Efficient frontier

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Mean-variance portfolio theory

a) Minimum standard deviation portfolio

Lets first define the necessary variables. Let Σ be the covariance matrix of the assets and w the (column) vector of asset weights. Now the portfolio variance in matrix notation is $w'\Sigma w$.

Finding the minimum variance portfolio is a constraint optimization problem which can expressed as

$$\min_w \sigma_p^2 = w'\Sigma w$$

$$s.t. \quad w'1 = 1$$

Now formulating the corresponding Lagrangian we get:

$$\begin{bmatrix} 2\Sigma & \mathbf{1} \\ \mathbf{1'} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$$

```
var_A <- 0.25**2
var_B <- 0.2**2
var_C <- 0.15**2

cov_AB <- -0.25*0.25*0.2
cov_AC <- 0.3*0.25*0.15
cov_BC <- 0.4*0.2*0.15

covm <- matrix(c(var_A,cov_AB,cov_AC,cov_AB,var_B,cov_BC,cov_AC,cov_BC,var_C),nrow=3,byrow=TRUE)
covm
```

```
##          [,1]    [,2]    [,3]
## [1,]  0.06250 -0.0125 0.01125
## [2,] -0.01250  0.0400 0.01200
## [3,]  0.01125  0.0120 0.02250
```

Now solving the Lagrangian (inverting the block matrix) with R

```
# Takes a N-dimensional covariance matrix as parameter (by def. square matrix)
# Solves Lagrangian system of equations. The system is written as block matrix.
# Returns vector of asset weights corresponding to the minimum variance portfolio
solve_min_variance_p <- function(covm) {
  N <- dim(covm)[1]
  top.mat <- cbind(2*covm,rep(1,N))
  bot.vec <- c(rep(1,N),0)
  mat <-rbind(top.mat,bot.vec)
  b <- c(rep(0,N),1)
  z <- solve(mat)%*%b
  return(z[1:N,1])
```

```
}

weights_minv <- solve_min_variance_p(covm)
weights_minv
```

```
## [1] 0.2565353 0.3610154 0.3824494
```

we get weights $w = (0.257, 0.361, 0.382)$ which sum to 1. With these weights the expected return of the portfolio is 14.37% and the standard deviation is 12.58%.

```
mu <- c(0.2,0.15,0.1)
r_minv <- mu%*%weights_minv
r_minv
```

```
##           [,1]
## [1,] 0.1437043
```

```
std_minv <- sqrt(t(weights_minv)%*%covm%*%weights_minv)
std_minv
```

```
##           [,1]
## [1,] 0.1257908
```

b) Efficient portfolio with expected return 18%

Now the optimization problem has another constraint: the portfolio return should be $\tilde{\mu}$:

$$\min_{w} \sigma_p^2 = w'\Sigma w$$
$$s.t. \quad w'\mathbf{1} = 1 \quad and \quad w'\mu = \tilde{\mu}$$

Formulating the corresponding Lagrangian we get:

$$\begin{bmatrix} 2\Sigma & \mu & \mathbf{1} \\ \mu' & 0 & 0 \\ \mathbf{1'} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \tilde{\mu} \\ 1 \end{bmatrix}$$

```
# Takes 3 parameters:
#   - Target return for the portfolio (mu_t)
#   - Vector of assets expected returns (mu)
#   - N-dimensional covariance matrix as parameter (by def. square matrix)
# Solves Lagrangian system of equations. The system is written as block matrix.
# Returns vector of asset weights corresponding to the efficient portfolio
solve_eff_p <- function(mu_t,mu,covm) {
  N <- dim(covm)[1]
  top.mat <- cbind(2*covm,mu,rep(1,N))
  mid.vec <- c(mu,0,0)
  bot.vec <- c(rep(1,N),0,0)
  mat <-rbind(top.mat,mid.vec,bot.vec)
  b <- c(rep(0,N),mu_t,1)
```

```
  z <- solve(mat)%*%b
  return(z[1:N,1])
}

weights_18 <- solve_eff_p(0.18,mu,covm)
weights_18
```

```
##
##  0.4996289  0.6007421 -0.1003711
```

```
r_18 <- mu%*%weights_18
r_18
```

```
##      [,1]
## [1,] 0.18
```

```
std_18 <- sqrt(t(weights_18)%*%covm%*%weights_18)
std_18
```

```
##           [,1]
## [1,] 0.1420738
```

The standard deviation of this portfolio is 14.21%.

c) Portfolio and risk-free asset

If we had a risk free asset available, we could combine an optimal portfolio and the risk free asset (One-fund theorem). The optimal portfolio in this case is found by maximizing the steepness of the tangency line to the efficient frontier:

$$\max_w \frac{\sum_{i=1}^n w_i(\mu_i - r_f)}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij}}}$$

The problem reduces to solving system of equations:

$$\mu_k - r_f = \sum_{i=1}^n v_i \sigma_{ik} \quad k = 1, 2, 3.$$

Where v_i is the unnormalized weight of the asset i. The weights (i.e. the optimal portfolio) is found by normalizing these weights.

```
# Solving the above system of equations
r_f <- 0.05
weights_onef <- solve(covm)%*%(mu-r_f)
weights_onef_scaled <- weights_onef/sum(weights_onef)
t(weights_onef_scaled)
```

```
##           [,1]     [,2]       [,3]
## [1,] 0.5981342 0.697883 -0.2960172
```

```
r_onef <- mu%*%weights_onef_scaled
r_onef
```

```
##           [,1]
## [1,] 0.1947076
```

```
std_onef <- sqrt(t(weights_onef_scaled)%*%covm%*%weights_onef_scaled)
std_onef
```

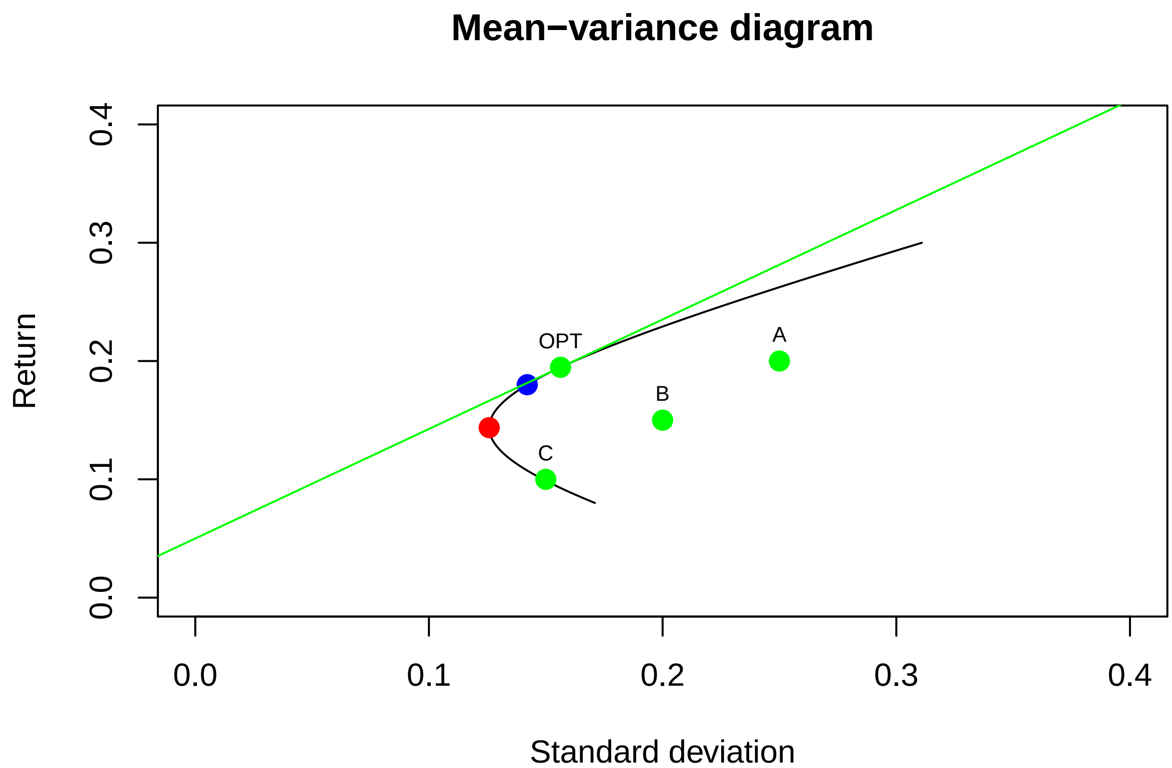
```
##         [,1]
## [1,] 0.15632
```

We find that the optimal portfolio's return is 19.47% and standard deviation is 15.63%.

d) Mean-variance diagram

```
# Solving the efficient portfolio for returns 0.08 ... 0.3
# i.e. forming the portfolio curve and efficient frontier
min_r <- 0.08; max_r <- 0.3
tmp <- numeric((max_r-min_r)*1000)
eff_frontier <- data.frame(tmp,tmp)
colnames(eff_frontier) <- c("Return", "Std")
i <- 1
for (r in seq(min_r,max_r,0.001)) {
  w <- solve_eff_p(r,mu,covm)
  std <- sqrt(t(w)%*%covm%*%w)
  eff_frontier[i,1] <- r
  eff_frontier[i,2] <- std
  i <- i + 1
}
```

```
plot(eff_frontier$Std,eff_frontier$Return,type="l",main="Mean-variance diagram",
     xlab="Standard deviation",ylab="Return",xlim=c(0,0.4),ylim=c(0,0.4))

lines(std_minv,r_minv,type="p",pch=20,col="red",cex=2) # Minimum variance
lines(std_18,r_18,type="p",pch=20,col="blue",cex=2) # Efficient portfolio for 18% target return

# Assets
lines(0.25,0.2,type="p",pch=20,col="green",cex=2) # A
lines(0.2,0.15,type="p",pch=20,col="green",cex=2) # B
lines(0.15,0.1,type="p",pch=20,col="green",cex=2) # C

# One-fund optimal portfolio
lines(std_onef,r_onef,type="p",pch=20,col="green",cex=2) # C

text(c(0.25,0.2,0.15,std_onef),c(0.2,0.15,0.1,r_onef),labels=c("A","B","C","OPT"),cex=0.7,pos=3)

abline(a=r_f,b=(r_onef-r_f)/std_onef,col="green")
```



The above plot contains the minimum variance portfolio (red) and the 18%-target-return portfolio (blue) from parts a) and b) respectively. Also we can see that the optimal portfolio can be found from the tangency line of the efficient frontier.