Understanding Analysis 2e Exercises

Chapter 1

Sequences and Series

1.2 The Limit of a Sequence

Exercise 1.2.1
What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $N \in \mathbf{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$.

Give an example of a vercongent sequence. Is there an example of a vercongent sequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

Solution
A vercongent sequence is just any sequence that is bounded. A vercongent sequence can be divergent; one example would be $a_n = (-1)^n$. It is easily possible for a sequence to verconge to two different values, given that ϵ is chosen to be sufficiently large.

Exercise 1.2.2
Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) $\lim \frac{2n^2}{n^3+3} = 0$.

(c) $\lim \frac{\sin\left(n^2\right)}{\sqrt[3]{n}} = 0$.

Solution
(a) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbf{N}$ with

$$N > \frac{3}{25\epsilon} - \frac{4}{5}.$$

To verify that this choice is appopriate, let $n \in \mathbf{N}$ satisfy $n \geq N$. Then

$$\begin{aligned}
n > \frac{3}{25\epsilon} - \frac{4}{5} &\implies \epsilon > \frac{\frac{3}{5}}{5n+4} \\
&\implies \epsilon > \left|\frac{\frac{3}{5}}{5n+4}\right| \\
&\implies \epsilon > \left|\frac{2n+1}{5n+4} - \frac{2n+\frac{8}{5}}{5n+4}\right| \\
&\implies \epsilon > \left|\frac{2n+1}{5n+4} - \frac{2}{5}\right|
\end{aligned}$$

(b) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbf{N}$ with

$$N > \frac{2}{\epsilon}.$$

To verify that this choice of N is appropriate, let $n \in \mathbf{N}$ satisfy $n \geq N$. Then

$$\begin{aligned}
n > \frac{2}{\epsilon} &\implies \frac{\epsilon}{2} > \frac{n^2}{n^3} > \frac{n^2}{n^3+3} \\
&\implies \epsilon > \left|\frac{2n^2}{n^3+3}\right|
\end{aligned}$$

(c) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbf{N}$ with

$$N > \frac{1}{\epsilon^3}.$$

To verify that this choice of N is appropriate, let $n \in \mathbf{N}$ satisfy $n \geq N$. Then

$$n > \frac{1}{\epsilon^3} \implies \epsilon > \frac{1}{\sqrt[3]{n}} > \left|\frac{\sin n^2}{\sqrt[3]{n}}\right|$$

Exercise 1.2.3
Describe what we would have to demonstrate in order to disprove each of the following statements.

(a) At every college in the United States, there is a student who is at least seven feet tall.

(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.

(c) There exists a college in the United States where every student is at least six feet tall.

Solution
(a) There exists one college in the United States where every student is less than seven feet tall.

(b) There exists one college in the United States where every professor gave at least one student a C or below.

(c) For every college in the United States, there is at least one student who is less than six feet tall.

Exercise 1.2.4
Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

(c) A divergent sequence such that for every $n \in \mathbf{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution
(a) $a_n = (-1)^n$.

(b) Assume a sequence with infinite ones has limit $b \neq 1$. If we choose $\epsilon < |b-1|$, then $|a_n - b| < |b-1|$. Since a has infinite ones, regardless of how $N \in \mathbf{N}$ is chosen, $a_n = 1$ for some $n \geq N$. However, this implies $|1-b| < |b-1|$, which is clearly a contradiction. Therefore there cannot be a sequence with infinite ones that does not either diverge or converge to 1 itself.

(c) $1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, \ldots$

Exercise 1.2.5
Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. For each sequence, find $\lim a_n$ and verify it with the definition of convergence.

(a) $a_n = [[5/n]]$,

(b) $a_n = [[(12 + 4n)/3n]]$.

Reflecting on these examples, comment on the statement following Definition 2.2.3 that "the smaller the ϵ-neighborhood, the larger N may have to be."

Solution
(a) Choose $n > 5$. Then $\left|[[\frac{5}{n}]]\right|$ always evaluates to 0.

(b) Choose $n > 6$. Then $\left|[[\frac{12+4n}{3n}]]\right|$ always evaluates to 1.
This statement is only true if the sequence converges gradually. In some cases, sequences can "jump" to their limit.

Exercise 1.2.6
Theorem 2.2.7 (Uniqueness of Limits). *The limit of a sequence, when it exists, must be unique.*

Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$.

Solution
Assume, for contradiction, that $a \neq b$. Let $a > b$, without loss of generality. Then $|a-b| = a-b > 0$. By definition, there exists N_a, $N_b \in \mathbf{N}$ such that

$$|a_n - b| < \epsilon \text{ for all } n \geq N_b,$$

$$|a_n - a| < \epsilon \text{ for all } n \geq N_a.$$

These two conditions are simultaneously true for $n \geq N$, where $N = \max(N_a, N_b)$. Now set $\epsilon < \frac{a-b}{2}$. This implies

$$b - \frac{a-b}{2} < a_n < b + \frac{a-b}{2} \implies b - \frac{a-b}{2} < a_n < \frac{a+b}{2}$$

$$a - \frac{a-b}{2} < a_n < a + \frac{a-b}{2} \implies \frac{a+b}{2} < a_n < a + \frac{a-b}{2}$$

However, this is impossible, as a_n cannot be greater than $\frac{a+b}{2}$ and less than $\frac{a+b}{2}$ at the same time.

Exercise 1.2.7
Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbf{R}$ if there exists an $N \in \mathbf{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbf{R}$ if, for every $N \in \mathbf{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

(b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2 . Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

Solution

(a) Frequently.

(b) Eventually implies frequently, so eventually is the stronger statement.

(c) Given any ϵ-neighborhood, a_n is eventually in that neighborhood.

(d) (x_n) is not necessarily eventually in $(1.9, 2.1)$, but it is frequently in $(1.9, 2.1)$.

Exercise 1.2.8
For some additional practice with nested quantifiers, consider the following invented definition:

Let's call a sequence (x_n) zero-heavy if there exists $M \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$.

(a) Is the sequence $(0, 1, 0, 1, 0, 1, \ldots)$ zero heavy?

(b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide a counterexample.

(c) If a sequence contains an infinite number of zeros, is it necessarily zeroheavy? If not, provide a counterexample.

(d) Form the logical negation of the above definition. That is, complete the sentence: A sequence is not zero-heavy if ....

Solution

(a) Yes.

(b) Yes. Fix $M \in \mathbf{N}$ and consider the intervals $N = 0, N = M + 1, N = 2M + 2 \ldots$

These all identify unique instances of zero in the sequence because they are disjoint. Since there are an infinite number of these instances, there must also be an infinite number of zeroes.

(c) No. Consider the sequence $0, 1, 0, 1, 1, 0, 1, 1, 1, \ldots$

Regardless of how M is chosen, there will be an M-long sequence of 1s in the sequence, which means that it is not zero-heavy, even though it has an infinite number of zeroes.

(d) A sequence is not zero-heavy if for all $M \in \mathbf{N}$, there exists $N \in \mathbf{N}$ such that for all n satisfying $N \leq n \leq N + M$, $x_n \neq 0$.

1.3 The Algebraic and Order Limit Theorems

Exercise 1.3.1
Let $x_n \geq 0$ for all $n \in \mathbf{N}$.

(a) If $(x_n) \to 0$, show that $\left(\sqrt{x_n}\right) \to 0$.

(b) If $(x_n) \to x$, show that $\left(\sqrt{x_n}\right) \to \sqrt{x}$.

Solution
(a) Given some $\epsilon > 0$, choose N such that $x_n < \epsilon^2$ for all $n \geq N$. Then $\sqrt{x}_n < \epsilon$ for all $n \geq N$.

(b) Fix $\epsilon > 0$; now consider $\delta = \epsilon\sqrt{x}$. For some $N \in \mathbf{N}$, $|x_n - x| < \delta$ for all $n \geq N$, which implies

$$|\sqrt{x}_n - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x}_n + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}} < \frac{\delta}{\sqrt{x}}.$$

From this, we have $|\sqrt{x}_n - \sqrt{x}| < \epsilon$ for all $n \geq N$ as desired.

Exercise 1.3.2
Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

(a) $\left(\frac{2x_n - 1}{3}\right) \to 1$;

(b) $(1/x_n) \to 1/2$.

(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)

Solution
(a) Let $\delta = \frac{3}{2}\epsilon$. Then

$$\begin{aligned} |x_n - 2| < \delta &\implies \frac{2}{3}|x_n - 2| < \epsilon \\ &\implies \left|\frac{2x_n - 4}{3}\right| < \epsilon \\ &\implies \left|\frac{2x_n - 1}{3} - 1\right| < \epsilon \text{ for all } n \geq N, \text{ as desired.} \end{aligned}$$

(b) Note that

$$\left|\frac{1}{x_n} - \frac{1}{2}\right| = \left|\frac{x_n - 2}{2x_n}\right|.$$

Choose N large enough such that $x_n > 1$ and $|x_n - 2| < \epsilon$ for all $n \geq N$. Then

$$\left|\frac{x_n - 2}{2x_n}\right| < \frac{|x_n - 2|}{2} < \frac{\epsilon}{2} < \epsilon.$$

Exercise 1.3.3 (Squeeze Theorem)
Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbf{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Solution
For any $\epsilon > 0$,

$$\begin{aligned} |y_n - l| &\leq |z_n - l| < \epsilon \text{ if } y_n - l \geq 0 \\ |y_n - l| &\leq |x_n - l| < \epsilon \text{ otherwise.} \end{aligned}$$

Exercise 1.3.4
Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

(a) $\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2} \right)$

(b) $\lim \left(\frac{(a_n+2)^2-4}{a_n} \right)$

(c) $\lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5} \right)$

Solution
(a)

$$\begin{aligned} \lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2} \right) &= \frac{\lim (1+2a_n)}{\lim(1+3a_n-4a_n^2)} \\ &= \frac{\lim 1 + 2 \lim a_n}{\lim 1 + 3 \lim a_n - 4 \lim a_n^2} \\ &= 1 \end{aligned}$$

(b)

$$\begin{aligned} \lim \left(\frac{(a_n+2)^2-4}{a_n} \right) &= \lim \left(\frac{a_n^2+4a_n}{a_n} \right) \\ &= \lim (a_n + 4) \\ &= 4 \end{aligned}$$

(c)

$$\begin{aligned} \lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5} \right) &= \lim \left(\frac{2+3a_n}{1+5a_n} \right) \\ &= \frac{2 + 3 \lim a_n}{1 + 5 \lim a_n} \\ &= 2 \end{aligned}$$

Exercise 1.3.5
Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots, x_n, y_n, \ldots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Solution
First we prove the forward direction. Let $\lim x_n = \lim y_n = l$, and fix $\epsilon > 0$. Then there exists an $N_0 \in \mathbf{N}$ such that $|x_n - l| < \epsilon$ and $|y_n - l| < \epsilon$ for all $n \geq N_0$. Setting $N = 2N_0 - 1$, however, leaves us with the tail $(x_n, y_n, x_{n+1}, y_{n+1}, \ldots)$, which clearly satisfies $|z_n - l| < \epsilon$.

For the backward direction, fix ϵ. Since $|z_n - l| < \epsilon$ for all $n \geq N$, $|x_n - l| < \epsilon$ and $|y_n - l| < \epsilon$ for all $n \geq \frac{N}{2} + 1$.

Exercise 1.3.6
Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

Solution
REVISE

$$
\begin{aligned}
b_n &= n - \sqrt{n^2 - 2n} \\
&= \sqrt{n^2} - \sqrt{n^2 - 2n} \\
&= \frac{2n}{\sqrt{n^2} + \sqrt{n^2 - 2n}} \\
&= \frac{2}{1 + \sqrt{1 - \frac{2}{n}}} \\
\lim b_n &= \frac{2}{\lim 1 + \sqrt{1 - \frac{2}{n}}} \\
&= \frac{2}{1 + \lim\left(\sqrt{1 - \frac{2}{n}}\right)} \\
&= \frac{2}{1 + \sqrt{\lim\left(1 - \frac{2}{n}\right)}} \\
&= \frac{2}{1 + 1} \\
&= 1
\end{aligned}
$$

Exercise 1.3.7
Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;

(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;

(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;

(e) two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

Solution

(a) $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$.

(b) This is impossible. By the Algebraic Limit Theorem, $\lim(x_n + y_n) - \lim(x_n) = \lim(y_n)$, and if $x_n + y_n$ and x_n converge, then y_n must have a limit as well.

(c) $b_n = 1/n$. Here, $1/b_n = n$, which diverges.

(d) There exist M_0 and M_1 such that $|b_n| < M_0$ and $|a_n - b_n| < M_1$. Therefore $|a_n - b_n| + |b_n| < M_0 + M_1$, and by the triangle inequality,

$$|a_n - b_n + b_n| < |a_n + b_n| + |b_n| < M_0 + M_1$$

Thus a_n is bounded by $M_0 + M_1$, and an unbounded sequence that meets the criteria is impossible.

(e) $a_n = 0$ and $b_n = (-1)^n$.

Exercise 1.3.8

Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p(x_n) \to p(x)$.

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

Solution

(a) Every polynomial has the form $p(x) = a_k x_n^k + \ldots + a_0$.

$$\begin{aligned} \lim p(x_n) &= \lim a_k x_n^k + \ldots + \lim a_0 \\ &= a_k \lim x_n^k + \ldots + \lim a_0 \\ &= a_k x^k + \ldots + a_0 \\ &= p(x) \end{aligned}$$

(b) $x_n = 1/n$, and let $f(x)$ be a function where if $x \neq 0$, $f(x) = 2$ and $f(x) = 3$ otherwise. Then $(x_n) \to 0$, $f(0) = 3$, but $f(x_n)$ really converges to 2.

Exercise 1.3.9

(a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim (a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b ?

(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Solution

(a) We aren't allowed to use the Algebraic Limit Theorem because it assumes that a_n and b_n converge.

Given that $|a_n| < M$ and $(b_n) \to 0$, then $|a_n b_n| < M|b_n| < M\epsilon$.

(b) We can say that $a_n b_n$ is bounded, but we can't say anything about its convergence.

(c) Convergent sequences are also bounded, so b_n satisfies the conditions in part (a).

Exercise 1.3.10

Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.

(a) If $\lim (a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.

(b) If $(b_n) \to b$, then $|b_n| \to |b|$.

(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.

(d) If $(a_n) \to 0$ and $|b_n - b| \le a_n$ for all $n \in \mathbf{N}$, then $(b_n) \to b$.

Solution

(a) This is necessarily true only if a_n and b_n are convergent, by the Algebraic Limit Theorem. Otherwise, consider $a_n = b_n = (-1)^n$.

(b) True, since $||b_n| - |b|| \le |b_n - b| < \epsilon$.

(c) By the Algebraic Limit Theorem, $\lim b_n = \lim(b_n - a_n) + \lim a_n = a$.

(d) True, since $|b_n - b| \le a_n < \epsilon$ for $n \ge N$.

Exercise 1.3.11 (Cesaro Means)

(a) Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

Solution

(a) Fix some $\epsilon > 0$. First, choose N such that $|x_n - x| < \epsilon/2$ for all $n > N$. Let M be the sum of the first N terms of $|x_n - x|$. Now choose some X such that

$$\begin{aligned}
\frac{2M}{\epsilon} - N < X &\implies M - N\epsilon < \frac{X\epsilon}{2} \\
&\implies M + \frac{X\epsilon}{2} < (N+X)\epsilon \\
&\implies |x_1 - x| + \ldots + |x_{N+X} - x| < M + \frac{X\epsilon}{2} < (N+X)\epsilon \\
&\implies |\frac{x_1 + \ldots + x_{N+X}}{N+X} - x| < \epsilon
\end{aligned}$$

Essentially, X is the number of terms $|x_n - x| < \epsilon/2$ that we need to "drag" the mean back under ϵ.

Then it is easy to see that for all $n \geq N + X$, $|y_n - x| < \epsilon$.

(b) $a_n = (-1)^n$, where the sequence of means converges to 0.

Exercise 1.3.12

A typical task in analysis is to decipher whether a property possessed by every term in a convergent sequence is necessarily inherited by the limit. Assume $(a_n) \to a$, and determine the validity of each claim. Try to produce a counterexample for any that are false.

(a) If every a_n is an upper bound for a set B, then a is also an upper bound for B.

(b) If every a_n is in the complement of the interval $(0, 1)$, then a is also in the complement of $(0, 1)$.

(c) If every a_n is rational, then a is rational.

Solution

(a) Since B has at least one upper bound, it is bounded and therefore has a supremum. Call this supremum b. If every a_n is an upper bound for B, $a_n \geq b$ for all $n \in \mathbf{N}$. By the Order Limit Theorem, $\lim a_n = a \geq b$, so a is an upper bound on B.

(b) This is false. Consider $a_n = -\frac{1}{n}$, which is always negative and therefore in the complement of $(0, 1)$ for all finite n, but tends to 0.

(c) This is false. Consider decimal approximations of $\sqrt{2}$ of increasing length: they are all rational, but tend to an irrational number.

Exercise 1.3.13 (Iterated Limits)

Given a doubly indexed array a_{mn} where $m, n \in \mathbf{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?

(a) Let $a_{mn} = m/(m + n)$ and compute the iterated limits

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) \quad \text{and} \quad \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right)$$

Define $\lim_{m,n\to\infty} a_{mn} = a$ to mean that for all $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \epsilon$.

(b) Let $a_{mn} = 1/(m + n)$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exist? How do these three values compare? Answer these same questions for $a_{mn} = mn/\left(m^2 + n^2\right)$.

(c) Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists but where neither iterated limit can be computed.

(d) Assume $\lim_{m,n\to\infty} a_{mn} = a$, and assume that for each fixed $m \in \mathbf{N}$, $\lim_{n\to\infty}(a_{mn}) \to b_m$. Show $\lim_{m\to\infty} b_m = a$.

(e) Prove that if $\lim_{m,n\to\infty} a_{mn}$ exists and the iterated limits both exist, then all three limits must be equal.

Solution

(a)

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) = 1$$

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right) = 0$$

(b) For $a_m n = 1/(m+n)$, all three limits are zero. For $a_m n = mn/(m^2+n^2)$, the combined limit does not exist because $a_m n = \frac{1}{2}$ whenever $M = N$, whereas the iterated limits are zero.

(c) **NOT ORIGINAL**

$$a_m n = \frac{(-1)^n}{m} + \frac{(-1)^m}{n}.$$

(d) Fix $\epsilon > 0$.

We are given two conditions: first, for any m, there exists S_m (dependent on m) such that

$$|a_{mn} - b_m| < \epsilon/2 \text{ for } n \geq S_m.$$

Second, there exists S such that

$$|a_{mn} - a| < \epsilon/2 \text{ for } m, n \geq S.$$

This implies that, for all $m \geq S$, there exists S_m such that the first condition is true; then choose $n \geq \max(S_m, S)$, and we have shown that for any choice of $m \geq S$, there exists an n such that both conditions are true. Thus we can say

$$|b_m - a| \leq |b_m - a_{mn}| + |a_{mn} - a| < \epsilon \text{ for } m \geq S.$$

(e) **REVISE**

This was already shown in (d).

1.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Exercise 1.4.1

(a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.

Solution

(a) x_n is monotone decreasing; and because $x_n < 4$, x_n is always greater than zero, which means that it is bounded and therefore convergent.

(b) The sequences x_n and x_{n+1} are just the same sequence, only offset by one term.

(c)

$$\begin{aligned}\lim x_{n+1} = \lim \left(\frac{1}{4 - x_n} \right) &\implies s = \frac{1}{4-s} \\ &\implies s^2 - 4s + 1 = 0 \\ &\implies s = 2 \pm \sqrt{3}\end{aligned}$$

However, s cannot be $2+\sqrt{3}$ as $2+\sqrt{3} > 3$, so $s = 2 - \sqrt{3}$.

Exercise 1.4.2

(a) Consider the recursively defined sequence $y_1 = 1$

$$y_{n+1} = 3 - y_n$$

and set $y = \lim y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y, we conclude $\lim y_n = 3/2$. What is wrong with this argument?

(b) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence?

Solution

(a) We don't know that it converges at all. In the previous exercise, we showed that the sequence was convergent before taking the limit.

(b) Yes, because it is bounded above and monotone increasing.

Exercise 1.4.3

(a) Show that

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

converges and find the limit.

(b) Does the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converge? If so, find the limit.

Solution

(a) To show that this sequence converges, we must show that it is bounded above and monotone increasing.

First, it is apparent that $y_2 \geq y_1$. Now, given that $y_{k+1} \geq y_k$, then $2 + y_{k+1} \geq 2 + y_k$, and $\sqrt{2 + y_{k+1}} \geq \sqrt{2 + y_k}$.

Similarly, it is apparent that $y_1 \leq 4$. Now, given that $y_k \leq 4$, it is easy to see that $y_k + 2 \leq 6$ and so $\sqrt{y_k + 2} \leq 4$. Finally, setting $y = \lim y_n$ gives $y = \sqrt{2 + y} \implies y = 2$.

(b) The sequence can be rewritten as $2^{\frac{1}{2}}, 2^{\frac{3}{4}}, 2^{\frac{7}{8}} \ldots$, which is clearly monotone increasing and bounded above. Now $y = \sqrt{2y} \implies y = 2$.

Exercise 1.4.4

(a) In Section 1.4 we used the Axiom of Completeness (AoC) to prove the Archimedean Property of **R** (Theorem 1.4.2). Show that the Monotone Convergence Theorem can also be used to prove the Archimedean Property without making any use of AoC.

(b) Use the Monotone Convergence Theorem to supply a proof for the Nested Interval Property (Theorem 1.4.1) that doesn't make use of AoC.

These two results suggest that we could have used the Monotone Convergence Theorem in place of AoC as our starting axiom for building a proper theory of the real numbers.

Solution

(a) **NOT ORIGINAL**
Assume, for contradiction, that **N** is bounded above. Then the sequence that defines the natural numbers, $x_{n+1} = x_n + 1$, must converge. Taking the limit of both sides gives $0 = 1$, which is clearly false.

(b) Given a sequence of nested intervals, consider the sequence of left-hand endpoints and the sequence of right-hand endpoints. They are monotone increasing and monotone decreasing respectively, and bounded by b_1 and a_1 respectively. Therefore they must converge, and the Order Limit Theorem, states that $a \leq b$. Now there must exist a $c \in \mathbf{R}$ such that $a \leq c \leq b$, and c is in every interval.

Exercise 1.4.5 (Calculating Square Roots)
Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

(a) Show that x_n^2 is always greater than or equal to 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

Solution

(a) It is trivial to see that $x_1^2 \geq 2$. Now,

$$\begin{aligned} x_n^2 \geq 2 &\implies (x_n^2 - 2)^2 \geq 0 \\ &\implies \frac{(x_n^2+2)^2}{4x_n^2} \geq 2 \\ &\implies \frac{1}{4}\left(x_n^2 + 4 + \frac{4}{x_n^2}\right) \geq 2 \\ &\implies x_{n+1}^2 \geq 2 \end{aligned}$$

To see that $x_n - x_{n+1} \geq 0$,

$$x_{n+1} - x_n = \frac{3}{4}x_n^2 - \frac{1}{x_n^2} - 1 \geq 0$$

for all $x_n^2 \geq 2$.

Finally, taking the limit of both sides gives $\frac{1}{2}c = \frac{1}{c} \implies c = \sqrt{2}$.

(b)

$$x_{n+1} = \left(1 - \frac{1}{c}\right)\left(x_n + \frac{1}{x_n(1-\frac{1}{c})}\right)$$

Exercise 1.4.6 (Arithmetic-Geometric Mean)

(a) Explain why $\sqrt{xy} \leq (x+y)/2$ for any two positive real numbers x and y. (The geometric mean is always less than the arithmetic mean.)

(b) Now let $0 \leq x_1 \leq y_1$ and define

$$x_{n+1} = \sqrt{x_n y_n} \quad \text{and} \quad y_{n+1} = \frac{x_n + y_n}{2}$$

Show $\lim x_n$ and $\lim y_n$ both exist and are equal.

Solution

(a)

$$\begin{aligned} \frac{(x-y)^2}{4} \geq 0 &\implies xy \leq \frac{(x+y)^2}{4} \\ &\implies \sqrt{xy} \leq \frac{x+y}{2} \end{aligned}$$

(b) First, (y_n) is monotone decreasing and bounded below by 0. This is because for any n, $x_n \leq y_n$ and so $\frac{x_n+y_n}{2} \leq \frac{2y_n}{2}$, and given that x_n and y_n are both greater than 0, their arithmetic mean must also be greater than 0. (x_n) is monotone increasing and bounded above by y_1. So $y = \lim y_n$ and $x = \lim x_n$ both exist; thus we can take

$$y = \frac{x+y}{2} \implies x = y.$$

Exercise 1.4.7 (Limit Superior)
Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

(b) The limit superior of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Solution

(a) (y_n) is decreasing and bounded because a_n is bounded.

(b) The limit of the sequence $y_n = \inf\{a_k \mid k \geq n\}$. It exists because the sequence is monotone increasing and bounded.

(c) Every term of the sequence of infimums is less than the corresponding term in the sequence of supremums, so by the Order Limit Theorem, $\liminf a_n \leq \limsup a_n$. One example of a sequence for which this inequality is strict is $x_n = (-1)^n$.

(d) $(\Rightarrow)$ If, for some n, both $\sup\{a_k \mid k \geq n\}$ and $\inf\{a_k \mid k \geq n\}$ are within ϵ of some c, then it is clear that a_k is within ϵ of that same c for $k \geq n$. Thus it has been demonstrated that the limit exists and is c.

$(\Leftarrow)$ Fix $\epsilon > 0$. Then there exists some N for which $|x_n - c| < \epsilon/2$ for all $n \geq N$. Now the sequence of infimums, i_n, for $n \geq N$ must be greater than or equal to $c - \epsilon/2$ since it is a lower bound on x_n; similarly, the sequence of supremums, s_n, must be less than or equal to $c + \epsilon/2$. Therefore $|i_n - c| < \epsilon$ and $|s_n - c| < \epsilon$ for all $n \geq N$.

Exercise 1.4.8
For each series, find an explicit formula for the sequence of partial sums and determine if the series converges.

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(c) $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$

(In (c), $\log(x)$ refers to the natural logarithm function from calculus.)

Solution

(a) $S_n = 1 - \frac{1}{2^n}$. This sequence converges to 1, as can be seen by applying the Archimedean Property.

(b) $S_n = 1 - \frac{1}{n+1}$. Again, this sequence converges to 1.

(c) $S_n = \log(n+1)$. This sequence diverges.

Exercise 1.4.9

Complete the proof of Theorem 2.4.6 by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Example 2.4.5 may be a useful reference.

Solution

First, divide the partial sum t_k by 2. This partial sum is still unbounded for all k as if it were bounded by M, then t_k would be bounded by $2M$. So the new partial sum is then

$$t_k = \frac{b_1}{2} + b_2 + 2b_4 + \ldots + 2^{k-1} b_{2^k}$$

Now consider

$$\begin{aligned} s_{2^k} &= (b_1) + (b_2) + (b_3 + b_4) + \ldots + (b_{2^{k-1}+1} + \ldots + b_{2^k}) \\ &\geq \frac{b_1}{2} + b_2 + 2b_4 + \ldots + 2^{k-1} b_{2^k} \\ &= t_k, \end{aligned}$$

which is unbounded. Therefore the partial sums must also be unbounded, and the sum diverges.

This completes the proof. We have proved that A' implies B'; this is the same as saying that B implies A since there is no way for B to be true without A being true.

Exercise 1.4.10 (Infinite Products)

A close relative of infinite series is the infinite product

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \cdots$$

which is understood in terms of its sequence of partial products

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \cdots b_m$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty} (1 + a_n) = (1 + a_1)(1 + a_2)(1 + a_3) \cdots, \qquad \text{where } a_n \geq 0$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = 1/n^2$ and make a conjecture about the convergence of this sequence.

(b) Show, in general, that the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1 + x \leq 3^x$ for positive x will be useful in one direction.)

Solution

(a) This is a telescoping product, most of the terms cancel

$$p_m = \prod_{n=1}^{m}(1 + 1/n) = \prod_{n=1}^{m} \frac{n+1}{n} = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{2} \cdots \frac{m+1}{m} = m + 1$$

therefore (p_m) diverges.

In the cast $a_n = 1/n^2$ we get

$$\prod_{n=1}^{\infty}(1 + 1/n^2) = \prod_{n=1}^{\infty} \frac{1+n^2}{n^2} = \frac{2}{1} \cdot \frac{5}{4} \cdot \frac{10}{9} \cdots$$

The growth seems slower, I conjecture it converges now.

(b) Using the inequality suggested we have $1 + a_n \leq 3^{a_n}$ letting $s_m = a_1 + \cdots + a_m$ we get

$$p_m = (1 + a_1) \cdots (1 + a_m) \leq 3^{a_1} 3^{a_2} \cdots 3^{a_m} = 3^{s_m}$$

Now if s_m converges it is bounded by some M meaning p_m is bounded by 3^M. and because $a_n \geq 0$ the partial products p_m are increasing, so they converge by the MCT. This shows s_m converging implies p_m converges.

For the other direction suppose $p_m \to p$. Distributing inside the products gives $p_2 = a_1 + a_2 + 1 + a_1 a_2 > s_2$ and in general $p_m > s_m$ implying that if p_m is bounded then s_n is bounded aswell. This completes the proof.

Summary: Convergence is if and only if because $s_m \leq p_m \leq 3^{s_m}$.

(By the way the inequality $1 + x \leq 3^x$ can be derived from $\log(1 + x) \leq x$ implying $1 + x \leq e^x$, I assume abbott rounded up to 3.)