Understanding Analysis 2e Exercises

Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1

(a) Prove that $\sqrt{3}$ is irrational. Does a similar similar argument work to show $\sqrt{6}$ is irrational?

(b) Where does the proof break down if we try to prove $\sqrt{4}$ is irrational?

Solution

(a) Assume, for contradiction, that there exists two integers p and q such that $p/q = \sqrt{3}$, and p/q is in lowest terms. Then $(p/q)^2 = 3$, and $p^2 = 3q^2$. Thus p is divisible by 3, and we can write $p = 3r$. Rearranging, however, we get $q^2 = 3r^2$, which implies that q is divisible by 3 as well and contradicts our assumption that p/q was in lowest terms. The same proof holds for $\sqrt{6}$.

(b) p^2 is divisible by 4 $\implies$ p is divisible by 4 does not hold.

Exercise 1.2.2

Show that there is no rational number satisfying $2^r = 3$.

Solution

If $2^r = 3$ and $r = p/q$ where p and q are integers, then $2^{p/q} = 3$ or $2^p = 3^q$. This is impossible, so r must not be a rational number.

Exercise 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution

(a) False. For example, consider $\bigcap_{n=0}^{\infty} A_n$ where $A_n = \{k2^n \mid k \in \mathbf{Z}^+\}$, where every set contains the all multiples of 2^n and is therefore infinite, but the only common element is 0.

(b) True. $\bigcap_{n=1}^{\infty} A_n$ must be a subset of every A_n. However, A_n is finite, and an infinite set cannot be the subset of a finite set.

(c) False. The set on the right includes all of C, whereas the set on the left includes only $A \cap C$. If $A \neq C$, then this falls apart.

(d) True. An element is in all three sets if and only if it is in both the left and the right set.

(e) True. If $x \in A \cap (B \cup C)$, then $x \in A \cap B$ or $x \in A \cap C$.

Exercise 1.2.4

Produce an infinite collection of sets $A_1, A_2, A_3, \ldots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$

Solution

Consider $A_n = \{k2^{n-1} \mid k \text{ is odd}\}$, all of which are obviously infinite. Each element of A_n is also some odd multiple of 2^{n-1}. Thus any element $x \in A_n$ cannot be in A_m for all $m < n$ as x can be expressed as an even multiple of 2^{m-1}, so A_n is disjoint. Any $k \in \mathbf{N}$ can be expressed as $2^a b$, where 2^a is the highest power of 2 that k divides. This implies that b is odd, so $k \in A_{a+1}$.

Exercise 1.2.5 (De Morgan's Laws)

Let A and B be subsets of $\mathbf{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution

(a) $x \in (A \cap B)^c \iff x \notin A \cap B$, which is to say that either $x \notin A$ or $x \notin B$. Therefore $x \in A^c \cup B^c$, which means that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) For the reverse, the same proof holds, as all statements in (a) are true in both directions.

(c) $x \in (A \cup B)^c \iff x \notin A \cup B \iff x$ is not in A or $B \iff x \in A^c \cap B^c$.

Exercise 1.2.6

(a) Verify the triangle inequality in the special case where a and b have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating $(a + b)^2 \leq (|a| + |b|)^2$.

(c) Prove $|a - b| \leq |a - c| + |c - d| + |d - b|$ for all a, b, c, and d.

(d) Prove $||a| - |b|| \leq |a - b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

Solution

(a) When a and b have the same sign, $|a + b| = |a| + |b|$ and so the triangle inequality is true.

(b) $(|a| + |b|)^2 = a^2 + 2|a||b| + b^2 \geq a^2 + 2ab + b^2 = (a + b)^2 \implies |a + b| \leq |a| + |b|$ as squaring a value and taking its positive square root is equivalent to taking its absolute value.

(c) By the triangle inequality, $|a-b| = |(a-c)+(c-d)+(d-b)| \leq |a-c|+|(c-d)+(d-b)| \leq |a - c| + |c - d| + |d - b|$.

(d) $(|a| - |b|)^2 = a^2 - 2|a||b| + b^2 \leq a^2 - 2ab + b^2 = (a - b)^2 \implies |a - b| \geq ||a| - |b||$.

Exercise 1.2.7

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbf{R} : 0 \leq x \leq 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbf{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

Solution

(a) $f(A) = [0, 4]$, and $f(B) = [1, 16]$. In this case,

$$f(A \cap B) = f([1, 2]) = [1, 4] = f(A) \cap f(B),$$

and

$$f(A \cup B) = f([0, 4]) = [0, 16] = f(A) \cup f(B).$$

(b) Let $A = [0, 1]$ and $B = [-1, 0]$. Then $f(A \cap B) = f(0) = 0$, but $f(A) \cap f(B) = [0, 1]$.

(c) Let $y \in g(A \cap B)$ be an arbitrary element, and $x \in A \cap B$ the element such that $f(x) = y$. Then, since x is in both A and B, $f(x) \in g(A) \cap g(B)$.

(d) Conjecture: for any arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g(A \cup B) = g(A) \cup g(B)$. To prove this, we show inclusion both ways. For any $y \in g(A \cup B)$, there exists $x \in A \cup B$ such that $f(x) = y$. Therefore, $y \in g(A) \cup g(B)$. For the reverse, if $y \in g(A) \cup g(B)$, then $x \in A \cup B$, and so $y \in g(A \cup B)$.

Exercise 1.2.8
Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* $(1-1)$ if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$ Give an example of each or state that the request is impossible:

(a) $f : \mathbf{N} \to \mathbf{N}$ that is $1-1$ but not onto.

(b) $f : \mathbf{N} \to \mathbf{N}$ that is onto but not $1-1$.

(c) $f : \mathbf{N} \to \mathbf{Z}$ that is $1-1$ and onto.

Solution
(a) $f(n) = n+1$ is $1-1$, but not onto.

(b) $f(n) = \lfloor \frac{n}{2} \rfloor + 1$ is onto, but not $1-1$.

(c) To construct a bijection from $\mathbf{N}$ to $\mathbf{Z}$, define

$$f(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ -\frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

Then the odd natural numbers map onto the natural numbers, whereas the even natural numbers map onto the negative integers and zero. This function is a bijection since it maps every natural number maps to a unique integer and every integer is mapped onto by some natural number.

Exercise 1.2.9
Given a function $f : D \to \mathbf{R}$ and a subset $B \subseteq \mathbf{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0,4]$ and B is the closed interval $[-1,1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbf{R}$.

Solution
(a) $f^{-1}(A) = [-2,2]$, $f^{-1}(B) = [-1,1]$. $f^{-1}(A \cap B) = [-1,1] = f^{-1}(A) \cap f^{-1}(B)$. $f^{-1}(A \cup B) = [-2,2] = f^{-1}(A) \cup f^{-1}(B)$.

(b) $x \in g^{-1}(A \cap B) \iff y \in A \cap B$ where $g(x) = y \iff y$ is in A and $B \iff x \in g^{-1}(A) \cap g^{-1}(B)$.

Similarly, $x \in g^{-1}(A \cup B) \iff y \in A \cup B$ where $g(x) = y \iff y$ is in A or $B \iff x \in g^{-1}(A) \cup g^{-1}(B)$.

Exercise 1.2.10
Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

(b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.

(c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

Solution
(a) False. If $a = b$, then $a < b + \epsilon$ for every $\epsilon > 0$.

(b) False. Same as part (a).

(c) True. If $a > b$, then there would be some $\epsilon \leq a - b$ for which $a < b + \epsilon$ does not hold, so $a < b + \epsilon \implies a \leq b$. Conversely, it is easy to see that $a \leq b \implies a < b + \epsilon$ for every $\epsilon > 0$.

Exercise 1.2.11
Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that..." in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbf{N}$ such that $a + 1/n < b$

(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbf{N}$.

(c) Between every two distinct real numbers there is a rational number.

Solution

Exercise 1.2.12
Let $y_1 = 6$, and for each $n \in \mathbf{N}$ define $y_{n+1} = (2y_n - 6)/3$

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbf{N}$.

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

Solution

Exercise 1.2.13
For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbf{N}$

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbf{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbf{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

Solution