Understanding Analysis 2e Exercises

Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1

(a) Prove that $\sqrt{3}$ is irrational. Does a similar similar argument work to show $\sqrt{6}$ is irrational?

(b) Where does the proof break down if we try to prove $\sqrt{4}$ is irrational?

Solution

(a) Assume, for contradiction, that there exists two integers p and q such that $p/q = \sqrt{3}$, and p/q is in lowest terms. Then $(p/q)^2 = 3$, and $p^2 = 3q^2$. Thus p is divisible by 3, and we can write $p = 3r$. Rearranging, however, we get $q^2 = 3r^2$, which implies that q is divisible by 3 as well and contradicts our assumption that p/q was in lowest terms. The same proof holds for $\sqrt{6}$.

(b) p^2 is divisible by 4 $\implies$ p is divisible by 4 does not hold.

Exercise 1.2.2

Show that there is no rational number satisfying $2^r = 3$.

Solution

If $2^r = 3$ and $r = p/q$ where p and q are integers, then $2^{p/q} = 3$ or $2^p = 3^q$. This is impossible, so r must not be a rational number.

Exercise 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution

(a) False. For example, consider $\bigcap_{n=0}^{\infty} A_n$ where $A_n = \{k2^n \mid k \in \mathbf{Z}^+\}$, where every set contains the all multiples of 2^n and is therefore infinite, but the only common element is 0.

(b) True. $\bigcap_{n=1}^{\infty} A_n$ must be a subset of every A_n. However, A_n is finite, and an infinite set cannot be the subset of a finite set.

(c) False. The set on the right includes all of C, whereas the set on the left includes only $A \cap C$. If $A \neq C$, then this falls apart.

(d) True. An element is in all three sets if and only if it is in both the left and the right set.

(e) True. If $x \in A \cap (B \cup C)$, then $x \in A \cap B$ or $x \in A \cap C$.

Exercise 1.2.4
Produce an infinite collection of sets $A_1, A_2, A_3, \dots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$

Solution
Consider $A_n = \{k2^{n-1} \mid k \text{ is odd}\}$, all of which are obviously infinite. Each element of A_n is also some odd multiple of 2^{n-1}. Thus any element $x \in A_n$ cannot be in A_m for all $m < n$ as x can be expressed as an even multiple of 2^{m-1}, so A_n is disjoint. Any $k \in \mathbf{N}$ can be expressed as $2^a b$, where 2^a is the highest power of 2 that k divides. This implies that b is odd, so $k \in A_{a+1}$.

Exercise 1.2.5 (De Morgan's Laws)
Let A and B be subsets of $\mathbf{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution

(a) $x \in (A \cap B)^c \iff x \notin A \cap B$, which is to say that either $x \notin A$ or $x \notin B$. Therefore $x \in A^c \cup B^c$, which means that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) For the reverse, the same proof holds, as all statements in (a) are true in both directions.

(c) $x \in (A \cup B)^c \iff x \notin A \cup B \iff x$ is not in A or $B \iff x \in A^c \cap B^c$.

Exercise 1.2.6

(a) Verify the triangle inequality in the special case where a and b have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$.

(c) Prove $|a-b| \leq |a-c|+|c-d|+|d-b|$ for all a, b, c, and d.

(d) Prove $||a|-|b|| \leq |a-b|$. (The unremarkable identity $a = a-b+b$ may be useful.)

Solution

(a) When a and b have the same sign, $|a+b| = |a|+|b|$ and so the triangle inequality is true.

(b) $(|a|+|b|)^2 = a^2+2|a||b|+b^2 \geq a^2+2ab+b^2 = (a+b)^2 \implies |a+b| \leq |a|+|b|$ as squaring a value and taking its positive square root is equivalent to taking its absolute value.

(c) By the triangle inequality, $|a-b| = |(a-c)+(c-d)+(d-b)| \leq |a-c|+|(c-d)+(d-b)| \leq |a-c|+|c-d|+|d-b|$.

(d) $(|a|-|b|)^2 = a^2-2|a||b|+b^2 \leq a^2-2ab+b^2 = (a-b)^2 \implies |a-b| \geq ||a|-|b||$.

Exercise 1.2.7

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0,2]$ (the closed interval $\{x \in \mathbf{R} : 0 \leq x \leq 2\}$) and $B = [1,4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbf{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

Solution

(a) $f(A) = [0,4]$, and $f(B) = [1,16]$. In this case,

$$f(A \cap B) = f([1,2]) = [1,4] = f(A) \cap f(B),$$

and

$$f(A \cup B) = f([0,4]) = [0,16] = f(A) \cup f(B).$$

(b) Let $A = [0,1]$ and $B = [-1,0]$. Then $f(A \cap B) = f(0) = 0$, but $f(A) \cap f(B) = [0,1]$.

(c) Let $y \in g(A \cap B)$ be an arbitrary element, and $x \in A \cap B$ the element such that $f(x) = y$. Then, since x is in both A and B, $f(x) \in g(A) \cap g(B)$.

(d) Conjecture: for any arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g(A \cup B) = g(A) \cup g(B)$. To prove this, we show inclusion both ways. For any $y \in g(A \cup B)$, there exists $x \in A \cup B$ such that $f(x) = y$. Therefore, $y \in g(A) \cup g(B)$. For the reverse, if $y \in g(A) \cup g(B)$, then $x \in A \cup B$, and so $y \in g(A \cup B)$.

Exercise 1.2.8
Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* $(1-1)$ if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$ Give an example of each or state that the request is impossible:

(a) $f : \mathbf{N} \to \mathbf{N}$ that is $1-1$ but not onto.

(b) $f : \mathbf{N} \to \mathbf{N}$ that is onto but not $1-1$.

(c) $f : \mathbf{N} \to \mathbf{Z}$ that is $1-1$ and onto.

Solution
(a) $f(n) = n + 1$ is $1-1$, but not onto.

(b) $f(n) = \lfloor \frac{n}{2} \rfloor + 1$ is onto, but not $1-1$.

(c) To construct a bijection from $\mathbf{N}$ to $\mathbf{Z}$, define

$$f(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ -\frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

Then the odd natural numbers map onto the natural numbers, whereas the even natural numbers map onto the negative integers and zero. This function is a bijection since it maps every natural number maps to a unique integer and every integer is mapped onto by some natural number.

Exercise 1.2.9
Given a function $f : D \to \mathbf{R}$ and a subset $B \subseteq \mathbf{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbf{R}$.

Solution
(a) $f^{-1}(A) = [-2, 2]$, $f^{-1}(B) = [-1, 1]$. $f^{-1}(A \cap B) = [-1, 1] = f^{-1}(A) \cap f^{-1}(B)$. $f^{-1}(A \cup B) = [-2, 2] = f^{-1}(A) \cup f^{-1}(B)$.

(b) $x \in g^{-1}(A \cap B) \iff y \in A \cap B$ where $g(x) = y \iff y$ is in A and $B \iff x \in g^{-1}(A) \cap g^{-1}(B)$.

Similarly, $x \in g^{-1}(A \cup B) \iff y \in A \cup B$ where $g(x) = y \iff y$ is in A or $B \iff x \in g^{-1}(A) \cup g^{-1}(B)$.

Exercise 1.2.10

Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

(b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.

(c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

Solution

(a) False. If $a = b$, then $a < b + \epsilon$ for every $\epsilon > 0$.

(b) False. Same as part (a).

(c) True. If $a > b$, then there would be some $\epsilon \leq a - b$ for which $a < b + \epsilon$ does not hold, so $a < b + \epsilon \implies a \leq b$. Conversely, it is easy to see that $a \leq b \implies a < b + \epsilon$ for every $\epsilon > 0$.

Exercise 1.2.11

Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that..." in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbf{N}$ such that $a + 1/n < b$.

(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbf{N}$.

(c) Between every two distinct real numbers there is a rational number.

Solution

(a) There exist real numbers satisfying $a < b$ such that $a + 1/n \geq b$ for all $n \in \mathbf{N}$. (Original)

(b) For all real numbers $x > 0$, there exists $n \in \mathbf{N}$ such that $x \geq 1/n$. (Negation)

(c) There exist two distinct real numbers without a rational number between them. (Original)

Exercise 1.2.12

Let $y_1 = 6$, and for each $n \in \mathbf{N}$ define $y_{n+1} = (2y_n - 6)/3$.

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbf{N}$.

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

Solution

First, $y_1 > -6$. Next, if $y_k > -6$, then $2y_k > -12$ and $(2y_k - 6)/3 > -6$. Therefore $y_k > -6$ implies $y_{k+1} > -6$, and so $y_n > -6$ for all $n \in \mathbf{N}$.

Exercise 1.2.13

For this exercise, assume Exercise **??** has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbf{N}$.

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbf{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where $\bigcap_{i=1}^n B_i \neq \emptyset$ is true for every $n \in \mathbf{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

Solution

(a) It is easy to show that $(\bigcup_{i=1}^n A_i)^c = A_i^c$ when $n = 1$.

It is given that

$$\left(\bigcup_{i=1}^{n} A_i \right)^c = \bigcap_{i=1}^{n} A_i^c.$$

Now, by De Morgan's law,

$$(A_1 \cup \ldots \cup A_{n+1})^c = (A_1 \cup \ldots \cup A_n)^c \cap A_{n+1}^c.$$

This, however, is equal to

$$A_1^c \cap \ldots \cap A_{n+1}^c,$$

and so De Morgan's law holding for n unions implies that it holds for $n + 1$ unions. Therefore it holds for n unions for all $n \in \mathbf{N}$.

(b) Consider the collection of sets $S_n = \{k2^n \mid k \in \mathbf{N}\}$.

For any finite n, $\bigcap_{i=1}^n S_i = S_n$. However, $\bigcap_{i=1}^{\infty} S_i = \emptyset$.

(c) Show that

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c.$$

x belongs to the set on the left $\iff$ $x \notin \bigcup_{i=1}^{\infty} A_i$ $\iff$ $x \notin A_i$ for all $i \in \mathbf{N}$ $\iff$ $x \in A_i^c$ for all $i \in \mathbf{N}$ $\iff$ x belongs to the set on the right.

1.3 The Axiom of Completeness

Exercise 1.3.1

(a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

Solution

(a) A real number s is the greatest lower bound for a set $A \subseteq \mathbf{R}$ if it meets the following two criteria:

s is a lower bound for A,

if b is any lower bound for A, $b \leq s$.

(b) Lemma: assume $s \in \mathbf{R}$ is a lower bound for a set $A \subseteq \mathbf{R}$. Then $s = \inf A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ such that $s + \epsilon > a$.

($\Rightarrow$) Assume that $s = \inf A$. Consider $s+\epsilon$. Because $s+\epsilon > s$, part (ii) of the definition of the infimum implies that $s + \epsilon$ is not a lower bound for A. Therefore there must be some $a \in A$ such that $a < s + \epsilon$.

($\Leftarrow$) Assume that s is a lower bound for A with the property that there exists $a \in A$ satisfying $s + \epsilon > a$ for all ϵ. This implies that any number greater than s cannot be a lower bound for A, and thus that any lower bound for A is less than or equal to s. Therefore s is the infimum of A.

Exercise 1.3.2

Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbf{Q}$ that contains its supremum but not its infimum.

Solution

(a) $\{2\}$.

(b) This is impossible. All finite sets must contain both their infimum and their supremum; they are the minimum and maximum elements, respectively.

(c) $A = \{x \mid 1 < x \leq 2\}$. $\sup A = 2 \in A$, and $\inf A = 1 \notin A$.

Exercise 1.3.3

(a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbf{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Solution

(a) By definition of the infimum, all elements of B must be less than or equal to $\inf A = p$; therefore, p must be an upper bound on B. Since $p \in B$ (the infimum is a lower bound itself), p must also be the supremum of B. Therefore $p = \sup B = \inf A$.

(b) If a set A is lower bounded, the set B of all lower bounds for A is nonempty. We also know that B is upper bounded - any element of A can serve as an upper bound. Therefore B must have a supremum, and $\sup B = \inf A$ as follows from (a).

Exercise 1.3.4

Let $A_1, A_2, A_3, \dots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup\left(\bigcup_{k=1}^{n} A_k\right)$.

(b) Consider $\sup\left(\bigcup_{k=1}^{\infty} A_k\right)$. Does the formula in (a) extend to the infinite case?

Solution

(a) $\sup(A_1 \cup A_2) = \max(\sup A_1, \sup A_2)$

$\sup\left(\bigcup_{i=1}^{n} A_k\right) = \max(\sup A_1, \dots, \sup A_n)$

(b) Yes. If it can be shown that $a \in \mathbf{R}$ is the supremum for all A_k, then it is the supremum for $\bigcup_{k=1}^{\infty} A_k$.

Exercise 1.3.5

As in Example 1.3.7, let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solution

(a) Let $s = \sup A$. Now for any cx where $c \geq 0$ and $x \in A$, it follows from $s \geq x$ that $cs \geq cx$. Therefore cs is an upper bound for cA.

For contradiction, assume that there exists a value $\epsilon > 0$ such that $cs - \epsilon$ is an upper bound on cA. In this case, $cs - \epsilon \geq cx$, for all $x \in A$, and $s - \frac{\epsilon}{c} \geq x$. This would imply that there is a value less than s that is an upper bound for A; however, this is impossible because $s = \sup A$. Thus, $cs = c \sup A$ is the least upper bound for cA.

(b) It is trivial to see that, by flipping the inequalities from (a), we get $c \inf A = \sup cA$ for $c < 0$.

Exercise 1.3.6

Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$.

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.

(b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.

(c) Finally, show $\sup(A+B) = s+t$.

(d) Construct another proof of this same fact using Lemma 1.3.8.

Solution

(a) $\sup A + \sup B \geq a + b$ follows from the fact that $\sup A \geq a$ and $\sup B \geq b$ for all $a \in A$ and $b \in B$.

(b) u is an upper bound for $A+B$, so $u \geq a+b$ for all $a \in A$ and $b \in B$. Fixing a, we have $u - a \geq b$ for all $b \in B$. Thus, $u - a$ is an upper bound for B, and so $t = \sup B \leq u - a$.

(c) It has been shown that $s+t$ is an upper bound for $A+B$; what remains to be shown is that it is the least upper bound.

From (b), $t \leq u - a$, where u is any upper bound for $A+B$. So $a \leq u - t$, and since $u - t$ is an upper bound for A, $u - t \geq s$. Therefore $s \leq u - t$, and $s + t \leq u$ as desired.

Exercise 1.3.7

Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

Solution

a must be the least upper bound; if $a - \epsilon$ were to be an upper bound, it would be less than a itself and therefore not be an upper bound.

Exercise 1.3.8

Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(a) $\{m/n : m, n \in \mathbf{N} \text{ with } m < n\}$.

(b) $\{(-1)^m/n : m, n \in \mathbf{N}\}$.

(c) $\{n/(3n+1) : n \in \mathbf{N}\}$

(d) $\{m/(m+n) : m, n \in \mathbf{N}\}$

Solution

(a) $\inf A = 0$, $\sup A = 1$

(b) $\inf B = -1$, $\sup B = 1$

(c) $\inf C = \frac{1}{4}$, $\sup C = \frac{1}{3}$

(d) $\inf D = 0$, $\sup D = 1$

Exercise 1.3.9

(a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

Solution

(a) $\sup A < \sup B$ implies that there is an element $b \in B$ such that $b > \sup A$. If no such element existed, then $\sup A$ would be an upper bound for B, which is impossible as $\sup A < \sup B$.

(b) If $\sup A = \sup B$ and B does not include its supremum, then this does not hold. For example, $A = (0, 1)$, $B = (0, 1)$.

Exercise 1.3.10 (Cut Property)

The Cut Property of the real numbers is the following:

If A and B are nonempty, disjoint sets with $A \cup B = \mathbf{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbf{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.

(b) Show that the implication goes the other way; that is, assume $\mathbf{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.

(c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbf{R}$ is replaced by $\mathbf{Q}$.

Solution

(a) Since B is nonempty, choose any element as an upper bound for A. By the Axiom of Completeness, let $c = \sup A$. Notice that every $b \in B$ is an upper bound for A, so $c \leq b$ for all $b \in B$; by definition, $a \leq p$ for all $a \in A$.

(b) **NOT ORIGINAL**

Let B be the set of all upper bounds of E and let $A = \mathbf{R} \setminus B$. Thus A and B are disjoint and $A \cup B = \mathbf{R}$.

Assume, for contradiction, that $\sup E$ does not exist. Since B does not have a smallest element, $E \cap B = \emptyset$. Therefore $E \subseteq A$. By the Cut Property, there must exist a $c \in \mathbf{R}$ such that $a \leq c$ and $c \leq b$. Since B does not have a smallest element, $c \notin B$. However, since $E \subseteq A$, c is an upper bound for E and must be in B, which is a contradiction.

(c) Let $A, B \subset \mathbf{Q}$, and $A = (-\infty, \sqrt{2})$ and $B = (\sqrt{2}, +\infty)$. The Cut Property does not hold; there is no rational number c for which $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

Exercise 1.3.11

Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

(c) If there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

Solution

(a) True. $\sup B$ must be an upper bound on A. By definition, $\sup A \leq \sup B$.

(b) True. Any real number c such that $\sup A < c < \inf B$ will suffice.

(c) False. Consider $(0, 2)$ and $(2, 3)$. $a < 2 < b$ for all $a \in A$ and $b \in B$, but $2 \not< 2$.

1.4 Consequences of Completeness

Exercise 1.4.1
Recall that $\mathbf{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbf{Q}$, then ab and $a+b$ are elements of $\mathbf{Q}$ as well.

(b) Show that if $a \in \mathbf{Q}$ and $t \in \mathbf{I}$, then $a+t \in \mathbf{I}$ and $at \in \mathbf{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbf{Q}$ is closed under addition and multiplication. Is $\mathbf{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s+t$ and st?

Solution

(a) Let $a = \frac{p}{q}$ and $b = \frac{n}{m}$, where $p, q, n, m \in \mathbf{Z}$. Then $ab = \frac{np}{mq} \in \mathbf{Q}$, and $a+b = \frac{mp+nq}{qm} \in \mathbf{Q}$.

(b) If $\frac{p}{q} + t = \frac{n}{m}$, then $t = \frac{n}{m} - \frac{p}{q}$ which contradicts (a).
Similarly, $\frac{p}{q}t = \frac{n}{m}$ implies $t = \frac{nq}{mp}$, which also contradicts (a).

(c) $\mathbf{I}$ is not closed under addition or multiplication. $\sqrt{2} \cdot \sqrt{2} = 2$, and $\sqrt{2} - \sqrt{2} = 0$.

Exercise 1.4.2
Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $s \in \mathbf{R}$ have the property that for all $n \in \mathbf{N}, s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.

Solution
To show that s is an upper bound for A, assume for contradiction that there did exist $a > s$ where $a \in A$. The Archimedean property states that there exists $n \in \mathbf{N}$ such that $\frac{1}{n} < a - s$. Then $s + \frac{1}{n} < a$, but this is impossible as $s + \frac{1}{n}$.

Now assume that there is an arbitrary upper bound $q < s$. Similarly, we set $n \in \mathbf{N}$ such that $s - \frac{1}{n} > q$, which must not be an upper bound. However, this is a contradiction because this would imply that there is an element of A that is larger than $s - \frac{1}{n}$ and thus q.

Exercise 1.4.3
Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

Solution
Assume, for contradiction, that

$$c \in \bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right).$$

But by the Archimedean property, there exists $n_0 \in \mathbf{N}$ such that $\frac{1}{n_0} < c$. Therefore, the intersection is the empty set.

Exercise 1.4.4
Let $a < b$ be real numbers and consider the set $T = \mathbf{Q} \cap [a, b]$. Show $\sup T = b$.

Solution
b is an upper bound for T since all elements of T are also elements of $[a, b]$.

Now consider an arbitrary upper bound p for T such that $p = b - \epsilon$, where $0 < \epsilon < b - a$. However, by Theorem 1.4.3, there must exist a $c \in \mathbf{Q}$ such that $b - \epsilon < c < b$; since $b - \epsilon$ and b are both in $[a, b]$, $c \in (a, b)$. But $c > p$, so p cannot be an upper bound. Therefore, all upper bounds for T are greater than or equal to b.

Exercise 1.4.5
Using Exercise 1.4.1, supply a proof that $\mathbf{I}$ is dense in $\mathbf{R}$ by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$. In other words show for every two real numbers $a < b$ there exists an irrational number t with $a < t < b$.

Solution
There exists a rational number s such that $a - \sqrt{2} < s < b - \sqrt{2}$. Therefore $a < s + \sqrt{2} < b$ and $s + \sqrt{2} \in \mathbf{I}$.

Exercise 1.4.6
Recall that a set B is dense in $\mathbf{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbf{R}$? Take $p \in \mathbf{Z}$ and $q \in \mathbf{N}$ in every case.

(a) The set of all rational numbers p/q with $q \leq 10$.

(b) The set of all rational numbers p/q with q a power of 2 .

(c) The set of all rational numbers p/q with $10|p| \geq q$.

Solution
(a) Not dense.

(b) Dense.

(c) Not dense.

Exercise 1.4.7
Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

Solution
Let $\alpha^2 > 2$, where α is the supremum. Then

$$\left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n}.$$

Choose n_0 large enough such that $\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}$. This implies $\frac{2\alpha}{n_0} < \alpha^2 - 2$, or $\alpha^2 - \frac{2\alpha}{n_0} > 2$. Therefore

$$\left(\alpha - \frac{1}{n_0}\right)^2 > \alpha^2 - \frac{2\alpha}{n_0} > 2.$$

This is a contradiction, as we have found a value less than α that is an upper bound for T, and as such α^2 cannot be greater than 2.

Exercise 1.4.8
Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset, \sup A = \sup B, \sup A \notin A$ and $\sup B \notin B$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in R : x \geq a\}$.)

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbf{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Solution

(a) $(0, 2)$ on $\mathbf{Q}$ and $(0, 2)$ on $\mathbf{R} \setminus \mathbf{Q}$.

(b) This is impossible. Following the Nested Interval and instead using open intervals, $a_n < x < b_n$, or (a_n, b_n). This has to be either empty or an infinite set.

(c) Consider the sequence of sets $A_n = [n, +\infty)$. Then the intersection of all of these sets from $n = 1$ to $n = \infty$ is the empty set.

(d) This is impossible.

Let I_n be a sequence of closed, bounded intervals that are not necessarily nested. Let $A_n = \bigcap_{N=1}^{n} I_N$ so that $A_1 = I_1$, $A_2 = I_1 \cap I_2$, ... Note that these intervals are nested, because $A \cap B \subseteq B$. By the Nested Interval Property,

$$\bigcap_{n=1}^{\infty} I_n = \bigcap_{n=1}^{\infty} \neq \emptyset,$$

which holds because $I_1 \cap I_2 \cap \ldots = I_1 \cap (I_1 \cap I_2) \cap \ldots$

1.5 Cardinality

Exercise 1.5.1
Finish the following proof for Theorem 1.5.7. Assume B is a countable set. Thus, there exists $f : \mathbf{N} \to B$, which is $1-1$ and onto. Let $A \subseteq B$ be an infinite subset of B. We must show that A is countable.

Let $n_1 = \min\{n \in \mathbf{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbf{N} \to A$ set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a 1-1 function g from $\mathbf{N}$ onto A.

Solution
REVISE
Let $n_k = \min\{n \in \mathbf{N} \mid f(n) \in A\}$, with $n \notin \{n_1, n_2, \ldots, n_{k-1}\}$. Then, because all elements of A are in B, all elements of A correspond to a natural number x such that $g(x) = f(n_k)$, $g(x)$ is $1-1$ and onto and therefore A is countable.

Exercise 1.5.2
Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbf{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbf{Q}$ is uncountable:

Assume, for contradiction, that $\mathbf{Q}$ is countable. Thus we can write $\mathbf{Q} = \{r_1, r_2, r_3, \ldots\}$ and, as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n = \emptyset$ while NIP implies $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. This contradiction implies $\mathbf{Q}$ must therefore be uncountable.

Solution
The Axiom of Completeness cannot be applied to rational numbers and therefore the Nested Interval Property does not hold on $\mathbf{Q}$.

Exercise 1.5.3

(a) Prove if $A_1, \ldots, A_m$ are countable sets then $A_1 \cup \cdots \cup A_m$ is countable.

(b) Explain why induction *cannot* be used to prove that if each A_n is countable, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

(c) Show how arranging $\mathbf{N}$ into the two-dimensional array

$$\begin{array}{llllll}
1 & 3 & 6 & 10 & 15 & \cdots \\
2 & 5 & 9 & 14 & \cdots & \\
4 & 8 & 13 & \cdots & & \\
7 & 12 & \cdots & & & \\
11 & \cdots & & & & \\
\vdots & & & & &
\end{array}$$

leads to a proof for the infinite case.

Solution

(a) First, we prove that $A_1 \cup A_2$ is countable. Define $B = A_2 \setminus A_1$; therefore, $A_1 \cup B = A_1 \cup A_2$. Now there exists bijections $f : \mathbf{N} \to A_1$ and $g : \mathbf{N} \to B$ since $B \subseteq A_2$. Construct a new function

$$h(n) = \begin{cases} f(\frac{n+1}{2}) & \text{if } n \text{ is odd} \\ g(\frac{n}{2}) & \text{if } n \text{ is even} \end{cases}$$

This is a bijection from $\mathbf{N}$ onto $A_1 \cup A_2$, which means that $A_1 \cup A_2$ is countable.

Proof by induction for the finite general case (which says that if A_1, A_2, $\ldots$, A_m are countable sets, their union is countable) follows easily.

(b) In general, induction is not a valid argument for the infinite case.

(c) Assign each set A_m to row m in the array. Then, to construct a bijection from $\mathbf{N}$ onto the infinite union, let m be the row and n be the column in which $x \in \mathbf{N}$ is located and define $h(x) = f_m(n)$. This is $1-1$ because each natural number has a unique row and column in the array, f_m for all $m \in \mathbf{N}$ is $1-1$, and all the sets are disjoint (to achieve this, simply let $B_m = A_m \setminus (A_1 \cup \ldots \cup A_{m-1} \cup A_{m+1} \cup \ldots)$). It is onto because all f_m are onto.

Exercise 1.5.4

(a) Show $(a, b) \sim \mathbf{R}$ for any interval (a, b).

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbf{R}$ as well.

(c) Using open intervals makes it more convenient to produce the required 1-1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1-1 onto function between the two sets.

Solution

(a) First, we prove that $(a, b) \sim (-\frac{\pi}{2}, \frac{\pi}{2})$. To do this, define the line through $(a, -\frac{\pi}{2})$ and $(b, -\frac{\pi}{2})$ as

$$y + \frac{\pi}{2} = \frac{\pi}{b-a}(x - a),$$

which is clearly bijective.

Now we prove that $(-\frac{\pi}{2}, \frac{\pi}{2}) \sim \mathbf{R}$. This can be done with the function $\tan x$, which is also a bijection.

Thus we have shown that $(a, b) \sim \mathbf{R}$.

(b) This can be done with $\log(x - a)$.

(c) Construct a new function

$$f(x) = \begin{cases} \frac{n+1}{n+2} & \text{if } x \text{ can be expressed as } \frac{n}{n+1} \text{ where } n \in \mathbf{N} \\ x & \text{otherwise} \end{cases}$$

This is a bijection from $[0, 1)$ to $(0, 1)$.

Exercise 1.5.5

(a) Why is $A \sim A$ for every set A?

(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an *equivalence relation*.

Solution

(a) Every element can be mapped onto itself, which is a bijection.

(b) In asserting $A \sim B$, two conditions are met:

   (i) every element in A is mapped onto only one element in B (function),

   (ii) every element in B is mapped onto only one element in A (injection, surjection).

   These are the same conditions that imply $B \sim A$.

(c) If $f : A \to B$ and $g : B \to C$ are bijections, then $h = g \circ f$ is a bijection from A to C.

Exercise 1.5.6

(a) Give an example of a countable collection of disjoint open intervals.

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

Solution

(a) The intervals $(n, n+1)$ for all $n \in \mathbf{N}$.

(b) Such a collection does not exist.

   By the density of $\mathbf{Q}$ in $\mathbf{R}$, there exists a rational in every open interval on $\mathbf{R}$. For any collection of disjoint intervals, each much contain at least one unique rational number. Choose any rational number for each interval to map onto; since $\mathbf{Q}$ countable and this new set is a subset of $\mathbf{Q}$, every collection of disjoint open intervals on $\mathbf{R}$ is countable as well.

Exercise 1.5.7

Consider the open interval $(0,1)$, and let S be the set of points in the open unit square; that is, $S = \{(x, y) : 0 < x, y < 1\}$.

(a) Find a 1-1 function that maps $(0,1)$ into, but not necessarily onto, S. (This is easy.)

(b) Use the fact that every real number has a decimal expansion to produce a $1-1$ function that maps S into $(0,1)$. Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expansion such as .235 represents the same real number as $.234999\ldots$.)

The Schröder-Bernstein Theorem discussed in Exercise 1.5.11 can now be applied to conclude that $(0,1) \sim S$.

Solution

(a) $f(x) = (x, \frac{1}{2})$.

(b) To map S into $(0, 1)$, "append" the decimal expansions as follows:

$$f(x, y) = x_1y_1x_2y_2x_3y_3\ldots$$

where x_n is the nth digit of the decimal expansion of x. Note that this does not terminate for any x or y since every real number has an infinitely repeating decimal expansion.

This is an injection because if two distinct points $(a, b) \neq (p, q)$, then $a \neq p$ or $b \neq q$, or that one of the digits differ in $f(x, y)$. Therefore $f(a, b) \neq f(p, q)$, which implies that $f(x, y)$ is an injection.

Exercise 1.5.8

Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Show B must be finite or countable.

Solution

Let $B_a = \{b \in B \mid b \geq a\}$. Notice that this set is finite for any $a > 0$, since if it was infinite, we could easily construct a subset of B that sums to greater than 2. Now it can shown with the Archimedean property that

$$B = \bigcup_{n=1}^{\infty} B \cap [1/n, 2],$$

and because this is the countable union of finite sets, B must be countable as well.

Exercise 1.5.9

A real number $x \in \mathbf{R}$ is called algebraic if there exist integers $a_0, a_1, a_2, \ldots, a_n \in \mathbf{Z}$, not all zero, such that

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers. Reread the last paragraph of Section 1.1. The final question posed here is closely related to the question of whether or not transcendental numbers exist.

(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{3} + \sqrt{2}$ are algebraic.

(b) Fix $n \in \mathbf{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.

(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

Solution

(a) $x^2 - 2 = 0$

$x^3 - 2 = 0$

$x^4 - 10x^2 + 1 = 0$

(b) Fix $n \in \mathbf{N}$. Then every polynomial of degree n has the form

$$a_n x_n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0.$$

Now $a \in \mathbf{Z}^{n+1}$, the set of all combinations of integer coefficients, is countable; it is the finite union of countable sets. Similarly, since each combination of integer coefficients corresponds to a finite number of roots, the set of the roots of all such polynomials A_n is countable because it is the countable union of finite sets.

(c) Since all the sets A_n are countable, their infinite union is also countable.

If the irrationals are the union of the algebraic and transcendental numbers, we know that the transcendental numbers must be uncountable.

Exercise 1.5.10

(a) Let $C \subseteq [0,1]$ be uncountable. Show that there exists $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable.

(b) Now let A be the set of all $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable, and set $\alpha = \sup A$. Is $C \cap [\alpha, 1]$ an uncountable set?

(c) Does the statement in (a) remain true if "uncountable" is replaced by "infinite"?

Solution

(a) Assume, for contradiction, that $C \cap [a,1]$ is countable for all $a \in (0,1)$. Now

$$C = \bigcup_{n=1}^{\infty} C \cap \left[\frac{1}{n}, 1\right] = C \cap [0,1]$$

is a countable union of countable sets, which implies that C is countable. However, this cannot be possible because C is uncountable.

(b) A similar argument to (a) using

$$\bigcup_{n=1}^{\infty} C \cap \left[\alpha + \frac{1}{n}, 1\right]$$

shows that $C \cap [\alpha, 1]$ is countable.

(c) No. Consider the set $S = \{1/2^n \mid n \in \mathbf{N}\}$. For all $c \in [0,1]$, $S \cap [c,1]$ is finite.

Exercise 1.5.11 (Schröder-Bernstein Theorem)
Assume there exists a 1-1 function $f : X \to Y$ and another 1-1 function $g : Y \to X$. Follow the steps to show that there exists a 1-1, onto function $h : X \to Y$ and hence $X \sim Y$. The strategy is to partition X and Y into components

$$X = A \cup A' \quad \text{and} \quad Y = B \cup B'$$

with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'.

(a) Explain how achieving this would lead to a proof that $X \sim Y$.

(b) Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbf{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbf{N}\}$ is a similar collection in Y.

(c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.

(d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A'.

Solution

(a) If $f : A \to B$ is onto and $g : B' \to A'$ is onto, then $A \cup A'$ is onto $B \cup B'$. Since both functions are also $1-1$, there exists a bijection from X onto Y.

(b) **REVISE**
$A_1 = X \setminus g(Y)$ is all of X that is not mapped onto by Y (if $A_1 = \emptyset$, then g is a surjection and $X \sim Y$ is already proven). We use strong induction to prove that A_n and $f(A_n)$ for all $n \in \mathbf{N}$ are pairwise disjoint sequences of sets.

For the base cases, A_1 and A_2 are disjoint because A_1 is not mapped onto by g, whereas A_2 is mapped onto by g and g is $1-1$. $f(A_1)$ and $f(A_2)$ are also disjoint because f is $1-1$.

Now assume that A_n is pairwise disjoint with $A_1 \ldots A_{n-1}$ and $f(A_n)$ is pairwise disjoint with $f(A_1) \ldots f(A_{n-1})$. $A_{n+1} = g(f(A_n))$ is pairwise disjoint with $g(f(A_1)) \ldots g(f(A_{n-1}))$ (or $A_2 \ldots A_n$) because g is $1-1$ and $f(A_1)$ to $f(A_{n-1})$ are all disjoint; it is disjoint with A_1 because g is $1-1$ and g does not map any of Y onto A_1. It follows easily that $f(A_{n+1})$ is pairwise disjoint with $f(A_1) \ldots f(A_n)$ because f is $1-1$.

(c) B is the range of A, so $f : A \to B$ is onto. That is, any element $y \in B$ belongs to $f(A_n)$ for some $n \in \mathbf{N}$, and f maps some element $x \in A_n$ to y.

(d) Suppose that $x \in X$ is not in A. That means that it is mapped onto by Y through g (it is not in $A_1 = X \setminus g(Y)$). Now, we know that $g'(x)$ exists; call it y. Then y cannot be in $f(A_n)$ for $n \in \mathbf{N}$, since if it were, x would be in A_{n+1} and therefore A. Thus y must be in B', and we have demonstrated that $g : B' \to A'$ is a surjection.

1.6 Cantor's theorem

Exercise 1.6.1
Show that $(0, 1)$ is uncountable if and only if $\mathbf{R}$ is uncountable.

Solution

$$f(x) = \tan\left(y + \frac{\pi}{2} = \pi x\right)$$

gives a bijection from $(0, 1)$ to $\mathbf{R}$.

Exercise 1.6.2
Let $f : \mathbf{N} \to \mathbf{R}$ be a way to list every real number (hence show $\mathbf{R}$ is countable).

Define a new number x with digits $b_1 b_2 \ldots$ given by

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2 \end{cases}$$

(a) Explain why the real number $x = .b_1 b_2 b_3 b_4 \ldots$ cannot be $f(1)$.

(b) Now, explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in \mathbf{N}$.

(c) Point out the contradiction that arises from these observations and conclude that $(0, 1)$ is uncountable.

Solution

(a) $a_1 1 \neq b_1$.

(b) $a_n n \neq b_n$.

(c) We have found a new number in $(0, 1)$; however, this contradicts our assumption that every real number was in our table and corresponded to a natural number.

Exercise 1.6.3
Supply rebuttals to the following complaints about the proof of Theorem 1.6.1.

(a) Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of $\mathbf{Q}$ must be countable, the proof of Theorem 1.6.1 must be flawed.

(b) Some numbers have two different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, $1/2$ can be written as $.5$ or as $.4999\ldots$ Doesn't this cause some problems?

Solution

(a) Every rational number having a decimal expansion does not imply every decimal expansion having a rational representation. In fact, because the constructed number has no repeating decimal pattern, it will not be in the set of rational numbers and $\mathbf{Q}$ remains countable.

(b) If we always assume the infinite representation, numbers with terminating decimal expansions are no different from any other real number.

Exercise 1.6.4
Let S be the set consisting of all sequences of 0 's and 1 's. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, \dots) : a_n = 0 \text{ or } 1\}$$

As an example, the sequence $(1, 0, 1, 0, 1, 0, 1, 0, \dots)$ is an element of S, as is the sequence $(1, 1, 1, 1, 1, 1, \dots)$. Give a rigorous argument showing that S is uncountable.

Solution
Organize all the sequences in a table indexed by $n \in \mathbf{N}$. Then for $a_n n$, the nth number of the nth sequence, add the opposite to our new sequence. This is a new sequence in S that does not have an index.

Exercise 1.6.5
(a) Let $A = \{a, b, c\}$. List the eight elements of $P(A)$. (Do not forget that $\emptyset$ is considered to be a subset of every set.)

(b) If A is finite with n elements, show that $P(A)$ has 2^n elements.

Solution
(a) $\emptyset$

$\{a\}$, $\{b\}$, $\{c\}$

$\{a, b\}$, $\{b, c\}$, $\{a, c\}$

$\{a, b, c\}$

(b) When constructing a subset, we have two options for each element: include it or leave it out, which corresponds to 2^n ways to construct a subset from a set of size n.

Exercise 1.6.6
(a) Using the particular set $A = \{a, b, c\}$, exhibit two different $1-1$ mappings from A into $P(A)$.

(b) Letting $C = \{1, 2, 3, 4\}$, produce an example of a $1 - 1$ map $g : C \to P(C)$.

(c) Explain why, in parts (a) and (b), it is impossible to construct mappings that are onto.

Solution
(a) $f(x) = \{x\}$, $f(x) = \{x, y\}$ where y is the next element in the set.

(b) $f(x) = \{x\}$.

(c) Because the $P(A)$ has more elements than A for the examples.

Exercise 1.6.7
SKIPPED

Exercise 1.6.8

(a) First, show that the case $a' \in B$ leads to a contradiction.

(b) Now, finish the argument by showing that the case $a' \notin B$ is equally unacceptable.

Solution

(a) $a' \in B$ does not work, because $f(a') = B$, and by definition we would have constructed B to not include a' if that were the case.

(b) $a' \notin B$ does not work, because then we would have constructed B to include a'.

Exercise 1.6.9

Using the various tools and techniques developed in the last two sections (including the exercises from Section 1.5), give a compelling argument showing that $P(\mathbf{N}) \sim \mathbf{R}$.

Solution

To show that $P(\mathbf{N}) \sim \mathbf{R}$, first consider $(0,1) \sim \mathbf{R}$. Note that our choice of base 10 is arbitrary; we can just as easily represent decimal expansions in binary. The bijection is constructed as follows: if the nth digit of the binary expansion of real number x is 1, then include n; if not, do not include n. This is an injection from $\mathbf{R}$ to $P(\mathbf{N})$ because if $a \neq b$, then there is an $n \in \mathbf{N}$ such that $a_n \neq b_n$. Therefore $f(a) \neq f(b)$, as only one of them includes n. It is onto because all subsets of $\mathbf{N}$ can be generated by determining which elements to leave out.

Exercise 1.6.10

As a final exercise, answer each of the following by establishing a $1-1$ correspondence with a set of known cardinality.

(a) Is the set of all functions from $\{0,1\}$ to $\mathbf{N}$ countable or uncountable?

(b) Is the set of all functions from $\mathbf{N}$ to $\{0,1\}$ countable or uncountable?

(c) Given a set B, a subset $\mathcal{A}$ of $P(B)$ is called an antichain if no element of $\mathcal{A}$ is a subset of any other element of $\mathcal{A}$. Does $P(\mathbf{N})$ contain an uncountable antichain?

Solution

(a) This is the same as $\mathbf{N^2}$, which is countable.

(b) This is the same proof from 1.6.4. It is uncountable.

(c) Yes. For example, construct an antichain A containing all subsets $A_k \subseteq \mathbf{N}$ such A_k contains either $2n$ or $2n-1$ but not both for any $n \in \mathbf{N}$. This is equivalent to the set of infinite sequences of 0s and 1s, with 0 denoting the even number picked and 1 denoting the odd number picked for each n; however, this set known to be uncountable, so A must also be uncountable.

Chapter 2

Sequences and Series

2.2 The Limit of a Sequence

Exercise 2.2.1
What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $N \in \mathbf{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$.

Give an example of a vercongent sequence. Is there an example of a vercongent sequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

Solution
A vercongent sequence is just any sequence that is bounded. A vercongent sequence can be divergent; one example would be $a_n = (-1)^n$. It is easily possible for a sequence to verconge to two different values, given that ϵ is chosen to be sufficiently large.

Exercise 2.2.2
Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) $\lim \frac{2n^2}{n^3+3} = 0$.

(c) $\lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

Solution
(a) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbf{N}$ with

$$N > \frac{3}{25\epsilon} - \frac{4}{5}.$$

To verify that this choice is appopriate, let $n \in \mathbf{N}$ satisfy $n \geq N$. Then

$$\begin{aligned} n > \frac{3}{25\epsilon} - \frac{4}{5} &\implies \epsilon > \frac{\frac{3}{5}}{5n+4} \\ &\implies \epsilon > \left| \frac{\frac{3}{5}}{5n+4} \right| \\ &\implies \epsilon > \left| \frac{2n+1}{5n+4} - \frac{2n+\frac{8}{5}}{5n+4} \right| \\ &\implies \epsilon > \left| \frac{2n+1}{5n+4} - \frac{2}{5} \right| \end{aligned}$$

(b) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbf{N}$ with

$$N > \frac{2}{\epsilon}.$$

To verify that this choice of N is appropriate, let $n \in \mathbf{N}$ satisfy $n \geq N$. Then

$$\begin{aligned} n > \frac{2}{\epsilon} &\implies \frac{\epsilon}{2} > \frac{n^2}{n^3} > \frac{n^2}{n^3+3} \\ &\implies \epsilon > \left| \frac{2n^2}{n^3+3} \right| \end{aligned}$$

(c) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbf{N}$ with

$$N > \frac{1}{\epsilon^3}.$$

To verify that this choice of N is appropriate, let $n \in \mathbf{N}$ satisfy $n \geq N$. Then

$$n > \frac{1}{\epsilon^3} \implies \epsilon > \frac{1}{\sqrt[3]{n}} > \left| \frac{\sin n^2}{\sqrt[3]{n}} \right|$$

Exercise 2.2.3

Describe what we would have to demonstrate in order to disprove each of the following statements.

(a) At every college in the United States, there is a student who is at least seven feet tall.

(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.

(c) There exists a college in the United States where every student is at least six feet tall.

Solution

(a) There exists one college in the United States where every student is less than seven feet tall.

(b) There exists one college in the United States where every professor gave at least one student a C or below.

(c) For every college in the United States, there is at least one student who is less than six feet tall.

Exercise 2.2.4
Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

(c) A divergent sequence such that for every $n \in \mathbf{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution
(a) $a_n = (-1)^n$.

(b) Assume a sequence with infinite ones has limit $b \neq 1$. If we choose $\epsilon < |b-1|$, then $|a_n - b| < |b-1|$. Since a has infinite ones, regardless of how $N \in \mathbf{N}$ is chosen, $a_n = 1$ for some $n \geq N$. However, this implies $|1-b| < |b-1|$, which is clearly a contradiction. Therefore there cannot be a sequence with infinite ones that does not either diverge or converge to 1 itself.

(c) $1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, \ldots$

Exercise 2.2.5
Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. For each sequence, find $\lim a_n$ and verify it with the definition of convergence.

(a) $a_n = [[5/n]]$,

(b) $a_n = [[(12+4n)/3n]]$.

Reflecting on these examples, comment on the statement following Definition 2.2.3 that "the smaller the ϵ-neighborhood, the larger N may have to be."

Solution
(a) Choose $n > 5$. Then $\left|[[\frac{5}{n}]]\right|$ always evaluates to 0.

(b) Choose $n > 6$. Then $\left|[[\frac{12+4n}{3n}]]\right|$ always evaluates to 1.
This statement is only true if the sequence converges gradually. In some cases, sequences can "jump" to their limit.

Exercise 2.2.6
Theorem 2.2.7 (Uniqueness of Limits). *The limit of a sequence, when it exists, must be unique.*

Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$.

Solution
Assume, for contradiction, that $a \neq b$. Let $a > b$, without loss of generality. Then $|a-b| = a-b > 0$. By definition, there exists N_a, $N_b \in \mathbf{N}$ such that

$$|a_n - b| < \epsilon \text{ for all } n \geq N_b,$$

$$|a_n - a| < \epsilon \text{ for all } n \geq N_a.$$

These two conditions are simultaneously true for $n \geq N$, where $N = \max(N_a, N_b)$. Now set $\epsilon < \frac{a-b}{2}$. This implies

$$b - \frac{a-b}{2} < a_n < b + \frac{a-b}{2} \implies b - \frac{a-b}{2} < a_n < \frac{a+b}{2}$$

$$a - \frac{a-b}{2} < a_n < a + \frac{a-b}{2} \implies \frac{a+b}{2} < a_n < a + \frac{a-b}{2}$$

However, this is impossible, as a_n cannot be greater than $\frac{a+b}{2}$ and less than $\frac{a+b}{2}$ at the same time.

Exercise 2.2.7
Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbf{R}$ if there exists an $N \in \mathbf{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbf{R}$ if, for every $N \in \mathbf{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?
(b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?
(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?
(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2 . Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

Solution
(a) Frequently.

(b) Eventually implies frequently, so eventually is the stronger statement.

(c) Given any ϵ-neighborhood, a_n is eventually in that neighborhood.

(d) (x_n) is not necessarily eventually in $(1.9, 2.1)$, but it is frequently in $(1.9, 2.1)$.

Exercise 2.2.8
For some additional practice with nested quantifiers, consider the following invented definition:

Let's call a sequence (x_n) zero-heavy if there exists $M \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$.

(a) Is the sequence $(0, 1, 0, 1, 0, 1, \ldots)$ zero heavy?

(b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide a counterexample.

(c) If a sequence contains an infinite number of zeros, is it necessarily zeroheavy? If not, provide a counterexample.

(d) Form the logical negation of the above definition. That is, complete the sentence: A sequence is not zero-heavy if ....

Solution

(a) Yes.

(b) Yes. Fix $M \in \mathbf{N}$ and consider the intervals $N = 0, N = M + 1, N = 2M + 2 \ldots$

These all identify unique instances of zero in the sequence because they are disjoint. Since there are an infinite number of these instances, there must also be an infinite number of zeroes.

(c) No. Consider the sequence $0, 1, 0, 1, 1, 0, 1, 1, 1, \ldots$

Regardless of how M is chosen, there will be an M-long sequence of 1s in the sequence, which means that it is not zero-heavy, even though it has an infinite number of zeroes.

(d) A sequence is not zero-heavy if for all $M \in \mathbf{N}$, there exists $N \in \mathbf{N}$ such that for all n satisfying $N \leq n \leq N + M$, $x_n \neq 0$.

2.3 The Algebraic and Order Limit Theorems

Exercise 2.3.1
Let $x_n \geq 0$ for all $n \in \mathbf{N}$.

(a) If $(x_n) \to 0$, show that $\left(\sqrt{x_n}\right) \to 0$.

(b) If $(x_n) \to x$, show that $\left(\sqrt{x_n}\right) \to \sqrt{x}$.

Solution

(a) Given some $\epsilon > 0$, choose N such that $x_n < \epsilon^2$ for all $n \geq N$. Then $\sqrt{x}_n < \epsilon$ for all $n \geq N$.

(b) Fix $\epsilon > 0$; now consider $\delta = \epsilon\sqrt{x}$. For some $N \in \mathbf{N}$, $|x_n - x| < \delta$ for all $n \geq N$, which implies

$$|\sqrt{x}_n - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x}_n + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}} < \frac{\delta}{\sqrt{x}}.$$

From this, we have $|\sqrt{x}_n - \sqrt{x}| < \epsilon$ for all $n \geq N$ as desired.

Exercise 2.3.2
Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

(a) $\left(\frac{2x_n - 1}{3}\right) \to 1$;

(b) $(1/x_n) \to 1/2$.

(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)

Solution

(a) Let $\delta = \frac{3}{2}\epsilon$. Then

$$\begin{aligned} |x_n - 2| < \delta &\implies \frac{2}{3}|x_n - 2| < \epsilon \\ &\implies \left|\frac{2x_n - 4}{3}\right| < \epsilon \\ &\implies \left|\frac{2x_n - 1}{3} - 1\right| < \epsilon \text{ for all } n \geq N, \text{ as desired.} \end{aligned}$$

(b) Note that

$$\left|\frac{1}{x_n} - \frac{1}{2}\right| = \left|\frac{x_n - 2}{2x_n}\right|.$$

Choose N large enough such that $x_n > 1$ and $|x_n - 2| < \epsilon$ for all $n \geq N$. Then

$$\left|\frac{x_n - 2}{2x_n}\right| < \frac{|x_n - 2|}{2} < \frac{\epsilon}{2} < \epsilon.$$

Exercise 2.3.3 (Squeeze Theorem)
Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbf{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Solution
For any $\epsilon > 0$,

$$|y_n - l| \leq |z_n - l| < \epsilon \text{ if } y_n - l \geq 0$$
$$|y_n - l| \leq |x_n - l| < \epsilon \text{ otherwise.}$$

Exercise 2.3.4
Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

(a) $\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$

(b) $\lim\left(\frac{(a_n+2)^2-4}{a_n}\right)$

(c) $\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$

Solution
(a)

$$\begin{aligned}\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right) &= \frac{\lim(1+2a_n)}{\lim(1+3a_n-4a_n^2)} \\ &= \frac{\lim 1 + 2\lim a_n}{\lim 1 + 3\lim a_n - 4\lim a_n^2} \\ &= 1\end{aligned}$$

(b)

$$\begin{aligned}\lim\left(\frac{(a_n+2)^2-4}{a_n}\right) &= \lim\left(\frac{a_n^2+4a_n}{a_n}\right) \\ &= \lim(a_n+4) \\ &= 4\end{aligned}$$

(c)

$$\begin{aligned}\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) &= \lim\left(\frac{2+3a_n}{1+5a_n}\right) \\ &= \frac{2+3\lim a_n}{1+5\lim a_n} \\ &= 2\end{aligned}$$

Exercise 2.3.5
Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots, x_n, y_n, \ldots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Solution
First we prove the forward direction. Let $\lim x_n = \lim y_n = l$, and fix $\epsilon > 0$. Then there exists an $N_0 \in \mathbf{N}$ such that $|x_n - l| < \epsilon$ and $|y_n - l| < \epsilon$ for all $n \geq N_0$. Setting $N = 2N_0 - 1$, however, leaves us with the tail $(x_n, y_n, x_{n+1}, y_{n+1}, \ldots)$, which clearly satisfies $|z_n - l| < \epsilon$.

For the backward direction, fix ϵ. Since $|z_n - l| < \epsilon$ for all $n \geq N$, $|x_n - l| < \epsilon$ and $|y_n - l| < \epsilon$ for all $n \geq \frac{N}{2} + 1$.

Exercise 2.3.6
Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

Solution
REVISE

$$\begin{aligned}
b_n &= n - \sqrt{n^2 - 2n} \\
&= \sqrt{n^2} - \sqrt{n^2 - 2n} \\
&= \frac{2n}{\sqrt{n^2} + \sqrt{n^2 - 2n}} \\
&= \frac{2}{1 + \sqrt{1 - \frac{2}{n}}} \\
\lim b_n &= \frac{2}{\lim 1 + \sqrt{1 - \frac{2}{n}}} \\
&= \frac{2}{1 + \lim\left(\sqrt{1 - \frac{2}{n}}\right)} \\
&= \frac{2}{1 + \sqrt{\lim\left(1 - \frac{2}{n}\right)}} \\
&= \frac{2}{1 + 1} \\
&= 1
\end{aligned}$$

Exercise 2.3.7
Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;

(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;

(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;

(e) two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

Solution

(a) $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$.

(b) This is impossible. By the Algebraic Limit Theorem, $\lim(x_n + y_n) - \lim(x_n) = \lim(y_n)$, and if $x_n + y_n$ and x_n converge, then y_n must have a limit as well.

(c) $b_n = 1/n$. Here, $1/b_n = n$, which diverges.

(d) There exist M_0 and M_1 such that $|b_n| < M_0$ and $|a_n - b_n| < M_1$. Therefore $|a_n - b_n| + |b_n| < M_0 + M_1$, and by the triangle inequality,

$$|a_n - b_n + b_n| < |a_n + b_n| + |b_n| < M_0 + M_1$$

Thus a_n is bounded by $M_0 + M_1$, and an unbounded sequence that meets the criteria is impossible.

(e) $a_n = 0$ and $b_n = (-1)^n$.

Exercise 2.3.8

Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p(x_n) \to p(x)$.

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

Solution

(a) Every polynomial has the form $p(x) = a_k x_n^k + \ldots + a_0$.

$$\begin{aligned} \lim p(x_n) &= \lim a_k x_n^k + \ldots + \lim a_0 \\ &= a_k \lim x_n^k + \ldots + \lim a_0 \\ &= a_k x^k + \ldots + a_0 \\ &= p(x) \end{aligned}$$

(b) $x_n = 1/n$, and let $f(x)$ be a function where if $x \neq 0$, $f(x) = 2$ and $f(x) = 3$ otherwise. Then $(x_n) \to 0$, $f(0) = 3$, but $f(x_n)$ really converges to 2.

Exercise 2.3.9

(a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b ?

(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Solution

(a) We aren't allowed to use the Algebraic Limit Theorem because it assumes that a_n and b_n converge.

Given that $|a_n| < M$ and $(b_n) \to 0$, then $|a_n b_n| < M|b_n| < M\epsilon$.

(b) We can say that $a_n b_n$ is bounded, but we can't say anything about its convergence.

(c) Convergent sequences are also bounded, so b_n satisfies the conditions in part (a).

Exercise 2.3.10

Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.

(a) If $\lim (a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.

(b) If $(b_n) \to b$, then $|b_n| \to |b|$.

(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.

(d) If $(a_n) \to 0$ and $|b_n - b| \le a_n$ for all $n \in \mathbf{N}$, then $(b_n) \to b$.

Solution

(a) This is necessarily true only if a_n and b_n are convergent, by the Algebraic Limit Theorem. Otherwise, consider $a_n = b_n = (-1)^n$.

(b) True, since $||b_n| - |b|| \le |b_n - b| < \epsilon$.

(c) By the Algebraic Limit Theorem, $\lim b_n = \lim(b_n - a_n) + \lim a_n = a$.

(d) True, since $|b_n - b| \le a_n < \epsilon$ for $n \ge N$.

Exercise 2.3.11 (Cesaro Means)

(a) Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

Solution

(a) Fix some $\epsilon > 0$. First, choose N such that $|x_n - x| < \epsilon/2$ for all $n > N$. Let M be the sum of the first N terms of $|x_n - x|$. Now choose some X such that

$$\begin{aligned}
\frac{2M}{\epsilon} - N < X &\implies M - N\epsilon < \frac{X\epsilon}{2} \\
&\implies M + \frac{X\epsilon}{2} < (N + X)\epsilon \\
&\implies |x_1 - x| + \ldots + |x_{N+X} - x| < M + \frac{X\epsilon}{2} < (N + X)\epsilon \\
&\implies |\frac{x_1 + \ldots + x_{N+X}}{N + X} - x| < \epsilon
\end{aligned}$$

Essentially, X is the number of terms $|x_n - x| < \epsilon/2$ that we need to "drag" the mean back under ϵ.

Then it is easy to see that for all $n \geq N + X$, $|y_n - x| < \epsilon$.

(b) $a_n = (-1)^n$, where the sequence of means converges to 0.

Exercise 2.3.12
A typical task in analysis is to decipher whether a property possessed by every term in a convergent sequence is necessarily inherited by the limit. Assume $(a_n) \to a$, and determine the validity of each claim. Try to produce a counterexample for any that are false.

(a) If every a_n is an upper bound for a set B, then a is also an upper bound for B.

(b) If every a_n is in the complement of the interval $(0, 1)$, then a is also in the complement of $(0, 1)$.

(c) If every a_n is rational, then a is rational.

Solution

(a) Since B has at least one upper bound, it is bounded and therefore has a supremum. Call this supremum b. If every a_n is an upper bound for B, $a_n \geq b$ for all $n \in \mathbf{N}$. By the Order Limit Theorem, $\lim a_n = a \geq b$, so a is an upper bound on B.

(b) This is false. Consider $a_n = -\frac{1}{n}$, which is always negative and therefore in the complement of $(0, 1)$ for all finite n, but tends to 0.

(c) This is false. Consider decimal approximations of $\sqrt{2}$ of increasing length: they are all rational, but tend to an irrational number.

Exercise 2.3.13 (Iterated Limits)
Given a doubly indexed array a_{mn} where $m, n \in \mathbf{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?

(a) Let $a_{mn} = m/(m + n)$ and compute the iterated limits

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) \quad \text{and} \quad \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right)$$

Define $\lim_{m,n\to\infty} a_{mn} = a$ to mean that for all $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \epsilon$.

(b) Let $a_{mn} = 1/(m + n)$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exist? How do these three values compare? Answer these same questions for $a_{mn} = mn/\left(m^2 + n^2\right)$.

(c) Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists but where neither iterated limit can be computed.

(d) Assume $\lim_{m,n\to\infty} a_{mn} = a$, and assume that for each fixed $m \in \mathbf{N}$, $\lim_{n\to\infty}(a_{mn}) \to b_m$. Show $\lim_{m\to\infty} b_m = a$.

(e) Prove that if $\lim_{m,n\to\infty} a_{mn}$ exists and the iterated limits both exist, then all three limits must be equal.

Solution

(a)

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) = 1$$

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right) = 0$$

(b) For $a_mn = 1/(m+n)$, all three limits are zero. For $a_mn = mn/(m^2+n^2)$, the combined limit does not exist because $a_mn = \frac{1}{2}$ whenever $M = N$, whereas the iterated limits are zero.

(c) **NOT ORIGINAL**

$$a_mn = \frac{(-1)^n}{m} + \frac{(-1)^m}{n}.$$

(d) Fix $\epsilon > 0$.

We are given two conditions: first, for any m, there exists S_m (dependent on m) such that

$$|a_{mn} - b_m| < \epsilon/2 \text{ for } n \geq S_m.$$

Second, there exists S such that

$$|a_{mn} - a| < \epsilon/2 \text{ for } m, n \geq S.$$

This implies that, for all $m \geq S$, there exists S_m such that the first condition is true; then choose $n \geq \max(S_m, S)$, and we have shown that for any choice of $m \geq S$, there exists an n such that both conditions are true. Thus we can say

$$|b_m - a| \leq |b_m - a_{mn}| + |a_{mn} - a| < \epsilon \text{ for } m \geq S.$$

(e) **REVISE**

This was already shown in (d).

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Exercise 2.4.1

(a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.

Solution

(a) x_n is monotone decreasing; and because $x_n < 4$, x_n is always greater than zero, which means that it is bounded and therefore convergent.

(b) The sequences x_n and x_{n+1} are just the same sequence, only offset by one term.

(c)

$$\begin{aligned} \lim x_{n+1} = \lim \left(\frac{1}{4 - x_n} \right) &\implies s = \frac{1}{4 - s} \\ &\implies s^2 - 4s + 1 = 0 \\ &\implies s = 2 \pm \sqrt{3} \end{aligned}$$

However, s cannot be $2 + \sqrt{3}$ as $2 + \sqrt{3} > 3$, so $s = 2 - \sqrt{3}$.

Exercise 2.4.2

(a) Consider the recursively defined sequence $y_1 = 1$

$$y_{n+1} = 3 - y_n$$

and set $y = \lim y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y, we conclude $\lim y_n = 3/2$. What is wrong with this argument?

(b) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence?

Solution

(a) We don't know that it converges at all. In the previous exercise, we showed that the sequence was convergent before taking the limit.

(b) Yes, because it is bounded above and monotone increasing.

Exercise 2.4.3

(a) Show that

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

converges and find the limit.

(b) Does the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converge? If so, find the limit.

Solution

(a) To show that this sequence converges, we must show that it is bounded above and monotone increasing.

First, it is apparent that $y_2 \geq y_1$. Now, given that $y_{k+1} \geq y_k$, then $2 + y_{k+1} \geq 2 + y_k$, and $\sqrt{2 + y_{k+1}} \geq \sqrt{2 + y_k}$.

Similarly, it is apparent that $y_1 \leq 4$. Now, given that $y_k \leq 4$, it is easy to see that $y_k + 2 \leq 6$ and so $\sqrt{y_k + 2} \leq 4$. Finally, setting $y = \lim y_n$ gives $y = \sqrt{2+y} \implies y = 2$.

(b) The sequence can be rewritten as $2^{\frac{1}{2}}, 2^{\frac{3}{4}}, 2^{\frac{7}{8}} \ldots$, which is clearly monotone increasing and bounded above. Now $y = \sqrt{2y} \implies y = 2$.

Exercise 2.4.4

(a) In Section 1.4 we used the Axiom of Completeness (AoC) to prove the Archimedean Property of $\mathbf{R}$ (Theorem 1.4.2). Show that the Monotone Convergence Theorem can also be used to prove the Archimedean Property without making any use of AoC.

(b) Use the Monotone Convergence Theorem to supply a proof for the Nested Interval Property (Theorem 1.4.1) that doesn't make use of AoC.

These two results suggest that we could have used the Monotone Convergence Theorem in place of AoC as our starting axiom for building a proper theory of the real numbers.

Solution

(a) **NOT ORIGINAL**
Assume, for contradiction, that $\mathbf{N}$ is bounded above. Then the sequence that defines the natural numbers, $x_{n+1} = x_n + 1$, must converge. Taking the limit of both sides gives $0 = 1$, which is clearly false.

(b) Given a sequence of nested intervals, consider the sequence of left-hand endpoints and the sequence of right-hand endpoints. They are monotone increasing and monotone decreasing respectively, and bounded by b_1 and a_1 respectively. Therefore they must converge, and the Order Limit Theorem, states that $a \leq b$. Now there must exist a $c \in \mathbf{R}$ such that $a \leq c \leq b$, and c is in every interval.

Exercise 2.4.5 (Calculating Square Roots)
Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

(a) Show that x_n^2 is always greater than or equal to 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

Solution

(a) It is trivial to see that $x_1^2 \geq 2$. Now,

$$\begin{aligned} x_n^2 \geq 2 &\implies (x_n^2 - 2)^2 \geq 0 \\ &\implies \frac{(x_n^2+2)^2}{4x_n^2} \geq 2 \\ &\implies \frac{1}{4}\left(x_n^2 + 4 + \frac{4}{x_n^2}\right) \geq 2 \\ &\implies x_{n+1}^2 \geq 2 \end{aligned}$$

To see that $x_n - x_{n+1} \geq 0$,

$$x_{n+1} - x_n = \frac{3}{4}x_n^2 - \frac{1}{x_n^2} - 1 \geq 0$$

for all $x_n^2 \geq 2$.

Finally, taking the limit of both sides gives $\frac{1}{2}c = \frac{1}{c} \implies c = \sqrt{2}$.

(b)

$$x_{n+1} = \left(1 - \frac{1}{c}\right)\left(x_n + \frac{1}{x_n(1-\frac{1}{c})}\right)$$

Exercise 2.4.6 (Arithmetic-Geometric Mean)

(a) Explain why $\sqrt{xy} \leq (x+y)/2$ for any two positive real numbers x and y. (The geometric mean is always less than the arithmetic mean.)

(b) Now let $0 \leq x_1 \leq y_1$ and define

$$x_{n+1} = \sqrt{x_n y_n} \quad \text{and} \quad y_{n+1} = \frac{x_n + y_n}{2}$$

Show $\lim x_n$ and $\lim y_n$ both exist and are equal.

Solution

(a)

$$\begin{aligned} \frac{(x-y)^2}{4} \geq 0 &\implies xy \leq \frac{(x+y)^2}{4} \\ &\implies \sqrt{xy} \leq \frac{x+y}{2} \end{aligned}$$

(b) First, (y_n) is monotone decreasing and bounded below by 0. This is because for any n, $x_n \leq y_n$ and so $\frac{x_n+y_n}{2} \leq \frac{2y_n}{2}$, and given that x_n and y_n are both greater than 0, their arithmetic mean must also be greater than 0. (x_n) is monotone increasing and bounded above by y_1. So $y = \lim y_n$ and $x = \lim x_n$ both exist; thus we can take

$$y = \frac{x+y}{2} \implies x = y.$$

Exercise 2.4.7 (Limit Superior)
Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

(b) The limit superior of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Solution

(a) (y_n) is decreasing and bounded because a_n is bounded.

(b) The limit of the sequence $y_n = \inf\{a_k \mid k \geq n\}$. It exists because the sequence is monotone increasing and bounded.

(c) Every term of the sequence of infimums is less than the corresponding term in the sequence of supremums, so by the Order Limit Theorem, $\liminf a_n \leq \limsup a_n$. One example of a sequence for which this inequality is strict is $x_n = (-1)^n$.

(d) $(\Rightarrow)$ If, for some n, both $\sup\{a_k \mid k \geq n\}$ and $\inf\{a_k \mid k \geq n\}$ are within ϵ of some c, then it is clear that a_k is within ϵ of that same c for $k \geq n$. Thus it has been demonstrated that the limit exists and is c.

$(\Leftarrow)$ Fix $\epsilon > 0$. Then there exists some N for which $|x_n - c| < \epsilon/2$ for all $n \geq N$. Now the sequence of infimums, i_n, for $n \geq N$ must be greater than or equal to $c - \epsilon/2$ since it is a lower bound on x_n; similarly, the sequence of supremums, s_n, must be less than or equal to $c + \epsilon/2$. Therefore $|i_n - c| < \epsilon$ and $|s_n - c| < \epsilon$ for all $n \geq N$.

Exercise 2.4.8
For each series, find an explicit formula for the sequence of partial sums and determine if the series converges.

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(c) $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$

(In (c), $\log(x)$ refers to the natural logarithm function from calculus.)

Solution

(a) $S_n = 1 - \frac{1}{2^n}$. This sequence converges to 1, as can be seen by applying the Archimedean Property.

(b) $S_n = 1 - \frac{1}{n+1}$. Again, this sequence converges to 1.

(c) $S_n = \log(n+1)$. This sequence diverges.

Exercise 2.4.9

Complete the proof of Theorem 2.4.6 by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Example 2.4.5 may be a useful reference.

Solution

First, divide the partial sum t_k by 2. This partial sum is still unbounded for all k as if it were bounded by M, then t_k would be bounded by $2M$. So the new partial sum is then

$$t_k = \frac{b_1}{2} + b_2 + 2b_4 + \ldots + 2^{k-1}b_{2^k}.$$

Now consider

$$\begin{aligned} s_{2^k} &= (b_1) + (b_2) + (b_3 + b_4) + \ldots + (b_{2^{k-1}+1} + \ldots + b_{2^k}) \\ &\geq \frac{b_1}{2} + b_2 + 2b_4 + \ldots + 2^{k-1}b_{2^k} \\ &= t_k, \end{aligned}$$

which is unbounded. Therefore the partial sums must also be unbounded, and the sum diverges.

This completes the proof. We have proved that A' implies B'; this is the same as saying that B implies A since there is no way for B to be true without A being true.

Exercise 2.4.10 (Infinite Products)

A close relative of infinite series is the infinite product

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \cdots$$

which is understood in terms of its sequence of partial products

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \cdots b_m$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty} (1 + a_n) = (1 + a_1)(1 + a_2)(1 + a_3) \cdots, \qquad \text{where } a_n \geq 0$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = 1/n^2$ and make a conjecture about the convergence of this sequence.

(b) Show, in general, that the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1 + x \leq 3^x$ for positive x will be useful in one direction.)

Solution

(a) $s_n = n + 1$. The sequence does not converge. For $1/n^2$, the first few terms are

$$\frac{2}{1} \times \frac{5}{4} \times \frac{10}{9} \cdots$$

which seems like it should converge.

(b) ($\Rightarrow$) First, note that

$$\sum_{n=1}^{m} a_n \leq \prod_{n=1}^{m} (1 + a_n)$$

since when the expression on the right is expanded, it will include the expression on the left. Since we are given that the expression on the right is bounded, and because $a_n \geq 0$, the partial sums of (a_n) converge by the Monotone Convergence Theorem.

($\Leftarrow$) It can be seen that

$$\prod_{n=1}^{m} (1 + a_n) \leq \prod_{n=1}^{m} (3^n)$$

for all m. Now the limit of the sequence on the right is given by 3^a where $a = \sum_{n=1}^{\infty} a_n$ which exists and, because the partial products are monotonically increasing, is greater than every partial product $\prod_{n=1}^{m} 3^n$ and by extension every partial product $\prod_{n=1}^{m}(1 + a_n)$. Thus we have shown a bound on the monotonically increasing sequence of partial products for $\prod_{n=1}^{\infty}(1 + a_n)$, and by the Monotone Convergence test it must converge.

2.5 Subsequences and the Bolzano–Weierstrass Theorem

Exercise 2.5.1
Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

(c) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$.

(d) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$, and no subsequences converging to points outside of this set.

Solution

(a) Impossible, since by the Bolzano-Weierstrass Theorem there exists a convergent subsequence of the bounded subsequence.

(b)
$$\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \ldots$$

(c) Consider the sequence
$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots$$
In this way, every rational number between 0 and 1 is represented in order of increasing denominator. Then it is easy to see that any subsequence $s_n = \frac{n}{c(n+1)}$ converges to $\frac{1}{c}$.

(d) This is impossible. A subsequence that converges to 0 can always be constructed from any sequence that satisfies the conditions.

Exercise 2.5.2
Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

(c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

(d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges.

Solution

(a) True, since the proper subsequence that begins with the second element converges, so must the original sequence.

(b) True. If a sequence converges, all subsequences must converge to the same limit; therefore, if a subsequence diverges, then the original sequence must also diverge.

(c) True. Consider the limit superior and the limit inferior, which both must converge since (x_n) is bounded. Although they are not subsequences themselves, two subsequences can be constructed that converge to these two values. From Exercise 1.2.7, if (x_n) is to be divergent, $\limsup a_n \neq \liminf a_n$.

(d) True, since if the subsequence is bounded, so is the original sequence.

Exercise 2.5.3

(a) Prove that if an infinite series converges, then the associative property holds. Assume $a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$ converges to a limit L (i.e., the sequence of partial sums $(s_n) \to L$). Show that any regrouping of the terms

$$(a_1 + a_2 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + (a_{n_2+1} + \cdots + a_{n_3}) + \cdots$$

leads to a series that also converges to L.

(b) Compare this result to the example discussed at the end of Section 2.1 where infinite addition was shown not to be associative. Why doesn't our proof in (a) apply to this example?

Solution

(a) Any regrouping of the terms is just a subsequence of the original sequence, so it must converge to the same limit.

(b) We proved that the original sequence converging implies that every subsequence converges, not the other way around.

Exercise 2.5.4

The Bolzano-Weierstrass Theorem is extremely important, and so is the strategy employed in the proof. To gain some more experience with this technique, assume the Nested Interval Property is true and use it to provide a proof of the Axiom of Completeness. To prevent the argument from being circular, assume also that $(1/2^n) \to 0$. (Why precisely is this last assumption needed to avoid circularity?)

Solution

Suppose that A is a nonempty set on $\mathbf{R}$ that is bounded above by M. First, choose any arbitrary point B that is less than M; divide A into two portions. If the portion greater than B is finite, then it must have a maximum and we have found our supremum; otherwise, we have a new interval $[B, M]$ that we divide in two. If we continue in this fashion (if the top half is empty, proceed with the bottom half), we have an infinite sequence of subsets that we can apply the NIP to; the reason we need the assumption that $(1/2^n) \to 0$ is because we need to show that the length of the intervals converges to zero and therefore there is only one value that must be the supremum.

Exercise 2.5.5
Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbf{R}$. Show that (a_n) must converge to a.

Solution
(a_n) must converge, because if it were to diverge, then by Exercise 2.5.3 there would exist two subsequences that converge to different limits. All that remains to be shown is that (a_n) converges to the same limit as its subsequences. Assume, for contradiction, that the subsequences converge to b, (a_n) converges to a, and $a \neq b$. Then if we choose $\epsilon = \left|\frac{a-b}{2}\right|$, it is impossible to find an element within that same neighborhood for b beyond the same N.

Exercise 2.5.6
Use a similar strategy to the one in Example 2.5.3 to show $\lim b^{1/n}$ exists for all $b \geq 0$ and find the value of the limit. (The results in Exercise 2.3.1 may be assumed.)

Solution
First, note that if $0 < b < 1$, then $b^{1/n}$ is monotone increasing and bounded above by 1; if $b > 1$, then $b^{1/n}$ is monotone decreasing and bounded below by 0. So the limit must exist in both cases, and the limit is trivial when $b = 1$ or $b = 0$. Now $b^{1/2n}$ is a subsequence that can also be expressed as the square root of the original sequence; therefore $l = \sqrt{l}$ and $l = 0$ or $l = 1$. Then it is easy to see that $l = 0$ when $b > 1$ and that $l = 1$ when $0 < b < 1$.

Exercise 2.5.7
Extend the result proved in Example 2.5.3 to the case $|b| < 1$; that is, show $\lim (b^n) = 0$ if and only if $-1 < b < 1$.

Solution
($\Rightarrow$) If $b \leq -1$ or $b > 1$, b^n diverges; if $b = 1$, it converges to 1.

($\Leftarrow$) Consider $|b_n|$ for $-1 < b < 1$; it is just the same sequence as b^n for $0 < b < 1$, and we already shown that it converges to 0. It is then easily seen that, for any $\epsilon > 0$,

$$||b_n| - |0|| < \epsilon \iff |b_n| < \epsilon.$$

Exercise 2.5.8
Another way to prove the Bolzano-Weierstrass Theorem is to show that every sequence contains a monotone subsequence. A useful device in this endeavor is the notion of a peak term. Given a sequence (x_n), a particular term x_m is a peak term if no later term in the sequence exceeds it; i.e., if $x_m \geq x_n$ for all $n \geq m$.

(a) Find examples of sequences with zero, one, and two peak terms. Find an example of a sequence with infinitely many peak terms that is not monotone.

(b) Show that every sequence contains a monotone subsequence and explain how this furnishes a new proof of the Bolzano-Weierstrass Theorem.

Solution
(a) $1/2, 3/4, 7/8 \ldots$ has no peak terms, $1, 1/2, 3/4, 7/8, \ldots$ has one peak term, and $2, 1, 1/2, 3/4, 7/8, \ldots$ has two peak terms. $(-1)^n$ has infinitely many peak terms but is not monotone.

(b) If a bounded sequence has infinite peak terms, then we have already found our monotone subsequence. If it has finite or no peak terms, consider the subsequence starting from the term after the last peak term, x_n, which cannot be a peak term. Then there exists an $n_1 \geq n$ such that $x_{n_1} \geq x_n$, and an $n_2 \geq n_1$ such that $x_{n_2} \geq x_{n_1}$, and so on. In this way we have constructed a monotone increasing subsequence. This subsequence must converge by the Monotone Convergence Theorem, which proves the Bolzano-Weierstrass Theorem.

Exercise 2.5.9
Let (a_n) be a bounded sequence, and define the set

$$S = \{x \in \mathbf{R} : x < a_n \text{ for infinitely many terms } a_n\}$$

Show that there exists a subsequence (a_{n_k}) converging to $s = \sup S$. (This is a direct proof of the Bolzano-Weierstrass Theorem using the Axiom of Completeness.)

Solution
Chooose $\epsilon > 0$. Now consider the ϵ-neighborhood around s. There cannot be an infinite number of terms greater than or equal to $s + \epsilon$ because that would mean s is no longer an upper bound for S. Similarly, there cannot be an infinite number of terms less than or equal to $s - \epsilon$ with only an finite number of terms greater than $\sup S - \epsilon$, since that would preclude $\sup S$ from being the least upper bound for S. Therefore there exists a subsequence of (a_n) that is within ϵ of s for every $\epsilon > 0$. What remains to be shown, however, is that there is a single subsequence that converges to s. To construct this subsequence, take $\epsilon = 1$ and find the first term within this range (which we know exists), then $\epsilon = 1/2, 1/4$, and so forth. We know we can always find a term that comes after the terms we have already used because they are all infinite subsequences.