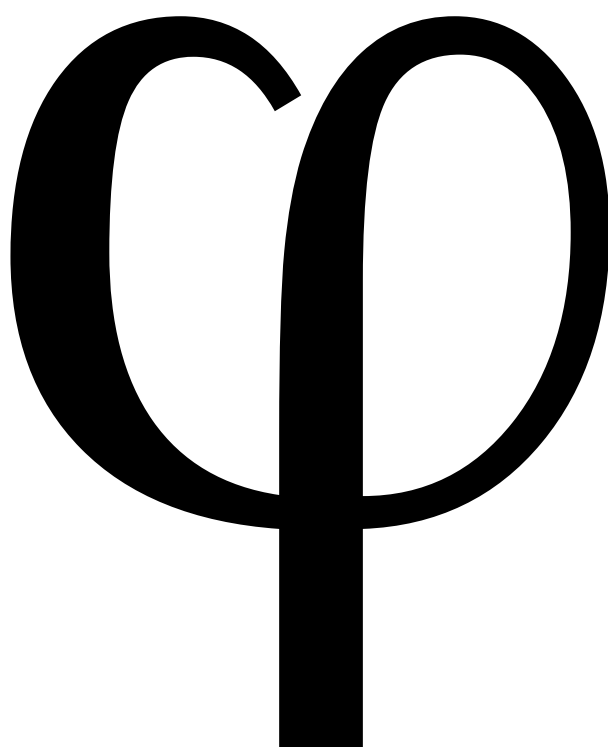


TOMMASO CORTONESI

# NUMBER THEORY REVEALED



Notes and Exercises

**Exercise 0.1.1.** For (a): the base step is obvious  $F_0 \in \mathbb{N}$ ; for the induction step, we assume that  $F_{n-1}, F_{n-2} \in \mathbb{N}$ , but  $\mathbb{N}$  is closed under the sum so we have  $F_{n-2} + F_{n-1} \in \mathbb{N} \Rightarrow F_n \in \mathbb{N}$ .

For (b), remember that:

$$\phi = \frac{1 + \sqrt{5}}{2}, 1 - \phi = \frac{1 - \sqrt{5}}{2} \text{ and } \phi^2 = \phi + 1.$$

The base step is trivial. For the induction step we assume:

$$F_{n-2} = \frac{1}{\sqrt{5}} \left( \phi^{n-2} - (1 - \phi)^{n-2} \right)$$

$$F_{n-1} = \frac{1}{\sqrt{5}} \left( \phi^{n-1} - (1 - \phi)^{n-1} \right)$$

and then

$$\begin{aligned} F_{n-2} + F_{n-1} &= \frac{1}{\sqrt{5}} \left[ \phi^{n-2} - (1 - \phi)^{n-2} + \phi \phi^{n-2} - (1 - \phi)(1 - \phi)^{n-2} \right] \\ &= \frac{1}{\sqrt{5}} \left[ \phi^{n-2}(1 + \phi) - (1 - \phi)^{n-2}(2 - \phi) \right] \\ &= \frac{1}{\sqrt{5}} (\phi^n - (1 - \phi)^n) = F_n, \end{aligned}$$

Because  $(1 - \phi)^2 = 2 - \phi$ .

■

**Exercise 0.1.2.** For (a): ■