Understanding Analysis 2e Exercises

Chapter 1

The Real Numbers

1.2 The Axiom of Completeness

Exercise 1.2.1

(a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

Solution

(a) A real number s is the greatest lower bound for a set $A \subseteq \mathbf{R}$ if it meets the following two criteria:

s is a lower bound for A,

if b is any lower bound for A, $b \leq s$.

(b) Lemma: assume $s \in \mathbf{R}$ is a lower bound for a set $A \subseteq \mathbf{R}$. Then $s = \inf A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ such that $s + \epsilon > a$.

($\Rightarrow$) Assume that $s = \inf A$. Consider $s+\epsilon$. Because $s+\epsilon > s$, part (ii) of the definition of the infimum implies that $s + \epsilon$ is not a lower bound for A. Therefore there must be some $a \in A$ such that $a < s + \epsilon$.

($\Leftarrow$) Assume that s is a lower bound for A with the property that there exists $a \in A$ satisfying $s + \epsilon > a$ for all ϵ. This implies that any number greater than s cannot be a lower bound for A, and thus that any lower bound for A is less than or equal to s. Therefore s is the infimum of A.

Exercise 1.2.2

Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbf{Q}$ that contains its supremum but not its infimum.

Solution

(a) 2.

(b) This is impossible. All finite sets must contain both their infimum and their supremum; they are the minimum and maximum elements, respectively.

(c) $A = x \mid 1 < x \leq 2$. $\sup A = 2 \in A$, and $\inf A = 1 \notin A$.

Exercise 1.2.3

(a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbf{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Solution

(a) By definition of the infimum, all elements of B must be less than or equal to $\inf A = p$; therefore, p must be an upper bound on B. Since $p \in B$ (the infimum is a lower bound itself), p must also be the supremum of B. Therefore $p = \sup B = \inf A$.

(b) If a set A is lower bounded, the set B of all lower bounds for A is nonempty. We also know that B is upper bounded - any element of A can serve as an upper bound. Therefore B must have a supremum, and $\sup B = \inf A$ as follows from (a).

Exercise 1.2.4

Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup\left(\bigcup_{k=1}^{n} A_k\right)$.

(b) Consider $\sup\left(\bigcup_{k=1}^{\infty} A_k\right)$. Does the formula in (a) extend to the infinite case?

Solution

(a) $\sup(A_1 \cup A_2) = \max(\sup A_1, \sup A_2)$

$\sup\left(\bigcup_{i=1}^{n} A_k\right) = \max(\sup A_1, \ldots, \sup A_n)$

(b) Yes. If it can be shown that $a \in \mathbf{R}$ is the supremum for all A_k, then it is the supremum for $\bigcup_{k=1}^{\infty} A_k$.

Exercise 1.2.5

As in Example 1.3.7, let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solution

(a) Let $s = \sup A$. Now for any cx where $c \geq 0$ and $x \in A$, it follows from $s \geq x$ that $cs \geq cx$. Therefore cs is an upper bound for cA.

For contradiction, assume that there exists a value $\epsilon > 0$ such that $cs - \epsilon$ is an upper bound on cA. In this case, $cs - \epsilon \geq cx$, for all $x \in A$, and $s - \frac{\epsilon}{c} \geq x$. This would imply that there is a value less than s that is an upper bound for A; however, this is impossible because $s = \sup A$. Thus, $cs = c \sup A$ is the least upper bound for cA.

(b) It is trivial to see that, by flipping the inequalities from (a), we get $c \inf A = \sup cA$ for $c < 0$.

Exercise 1.2.6
Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$.

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.

(b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.

(c) Finally, show $\sup(A + B) = s + t$.

(d) Construct another proof of this same fact using Lemma 1.3.8.

Solution

(a) $\sup A + \sup B \geq a + b$ follows from the fact that $\sup A \geq a$ and $\sup B \geq b$ for all $a \in A$ and $b \in B$.

(b) u is an upper bound for $A + B$, so $u \geq a + b$ for all $a \in A$ and $b \in B$. Fixing a, we have $u - a \geq b$ for all $b \in B$. Thus, $u - a$ is an upper bound for B, and so $t = \sup B \leq u - a$.

(c) It has been shown that $s + t$ is an upper bound for $A + B$; what remains to be shown is that it is the least upper bound.

From (b), $t \leq u - a$, where u is any upper bound for $A + B$. So $a \leq u - t$, and since $u - t$ is an upper bound for A, $u - t \geq s$. Therefore $s \leq u - t$, and $s + t \leq u$ as desired.

Exercise 1.2.7
Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

Solution
a must be the least upper bound; if $a - \epsilon$ were to be an upper bound, it would be less than a itself and therefore not be an upper bound.

Exercise 1.2.8
Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(a) $\{m/n : m, n \in \mathbf{N} \text{ with } m < n\}$.

(b) $\{(-1)^m/n : m, n \in \mathbf{N}\}$.

(c) $\{n/(3n+1) : n \in \mathbf{N}\}$

(d) $\{m/(m+n) : m, n \in \mathbf{N}\}$

Solution

(a) $\inf A = 0$, $\sup A = 1$

(b) $\inf B = -1$, $\sup B = 1$

(c) $\inf C = \frac{1}{4}$, $\sup C = \frac{1}{3}$

(d) $\inf D = 0$, $\sup D = 1$

Exercise 1.2.9

(a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

Solution

(a) $\sup A < \sup B$ implies that there is an element $b \in B$ such that $b > \sup A$. If no such element existed, then $\sup A$ would be an upper bound for B, which is impossible as $\sup A < \sup B$.

(b) If $\sup A = \sup B$ and B does not include its supremum, then this does not hold. For example, $A = (0, 1)$, $B = (0, 1)$.

Exercise 1.2.10 (Cut Property)

The Cut Property of the real numbers is the following:

If A and B are nonempty, disjoint sets with $A \cup B = \mathbf{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbf{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.

(b) Show that the implication goes the other way; that is, assume $\mathbf{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.

(c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbf{R}$ is replaced by $\mathbf{Q}$.

Solution

(a) Since B is nonempty, choose any element as an upper bound for A. By the Axiom of Completeness, let $c = \sup A$. Notice that every $b \in B$ is an upper bound for A, so $c \leq b$ for all $b \in B$; by definition, $a \leq p$ for all $a \in A$.

(b) **NOT ORIGINAL**
Let B be the set of all upper bounds of E and let $A = \mathbf{R} \setminus B$. Thus A and B are disjoint and $A \cup B = \mathbf{R}$.

Assume, for contradiction, that $\sup E$ does not exist. Since B does not have a smallest element, $E \cap B = \emptyset$. Therefore $E \subseteq A$. By the Cut Property, there must exist a $c \in \mathbf{R}$ such that $a \leq c$ and $c \leq b$. Since B does not have a smallest element, $c \notin B$. However, since $E \subseteq A$, c is an upper bound for E and must be in B, which is a contradiction.

(c) Let $A, B \subset \mathbf{Q}$, and $A = (-\infty, \sqrt{2})$ and $B = (\sqrt{2}, +\infty)$. The Cut Property does not hold; there is no rational number c for which $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

Exercise 1.2.11
Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

(c) If there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

Solution
(a) True. $\sup B$ must be an upper bound on A. By definition, $\sup A \leq \sup B$.

(b) True. Any real number c such that $\sup A < c < \inf B$ will suffice.

(c) False. Consider $(0, 2)$ and $(2, 3)$. $a < 2 < b$ for all $a \in A$ and $b \in B$, but $2 \not< 2$.