Understanding Analysis 2e Exercises

Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1

(a) Prove that $\sqrt{3}$ is irrational. Does a similar similar argument work to show $\sqrt{6}$ is irrational?

(b) Where does the proof break down if we try to prove $\sqrt{4}$ is irrational?

Solution

(a) Assume, for contradiction, that there exists two integers p and q such that $p/q = \sqrt{3}$, and p/q is in lowest terms. Then $(p/q)^2 = 3$, and $p^2 = 3q^2$. Thus p is divisible by 3, and we can write $p = 3r$. Rearranging, however, we get $q^2 = 3r^2$, which implies that q is divisible by 3 as well and contradicts our assumption that p/q was in lowest terms. The same proof holds for $\sqrt{6}$.

(b) p^2 is divisible by 4 $\implies$ p is divisible by 4 does not hold.

Exercise 1.2.2

Show that there is no rational number satisfying $2^r = 3$.

Solution

If $2^r = 3$ and $r = p/q$ where p and q are integers, then $2^{p/q} = 3$ or $2^p = 3^q$. This is impossible, so r must not be a rational number.

Exercise 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution

(a) False. For example, consider $\bigcap_{n=0}^{\infty} A_n$ where $A_n = \{k2^n \mid k \in \mathbf{Z}^+\}$, where every set contains the all multiples of 2^n and is therefore infinite, but the only common element is 0.

(b) True. $\bigcap_{n=1}^{\infty} A_n$ must be a subset of every A_n. However, A_n is finite, and an infinite set cannot be the subset of a finite set.

(c) False. The set on the right includes all of C, whereas the set on the left includes only $A \cap C$. If $A \neq C$, then this falls apart.

(d) True. An element is in all three sets if and only if it is in both the left and the right set.

(e) True. If $x \in A \cap (B \cup C)$, then $x \in A \cap B$ or $x \in A \cap C$.

Exercise 1.2.4
Produce an infinite collection of sets $A_1, A_2, A_3, \ldots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$

Solution
Consider $A_n = \{k2^{n-1} \mid k \text{ is odd}\}$, all of which are obviously infinite. Each element of A_n is also some odd multiple of 2^{n-1}. Thus any element $x \in A_n$ cannot be in A_m for all $m < n$ as x can be expressed as an even multiple of 2^{m-1}, so A_n is disjoint. Any $k \in \mathbf{N}$ can be expressed as $2^a b$, where 2^a is the highest power of 2 that k divides. This implies that b is odd, so $k \in A_{a+1}$.

Exercise 1.2.5 (De Morgan's Laws)
Let A and B be subsets of $\mathbf{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution

(a) $x \in (A \cap B)^c \iff x \notin A \cap B$, which is to say that either $x \notin A$ or $x \notin B$. Therefore $x \in A^c \cup B^c$, which means that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) For the reverse, the same proof holds, as all statements in (a) are true in both directions.

(c) $x \in (A \cup B)^c \iff x \notin A \cup B \iff x \text{ is not in } A \text{ or } B \iff x \in A^c \cap B^c$.

Exercise 1.2.6

(a) Verify the triangle inequality in the special case where a and b have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$.

(c) Prove $|a-b| \leq |a-c|+|c-d|+|d-b|$ for all a, b, c, and d.

(d) Prove $||a|-|b|| \leq |a-b|$. (The unremarkable identity $a = a-b+b$ may be useful.)

Solution

(a) When a and b have the same sign, $|a+b| = |a|+|b|$ and so the triangle inequality is true.

(b) $(|a|+|b|)^2 = a^2+2|a||b|+b^2 \geq a^2+2ab+b^2 = (a+b)^2 \implies |a+b| \leq |a|+|b|$ as squaring a value and taking its positive square root is equivalent to taking its absolute value.

(c) By the triangle inequality, $|a-b| = |(a-c)+(c-d)+(d-b)| \leq |a-c|+|(c-d)+(d-b)| \leq |a-c|+|c-d|+|d-b|$.

(d) $(|a|-|b|)^2 = a^2-2|a||b|+b^2 \leq a^2-2ab+b^2 = (a-b)^2 \implies |a-b| \geq ||a|-|b||$.

Exercise 1.2.7

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0,2]$ (the closed interval $\{x \in \mathbf{R} : 0 \leq x \leq 2\}$) and $B = [1,4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbf{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

Solution

(a) $f(A) = [0,4]$, and $f(B) = [1,16]$. In this case,

$$f(A \cap B) = f([1,2]) = [1,4] = f(A) \cap f(B),$$

and

$$f(A \cup B) = f([0,4]) = [0,16] = f(A) \cup f(B).$$

(b) Let $A = [0,1]$ and $B = [-1,0]$. Then $f(A \cap B) = f(0) = 0$, but $f(A) \cap f(B) = [0,1]$.

(c) Let $y \in g(A \cap B)$ be an arbitrary element, and $x \in A \cap B$ the element such that $f(x) = y$. Then, since x is in both A and B, $f(x) \in g(A) \cap g(B)$.

(d) Conjecture: for any arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g(A \cup B) = g(A) \cup g(B)$. To prove this, we show inclusion both ways. For any $y \in g(A \cup B)$, there exists $x \in A \cup B$ such that $f(x) = y$. Therefore, $y \in g(A) \cup g(B)$. For the reverse, if $y \in g(A) \cup g(B)$, then $x \in A \cup B$, and so $y \in g(A \cup B)$.

Exercise 1.2.8
Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* $(1-1)$ if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$ Give an example of each or state that the request is impossible:

(a) $f : \mathbf{N} \to \mathbf{N}$ that is $1-1$ but not onto.

(b) $f : \mathbf{N} \to \mathbf{N}$ that is onto but not $1-1$.

(c) $f : \mathbf{N} \to \mathbf{Z}$ that is $1-1$ and onto.

Solution
(a) $f(n) = n + 1$ is $1-1$, but not onto.

(b) $f(n) = \lfloor \frac{n}{2} \rfloor + 1$ is onto, but not $1-1$.

(c) To construct a bijection from $\mathbf{N}$ to $\mathbf{Z}$, define

$$f(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ -\frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

Then the odd natural numbers map onto the natural numbers, whereas the even natural numbers map onto the negative integers and zero. This function is a bijection since it maps every natural number maps to a unique integer and every integer is mapped onto by some natural number.

Exercise 1.2.9
Given a function $f : D \to \mathbf{R}$ and a subset $B \subseteq \mathbf{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbf{R}$.

Solution
(a) $f^{-1}(A) = [-2, 2]$, $f^{-1}(B) = [-1, 1]$. $f^{-1}(A \cap B) = [-1, 1] = f^{-1}(A) \cap f^{-1}(B)$. $f^{-1}(A \cup B) = [-2, 2] = f^{-1}(A) \cup f^{-1}(B)$.

(b) $x \in g^{-1}(A \cap B) \iff y \in A \cap B$ where $g(x) = y \iff y$ is in A and $B \iff x \in g^{-1}(A) \cap g^{-1}(B)$.

Similarly, $x \in g^{-1}(A \cup B) \iff y \in A \cup B$ where $g(x) = y \iff y$ is in A or $B \iff x \in g^{-1}(A) \cup g^{-1}(B)$.

Exercise 1.2.10

Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

(b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.

(c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

Solution

(a) False. If $a = b$, then $a < b + \epsilon$ for every $\epsilon > 0$.

(b) False. Same as part (a).

(c) True. If $a > b$, then there would be some $\epsilon \leq a - b$ for which $a < b + \epsilon$ does not hold, so $a < b + \epsilon \implies a \leq b$. Conversely, it is easy to see that $a \leq b \implies a < b + \epsilon$ for every $\epsilon > 0$.

Exercise 1.2.11

Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that..." in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbf{N}$ such that $a + 1/n < b$.

(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbf{N}$.

(c) Between every two distinct real numbers there is a rational number.

Solution

(a) There exist real numbers satisfying $a < b$ such that $a+1/n \geq b$ for all $n \in \mathbf{N}$. (Original)

(b) For all real numbers $x > 0$, there exists $n \in \mathbf{N}$ such that $x \geq 1/n$. (Negation)

(c) There exist two distinct real numbers without a rational number between them. (Original)

Exercise 1.2.12

Let $y_1 = 6$, and for each $n \in \mathbf{N}$ define $y_{n+1} = (2y_n - 6)/3$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbf{N}$.

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

Solution
First, $y_1 > -6$. Next, if $y_k > -6$, then $2y_k > -12$ and $(2y_k - 6)/3 > -6$. Therefore $y_k > -6$ implies $y_{k+1} > -6$, and so $y_n > -6$ for all $n \in \mathbf{N}$.

Exercise 1.2.13
For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbf{N}$.

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbf{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbf{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

Solution
(a) It is easy to show that $(\bigcup_{i=1}^{n} A_i)^c = A_i^c$ when $n = 1$.

It is given that

$$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c.$$

Now, by De Morgan's law,

$$(A_1 \cup \ldots \cup A_{n+1})^c = (A_1 \cup \ldots \cup A_n)^c \cap A_{n+1}^c.$$

This, however, is equal to

$$A_1^c \cap \ldots \cap A_{n+1}^c,$$

and so De Morgan's law holding for n unions implies that it holds for $n+1$ unions. Therefore it holds for n unions for all $n \in \mathbf{N}$.

(b) Consider the collection of sets $S_n = \{k2^n \mid k \in \mathbf{N}\}$.

For any finite n, $\bigcap_{i=1}^{n} S_i = S_n$. However, $\bigcap_{i=1}^{\infty} S_i = \emptyset$.

(c) Show that

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c.$$

x belongs to the set on the left $\iff$ $x \notin \bigcup_{i=1}^{\infty} A_i$ $\iff$ $x \notin A_i$ for all $i \in \mathbf{N}$ $\iff$ $x \in A_i^c$ for all $i \in \mathbf{N}$ $\iff$ x belongs to the set on the right.

1.3 The Axiom of Completeness

Exercise 1.3.1

(a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

Solution

(a) A real number s is the greatest lower bound for a set $A \subseteq \mathbf{R}$ if it meets the following two criteria:

s is a lower bound for A,

if b is any lower bound for A, $b \leq s$.

(b) Lemma: assume $s \in \mathbf{R}$ is a lower bound for a set $A \subseteq \mathbf{R}$. Then $s = \inf A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ such that $s + \epsilon > a$.

($\Rightarrow$) Assume that $s = \inf A$. Consider $s+\epsilon$. Because $s+\epsilon > s$, part (ii) of the definition of the infimum implies that $s + \epsilon$ is not a lower bound for A. Therefore there must be some $a \in A$ such that $a < s + \epsilon$.

($\Leftarrow$) Assume that s is a lower bound for A with the property that there exists $a \in A$ satisfying $s + \epsilon > a$ for all ϵ. This implies that any number greater than s cannot be a lower bound for A, and thus that any lower bound for A is less than or equal to s. Therefore s is the infimum of A.

Exercise 1.3.2

Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbf{Q}$ that contains its supremum but not its infimum.

Solution

(a) $\{2\}$.

(b) This is impossible. All finite sets must contain both their infimum and their supremum; they are the minimum and maximum elements, respectively.

(c) $A = \{x \mid 1 < x \leq 2\}$. $\sup A = 2 \in A$, and $\inf A = 1 \notin A$.

Exercise 1.3.3

(a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbf{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Solution

(a) By definition of the infimum, all elements of B must be less than or equal to $\inf A = p$; therefore, p must be an upper bound on B. Since $p \in B$ (the infimum is a lower bound itself), p must also be the supremum of B. Therefore $p = \sup B = \inf A$.

(b) If a set A is lower bounded, the set B of all lower bounds for A is nonempty. We also know that B is upper bounded - any element of A can serve as an upper bound. Therefore B must have a supremum, and $\sup B = \inf A$ as follows from (a).

Exercise 1.3.4

Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup\left(\bigcup_{k=1}^{n} A_k\right)$.

(b) Consider $\sup\left(\bigcup_{k=1}^{\infty} A_k\right)$. Does the formula in (a) extend to the infinite case?

Solution

(a) $\sup(A_1 \cup A_2) = \max(\sup A_1, \sup A_2)$

$\sup\left(\bigcup_{i=1}^{n} A_k\right) = \max(\sup A_1, \ldots, \sup A_n)$

(b) Yes. If it can be shown that $a \in \mathbf{R}$ is the supremum for all A_k, then it is the supremum for $\bigcup_{k=1}^{\infty} A_k$.

Exercise 1.3.5

As in Example 1.3.7, let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solution

(a) Let $s = \sup A$. Now for any cx where $c \geq 0$ and $x \in A$, it follows from $s \geq x$ that $cs \geq cx$. Therefore cs is an upper bound for cA.

For contradiction, assume that there exists a value $\epsilon > 0$ such that $cs - \epsilon$ is an upper bound on cA. In this case, $cs - \epsilon \geq cx$, for all $x \in A$, and $s - \frac{\epsilon}{c} \geq x$. This would imply that there is a value less than s that is an upper bound for A; however, this is impossible because $s = \sup A$. Thus, $cs = c \sup A$ is the least upper bound for cA.

(b) It is trivial to see that, by flipping the inequalities from (a), we get $c \inf A = \sup cA$ for $c < 0$.

Exercise 1.3.6

Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$.

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.

(b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.

(c) Finally, show $\sup(A+B) = s+t$.

(d) Construct another proof of this same fact using Lemma 1.3.8.

Solution

(a) $\sup A + \sup B \geq a+b$ follows from the fact that $\sup A \geq a$ and $\sup B \geq b$ for all $a \in A$ and $b \in B$.

(b) u is an upper bound for $A+B$, so $u \geq a+b$ for all $a \in A$ and $b \in B$. Fixing a, we have $u-a \geq b$ for all $b \in B$. Thus, $u-a$ is an upper bound for B, and so $t = \sup B \leq u-a$.

(c) It has been shown that $s+t$ is an upper bound for $A+B$; what remains to be shown is that it is the least upper bound.

From (b), $t \leq u-a$, where u is any upper bound for $A+B$. So $a \leq u-t$, and since $u-t$ is an upper bound for A, $u-t \geq s$. Therefore $s \leq u-t$, and $s+t \leq u$ as desired.

Exercise 1.3.7

Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

Solution

a must be the least upper bound; if $a-\epsilon$ were to be an upper bound, it would be less than a itself and therefore not be an upper bound.

Exercise 1.3.8

Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(a) $\{m/n : m,n \in \mathbf{N} \text{ with } m<n\}$.

(b) $\{(-1)^m/n : m,n \in \mathbf{N}\}$.

(c) $\{n/(3n+1) : n \in \mathbf{N}\}$

(d) $\{m/(m+n) : m,n \in \mathbf{N}\}$

Solution

(a) $\inf A = 0$, $\sup A = 1$

(b) $\inf B = -1$, $\sup B = 1$

(c) $\inf C = \frac{1}{4}$, $\sup C = \frac{1}{3}$

(d) $\inf D = 0$, $\sup D = 1$

Exercise 1.3.9

(a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

Solution

(a) $\sup A < \sup B$ implies that there is an element $b \in B$ such that $b > \sup A$. If no such element existed, then $\sup A$ would be an upper bound for B, which is impossible as $\sup A < \sup B$.

(b) If $\sup A = \sup B$ and B does not include its supremum, then this does not hold. For example, $A = (0, 1)$, $B = (0, 1)$.

Exercise 1.3.10 (Cut Property)

The Cut Property of the real numbers is the following:

If A and B are nonempty, disjoint sets with $A \cup B = \mathbf{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbf{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.

(b) Show that the implication goes the other way; that is, assume $\mathbf{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.

(c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbf{R}$ is replaced by $\mathbf{Q}$.

Solution

(a) Since B is nonempty, choose any element as an upper bound for A. By the Axiom of Completeness, let $c = \sup A$. Notice that every $b \in B$ is an upper bound for A, so $c \leq b$ for all $b \in B$; by definition, $a \leq p$ for all $a \in A$.

(b) **NOT ORIGINAL**

Let B be the set of all upper bounds of E and let $A = \mathbf{R} \setminus B$. Thus A and B are disjoint and $A \cup B = \mathbf{R}$.

Assume, for contradiction, that $\sup E$ does not exist. Since B does not have a smallest element, $E \cap B = \emptyset$. Therefore $E \subseteq A$. By the Cut Property, there must exist a $c \in \mathbf{R}$ such that $a \leq c$ and $c \leq b$. Since B does not have a smallest element, $c \notin B$. However, since $E \subseteq A$, c is an upper bound for E and must be in B, which is a contradiction.

(c) Let $A, B \subset \mathbf{Q}$, and $A = (-\infty, \sqrt{2})$ and $B = (\sqrt{2}, +\infty)$. The Cut Property does not hold; there is no rational number c for which $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

Exercise 1.3.11

Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

(c) If there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

Solution

(a) True. $\sup B$ must be an upper bound on A. By definition, $\sup A \leq \sup B$.

(b) True. Any real number c such that $\sup A < c < \inf B$ will suffice.

(c) False. Consider $(0, 2)$ and $(2, 3)$. $a < 2 < b$ for all $a \in A$ and $b \in B$, but $2 \not< 2$.

1.4 Consequences of Completeness

Exercise 1.4.1
Recall that $\mathbf{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbf{Q}$, then ab and $a+b$ are elements of $\mathbf{Q}$ as well.

(b) Show that if $a \in \mathbf{Q}$ and $t \in \mathbf{I}$, then $a+t \in \mathbf{I}$ and $at \in \mathbf{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbf{Q}$ is closed under addition and multiplication. Is $\mathbf{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s+t$ and st?

Solution

(a) Let $a = \frac{p}{q}$ and $b = \frac{n}{m}$, where $p, q, n, m \in \mathbf{Z}$. Then $ab = \frac{np}{mq} \in \mathbf{Q}$, and $a+b = \frac{mp+nq}{qm} \in \mathbf{Q}$.

(b) If $\frac{p}{q} + t = \frac{n}{m}$, then $t = \frac{n}{m} - \frac{p}{q}$ which contradicts (a).
Similarly, $\frac{p}{q}t = \frac{n}{m}$ implies $t = \frac{nq}{mp}$, which also contradicts (a).

(c) $\mathbf{I}$ is not closed under addition or multiplication. $\sqrt{2} \cdot \sqrt{2} = 2$, and $\sqrt{2} - \sqrt{2} = 0$.

Exercise 1.4.2
Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $s \in \mathbf{R}$ have the property that for all $n \in \mathbf{N}, s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.

Solution
To show that s is an upper bound for A, assume for contradiction that there did exist $a > s$ where $a \in A$. The Archimedean property states that there exists $n \in \mathbf{N}$ such that $\frac{1}{n} < a - s$. Then $s + \frac{1}{n} < a$, but this is impossible as $s + \frac{1}{n}$.

Now assume that there is an arbitrary upper bound $q < s$. Similarly, we set $n \in \mathbf{N}$ such that $s - \frac{1}{n} > q$, which must not be an upper bound. However, this is a contradiction because this would imply that there is an element of A that is larger than $s - \frac{1}{n}$ and thus q.

Exercise 1.4.3
Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

Solution
Assume, for contradiction, that

$$c \in \bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right).$$

But by the Archimedean property, there exists $n_0 \in \mathbf{N}$ such that $\frac{1}{n_0} < c$. Therefore, the intersection is the empty set.

Exercise 1.4.4
Let $a < b$ be real numbers and consider the set $T = \mathbf{Q} \cap [a, b]$. Show $\sup T = b$.

Solution
b is an upper bound for T since all elements of T are also elements of $[a, b]$.

Now consider an arbitrary upper bound p for T such that $p = b - \epsilon$, where $0 < \epsilon < b - a$. However, by Theorem 1.4.3, there must exist a $c \in \mathbf{Q}$ such that $b - \epsilon < c < b$; since $b - \epsilon$ and b are both in $[a, b]$, $c \in (a, b)$. But $c > p$, so p cannot be an upper bound. Therefore, all upper bounds for T are greater than or equal to b.

Exercise 1.4.5
Using Exercise 1.4.1, supply a proof that $\mathbf{I}$ is dense in $\mathbf{R}$ by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$. In other words show for every two real numbers $a < b$ there exists an irrational number t with $a < t < b$.

Solution
There exists a rational number s such that $a - \sqrt{2} < s < b - \sqrt{2}$. Therefore $a < s + \sqrt{2} < b$ and $s + \sqrt{2} \in \mathbf{I}$.

Exercise 1.4.6
Recall that a set B is dense in $\mathbf{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbf{R}$? Take $p \in \mathbf{Z}$ and $q \in \mathbf{N}$ in every case.

(a) The set of all rational numbers p/q with $q \leq 10$.

(b) The set of all rational numbers p/q with q a power of 2 .

(c) The set of all rational numbers p/q with $10|p| \geq q$.

Solution
(a) Not dense.

(b) Dense.

(c) Not dense.

Exercise 1.4.7
Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

Solution
Let $\alpha^2 > 2$, where α is the supremum. Then

$$\left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n}.$$

Choose n_0 large enough such that $\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}$. This implies $\frac{2\alpha}{n_0} < \alpha^2 - 2$, or $\alpha^2 - \frac{2\alpha}{n_0} > 2$. Therefore

$$\left(\alpha - \frac{1}{n_0}\right)^2 > \alpha^2 - \frac{2\alpha}{n_0} > 2.$$

This is a contradiction, as we have found a value less than α that is an upper bound for T, and as such α^2 cannot be greater than 2.

Exercise 1.4.8
Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset, \sup A = \sup B, \sup A \notin A$ and $\sup B \notin B$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in R : x \geq a\}$.)

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbf{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Solution

(a) $(0, 2)$ on $\mathbf{Q}$ and $(0, 2)$ on $\mathbf{R} \setminus \mathbf{Q}$.

(b) This is impossible. Following the Nested Interval and instead using open intervals, $a_n < x < b_n$, or (a_n, b_n). This has to be either empty or an infinite set.

(c) Consider the sequence of sets $A_n = [n, +\infty)$. Then the intersection of all of these sets from $n = 1$ to $n = \infty$ is the empty set.

(d) This is impossible.

Let I_n be a sequence of closed, bounded intervals that are not necessarily nested. Let $A_n = \bigcap_{N=1}^{n} I_N$ so that $A_1 = I_1$, $A_2 = I_1 \cap I_2$, ...Note that these intervals are nested, because $A \cap B \subseteq B$. By the Nested Interval Property,

$$\bigcap_{n=1}^{\infty} I_n = \bigcap_{n=1}^{\infty} \neq \emptyset,$$

which holds because $I_1 \cap I_2 \cap \ldots = I_1 \cap (I_1 \cap I_2) \cap \ldots$

1.5 Cardinality

Exercise 1.5.1
Finish the following proof for Theorem 1.5.7. Assume B is a countable set. Thus, there exists $f : \mathbf{N} \to B$, which is $1-1$ and onto. Let $A \subseteq B$ be an infinite subset of B. We must show that A is countable.

Let $n_1 = \min\{n \in \mathbf{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbf{N} \to A$ set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a 1-1 function g from $\mathbf{N}$ onto A.

Solution
REVISE
Let $n_k = min\{n \in \mathbf{N} \mid f(n) \in A\}$, with $n \notin \{n_1, n_2, \ldots, n_{k-1}\}$. Then, because all elements of A are in B, all elements of A correspond to a natural number x such that $g(x) = f(n_k)$, $g(x)$ is $1-1$ and onto and therefore A is countable.

Exercise 1.5.2
Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbf{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbf{Q}$ is uncountable:

Assume, for contradiction, that $\mathbf{Q}$ is countable. Thus we can write $\mathbf{Q} = \{r_1, r_2, r_3, \ldots\}$ and, as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n = \emptyset$ while NIP implies $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. This contradiction implies $\mathbf{Q}$ must therefore be uncountable.

Solution
The Axiom of Completeness cannot be applied to rational numbers and therefore the Nested Interval Property does not hold on $\mathbf{Q}$.

Exercise 1.5.3

(a) Prove if $A_1, \ldots, A_m$ are countable sets then $A_1 \cup \cdots \cup A_m$ is countable.

(b) Explain why induction *cannot* be used to prove that if each A_n is countable, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

(c) Show how arranging $\mathbf{N}$ into the two-dimensional array

$$\begin{array}{llllll} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \cdots & \\ 4 & 8 & 13 & \cdots & & \\ 7 & 12 & \cdots & & & \\ 11 & \cdots & & & & \\ \vdots & & & & & \end{array}$$

leads to a proof for the infinite case.

Solution

(a) First, we prove that $A_1 \cup A_2$ is countable. Define $B = A_2 \setminus A_1$; therefore, $A_1 \cup B = A_1 \cup A_2$. Now there exists bijections $f : \mathbf{N} \to A_1$ and $g : \mathbf{N} \to B$ since $B \subseteq A_2$. Construct a new function

$$h(n) = \begin{cases} f(\frac{n+1}{2}) & \text{if } n \text{ is odd} \\ g(\frac{n}{2}) & \text{if } n \text{ is even} \end{cases}$$

This is a bijection from $\mathbf{N}$ onto $A_1 \cup A_2$, which means that $A_1 \cup A_2$ is countable.

Proof by induction for the finite general case (which says that if A_1, A_2, …, A_m are countable sets, their union is countable) follows easily.

(b) Induction does not hold for the infinite case.

(c) Assign each set A_m to row m in the array. Then, to construct a bijection from $\mathbf{N}$ onto the infinite union, let m be the row and n be the column in which $x \in \mathbf{N}$ is located and define $h(x) = f_m(n)$. This is $1-1$ because each natural number has a unique row and column in the array, f_m for all $m \in \mathbf{N}$ is $1-1$, and all the sets are disjoint (to achieve this, simply let $B_m = A_m \setminus (A_1 \cup \ldots \cup A_{m-1} \cup A_{m+1} \cup \ldots)$). It is onto because all f_m are onto.

Exercise 1.5.4

(a) Show $(a, b) \sim \mathbf{R}$ for any interval (a, b).

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbf{R}$ as well.

(c) Using open intervals makes it more convenient to produce the required 1-1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1-1 onto function between the two sets.

Solution

(a) First, we prove that $(a, b) \sim (-\frac{\pi}{2}, \frac{\pi}{2})$. To do this, define the line through $(a, -\frac{\pi}{2})$ and $(b, -\frac{\pi}{2})$ as

$$y + \frac{\pi}{2} = \frac{\pi}{b-a}(x-a),$$

which is clearly bijective.

Now we prove that $(-\frac{\pi}{2}, \frac{\pi}{2}) \sim \mathbf{R}$. This can be done with the function $\tan x$, which is also a bijection.

Thus we have shown that $(a, b) \sim \mathbf{R}$.

(b) This can be done with $\log(x - a)$.

(c) **TODO**

Exercise 1.5.5

(a) Why is $A \sim A$ for every set A?

(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an *equivalence relation*.

Solution

(a) Every element can be mapped onto itself, which is a bijection.

(b) In asserting $A \sim B$, two conditions are met:

(i) every element in A is mapped onto only one element in B (function),

(ii) every element in B is mapped onto only one element in A (injection, surjection).

These are the same conditions that imply $B \sim A$.

(c) If $f : A \to B$ and $g : B \to C$ are bijections, then $h = g \circ f$ is a bijection from A to C.

Exercise 1.5.6

(a) Give an example of a countable collection of disjoint open intervals.

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

Solution

(a) The intervals $(n, n+1)$ for all $n \in \mathbf{N}$.

(b) Such a collection does not exist.

By the density of $\mathbf{Q}$ in $\mathbf{R}$, there exists a rational in every open interval on $\mathbf{R}$. For any collection of disjoint intervals, each much contain at least one unique rational number. Choose any rational number for each interval to map onto; since $\mathbf{Q}$ countable and this new set is a subset of $\mathbf{Q}$, every collection of disjoint open intervals on $\mathbf{R}$ is countable as well.

Exercise 1.5.7

Consider the open interval $(0, 1)$, and let S be the set of points in the open unit square; that is, $S = \{(x, y) : 0 < x, y < 1\}$.

(a) Find a 1-1 function that maps $(0, 1)$ into, but not necessarily onto, S. (This is easy.)

(b) Use the fact that every real number has a decimal expansion to produce a $1-1$ function that maps S into $(0, 1)$. Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expansion such as .235 represents the same real number as .234999 . . . .)

The Schröder-Bernstein Theorem discussed in Exercise 1.5.11 can now be applied to conclude that $(0, 1) \sim S$.

Solution

(a) $f(x) = (x, \frac{1}{2})$.

(b) To map S into $(0,1)$, "append" the decimal expansions as follows:

$$f(x,y) = x_1y_1x_2y_2x_3y_3\ldots$$

where x_n is the nth digit of the decimal expansion of x. Note that this does not terminate for any x or y since every real number has an infinitely repeating decimal expansion.

This is an injection because if two distinct points $(a,b) \neq (p,q)$, then $a \neq p$ or $b \neq q$, or that one of the digits differ in $f(x,y)$. Therefore $f(a,b) \neq f(p,q)$, which implies that $f(x,y)$ is an injection.

Exercise 1.5.8
Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Show B must be finite or countable.

Solution
TODO

Exercise 1.5.9
A real number $x \in \mathbf{R}$ is called algebraic if there exist integers $a_0, a_1, a_2, \ldots, a_n \in \mathbf{Z}$, not all zero, such that

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers. Reread the last paragraph of Section 1.1. The final question posed here is closely related to the question of whether or not transcendental numbers exist.

(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{3}+\sqrt{2}$ are algebraic.

(b) Fix $n \in \mathbf{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.

(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

Solution

(a) $x^2 - 2 = 0$

$x^3 - 2 = 0$

$x^4 - 10x^2 + 1 = 0$

(b) Fix $n \in \mathbf{N}$. Then every polynomial of degree n has the form

$$a_nx_n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0.$$

Now $a \in \mathbf{Z}^{n+1}$, the set of all combinations of integer coefficients, is countable; it is the finite union of countable sets. Similarly, since each combination of integer coefficients corresponds to a finite number of roots, the set of the roots of all such polynomials A_n is countable because it is the countable union of finite sets.

(c) Since all the sets A_n are countable, their infinite union is also countable.

If the irrationals are the union of the algebraic and transcendental numbers, we know that the transcendental numbers must be uncountable.

Exercise 1.5.10

(a) Let $C \subseteq [0,1]$ be uncountable. Show that there exists $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable.

(b) Now let A be the set of all $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable, and set $\alpha = \sup A$. Is $C \cap [\alpha,1]$ an uncountable set?

(c) Does the statement in (a) remain true if "uncountable" is replaced by "infinite"?

Solution

(a) Assume, for contradiction, that $C \cap [a,1]$ is countable for all $a \in (0,1)$. Now $(C \cap [a,1]) \cup (C \cap [0,a]) = C$.

Exercise 1.5.11 (Schröder-Bernstein Theorem)

Assume there exists a 1-1 function $f : X \to Y$ and another 1-1 function $g : Y \to X$. Follow the steps to show that there exists a 1-1, onto function $h : X \to Y$ and hence $X \sim Y$. The strategy is to partition X and Y into components

$$X = A \cup A' \quad \text{and} \quad Y = B \cup B'$$

with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'.

(a) Explain how achieving this would lead to a proof that $X \sim Y$.

(b) Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbf{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbf{N}\}$ is a similar collection in Y.

(c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.

(d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A'.

Solution