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INVITED-TALK

Two Views of ℙ3

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06 January 2026
Total Citations: 0
Total Downloads: 78

Published: 24 July 2023

Citation in BibTeX format

ISSAC 2023: International Symposium on Symbolic and Algebraic Computation 2023
July 24 - 27, 2023
Tromsø, Norway

ISSAC '23: Proceedings of the 2023 International Symposium on Symbolic and Algebraic Computation (July 2023)
https://doi.org/10.1145/3597066.3597146
ISBN: 9798400700392

Two Views of $\mathbb{P}^3$

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ABSTRACT

The reconstruction of a 3-dimensional scene from (noisy) camera images is a routine task in computer vision. This problem encompasses a number of interesting mathematical questions with deep and old roots in projective geometry, which makes them amenable to tools from algebraic geometry. In this article I will address two foundational questions that revolve around the existence of a reconstruction from two camera images. The results involve real and semialgebraic geometry, as well as classical algebraic geometry and invariant theory. This article is a written account of my talk at ISSAC 2023.

KEYWORDS

computer vision, multiview geometry, projective reconstruction, chiral reconstruction, cubic surfaces

ACM Reference Format:
Rekha R. Thomas. 2023. Two Views of $\mathbb{P}^3$. In *International Symposium on Symbolic and Algebraic Computation 2023 (ISSAC 2023), July 24–27, 2023, Tromsø, Norway*. ACM, New York, NY, USA, 6 pages. https://doi.org/10.1145/3597066.3597146

1 INTRODUCTION

The modern camera evolved from the *camera obscura*, commonly called the *pinhole camera*. This a dark box with a hole at the center of one side through which a well-lit scene is projected onto the back wall of the box, creating an image that is both laterally and vertically inverted. The concept of the pinhole can be found in ancient Chinese and Greek literature, and the first physical model is said to have been built during the Renaissance.

Physically, the pinhole camera images a *world point* in $\mathbb{R}^3$ to its *image* on the camera plane, a copy of $\mathbb{R}^2$. This image is the intersection of the camera plane with the line connecting the world point and the camera center. Mathematically, the pinhole camera, or more generally a *projective camera*, is a rational linear map from $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ that sends a world point $q \mapsto Aq$ where A is a rank 3 real matrix of size 3×4 defined up to scale. Here $\mathbb{P}$ stands for the real projective line. Identifying the map with A, the point $x \sim Aq$ is called the image of q in the camera A, and the notation $Aq \sim x$ means that Aq and x are the same point in $\mathbb{P}^2$. The only input at which the map is not defined is the unique $c \in \mathbb{P}^3$ such that $Ac = 0$. The point c is the *center* of the camera A.

The simplest images are points, and a *3-dimensional reconstruction* of it is a set of world points and cameras such that the image of the ith world point in the jth camera is the image point labeled i in the jth camera plane (view). In this article we will be concerned with reconstructions from just two camera images which is already rather rich and interesting. The aim is to give a high level exposition of recent results on two view image reconstruction. Details can be found in the sources referenced.

The problem of image reconstruction falls under the umbrella of *multiview geometry* in computer vision. We refer the reader to the classic book [4] for the basics of multiview geometry.

2 PROJECTIVE RECONSTRUCTION

Projective reconstructions are defined and studied in Chapters 9 and 10 of [4]. Given a set of points in one camera plane, a set $\{x_i\} \subset \mathbb{P}^2$, it is easy to find a camera A and world points $\{q_i\} \subset \mathbb{P}^3$ such that $Aq_i \sim x_i$ for all i. However, given a collection of matched points in two camera planes, $\{(x_i, y_i)\} \subset \mathbb{P}^2 \times \mathbb{P}^2$, it becomes non-trivial to decide whether there exits a pair of cameras $A_1, A_2 \in \mathbb{P}(\mathbb{R}^{3\times 4})$ and world points $\{q_i\} \subset \mathbb{P}^3$ such that $A_1 q_i \sim x_i$ and $A_2 q_i \sim y_i$ for each i.

Question: Existence of a projective reconstruction:
Given $\{(x_i, y_i)\}_{i=1}^k \subset \mathbb{P}^2 \times \mathbb{P}^2$ does there exist two projective cameras $A_1, A_2 \in \mathbb{P}(\mathbb{R}^{3\times 4})$ and k world points $q_i \in \mathbb{P}^3$ such that $A_1 q_i \sim x_i$ and $A_2 q_i \sim y_i$ for each i?

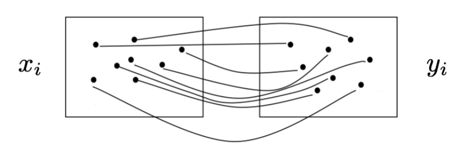

Figure 1: Do these point correspondences have a projective reconstruction?

Suppose $\{(x_i, y_i)\}_{i=1}^k \subset \mathbb{P}^2 \times \mathbb{P}^2$ has a projective reconstruction with cameras A_1, A_2 and world point $\{q_i\}_{i=1}^k \subset \mathbb{P}^3$. Then a first observation is that this reconstruction is unique only up to a projective transformation. Indeed, for each $H \in \mathrm{PGL}(4)$, since $A_i q_j = A_i H^{-1} H q_j$, the cameras $A_1 H^{-1}, A_2 H^{-1}$ and the world points $\{Hq_i\}_{i=1}^k$ provide another projective reconstruction of the images.

If (x, y) constitute the images of q in cameras A_1 and A_2, then

$$A_1 q \sim x, \qquad A_2 q \sim y, \tag{1}$$

or equivalently, there exists scalars $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$A_1 q = \lambda_1 x, \qquad A_2 q = \lambda_2 y \tag{2}$$

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ISSAC 2023, July 24–27, 2023, Tromsø, Norway
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ACM ISBN 979-8-4007-0039-2/23/07.
https://doi.org/10.1145/3597066.3597146

which can be written in matrix notation as

$$\begin{bmatrix} A_1 & x & 0 \\ A_2 & 0 & y \end{bmatrix} \begin{pmatrix} q \\ -\lambda_1 \\ -\lambda_2 \end{pmatrix} = 0. \tag{3}$$

The matrix in (3) has size 6×6 and so (x, y) has a reconstruction only if the matrix has a non-trivial nullspace or equivalently,

$$\det \begin{bmatrix} A_1 & x & 0 \\ A_2 & 0 & y \end{bmatrix} = 0. \tag{4}$$

The above determinant is a bilinear form in x and y, and so there exists a matrix $F \in \mathbb{R}^{3\times 3}$ such that (4) is

$$y^\top F x = 0. \tag{5}$$

The matrix F has rank 2 and is called a *fundamental matrix* of the image pair (x, y). It is equivalent to knowing the cameras A_1, A_2 in the sense that F can be computed from A_1, A_2 and conversely, F gives rise to two cameras A_1, A_2 up to projective equivalence [4, Chapter 9]. Running over all image pairs, if $\{(x_i, y_i)\}_{i=1}^k$ has a projective reconstruction, then there exists a $F \in \mathbb{P}(\mathbb{R}^{3\times 3})$, $\text{rank}(F) = 2$, such that

$$y_i^\top F x_i = 0 \ \forall \ i = 1, \ldots, k. \tag{6}$$

These equations are called the *epipolar equations* of $\{(x_i, y_i)\}_{i=1}^k$ and can be rewritten as

$$\underbrace{\begin{pmatrix} x_1^\top \otimes y_1^\top \\ x_2^\top \otimes y_2^\top \\ \vdots \\ x_k^\top \otimes y_k^\top \end{pmatrix}}_{Z_k} \text{vec}(F) = 0. \tag{7}$$

Here $\text{vec}(F) \in \mathbb{R}^9$ is the vectorization of F obtained by concatenating the columns of F and Z_k is the $k \times 9$ matrix whose rows are $x_i^\top \otimes y_i^\top = (x_{i1}y_{i1}, x_{i1}y_{i2}, x_{i1}y_{i3}, x_{i2}y_{i1}, \ldots, x_{i3}y_{i3})$. We may assume that the image points are finite which allows to fix $x_{i3} = y_{i3} = 1$ for all i which simplifies the structure of Z_k. To summarize, a necessary condition for $\{(x_i, y_i)\}_{i=1}^k$ to have a projective reconstruction is the existence of a rank two $F \in \mathbb{P}(\mathbb{R}^{3\times 3})$ such that $Z_k \text{vec}(F) = 0$.

The converse is almost true. A $F \in \mathbb{P}(\mathbb{R}^{3\times 3})$ of rank 2 is (x, y)-regular if $Fx = 0$ if and only if $y^\top F = 0$. Equivalently, either x and y generate the right and left nullspaces of F or neither generates a nullspace of F.

Theorem 2.1 (Projective Reconstruction Theorem). *[6] There exists a projective reconstruction of $\{(x_i, y_i)\}_{i=1}^k$ if and only if there exists a $F \in \mathbb{P}(\mathbb{R}^{3\times 3})$ of rank 2 satisfying the epipolar equations (6) that is (x_i, y_i)-regular for all $i = 1, \ldots, k$.*

The two view projective reconstruction pipeline is therefore, as follows:

(1) Given (finite) points $\{(x_i, y_i)\}_{i=1}^k \subset \mathbb{P}^2 \times \mathbb{P}^2$, solve $Z_k u = 0$.
(2) If a solution u exists, then form the matrix $U \in \mathbb{R}^{3\times 3}$ such that $u = \text{vec}(U)$.
(3) If $\text{rank}(U) = 2$ then $U = F$ is a fundamental matrix. Use it to compute two cameras A_1, A_2 (up to the action of PGL(4)).
(4) From the cameras, find q_i such that $A_1 q_i \sim x_i$ and $A_2 q_i \sim y_i$ for $i = 1, \ldots, k$.

We refer the reader to [4] for steps (3) and (4). From steps (1) and (2) we see that the existence of a reconstruction hinges crucially on the existence of a solution to $Z_k u = 0$ such that U is a real matrix of rank 2. Generically, $\text{rank}(Z_k) = k$ and so its nullspace $\mathcal{N}(Z_k) \cong \mathbb{P}^{8-k} \subset \mathbb{P}^8 \cong \mathbb{P}(\mathbb{R}^{3\times 3})$. When $k = 7$, $\mathcal{N}(Z_k) \cong \mathbb{P}^1$ will generically intersect the cubic hypersurface $Det = \{U \in \mathbb{P}(\mathbb{R}^{3\times 3}) : \det(U) = 0\}$ in 3 points of which at least one is real. If one of the real solutions has rank 2 then we have a fundamental matrix F. The standard algorithm in vision for finding F is an algebraic relaxation of the above requirements, that ignores the rank condition.

The 7-point Algorithm
• Input 7 image correspondences $\{(x_i, y_i)\}_{i=1}^7$. Compute Z_7.   • Find a basis $\{A, B\} \subset \mathbb{R}^{3\times 3}$ of $\mathcal{N}(Z_7)$.   • If either A or B have rank 2, output the one of rank 2.   • Else output a real point in $\{A + tB : t \in \mathbb{R}\} \cap Det$.

Since the line $\mathcal{N}(Z_7)$ generically intersects Det in three real solutions of rank 2, the common belief is that the 7-point algorithm will always find a fundamental matrix. However, this is not always true. In [1] the question of when $\mathcal{N}(Z_k)$ contains a real matrix of rank 2 was investigated, and a high level summary of the answers is as follows.

Theorem 2.2. *[1]*

(1) *Given $\{(x_i, y_i)\}_{i=1}^k$, there is always a real matrix of rank 2 in $\mathcal{N}(Z_k)$ if $k \leq 5$.*
(2) *When $k \geq 6$ there may be no real matrices of rank 2 in $\mathcal{N}(Z_k)$.*

Thus the 7-point algorithm does not always succeed in finding a fundamental matrix for the input points. An example of 7 point correspondences for which there this no fundamental matrix can be seen in Figure 2.

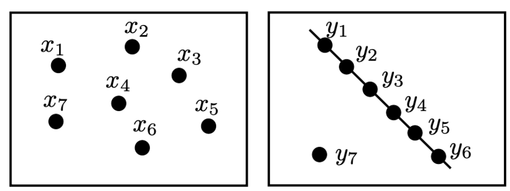

Figure 2: Point pairs with no fundamental matrix [1].

By Theorem 2.1, we need a bit more than a fundamental matrix to reconstruct image correspondences. However, the existence of a fundamental matrix is the key step and Theorem 2.2 addresses that. [1] studies the existence of a fundamental matrix in detail. The key tool in that paper is the following lemma.

Lemma 2.3. *Let $\mathcal{L} \subset \mathbb{P}(\mathbb{R}^{3\times 3})$ be a positive dimensional subspace with at least one matrix of rank 3. If the determinant restricted to $\mathcal{L}$ is not a power of a linear form then $\mathcal{L}$ contains a matrix of rank 2.*

The proofs in Theorem 2.2 have two difficult requirements – a real F that is also of rank 2. Establishing this is relatively straightforward when $k \leq 4$. The tricky case is $k = 5$, where the above lemma can be used to show that $\mathcal{N}(Z_5)$ always has a real matrix of rank 2.

3 CHIRAL RECONSTRUCTION

A projective reconstruction of a collection of image correspondences is a solution to the algebraic image formation equations $A_1 q_i \sim x_i, A_2 q_i \sim y_i$ which is oblivious to directionality, and carries no mandate that the reconstructed points must be visible in the cameras. Therefore, it may reconstruct some of the world points to be in front of the cameras and others behind. Since cameras cannot image points that are behind them, a meaningful reconstruction needs to obey physical laws which impose semialgebraic restrictions. A *chiral reconstruction* is a projective reconstruction in which the world points are required to be *in front of* the cameras.

Question: Existence of a chiral reconstruction:
Given $\{(x_i, y_i)\}_{i=1}^k \subset \mathbb{P}^2 \times \mathbb{P}^2$ does there exist two projective cameras $A_1, A_2 \in \mathbb{P}(\mathbb{R}^{3\times 4})$ and k world points $q_i \subset \mathbb{P}^3$ such that $A_1 q_i \sim x_i$ and $A_2 q_i \sim y_i$ for each i, and the world points lie *in front* of the two cameras?

The study of chirality was initiated in [3] with follow up work by Werner, Pajdla and others [8–11]. These resuts were revisited and extended in the recent papers [2] and [7]. We now briefly explain how chirality is modeled.

3.1 The notion of projective depth

Consider $L_\infty = \{q \in \mathbb{P}^3 : n_\infty^\top q = 0\}$, the hyperplane at infinity in $\mathbb{P}^3$, as an oriented hyperplane in $\mathbb{R}^4$ with fixed normal $n_\infty = (0, 0, 0, 1)^\top$. A camera A is said to be *finite* if its center c is a finite point in $\mathbb{P}^3$, i.e., $c \notin L_\infty$. This is equivalent to saying that the camera matrix can be partitioned as $A = \begin{bmatrix} G & t \end{bmatrix}$ where $\det(G) \neq 0$. It is known that a collection of point pairs has a projective reconstruction if and only if it has a reconstruction with finite cameras and world points [6, Theorem 3.1], and so we will restrict our attention to finite reconstructions. Note that if a world point q is finite, then $n_\infty^\top q \neq 0$.

The *principal plane* of camera $A = \begin{bmatrix} G & t \end{bmatrix}$ is the hyperplane $L_A := \{q \in \mathbb{P}^3 : A_{3,\bullet} q = 0\}$, where $A_{3,\bullet}$ is the third row of A. The plane L_A consists of those points in $\mathbb{P}^3$ that image to infinite points in $\mathbb{P}^2$. The camera center c lies on L_A. We regard L_A as an oriented hyperplane in $\mathbb{R}^4$ with normal vector $n_A := \det(G) A_{3\bullet}^\top$, called the *principal ray* of A. Multiplying by $\det(G)$ ensures that the direction of the principal ray remains invariant under scaling A. We now explain how this invariant direction distinguishes the *front* of the camera, see [4, Chapter 6.2.3].

The physical intuition for the *front of the camera* is best gotten from the dehomogenized setting where we work with $\tilde{q} \in \mathbb{R}^3$ instead of $q = (\tilde{q}^\top, 1) \in \mathbb{P}^3$. Since the principal plane had the equation $\det(G) A_{3\bullet} q = 0$, its dehomogenization has the equation:

$$\det(G) G_{3,\bullet} \tilde{q} = -\det(G) t_3, \tag{8}$$

and so, $\tilde{n}_A := \det(G) G_{3,\bullet}$ is the dehomogenized principal ray. Suppose $q = (\tilde{q}, 1)$ is imaged by A to $x = (x_1, x_2, 1)$. Then the image formation equation says that $Aq = \lambda x$ for some scalar λ. Using the fact that $A_{3,\bullet} c = 0$ and that $x_3 = 1$, the last components on either side of $Aq = \lambda x$ say that

$$\lambda = A_{3,\bullet} q = A_{3,\bullet}(q - c). \tag{9}$$

On the other hand, since the last coordinates of both q and c are 1, (9) implies that

$$\lambda = G_{3,\bullet}(\tilde{q} - \tilde{c}) \tag{10}$$

where $c = (\tilde{c}, 1)$. Therefore,

$$\frac{\det(G)}{|\det(G)| \|G_{3,\bullet}\|} \lambda = \frac{\det(G) G_{3,\bullet}}{|\det(G)| \|G_{3,\bullet}\|} (\tilde{q} - \tilde{c}). \tag{11}$$

The expression

$$\frac{\det(G) G_{3,\bullet}}{|\det(G)| \|G_{3,\bullet}\|} \tag{12}$$

is the unit vector in the direction of the dehomogenized principal ray $\tilde{n}_A$ and so the expression on the right hand side of (11) is the projection of $\tilde{q} - \tilde{c}$, the line segment from $\tilde{c}$ to $\tilde{q}$, onto the dehomogenized principal ray, see Figure 3. Using this interpretation, the expression on the right hand side of (11) is the *depth* of q in A, and we would like to say that if it is positive then $\tilde{q}$ is in front of the camera, and if it is negative then $\tilde{q}$ is behind the camera.

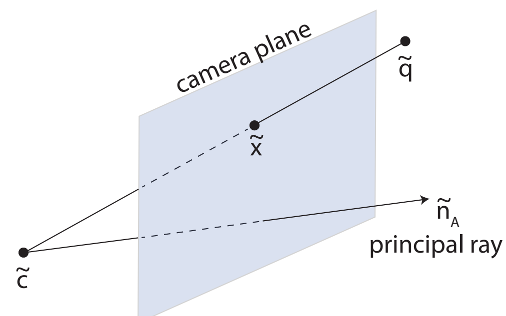

Figure 3: The front of the camera is distinguished by the direction of the dehomogenized principal ray $\tilde{n}_A$.

However, since we are working in projective space we need to make sure that the sign of depth is unchanged under scalings of A and q. Using (9) the right hand side of (11) is

$$\frac{\det(G)}{|\det(G)| \|G_{3,\bullet}\|} \lambda = \frac{\det(G) A_{3,\bullet} q}{|\det(G)| \|G_{3,\bullet}\|} = \frac{n_A^\top q}{|\det(G)| \|G_{3,\bullet}\|}. \tag{13}$$

The last expression is invariant under scaling A. To make it invariant under scaling q we can divide by $q_4 = n_\infty^\top q$ and define the *depth of a projective point q in the projective camera A* as

$$\operatorname{depth}(q; A) := \left(\frac{1}{|\det(G)| \|G_{3,\bullet}\|} \right) \frac{(n_A^\top q)}{(n_\infty^\top q)}. \tag{14}$$

Note that $\operatorname{depth}(q; A)$ is zero if and only if $n_A^\top q = 0$ which happens if and only if q lies on the principal plane L_A. Otherwise, $n_A^\top q \neq 0$ and $\operatorname{sign}(\operatorname{depth}(q; A)) = \operatorname{sign}((n_A^\top q)(n_\infty^\top q))$ is either positive or negative. This sign is the *chirality* of q in A. We declare that q is *in front of* the camera A if $\operatorname{depth}(q; A) > 0$ or equivalently, q has chirality 1 with respect to A.

3.2 Back to chiral reconstructions

The notion of chirality was introduced in [3] where Hartley considered two cameras as we are doing in this paper. This was extended to multiple cameras in [2] where the *chiral domain* of a camera arrangement was defined. For two cameras the definition is as follows.

Definition 3.1. [2] Let (A_1, A_2) be a pair of finite projective cameras. Then the *chiral domain* of (A_1, A_2), is the Euclidean closure in $\mathbb{P}^3$ of the set

$$\{q \in \mathbb{P}^3 \mid q \text{ finite},\ \operatorname{depth}(q, A_1) > 0,\ \operatorname{depth}(q, A_2) > 0\}.$$

Taking the Euclidean closure of the chiral domain of (A_1, A_2), leads to the following formal definition.

Definition 3.2. A *chiral reconstruction* of $\{(x_i, y_i)\}_{i=1}^k$ is a projective reconstruction $(A_1, A_2, \{q_1, \ldots, q_k\})$ of the point correspondences with finite non-coincident cameras such that for all i,

$$(n_\infty^\top q_i)(n_1^\top q_i) \geq 0,\ (n_\infty^\top q_i)(n_2^\top q_i) \geq 0,\ (n_1^\top q_i)(n_2^\top q_i) \geq 0 \tag{15}$$

where $n_\infty = (0, 0, 0, 1)^\top$ and n_i is the principal ray of camera A_i.

The chiral domain of (A_1, A_2) is the part of $\mathbb{P}^3$ that the cameras can image. Therefore a chiral reconstruction must find world points in the chiral domain. Armed with these tools, we can now answer the existence question for a chiral reconstruction of k point pairs. The main results from [7] are summarized below at a high level.

Theorem 3.3. *[7] Suppose we are given a collection of image correspondences $\{(x_i, y_i)\}_{i=1}^k$.*

1. *If $k \leq 3$ then there is always has a chiral reconstruction.*
2. *If $k = 4$ then there is a chiral reconstruction unless the configurations are of two specific non-generic types, in which case a chiral reconstruction may not exist.*
3. *If $k \geq 5$, then a chiral reconstruction may fail to exist with positive probability (in particular, even if the point correspondences are in general position).*

In the case of 4 point pairs, there are 6 possible combinatorial types as you see in Figures 4 and 5 taken from [7]. Those in Figure 4 always admit a chiral reconstruction while those in Figure 5 may fail to have a chiral reconstruction. The distinction is based on the comparative geometry of the two images. If the points in each image are arranged in the columns of two 4×4 matrices and the ranks of these matrices coincide, then there is always a chiral reconstruction [7, Theorems 5.13 and 5.14].

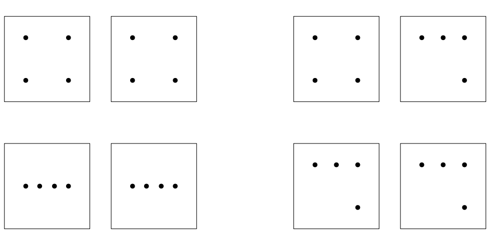

Figure 4: Pairs of 4 point correspondences that always have a chiral reconstruction.

Figure 5: Pairs of 4 point correspondences that may fail to have a chiral reconstruction.

4 CUBIC SURFACES AND SCHLÄFLI DOUBLE SIXES

We now look more closely at the case of 5 point pairs $\{(x_i, y_i)\}_{i=1}^5$, from the previous section. Generically, $\operatorname{rank}(Z_5) = 5$ and so $\mathcal{N}(Z_5) \cong \mathbb{P}^3 \subset \mathbb{P}^8$. This means that $\mathcal{N}(Z_5) \cap Det =: S$ is a cubic surface in $\mathcal{N}(Z_5) \cong \mathbb{P}^3$, and for generic input, S misses the singular locus of Det which is the variety of rank one matrices. Therefore, for generic input, all points on S are fundamental matrices for $\{(x_i, y_i)\}_{i=1}^5$. A determinantal representation of a cubic surface in $\mathbb{P}^3$ is a matrix pencil $M(z) := M_1 z_1 + M_2 z_2 + M_3 z_3 + M_4 z_4$ such that the defining equation of the surface is $\det(M(z)) = 0$. Our cubic surface S has a natural determinantal representation since $S = \mathcal{N}(Z_5) \cap Det$ which means that we can choose $\{M_1, \ldots, M_4\}$ to be a basis of $\mathcal{N}(Z_5)$.

It is a classical result in algebraic geometry that every smooth cubic surface contains 27 complex lines. We will now produce 12 real lines on the cubic surface S from our input points $\{(x_i, y_i)\}_{i=1}^5$, and eventually, 27 real lines. For each x_i and y_i define

$$\ell_{x_i} = \{M \in \mathcal{N}(Z_5) \ :\ M x_i = 0\}, \tag{16}$$

$$\ell'_{y_j} := \{M \in \mathcal{N}(Z_5) \ :\ y_i^\top M = 0\}. \tag{17}$$

By definition $\ell_{x_i} \subset \mathcal{N}(Z_5)$ and since all matrices in it have a non-trivial right nullspace, $\ell_{x_i} \subset Det$, and so $\ell_{x_i} \subset S$. There are 3 conditions on ℓ_{x_i} from $M x_i = 0$ and 5 from $y_j^\top M x_j = 0$ for $j = 1, \ldots, 5$, but one of these 5 is redunant since if $M x_i = 0$, then $y_i^\top M x_i = 0$. Therefore, there are 7 conditions in all and ℓ_{x_i} is a line on S. This argument shows that $\{\ell_{x_i}\}$ and $\{\ell_{y'_j}\}$ are two sets of 5 lines contained in S. Further, these lines are all real.

It turns out that there is a sixth pair of lines that come along for the ride. The Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$ identifies $\mathbb{P}^2 \times \mathbb{P}^2$ with the set of all rank 1 matrices in $\mathbb{P}(\mathbb{R}^{3\times 3})$ via $(u, v) \mapsto u^\top \otimes v^\top$. This is a variety of dimension 4 and degree 6 in $\mathbb{P}^8$. Therefore, generically, the subspace

$$\mathcal{L} = \operatorname{span}\{x_i^\top \otimes y_i^\top \ :\ i = 1, \ldots, 5\} \cong \mathbb{P}^4 \subset \mathbb{P}^8$$

intersects $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ in 6 points. Five of these intersection points are $x_i^\top \otimes y_i^\top$ for $i = 1, \ldots, 5$. If we denote the 6th point as $x_6^\top \otimes y_6^\top$, then ℓ_{x_6} and ℓ'_{y_6} are also lines on S. Summarizing, we have produced two sets of 6 real lines $\{\ell_{x_i}\}_{i=1}^6$ and $\{\ell'_{y_j}\}_{j=1}^6$ on S.

Definition 4.1. Two sets of 6 lines $\{\ell_i\}_{i=1}^6$ and $\{\ell'_j\}_{j=1}^6$ in $\mathbb{P}^3$ form a *Schläfli double six* if (i) the lines in each set are mutually skew, and (ii) ℓ_i intersects ℓ'_j if and only if $i \neq j$.

The following result was shown in [7].

Lemma 4.2. *The two sets of 6 real lines $\{\ell_{x_i}\}_{i=1}^6$ and $\{\ell'_{y_j}\}_{j=1}^6$ form a Schläfli double six on the cubic surface $S = \mathcal{N}(Z_5) \cap Det$.*

There is a great deal of combinatorics underlying the lines on a smooth cubic surface. In particular, every smooth cubic surface contains 36 Schläfli double sixes and a Schläfli double six uniquely determines the cubic surface on which it lies. See [5, Chapter 5] for more information.

We now look back at the inequalities (15) that need to be satisfied by a chiral reconstruction of our input $\{(x_i, y_i)\}_{i=1}^5$. These inequalities, when translated from conditions on $q_i, n_i, n_\infty \in \mathbb{P}^3$ to fundamental matrices F that live in $\mathbb{P}(\mathbb{R}^{3\times 3})$, take the form

$$g_i(F)g_j(F) \geq 0, \forall\, 1 \leq i < j \leq 5 \tag{18}$$

where $g_i g_j$ are polynomials of degree 6. So the fundamental matrices that correspond to chiral reconstructions are the points on S that satisfy the 10 semialgebraic conditions in (18). At first sight this seems daunting because the polynomials in (18) have high degree and the regions they cut out are on a cubic surface. However, [7] makes the crucial observation that each g_i changes sign exactly as F crosses ℓ_{x_i} or ℓ'_{y_i}. Therefore, the F's that correspond to chiral reconstructions of the input lie in certain cells in the decomposition of S created by the line configuration $(\{\ell_{x_i}\}_{i=1}^5, \{\ell'_{y_j}\}_{j=1}^5)$. Note that the 6th lines are not needed; indeed, they were not part of the input. We call the union of these special cells, the *chiral region* of the input point pairs. In [7] there are additional results on how to find these chiral cells. It suffices to check the 20 intersection points of the lines in $(\{\ell_{x_i}\}_{i=1}^5, \{\ell'_{y_j}\}_{j=1}^5)$.

Another classical fact in algebraic geometry is that every smooth cubic surface is the blow up of $\mathbb{P}^2$ in 6 generic points, i.e., points where no 3 are collinear and all 6 are not on a conic. The blow up construction is done as follows. Given 6 generic points $p_1, \dots, p_6 \in \mathbb{P}^2$, consider the vector space of homogeneous cubic polynomials in the coordinates $u = (u_1 : u_2 : u_3)$ of $\mathbb{P}^2$ that vanish on the 6 points. This vector space has dimension 4 and let c_1, c_2, c_3, c_4 be a basis. Then the rational map

$$\begin{aligned} \mathbb{P}^2 &\dashrightarrow \mathbb{P}^3 \\ u &\mapsto (c_1(u) : c_2(u) : c_3(u) : c_4(u)) \end{aligned} \tag{19}$$

blows up $\mathbb{P}^2$ at $p_1, \dots, p_6$ and its image is a smooth cubic surface $\mathcal{S}$ that is birationally equivalent to $\mathbb{P}^2$. This map is defined everywhere except at $p_1, \dots, p_6$ and the exceptional curves of the map form 6 skew lines $\{\ell_i\}$ on the cubic surface $\mathcal{S}$. This means that ℓ_i is the preimage of p_i under the blow down morphism from $\mathcal{S}$ to $\mathbb{P}^2$. The lines $\{\ell_i\}$ are one half of a Schläfli double six. The 6 lines $\{\ell'_j\}$ in the other half of the double six blow down to 6 conics in $\mathbb{P}^2$ with ℓ'_j going to the conic C_j through $\{p_1, \dots, p_6\} \setminus \{p_j\}$. The remaining 15 lines on the cubic surface are the preimages under the blow down, of the lines through pairs of points p_i and p_j.

We now explain how the chiral region of $\{(x_i, y_i)\}_{i=1}^5$, on the cubic surface $S = \mathcal{N}(Z_5) \cap Det$, can be visualized in the input space. Denote the $\mathbb{P}^2$ containing the x_i's as $\mathbb{P}^2_x$ and the $\mathbb{P}^2$ containing the y_i's as $\mathbb{P}^2_y$. For $F \in S$, let x and y be the unique projective points such that $Fx = 0$ and $y^\top F = 0$. Then the maps that send $F \mapsto x \in \mathbb{P}^2_x$ and $F \mapsto y \in \mathbb{P}^2_y$ are morphisms from S to the two camera planes.

$$\begin{array}{ccc} & S & \\ \swarrow & & \searrow \\ \mathbb{P}^2_x & & \mathbb{P}^2_y \end{array}$$

Recall that the line configuration $\{\ell_{x_i}\}_{i=1}^6, \{\ell'_{y_j}\}_{j=1}^6$ forms a Schläfli double six on S. Further, $\ell_{x_i} \mapsto x_i$ and $\ell'_{y_j} \mapsto y_j$ under the above mentioned morphisms. Therefore, the cubic surface S is the blow up of $\mathbb{P}^2_x$ at $x_1, \dots, x_6$ and also the blow up of $\mathbb{P}^2_y$ at $y_1, \dots, y_6$, and the maps $F \mapsto x \in \mathbb{P}^2_x$ and $F \mapsto y \in \mathbb{P}^2_y$ are the blow down maps from S to $\mathbb{P}^2_x$ and $\mathbb{P}^2_y$. Recall that every point $F \in S$ is a fundamental matrix. In vision, the unique x such that $Fx = 0$ and the unique y such that $y^\top F = 0$ are called the right and left *epipoles* of F.

Under the blow down to $\mathbb{P}^2_x$, ℓ'_{y_j} maps to the conic $C_j \subset \mathbb{P}^2_x$ that passes through all x_i except x_j. Similarly, ℓ_{x_i} maps to the conic $C'_i \subset \mathbb{P}^2_y$ that passes through all points y_j except y_i. These observations imply that the chiral regions, which are bounded by lines on S, are now bounded by conics in $\mathbb{P}^2_x \times \mathbb{P}^2_y$. Therefore, they can be visualized in $\mathbb{P}^2_x \times \mathbb{P}^2_y$ as regions of the allowed epipoles of the fundamental matrices that lead to a chiral reconstruction.

Example 4.3. [7] Consider the five point pairs $\{(x_i, y_i)\}_{i=1}^5$ where the x_i's are the columns of X and the y_i are the columns of Y shown below:

$$X = \begin{pmatrix} 0 & 0 & 4 & 2 & 2 \\ 0 & 4 & 0 & 1 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad Y = \begin{pmatrix} 2 & 2 & 4 & 0 & 4 \\ 1 & 3 & 0 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The shaded regions in Figures 6 and 7 constitute the chiral region in $\mathbb{P}^2_x \times \mathbb{P}^2_y$ obtained by blowing down from the cubic surface S. Notice that in Figure 6, the conic C_6 cuts through the chiral region. This does not pose a problem since this conic should be disregarded as the lines indexed by 6 do not contribute to the construction of the chiral region on S. The same comment applies also to Figure 7.

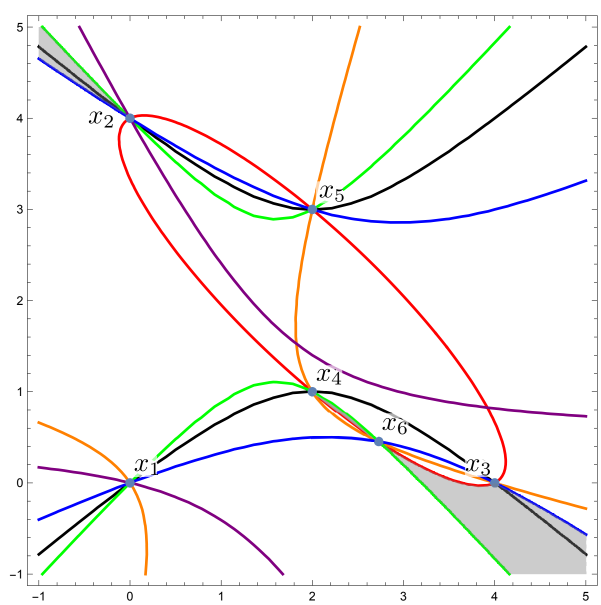

Figure 6: Shaded area represents region of allowed right epipoles for a chiral reconstruction.

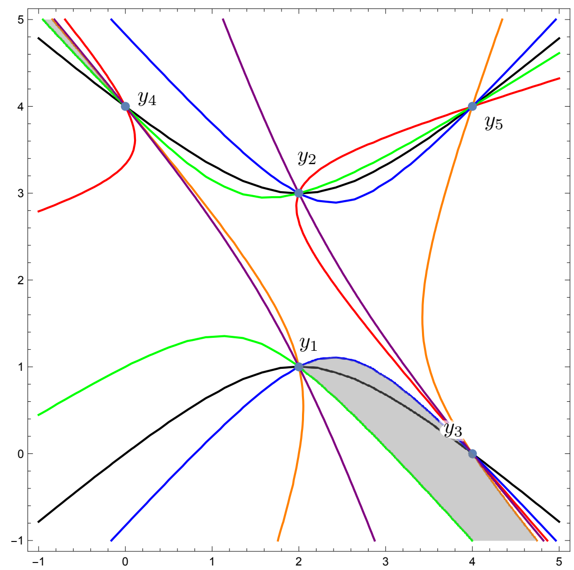

Figure 7: Shaded area represents region of allowed left epipoles for a chiral reconstruction.

As a concluding remark to this section we note that the cubic surface $S = \mathcal{N}(Z_5) \cap Det$ has 27 real lines. We already produced 12 real lines ℓ_{x_i} and ℓ'_{y_j}. The remaining 15 lines are the preimages of the lines through pairs of x points (or y points) under the blow up maps. There are 5 possible real topologies for cubic surfaces, those that come with 27, 15 or 7 real lines, and two types that come with 3 real lines. The cubic surfaces that arise as above from vision are always of the first type and have 27 real lines.

ACKNOWLEDGMENTS

The author thanks her long term collaborator, Sameer Agarwal, for all things computer vision related. She also thanks all her students and collaborators whose results are mentioned in this paper. Figure 2 is from [1]. Figures 4, 5, 6 and 7 are from [7], and Figure 3 was made by Catherine Babecki.

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