^2$

If Birge's results are accepted, the uncertainties of  $c$  and  $\mathfrak{R}$  are negligible, and  $\mathfrak{F}$  has a probable error of 1 part in 10 000. The highest power of  $\mathfrak{F}$  in any of the listed constants is  $\mathfrak{F}^2$ ; their probable errors should therefore not exceed 1 part in 5000.†

A more serious complication arises because the so-called "observational values" of the constants are not derived wholly from observation, but are a combination of observational data with a theory which I regard as superseded. Thus a straightforward comparison with Birge's constants would not be an observational test of my theory, but a comparison partly with observation and partly with the theory which it condemns. It has to be recognized that the current procedure of computing constants from observational data has acquired the status of a vested interest: and, as far as practicable, we accept it as furnishing the official definition of the constants. We therefore go out of our way to adapt the theoretical formulae (§ 2) to the official definition—so that the calculated constants represent the results which Birge presumably would have obtained by his procedure of reduction if the observational data had been perfect. But the current procedure is not entirely self-consistent, and in the most accurate comparisons we cannot avoid applying small corrections to certain of Birge's

\* By a "gesticulatory definition" I mean pointing to an object and saying that the metre (kilogram, day) is the extension (inertia, period) of *that* object.

† Except the last three constants in table 3, for which the theory has not yet been extended to this degree of refinement.

values in order to obtain a genuine comparison of theory and observation. This inconsistency of the official definitions is perhaps rather fortunate, since it relieves us of the obligation of carrying respect for tradition to an embarrassing extreme.

## § 2. THE THEORETICAL FORMULAE

The formulae used in calculating the principal constants are—

$$\begin{aligned}
 (1) \quad e^2 &= \frac{1}{137} \frac{hc}{2\pi}, & (2) \quad M &= \frac{136}{10} m_0, & (3) \quad \mu' &= \frac{1}{136} m_0, \\
 (4) \quad M &= m_p + m_e, & (5) \quad \mu &= m_p m_e / (m_p + m_e), \\
 (6) \quad \mu &= \beta^2 \mu', & (7) \quad \beta &= \frac{137}{136}, \\
 (8) \quad m_0 &= \frac{\frac{3}{4} \beta^2 \hbar \sqrt{(\frac{2}{3} N)}}{2\pi R_0 c}, & (9) \quad \frac{N_0}{R_0} &= \frac{\pi c^2}{\kappa M}, \\
 (10) \quad N_0 &= \frac{3}{4} N, & (11) \quad N &= 2 \times 136 \times 2^{256}, \\
 (12) \quad \kappa &= \frac{136 \cdot 137}{10} \frac{\pi \delta^2 c^2 \beta^2}{\sqrt{(\frac{2}{3} N)}}, \\
 (13) \quad \delta &= e/Mc, & (14) \quad \mathfrak{R} &= \frac{1}{2} \left( \frac{1}{137} \right)^2 \frac{\mu c}{\hbar}.
 \end{aligned}$$

Most of the notation is defined in table 3. Of the remaining symbols,  $c$ ,  $\mathfrak{M}$ ,  $\delta$  have already been defined: and  $m_0$  (the mass of a comparison particle),  $N$  (the cosmical number) and  $\mu'$  may here be regarded merely as auxiliaries in the calculation.

Formulae (1), (2), (3), (4), (5), (8), (9) are *P. & E.* (15.65), (12.63), (12.62), (14.71), (14.72) with certain changes explained below; formulae (6), (7), (10) relate to these changes. The result (11) is given in *P. & E.*, §§ 16.5, 16.7. Formula (12) is redundant, being derivable from the others, but is given here as the most direct expression for the constant of gravitation  $\kappa$  in terms of atomic constants. Formula (13) follows from the definition of  $\delta$ , since  $M$  is the mass of a hydrogen atom: and (14) is the ordinary expression for the Rydberg constant simplified by using (1).

From formulae (2)–(6) we find that  $m_p$ ,  $m_e$  are the roots of the equation

$$10m^2 - 136mm_0 + \beta^2 m_0^2 = 0. \quad \dots\dots(15)$$

Apart from the  $\beta$  factors, (12) and (15) are *P. & E.* (14.75) and (12.47).

The changes that have been made in the *P. & E.* formulae are: firstly, a factor  $\frac{3}{4}$  has been inserted in (8) and (10). This emendation was announced tentatively by Eddington and Thaxton (1940). I have since been able to simplify the theory, and it has become evident that the factor was wrongly omitted in my earlier work. Both in (8) and (10) it is introduced in reducing the spherical space of relativity theory to the flat space postulated in quantum theory. The number  $N_0$  of elementary particles is now  $\frac{3}{4}$  of the cosmical number  $N$ , which stands for the number of independent quadruple wave functions at any point. These wave functions are non-integrable in spherical space: and it is the non-integrability that causes the break-down of the simple exclusion principle which assigns one particle to each wave function.

Secondly, coefficients depending on the *Bond factor*  $\beta = 137/136$  appear in (6) and (8), and consequentially in (12) and (15). In principle  $\beta^{-1}$  has the same origin as the factor  $\frac{3}{4}$ , both being of the form  $n/(n+1)$  and arising out of the extension of a probability distribution from  $n$  to  $(n+1)$  dimensions. The additional variate is introduced, in the one case when we represent the inertial-gravitational field quantum-mechanically instead of by an extraneous concept (curvature), and in the other case when we represent the electric field quantum-mechanically by an interchange co-ordinate. The primary 'introduction of  $\beta$  has already been noticed in *P. & E.*, §§ 15.8, 15.9; but the formulae there given correspond to  $\mu = \beta\mu'$ . Here the same factor is distributed,  $\beta^{\frac{1}{2}}$  being included in (6) and  $\beta^{\frac{1}{2}}$  in (8). This change is not an amendment of the theory, but an adaptation of it to give the officially defined constants instead of the constants which the theory itself points out as physically significant. It would be unsuitable in this paper to enter deeply into theoretical questions: but, since the  $\beta$  factors especially concern the comparison of theory with observation, we shall in § 6 go far enough to show how the factors  $\beta^{\frac{1}{2}}$ ,  $\beta^{\frac{1}{2}}$  have been determined.

The calculation of the constants proceeds as follows:—From (2), (3), (6) we obtain  $M/\mu$ . Combining this with (14) we obtain  $h/Mc$ . Since (13) gives  $e/Mc$ , we obtain  $h/e$ : and by (1) we can then find  $h$  and  $e$  separately. The remaining steps are straightforward.

From  $R_0$  we derive an interesting constant  $V_0$ , which is the limiting speed of recession of the galaxies. The well-known formula is

$$V_0 = c/R_0\sqrt{3} \text{ cm. sec.}^{-1} \text{ per cm. distance.} \quad \dots\dots(16)$$

In table 3,  $V_0$  is given in the usual astronomical measure, km. sec.<sup>-1</sup> per megaparsec. Thanks to H. S. Jones's recent determination of the solar parallax, the result remains accurate to 1 part in 10 000. The observed speed is (as it should be) rather less than the calculated limiting speed; but the most optimistic estimate cannot put the probable error at less than 5 per cent.

The theory (Eddington, 1937) also gives two constants important in nuclear physics, namely the constants  $k$  and  $A$  in the expression

$$E = -Ae^{-r/k^{\frac{1}{2}}} \quad \dots\dots(17)$$

for the non-Coulombian energy of two protons distant  $r$  apart. The formulae are

$$k = \frac{R_0}{\sqrt{N_0}}, \quad A = \left(\frac{4}{3}\right)^{\frac{1}{2}} \frac{4}{\pi^{\frac{1}{2}}} \frac{m_p}{m_0} \frac{e^2}{k}. \quad \dots\dots(18)$$

In the original investigation,  $A$  was given without the  $(\frac{4}{3})^{\frac{1}{2}}$  factor, which has the same origin as the  $\frac{3}{4}$  factor in (8) and (10). The  $\beta$  factor, if any, has not yet been considered: it can scarcely exceed  $\beta^{\frac{1}{2}}$  in  $k$ , but may be more serious in  $A$ . Provisionally the uncertainties of  $k$  and  $A$  may be taken as 1 in 800 and 1 in 100, respectively. In the course of a general revision and simplification of the theory I have not yet dealt with  $A$ , and it is therefore more provisional than the other constants.

For completeness it may be added that the theory (Eddington, 1940 a) gives for the mass of a neutron  $M + \frac{3}{2}\mu$ , or very nearly  $m_p + \frac{5}{2}m_e$ . More tentatively

it gives  $173.8 m_e$  for the mass of a mesotron, and  $2.36 m_p$  for the mass of a "heavy mesotron" which decays into a proton or negatron. These results are not included in table 3.

### § 3. COMPARISON OF THEORY AND OBSERVATION

Setting aside the cosmological constants ( $R_0$ ,  $N_0$ ), the equations (1) to (14) boil down into a theoretical determination of three independent ratios  $hc/2\pi e^2$ ,  $m_p/m_e$ ,  $\kappa/\beta^2 c^2$ . These, being purely numerical, do not involve the conversion constants, and therefore afford the most direct comparison of theory with observation. These are given in table 2 and compared with the results given by Birge, or obtained directly from his values of the constants.

The agreement seems not quite satisfactory, since the residuals of  $A$  and  $B$  are nearly twice their probable errors. Two such residuals out of three would be counted a significant discordance if they were independent. But the observed values of  $A$  and  $B$  both involve the observed constant  $e/m_e c$ ; and we find on examination that both residuals have the same source, namely a slightly discordant value of this constant. This is made evident by forming the combination  $AB^\dagger$ , the observational value of which is independent of  $e/m_e c$ .\* The low probable

Table 2

Ratio	Calculated	Observed
$A = hc/2\pi e^2$	137.000	$137.030 \pm .016$
$B = m_p/m_e$	1837.46	$1836.56 \pm .56$
$C = \kappa/\beta^2 c^2$	$8.0939.10^{-37}$	$(8.0981 \pm .0060)10^{-37}$
$AB^\dagger$	1678.01	$1678.10 \pm .15$

error (1 part in 10 000) makes the test very stringent, and the agreement is perfect.

When the correlation of the residuals is allowed for, the agreement in table 2 is just about as close as the theory of probability admits.\* But although the comparison does not indicate any real need for correcting the observed values, the theory insists that certain small corrections should be applied.

Birge's adopted value of  $e/m_e c$  is the mean of a deflection value and a spectroscopic value which are not very accordant. We shall find (§ 7) that the theory predicts a discordance of the two values with one another, and also a discordance with the value implied by the official definition of the other constants. Distinguishing the deflection and spectroscopic results (uncorrected) by the suffixes  $d$  and  $s$ , the theory in § 7 gives

$$(e/m_e c)_d = \beta^{-\frac{1}{24}} e/m_e c, \quad (e/m_e c)_s = \beta^{-\frac{1}{12}} e/m_e c. \quad \dots\dots (19)$$

Birge's values of  $e/m_e c$  when corrected in accordance with (19) become

$$\begin{array}{ll} \text{spectroscopic} & (1.75987 \pm .00028)10^7, \\ \text{deflection} & (1.76013 \pm .00024)10^7. \end{array}$$

\* It involves  $\beta$ ,  $\beta$  and the Avogadro number.

† It is an even chance that the worst of three residuals is less than 1.87 times its probable error.

Adopting the mean,  $e/m_e c = 1.76000 \cdot 10^7$  instead of Birge's uncorrected mean (1.7592), we obtain for the ratios  $A$  and  $B$  in table 2 the revised comparison:

	Calculated	Observed
$hc/2\pi e^2$	137.000	137.009
$m_p/m_e$	1837.46	1837.40

I do not venture to give the probable errors of the observed values, which require technical discussion; but, although they are not quite so small as the agreement suggests, they are considerably smaller than in table 2, owing to the removal of the disagreement between the spectroscopic and deflection methods.

#### § 4. VALUES OF THE CONSTANTS

The calculated values of the most important constants are given in table 3. All except nos. 16, 17 are in C.G.S. units. The authority for the observed values is Birge's *Report*, except no. 17 (Hubble and Humason, 1931) and nos. 18, 20 (Thaxton, 1940). The probable error, given in the last column, is in units of the last decimal place in the preceding column. The observed values are taken directly from Birge and the other authorities without correction.

Table 3					
Ref. no.	Constant	Description	Calculated	Observed	P.E.
1	$e$	elementary charge (e.s.u.)	$4.80333 \cdot 10^{-10}$	4.8025	10
2	$h$	Planck constant	$6.62504 \cdot 10^{-27}$	6.6242	24
3	$M$	mass of $^1\text{H}$ atom	$1.67368 \cdot 10^{-24}$	1.67339	31
4	$\mu$	$m_p m_e / (m_p + m_e)$	$9.09873 \cdot 10^{-28}$	..	..
5	$m_e$	mass of electron	$9.10368 \cdot 10^{-28}$	9.1066	32
6	$m_p$	mass of proton	$1.67277 \cdot 10^{-24}$	1.67248	31
7	$\kappa$	constant of gravitation	$6.66649 \cdot 10^{-8}$	6.670	5
8	$hc/2\pi e^2$	fine-structure constant	137	137.030	16
9	$h/e$	.. . . .	$1.37926 \cdot 10^{-17}$	1.37933	23
10	$m_p/m_e$	mass-ratio	1837.46	1836.56	56
11	$e/m_e c$	(deflection method)	$1.76006 \cdot 10^7$	1.75959	24
12	$e/m_e c$	(spectroscopic method)	$1.76006 \cdot 10^7$	1.75880	28
13	$N_0$	particles in universe	$2.36216 \cdot 10^{79}$	..	..
14	$M_0$	mass of universe	$1.97675 \cdot 10^{55}$	..	..
15	$R_0$	Einstein radius of space	$9.33544 \cdot 10^{26}$	..	..
16	$R_0$	(in megaparsecs)	302.38	..	..
17	$V_0$	nebular speed (km. sec. <sup>-1</sup> mp <sup>-1</sup> )	572.36	560	..
18	$k$	nuclear range-constant	$1.92079 \cdot 10^{-13}$	1.9	..
19	$A$	nuclear energy-constant	$4.2546 \cdot 10^{-5}$	..	..
20	$A/m_e c^2$	.. . . .	52.006	51.4	..

The main purpose of table 3 is to give explicitly the calculated values. With the exception of nos. 8, 10, 13, they involve the conversion constants in table 1; but the effect of any future changes in the adopted values of  $\beta$ ,  $\eta$ ,  $c$  is readily found from the physical dimensions of the constant considered, as explained in § 1.

Nos. 1 to 12 are all obtained by combining the three ratios in table 2 with the three conversion constants. Thus the comparison with Birge's values adds

nothing essentially new to the discussion in § 3: the same agreement and the same mild discordance with the uncorrected observed values is shown (with much redundancy) in another form.

Besides the accurate comparisons, the observational determinations of nebular recession and of the two nuclear constants provide independent but much rougher comparisons (nos. 17, 18, 20). It is worth remarking that by equations (16) and (18)

$$kV_0 = c/\sqrt{(3N_0)}, \quad (N_0 = \frac{2}{3} \times 136 \times 2^{256}), \quad \dots\dots(20)$$

so that the range-constant of the proton-proton force can be found directly from the speed of recession of the galaxies, or vice versa, with no other observational data except the velocity of light.\*

An improvement in the observational determination of  $\kappa$  is greatly to be desired. It has been suggested to me that the observed value should differ from my calculated value by an amount depending on the mean packing-fraction of the elements of which the universe is composed: but I find that the suggestion is fallacious. In the theoretical calculation the universe was represented as a uniform static zero-temperature distribution of protons and electrons. It is, of course, legitimate to rearrange the matter of the universe so as to simplify the theoretical determination, just as it is legitimate to rearrange the matter in the laboratory so as to simplify the observational determination: but in such rearrangement the experimenter cannot, and the theorist must not, violate the conservation of energy. There is therefore an implicit assumption that the energy of free radiation, cosmic rays, etc., in space is just sufficient to transmute the complex elements back into hydrogen, restore the gravitational energy lost in the formation of condensations, and leave a uniform distribution at zero temperature. For theoretical reasons I believe this to be true; but I should now describe it as a *consequence* of the formula for  $\kappa$ , and not an assumption that must be made in order to derive it. In formula (12) all reference to the "universe" is eliminated, and the ratio  $\kappa/\delta^2 c^2$  of locally measured constants is expressed in terms of the number  $N$  of independent quadruple wave functions introduced by our system of analysis. The fact that the wave functions are non-integrable emphasizes that  $N$  (unlike  $N_0$ ) is a local concept. We should ordinarily say that the remote environment has no effect on  $\kappa/\delta^2 c^2$ , because it has the same effect on  $\delta^2 c^2$  as on  $\kappa$ .

### § 5. THE PROBLEM OF CONSISTENCY

Mass, momentum, energy, electric charge, etc., are primarily defined in molar physics. In some cases the molar definition can be adapted to cover microscopic quantities. Thus the mass  $M$  of a hydrogen atom is  $1/n$  of the molar mass of a quantity of hydrogen composed of  $n$  atoms: and the mass of any *neutral* atom or particle can be defined similarly. The elementary charge  $e$  is  $1/n$  of the molar electric charge of a body from which  $n$  electrons have been removed. Microscopic quantities defined in this way are said to be "molarly controlled". There is one important microscopic constant which is not molarly controlled, namely

\* For an accurate determination, a correction must be applied to the observed nebular recession to obtain the limiting speed of recession. Ideally the correction can be found by astronomical determinations of the average density of matter in space, but these are at present too inaccurate.



$m_e$ . The mass of an electron is obviously not defined as  $1/n$  of the molarly measured mass of an aggregation of  $n$  electrons: the observer (if he escaped electrocution) would have no idea how to set about so unprecedented a measurement.

In quantum theory a fresh start is made, and the terms mass, energy, charge, etc., are applied to *analogues* of the molar quantities, which play a corresponding part in the equations of quantum dynamics to that played by the originals in classical dynamics. Whether the analogy is to be construed as identity (for such quantities as are molarly controlled) can only be ascertained when quantum theory is extended into a unified microscopic and molar theory as in *P. & E.* We must not assume the identity prematurely.

There is usually a definite understanding that, whatever other divergences may exist, lengths and periods in microscopic structures are molarly controlled. This is implied when we assume that the frequencies of light waves emitted by an atom are equal to the frequencies of oscillating sources in its structure; for the undulatory theory of light is part of molar theory. This assumption or convention was the origin of Heisenberg's theory, which superseded Bohr's theory of the atom; and it is too deeply rooted in wave mechanics to be disturbed. Molar control of lengths and periods is imposed through the formula (14) for the Rydberg constant.

We naturally try to find a set of physical constants which, besides satisfying molar control of lengths and periods, shall (as a minimum) be consistent with

- (1) the analogy between quantum dynamics and molar dynamics;
- (2) molar control of  $M$  and other neutral masses;
- (3) molar control of  $e$ ;

but it is by no means clear that even these minimum conditions can be satisfied simultaneously. Accepting (1), the measure-scale of mass can certainly be adjusted to satisfy (2); but it appears that a rather different scale is required to satisfy (3). This is the cause of the corrections introduced in equation (19).

For the theorist, it is impossible to sacrifice (1), and evidently (3) is the condition to be dropped. This means that the Faraday constant  $\mathcal{F} = e/Mc$  used in microscopic physics will not agree with the constant found by molar observation, and there will be corresponding differences in the constants  $e$  and  $h$ . This, however, is a small price to pay for the far-reaching developments of quantum mechanics which can only be utilized by accepting condition (1).

But, notwithstanding its advantages, I clearly cannot adopt this system of definition of the constants in the present context. To tamper with the Faraday constant would bar out all comparison with Birge's constants at the very outset. I must follow him in accepting conditions (2) and (3) as a *sine qua non*. This will lead to trouble sooner or later, when we try to utilize results predicted by quantum theory on the basis of condition (1). The trouble does not begin immediately, because we have a constant  $m_e$  which is not molarly controlled, and is (on this system) virtually defined so as to satisfy the most immediate requirements of condition (1). The formulae of § 2 and the calculated values in table 3 relate to this system of constants, which we shall call the "official system". It will be treated in § 6. The "trouble" which arises in the adapta-

tion to the official system of results which are based on condition (1) will be treated in § 7.

To sum up:—We accept the ruling that in microscopic physics lengths, periods and the masses of electrically neutral objects must agree with the molar definitions.\* As regards electric charges and the masses of charged particles, we have a choice of two evils. The quantities which appear in the equations of quantum theory are the analogues (not the direct equivalents) of molar charge and mass, and the theorist would naturally prefer that this should be the recognized definition of constants such as  $m_e$  and  $e$ . But in the official system of constants,  $e$  is defined as the direct equivalent of molar charge. (There is no direct molar equivalent of  $m_e$ ). Thus the results of quantum theory normally require a correction to reduce them to the official system of constants. Although the formula for  $\mathfrak{R}$  is a result of quantum theory, it is in an exceptional position because of the adoption of  $\mathfrak{R}$  as a conversion constant. This requires that the quantity determined by it, namely  $\mu c/h$ , shall be the same in both systems of constants.

#### § 6. THE $\beta$ -COEFFICIENTS

The constant denoted by  $m_p/m_e$  is usually defined in the way indicated by the notation—a definition which would have had no meaning before the isolation of the electron. But the same constant can be described, independently of the atomic structure of electricity, as the ratio of the *densities* of the electrically positive and negative material in molar hydrogen. The related constant  $M/\mu$ † can similarly be described as a density-ratio of external and internal energies in molar hydrogen. Relativity mechanics is based on the energy tensor, which includes the density as a component; so that in relativistic theory  $m_p/m_e$  and  $M/\mu$  appear primarily as density-ratios, and their interpretation as mass-ratios comes later. By rearrangement of the theory in *P. & E.*, I now obtain equations (2) and (3), which yield  $M/\mu' = 136^2/10$ , *before treating* microscopic problems.

In passing over into microscopic physics, we introduce the extra variate referred to in § 2, and this determines the scale of microscopic structure. Like all observables, it has an uncertainty: and its introduction involves an extra dimension of the domain of probability-distribution of the complete energy tensor, extending the number of degrees of freedom from 136 to 137. We find that this alters the measure of the energy tensor, and therefore of the density, in the ratio  $\beta$ ; so that the density  $\mu'$  is replaced in microscopic theory by the density  $\mu_0 = \beta\mu'$ . This is a distinction between the analogue  $\mu_0$  and the equivalent  $\mu'$  of the molar density.

My earlier work in *P. & E.* did not go beyond this point; but it is clear that when we modify the definition of density to suit microscopic theory some further adjustment will become necessary. Either the constant of gravitation must be modified, or the metric of space will be changed. By accepting the condition of molar control of  $e$ , we settle the way in which this adjustment is to be made. For  $e^2$  is of dimensions  $ML^3T^{-2}$ ; and in order that its measure may be unaltered, the change of measure of density (dimensions  $ML^{-3}$ ) in the ratio  $\beta$  must be

\* This does not apply to constants or systems of measurement employed in auxiliary theoretical investigations (e.g.  $\mu_0$ ,  $\mathfrak{R}'$  in § 6) which are eliminated in the final formulae.

† To six significant figures  $M/\mu = m_p/m_e + 2$ .

coupled with a change of  $L^6T^{-2}$  in the ratio  $\beta^{-1}$ . To preserve relativistic symmetry  $L$  and  $T$  must be altered in the same ratio; hence the increase of density in the ratio  $\beta$  is coupled with a decrease of lengths in the ratio  $\beta^{-\frac{1}{3}}$ . It may be noticed that this combination makes Action invariant—a highly satisfactory property from the ordinary dynamical point of view.

The constant  $\mathfrak{R}$  in table 1 is found by a molar measurement of light in its molar aspect (undulatory theory). Being the reciprocal of a length, it is transformed into a constant  $\mathfrak{R}' = \beta^{\frac{1}{3}}\mathfrak{R}$  when  $\mu'$  is transformed into  $\mu_0 = \beta\mu'$ . The relation found in the theory of the internal wave functions of a hydrogen atom is therefore primarily

$$\mathfrak{R}' = \frac{1}{2} \left( \frac{1}{137} \right)^2 \frac{\mu_0 c}{h}, \quad \dots\dots (21)$$

since  $h$  (which has the fixed relation (1) to  $e$ ) is an invariant of the transformation. This gives

$$\mathfrak{R} = \frac{1}{2} \left( \frac{1}{137} \right)^2 \frac{\beta^{\frac{1}{3}} \mu' c}{h}. \quad \dots\dots (22)$$

Comparing with (14), we have  $\mu = \beta^{\frac{2}{3}}\mu'$ , which is the equation (6) that has been used in our calculations. We notice that molar control of  $e$  (and  $h$ ) is secured by the choice of transformation; molar control of period and length is imposed by making (22) agree with (14); and molar control of mass is in this problem irrelevant, since we are not dealing with a neutral particle.

Equation (8) is made comparable with (21) by writing it in the form

$$R_0^{-1} = \text{numerical constant} \times \frac{m_0 c}{h}.$$

The theory gives primarily a constant  $R_0'^{-1} = \beta^{\frac{1}{3}}R_0^{-1}$ , just as it gives  $\mathfrak{R}' = \beta^{\frac{1}{3}}\mathfrak{R}$ ; so that the factor  $\beta^{\frac{1}{3}}$  is introduced when we employ the ordinary Einstein radius  $R_0$ . There is no  $\beta$  factor in (9), which is a well known equation of molar relativity theory. The factor  $\beta^{\frac{1}{3}}$  in (8) is carried through into (12), which is derived from (8), (9) and (13).\*

This shows how the  $\beta$ -coefficients in the equations of § 2 have been determined, and we proceed to the explanation of the  $\beta$ -coefficients in equation (19).

#### § 7. THE CONSTANT $e/m_e c$

In a comprehensive treatment we should begin with the complete set of theoretical relations expressed in terms of the constants which the theory itself selects, and then, considering each kind of experimental measurement in turn, investigate the precise expression of the result of the measurement in terms of those constants. The existing treatment is an attempt at a short cut. Out of the wide variety of experimental measurements, three or four are selected on account of the great practical accuracy attained: constants directly related to these selected measurements are adopted, and the theorist is expected to transform his formulae so that they are expressed in terms of those constants. This

\* Since the observed value of  $\kappa$  is relatively inaccurate, it is specially important to control the foregoing calculation of the  $\beta$ -coefficient in (12). I may therefore add that the simplest calculation employs the system of constants in § 7. The coefficient is then  $\beta^{\frac{1}{3}}$ ; but the corresponding charge is  $e_s$ , so that  $\tilde{\kappa}$  must be replaced by  $\tilde{\kappa}_s$ . By (28) we have  $\beta^{\frac{1}{3}} \tilde{\kappa}_s^2 = \beta^{\frac{1}{3}} \tilde{\kappa}^2$ , so that the formula is confirmed.

is a troublesome policy in the long run; for sooner or later one of the non-selected experimental measurements becomes the centre of attention. The special adaptation of the theory to the selected measurements becomes irrelevant, and we have to transform the formulae back to their original form before we can relate them to the new measurement. At present the kinds of measurement accurate enough to be used in determining the constants are very limited, and the trouble is localized: it is concerned with  $\mu$ , and with  $m_e$  which is derived from  $\mu$ .

A "spectroscopic" value of  $M/\mu$  (or  $m_p/m_e$ ) can be deduced from the ratio of the Rydberg constant for deuterium (or helium) to the Rydberg constant for hydrogen. Since this is a deduction of one purely numerical ratio from another, the theory is particularly simple; only we must use the primary theoretical formulae which satisfy condition (1) of § 5, and not the formulae adapted for purposes irrelevant to this particular measurement. The following remarks therefore relate to the primary theory.

To avoid unnecessary complication we employ "natural units" (Eddington, 1940) such that

$$c = 1, \quad \kappa h^2 = 1. \quad \dots\dots(23)$$

This leaves one disposable unit; and it is easily found that when the unit is changed so that densities are changed in the ratio  $\beta$ , masses are changed in the ratio  $\beta^{\frac{1}{2}}$ , and lengths in the ratio  $\beta^{-\frac{1}{2}}$ . Also  $h$  and  $e$  change in the ratios  $\beta^{\frac{1}{2}}$ ,  $\beta^{\frac{1}{2}}$ . Proceeding as in § 6, we have in place of (22)

$$\Re = \frac{1}{2} \left( \frac{1}{137} \right)^2 \frac{\mu' c \beta^{\frac{1}{2}}}{h_s}, \quad \dots\dots(24)$$

where  $h_s$  is the natural theoretical value of  $h$  in microscopic theory (the symbol  $h$  having already been appropriated to an adapted value). Comparing with (22), we have

$$h_s = \beta^{\frac{1}{12}} h. \quad \dots\dots(25)$$

The difference is that  $h$  corresponds to conditions (2) and (3) in § 5, and  $h_s$  corresponds to conditions (1) and (2). Since  $e$  is always related to  $h$  by

$hc/2\pi e^2 = 137$ , we have  $e_s = \beta^{\frac{1}{24}} e$ . The value  $e$  is the equivalent of molar charge (i.e. it is molarly controlled), and the value  $e_s$  is the analogue of molar charge

employed in quantum dynamics. The factor  $\beta^{\frac{1}{12}}$  in (25) is the result of taking out the transformation of length in the ratio  $\beta^{\frac{1}{2}}$  imposed by condition (3) before putting in the transformation in the ratio  $\beta^{\frac{1}{2}}$  imposed by condition (1). Since condition (2) is common to the two systems,  $M$  is unchanged; and  $\mu'$  is also unchanged, since the relation  $\mu' = 10M/136^2$  is independent of microscopic theory as explained at the beginning of § 6.

The "observed" spectroscopic value of  $M/\mu$  or  $m_p/m_e$  is computed on the assumption that the classical formulae (4) and (5) for transforming a two-particle system into external and internal particles apply. This assumption commits us to condition (1); for evidently a classical formula will not apply to quantities in quantum theory unless they are the analogues of the classical quantities. On

this understanding it is easily proved that the assumption is valid.\* Thus the uncorrected spectroscopic constants  $(M/\mu)_s$  and  $(m_p/m_e)_s$  correspond to the values  $h_s$ ,  $e_s$ , and are inconsistent with the system of constants in table 3 which corresponds to  $h$  and  $e$ . In particular, the corresponding Faraday constant

$\mathfrak{F}_s = e_s/Mc$  is  $\beta^{\frac{1}{24}} \mathfrak{F}$ .

By (13), (14) and (1)

$$\mathfrak{F}^2 \mathfrak{M} = \frac{1}{2} \left( \frac{1}{137} \right)^2 \frac{e^2}{hc} \frac{\mu}{M^2} = \frac{1}{4\pi} \left( \frac{1}{137} \right)^3 \frac{\mu}{M^2}. \quad \dots\dots (26)$$

Since  $(M/\mu)_s$  corresponds to  $\mathfrak{F}_s$  instead of  $\mathfrak{F}$ , and the other constants are unchanged, we have

$$\frac{(M/\mu)_s}{M/\mu} = \frac{\mathfrak{F}_s^2}{\mathfrak{F}^2} = \beta^{-\frac{1}{12}}. \quad \dots\dots (27)$$

and  $(m_p/m_e)_s$  differs from  $m_p/m_e$  in nearly the same ratio. The collected comparison is

$$h_s = \beta^{\frac{1}{12}} h, \quad \mu_s = \beta^{\frac{1}{12}} \mu, \quad e_s = \beta^{\frac{1}{24}} e, \quad \mathfrak{F}_s = \beta^{\frac{1}{24}} \mathfrak{F}. \quad \dots\dots (28)$$

Applying these factors to the constants already calculated, we obtain

Table 4

$\mathfrak{F}_s = 9576.48$	
$e_s = 4.80480 \cdot 10^{-10}$	
$h_s = 6.62908 \cdot 10^{-27}$	
$(m_p/m_e)_s = 1836.34$	
$(e/m_s c)_s = 1.75899 \cdot 10^7$	(Birge $1.75880 \pm .00028$ )
$(e/m_e c)_{ss} = 1.75953 \cdot 10^7$	(Birge $1.75959 \pm .00024$ )

The last two quantities require explanation. Birge does not give the observational value of  $(m_p/m_e)_s$  explicitly, but combines it with the ordinary Faraday constant so as to obtain "the spectroscopic value of  $e/m_e c$ ". This is a preposterously hybrid constant, since the  $e$  of the Faraday constant and the  $m_e$  of the spectroscopic result belong to different systems. However, we can calculate it for comparison with Birge. The result, denoted by  $(e/m_e c)_s$ , agrees excellently.

The *pure* spectroscopic value, which we denote by  $(e/m_e c)_{ss}$ , is obtained by combining  $(m_p/m_e)_s$  with  $\mathfrak{F}_s$ . This is the value which we should expect to obtain by a measurement such as the deflection experiment which determines  $e/m_e c$  directly. This conclusion requires careful verification, since the comparison of a dimensional quantity with observed measurements is a more complicated matter than the comparison of a numerical ratio  $m_p/m_e$ . The scrutiny is as follows :—

The quantities actually measured in the deflection experiment are molar electromagnetic fields  $X$ ,  $H$  and the curvature of track of a wave-packet. The curvature measurement applies directly to the theoretical equations, since lengths in those equations are molarly controlled. But it might be questioned

\* Valid in so far as the only relevant difference between the proton and the deuteron is the difference of mass. I do not profess to have examined whether the effect of the complexity of the nucleus has been rigorously allowed for in the reductions.

whether  $X$  and  $H$  can be used as they stand, since we have seen that  $\mathfrak{F}$  is inapplicable. Consider a modification of the experiment in which the fields are produced by charged bodies whose positions, velocities and molar charges  $ne$  are measured, so that  $n$  is determined from the molar measurement of charge by using the known constant  $e$ . The positions, velocities and  $n$  are transferred directly into the microscopic equations, and we can deduce from the experiment the pure spectroscopic constant  $(e/m_e c)_{ss}$ . It is clear that the two forms of experiment must give the same result, for a discrepancy would imply that it makes a difference whether we measure  $X$  and  $H$  or measure the molar charge and velocity which produce them; that is to say, it would be a discrepancy in Maxwell's theory. Thus  $X$  and  $H$  require no correction, the adjustment to microscopic theory being automatically provided for in the fact that they are computed (or measured) as though the body producing them had a charge  $ne$  instead of  $ne_s$ .

Since  $(e/m_e c)_{ss} = (e/m_e c)_d$ , we have inserted Birge's value of  $(e/m_e c)_d$  for comparison in table 4. In § 3 the comparison was made the reverse way by applying corrections to Birge's values of  $(e/m_e c)_s$  and  $(e/m_e c)_d$  to obtain the official constant  $e/m_e c$ .\*

In table 2 (p. 120) of Birge's *Report*, twelve determinations of  $e/m_e c$  by seven distinct methods are collected, and the table conveniently indicates how the results depend on the Faraday constant. We have not scrutinized each method separately, but there seems no reason to suspect any theoretical error except the use of an inconsistent Faraday constant. A constant larger in the ratio  $\mathfrak{F}_s/\mathfrak{F}$  or  $\beta^{\frac{1}{24}}$  should have been employed. This correction should make all the determinations agree. The amended values correspond to  $(e/m_e c)_d$ , and a further multiplication by  $\beta^{\frac{1}{24}}$  is necessary to reduce them to official reckoning.

#### § 8. THE CONSTANT $h/e$

Another question of practical importance is whether the "direct value" of  $h/e$  should agree with the indirect value. According to Birge, the latest experiments give no indication of discrepancy; but in view of the stop-press character of this information, and the many vicissitudes of opinion, we desire an independent theoretical answer.

I assume that "direct" refers to a determination  $h/e = \delta V / c \delta \lambda^{-1}$ , where  $\delta \lambda^{-1}$  is the measured change of spectral wave-number due to the fall of an electron through a measured potential difference  $\delta V$ . We remark first that the equation does not introduce the constants  $\mu$ ,  $m_e$ , which are the root of the divergence of the theoretical and official systems. The theory of the experiment is in fact independent of the quantum dynamics of the atom, which only appears as a catalyst in the conversion of a molarly measured energy  $eV$  into a molarly measured energy  $h\nu$ . Also, if the experiment could be satisfactorily carried out with hydrogen, the whole theory of it would fall within the part of quantum theory where the official constants directly apply. In any of these ways we can see that the direct value will agree with the official indirect value in table 3. Alternatively

\* The formulae (19) are not quite exact, owing to the difference of transformation of  $M/\mu$  and  $m_p/m_e$ ; but the inaccuracy does not affect the first six significant figures.

we can eliminate  $h$  by the fixed relation  $hc/2\pi e^2 = 137$ , and regard the experiment as a determination of  $e$  by the equation

$$e = \frac{1}{137} \frac{\delta V}{2\pi\delta\lambda^{-1}},$$

and it is clear that  $e$  will be in the same molar electrical measure in which  $V$  is expressed.

The theory therefore predicts agreement of the direct and indirect values.

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## NOTE ON SOME RECENT PAPERS ON PHYSICAL QUANTITIES AND THEIR DIMENSIONS

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**ABSTRACT.** Examination of recent papers on dimensions and physical quantities suggests a need for more careful definition of these quantities. This matter was dealt with in an earlier paper, but is now expanded and treated in more detail. It is maintained that if definitions were more precise there would not be the same confusion about the dimensions of certain quantities. There may still be differences of opinion on the most appropriate definition of a particular quantity, but if the definition is adequate then there is no doubt about the dimensions of the quantity as deduced from the definition. In this connection there is discussed the use of quantities defined as ratios, such as specific gravity (or relative density), and their more fundamental equivalents, such as density. Further, a logical set of definitions leads to the conclusion that the introduction of certain universal constants is unnecessary—for example, the Newtonian constant  $N$  in the equation for force,  $F = Nmf$ . Finally, there is discussed briefly the definition of the intensive quantity, temperature, in terms of the extensive quantity, energy.

#### § 1. INTRODUCTION

FROM consideration of the recent papers on dimensions and on the nature of physical quantities\*, there are two points which emerge:

(a) That unnecessary quantities are introduced into physics, partly because of lack or ambiguity of definition, and

(b) That much of the discussion on the dimensions of physical quantities is misplaced in that the difference of opinion on dimensions derives from differences in the definition of the quantities, and the emphasis of much of the discussion

\* Brown (1941), Yarnold (1942), Benham (1942) and Hansel (1942).