Understanding Analysis 2e Exercises

Chapter 1

Sequences and Series

1.2 The Limit of a Sequence

Exercise 1.2.1
What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $N \in \mathbf{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$.

Give an example of a vercongent sequence. Is there an example of a vercongent sequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

Solution
A vercongent sequence is just any sequence that is bounded. A vercongent sequence can be divergent; one example would be $a_n = (-1)^n$. It is easily possible for a sequence to verconge to two different values, given that ϵ is chosen to be sufficiently large.

Exercise 1.2.2
Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) $\lim \frac{2n^2}{n^3+3} = 0$.

(c) $\lim \frac{\sin\left(n^2\right)}{\sqrt[3]{n}} = 0$.

Solution
(a) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbf{N}$ with

$$N > \frac{3}{25\epsilon} - \frac{4}{5}.$$

To verify that this choice is appopriate, let $n \in \mathbf{N}$ satisfy $n \geq N$. Then

$$
\begin{aligned}
n > \frac{3}{25\epsilon} - \frac{4}{5} &\implies \epsilon > \frac{\frac{3}{5}}{5n+4} \\
&\implies \epsilon > \left| \frac{\frac{3}{5}}{5n+4} \right| \\
&\implies \epsilon > \left| \frac{2n+1}{5n+4} - \frac{2n+\frac{8}{5}}{5n+4} \right| \\
&\implies \epsilon > \left| \frac{2n+1}{5n+4} - \frac{2}{5} \right|
\end{aligned}
$$

(b) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbf{N}$ with

$$N > \frac{2}{\epsilon}.$$

To verify that this choice of N is appropriate, let $n \in \mathbf{N}$ satisfy $n \geq N$. Then

$$
\begin{aligned}
n > \frac{2}{\epsilon} &\implies \frac{\epsilon}{2} > \frac{n^2}{n^3} > \frac{n^2}{n^3+3} \\
&\implies \epsilon > \left| \frac{2n^2}{n^3+3} \right|
\end{aligned}
$$

(c) Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbf{N}$ with

$$N > \frac{1}{\epsilon^3}.$$

To verify that this choice of N is appropriate, let $n \in \mathbf{N}$ satisfy $n \geq N$. Then

$$n > \frac{1}{\epsilon^3} \implies \epsilon > \frac{1}{\sqrt[3]{n}} > \left| \frac{\sin n^2}{\sqrt[3]{n}} \right|$$

Exercise 1.2.3
Describe what we would have to demonstrate in order to disprove each of the following statements.

(a) At every college in the United States, there is a student who is at least seven feet tall.

(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.

(c) There exists a college in the United States where every student is at least six feet tall.

Solution
(a) There exists one college in the United States where every student is less than seven feet tall.

(b) There exists one college in the United States where every professor gave at least one student a C or below.

(c) For every college in the United States, there is at least one student who is less than six feet tall.

Exercise 1.2.4
Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

(c) A divergent sequence such that for every $n \in \mathbf{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution
(a) $a_n = (-1)^n$.

(b) Assume a sequence with infinite ones has limit $b \neq 1$. If we choose $\epsilon < |b-1|$, then $|a_n - b| < |b-1|$. Since a has infinite ones, regardless of how $N \in \mathbf{N}$ is chosen, $a_n = 1$ for some $n \geq N$. However, this implies $|1-b| < |b-1|$, which is clearly a contradiction. Therefore there cannot be a sequence with infinite ones that does not either diverge or converge to 1 itself.

(c) $1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, \ldots$

Exercise 1.2.5
Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. For each sequence, find $\lim a_n$ and verify it with the definition of convergence.

(a) $a_n = [[5/n]]$,

(b) $a_n = [[(12 + 4n)/3n]]$.

Reflecting on these examples, comment on the statement following Definition 2.2.3 that "the smaller the ϵ-neighborhood, the larger N may have to be."

Solution
(a) Choose $n > 5$. Then $\left|\left[\left[\frac{5}{n}\right]\right]\right|$ always evaluates to 0.

(b) Choose $n > 6$. Then $\left|\left[\left[\frac{12+4n}{3n}\right]\right]\right|$ always evaluates to 1.

This statement is only true if the sequence converges gradually. In some cases, sequences can "jump" to their limit.

Exercise 1.2.6
Theorem 2.2.7 (Uniqueness of Limits). *The limit of a sequence, when it exists, must be unique.*

Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$.

Solution
Assume, for contradiction, that $a \neq b$. Let $a > b$, without loss of generality. Then $|a-b| = a-b > 0$. By definition, there exists N_a, $N_b \in \mathbf{N}$ such that

$$|a_n - b| < \epsilon \text{ for all } n \geq N_b,$$

$$|a_n - a| < \epsilon \text{ for all } n \geq N_a.$$

These two conditions are simultaneously true for $n \geq N$, where $N = \max(N_a, N_b)$. Now set $\epsilon < \frac{a-b}{2}$. This implies

$$b - \frac{a-b}{2} < a_n < b + \frac{a-b}{2} \implies b - \frac{a-b}{2} < a_n < \frac{a+b}{2}$$

$$a - \frac{a-b}{2} < a_n < a + \frac{a-b}{2} \implies \frac{a+b}{2} < a_n < a + \frac{a-b}{2}$$

However, this is impossible, as a_n cannot be greater than $\frac{a+b}{2}$ and less than $\frac{a+b}{2}$ at the same time.

Exercise 1.2.7
Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbf{R}$ if there exists an $N \in \mathbf{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbf{R}$ if, for every $N \in \mathbf{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

(b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2 . Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

Solution

(a) Frequently.

(b) Eventually implies frequently, so eventually is the stronger statement.

(c) Given any ϵ-neighborhood, a_n is eventually in that neighborhood.

(d) (x_n) is not necessarily eventually in $(1.9, 2.1)$, but it is frequently in $(1.9, 2.1)$.

Exercise 1.2.8
For some additional practice with nested quantifiers, consider the following invented definition:

Let's call a sequence (x_n) zero-heavy if there exists $M \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$.

(a) Is the sequence $(0, 1, 0, 1, 0, 1, \ldots)$ zero heavy?

(b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide a counterexample.

(c) If a sequence contains an infinite number of zeros, is it necessarily zeroheavy? If not, provide a counterexample.

(d) Form the logical negation of the above definition. That is, complete the sentence: A sequence is not zero-heavy if ....

Solution

(a) Yes.

(b) Yes. Fix $M \in \mathbf{N}$ and consider the intervals $N = 0, N = M + 1, N = 2M + 2 \ldots$

These all identify unique instances of zero in the sequence because they are disjoint. Since there are an infinite number of these instances, there must also be an infinite number of zeroes.

(c) No. Consider the sequence $0, 1, 0, 1, 1, 0, 1, 1, 1, \ldots$

Regardless of how M is chosen, there will be an M-long sequence of 1s in the sequence, which means that it is not zero-heavy, even though it has an infinite number of zeroes.

(d) A sequence is not zero-heavy if for all $M \in \mathbf{N}$, there exists $N \in \mathbf{N}$ such that for all n satisfying $N \leq n \leq N + M$, $x_n \neq 0$.

1.3 The Algebraic and Order Limit Theorems

Exercise 1.3.1
Let $x_n \geq 0$ for all $n \in \mathbf{N}$.

(a) If $(x_n) \to 0$, show that $\left(\sqrt{x_n}\right) \to 0$.

(b) If $(x_n) \to x$, show that $\left(\sqrt{x_n}\right) \to \sqrt{x}$.

Solution
(a) Given some $\epsilon > 0$, choose N such that $x_n < \epsilon^2$ for all $n \geq N$. Then $\sqrt{x}_n < \epsilon$ for all $n \geq N$.

(b) Fix $\epsilon > 0$; now consider $\delta = \epsilon\sqrt{x}$. For some $N \in \mathbf{N}$, $|x_n - x| < \delta$ for all $n \geq N$, which implies

$$|\sqrt{x}_n - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x}_n + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}} < \frac{\delta}{\sqrt{x}}.$$

From this, we have $|\sqrt{x}_n - \sqrt{x}| < \epsilon$ for all $n \geq N$ as desired.

Exercise 1.3.2
Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

(a) $\left(\frac{2x_n - 1}{3}\right) \to 1$;

(b) $(1/x_n) \to 1/2$.

(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)

Solution
(a) Let $\delta = \frac{3}{2}\epsilon$. Then

$$\begin{aligned}
|x_n - 2| < \delta &\implies \frac{2}{3}|x_n - 2| < \epsilon \\
&\implies \left|\frac{2x_n - 4}{3}\right| < \epsilon \\
&\implies \left|\frac{2x_n - 1}{3} - 1\right| < \epsilon \text{ for all } n \geq N\text{, as desired.}
\end{aligned}$$

(b) Note that

$$\left|\frac{1}{x_n} - \frac{1}{2}\right| = \left|\frac{x_n - 2}{2x_n}\right|.$$

Choose N large enough such that $x_n > 1$ and $|x_n - 2| < \epsilon$ for all $n \geq N$. Then

$$\left|\frac{x_n - 2}{2x_n}\right| < \frac{|x_n - 2|}{2} < \frac{\epsilon}{2} < \epsilon.$$

Exercise 1.3.3 (Squeeze Theorem)
Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbf{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Solution
For any $\epsilon > 0$,

$$|y_n - l| \leq |z_n - l| < \epsilon \text{ if } y_n - l \geq 0$$
$$|y_n - l| \leq |x_n - l| < \epsilon \text{ otherwise.}$$

Exercise 1.3.4
Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

(a) $\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$

(b) $\lim \left(\frac{(a_n+2)^2-4}{a_n}\right)$

(c) $\lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$

Solution
(a)

$$\begin{aligned}\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right) &= \frac{\lim (1+2a_n)}{\lim(1+3a_n-4a_n^2)} \\ &= \frac{\lim 1 + 2\lim a_n}{\lim 1 + 3\lim a_n - 4\lim a_n^2} \\ &= 1\end{aligned}$$

(b)

$$\begin{aligned}\lim \left(\frac{(a_n+2)^2-4}{a_n}\right) &= \lim \left(\frac{a_n^2+4a_n}{a_n}\right) \\ &= \lim (a_n+4) \\ &= 4\end{aligned}$$

(c)

$$\begin{aligned}\lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) &= \lim \left(\frac{2+3a_n}{1+5a_n}\right) \\ &= \frac{2+3\lim a_n}{1+5\lim a_n} \\ &= 2\end{aligned}$$

Exercise 1.3.5
Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots, x_n, y_n, \ldots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Solution

First we prove the forward direction. Let $\lim x_n = \lim y_n = l$, and fix $\epsilon > 0$. Then there exists an $N_0 \in \mathbf{N}$ such that $|x_n - l| < \epsilon$ and $|y_n - l| < \epsilon$ for all $n \geq N_0$. Setting $N = 2N_0 - 1$, however, leaves us with the tail $(x_n, y_n, x_{n+1}, y_{n+1}, \ldots)$, which clearly satisfies $|z_n - l| < \epsilon$.

For the backward direction, fix ϵ. Since $|z_n - l| < \epsilon$ for all $n \geq N$, $|x_n - l| < \epsilon$ and $|y_n - l| < \epsilon$ for all $n \geq \frac{N}{2} + 1$.

Exercise 1.3.6

Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

Solution

REVISE

$$\begin{aligned}
b_n &= n - \sqrt{n^2 - 2n} \\
&= \sqrt{n^2} - \sqrt{n^2 - 2n} \\
&= \frac{2n}{\sqrt{n^2} + \sqrt{n^2 - 2n}} \\
&= \frac{2}{1 + \sqrt{1 - \frac{2}{n}}} \\
\lim b_n &= \frac{2}{\lim 1 + \sqrt{1 - \frac{2}{n}}} \\
&= \frac{2}{1 + \lim\left(\sqrt{1 - \frac{2}{n}}\right)} \\
&= \frac{2}{1 + \sqrt{\lim\left(1 - \frac{2}{n}\right)}} \\
&= \frac{2}{1 + 1} \\
&= 1
\end{aligned}$$

Exercise 1.3.7

Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;

(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;

(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;

(e) two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

Solution

(a) $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$.

(b) This is impossible. By the Algebraic Limit Theorem, $\lim(x_n + y_n) - \lim(x_n) = \lim(y_n)$, and if $x_n + y_n$ and x_n converge, then y_n must have a limit as well.

(c) $b_n = 1/n$. Here, $1/b_n = n$, which diverges.

(d) There exist M_0 and M_1 such that $|b_n| < M_0$ and $|a_n - b_n| < M_1$. Therefore $|a_n - b_n| + |b_n| < M_0 + M_1$, and by the triangle inequality,

$$|a_n - b_n + b_n| < |a_n + b_n| + |b_n| < M_0 + M_1$$

Thus a_n is bounded by $M_0 + M_1$, and an unbounded sequence that meets the criteria is impossible.

(e) $a_n = 0$ and $b_n = (-1)^n$.

Exercise 1.3.8

Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p\,(x_n) \to p(x)$.

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f\,(x_n)$ converges, but not to $f(x)$.

Solution

(a) Every polynomial has the form $p(x) = a_k x_n^k + \ldots + a_0$.

$$\begin{aligned}\lim p(x_n) &= \lim a_k x_n^k + \ldots + \lim a_0 \\ &= a_k \lim x_n^k + \ldots + \lim a_0 \\ &= a_k x^k + \ldots + a_0 \\ &= p(x)\end{aligned}$$

(b) $x_n = 1/n$, and let $f(x)$ be a function where if $x \neq 0$, $f(x) = 2$ and $f(x) = 3$ otherwise. Then $(x_n) \to 0$, $f(0) = 3$, but $f(x_n)$ really converges to 2.

Exercise 1.3.9

(a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim (a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b ?

(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Solution

(a) We aren't allowed to use the Algebraic Limit Theorem because it assumes that a_n and b_n converge.

Given that $|a_n| < M$ and $(b_n) \to 0$, then $|a_n b_n| < M|b_n| < M\epsilon$.

(b) We can say that $a_n b_n$ is bounded, but we can't say anything about its convergence.

(c) Convergent sequences are also bounded, so b_n satisfies the conditions in part (a).

Exercise 1.3.10

Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.

(a) If $\lim (a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.

(b) If $(b_n) \to b$, then $|b_n| \to |b|$.

(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.

(d) If $(a_n) \to 0$ and $|b_n - b| \le a_n$ for all $n \in \mathbf{N}$, then $(b_n) \to b$.

Solution

(a) This is necessarily true only if a_n and b_n are convergent, by the Algebraic Limit Theorem. Otherwise, consider $a_n = b_n = (-1)^n$.

(b) True, since $||b_n| - |b|| \le |b_n - b| < \epsilon$.

(c) By the Algebraic Limit Theorem, $\lim b_n = \lim(b_n - a_n) + \lim a_n = a$.

(d) True, since $|b_n - b| \le a_n < \epsilon$ for $n \ge N$.

Exercise 1.3.11 (Cesaro Means)

(a) Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

Solution

(a) Fix some $\epsilon > 0$. First, choose N such that $|x_n - x| < \epsilon/2$ for all $n > N$. Let M be the sum of the first N terms of $|x_n - x|$. Now choose some X such that

$$\begin{aligned}
\frac{2M}{\epsilon} - N < X &\implies M - N\epsilon < \frac{X\epsilon}{2} \\
&\implies M + \frac{X\epsilon}{2} < (N+X)\epsilon \\
&\implies |x_1 - x| + \ldots + |x_{N+X} - x| < M + \frac{X\epsilon}{2} < (N+X)\epsilon \\
&\implies |\frac{x_1 + \ldots + x_{N+X}}{N+X} - x| < \epsilon
\end{aligned}$$

Essentially, X is the number of terms $|x_n - x| < \epsilon/2$ that we need to "drag" the mean back under ϵ.

Then it is easy to see that for all $n \geq N + X$, $|y_n - x| < \epsilon$.

(b) $a_n = (-1)^n$, where the sequence of means converges to 0.

Exercise 1.3.12

A typical task in analysis is to decipher whether a property possessed by every term in a convergent sequence is necessarily inherited by the limit. Assume $(a_n) \to a$, and determine the validity of each claim. Try to produce a counterexample for any that are false.

(a) If every a_n is an upper bound for a set B, then a is also an upper bound for B.

(b) If every a_n is in the complement of the interval $(0, 1)$, then a is also in the complement of $(0, 1)$.

(c) If every a_n is rational, then a is rational.

Solution

(a) Since B has at least one upper bound, it is bounded and therefore has a supremum. Call this supremum b. If every a_n is an upper bound for B, $a_n \geq b$ for all $n \in \mathbf{N}$. By the Order Limit Theorem, $\lim a_n = a \geq b$, so a is an upper bound on B.

(b) This is false. Consider $a_n = -\frac{1}{n}$, which is always negative and therefore in the complement of $(0, 1)$ for all finite n, but tends to 0.

(c) This is false. Consider decimal approximations of $\sqrt{2}$ of increasing length: they are all rational, but tend to an irrational number.

Exercise 1.3.13 (Iterated Limits)

Given a doubly indexed array a_{mn} where $m, n \in \mathbf{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?

(a) Let $a_{mn} = m/(m + n)$ and compute the iterated limits

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) \quad \text{and} \quad \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right)$$

Define $\lim_{m,n\to\infty} a_{mn} = a$ to mean that for all $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \epsilon$

(b) Let $a_{mn} = 1/(m + n)$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exist? How do these three values compare? Answer these same questions for $a_{mn} = mn/\left(m^2 + n^2\right)$

(c) Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists but where neither iterated limit can be computed.

(d) Assume $\lim_{m,n\to\infty} a_{mn} = a$, and assume that for each fixed $m \in \mathbf{N}$, $\lim_{n\to\infty}(a_{mn}) \to b_m$. Show $\lim_{m\to\infty} b_m = a$

(e) Prove that if $\lim_{m,n\to\infty} a_{mn}$ exists and the iterated limits both exist, then all three limits must be equal.

Solution
TODO