TOMMASO CORTONESI

NUMBER THEORY REVEALED

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Notes and Exercises

Exercise 0.1.1. For (a): the base step is obvious $F_0 \in \mathbb{N}$; for the induction step, we assume that $F_{n-1}, F_{n-2} \in \mathbb{N}$, but $\mathbb{N}$ is closed under the sum so we have $F_{n-2} + F_{n-1} \in \mathbb{N} \Rightarrow F_n \in \mathbb{N}$.
For (b), remember that:

$$\phi = \frac{1+\sqrt{5}}{2}, 1 - \phi = \frac{1-\sqrt{5}}{2} \text{ and } \phi^2 = \phi + 1.$$

The base step is trivial. For the induction step we assume:

$$F_{n-2} = \frac{1}{\sqrt{5}}\left(\phi^{n-2} - (1-\phi)^{n-2}\right)$$
$$F_{n-1} = \frac{1}{\sqrt{5}}\left(\phi^{n-1} - (1-\phi)^{n-1}\right)$$

and then

$$\begin{aligned} F_{n-2} + F_{n-2} = \frac{1}{\sqrt{5}}\left[\phi^{n-2} - (1-\phi)^{n-2} + \phi\phi^{n-2} - (1-\phi)(1-\phi)^{n-2}\right] \\ = \frac{1}{\sqrt{5}}\left[\phi^{n-2}(1+\phi) - (1-\phi)^{n-2}(2-\phi)\right] \\ = \frac{1}{\sqrt{5}}\left(\phi^n - (1-\phi)^n\right) = F_n, \end{aligned}$$

Because $(1-\phi)^2 = 2 - \phi$.
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Exercise 0.1.2. For (a): ■